UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

Logarithmic interpolation methods, measure of non-compactness of bilinear operators and function spaces of Lorentz-Sobolev type

(Métodos logarítmicos de interpolación, medida de no compacidad de operadores bilineales y espacios de funciones de tipo Lorentz-Sobolev)

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Blanca Fernández Besoy

Director

Fernando Cobos Díaz

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Memoria para optar al grado de doctor
con mención de Doctorado Internacional
presentada por

Blanca Fernández Besoy

Bajo la dirección del doctor

Fernando Cobos Díaz

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Además, he participado desde enero de 2018 como miembro del equipo en el proyecto de título *Interpolación, Espacios de Funciones y Compacidad de Operadores* de referencia MTM2017-84058-P financiado por el Ministerio de Economía, Industria y Competitividad, y desde mayo de 2020 como miembro del Grupo de Investigación UCM *Interpolación de Operadores y Espacios de Funciones* con referencia UCM-970905.

Entre septiembre y diciembre de 2019, realicé una estancia de investigación en la Universidad Friedrich Schiller de Jena (Alemania) bajo la supervisión de la Profesora D. D. Harsoke. Esta estancia fue subvencionada por una beca FPU para estancias breves. Durante estos tres meses pude trabajar y ampliar mis conocimientos con expertos de referencia internacional como la Profesora D. D. Haroske y el Profesor H. Triebel, además tuve la oportunidad de participar en el Seminario organizado por el grupo “Funktionenräume”.

Fruto del trabajo desarrollado en la tesis son los siguientes artículos:

- Artículos publicados o aceptados.
  - B. F. Besoy. “On compactness theorems for logarithmic interpolation methods”. In: *Proceedings of the Conference Function Spaces XII*. vol. 119. Banach Center Pubication, 2019, pp. 33–45. DOI: 10.4064/bc119-2
  - B. F. Besoy and F. Cobos. “Duality for logarithmic interpolation spaces when $0 < q < 1$ and applications”. In: *J. Math. Anal. Appl.* 466 (2018), pp. 373–399. DOI: 10.1016/j.jmaa.2018.05.082
  - B. F. Besoy and F. Cobos. “Interpolation of the measure of non-compactness of bilinear operators among quasi-Banach spaces”. In: *J. Approx. Theory* 243 (2019), pp. 25–44. DOI: 10.1016/j.jat.2019.03.004
  - B. F. Besoy and F. Cobos. “Logarithmic interpolation methods and measure of non-compactness”. In: *Quart. J. Math.* 71 (2020), pp. 73–95. DOI: 10.1093/qmathj/haz041
  - B. F. Besoy and F. Cobos. “The equivalence theorem for logarithmic interpolation spaces in the quasi-Banach case”. In: *Z. Anal. Anwend.* 40 (2021). To appear
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  - B. F. Besoy, F. Cobos, and L. M. Fernández-Cabrera. “On the structure of a limit class of logarithmic interpolation spaces”. In: *Mediterr. J. Math.* 17:168 (2020). DOI: 10.1007/s00009-020-01602-7
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  - B. F. Besoy, F. Cobos, and H. Triebel. “On function spaces of Lorentz-Sobolev type”. Preprint. 2020

- B. F. Besoy, D. D. Haroske, and H. Triebel. "Traces of some weighted function spaces and related non-standard real interpolation of Besov spaces". Preprint. 2020. URL: https://arxiv.org/abs/2009.03656

- Trabajos en preparación.

  - B. F. Besoy and F. Cobos. "Function spaces of Lorentz-Sobolev type: atomic decompositions, characterization in terms of wavelets and multiplications". In preparation

Contents

Resumen

El hilo conductor y tema central de esta memoria es la Teoría de Interpolación. Sin embargo, como indica su título, podemos diferenciar en ella tres partes: la primera comprende los Capítulos 3-7 y se centra en estudiar los llamados métodos de interpolación logarítmicos. En cuanto a la segunda, consta del Capítulo 8 y se enmarca en la investigación de propiedades de interpolación de operadores bilineales, esta vez por el método real y algunas de sus variantes. Por último, la tercera parte, que comprende los Capítulos 9 y 10, se enfoca en la investigación de los espacios de funciones de tipo Lorentz-Sobolev, en concreto, en los espacios de Besov-Lorentz y Triebel-Lizorkin-Lorentz y el estudio de algunas de sus propiedades a través de distintos resultados de interpolación.

La Teoría de Interpolación es una rama del Análisis Funcional que tiene importantes aplicaciones en Ecuaciones en Derivadas Parciales, Análisis Armónico, Teoría de Aproximación, Espacios de Funciones y Teoría de Operadores, entre otras áreas de las matemáticas. Manuales de referencia en este tema son, por ejemplo, los libros de Bennett y Sharpley [6], Bergh y Löfström [11], Butzer y Berens [23], Brudnyĭ y Krugljak [22], König [84] y Triebel [110].

Esta teoría surge a comienzos del siglo XX con el Teorema de Riesz-Thorin y el Teorema de Marcinkiewicz, en el contexto de los espacios de Lebesgue L_p y los espacios L_p-débiles. En concreto, el Teorema de Riesz-Thorin prueba que si T es un operador linear y continuo de L_{p_0} en L_{q_0} y de L_{p_1} en L_{q_1} con $1 \leq p_0, q_0, p_1, q_1 \leq \infty$, entonces T actúa continuamente de L_p en L_q siendo $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ y $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ con $0 < \theta < 1$. El Teorema de Marcinkiewicz tiene una formulación similar pero los espacios de llegada se sustituyen por los correspondientes espacios L_p-débiles.

Motivados por estos resultados, a partir de los años 60, autores como Lions, Peetre, Aronszajn, Gagliardo, Calderón, Krein y Triebel comenzaron a desarrollar una teoría abstracta de interpolación de espacios de Banach, donde el problema que se trataba de resolver es el siguiente: dados dos pares compatibles de espacios de Banach $\bar{A} = (A_0, A_1)$ y $\bar{B} = (B_0, B_1)$, encontrar espacios de Banach A entre $A_0 \cap A_1$ y $A_0 + A_1$ y B entre $B_0 \cap B_1$ y $B_0 + B_1$, de forma que cualquier operador lineal que sea continuo de A_0 en B_0 y de A_1 en B_1, actúe también continuamente de A en B. De hecho, se desarrollan métodos de interpolación (denotados por $\mathcal{F}$) que aplicados a cualesquiera dos pares compatibles de espacios de Banach $\bar{A}$ y $\bar{B}$ generan espacios de interpolación $\mathcal{F}(\bar{A})$ y $\mathcal{F}(\bar{B})$.

Dentro de los métodos de interpolación destacan el método complejo y el método real. El método de interpolación complejo fue introducido por Calderón en [26] y está motivado por algunas de las ideas desarrolladas en la prueba del Teorema de Riesz-Thorin. En cuanto al método de interpolación real, fue establecido por Lions y Peetre en [89] y está asociado al Teorema de Marcinkiewicz. Aunque el método complejo será de utilidad en el último capítulo de esta memoria (ver los Teoremas 10.10 y 10.21), será el método real y algunas de sus variantes los que jueguen un papel central a lo largo de todo el trabajo.

Dado $\bar{A} = (A_0, A_1)$ un par compatible de espacios de Banach, $0 < \theta < 1$ y $1 \leq q \leq \infty$, el espacio de interpolación real se puede definir mediante el K-funcional de Peetre

$$K(t, a) = K(t, a; A_0, A_1) = \inf\{\|a_0|A_0\| + t\|a_1|A_1\| : a = a_0 + a_1,\ a_j \in A_j,\ j = 0, 1\},\ a \in A_0 + A_1,$$

como la colección de todos los elementos en $A_0 + A_1$ para los que la norma

$$\|a|(A_0, A_1)_{\theta,q}\| = \Big(\int_0^\infty [t^{-\theta} K(t,a)]^q \frac{dt}{t} \Big)^{1/q}$$

es finita. Resulta que esta norma tiene una representación discreta equivalente de la forma

$$\|a|(A_0, A_1)_{\theta,q}\| \sim \|(K(2^k, a))_{k\in\mathbb{Z}} | \ell_q(2^{-k\theta})\| = \Big(\sum_{k=-\infty}^{\infty} [2^{-k\theta} K(2^k, a)]^q \Big)^{1/q}.$$

El K-funcional de Peetre está relacionado con la norma en $A_0 + A_1$. De forma análoga, se define el J-funcional, relacionado esta vez con la norma en $A_0 \cap A_1$, como

$$J(t,a) = J(t,a; A_0, A_1) = \max\{\|a|A_0\|, t\|a|A_1\|\},\ a \in A_0 \cap A_1,\ t > 0.$$

El espacio $(A_0, A_1)_{\theta,q}$ también puede ser descrito por medio del J-funcional como la colección de todos los elementos en $A_0 + A_1$ que admiten una descomposición de la forma $a = \sum_{m=-\infty}^{\infty} u_m$ con $u_m \in A_0 \cap A_1$ tal que $\Big(\sum_{m=-\infty}^{\infty} [2^{-m\theta} J(2^m, u_m)]^q \Big)^{1/q}$ es finito y, además,

$$\|a|(A_0, A_1)_{\theta,q}\| \sim \inf \Big\{ \Big(\sum_{m=-\infty}^{\infty} [2^{-m\theta} J(2^m, u_m)]^q \Big)^{1/q},\ a = \sum_m u_m,\ u_m \in A_0 \cap A_1 \Big\}.$$

En el libro de Butzer y Berens [23] se prueba que el método real también tiene sentido cuando $q = \infty$ y $\theta = 0, 1$. A los espacios $(A_0, A_1)_{0,\infty}$ y $(A_0, A_1)_{1,\infty}$ se los conoce como completados de Gagliardo de A_0 y A_1, respectivamente, y se suelen denotar por $A_0^\sim$ y $A_1^\sim$. De hecho, el método de interpolación real tiene sentido en contextos muy generales, por ejemplo, pares de espacios cuasi-Banach y/o q tomando valores entre 0 e infinito. Por supuesto, en algunos de estos casos el espacio de interpolación obtenido no es Banach sino sólo cuasi-Banach. Un ejemplo notable de espacios obtenidos por interpolación real es el de los espacios de Lorentz $L_{p,q}$, que resultan al interpolar el par (L_{p_0}, L_∞). Concretamente

$$(L_{p_0}, L_\infty)_{\theta,q} = L_{p,q},$$

con $\frac{1}{p} = \frac{1-\theta}{p_0}$, $0 < p_0 < \infty$, $0 < q \le \infty$ y $0 < \theta < 1$.

Una de las generalizaciones más destacadas del método de interpolación real es el método de interpolación real general, descrito en los libros de Peetre [99] y Brudnyĭ y Krugljak [22] y los trabajos de Nilsson [95] y Cwikel y Peetre [51], donde el papel del espacio $\ell_q(2^{-k\theta})$ en la definición en forma discreta del método real se sustituye por un retículo cuasi-Banach de sucesiones cualquiera Γ. Esta construcción es muy general y engloba otras variantes relevantes del método real como el método real con un parámetro funcional que fue estudiado por Gustavson [72] y Persson [103], entre otros autores, y que tiene propiedades muy similares a las del método real clásico. Un caso particular del método real con parámetro funcional son los métodos de interpolación logarítmicos $(A_0, A_1)_{\theta,q,\mathbb{A}}$ cuya cuasinorma viene definida por

$$\|a|(A_0, A_1)_{\theta,q,\mathbb{A}}\| = \Big(\int_0^\infty [t^{-\theta} \ell^{\mathbb{A}}(t) K(t,a)]^q \Big)^{1/q} \sim \Big(\sum_{k=-\infty}^{\infty} [2^{-k\theta} \ell^{\mathbb{A}}(2^k) K(2^k, a)]^q \Big)^{1/q},$$

siendo $0 < \theta < 1$, $0 < q \le \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ y

$$\ell^{\mathbb{A}}(t) = \begin{cases} (1 - \log t)^{\alpha_0} & \text{si} \quad 0 < t < 1, \\ (1 + \log t)^{\alpha_\infty} & \text{si} \quad 1 \le t < \infty. \end{cases}$$

Los espacios de interpolación logarítmicos han sido estudiados en los trabajos de Gustavsson [72], Doktorskii [53], Evans y Opic [56], Evans, Opic y Pick [59] y Cobos y Segurado [48]. Bajo las hipótesis anteriores, el espacio $(A_0, A_1)_{\theta,q,\mathbb{A}}$ puede ser también definido equivalentemente por medio del J-funcional como en el caso del método real, pero ahora la cuasinorma involucrada es

$$\|a|(A_0,A_1)_{\theta,q,\mathbb{A}}\| \sim \inf\Big\{\Big(\sum_{m=-\infty}^{\infty}[2^{-m\theta}\ell^{\mathbb{A}}(2^m)J(2^m,u_m)]^q\Big)^{1/q},\ a=\sum_m u_m,\ u_m\in A_0\cap A_1\Big\}.$$

Cuando aplicamos el método de interpolación logarítmico con parámetros $0<\theta<1$, $0<q\leq\infty$ y $\mathbb{A}\in\mathbb{R}^2$ al par (L_p, L_∞) con $0<p<\infty$, obtenemos el espacio de Lorentz-Zygmund generalizado $L_{\frac{p}{1-\theta},q,\mathbb{A}}$ donde la cuasinorma viene definida por

$$\|f|L_{\frac{p}{1-\theta},q,\mathbb{A}}\| = \Big(\int_0^\infty [t^{\frac{1-\theta}{p}}\ell^{\mathbb{A}}(t)f^*(t)]^q\frac{dt}{t}\Big)^{1/q},$$

siendo f^* la reordenada decreciente de f. Los espacios de Lorentz-Zygmund generalizados fueron introducidos por Opic y Pick en [97] y contienen la célebre escala de los espacios de Lorentz-Zygmund: $L_{p,q}(\mathrm{LogL})_\alpha = L_{p,q,(\alpha,\alpha)}$ (ver [5]). Si $\alpha=0$ obtenemos los espacios de Lorentz $L_{p,q}$.

Evans, Opic y Pick probaron en [59] que también tiene sentido considerar los espacios de interpolación logarítmicos $(A_0,A_1)_{\theta,q,\mathbb{A}}$ si

$$\begin{cases}\theta=0, & \alpha_\infty+\frac{1}{q}<0;\\ \theta=0, & q=\infty,\quad \alpha_\infty=0;\\ \theta=1, & \alpha_0+\frac{1}{q}<0;\\ \theta=1, & q=\infty,\quad \alpha_0=0.\end{cases} \tag{1}$$

En estos caso $(A_0,A_1)_{\theta,q,\mathbb{A}}$ no satisface la definición de espacio de interpolación con parámetro funcional. Cobos y Segurado en [48] estudiaron propiedades de los métodos de interpolación logarítmicos cuando $\theta=0$ ó $\theta=1$, $1\leq q\leq\infty$ y actúan sobre pares de espacios Banach, viendo importantes cambios con respecto al caso $0<\theta<1$, en particular en los teoremas de equivalencia, las fórmulas de dualidad y los resultados de interpolación de operadores compactos. El objetivo de la primera parte del trabajo (Capítulos 3-7) es estudiar las propiedades de estos casos límites de los métodos logarítmicos pero permitiendo ahora que q tome cualquier valor entre 0 e ∞ y, cuando tenga sentido, trabajando con pares de espacios cuasi-Banach en lugar de pares Banach.

En el Capítulo 3 estudiamos el dual de los métodos de interpolación logarítmicos para $\theta=0,1$ y $0<q<1$. Como el dual de un espacio cuasi-Banach puede ser $\{0\}$ trabajamos en esta ocasión con pares de espacios de Banach. Las fórmulas de dualidad de un método de interpolación son una propiedad de gran relevancia debido a sus aplicaciones potenciales. Para el método de interpolación real, Lions y Peetre probaron en su trabajo fundacional [89] que

$$(A_0,A_1)^*_{\theta,q} = (A_0^*,A_1^*)_{\theta,q'} \quad \text{con} \quad \frac{1}{q}+\frac{1}{q'}=1$$

suponiendo que (A_0,A_1) es un par regular de espacios de Banach, es decir, $A_0\cap A_1$ es denso en A_0 y A_1, y que $1\leq q<\infty$. Más adelante, Peetre amplió en [100] este resultado probando que si $0<q<1$ entonces

$$(A_0,A_1)^*_{\theta,q} = (A_0^*,A_1^*)_{\theta,\infty}.$$

En su artículo, Peetre utiliza el concepto de envolvente Banach de un espacio cuasi-Banach que es un espacio de Banach que lo contiene y cuyos duales coinciden. En nuestro caso utilizaremos también esta idea, pero para calcular la envolvente Banach de $(A_0, A_1)_{\theta,q,\mathbb{A}}$ cuando $\theta = 0, 1$ y $0 < q < 1$, primero necesitaremos describir el espacio en términos del J-funcional. Estos resultados, que conforman los Teoremas 3.5, 3.6, 3.8 y 3.9, se basan en algunas ideas de Cobos y Segurado [48] que describieron el dual para $1 \leq q \leq \infty$ y en el trabajo de Nilsson [95] sobre el método real general. En particular, mostramos que bajo ciertas hipótesis sobre los exponentes de los logaritmos

$$(A_0, A_1)_{\theta,q,(\alpha_0,\alpha_\infty)} = (A_0^{\sim}, A_1^{\sim})^J_{\theta,q,(\alpha_0+1/q,\alpha_\infty+1/q)}, \quad \theta = 0, 1, \quad 0 < q < 1,$$

siendo $A_j^{\sim}$ el completado de Gagliardo de A_j descrito anteriormente. Estos resultados se diferencian de los correspondientes al caso $1 \leq q \leq \infty$ en la aparición de los completados de Gagliardo y el desplazamiento de los exponentes de los logaritmos en $1/q$, ya que cuando $1 \leq q \leq \infty$, este desplazamiento es siempre de una unidad independientemente de la q (ver [48]). Mientras que la aparición de los completados de Gagliardo no se refleja en los resultados de dualidad (Teoremas 3.11 y 3.12), la traslacion de los exponentes sí lo hace. Probamos que bajo ciertas hipótesis en los exponentes de los logaritmos

$$(A_0, A_1)^*_{\theta,q,(\alpha_0,\alpha_\infty)} = (A_0^*, A_1^*)_{\theta,\infty,(-\alpha_\infty-1/q,-\alpha_0-1/q)}, \quad \theta = 0, 1, \quad 0 < q < 1.$$

Este resultado difiere con respecto a lo que ocurre cuando $1 \leq q \leq \infty$, donde q no juega ningún papel en los índices del logaritmo del espacio dual, pero sí aparece en el índice de la integral en la forma de q' con $\frac{1}{q} + \frac{1}{q'} = 1$, mientras que en nuestro caso es siempre infinito.

Concluimos el Capítulo 3 con algunas aplicaciones de las fórmulas de dualidad obtenidas. Consideramos $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^n)$ con $1 < p < \infty$, $0 < q \leq \infty$ y $b + 1/q \geq 0$, los espacios de Besov con suavidad logarítmica definidos mediante el módulo de suavidad. Completando los resultados de Cobos y Domínguez en [32], en el Teorema 3.15 describimos el dual de estos espacios cuando $0 < q < 1$, por medio de los espacios de Lipschitz $\mathrm{Lip}^{(1,-\alpha)}_{p,\infty}$ estudiados por Haroske en [73, 74].

Por último consideramos las clases de operadores lineales y continuos actuando entre espacios de Hilbert $S_{\infty,q,b}$ y $S_{\pi,q,b}$, que en particular contienen a los ideales de Macaev $S_{\infty,1}$ y S_π, respectivamente. En los Teoremas 3.19, 3.20 y 3.22, se prueba la relación dual que existe entre estas dos clases. En este caso los resultados son novedosos para el rango completo de parámetros $0 < q \leq \infty$.

Los resultados del Capítulo 3 forman el trabajo conjunto [13] que apareció en el J. Math. Anal. Appl. enviado por el Prof. A. Cianchi.

Los resultados de dualidad probados en el Capítulo 3 sólo pueden ser aplicados a pares regulares de espacios de Banach en los que la intersección es densa en cada uno de los espacios. Este no es el caso del par (L_1, L_∞) ya que $L_1 \cap L_\infty$ no es denso en L_∞. Sin embargo, si aplicásemos formalmente los resultados a este par, obtendríamos las fórmulas correctas para los duales de los espacios de Lorentz-Zygmund generalizados $L_{(p,q,\mathbb{A})}$ que fueron calculadas por Opic y Pick en [97] por métodos directos. El objetivo del Capítulo 4 es clarificar esta coincidencia.

En ese capítulo trabajamos con espacios de funciones medibles denominados espacios de Banach de funciones (ver la Definición 4.5). Si X es un espacio de Banach de funciones se define su espacio asociado X' como el conjunto de todas las funciones medibles g tales que $\int |fg| < \infty$ para todo $f \in X$ y lo dotamos de la norma

$$\|g|X'\| = \sup\left\{ \int |fg| : \|f|X\| \leq 1 \right\}.$$

Esta definición continúa teniendo sentido cuando X es un espacio cuasi-Banach que satisface las propiedades en la Definición 4.5. La noción de espacio asociado y espacio dual están estrechamente relacionadas, de hecho, si X es un espacio de Banach de funciones con una norma absolutamente continua, entonces su dual y su asociado coinciden (ver [6]).

Siguiendo algunas de las ideas de Fernández-Cabrera en [63], en la Sección 4.2 se calcula el asociado de los espacios de interpolación logarítmicos $(X_0, X_1)_{\theta,q,\mathbb{A}}$ cuando $0 \leq \theta \leq 1$, $0 < q \leq \infty$ y (X_0, X_1) es un par de espacios de Banach de funciones de los que al menos uno de ellos tiene norma absolutamente continua. Para ello, primero se prueba en el Lema 4.7, la relación dual que existe entre el asociado de la suma e intersección de espacios y que se refleja en el Teorema 4.8, donde al calcular el espacio asociado de un J-espacio logarítmico se obtiene otro espacio logarítmico definido esta vez por el K-funcional. Por lo tanto, para definir el asociado de un espacio logarítmico, son de nuevo fundamentales los teoremas de equivalencia del Capítulo 3 y algunos nuevos que se prueban en la Sección 4.1. El resultado final se corresponde con el Teorema 4.12 y prueba que bajo ciertas hipótesis sobre los exponentes de los logaritmos

$$(X_0, X_1)'_{\theta,q,(\alpha_0,\alpha_\infty)} = \begin{cases} (X'_0, X'_1)_{\theta,q^*,(-\alpha_\infty,-\alpha_0)} & \text{si} \quad 0 < \theta < 1, \\ (X'_0, X'_1)_{\theta,q^*,(-\alpha_\infty-1/\min\{1,q\},-\alpha_0-1/\min\{1,q\})} & \text{si} \quad \theta = 0, 1, \end{cases}$$

donde (X_0, X_1) es un par de espacios de Banach de funciones, al menos uno de ellos con norma absolutamente continua y $q^* = \begin{cases} \frac{q}{q-1} & \text{si} \quad 1 < q \leq \infty, \\ \infty & \text{si} \quad 0 < q \leq 1, \end{cases}$. Los parámetros en esta fórmula coinciden con los parámeteros en la fórmula de dualidad, pero el resultado para los espacios asociados no requiere a los espacios ser regulares y además es también válido para $q = \infty$.

Finalizamos el Capítulo 4 aplicando las fórmulas de espacios asociados obtenidas en el Teorema 4.12 para calcular el asociado de

$$L_{(p,q,\mathbb{A})}(\Omega) = (L_1(\Omega), L_\infty(\Omega))_{1-1/p,q,\mathbb{A}}(\Omega),$$

siendo (Ω, μ) un espacio de medida σ-finito y no atómico, $1 \leq p \leq \infty$ y $0 < q \leq \infty$. Obtenemos en los Teoremas 4.13, 4.14 y 4.15 las mismas fórmulas que Opic y Pick obtuvieron mediante cálculos directos en [97]. Finalmente, definimos los espacios de sucesiones $\ell_{p,q,\alpha}$ que son la versión discreta de los espacios $L_{(p,q,\mathbb{A})}$ considerando el espacio de medida de los naturales con la medida de contar. En el Teorema 4.16 calculamos el asociado de los espacios $\ell_{p,q,\alpha}$, problema que no fue tratado por Opic and Pick.

Los resultados del Capítulo 4 forman el artículo conjunto [18] que ha aparecido en Ann. Acad. Sci. Fenn. Math.

Según hemos visto en (1), tiene sentido considerar los espacios logarítmicos $(A_0, A_1)_{0,\infty,(\alpha_0,0)}$ y $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$, sin embargo en los resultados de dualidad en el Capítulo 3 y los resultados sobre espacios asociados del Capítulo 4, no se incluye este caso. El motivo es que todos estos resultados se apoyan en la representación del espacio por medio del J-funcional y, aunque aparentemente la J-descripción de estos espacios se dio en [48], la realidad es que los argumentos fallaban en este caso límite.

Esto ha motivado que en el Capítulo 5 desarrollemos el estudio de los espacios $(A_0, A_1)_{0,\infty,(\alpha_0,0)}$ y $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$. De hecho, sólo estudiaremos estos últimos puesto que debido a la simetría de los pesos logarítmicos y del K-funcional se tiene

$$(A_0, A_1)_{0,\infty,(\alpha_0,0)} = (A_1, A_0)_{1,\infty,(0,\alpha_0)}.$$

Comenzamos probando que si $\alpha_\infty \leq 0$ entonces $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ coincide con el completado de Gagliardo $A_1^\sim$, por lo que sólo nos ocupamos del caso $\alpha_\infty > 0$. En estas hipótesis el Teorema 5.5 prueba que $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ admite una representación mediante el J-funcional con una norma mixta

$$\|a|(A_0,A_1)_{1,\infty,(0,\alpha_\infty)}\| \sim \inf\left\{\int_0^1 \frac{J(t,u(t))}{t}\frac{dt}{t} + \sup_{1<t<\infty}\frac{\ell^{\alpha_\infty+1}(t)J(t,u(t))}{t} : a = \int_0^\infty u(t)\frac{dt}{t}\right\}.$$

Aunque este J-espacio no se corresponda con ninguno del tipo logarítmico de nuestra escala, es suficiente para obtener resultados sobre los espacios asociado y dual.

En el Teorema 5.10 se complementan los resultados del Teorema 4.12, probando que si (X_0, X_1) es un par de espacios de Banach de funciones donde al menos uno de ellos tiene norma absolutamente continua y $\alpha_\infty > 0$, entonces

$$((X_0,X_1)_{1,\infty,(0,\alpha_\infty)})' = (X_0', X_1')_{1,1,(-\alpha_\infty-1,\alpha)},$$

para cualquier $\alpha < -1$. Como aplicación se obtiene el espacio asociado del espacio de Lorentz-Zygmund generalizado $L_{(\infty,\infty,(0,\alpha_\infty))}(\Omega)$.

En el Teorema 5.12 se estudia el dual del cierre de $A_0 \cap A_1$ en $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ para cualquier par de Banach regular (A_0, A_1) y finalmente, en la Proposición 5.13 se prueba que el cierre de $A_0 \cap A_1$ en $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ coincide con aquellos elementos de este espacio que verifican

$$t^{-1}\ell^{\alpha_\infty}(t)K(t,a) \xrightarrow{t\to\infty} 0.$$

Los resultados del Capítulo 5 forman el artículo conjunto [19] que ha sido publicado en Mediterr. J. Math.

En el Capítulo 6 estudiamos algunas cuestiones relacionadas con los métodos de interpolación logarítmicos y operadores compactos. El estudio de las propiedades de interpolación de operadores compactos tiene su origen en 1960 con el teorema que probó Kranosel'skiĭ, mostrando que en las hipótesis del Teorema de Riesz-Thorin, si además suponemos que una de las restricciones es compacta, entonces el operador interpolado es también compacto. Esto motivó la investigación sobre las propiedades de interpolación de operadores compactos por métodos de interpolación abstractos. Para el método real, el resultado final se probó en 1992 por Cwikel [50] y Cobos, Kühn y Schonbek [42] para el caso de espacios de Banach, y fue extendido posteriormente por Cobos y Persson [45] al contexto cuasi-Banach. El estudio de las propiedades de interpolación de operadores compactos por el método complejo sin condiciones auxiliares sigue abierto.

Dado un operador lineal y continuo actuando entre espacios cuasi-Banach hay distintas formas de medir cómo de lejos está ese operador de ser compacto. Una de las más relevantes es la medida de no-compacidad: Si A y B son espacios cuasi-Banach y $T : A \longrightarrow B$ es un operador lineal y continuo definimos su medida de no-compacidad ($\beta(T : A \longrightarrow B)$) como el ínfimo de todos los $\sigma > 0$ para los que existe un subconjunto finito $\{b_1, ..., b_s\} \subset B$ tal que

$$T(U_A) \subseteq \bigcup_{k=1}^{s}\{b_k + \sigma U_B\}.$$

Obsérvese que un operador es compacto si, y sólo si, su medida de no compacidad es nula.

Cobos, Fernández-Martínez y Martínez [41] probaron que si (A_0, A_1) y (B_0, B_1) son pares de espacios de Banach, $0 < \theta < 1$, $1 \le q \le \infty$ y T un operador lineal, que es continuo de A_0 en B_0 y de A_1 en B_1, entonces

$$\beta(T : (A_0,A_1)_{\theta,q} \to (B_0,B_1)_{\theta,q}) \le C\beta(T : A_0 \to B_0)^{1-\theta}\beta(T : A_1 \to B_1)^{\theta}.$$

Esto fue extendido posteriormente por Fernández-Martínez [66] al caso de pares de espacios cuasi-Banach y $0 < q \leq \infty$.

En 2014, Edmunds y Opic [56] obtuvieron una variante límite del Teorema de Kranosel'skiĭ involucrando espacios de Lorentz-Zygmund generalizados sobre espacios de medida finita y siempre en el contexto de espacios de Banach. Esto motivó la búsqueda de resultados abstractos de interpolación de operadores compactos por métodos logarítmicos cuando $\theta = 0, 1$ (ver [38, 48]). De hecho como una consecuencia de los resultados en [48] se obtiene una extensión del Teorema de Edmunds y Opic para espacios de Lorentz-Zygmund generalizados sobre espacios de medida σ-finitos.

El objetivo del Capítulo 6 es estudiar la interpolación de la medida de no-compacidad por métodos logarítmicos de interpolación cuando $\theta = 0, 1$ y actúan entre espacios cuasi-Banach, generalizando así lo probado en [38, 39]. Los resultados principales son los Teoremas 6.11 y 6.12 que para $\theta = 0$ prueban lo siguiente:

Teorema 1. *Sean $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ pares compatibles de espacios cuasi-Banach y T un operador lineal continuo de A_0 en B_0 y de A_1 en B_1. Supongamos que $0 < q \leq \infty$ y $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfacen que*

$$\alpha_\infty + 1/q < 0 \leq \alpha_0 + \frac{1}{q} \quad \text{si} \quad 0 < q < \infty \qquad \text{ó} \qquad \alpha_\infty \leq 0 < \alpha_0 \quad \text{si} \quad q = \infty.$$

Entonces,

a) $\beta(T : (A_0, A_1)_{0,q,\mathbb{A}} \to (B_0, B_1)_{0,q,\mathbb{A}}) = 0$ *si* $\beta(T : A_0 \to B_0) = 0$,

b) $\beta(T : (A_0, A_1)_{0,q,\mathbb{A}} \to (B_0, B_1)_{0,q,\mathbb{A}}) \leq C\beta(T : A_0 \to B_0)$ *si* $\beta(T : A_1 \to B_1) = 0$,

c) $\beta(T : (A_0, A_1)_{0,q,\mathbb{A}} \to (B_0, B_1)_{0,q,\mathbb{A}}) \leq C\beta(T : A_0 \to B_0)\Big(1 + \Big(\log \frac{\beta(T:A_1 \to B_1)}{\beta(T:A_0 \to B_0)}\Big)^+\Big)^{\alpha_0^+ - \alpha_\infty}$ *si* $\beta(T : A_j \to B_j) > 0$, $j = 0, 1$.

Observemos que de aquí se deduce que si la primera restricción es compacta entonces el operador interpolado por el método logarítmico con $\theta = 0$ es también compacto. Sin embargo, al contrario de lo que ocurre con el método real, si la compacidad se supone en la segunda restricción no podemos asegurar que el operador interpolado sea compacto, como se muestra en la Observación 6.15.

Finalizamos el capítulo aplicando estos resultados para estimar la medida de no-compacidad de operadores actuando entre espacios de Lorentz-Zygmund generalizados (ver los Teoremas 6.16 y 6.18). En particular, en los Corolarios 6.17 y 6.19, obtenemos la siguiente extensión de los resultados de Edmunds y Opic a espacios cuasi-Banach:

Teorema 2. *Sean (R, μ) y (S, ν) espacios de medida σ-finitos. Tomemos $1 < p_0 < p_1 \leq \infty$, $0 < q_0 < q_1 \leq \infty$, $0 < q < \infty$ y $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ con $\alpha_\infty + 1/q < 0 < \alpha_0 + 1/q$. Sea T un operador lineal tal que*

$$T : L_{p_0}(R) \longrightarrow L_{q_0}(S) \quad \text{es compacto y ,}$$
$$T : L_{p_1}(R) \longrightarrow L_{q_1}(S) \quad \text{es continuo.}$$

Entonces $T : L_{p_0,q,\mathbb{A}+\frac{1}{\min\{p_0,q\}}}(R) \to L_{q_0,q,\mathbb{A}+\frac{1}{\max\{q_0,q\}}}(S)$ es también compacto.

Los resultados del Capítulo 6 forman mi artículo [12] en Banach Center Publ. y el artículo conjunto [15] que ha aparecido en Quart. J. Math.

La representación por medio del J-funcional de los espacios $(A_0, A_1)_{\theta,q,\mathbb{A}}$ juega un papel fundamental en el desarrollo de los Capítulos 3-6. Si $0 < \theta < 1$, entonces $(A_0, A_1)_{\theta,q,\mathbb{A}}$ es equivalente a un espacio de interpolación con parámetro funcional y, por lo tanto,

$$(A_0, A_1)_{\theta,q,\mathbb{A}} = (A_0, A_1)^J_{\theta,q,\mathbb{A}},$$

para cualquier par de espacio cuasi-Banach $\bar{A} = (A_0, A_1)$. Sin embargo, como hemos visto anteriormente los casos $\theta = 0$ y $\theta = 1$ son más interesantes. Nos centraremos en el caso $\theta = 1$ por simplicidad, pero los resultados se pueden extender fácilmente al caso $\theta = 0$. Si $\bar{A} = (A_0, A_1)$ es un par de espacios de Banach, la representación por medio del J-funcional de $(A_0, A_1)_{1,q,\mathbb{A}}$ con $1 \leq q \leq \infty$ fue probada por Cobos y Segurado en [48], el caso $0 < q < 1$ se corresponde con el Teorema 3.5 y el caso límite $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ con el Teorema 5.5.

Lo que nos planteamos en el Capítulo 7 es qué ocurre cuando trabajamos con pares de espacios cuasi-Banach en lugar de espacios de Banach. Como herramienta para la interpolación de la medida de no-compacidad, en el Teorema 6.1 se prueba que si $\bar{A} = (A_0, A_1)$ es un par de espacios cuasi-Banach p-normados y $0 < q \leq \infty$, entonces bajo ciertas hipótesis en los exponentes de los logaritmos se verifica que

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^{\sim}, A_1^{\sim})_{\Lambda;J},$$

donde $\Lambda = (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}$. La pregunta ahora es si bajo las mismas hipótesis el espacio $(A_0, A_1)_{1,q,\mathbb{A}}$ es equivalente a un J-espacio logarítmico para cualquier par de espacios cuasi-Banach p-normados $\bar{A} = (A_0, A_1)$.

La respuesta depende de la relación entre p, q y los exponentes de los logaritmos. Si $0 < q \leq p \leq 1$, el Teorema 7.18 prueba que sí existe esta representación con una traslación de $1/q$ en los exponentes de los logaritmos. Sin embargo, si $0 < p < q$, en la Sección 7.2 se prueba que en algunos casos es imposible encontrar una representación de este tipo. En este caso se buscan los mejores pares de exponentes $\mathbb{M}$ y $\mathbb{B}$ tales que

$$(A_0, A_1)^J_{1,q,\mathbb{B}} \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow (A_0, A_1)^J_{1,q,\mathbb{M}},$$

para cualquier par de espacios cuasi-Banach p-normados. Según los Teoremas 7.9 y 7.10, bajo ciertas hipótesis si $\mathbb{A} = (\alpha_0, \alpha_\infty)$, los mejores exponentes para la segunda inclusión son $\mathbb{M} = (\alpha_0 + \frac{1}{\min\{1,q\}}, \alpha_\infty + \frac{1}{\min\{1,q\}})$ y según el Teorema 7.11 y la Proposición 7.15 los mejores exponentes posibles para la primera inclusión son $\mathbb{B} = (\alpha_0 + 1/p, \alpha_\infty + 1/p)$. Una representación gráfica de la situación completa puede encontrarse en las Figuras 7.1 y 7.2 de la página 134.

Los resultados del Capítulo 7 forman el artículo conjunto [16] que ha sido aceptado para su publicación en Z. Anal. Anwend.

La segunda parte de esta memoria, que comprende el Capítulo 8, estudia la interpolación de la medida de no-compacidad de operadores bilineales. Las propiedades de interpolación de operadores bilineales ya se estudiaron tanto en el trabajo pionero sobre el método real de Lions y Peetre [89] como en el de Calderón [26] para el método complejo. Para el método real el resultado de Lions y Peetre, posteriormente generalizado a espacios cuasi-Banach por Karadzhov [82] y König [83], dice lo siguiente:

Teorema 3. *Sean $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ pares de espacios cuasi-Banach y sea $\bar{E} = (E_0, E_1)$ un par de espacios cuasi-Banach r-normados (para algún $0 < r \leq 1$). Sea $0 < \theta < 1$, $0 < q_0, q_1 \leq \infty$ y $0 < q \leq \infty$ satisfaciendo*

$$\frac{1}{q} = \begin{cases} \frac{1}{q_0} + \frac{1}{q} - \frac{1}{r} & \text{si} \quad q_0, q_1 \geq r; \\ \frac{1}{\max\{q_0, q_1\}} & \text{si} \quad q_0 < r \quad \text{ó} \quad q_1 < r. \end{cases}$$

Si T es un operador bilineal tal que $T : (A_0 + A_1) \times (B_0 + B_1) \longrightarrow (E_0 + E_1)$ es continuo y cada una de las restricciones $T : A_j \times B_j \longrightarrow E_j$ es continua para $j = 0, 1$, entonces

$$T : (A_0, A_1)_{\theta,q_0} \times (B_0, B_1)_{\theta,q_1} \longrightarrow (E_0, E_1)_{\theta,q} \text{ es continuo también.}$$

Recientemente, en los trabajos de Bényi y Torres [10], Bényi y Oh [9] y Hu [79], entre otros, se han mostrado ejemplos de operadores bilineales compactos que aparecen de forma natural en Análisis Armónico. Este hecho motivó el estudio de las propiedades de interpolación de operadores bilineales compactos, algo que Calderón ya había considerado en [26] para el método complejo. En cuanto al método de interpolación real algunos resultados pueden encontrarse en [62, 65, 64, 40]. En el último artículo citado, Cobos, Fernández-Cabrera y Martínez probaron que si en el teorema anterior se supone que al menos una de las restricciones es compacta, entonces el operador bilineal interpolado es también compacto. El siguiente paso natural es buscar resultados cuantitativos relacionados con la medida de no-compacidad del operador. Mastyło y Silva en [93] probaron, entre otras cosas, que

$$\beta(T : \bar{A}_{\theta,q_0} \times \bar{B}_{\theta,q_1} \longrightarrow \bar{E}_{\theta,q}) \leq C\beta(T : A_0 \times B_0 \longrightarrow E_0)^{1-\theta}\beta(T : A_1 \times B_1 \longrightarrow E_1)^{\theta} \tag{2}$$

siendo los pares involucrados espacios de Banach, $1 \leq q_0, q_1 < \infty$, $1 < q < \infty$ y $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1} - 1$. En el Capítulo 8 extendemos la estimación (2) al contexto cuasi-Banach del Teorema 3 (ver el Teorema 8.9), incluyendo en particular los casos $q_0 = \infty$, $q_1 = \infty$, $q = 1$ y $q = \infty$ que no se trataban en [93]. De hecho, durante todo el capítulo se trabaja con el método real general obteniendo en el Teorema 8.7 estimaciones para la medida de no-compacidad en este contexto. En particular, obtenemos también el siguiente resultado para métodos de interpolación con un parámetro funcional (ver el Teorema 8.8):

Teorema 4. *Sean $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$, $0 < q_0, q_1, q \leq \infty$ y sea T como en el Teorema 3. Sean ρ_0, ρ_1, ρ_2 parámetros funcionales tales que para una constante L se verifica*

$$\rho_0(t)\rho_1(s) \leq L\rho_2(ts), \quad t, s > 0.$$

Entonces si $\beta_j = \beta(T : A_j \times B_j \longrightarrow E_j)$, $j = 0, 1$, se tiene:

a) $\beta(T : \bar{A}_{\rho_0,q_0} \times \bar{B}_{\rho_1,q_1} \longrightarrow \bar{E}_{\rho_2,q}) = 0$, *si* $\beta_j = 0$, $j = 0$ *ó* $j = 1$.

b) $\beta(T : \bar{A}_{\rho_0,q_0} \times \bar{B}_{\rho_1,q_1} \longrightarrow \bar{E}_{\rho_2,q}) \leq C\beta_0 s_{\rho_1}(\beta_1/\beta_0)$ *si* $\beta_j > 0$, $j = 0, 1$.

Aquí C es una constante independiente de T y $s_{\rho_1}(t) = \sup\{\rho_1(ts)/\rho_1(s) : s > 0\}$.

Los resultados del Capítulo 8 forman el trabajo conjunto [14] que apareció en J. Approx. Theory comunicado por el Prof. P. Nevai.

La tercera y última parte de esta memoria (Capítulos 9 y 10) se centra en el estudio de espacios de funciones de tipo Lorentz-Sobolev, en particular, espacios de Besov-Lorentz $B^s_q L_{p,r}(\mathbb{R}^n)$ y espacios de Triebel-Lizorkin-Lorentz $F^s_q L_{p,r}(\mathbb{R}^n)$. Estos espacios surgen al remplazar en la definición mediante particiones diádicas de la unidad de los espacios de Besov y Triebel-Lizorkin clásicos los espacios de Lebesgue $L_p(\mathbb{R}^n)$ por espacios de Lorentz $L_{p,r}(\mathbb{R}^n)$ (ver Definición 9.1). Triebel introdujo estos espacios para $1 < p, q < \infty$ y $1 \leq r \leq \infty$ en su libro [110] como herramienta para describir el resultado de interpolar el par $(A^s_{p_0,q}, A^s_{p_1,q})$, $A \in \{B, F\}$ por el método real bajo ciertas hipótesis. Sin embargo, ya desde 1974, distintos tipos de espacios con suavidad Lorentz habían aparecido en la literatura en contextos muy variados. Por ejemplo, los espacios Hardy-Lorentz han sido estudiados por Fefferman, Riviere y Sagher en [60] y Almeida y Caetano [3, 2]; los espacios $F^s_2 L_{p,r}(\mathbb{R}^n)$ aparecen en los artículos de Stein [108], Caetano [25], Cianchi y Pick [28], entre otros; y los espacios de

Triebel-Lizorkin débiles $F^s_q L_{p,\infty}(\mathbb{R}^n)$ y espacios de Besov débiles $B^s_q L_{p,\infty}(\mathbb{R}^n)$ aparecen en el libro de Edmunds y Triebel [57]. Más recientemente, Seeger y Trebels [107] han estudiado las inclusiones entre los espacios $A^s_q L_{p,r}(\mathbb{R}^n)$, $A \in \{B, F\}$ y Hobus y Saal [78] han investigado algunas propiedades de los espacios $F^s_q L_{p,r}(\mathbb{R}^n)$ como herramienta para estudiar las ecuaciones de Navier-Stokes. Nuestro objetivo es continuar con el estudio de diversas propiedades de estos espacios.

En el Capítulo 9 damos una caracterización por medio de wavelets de los espacios $A^s_q L_{p,r}(\mathbb{R}^n)$, $A \in \{B, F\}$ cuando $0 < p < \infty$, $0 < q, r \leq \infty$ y $s \in \mathbb{R}$ (ver el Teorema 9.16). Esta caracterización fue probada por Yang, Cheng y Peng [121] para los espacios $F^s_q L_{p,r}(\mathbb{R}^n)$ pero el resultado es poco accesible al encontrarse este trabajo en chino. Por otro lado, el caso de los espacios de Besov-Lorentz ha sido considerado por Almeida [1, Corollary 3.2], pero sólo en el caso $B^s_q L_{p,q}(\mathbb{R}^n)$. Nuestro resultado es nuevo para los espacios $B^s_q L_{p,r}(\mathbb{R}^n)$ con $r \neq q$. Seguimos una estrategia diferente a la de [121] y [1], basada en un artículo reciente de Haroske, Skandera y Triebel [77], para lo cual comenzamos dando una descomposición en términos de átomos (ver el Teorema 9.11 y la Proposición 9.12).

La descomposición de los espacios $A^s_q L_{p,r}(\mathbb{R}^n)$, $A \in \{B, F\}$ por medio de wavelets establece un isomorfismo entre estos espacios de distribuciones y los espacios de sucesiones $b^s_q L_{p,r}$ y $f^s_q L_{p,r}$, respectivamente, introducidos en la Definición 9.13. Esto hace que el problema de interpolar los espacios $A^s_q L_{p,r}(\mathbb{R}^n)$ se pueda reducir al problema más sencillo de interpolar los espacios de sucesiones correspondientes, lo que nos ha permitido obtener distintas fórmulas de interpolación para los espacios $A^s_q L_{p,r}(\mathbb{R}^n)$, $A \in \{B, F\}$ que quedan recogidas en los Teoremas 9.17, 9.21, 9.22 y 9.23.

Estas fórmulas permiten transferir ciertas propiedades de los espacios de Triebel-Lizorkin clásicos a los espacios $F^s_q L_{p,r}(\mathbb{R}^n)$. En particular, se prueba que $F^0_2 L_{p,r}(\mathbb{R}^n)$ coincide con el espacio de Hardy-Lorentz local $h_{p,r}(\mathbb{R}^n)$ cuando $0 < p, r < \infty$. Además, en el Teorema 9.18 se prueba la siguiente mejora de suavidad debida al semigrupo de Gauss-Weierstrass

$$W_t f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)dy, \quad f \in \mathcal{S}'(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

Teorema 5. *Sea $d \geq 0$, $0 < p < \infty$, $s \in \mathbb{R}$ y $0 < q, r \leq \infty$. Existe una constante $c > 0$ tal que para toda t con $0 < t \leq 1$ y para toda $f \in F^s_q L_{p,r}(\mathbb{R}^n)$,*

$$t^{d/2} \|W_t f | F^{s+d}_q L_{p,r}(\mathbb{R}^n)\| \leq c \|f | F^s_q L_{p,r}(\mathbb{R}^n)\|.$$

Finalizamos el Capítulo 9 probando en el Teorema 9.24 una caracterización de los espacios de Besov-Lorentz $B^s_q L_{p,r}(\mathbb{R}^n)$ como espacios de aproximación por medio de wavelets y del espacio $F^0_2 L_{p,r}(\mathbb{R}^n)$.

El Capítulo 10 tiene por objetivo estudiar la extensión de algunas propiedades de los espacios de Triebel-Lizorkin y espacios de Besov clásicos al contexto de los espacios $A^s_q L_{p,r}(\mathbb{R}^n)$, $A \in \{B, F\}$. En concreto, nos centraremos en algunos problemas clave estudiados en [112, Chapter 4] y que son de interés por sus aplicaciones al campo de las EDPs. Estos problemas son: invarianza de $A^s_q L_{p,r}(\mathbb{R}^n)$ con respecto a difeomorfismos de $\mathbb{R}^n$ en sí mismo, existencia de operadores extensión lineales de $A^s_q L_{p,r}(\mathbb{R}^n_+)$ en $A^s_q L_{p,r}(\mathbb{R}^n)$, multiplicadores puntuales, propiedades de multiplicación de $A^s_q L_{p,r}(\mathbb{R}^n)$ y trazas de estos espacios en hiperplanos.

Los tres primeros problemas mencionados se estudian en la Sección 10.1, en particular, en los Teoremas 10.1, 10.2 y 10.3, respectivamente. La prueba de estos resultados es una aplicación directa de las fórmulas de interpolación obtenidas en el Capítulo 9.

En la Sección 10.2 estudiamos algunas propiedades de multiplicación para los espacios $A^s_q L_{p,r}(\mathbb{R}^n)$, $A \in \{B, F\}$. Aquí las técnicas de interpolación empleadas son más refinadas y requieren del uso de teoremas de interpolación de operadores bilineales, del método de interpolación complejo y algunas caracterizaciones de $B^s_q L_{p,r}(\mathbb{R}^n)$ como espacios de aproximación.

Triebel muestra en sus libros [115, 117] que las propiedades de multiplicación de los espacios de Triebel-Lizorkin clásicos son importantes por sus aplicaciones a las ecuaciones de Navier-Stokes y ecuaciones del calor no lineales. En la Sección 10.2.1 estudiamos las propiedades de la multiplicación en los espacios $F^s_q L_{p,r}(\mathbb{R}^n)$. En particular, en los Teoremas 10.9 y 10.10 se dan condiciones suficientes para que estos espacios sean álgebras con respecto a la multiplicación. El resultado es el siguiente:

Teorema 6. *Si se verifica*

$$\begin{aligned} &0 < p < \infty,\ s > n/p,\ 0 < q \le \infty,\ 0 < r \le \min\{1, q\},\ r < p, \quad \text{ó} \\ &1 < r < p < \infty,\ s > n/p,\ 1 \le q \le \infty, \end{aligned}$$

el espacio $F^s_q L_{p,r}(\mathbb{R}^n)$ es un álgebra con respecto a la multiplicación.

En la Sección 10.2.2 se estudia el caso de los espacios $B^s_q L_{p,r}(\mathbb{R}^n)$. En los Teoremas 10.20 y 10.21 se dan condiciones suficientes para los espacios de Besov-Lorentz sean álgebras multiplicativas. En concreto se prueba:

Teorema 7. *Si se verifica*

$$0 < p < \infty,\ s > n/p,\ 0 < q \le \infty,\ 0 < r < p,$$

entonces $B^s_q L_{p,r}(\mathbb{R}^n)$ es un álgebra con respecto a la multiplicación.

La prueba de este último resultado se basa en la caracterización de $B^s_q L_{p,r}(\mathbb{R}^n)$ como espacio de aproximación (ver el Teorema 10.16), esta vez por medio del espacio $F^0_2 L_{p,r}(\mathbb{R}^n)$ y de funciones enteras de tipo exponencial. Aquí juega un papel clave la equivalencia probada en el Teorema 10.15 que nos permite sustituir el espacio de Lorentz $L_{p,r}(\mathbb{R}^n)$ en la definición de $B^s_q L_{p,r}(\mathbb{R}^n)$ por el espacio de Hardy-Lorentz $h_{p,r}(\mathbb{R}^n)$. Es decir,

$$B^s_q L_{p,r}(\mathbb{R}^n) = B^s_q h_{p,r}(\mathbb{R}^n), \quad s \in \mathbb{R},\ 0 < q \le \infty,\ 0 < p, r < \infty.$$

Finalmente, la Sección 10.3 estudia el problema de la traza en hiperplanos para los espacios con suavidad Lorentz. En el caso de los espacios $F^s_q L_{p,r}(\mathbb{R}^n)$, el Teorema 10.23 muestra que si $s - 1/p > (n-1)(1/p - 1)_+$ entonces

$$\text{tr}\ F^s_q L_{p,r}(\mathbb{R}^n) = (B^{s-1/p_0}_{p_0,p_0}(\mathbb{R}^n), B^{s-1/p_1}_{p_1,p_1}(\mathbb{R}^n))_{\theta,r}, \tag{3}$$

para cualesquiera $0 < p_0 < p < p_1$ tales que $s - 1/p_j > (n-1)(1/p_j - 1)_+$ y $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Cuando $r = p$ el espacio de interpolación anterior coincide con $B^{s-1/p}_{p,p}(\mathbb{R}^n)$, lo que se corresponde al resultado sobre trazas para espacios de Triebel-Lizorkin clásicos. Sin embargo, para $r \neq p$ la descripción del espacio de interpolación (3) es un problema abierto que ya enunció Peetre en su libro [101, p.110]. En el Teorema 10.24 caracterizamos este espacio por medio de wavelets y, con mayor generalidad, en el Teorema 10.25 obtenemos una caracterización por wavelets de los espacios

$$(B^{s-\frac{\alpha}{p_0}}_{p_0,p_0}(\mathbb{R}^n), B^{s-\frac{\alpha}{p_1}}_{p_1,p_1}(\mathbb{R}^n))_{\theta,r}$$

con $\alpha \in \mathbb{R}$ y el resto de parámetros como antes. Esto nos permite probar que, como se esperaba, la traza de $F^s_q L_{p,r}(\mathbb{R}^n)$ sólo depende de s, p y r. Finalmente en el Teorema 10.26 mostramos que bajo ciertas hipótesis tr $B^s_q L_{p,q}(\mathbb{R}^n) = B^{s-1/p}_q L_{p,q}(\mathbb{R}^{n-1})$.

Los resultados de los Capítulos 9 y 10 forman los artículos conjuntos [20, 21] y el artículo en preparación [17].

Chapter 1

Introduction

The guiding theme and main topic of this monograph is Interpolation Theory. However, as it is suggested by the title, we can distinguish three different parts: the first one covers Chapters 3-7 and it focuses on the study of the so-called logarithmic interpolation methods. As for the second one, it consists of Chapter 8 and concentrates on the research of some properties related to the interpolation of bilinear operators, this time by the real method and some of its variants. Finally, the third part, containing Chapters 9 and 10, examines function spaces of Lorentz-Sobolev type, in particular, Besov-Lorentz and Triebel-Lizorkin-Lorentz spaces and it studies some of its properties by means of different interpolation results.

Interpolation Theory is a branch of Functional Analysis with important applications to Partial Differential Equations, Harmonic Analysis, Approximation Theory, Function Spaces and Operators Theory, among other areas in mathematics. Reference sources for the subject are, for example, the books by Bennett and Sharpley [6], Bergh and Löfström [11], Butzer and Berens [23], Brudnyĭ and Krugljak [22], König [84] and Triebel [110].

This theory has its origin at the beginning of the 20th century with the Riesz-Thorin Theorem and the Marcinkiewicz Theorem, in the context of L_p and weak-L_p spaces. In particular, the Riesz-Thorin Theorem proves that if T is a linear and continuous operator from L_{p_0} into L_{q_0} and from L_{p_1} into L_{q_1} with $1 \leq p_0, q_0, p_1, q_1 \leq \infty$, then T is continuous from L_p into L_q being $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ with $0 < \theta < 1$. The Marcinkiewicz Theorem has a similar formulation but the target spaces are replaced by the corresponding weak-L_p spaces.

Encouraged by these results, in the 60's, authors like Lions, Peetre, Aronszajn, Gagliardo, Calderón, Krein and Triebel started to develop the abstract theory of interpolation of Banach spaces. The problem they were trying to solve is the following: given two Banach couples $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$, find Banach spaces A between $A_0 \cap A_1$ and B between $B_0 \cap B_1$ and $B_0 + B_1$ such that any linear and continuous operator from A_0 to B_0 and from A_1 to B_1 is also continuous from A to B. In fact, they develop interpolation methods (denoted by $\mathcal{F}$), that when applied to any two Banach couples $\bar{A}$ and $\bar{B}$ they generate interpolation spaces $\mathcal{F}(\bar{A})$ and $\mathcal{F}(\bar{B})$.

Among interpolation methods, the real and complex methods stand out. The complex interpolation method was introduced by Calderón in [26] and it is motivated by some of the ideas developed in the proof of Riesz-Thorin Theorem. On the other hand, the real method was established by Lions and Peetre in [89] and it is linked to the Marcinkiewicz Theorem. Although the complex method will be useful in the last chapter of this thesis (see Theorems 10.10 and 10.21), the real method and some of its variants will play the main role during the whole report.

Given a Banach couple $\bar{A} = (A_0, A_1)$, $0 < \theta < 1$ and $1 \leq q \leq \infty$, the real interpolation method can be defined by means of the Peetre's K-functional

$$K(t,a) = K(t,a;A_0,A_1) = \inf\{\|a_0|A_0\| + t\|a_1|A_1\| : a = a_0 + a_1,\, a_j \in A_j,\, j = 0,1\},\ a \in A_0 + A_1,$$

as the collection of all elements in $A_0 + A_1$ with finite norm

$$\|a|(A_0,A_1)_{\theta,q}\| = \Big(\int_0^\infty [t^{-\theta}K(t,a)]^q \frac{dt}{t}\Big)^{1/q}.$$

It turns out that this norm admits an equivalent discrete representation as

$$\|a|(A_0,A_1)_{\theta,q}\| \sim \|(K(2^k,a))_{k\in\mathbb{Z}}|\ell_q(2^{-k\theta})\| = \Big(\sum_{k=-\infty}^{\infty}[2^{-k\theta}K(2^k,a)]^q\Big)^{1/q}.$$

Peetre's K-functional is related to the norm of $A_0 + A_1$. Analogously, we can define the J-functional, related to the norm in $A_0 \cap A_1$, as

$$J(t,a) = J(t,a;A_0,A_1) = \max\{\|a|A_0\|, t\|a|A_1\|\}, \quad a\in A_0\cap A_1,\ t>0.$$

The space $(A_0,A_1)_{\theta,q}$ can also be described by means of the J-functional as the collection of all elements in $A_0 + A_1$ that admit a decomposition as $a = \sum_{m=-\infty}^{\infty} u_m$ with $u_m \in A_0\cap A_1$ such that $\Big(\sum_{m=-\infty}^{\infty}[2^{-m\theta}J(2^m,u_m)]^q\Big)^{1/q}$ is finite and, in addition,

$$\|a|(A_0,A_1)_{\theta,q}\| \sim \inf\Big\{\Big(\sum_{m=-\infty}^{\infty}[2^{-m\theta}J(2^m,u_m)]^q\Big)^{1/q},\ a=\sum_m u_m,\ u_m\in A_0\cap A_1\Big\}.$$

In the book by Butzer and Berens [23] it is shown that the real method also makes sense when $q=\infty$ and $\theta = 0,1$. Spaces $(A_0,A_1)_{0,\infty}$ and $(A_0,A_1)_{1,\infty}$ are known as Gagliardo completions of A_0 and A_1, respectively, and they are usually denoted by $A_0^\sim$ and $A_1^\sim$. In fact, the real method makes sense in very general contexts, for example, for quasi-Banach couples and/or the parameter q taking values between 0 and ∞. Certainly, in some of these cases the resulting interpolation space is not Banach but only quasi-Banach. A remarkable example of real interpolation spaces is the family of Lorentz spaces $L_{p,q}$, that appear when interpolating the couple (L_{p_0}, L_∞). More specifically,

$$(L_{p_0},L_\infty)_{\theta,q} = L_{p,q},$$

with $\frac{1}{p} = \frac{1-\theta}{p_0}$, $0<p_0<\infty$, $0<q\le\infty$ and $0<\theta<1$.

One of the most relevant extensions of the real method is the general real method described in the books by Peetre [99] and Brudnyĭ and Krugljak [22] and the papers by Nilsson [95] and Cwikel and Peetre [51], where the sequence space $\ell_q(2^{-k\theta})$ in the discrete definition of the real method is replaced by a more general quasi-Banach lattice Γ. This construction is quite general and it comprehends other relevant variations of the real method such as the real method with a function parameter that was studied by Gustavson [72] and Persson [103], among other authors, and which behaves in a similar way to the classical real method. Particular cases of the real method with a function parameter are logarithmic interpolation methods $(A_0,A_1)_{\theta,q,\mathbb{A}}$ quasi-normed by

$$\|a|(A_0,A_1)_{\theta,q,\mathbb{A}}\| = \Big(\int_0^\infty [t^{-\theta}\ell^{\mathbb{A}}(t)K(t,a)]^q\Big)^{1/q} \sim \Big(\sum_{k=-\infty}^{\infty}[2^{-k\theta}\ell^{\mathbb{A}}(2^k)K(2^k,a)]^q\Big)^{1/q},$$

being $0<\theta<1$, $0<q\le\infty$, $\mathbb{A} = (\alpha_0,\alpha_\infty)\in\mathbb{R}^2$ and

$$\ell^{\mathbb{A}}(t) = \begin{cases}(1-\log t)^{\alpha_0} & \text{if} \quad 0<t<1,\\ (1+\log t)^{\alpha_\infty} & \text{if} \quad 1\le t<\infty.\end{cases}$$

Logarithmic interpolation spaces have been studied in the works by Gustavsson [72], Doktorskii [53], Evans and Opic [56], Evans, Opic and Pick [59] and Cobos and Segurado [48]. Under the previous hypothesis, the space $(A_0, A_1)_{\theta,q,\mathbb{A}}$ can also be defined by means of the J-functional as in the case of the real method, but now the appropriate quasi-norm is

$$\|a|(A_0, A_1)_{\theta,q,\mathbb{A}}\| \sim \inf\Big\{\Big(\sum_{m=-\infty}^{\infty}[2^{-m\theta}\ell^{\mathbb{A}}(2^m)J(2^m, u_m)]^q\Big)^{1/q},\ a = \sum_m u_m,\ u_m \in A_0 \cap A_1\Big\}.$$

When we apply the logarithmic interpolation method with parameters $0 < \theta < 1, 0 < q \le \infty$ and $\mathbb{A} \in \mathbb{R}^2$ to the couple (L_p, L_∞) with $0 < p < \infty$, we obtain the generalized Lorentz-Zygmund space $L_{\frac{p}{1-\theta},q,\mathbb{A}}$ where the quasi-norm is defined by

$$\|f|L_{\frac{p}{1-\theta},q,\mathbb{A}}\| = \Big(\int_0^\infty [t^{\frac{1-\theta}{p}}\ell^{\mathbb{A}}(t)f^*(t)]^q\frac{dt}{t}\Big)^{1/q},$$

being f^* the non-increasing rearrangement of f. Generalized Lorentz-Zygmund spaces were introduced by Opic and Pick in [97] and they contain the well-known scale of Lorentz-Zygmund spaces: $L_{p,q}(\text{LogL})_\alpha = L_{p,q,(\alpha,\alpha)}$ (see [5]). If $\alpha = 0$ we get Lorentz spaces $L_{p,q}$.

Evans, Opic and Pick proved in [59] that it also makes sense to consider the logarithmic interpolation spaces $(A_0, A_1)_{\theta,q,\mathbb{A}}$ when

$$\begin{cases} \theta = 0, & \alpha_\infty + \frac{1}{q} < 0; \\ \theta = 0, & q = \infty, \quad \alpha_\infty = 0; \\ \theta = 1, & \alpha_0 + \frac{1}{q} < 0; \\ \theta = 1, & q = \infty, \quad \alpha_0 = 0. \end{cases} \tag{1.1}$$

In these cases, $(A_0, A_1)_{\theta,q,\mathbb{A}}$ does not satisfy the definition of interpolation space with a function parameter. Cobos and Segurado in [48] studied some properties of logarithmic interpolation methods when $\theta = 0$ or $\theta = 1, 1 \le q \le \infty$ and they act on Banach couples. They found important changes with respect to the case $0 < \theta < 1$ in equivalence theorems, duality formulae and interpolation results for compact operators. The goal of the first part of this thesis (Chapters 3-7) is to study properties of these limit cases for the logarithmic interpolation methods but now allowing the parameter q to take any value between 0 and ∞ and, when it makes sense, working with quasi-Banach couples instead of Banach.

In Chapter 3 we study dual formulae for logarithmic interpolation methods when $\theta = 0, 1$ and $0 < q < 1$. As the dual of a quasi-Banach space can be $\{0\}$ we work here with Banach couples. Duality formulae are an important property of interpolation methods due to its possible applications. In the case of the classical real method, Lions and Peetre proved in their original work [89] that

$$(A_0, A_1)^*_{\theta,q} = (A_0^*, A_1^*)_{\theta,q'} \quad \text{with} \quad \frac{1}{q} + \frac{1}{q'} = 1$$

provided that (A_0, A_1) is a regular Banach couple, that is to say $A_0 \cap A_1$ is dense in both A_0 and A_1, and $1 \le q < \infty$. Later, Peetre extended in [100] this result proving that if $0 < q < 1$, then

$$(A_0, A_1)^*_{\theta,q} = (A_0^*, A_1^*)_{\theta,\infty}.$$

In his paper, Peetre uses the concept of Banach envelope of a quasi-Banach space that is a Banach space that contains it and whose duals coincide. We use this idea in our setting, but in order to compute the Banach envelope of $(A_0, A_1)_{\theta,q,\mathbb{A}}$ when $\theta = 0, 1$ and $0 < q < 1$, we need first to describe the space by means of the J-functional. These results correspond to Theorems 3.5, 3.6, 3.8 and 3.9, and are based on some ideas of Cobos and Segurado [48] that described the dual when $1 \leq q \leq \infty$ and on the work by Nilsson [95] about the general real method. In particular, we show that under some conditions on the logarithmic exponents

$$(A_0, A_1)_{\theta,q,(\alpha_0,\alpha_\infty)} = (A_0^{\sim}, A_1^{\sim})^J_{\theta,q,(\alpha_0+1/q,\alpha_\infty+1/q)}, \quad \theta = 0, 1, \quad 0 < q < 1,$$

being $A_j^{\sim}$ the Gagliargo completion of A_j described before. These results differ from the corresponding to $1 \leq q \leq \infty$ in the appearance of the Gagliardo completions and in the shift of the logarithmic exponents. Here the shift is of $1/q$ while for $1 \leq q \leq \infty$ the exponents are always shifted one unity independently of q (see [48]). While the Gagliardo completions do not appear later on the duality formulae (Theorems 3.11 and 3.12), the shift on logarithmic exponents does. We prove that under certain hypothesis for the logarithmic exponents

$$(A_0, A_1)^*_{\theta,q,(\alpha_0,\alpha_\infty)} = (A_0^*, A_1^*)_{\theta,\infty,(-\alpha_\infty-1/q,-\alpha_0-1/q)}, \quad \theta = 0, 1, \quad 0 < q < 1.$$

This result contrasts with the corresponding one for $1 \leq q \leq \infty$, where the parameter q does not play any role in the logarithmic exponents of the dual space, but it does appear in the integral index as q' with $\frac{1}{q} + \frac{1}{q'} = 1$ while in our case this is always infinity.

We finish Chapter 3 with some applications for the previous duality formulae. We consider $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^n)$ with $1 < p < \infty$, $0 < q \leq \infty$ and $b + 1/q \geq 0$, the Besov spaces with logarithmic smoothness defined by means of the modulus of smoothness. In Theorem 3.15, we give a description for the dual of these spaces when $0 < q < 1$ in terms of the Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,\infty}$ studied by Haroske in [73, 74]. This completes the results of Cobos and Domínguez in [32].

Finally, we consider the spaces of continuous linear operators acting between Hilbert spaces $S_{\infty,q,b}$ and $S_{\pi,q,b}$ that, in particular, contain the Macaev's ideals $S_{\infty,1}$ and S_π, respectively. In Theorems 3.19, 3.20 and 3.22, we prove the dual relationship between these two classes of spaces. These results are new for the whole range of parameters $0 < q \leq \infty$.

The results in Chapter 3 form the joint work [13] that appeared in the J. Math. Anal. Appl., sent by Prof. A. Cianchi.

Duality results obtained in Chapter 3 can only be applied to regular Banach couples where the intersection is dense in both elements of the couple. This is not the case for the couple (L_1, L_∞) since $L_1 \cap L_\infty$ is not dense in L_∞. However, if we disregards this fact and apply duality results to this couple we get the correct formulae for the duals of generalized Lorentz-Zygmund spaces $L_{(p,q,\mathbb{A})}$ that were computed by Opic and Pick in [97] using direct methods.The aim of Chapter 4 is to clarify this coincidence.

In this chapter we work with spaces of measurable functions called Banach function spaces (see Definition 4.5). If X is a Banach function space we define its associate space as the set of all measurable functions g such that $\int |fg| < \infty$ for every $f \in X$, and we equip it with the norm

$$\|g|X'\| = \sup\Big\{ \int |fg| : \|f|X\| \leq 1 \Big\}.$$

This definition still makes sense when X is a quasi-Banach space satisfying the properties in Definition 4.5. The concepts of associate and dual spaces are closely related, in fact, if X is a Banach function space with an absolutely continuous norm, then its dual and associate space coincide (see [6]).

Following some of the ideas that Fernández-Cabrera developed in [63], in Section 4.2 we compute the associate of logarithmic interpolation spaces $(X_0, X_1)_{\theta,q,\mathbb{A}}$ when $0 \leq \theta \leq 1$, $0 < q \leq \infty$ and

(X_0, X_1) is a couple of Banach function spaces where at least one of them has an absolutely continuous norm. For this aim, we first prove in Lemma 4.7 the dual relation between the associate of the sum and intersection of Banach function spaces. This relationship is reflected when computing the associate of a J-logarithmic space since we obtain another logarithmic space but this time defined by means of the K-functional (see Theorem 4.8). Therefore, in order to characterize the associate of a logarithmic space, the equivalence results obtained in Chapter 3 are again essential along with some new ones proved in Section 4.1. The final result corresponds to Theorem 4.12, which prove that under certain hypothesis for the logarithmic exponents

$$(X_0, X_1)'_{\theta,q,(\alpha_0,\alpha_\infty)} = \begin{cases} (X_0', X_1')_{\theta,q^*,(-\alpha_\infty,-\alpha_0)} & \text{if} \quad 0 < \theta < 1, \\ (X_0', X_1')_{\theta,q^*,(-\alpha_\infty-1/\min\{1,q\},-\alpha_0-1/\min\{1,q\})} & \text{if} \quad \theta = 0, 1, \end{cases}$$

where (X_0, X_1) is a couple of Banach function spaces, at least one of them having absolutely continuous norm, and $q^* = \begin{cases} \frac{q}{q-1} & \text{si} \quad 1 < q \le \infty, \\ \infty & \text{si} \quad 0 < q \le 1, \end{cases}$. The parameters in this formula coincide with the parameters in the duality formula, however the result for associate spaces does not require the couple to be regular and, moreover, it is also valid for $q = \infty$.

We close Chapter 4 with the application of formulae for associate spaces obtained in Theorem 4.12 to compute the associate of

$$L_{(p,q,\mathbb{A})}(\Omega) = (L_1(\Omega), L_\infty(\Omega))_{1-1/p,q,\mathbb{A}}(\Omega),$$

being (Ω, μ) a non-atomic, σ-finite measure space, $1 \le p \le \infty$ and $0 < q \le \infty$. In Theorems 4.13, 4.14 and 4.15, we obtain the same formulae that Opic and Pick proved by means of direct calculations. Finally, we introduce the sequence spaces $\ell_{p,q,\alpha}$ that are a discrete version of $L_{(p,q,\mathbb{A})}$ when considering as measure space the natural numbers with the counting measure. In Theorem 4.16, we compute the associate space for $\ell_{p,q,\alpha}$, a problem that was not studied by Opic and Pick.

The results in Chapter 4 form the joint paper [18] that appeared in Ann. Acad. Sci. Fenn. Math.

According to (1.1), it makes sense to consider logarithmic spaces

$$(A_0, A_1)_{0,\infty,(\alpha_0,0)} \quad \text{and} \quad (A_0, A_1)_{1,\infty,(0,\alpha_\infty)},$$

however in the duality results in Chapter 3 and the results on associate spaces in Chapter 4, these cases are not included. The reason is that these results are based on the representation of the spaces by means of the J-functional and, although the J-representation of these spaces was apparently given in [48], the truth is that arguments failed in this limiting case.

This motivated the research in Chapter 5 on spaces $(A_0, A_1)_{0,\infty,(\alpha_0,0)}$ and $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$. In fact, we only study the last ones since due to symmetry on the logarithmic weights and the K-functional we have

$$(A_0, A_1)_{0,\infty,(\alpha_0,0)} = (A_1, A_0)_{1,\infty,(0,\alpha_0)}.$$

We start proving that if $\alpha_\infty \le 0$ then $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ coincides with the Gagliardo completion $A_1^\sim$, therefore we only treat the case $\alpha_\infty > 0$. Under this hypothesis Theorem 5.5 proves that $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ admits a representation by means of the J-functional with a mixed norm

$$\|a|(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}\| \sim \inf\left\{ \int_0^1 \frac{J(t,u(t))}{t}\frac{dt}{t} + \sup_{1<t<\infty} \frac{\ell^{\alpha_\infty+1}(t)J(t,u(t))}{t} : a = \int_0^\infty u(t)\frac{dt}{t} \right\}.$$

Although this J-space is not a logarithmic space in our scale, this representation is enough to get results about the dual and associate spaces.

In Theorem 5.10, we complement the results in Theorem 4.12, proving that if (X_0, X_1) is a couple of Banach function spaces where at least one of them has an absolutely continuous norm and $\alpha_\infty > 0$, then

$$((X_0, X_1)_{1,\infty,(0,\alpha_\infty)})' = (X_0', X_1')_{1,1,(-\alpha_\infty-1,\alpha)},$$

for every $\alpha < -1$. As an application, we obtain the associate space for the generalized Lorentz-Zygmund space $L_{(\infty,\infty,(0,\alpha_\infty))}(\Omega)$.

Theorem 5.12 studies the dual of the closure of $A_0 \cap A_1$ in $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ for any regular Banach couple (A_0, A_1) and finally, in Proposition 5.13 it is shown that the closure of $A_0 \cap A_1$ in $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ coincide with those elements in the space satisfying

$$t^{-1}\ell^{\alpha_\infty}(t)K(t,a) \xrightarrow{t\to\infty} 0.$$

The results in Chapter 5 form the joint article [19] that has been published in Mediterr. J. Math.

In Chapter 6 we study some problems related to logarithmic interpolation methods and compact operators. The research on interpolation properties of compact operators has its origin in 1960 with Kranosel'skiĭ Theorem, proving that under the hypothesis in the Riesz-Thorin Theorem if, in addition, we assume that one of the restrictions is compact, then the interpolated operator is also compact. This motivated the research on interpolation properties of compact operators by abstract interpolation methods. For the real method, the final result was achieved in 1992 by Cwikel [50] and Cobos, Kühn and Schonbek [42] for the Banach case, and it was extended later to quasi-Banach spaces by Cobos and Persson [45]. The study of interpolation properties of compact operators by the complex method without auxiliary conditions is still open.

Given a linear and continuous operator acting between quasi-Banach spaces there are different ways of measuring how far this operator is from being compact. One of the most relevant ones is the measure of non-compactness: If A and B are quasi-Banach spaces and $T : A \longrightarrow B$ is a linear and continuous operator, we define its measure of non-compactness ($\beta(T : A \longrightarrow B)$) as the infimum of all $\sigma > 0$ for which there exists a finite subset $\{b_1, ..., b_s\} \subset B$ such that

$$T(U_A) \subseteq \bigcup_{k=1}^{s} \{b_k + \sigma U_B\}.$$

Notice that a linear and continuous operator is compact if, and only if, its measure of non-compactness is zero.

Cobos, Fernández-Martínez and Martínez [41] proved that if (A_0, A_1) and (B_0, B_1) are Banach couples, $0 < \theta < 1, 1 \le q \le \infty$ and T is a linear operator acting continuously from A_0 into B_0 and from A_1 into B_1, then

$$\beta(T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}) \le C\beta(T : A_0 \to B_0)^{1-\theta}\beta(T : A_1 \to B_1)^{\theta}.$$

This was extended later by Fernández-Martínez [66] to quasi-Banach couples and $0 < q \le \infty$.

In 2014, Edmunds and Opic [56] obtained a limiting variant of Kranosel'skiĭ Theorem involving generalized Lorentz-Zygmund spaces on finite measure spaces and always on the Banach context. This motivated the search for abstract results on interpolation of compact operators by logarithmic methods when $\theta = 0, 1$ (see [38, 48]). In fact, as a consequence of the results in [48], an extension of the Edmunds and Opic's Theorem was obtained for generalized Lorentz-Zygmund spaces on σ-finite measure spaces.

The aim of Chapter 6 is to study the interpolation of the measure of non-compactness by logarithmic methods acting on quasi-Banach couples and $\theta = 0$ or $\theta = 1$, generalizing the results in [38, 39]. For $\theta = 0$, Theorems 6.11 and 6.12 show the following:

Theorem 1.1. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let T be a linear operator that acts continuously from A_0 to B_0 and from A_1 to B_1. Assume that $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfy that

$$\alpha_\infty + 1/q < 0 \leq \alpha_0 + \frac{1}{q} \quad \text{if} \quad 0 < q < \infty \qquad \text{or} \qquad \alpha_\infty \leq 0 < \alpha_0 \quad \text{if} \quad q = \infty.$$

Then,

a) $\beta(T : (A_0, A_1)_{0,q,\mathbb{A}} \to (B_0, B_1)_{0,q,\mathbb{A}}) = 0$ if $\beta(T : A_0 \to B_0) = 0$,

b) $\beta(T : (A_0, A_1)_{0,q,\mathbb{A}} \to (B_0, B_1)_{0,q,\mathbb{A}}) \leq C\beta(T : A_0 \to B_0)$ if $\beta(T : A_1 \to B_1) = 0$,

c) $\beta(T : (A_0, A_1)_{0,q,\mathbb{A}} \to (B_0, B_1)_{0,q,\mathbb{A}}) \leq C\beta(T : A_0 \to B_0)\Big(1 + \Big(\log \frac{\beta(T:A_1 \to B_1)}{\beta(T:A_0 \to B_0)}\Big)^+\Big)^{\alpha_0^+ - \alpha_\infty}$ if $\beta(T : A_j \to B_j) > 0, j = 0, 1$.

We deduce from here that compactness in the first restriction implies that the interpolated operator is also compact. However, contrary to what happens for the real method, if compactness is assumed in the second restriction we cannot guarantee that the interpolated operator is compact, as it is shown in Remark 6.15.

We close Chapter 6 applying these results to estimate the measure of non-compactness of operators acting between generalized Lorentz-Zygmund spaces (see Theorems 6.16 and 6.18). In particular, in Corollaries 6.17 and 6.19, we obtain the following extension of Edmunds and Opic's results to quasi-Banach spaces:

Theorem 1.2. Let (R, μ) and (S, ν) be σ-finite measure spaces. Take $1 < p_0 < p_1 \leq \infty$, $0 < q_0 < q_1 \leq \infty$, $0 < q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ with $\alpha_\infty + 1/q < 0 < \alpha_0 + 1/q$. Let T be a linear operator such that

$$\begin{aligned} T : L_{p_0}(R) &\longrightarrow L_{q_0}(S) \quad \text{is compact and,} \\ T : L_{p_1}(R) &\longrightarrow L_{q_1}(S) \quad \text{is continuous.} \end{aligned}$$

Then $T : L_{p_0,q,\mathbb{A}+\frac{1}{\min\{p_0,q\}}}(R) \to L_{q_0,q,\mathbb{A}+\frac{1}{\max\{q_0,q\}}}(S)$ is also compact.

The results in Chapter 6 form the paper [12] in Banach Center Publ. and the joint work [15] that appeared in Quart. J. Math.

The representation by means of the J-functional of spaces $(A_0, A_1)_{\theta,q,\mathbb{A}}$ plays a main role in Chapters 3-6. If $0 < \theta < 1$, then $(A_0, A_1)_{\theta,q,\mathbb{A}}$ is equivalent to an interpolation space with function parameter and, therefore,

$$(A_0, A_1)_{\theta,q,\mathbb{A}} = (A_0, A_1)^J_{\theta,q,\mathbb{A}}\,,$$

for every quasi-Banach couple $\bar{A} = (A_0, A_1)$. However, as we have seen before the cases $\theta = 0$ and $\theta = 1$ are more interesting. For simplicity, we will focus on the case $\theta = 1$, but the results can be easily extended to $\theta = 0$. If $\bar{A} = (A_0, A_1)$ is a Banach couple, the representation by means of the J-functional of $(A_0, A_1)_{1,q,\mathbb{A}}$ with $1 \leq q \leq \infty$ was proved by Cobos and Segurado in [48], the case $0 < q < 1$ corresponds to Theorem 3.5 and the limiting case $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ corresponds to Theorem 5.5.

In Chapter 7 we investigate what happens when we work with quasi-Banach couples instead of Banach couples. As a tool for the results on the interpolation of the measure of non-compactness, we proved in Theorem 6.1 that if $\bar{A} = (A_0, A_1)$ is a p-normed quasi-Banach couple and $0 < q \leq \infty$, then under certain hypothesis on the logarithmic exponents

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^{\sim}, A_1^{\sim})_{\Lambda;J},$$

where $\Lambda = (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}$. The question now is whether under the same hypothesis the space $(A_0, A_1)_{1,q,\mathbb{A}}$ is equivalent to a logarithmic J-space for any p-normed quasi-Banach couple $\bar{A} = (A_0, A_1)$.

The answer depends on the relationship between the parameters p, q and the logarithmic exponents. If $0 < q \leq p \leq 1$, Theorem 7.18 proves that this representation exists with a shift of $1/q$ in the logarithmic exponents. However, if $0 < p < q$, in Section 7.2 it is shown that for some cases it is impossible to find a representation of this type. In this case, we look for the best possible pairs of exponents $\mathbb{M}$ and $\mathbb{B}$ such that

$$(A_0, A_1)^J_{1,q,\mathbb{B}} \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow (A_0, A_1)^J_{1,q,\mathbb{M}},$$

for any p-normed quasi-Banach couple. According to Theorems 7.9 and 7.10, under certain hypothesis if $\mathbb{A} = (\alpha_0, \alpha_\infty)$ the best possible exponents for the second embedding are $\mathbb{M} = (\alpha_0 + \frac{1}{\min\{1,q\}}, \alpha_\infty + \frac{1}{\min\{1,q\}})$, and according to Theorem 7.11 and Proposition 7.15 the best possible exponents for the fist embedding are $\mathbb{B} = (\alpha_0 + 1/p, \alpha_\infty + 1/p)$. A graphic representation of the whole situation can be found in Figures 7.1 and 7.2 in page 134.

The results in Chapter 7 form the joint paper [16] that has been accepted for its publication in Z. Anal. Anwend.

The second part of this monograph consists of Chapter 8 and studies the interpolation of the measure of non-compactness for bilinear operators. The interpolation properties of bilinear operators were studied in the pioneering work about the real method of Lions and Peetre [89] and the paper by Calderón [26] on the complex method. For the real method, the Lions and Peetre's result, later generalized to quasi-Banach spaces by Karadzhov [82] and König [83], reads as follows:

Theorem 1.3. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let $\bar{E} = (E_0, E_1)$ be an r-normed quasi-Banach couple (with $0 < r \leq 1$). Let $0 < \theta < 1$, $0 < q_0, q_1 \leq \infty$ and $0 < q \leq \infty$ verifying

$$\frac{1}{q} = \begin{cases} \frac{1}{q_0} + \frac{1}{q} - \frac{1}{r} & \text{if} \quad q_0, q_1 \geq r; \\ \frac{1}{\max\{q_0,q_1\}} & \text{if} \quad q_0 < r \quad \text{or} \quad q_1 < r. \end{cases}$$

If T is a bilinear operator such that $T : (A_0 + A_1) \times (B_0 + B_1) \longrightarrow (E_0 + E_1)$ is continuous and the restrictions $T : A_j \times B_j \longrightarrow E_j$ are continuous for $j = 0, 1$, then

$$T : (A_0, A_1)_{\theta,q_0} \times (B_0, B_1)_{\theta,q_1} \longrightarrow (E_0, E_1)_{\theta,q} \text{ is also continuous.}$$

Recently, Bényi and Torres [10], Bényi and Oh [9] and Hu [79], among other authors, have studied examples of bilinear compact operators that appear naturally in Harmonic Analysis. This fact motivated the research on interpolation properties of bilinear compact operators, something that Calderón had already considered in [26] for the complex method. As for the real interpolation method, some results can be found in [62, 65, 64, 40]. In the last cited paper, Cobos, Fernández-Cabrera and Martínez proved that if we assume that one restriction in the previous theorem is compact, then the interpolated bilinear operator is also compact. The next natural step is to search for quantitative results related to the measure of non-compactness. Mastyło and Silva in [93] proved, among other things, that

$$\beta(T : \bar{A}_{\theta,q_0} \times \bar{B}_{\theta,q_1} \longrightarrow \bar{E}_{\theta,q}) \leq C\beta(T : A_0 \times B_0 \longrightarrow E_0)^{1-\theta}\beta(T : A_1 \times B_1 \longrightarrow E_1)^{\theta} \tag{1.2}$$

being the involved spaces Banach, $1 \leq q_0, q_1 < \infty$, $1 < q < \infty$ and $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1} - 1$. In Chapter 8, we extend the estimate (1.2) to the quasi-Banach context of Theorem 1.3 (see Theorem 8.9), including, in particular, the cases $q_0 = \infty$, $q_1 = \infty$, $q = 1$ and $q = \infty$ that are not treated in [93]. In fact, throughout the whole chapter we work with the general real method obtaining in Theorem 8.7 estimates for the measure of non-compactness in this context. In particular, we also get the next result for interpolation methods with a function parameter (see Theorem 8.8):

Theorem 1.4. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$, $0 < q_0, q_1, q \leq \infty$ and T be as in Theorem 1.3. Let ρ_0, ρ_1, ρ_2 be function parameters such that there exists a constant L satisfying

$$\rho_0(t)\rho_1(s) \leq L\rho_2(ts), \quad t, s > 0.$$

Then, if $\beta_j = \beta(T : A_j \times B_j \longrightarrow E_j)$, $j = 0, 1$, we have:

a) $\beta(T : \bar{A}_{\rho_0,q_0} \times \bar{B}_{\rho_1,q_1} \longrightarrow \bar{E}_{\rho_2,q}) = 0$, if $\beta_j = 0$, $j = 0$ ó $j = 1$.

b) $\beta(T : \bar{A}_{\rho_0,q_0} \times \bar{B}_{\rho_1,q_1} \longrightarrow \bar{E}_{\rho_2,q}) \leq C\beta_0 s_{\rho_1}(\beta_1/\beta_0)$ if $\beta_j > 0$, $j = 0, 1$.

Here C is a constant independet of T and $s_{\rho_1}(t) = \sup\{\rho_1(ts)/\rho_1(s) : s > 0\}$.

The results in Chapter 8 form the joint work [14] that appeared in J. Approx. Theory communicated by Prof. P. Nevai.

The third and last part of this thesis (Chapters 9 and 10)) focuses on the study of function spaces of Lorentz-Sobolev type, in particular, Besov-Lorentz spaces $B^s_q L_{p,r}(\mathbb{R}^n)$ and Triebel-Lizorkin-Lorentz spaces $F^s_q L_{p,r}(\mathbb{R}^n)$. These spaces arise when we replace the Lebesgue spaces $L_p(\mathbb{R}^n)$ by Lorentz spaces $L_{p,r}(\mathbb{R}^n)$ in the definition of Besov and Triebel-Lizorkin spaces by means of a dyadic resolution of unity (see Definition 9.1). Triebel introduced these spaces for $1 < p, q < \infty$ and $1 \leq r \leq \infty$ in his book [110] as a tool to describe the result of interpolating the couple $(A^s_{p_0,q}, A^s_{p_1,q})$, $A \in \{B, F\}$ by the real method under certain hypothesis. However, since 1974 different type of spaces with Lorentz smoothness had already appeared in the literature in very different contexts. For example, Hardy-Lorentz spaces have been studied by Fefferman, Riviere and Sagher in [60] and Almeida and Caetano [3, 2]; spaces $F^s_2 L_{p,r}(\mathbb{R}^n)$ appear in the papers by Stein [108], Caetano [25] and Cianchi and Pick [28], among others; and weak Triebel-Lizorkin spaces $F^s_q L_{p,\infty}(\mathbb{R}^n)$ and weak Besov spaces $B^s_q L_{p,\infty}(\mathbb{R}^n)$ appear in the book by Edmunds and Triebel [57]. More recently, Seeger and Trebels [107] studied the embeddings between spaces $A^s_q L_{p,r}(\mathbb{R}^n)$, $A \in \{B, F\}$ and Hobus and Saal [78] investigated some properties of spaces $F^s_q L_{p,r}(\mathbb{R}^n)$ as a tool for the study of Navier-Stokes equations. Our objective is to continue with the study of different properties of these spaces.

In Chapter 9 we give a characterization by means of wavelets for $A^s_q L_{p,r}(\mathbb{R}^n)$, $A \in \{B, F\}$ when $0 < p < \infty$, $0 < q, r \leq \infty$ and $s \in \mathbb{R}$ (see Theorem 9.16). This characterization was proved by Yang, Cheng and Peng [121] for spaces $F^s_q L_{p,r}(\mathbb{R}^n)$ but the result is not very accessible being the paper in Chinese. On the other hand, the case of Besov-Lorentz spaces has been considered by Almeida [1, Corollary 3.2], but only in the case $B^s_q L_{p,q}(\mathbb{R}^n)$. Our result is new for spaces $B^s_q L_{p,r}(\mathbb{R}^n)$ with $r \neq q$. We follow a different strategy from the one in [121] and [1], based on a recent paper by Haroske, Skandera and Triebel [77], and we start giving a decomposition in terms of atoms (see Theorem 9.11 and Proposition 9.12).

The wavelet decomposition of spaces $A^s_q L_{p,r}(\mathbb{R}^n)$, $A \in \{B, F\}$ establishes an isomorphism between these spaces of distributions and the sequence spaces $b^s_q L_{p,r}$ and $f^s_q L_{p,r}$, respectively, introduced in Definition 9.13. This reduces the problem of interpolating spaces $A^s_q L_{p,r}(\mathbb{R}^n)$ to the more simple problem of interpolating the corresponding sequence spaces and allows us to obtain different interpolation formulae for spaces $A^s_q L_{p,r}(\mathbb{R}^n)$, $A \in \{B, F\}$, that are collected in Theorems 9.17, 9.21, 9.22 and 9.23.

These formulae allow us to transfer certain properties from classical Triebel-Lizorkin spaces to spaces $F^s_q L_{p,r}(\mathbb{R}^n)$. In particular, it is proved that $F^0_2 L_{p,r}(\mathbb{R}^n)$ coincides with the local Hardy-Lorentz space $h_{p,r}(\mathbb{R}^n)$ when $0 < p, r < \infty$. Moreover, Theorem 9.18 proves the following smoothness improvement due to the Gauss-Weierstrass semigroup

$$W_t f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)dy, \quad f \in \mathcal{S}'(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

Theorem 1.5. Let $d \geq 0$, $0 < p < \infty$, $s \in \mathbb{R}$ and $0 < q, r \leq \infty$. There exists a constant $c > 0$ such that for every t with $0 < t \leq 1$ and every $f \in F^s_q L_{p,r}(\mathbb{R}^n)$,

$$t^{d/2} \| W_t f | F^{s+d}_q L_{p,r}(\mathbb{R}^n) \| \leq c \| f | F^s_q L_{p,r}(\mathbb{R}^n) \|.$$

We close Chapter 9 giving in Theorem 9.24 a characterization of Besov-Lorentz spaces $B^s_q L_{p,r}(\mathbb{R}^n)$ as approximation spaces by means of wavelets and the space $F^0_2 L_{p,r}(\mathbb{R}^n)$.

The aim of Chapter 10 is to study the extension of some properties of classical Besov and Triebel-Lizorkin spaces to the context of spaces $A^s_q L_{p,r}(\mathbb{R}^n)$, $A \in \{B, F\}$. In particular, we focus on some key problems studied in [112, Chapter 4] and that are interesting because of its applications to PDE's. These problems are: invariance of $A^s_q L_{p,r}(\mathbb{R}^n)$ with respect to a diffeomorphism from $\mathbb{R}^n$ into itself, the existence of linear extension operators from $A^s_q L_{p,r}(\mathbb{R}^n_+)$ in $A^s_q L_{p,r}(\mathbb{R}^n)$, pointwise multipliers, multiplication properties of $A^s_q L_{p,r}(\mathbb{R}^n)$ and traces of these spaces on hyperplanes.

In Section 10.1, we study the first three problems above mentioned, in particular, in Theorems 10.1, 10.2 and 10.3, respectively. The proof of these results is a direct application of interpolation formulae obtained in Chapter 9.

In Section 10.2 we study some multiplication properties for spaces $A^s_q L_{p,r}(\mathbb{R}^n)$, $A \in \{B, F\}$. Here the interpolation techniques employed are more subtle and they require the use of interpolation theorems for bilinear operators, the complex interpolation method and the characterization of $B^s_q L_{p,r}(\mathbb{R}^n)$ as an approximation space.

Triebel shows in his books [115, 117] that the multiplication properties of classical Triebel-Lizorkin spaces are important for their applications to Navier-Stokes equations and non-linear heat equations. In Section 10.2.1, we study multiplication properties for $F^s_q L_{p,r}(\mathbb{R}^n)$ spaces. In particular, in Theorems 10.9 and 10.10 we give sufficient conditions for these spaces to be multiplication algebras. The result reads as follows:

Theorem 1.6. Under the hypothesis

$$0 < p < \infty,\ s > n/p,\ 0 < q \leq \infty,\ 0 < r \leq \min\{1, q\},\ r < p, \quad \text{or}$$
$$1 < r < p < \infty,\ s > n/p,\ 1 \leq q \leq \infty,$$

the space $F^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra.

In Section 10.2.2, we study the case of spaces $B^s_q L_{p,r}(\mathbb{R}^n)$. In Theorems 10.20 and 10.21, we give sufficient conditions for Besov-Lorentz spaces to be multiplication algebras. In particular we prove:

Theorem 1.7. Under the hypothesis

$$0 < p < \infty,\ s > n/p,\ 0 < q \leq \infty,\ 0 < r < p,$$

the space $B^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra.

The proof of this last result is based on the characterization of $B^s_q L_{p,r}(\mathbb{R}^n)$ as an approximation space (see Theorem 10.16), this time by means of the space $F^0_2 L_{p,r}(\mathbb{R}^n)$ and entire functions of exponential type. Here, the equivalence proved in Theorem 10.15 is crucial since it allows us to replace the Lorentz space $L_{p,r}(\mathbb{R}^n)$ in the definition of $B^s_q L_{p,r}(\mathbb{R}^n)$ by the local Hardy-Lorentz space $h_{p,r}(\mathbb{R}^n)$. That is to say,

$$B^s_q L_{p,r}(\mathbb{R}^n) = B^s_q h_{p,r}(\mathbb{R}^n), \quad s \in \mathbb{R},\ 0 < q \leq \infty,\ 0 < p, r < \infty.$$

Finally, in Section 10.3 we study the problem of traces on hyperplanes for Lorentz smoothness spaces. For spaces $F^s_q L_{p,r}(\mathbb{R}^n)$, Theorem 10.23 shows that if $s - 1/p > (n-1)(1/p - 1)_+$ then

$$\text{tr}\ F^s_q L_{p,r}(\mathbb{R}^n) = (B^{s-1/p_0}_{p_0,p_0}(\mathbb{R}^n), B^{s-1/p_1}_{p_1,p_1}(\mathbb{R}^n))_{\theta,r}, \tag{1.3}$$

for any $0 < p_0 < p < p_1$ such that $s - 1/p_j > (n-1)(1/p_j - 1)_+$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. For $r = p$ the last interpolation space coincides with $B^{s-1/p}_{p,p}(\mathbb{R}^n)$, and the previous result corresponds with the already known result for classical Triebel-Lizorkin spaces. However, for $r \neq p$ the description of the interpolation space (1.3) is an open problem already stated by Peetre in [101, p.110]. In Theorem 10.24 we characterize this space by means of wavelets and, with more generality, in Theorem 10.25 we obtain a characterization by means of wavelets for the space

$$(B^{s-\frac{\alpha}{p_0}}_{p_0,p_0}(\mathbb{R}^n), B^{s-\frac{\alpha}{p_1}}_{p_1,p_1}(\mathbb{R}^n))_{\theta,r}$$

with $\alpha \in \mathbb{R}$ and the other parameters as before. This allows us to prove that, as expected, the trace of $F^s_q L_{p,r}(\mathbb{R}^n)$ only depend on s, p and r. Finally, in Theorem 10.26 we show that under certain hypothesis $\text{tr}\ B^s_q L_{p,q}(\mathbb{R}^n) = B^{s-1/p}_q L_{p,q}(\mathbb{R}^{n-1})$.

The results in Chapters 9 and 10 form the joint articles [20, 21] and the paper in preparation [17].

Chapter 2

Preliminaries

This chapter aims to provide an introduction to the main notations and concepts that are going to be used subsequently. We start by recalling some properties of quasi-Banach spaces.

Let X be a linear space over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), we call the function $\|\cdot|X\| : X \longrightarrow [0,\infty)$ a *quasi-norm* if the following properties hold:

(1) $\|x|X\| = 0$ if, and only if, $x = 0$.

(2) $\|\lambda x|X\| = |\lambda| \cdot \|x|X\|$ for every scalar λ and every $x \in X$.

(3) There exists a constant $c_X \geq 1$ such that $\|x+y|X\| \leq c_X(\|x|X\| + \|y|X\|)$ for every $x, y \in X$.

We say that $(X, \|\cdot|X\|)$ or simply X is a *quasi-normed space* (or a *normed space* if $c_X = 1$). If a quasi-normed space is complete, we say that it is a *quasi-Banach space*.

On the other hand, for $0 < p \leq 1$ we call the function $\|\cdot|X\| : X \longrightarrow [0,\infty)$ a *p-norm* if it satisfies properties (1), (2) and

(4) $\|x+y|X\|^p \leq \|x|X\|^p + \|y|X\|^p$, for any $x, y \in X$.

We say now that X is a *p-normed space*. Note that any p-norm is also an r-norm for any $0 < r \leq p$.

It is straightforward that any p-normed space X is quasi-normed with $c_X = 2^{1/p-1}$. In the other direction, the Aoki-Rolewicz theorem states that if $(X, \|\cdot|X\|)$ is a quasi-normed space with constant $c_X \geq 1$ in the quasi-triangle inequality, there is another quasi-norm on X equivalent to $\|\cdot|X\|$ which is a p-norm with $\frac{1}{p} = 1 + \log_2 c_X$ (see, for example, [11, Lemma 3.10.1]).

Given two quasi-Banach spaces X and Y we denote by $\mathcal{L}(X,Y)$ (or simply $\mathcal{L}(X)$ if $X = Y$) the space of continuous and linear operators from X to Y. As in the case of Banach spaces, a linear operator $T : X \longrightarrow Y$ is continuous if, and only if, it is bounded

$$\|T|\mathcal{L}(X,Y)\| := \sup\{\|Tx|Y\| : \|x|X\| \leq 1\} < \infty.$$

Furthermore, $(\mathcal{L}(X,Y), \|\cdot|\mathcal{L}(X,Y)\|)$ is also a quasi-Banach space with constant c_Y in the quasi-triangle inequality. In particular, if $(Y, \|\cdot|Y\|)$ is a Banach space, then $\mathcal{L}(X,Y)$ is a Banach space. This implies that the dual space $X^* := \mathcal{L}(X,\mathbb{K})$ of any quasi-Banach space X is always a Banach space.

Analogously, given three quasi-Banach spaces X, Y and Z we put $\mathcal{B}(X \times Y, Z)$ for the set of all continuous and bilinear operators from $X \times Y$ in Z. A bilinear operator $T : X \times Y \longrightarrow Z$ is continuous if, and only if, it is bounded

$$\|T|\mathcal{B}(X \times Y, Z)\| := \sup\{\|T(x,y)|Z\| : \|x|X\| \leq 1,\ \|y|Y\| \leq 1\} < \infty.$$

From now on, we follow the usual notation concerning the symbols $\lesssim$ and $\sim$: If U and V are quantities depending on certain parameters, we put $U \lesssim V$ if $U \leq cV$ with a constant c independent of the significant parameters. We put $U \sim V$ if $U \lesssim V$ and $V \lesssim U$.

2.1 Interpolation methods

By a *quasi-Banach couple* $\bar{A} = (A_0, A_1)$ we mean two quasi-Banach spaces A_0, A_1 which are continuously embedded in the same Hausdorff topological vector space $\mathcal{A}$. If A_0 and A_1 are p-normed quasi-Banach spaces for some $0 < p \leq 1$, we say that $\bar{A} = (A_0, A_1)$ is a *p-normed quasi-Banach couple*. It follows, from the properties mentioned before, that any quasi-Banach couple is a p-normed quasi-Banach couple for a convenient $0 < p \leq 1$. When A_0 and A_1 are Banach spaces we simply say that $\bar{A} = (A_0, A_1)$ is a *Banach couple*.

If $\bar{A} = (A_0, A_1)$ is a quasi-Banach couple, then it makes sense to consider $A_0 \cap A_1$ and $A_0 + A_1$ which are also quasi-Banach spaces with the following quasi-norms:

$$\|a|A_0 \cap A_1\| = \max\{\|a|A_0\|, \|a|A_1\|\}, \quad a \in A_0 \cap A_1, \tag{2.1}$$

$$\|a|A_0 + A_1\| = \inf\{\|a_0|A_0\| + \|a_1|A_1\| : a = a_0 + a_1, a_j \in A_j, j = 0, 1\}, \quad a \in A_0 + A_1. \tag{2.2}$$

If $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ are quasi-Banach couples, we write $T \in \mathcal{L}(\bar{A}, \bar{B})$ (or simply $\mathcal{L}(\bar{A})$ if $\bar{A} = \bar{B}$) if T is a linear operator, $T : A_0 + A_1 \longrightarrow B_0 + B_1$, with continuous restrictions $T : A_j \longrightarrow B_j$, $j = 0, 1$. We say that a quasi-Banach space A is an *intermediate space* with respect to the quasi-Banach couple $\bar{A}$ if

$$A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1$$

where $\hookrightarrow$ denotes a continuous inclusion. If, in addition, for any $T \in \mathcal{L}(\bar{A})$ the restriction $T : A \longrightarrow A$ is continuous, we say that A is an *interpolation space* with respect to $\bar{A}$. More generally, let $\bar{A}$ and $\bar{B}$ two quasi-Banach couples, we say that two quasi-Banach spaces A and B are *interpolation spaces* with respect to $\bar{A}$ and $\bar{B}$ if A and B are intermediate spaces associated to $\bar{A}$ and $\bar{B}$, respectively, and if $T \in \mathcal{L}(\bar{A}, \bar{B})$ implies $T \in \mathcal{L}(A, B)$.

We say that $\mathcal{F}$ is an *interpolation method* if $\mathcal{F}$ is a procedure that associates to each quasi-Banach couple $\bar{A}$ an intermediate space with respect to $\bar{A}$ in such a way that given any other quasi-Banach couple $\bar{B}$ the spaces $\mathcal{F}(\bar{A})$ and $\mathcal{F}(\bar{B})$ are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$. We say that $\mathcal{F}$ is an *exact interpolation method* if it is an interpolation method and for every quasi-Banach couples $\bar{A}$, $\bar{B}$ and $T \in \mathcal{L}(\bar{A}, \bar{B})$ we have that

$$\|T|\mathcal{L}(\mathcal{F}(\bar{A}), \mathcal{F}(\bar{B}))\| \leq \max\{\|T|\mathcal{L}(A_0, B_0)\|, \|T|\mathcal{L}(A_1, B_1)\|\}.$$

In what follows we are going to focus mainly on the so-called real interpolation method (and some of its variants), but we will say a few words on the complex interpolation method too.

2.1.1 The real method

Let $\bar{A} = (A_0, A_1)$ be a (p-normed) quasi-Banach couple. For $t > 0$ and $a \in A_0 + A_1$, the *Peetre's K-functional* is defined by

$$K(t, a) = K(t, a; A_0, A_1) = \inf\{\|a_0|A_0\| + t\|a_1|A_1\| : a = a_0 + a_1,\ a_j \in A_j,\ j = 0, 1\}.$$

If $t=1$, $K(1,\cdot)$ coincides with the quasi-norm of A_0+A_1 given in (2.2). We can also consider the equivalent p-norm

$$K_p(t,a)=K_p(t,a;A_0,A_1)=\inf\{(\|a_0|A_0\|^p+t^p\|a_1|A_1\|^p)^{1/p}:a=a_0+a_1,\ a_j\in A_j,\ j=0,1\}.$$

It turns out that

$$K(t,a)\le K_p(t,a)\le 2^{1/p-1}K(t,a),\quad a\in A_0+A_1,\quad t>0.$$

It can be easily seen that $K(t,a)$ is increasing and $K(t,a)/t$ is decreasing on t for any $a\in A_0+A_1$. Furthermore,

$$K(t,a)\le\max\{1,t/s\}K(s,a),\qquad \text{for any } 0<t,s<\infty \text{ and } a\in A_0+A_1, \tag{2.3}$$
$$K(t,a;A_0,A_1)=tK(1/t,a;A_1,A_0),\qquad \text{for any } t>0 \text{ and } a\in A_0+A_1. \tag{2.4}$$

Similar inequalities hold for $K_p(t,a)$.

Let $\bar{A}=(A_0,A_1)$ be a (p-normed) quasi-Banach couple, $0<\theta<1$ and $0<q\le\infty$, the *real interpolation space* $(A_0,A_1)_{\theta,q}$ consists of all $a\in A_0+A_1$ with finite quasi-norm

$$\|a|(A_0,A_1)_{\theta,q}\|=\Big(\int_0^\infty[t^{-\theta}K(t,a)]^q\frac{dt}{t}\Big)^{1/q}\sim\Big(\int_0^\infty[t^{-\theta}K_p(t,a)]^q\frac{dt}{t}\Big)^{1/q}. \tag{2.5}$$

When $q=\infty$ the integral should be replaced by the supremum. The real interpolation space $(A_0,A_1)_{\theta,q}$ admits also discrete quasi-norms equivalent to (2.5),

$$\|a|(A_0,A_1)_{\theta,q}\|\sim\Big(\sum_{k=-\infty}^{\infty}[2^{-k\theta}K(2^k,a)]^q\Big)^{1/q}\sim\Big(\sum_{k=-\infty}^{\infty}[2^{-k\theta}K_p(2^k,a)]^q\Big)^{1/q}. \tag{2.6}$$

Again when $q=\infty$ the sum should be replaced by the supremum. In the above hypothesis $(A_0,A_1)_{\theta,q}$ is a quasi-Banach space (a Banach space when $\bar{A}$ is a Banach couple and $1\le q\le\infty$). Furthermore $(\cdot,\cdot)_{\theta,q}$ is an interpolation method with the following property.

Theorem 2.1. Let $\bar{A}=(A_0,A_1)$ and $\bar{B}=(B_0,B_1)$ be two quasi-Banach couples and $T\in\mathcal{L}(\bar{A},\bar{B})$. If $0<\theta<1$ and $0<q\le\infty$, then the restriction $T:(A_0,A_1)_{\theta,q}\longrightarrow(B_0,B_1)_{\theta,q}$ satisfies

$$\|T|\mathcal{L}((A_0,A_1)_{\theta,q},(B_0,B_1)_{\theta,q})\|\le\|T|\mathcal{L}(A_0,B_0)\|^{1-\theta}\|T|\mathcal{L}(A_1,B_1)\|^{\theta}.$$

The real interpolation method was first introduced by Lions and Peetre [89] in 1964 and it has been extensively studied later. See, for example, [6, 11, 22, 23, 110] for full details on this method and its properties.

When $q=\infty$ the parameter θ in the definition of the real method can also take the values 0 or 1. For $j=0,1$, the space $(A_0,A_1)_{j,\infty}$ is called the *Gagliardo completion* of A_j and will be sometimes denoted by $A_j^\sim$ (see [6, 11]). It turns out that $a\in A_0+A_1$ belongs to $A_j^\sim$ if, and only if, there is a sequence $(a_n)\subseteq A_j$ with

$$\sup_n\|a_n|A_j\|<\infty\quad\text{and}\quad\lim_n\|a-a_n|A_0+A_1\|=0. \tag{2.7}$$

Furthermore, minor modifications in the arguments for the Banach case of [6, Theorem V.1.5] show that

$$K(t,a;A_0^\sim,A_1^\sim)\le K(t,a;A_0,A_1)\le\max\{c_{A_0},c_{A_1}\}K(t,a;A_0^\sim,A_1^\sim),\quad t>0. \tag{2.8}$$

We call the quasi-Banach couple $\bar{A}$ *mutually closed* if $A_j^{\sim} = A_j$, for $j = 0, 1$.

The definition of the real interpolation method that we have seen before relies on the K-functional which is linked to the sum of spaces, however it can also be defined by means of the *J-functional* that is related to the intersection and, for a (p-normed) quasi-Banach couple $\bar{A}$, is defined by

$$J(t,a) = J(t,a;A_0,A_1) = \max\{\|a|A_0\|, t\|a|A_1\|\}, \quad t > 0,\ a \in A_0 \cap A_1.$$

The J-functional gives equivalent p-norms on $A_0 \cap A_1$ that coincides with (2.1) when $t = 1$. For any $a \in A_0 \cap A_1$, it can be easily proved that $J(t,a)$ is increasing and $J(t,a)/t$ is decreasing on t. Moreover

$$J(t,a) \le \max\{1, t/s\} J(s,a), \qquad 0 < t,s < \infty,\ a \in A_0 \cap A_1, \tag{2.9}$$
$$K(t,a) \le \min\{1, t/s\} J(s,a), \qquad 0 < t,s < \infty,\ a \in A_0 \cap A_1, \tag{2.10}$$
$$J(t,a;A_0,A_1) = tJ(1/t,a;A_1,A_0), \qquad 0 < t < \infty,\ a \in A_0 \cap A_1. \tag{2.11}$$

Assume that $\bar{A} = (A_0, A_1)$ is a quasi-Banach couple, $0 < \theta < 1$ and $0 < q \le \infty$. It follows from the equivalence theorem [11, Theorem 3.11.3] that $(A_0, A_1)_{\theta,q}$ agrees with the collection of those $a \in A_0 + A_1$ for which there is a sequence $(u_m)_{m\in\mathbb{Z}}$ such that $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$) and

$$\Big(\sum_{m=-\infty}^{\infty} [2^{-m\theta} J(2^m, u_m)]^q \Big)^{1/q} < \infty.$$

Moreover,

$$\|a|(A_0,A_1)^J_{\theta,q}\| = \inf\Big\{ \Big(\sum_{m=-\infty}^{\infty} [2^{-m\theta} J(2^m, u_m)]^q \Big)^{1/q} : a = \sum_{m=-\infty}^{\infty} u_m,\ u_m \in A_0 \cap A_1 \Big\} \tag{2.12}$$

is an equivalent quasi-norm to $\|\cdot|(A_0,A_1)_{\theta,q}\|$. If $\bar{A}$ is a Banach couple and $1 \le q \le \infty$ we can also consider a continuous quasi-norm equivalent to (2.12) defined as follows:

$$\|a|(A_0,A_1)^J_{\theta,q}\| \sim \inf\Big\{ \Big(\int_0^\infty [t^{-\theta} J(t, u(t))]^q \frac{dt}{t} \Big)^{1/q} : a = \int_0^\infty u(t)\frac{dt}{t},\ u(t) : (0,\infty) \longrightarrow A_0 \cap A_1 \Big\}$$

understanding that $u(t)$ is a strongly measurable function with values in $A_0 \cap A_1$ and $a = \int_0^\infty u(t)\frac{dt}{t}$ as a Bochner integral with convergence in $A_0 + A_1$.

Real interpolation spaces satisfy the following property.

Proposition 2.2. Let (A_0, A_1) be a p-normed quasi-Banach couple ($0 < p \le 1$), $0 < \theta < 1$ and $0 < q \le \infty$. Then the space $(A_0, A_1)_{\theta,q}$ admits an equivalent $\min\{p,q\}$-norm.

Proof. Let $\|\cdot|(A_0,A_1)_{\theta,q}\|^\diamond$ be the continuous quasi-norm on $(A_0,A_1)_{\theta,q}$ using $K_p(t,\cdot)$. If $x, y \in A_0 + A_1$, we get

$$\|x+y|(A_0,A_1)_{\theta,q}\|^\diamond \le \Big(\int_0^\infty t^{-\theta q} \big(K_p(t,x)^p + K_p(t,y)^p\big)^{q/p} \frac{dt}{t} \Big)^{1/q}$$

(the integral should be replaced by the supremum if $q = \infty$). If $q \geq p$ then it follows from the triangle inequality that

$$(\|x+y|(A_0,A_1)_{\theta,q}\|^\diamond)^p \leq \Big(\int_0^\infty (t^{-\theta p}K_p(t,x)^p)^{q/p}\frac{dt}{t}\Big)^{p/q} + \Big(\int_0^\infty (t^{-\theta p}K_p(t,y)^p)^{q/p}\frac{dt}{t}\Big)^{p/q}$$
$$= (\|x|(A_0,A_1)_{\theta,q}\|^\diamond)^p + (\|y|(A_0,A_1)_{\theta,q}\|^\diamond)^p.$$

Hence, $\|\cdot|(A_0,A_1)_{\theta,q}\|^\diamond)$ is a p-norm. If $0<q<p\leq 1$ then A_0 and A_1 are also q-normed quasi-Banach spaces. Using this time $K_q(t,a)$ to define $\|\cdot|(A_0,A_1)_{\theta,q}\|^\diamond$, we obtain

$$(\|x+y|(A_0,A_1)_{\theta,q}\|^\diamond)^q \leq \int_0^\infty t^{-\theta q}K_q(t,x)^q\frac{dt}{t} + \int_0^\infty t^{-\theta q}K_q(t,y)^q\frac{dt}{t}$$
$$= (\|x|(A_0,A_1)_{\theta,q}\|^\diamond)^q + (\|y|(A_0,A_1)_{\theta,q}\|^\diamond)^q.$$

This shows that $\|\cdot|(A_0,A_1)_{\theta,q}\|^\diamond$ is a q-norm. □

The *reiteration theorem* (see, for example, [11, Theorem 3.11.5]) describes what happens when we apply repeatedly the real interpolation method. In particular, if $\bar{A} = (A_0,A_1)$ is a quasi-Banach couple, $0<q_0,q_1,q\leq\infty$, $0<\theta_0\neq\theta_1<1$, $0<\eta<1$ and $\theta=(1-\eta)\theta_0+\eta\theta_1$. Then

$$\big((A_0,A_1)_{\theta_0,q_0},(A_0,A_1)_{\theta_1,q_1}\big)_{\eta,q} = (A_0,A_1)_{\theta,q} \quad \text{with equivalent quasi-norms.} \tag{2.13}$$

Not only continuous linear operators can be interpolated by the real method, as we have seen in Theorem 2.1, but also continuous bilinear operators. Let $\bar{A}=(A_0,A_1)$, $\bar{B}=(B_0,B_1)$ and $\bar{E}=(E_0,E_1)$ be quasi-Banach couples. By $T:\bar{A}\times\bar{B}\longrightarrow\bar{E}$, we mean that $T\in\mathcal{B}((A_0+A_1)\times(B_0+B_1),E_0+E_1)$ and the restriction of T to $A_j\times B_j$ belongs to $\mathcal{B}(A_j\times B_j,E_j)$, for $j=0$ and $j=1$. The following result concerning the interpolation of bilinear operators was proved in the seminal paper by Lions and Peetre [89] for the Banach case and extended to the quasi-Banach setting by Karadzhov [82] and König [83]. See [64, Theorem 3.1] and [40, Theorem 3.1] for the norm estimate.

Theorem 2.3. Let $\bar{A}=(A_0,A_1)$, $\bar{B}=(B_0,B_1)$ be quasi-Banach couples and let $\bar{E}=(E_0,E_1)$ be an r-normed quasi-Banach couple. Let $0<\theta<1$, $0<q_0,q_1\leq\infty$ and put

$$\frac{1}{q} = \begin{cases} \frac{1}{q_0}+\frac{1}{q_1}-\frac{1}{r} & \text{if} \quad q_0,q_1\geq r, \\ \frac{1}{\max\{q_0,q_1\}} & \text{if} \quad q_0<r \quad \text{or} \quad q_1<r. \end{cases} \tag{2.14}$$

If $T:\bar{A}\times\bar{B}\longrightarrow\bar{E}$, then T defines a continuous bilinear map $T:\bar{A}_{\theta,q_0}\times\bar{B}_{\theta,q_1}\longrightarrow\bar{E}_{\theta,q}$ and

$$\|T|\mathcal{B}(\bar{A}_{\theta,q_0}\times\bar{B}_{\theta,q_1},\bar{E}_{\theta,q})\| \leq C\|T|\mathcal{B}(A_0\times B_0,E_0)\|^{1-\theta}\|T|\mathcal{B}(A_1\times B_1,E_1)\|^{\theta}.$$

One can also interpolate continuous bilinear operators defined just on the product of the intersection of quasi-Banach couples instead of the product of the sum, as it can be seen in the following result.

Theorem 2.4. Let (A_0,A_1), (B_0,B_1) be quasi-Banach couples and let (E_0,E_1) be an r-Banach couple, $0<r\leq 1$. Let $0<\theta<1$ and $0<q_0,q_1,q\leq\infty$ verifying (2.14). If T is a bilinear operator defined on $(A_0\cap A_1)\times(B_0\cap B_1)$ with values in $E_0\cap E_1$ such that

$$\|T(a,b)|E_j\| \leq M_j\|a|A_j\|\cdot\|b|B_j\|, \quad a\in A_0\cap A_1,\ b\in B_0\cap B_1,\ j=0,1$$

then

$$\|T(a,b)|(E_0,E_1)_{\theta,q}\| \le CM_0^{1-\theta}M_1^{\theta}\|a|(A_0,A_1)_{\theta,q_0}\| \cdot \|b|(B_0,B_1)_{\theta,q_1}\|, \quad a \in A_0 \cap A_1,\ b \in B_0 \cap B_1.$$

Furthermore, if $q_j < \infty$ for $j = 0,1$, the operator T may be uniquely extended to a bounded bilinear operator from $(A_0,A_1)_{\theta,q_0} \times (B_0,B_1)_{\theta,q_1}$ to $(E_0,E_1)_{\theta,q}$.

We show now one of the most well-known applications of the real method to specific quasi-Banach couples. Let (Ω,μ) be a σ-finite measure space. Given a μ-measurable function on Ω one can define its *non-increasing rearrangement* as

$$f^*(t) = \inf\{s > 0 : \mu(\{\omega \in \Omega : |f(\omega)| > s\}) \le t\}. \tag{2.15}$$

Let $0 < p < \infty$, $0 < q \le \infty$, the *Lorentz space* $L_{p,q}(\Omega)$ consists of all (classes of) measurable functions f having a finite quasi-norm

$$\|f|L_{p,q}(\Omega)\| = \begin{cases} \left(\int_0^\infty [t^{1/p}f^*(t)]^q \frac{dt}{t}\right)^{1/q} & \text{if} \quad 0<q<\infty, \\ \sup_{t>0} t^{1/p}f^*(t) & \text{if} \quad q=\infty. \end{cases} \tag{2.16}$$

The space $L_{p,q}(\Omega)$ also admits the equivalent quasi-norm

$$\|f|L_{p,q}(\Omega)\| \sim \begin{cases} \left(\int_0^\infty [t\mu(\{\omega \in \Omega : |f(\omega)| > t\})^{1/p}]^q \frac{dt}{t}\right)^{1/q} & \text{if} \quad 0<q<\infty, \\ \sup_{t>0} t\mu(\{\omega \in \Omega : |f(\omega)| > t\})^{1/p} & \text{if} \quad q=\infty. \end{cases}$$

It turns out, that if $p = q$, then the Lorentz space $L_{p,p}(\Omega)$ coincides with the *Lebesgue space* $L_p(\Omega)$ of p-integrable μ-measurable functions on Ω. If $0 < p_0 \ne p_1 \le \infty$, $0 < q \le \infty$ and $0 < \theta < 1$ it turns out that $(L_{p_0}(\Omega), L_{p_1}(\Omega))$ is a quasi-Banach couple and

$$(L_{p_0}(\Omega), L_{p_1}(\Omega))_{\theta,q} = L_{p,q}(\Omega) \quad \text{with } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \tag{2.17}$$

with equivalent quasi-norms. Using now reiteration formula (2.13) we obtain that, if in addition, $0 < q_0, q_1 \le \infty$ and $0 < p_0, p_1 < \infty$ then

$$(L_{p_0,q_0}(\Omega), L_{p_1,q_1}(\Omega))_{\theta,q} = L_{p,q}(\Omega), \quad \text{with } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \tag{2.18}$$

with equivalent quasi-norms (see, for example, [11, Theorem 5.2.1]).

We close this brief introduction to the real method by pointing out its relation with abstract approximation spaces (see [24, 105, 104, 52, 33] and the references given in there).

Let $(X, \|\cdot|X\|)$ be a quasi-Banach space and let $(A_k)_{k=0}^\infty$ be a sequence of subsets of X satisfying the following conditions:

(1) $A_0 = \{0\} \subseteq A_1 \subseteq A_2 \subseteq ... \subseteq A_k \subseteq ... \subseteq X$.

(2) $\lambda A_k \subseteq A_k$ for any $\lambda \in \mathbb{K}$ and any $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

(3) $A_k + A_m \subseteq A_{k+m}$ for any $k, m \in \mathbb{N}_0$.

If $x \in X$, we put $E_0(x) = \|x|X\|$ and define the *k-th approximation number* as

$$E_k(x) = E_k^A(x)_X = \inf\{\|x - a|X\| : a \in A_k\}, \quad k \in \mathbb{N}.$$

Let $\alpha > 0$ and $0 < q \leq \infty$. The *approximation space* $X_q^\alpha = (X; A_k)_q^\alpha$ is formed by all those $x \in X$ having finite quasi-norm

$$\|x|X_q^\alpha\| = \begin{cases} \left(\sum_{k=1}^{\infty}[k^\alpha E_{k-1}(x)]^q k^{-1}\right)^{1/q} & \text{if} \quad 0 < q < \infty, \\ \sup_{k\geq 1}\{k^\alpha E_{k-1}(x)\} & \text{if} \quad q = \infty. \end{cases} \tag{2.19}$$

We can also consider the following equivalent quasi-norm on X_q^α

$$\|x|X_q^\alpha\|^\diamond = \left(\|x|X\| + \sum_{k=1}^{\infty} 2^{k\alpha q} E_{2^k}(x)^q\right)^{1/q}, \tag{2.20}$$

with the usual modification if $q = \infty$. See, for example, [105, Proposition 2].

When interpolating approximation spaces by the real interpolation method, the following identities hold.

Theorem 2.5. Let $0 < \alpha_0 \neq \alpha_1 < \infty$, $0 < p_0, p_1, q \leq \infty$, $0 < \theta < 1$ and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, then we have

$$(X_{p_0}^{\alpha_0}, X_{p_1}^{\alpha_1})_{\theta,q} = X_q^\alpha.$$

Furthermore,

$$(X, X_{p_1}^{\alpha_1})_{\theta,q} = X_q^{\theta\alpha_1}.$$

See, for example, [102] and [24], or [31, Proposition 2.7] for the last part.

2.1.2 The general real method

Now we introduce the *general real interpolation method* that extends the real method by changing the weighted ℓ_q quasi-norm in its discrete definition (2.6) by a more general quasi-Banach sequence lattice Γ. See the books by Peetre [99] and Brudnyĭ and Krugljak [22] and the papers by Nilsson [95] and Cwikel and Peetre [51] for the basic theory on this interpolation method. We follow mainly the approach of Nilsson [95].

A quasi-Banach space $(\Gamma, \|\cdot|\Gamma\|)$ of scalar valued sequences with $\mathbb{Z}$ as index set is said to be a *quasi-Banach sequence lattice* if it verifies the following properties:

(1) Γ contains all sequences with only finitely many non-zero coordinates.

(2) Whenever $|\xi_m| \leq |\eta_m|$ for each $m \in \mathbb{Z}$ and $(\eta_m)_{m\in\mathbb{Z}} \in \Gamma$, then $(\xi_m)_{m\in\mathbb{Z}} \in \Gamma$ and $\|(\xi_m)|\Gamma\| \leq \|(\eta_m)|\Gamma\|$.

For example, if $0 < q \leq \infty$ and $(\lambda_m)_{m\in\mathbb{Z}}$ is a sequence of positive real numbers, the space $\ell_q(\lambda_m)$ consisting of all scalar valued sequences $(\xi_m)_{m\in\mathbb{Z}}$ with finite quasi-norm

$$\|(\xi_m)|\ell_q(\lambda_m)\| = \begin{cases} \left(\sum_{m=-\infty}^{\infty} \lambda_m^q |\xi_m|^q\right)^{1/q} & \text{if} \quad 0 < q < \infty, \\ \sup_{m\in\mathbb{Z}} \lambda_m|\xi_m| & \text{if} \quad q = \infty, \end{cases}$$

is a quasi-Banach sequence lattice. If $\lambda_m = 1$ for every $m \in \mathbb{Z}$ we just write ℓ_q (or $\ell_q(\mathbb{N})$ if the index set is $\mathbb{N}$ instead of $\mathbb{Z}$).

We say that a quasi-Banach sequence lattice Γ is *K-non trivial* if

$$\|(\min\{1, 2^m\})|\Gamma\| < \infty \tag{2.21}$$

and for $0 < p \leq 1$ we say that Γ is (p, J)-*non trivial* if $\Gamma \hookrightarrow \ell_p + \ell_p(2^{-m})$, that is

$$\sup\Big\{\Big(\sum_{m=-\infty}^{\infty}[\min\{1,2^{-m}\}|\xi_m|]^p\Big)^{1/p} : \|(\xi_m)|\Gamma\| \leq 1\Big\} < \infty. \tag{2.22}$$

Clearly, if Γ is (p, J)-non trivial then Γ is also (r, J)-non trivial for any $p \leq r \leq 1$.

Let Γ be a quasi-Banach sequence lattice and $\bar{A} = (A_0, A_1)$ a (p-normed) quasi-Banach couple, we can define the space $(A_0, A_1)_{\Gamma;K} = \bar{A}_{\Gamma;K}$ as the set of all $a \in A_0 + A_1$ with finite quasi-norm

$$\|a|\bar{A}_{\Gamma;K}\| = \|(K(2^m, a))|\Gamma\|.$$

Analogously, we can define the space $(A_0, A_1)_{\Gamma;J} = \bar{A}_{\Gamma;J}$ as the set of all $a \in A_0 + A_1$ for which there exists a sequence $(u_m)_{m\in\mathbb{Z}} \subseteq A_0 \cap A_1$ such that $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$) and $\|(J(2^m, u_m))|\Gamma\|$ is finite. We define the quasi-norm

$$\|a|\bar{A}_{\Gamma;J}\| = \inf\Big\{\|(J(2^m, u_m))|\Gamma\| : a = \sum_{m=-\infty}^{\infty} u_m,\ u_m \in A_0 \cap A_1\Big\}.$$

Theorem 2.6. Let Γ be a quasi-Banach sequence lattice, $0 < p \leq 1$ and $\bar{A} = (A_0, A_1)$ be a (p-normed) quasi-Banach couple.

- If Γ is K-non trivial, then $\bar{A}_{\Gamma;K}$ is an exact interpolation space with respect to $\bar{A}$. Otherwise, $\bar{A}_{\Gamma;K} = \{0\}$.
- If Γ is (p, J)-non trivial, then $\bar{A}_{\Gamma;J}$ is an exact interpolation space with respect to $\bar{A}$. Otherwise, there exists a p-normed quasi-Banach couple $\bar{B} = (B_0, B_1)$ such that $(B_0, B_1)_{\Gamma;J}$ is not embedded in $B_0 + B_1$.

Proof. Assume first that Γ is K-non trivial. Then, from the properties of quasi-Banach sequence lattices, we have that for $e_0 = (\delta_m^0)_{m\in\mathbb{Z}}$ the sequence with 1 in the position of index 0 and 0 elsewhere

$$\|a|\bar{A}_{\Gamma;K}\| = \|(K(2^k, a))|\Gamma\| \geq \|K(1, a)e_0|\Gamma\| = K(1, a)\|e_0|\Gamma\| = \|a|A_0 + A_1\| \cdot \|e_0|\Gamma\|, \quad a \in \bar{A}_{\Gamma;K}.$$

This implies $\bar{A}_{\Gamma;K} \hookrightarrow A_0 + A_1$. On the other hand, using (2.10) we get that

$$\|a|\bar{A}_{\Gamma;K}\| \leq \|(\min\{1, 2^m\}J(1, a))|\Gamma\| = \|a|A_0 \cap A_1\| \cdot \|(\min\{1, 2^m\})|\Gamma\|, \quad a \in A_0 \cap A_1.$$

As $\|(\min\{1, 2^m\})|\Gamma\|$ is finite, the last estimation implies that $A_0 \cap A_1 \hookrightarrow \bar{A}_{\Gamma;K}$. Now we proof the interpolation property. Let $T \in \mathcal{L}(\bar{A})$, then for any $a \in A_0 + A_1$

$$K(2^k, Ta) \leq \max\{\|T|\mathcal{L}(A_0)\|, \|T|\mathcal{L}(A_1)\|\}K(2^k, a), \quad k \in \mathbb{Z}.$$

This implies that $T : \bar{A}_{\Gamma;K} \longrightarrow \bar{A}_{\Gamma;K}$ and

$$\|Ta|\bar{A}_{\Gamma;K}\| \leq \max\{\|T|\mathcal{L}(A_0)\|, \|T|\mathcal{L}(A_1)\|\}\|a|\bar{A}_{\Gamma;K}\|, \quad a \in \bar{A}_{\Gamma;K}.$$

Suppose now that Γ is not K-non trivial. This means that $\|(\min\{1, 2^m\})|\Gamma\| = \infty$. In this case, if $a \in \bar{A}_{\Gamma;K}$, then using (2.3) we get

$$\infty > \|a|\bar{A}_{\Gamma;K}\| = \|(K(2^m, a))|\Gamma\| > \|(K(1, a)\min\{1, 2^m\})|\Gamma\| = \|a|A_0 + A_1\| \cdot \|(\min\{1, 2^m\})|\Gamma\|,$$

which implies that $\|a|A_0 + A_1\| = 0$ and, consequently, $a = 0$.

In what remains we work with J-spaces. Suppose that Γ is (p, J)-non trivial, then we can decompose any $a \in A_0 \cap A_1$ as $a = \sum_{m=-\infty}^{\infty} u_m$ where $u_0 = a$ and $u_m = 0$ for any $m \neq 0$. Thus

$$\|a|\bar{A}_{\Gamma;J}\| \leq \|J(1,a)e_0|\Gamma\| = J(1,a)\|e_0|\Gamma\| = \|a|A_0 \cap A_1\| \cdot \|e_0|\Gamma\|, \quad a \in A_0 \cap A_1,$$

and $A_0 \cap A_1 \hookrightarrow \bar{A}_{\Gamma;J}$. Now assume that $a \in \bar{A}_{\Gamma;J}$ and $(u_m)_{m\in\mathbb{Z}} \subset A_0 \cap A_1$ is a J-decomposition for a. Then using (2.10)

$$\begin{aligned}\|a|A_0 + A_1\| \leq K_p(1,a) &\leq \Big(\sum_{m=-\infty}^{\infty} K_p(1,u_m)^p\Big)^{1/p} \leq \Big(\sum_{m=-\infty}^{\infty} \min\{1,2^{-m}\}^p J(2^m,u_m)^p\Big)^{1/p} \\ &\leq \sup\Big\{\Big(\sum_{m=-\infty}^{\infty} [\min\{1,2^{-m}\}|\xi_m|]^p\Big)^{1/p} : \|(\xi_m)|\Gamma\| \leq 1\Big\}\|(J(2^m,u_m))|\Gamma\|.\end{aligned}$$

Taking the infimum over all the possible decompositions for a we derive that $\bar{A}_{\Gamma;J} \hookrightarrow A_0 + A_1$. The proof of the interpolation property follows the same ideas that we used for K-spaces.

Finally, if Γ is not (p, J)-non trivial then there exists a p-normed quasi-Banach couple $\bar{B}$ such that $\bar{B}_{\Gamma;J}$ is not embedded in $B_0 + B_1$. Indeed, consider $\bar{B} = (\ell_p, \ell_p(2^{-m}))$. If Γ is not (p, J)-non-trivial then given any $N \in \mathbb{N}$, there is $x = (x_m) \in \Gamma$ and $L_N \in \mathbb{N}$ such that

$$\|x|\Gamma\| \leq 1 \quad \text{and} \quad \Big(\sum_{|m|\leq L_N} \big[\min\{1,2^{-m}\}|x_m|\big]^p\Big)^{1/p} > N.$$

Put $y_m = x_m$ if $|m| \leq L_N$ and $y_m = 0$ otherwise, and let $y = (y_m)$. Let $e_k = (\delta_m^k)$ where δ_m^k is the Kronecker's delta and write $u_k = y_k e_k$, $k \in \mathbb{Z}$. Then we have that $y = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $\ell_p + \ell_p(2^{-m})$) with $J(2^k, u_k; \ell_p, \ell_p(2^{-m})) = |y_k|$. Hence

$$\|y|(\ell_p, \ell_p(2^{-m}))_{\Gamma;J}\| \leq \|y|\Gamma\| \leq \|x|\Gamma\| \leq 1$$

and

$$\|y|\ell_p + \ell_p(2^{-m})\| \sim \Big(\sum_{|m|\leq L_N} \big[\min\{1,2^{-m}\}|x_m|\big]^p\Big)^{1/p} > N.$$

This yields that $(\ell_p, \ell_p(2^{-m}))_{\Gamma;J}$ is not continuously embedded in $\ell_p + \ell_p(2^{-m})$. □

We wonder now under which conditions the interpolation spaces $(A_0, A_1)_{\Gamma;K}$ and $(A_0, A_1)_{\Gamma;J}$ coincides for any quasi-Banach couple $\bar{A}$. The first thing we should notice is that, by construction, any element of $\bar{A}_{\Gamma;J}$ can be approximated in $A_0 + A_1$ by elements in $A_0 \cap A_1$. We call

$$(A_0 + A_1)^\circ = \{a \in A_0 + A_1 : \text{ there is } (v_k)_{k=0}^{\infty} \subseteq A_0 \cap A_1 \text{ such that } \|a - v_k|A_0 + A_1\| \xrightarrow{k\to\infty} 0\}.$$

From the previous comment, it follows that the inclusion $\bar{A}_{\Gamma;K} \subseteq (A_0 + A_1)^\circ$ is a necessary condition for $\bar{A}_{\Gamma;K} = \bar{A}_{\Gamma;J}$. The complete result reads as follows.

Theorem 2.7. Let $0 < p \leq 1$ and Γ be a quasi-Banach sequence lattice K-non trivial and (p, J)-non trivial. Then

$$\bar{A}_{\Gamma;K} \hookrightarrow \bar{A}_{\Gamma;J} \quad \text{for every } p\text{-normed quasi-Banach couple } \bar{A}.$$

Furthermore,

$$\bar{A}_{\Gamma;J} \hookrightarrow \bar{A}_{\Gamma;K} \quad \text{for every } p\text{-normed quasi-Banach couple } \bar{A},$$

if, and only if, $\Gamma \hookrightarrow (\ell_p, \ell_p(2^{-m}))_{\Gamma;K}$.

Proof. We show first that $\bar{A}_{\Gamma;K} \subseteq (A_0 + A_1)^\circ$ for any quasi-Banach couple $\bar{A}$. Indeed, it follows from [95, Lemma 2.4] that $a \in (A_0 + A_1)^\circ$ if, and only if, $\min\{1, 2^{-k}\}K(2^k, a) \to 0$ as $k \to \pm\infty$. As Γ is (p, J)-non trivial we have that for any $a \in \bar{A}_{\Gamma;K}$

$$\Big(\sum_{k=-\infty}^{\infty} [\min\{1, 2^{-k}\}K(2^k, a)]^p\Big)^{1/p} = \|(K(2^k, a))|\ell_p + \ell_p(2^{-k})\|$$
$$\lesssim \|(K(2^k, a))|\Gamma\| = \|a|\bar{A}_{\Gamma;K}\| < \infty,$$

and consequently $\min\{1, 2^{-k}\}K(2^k, a) \to 0$ as $k \to \pm\infty$ and $\bar{A}_{\Gamma;K} \subseteq (A_0 + A_1)^\circ$. Then, *the fundamental lemma of interpolation theory* (see, [11, Lemma 3.3.2]) implies that for any $a \in \bar{A}_{\Gamma;K}$ there exists a sequence $(u_m) \subset A_0 \cap A_1$ such that $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$) and

$$J(2^m, u_m) \lesssim K(2^m, a), \quad \text{for every } m \in \mathbb{Z}.$$

Therefore,

$$\|a|\bar{A}_{\Gamma;J}\| \le \|(J(2^m, u_m))|\Gamma\| \lesssim \|(K(2^m, a))|\Gamma\| = \|a|\bar{A}_{\Gamma;K}\|.$$

This proves that $\bar{A}_{\Gamma;K} \hookrightarrow \bar{A}_{\Gamma;J}$.

On the other hand, if $\Gamma \hookrightarrow (\ell_p, \ell_p(2^{-m}))_{\Gamma;K}$ then for any $a \in \bar{A}_{\Gamma;J}$ and any J-representation $(u_m) \subset A_0 \cap A_1$ of a we have that

$$K_p(2^k, a) \le \Big(\sum_{m=-\infty}^{\infty} K_p(2^k, u_m)^p\Big)^{1/p} \le \Big(\sum_{m=-\infty}^{\infty} [\min\{1, 2^{k-m}\}J(2^m, u_m)]^p\Big)^{1/p}$$
$$= K_p(2^k, (J(2^m, u_m)); \ell_p, \ell_p(2^{-m})).$$

Thus

$$\|a|\bar{A}_{\Gamma;K}\| \sim \|(K_p(2^k, a))|\Gamma\| \le \|(K_p(2^k, (J(2^m, u_m)); \ell_p, \ell_p(2^{-m})))|\Gamma\|$$
$$= \|(J(2^m, u_m))|(\ell_p, \ell_p(2^{-m}))_{\Gamma;K}\| \lesssim \|(J(2^m, u_m))|\Gamma\|.$$

Taking the infimum over all the possible J-representations of a we derive that $\|a|\bar{A}_{\Gamma;K}\| \lesssim \|a|\bar{A}_{\Gamma;J}\|$ and, consequently, $\bar{A}_{\Gamma;J} \hookrightarrow \bar{A}_{\Gamma;K}$.

Reciprocally, assume that $\bar{A}_{\Gamma;J} \hookrightarrow \bar{A}_{\Gamma;K}$ for any p-normed quasi-Banach couple. In particular,

$$(\ell_p, \ell_p(2^{-m}))_{\Gamma;J} \hookrightarrow (\ell_p, \ell_p(2^{-m}))_{\Gamma;K}.$$

So, in order to prove that $\Gamma \hookrightarrow (\ell_p, \ell_p(2^{-m}))_{\Gamma;K}$, we just need to show that $\Gamma \hookrightarrow (\ell_p, \ell_p(2^{-m}))_{\Gamma;J}$. Let $\xi = (\xi_m)_{m=-\infty}^{\infty} \in \Gamma$, then $\xi = \sum_{m=-\infty}^{\infty} u_m$ with $u_m = \xi_m e_m \in \ell_p \cap \ell_p(2^{-m})$. Thereby

$$\|\xi|(\ell_p, \ell_p(2^{-m}))_{\Gamma;J}\| \le \|(J(2^m, \xi_m e_m; \ell_p, \ell_p(2^{-m})))|\Gamma\| = \|(\xi_m)|\Gamma\|,$$

which implies that $\Gamma \hookrightarrow (\ell_p, \ell_p(2^{-m}))_{\Gamma;J}$. □

Some examples

For $\Gamma = \ell_q(2^{-m\theta})$ with $0 < q \le \infty$ and $0 < \theta < 1$, $\bar{A}_{\Gamma;K}$ and $\bar{A}_{\Gamma;J}$ coincide and they are equal to the real interpolation space $\bar{A}_{\theta,q}$ for every quasi-Banach couple $\bar{A}$.

Let $\rho : (0, \infty) \longrightarrow (0, \infty)$ be a *function parameter*, that is to say,

(1) $\rho(t)$ increases from 0 to ∞.

(2) $\rho(t)/t$ decreases from ∞ to 0.

(3) $s_\rho(t) = \sup\{\rho(ts)/\rho(s) : s > 0\} < \infty$, for every $t > 0$.

(4) $\min\{1, 1/t\} s_\rho(t) \to 0$ as $t \to 0$ and $t \to \infty$.

For $0 < q \leq \infty$ and $\Gamma = \ell_q(1/\rho(2^m))$, K and J-spaces coincide and they agree with the *real method with a function parameter* $(A_0, A_1)_{\rho,q}$ that is defined as the real method but replacing t^θ by the function parameter $\rho(t)$ in (2.5). This method was studied by Gustavsson [72] and Persson [103], among other authors, and shares its main properties with the real interpolation method.

Now we are going to focus on logarithmic perturbations of the real interpolation method that we are going to study in Chapters 3-7. Let $\ell(t) = (1 + |\log t|)$ and $\ell\ell(t) = 1 + \log(1 + |\log t|)$. For $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ put

$$\ell^{\mathbb{A}}(t) = \ell^{(\alpha_0,\alpha_\infty)}(t) = \begin{cases} \ell^{\alpha_0}(t) & \text{if} \quad 0 < t \leq 1, \\ \ell^{\alpha_\infty}(t) & \text{if} \quad 1 < t < \infty, \end{cases}$$

and define $\ell\ell^{\mathbb{A}}(t)$ similarly. If $0 < \theta < 1$, $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ the function $\rho(t) = t^\theta \ell^{-\mathbb{A}}(t)$ is equivalent to a function parameter and, therefore, for any quasi-Banach couple $\bar{A} = (A_0, A_1)$ the interpolation space $(A_0, A_1)_{\theta,q,\mathbb{A}} = \bar{A}_{\theta,q,\mathbb{A}}$ defined as the set of all elements $a \in A_0 + A_1$ with finite quasi-norm

$$\|a|\bar{A}_{\theta,q,\mathbb{A}}\| = \Big(\int_0^\infty [t^{-\theta} K(t,a) \ell^{\mathbb{A}}(t)]^q \Big)^{1/q}$$

coincides with the spaces $\bar{A}_{\Gamma;K}$ and $\bar{A}_{\Gamma;J}$ for $\Gamma = \ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m))$. These interpolation methods where introduced by Evans and Opic in [58] and Evans, Opic and Pick in [59].

When we apply logarithmic interpolation methods to Lebesgue spaces, we get the so-called *Generalized Lorentz-Zygmund spaces* introduced by Opic and Pick in [97]. Let (Ω, μ) be a non-atomic σ-finite measure space, $0 < p, q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, the generalized Lorentz-Zygmund space $L_{p,q,\mathbb{A}}(\Omega)$ is the set of all measurable functions on Ω such that

$$\|f|L_{p,q,\mathbb{A}}(\Omega)\| = \Big(\int_0^\infty [t^{1/p} \ell^{\mathbb{A}}(t) f^*(t)]^q \frac{dt}{t} \Big)^{1/q} \tag{2.23}$$

is finite. When $q = \infty$ the integral should be replaced by the supremum.

We denote by $L_{(p,q,\mathbb{A})}(\Omega)$ to the collection of all measurable functions on Ω such that

$$\|f|L_{(p,q,\mathbb{A})}(\Omega)\| = \Big(\int_0^\infty [t^{1/p-1} \ell^{\mathbb{A}}(t) \int_0^t f^*(s) ds]^q \frac{dt}{t} \Big)^{1/q}. \tag{2.24}$$

According to [97, Lemma 3.5], in order to avoid the space $L_{p,q,\mathbb{A}}(\Omega)$ be the trivial space $\{0\}$ we should assume that

$$\begin{cases} 0 < p < \infty; \\ p = \infty, \quad \alpha_0 + 1/q < 0; \\ p = \infty, \quad q = \infty, \quad \alpha_0 = 0. \end{cases} \tag{2.25}$$

On the other hand, in order to prevent the space $L_{(p,q,\mathbb{A})}(\Omega)$ from being the trivial space $\{0\}$ we should impose that

$$\begin{cases} 1 < p < \infty; \\ p = \infty, \quad \alpha_0 + 1/q < 0; \\ p = \infty, \quad q = \infty, \quad \alpha_0 = 0; \\ p = 1, \quad \alpha_\infty + 1/q < 0; \\ p = 1, \quad q = \infty, \quad \alpha_\infty = 0. \end{cases} \tag{2.26}$$

If $1 < p \leq \infty$, $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfy (2.26) then $L_{p,q,\mathbb{A}}(\Omega) = L_{(p,q,\mathbb{A})}(\Omega)$ with equivalence of quasi-norms. See [97, Theorem 3.8].

For $0 < \theta < 1$, $0 < q \leq \infty$, $0 < p < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, according to [58, Corollary 8.4], we have the following interpolation formula:

$$(L_p(\Omega), L_\infty(\Omega))_{\theta,q,\mathbb{A}} = L_{\frac{p}{1-\theta},q,\mathbb{A}}(\Omega). \tag{2.27}$$

The generalized Lorentz-Zygmund scale contains other well-known spaces. If $\alpha_0 = \alpha_\infty$, then the space $L_{p,q,(\alpha_0,\alpha_0)}(\Omega)$ coincides with the *Lorentz-Zygmund spaces* $L_{p,q}(\mathrm{LogL})_{\alpha_0}(\Omega)$ (see, for example, [5]). If $\alpha_0 = \alpha_\infty = 0$, then $L_{p,q,(0,0)}(\Omega)$ is the Lorentz space $L_{p,q}(\Omega)$ defined in (2.16).

An important feature of logarithmic interpolation spaces, as we are going to show next, is that the quasi-Banach sequence lattice $\Gamma = \ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m))$ is also K-non trivial when $\theta = 0$ or $\theta = 1$ and the exponents of the logarithm satisfy certain conditions. In these cases $(\cdot,\cdot)_{\theta,q,\mathbb{A}}$ does not satisfy the conditions of an interpolation method with a function parameter but still has some special properties (see [48]). In order to study these limiting cases sometimes it will be useful to consider more general quasi-Banach lattices with a double logarithm.

Theorem 2.8. Let $\theta \in \mathbb{R}$, $0 < q \leq \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. The quasi-Banach sequence lattice $\Gamma = \ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))$ is K-non trivial if, and only if, one of the following properties is fulfilled:

$$\begin{cases} 0 < \theta < 1; \\ \theta = 0, \quad \alpha_\infty + 1/q < 0 \quad \text{or} \quad \alpha_\infty + 1/q = 0 \text{ and } \beta_\infty + 1/q < 0; \\ \theta = 0, \quad q = \infty, \quad \alpha_\infty = \beta_\infty = 0; \\ \theta = 1, \quad \alpha_0 + 1/q < 0 \quad \text{or} \quad \alpha_0 + 1/q = 0 \text{ and } \beta_0 + 1/q < 0; \\ \theta = 1, \quad q = \infty, \quad \alpha_0 = \beta_0 = 0. \end{cases} \tag{2.28}$$

Proof. We need to study under which hypothesis $\|(\min\{1,2^m\})|\ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))\| < \infty$ or, equivalently, under which assumptions

$$\sum_{m=-\infty}^{0} [2^{m(1-\theta)}\ell^{\alpha_0}(2^m)\ell\ell^{\beta_0}(2^m)]^q \sim \int_0^1 [t^{1-\theta}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)]^q \frac{dt}{t} < \infty \quad \text{and}$$
$$\sum_{m=0}^{\infty} [2^{-m\theta}\ell^{\alpha_\infty}(2^m)\ell\ell^{\beta_\infty}(2^m)]^q \sim \int_1^\infty [t^{-\theta}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)]^q \frac{dt}{t} < \infty.$$

Now the result follows from direct computations. □

Note that for $\mathbb{A} = \mathbb{B} = (0,0)$ the logarithmic interpolation method $(\cdot,\cdot)_{\theta,q,\mathbb{A},\mathbb{B}}$ coincides with the real method $(\cdot,\cdot)_{\theta,q}$. The previous theorem proves that it only makes sense to consider the real method as usual (when $0 < \theta < 1$) or in the case of Gagliardo completions (when $q = \infty$ and $\theta = 0, 1$).

Theorem 2.9. Let $\theta \in \mathbb{R}$, $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ satisfying (2.28). $(A_0, A_1)_{\theta,q,\mathbb{A},\mathbb{B}} \subseteq (A_0 + A_1)^\circ$, for every quasi-Banach couple $\bar{A} = (A_0, A_1)$ if, and only if, one of the following conditions is fulfilled:

$$\begin{cases} 0 < \theta < 1; \\ \theta = 1, \quad 0 < q < \infty \quad \text{and} \quad \alpha_\infty + 1/q > 0 \quad \text{or} \quad \alpha_\infty + 1/q = 0 \text{ and } \beta_\infty + 1/q \geq 0; \\ \theta = 1, \quad q = \infty \quad \text{and} \quad \alpha_\infty > 0 \quad \text{or} \quad \alpha_\infty = 0 \text{ and } \beta_\infty > 0; \\ \theta = 0, \quad 0 < q < \infty \quad \text{and} \quad \alpha_0 + 1/q > 0 \quad \text{or} \quad \alpha_0 + 1/q = 0 \text{ and } \beta_0 + 1/q \geq 0; \\ \theta = 0, \quad q = \infty \quad \text{and} \quad \alpha_0 > 0 \quad \text{or} \quad \alpha_0 = 0 \text{ and } \beta_0 > 0. \end{cases} \tag{2.29}$$

Proof. Let $a \in \bar{A}_{\theta,q,\mathbb{A},\mathbb{B}}$. Then

$$\|a|\bar{A}_{\theta,q,\mathbb{A},\mathbb{B}}\| \sim \left(\int_0^\infty [t^{-\theta} \ell^{\mathbb{A}}(t) \ell\ell^{\mathbb{B}}(t) K(t,a)]^q \frac{dt}{t} \right)^{1/q} < \infty$$

but, according to (2.28) and (2.29)

$$\int_0^1 [t^{-\theta} \ell^{\alpha_0}(t) \ell\ell^{\beta_0}(t)]^q \frac{dt}{t} = \infty \quad \text{and} \quad \int_1^\infty [t^{(1-\theta)} \ell^{\alpha_\infty}(t) \ell\ell^{\beta_\infty}(t)]^q \frac{dt}{t} = \infty.$$

Therefore $\lim_{t\to 0} K(t,a) = 0$ and $\lim_{t\to\infty} \frac{K(t,a)}{t} = 0$, which implies that $a \in (A_0 + A_1)^\circ$ (see [95, Lemma 2.4]).

Assume now that the parameters verify (2.28) but do not satisfy (2.29), that is to say,

$$\begin{cases} \theta = 1, \quad 0 < q < \infty \quad \text{and} \quad \alpha_\infty + 1/q < 0 \quad \text{or} \quad \alpha_\infty + 1/q = 0 \text{ and } \beta_\infty + 1/q < 0; \\ \theta = 1, \quad q = \infty, \quad \text{and} \quad \alpha_\infty < 0 \quad \text{or} \quad \alpha_\infty = 0 \text{ and } \beta_\infty \leq 0; \\ \theta = 0, \quad 0 < q < \infty \quad \text{and} \quad \alpha_0 + 1/q < 0 \quad \text{or} \quad \alpha_0 + 1/q = 0 \text{ and } \beta_0 + 1/q < 0; \\ \theta = 0, \quad q = \infty \quad \text{and} \quad \alpha_0 < 0 \quad \text{or} \quad \alpha_0 = 0 \text{ and } \beta_0 \leq 0. \end{cases}$$

Assume first that $\theta = 1$. Let $\xi = (1, 1, 1, \ldots)$ and $\bar{A} = (\ell_1(\mathbb{N}), \ell_\infty(\mathbb{N}))$. According to [110, Theorem 1.18.3] we have that $K(2^m, \xi) = 2^m$ for $m \in \mathbb{Z}$. Then ξ belongs to $(\ell_1(\mathbb{N}), \ell_\infty(\mathbb{N}))_{1,q,\mathbb{A},\mathbb{B}}$ since

$$\|\xi|(\ell_1(\mathbb{N}), \ell_\infty(\mathbb{N}))_{1,q,\mathbb{A},\mathbb{B}}\| = \left(\sum_{m=-\infty}^{0} \ell^{\alpha_0 q}(2^m) \ell\ell^{\beta_0 q}(2^m) + \sum_{m=1}^{\infty} \ell^{\alpha_\infty q}(2^m) \ell\ell^{\beta_\infty q}(2^m) \right)^{1/q} < \infty.$$

However, $\lim_{m\to\infty} K(2^m, \xi; \ell_1(\mathbb{N}), \ell_\infty(\mathbb{N}))/2^m = 1$ and this implies that $\xi \notin (\ell_1(\mathbb{N}) + \ell_\infty(\mathbb{N}))^\circ$.

On the other hand, if $\theta = 0$ we proceed in the same way with ξ as before and the Banach couple $(\ell_\infty(\mathbb{N}), \ell_1(\mathbb{N}))$. Then, according to (2.4)

$$K(2^m, \xi; \ell_\infty(\mathbb{N}), \ell_1(\mathbb{N})) = 2^m K(2^{-m}, \xi; \ell_1(\mathbb{N}), \ell_\infty(\mathbb{N})) = 1.$$

As before ξ belongs to $(\ell_\infty(\mathbb{N}), \ell_1(\mathbb{N}))_{0,q,\mathbb{A},\mathbb{B}}$. However $\lim_{m\to-\infty} K(2^m, \xi; \ell_\infty(\mathbb{N}), \ell_1(\mathbb{N})) = 1$ and, therefore ξ does not belong to $(\ell_\infty(\mathbb{N}) + \ell_1(\mathbb{N}))^\circ$. □

Now we study under which hypothesis the quasi-Banach lattice $\Gamma = \ell_q(2^{-m\theta} \ell^{\mathbb{A}}(2^m) \ell\ell^{\mathbb{B}}(2^m))$ is (p, J)-non trivial.

Theorem 2.10. Let $0 < p \le 1, 0 < q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty), \mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. The necessary and sufficient condition for the continuous embedding

$$\ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)) \hookrightarrow \ell_p + \ell_p(2^{-m})$$

is that

$$\begin{cases} 0 < \theta < 1; & \\ \theta = 1, \quad \alpha_\infty > 0 \quad \text{or} \quad \alpha_\infty = 0 \text{ and } \beta_\infty \ge 0 & \text{if} \quad 0 < q \le p; \\ \theta = 1, \quad \alpha_\infty + \frac{1}{q} - \frac{1}{p} > 0 \quad \text{or} \quad \alpha_\infty + \frac{1}{q} - \frac{1}{p} = 0 \text{ and } \beta_\infty + \frac{1}{q} - \frac{1}{p} > 0 & \text{if} \quad p < q \le \infty; \\ \theta = 0, \quad \alpha_0 > 0 \quad \text{or} \quad \alpha_0 = 0 \text{ and } \beta_0 \ge 0 & \text{if} \quad 0 < q \le p; \\ \theta = 0, \quad \alpha_0 + \frac{1}{q} - \frac{1}{p} > 0 \quad \text{or} \quad \alpha_0 + \frac{1}{q} - \frac{1}{p} = 0 \text{ and } \beta_0 + \frac{1}{q} - \frac{1}{p} > 0 & \text{if} \quad p < q \le \infty. \end{cases} \tag{2.30}$$

Proof. Let $q^* = \begin{cases} \infty & \text{if} \quad 0 < q \le p; \\ \frac{pq}{q-p} & \text{if} \quad 0 < p < q < \infty; \\ p & \text{if} \quad q = \infty. \end{cases}$

Take any $x = (x_m) \in \ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))$. Using Hölder's inequality we get

$$\begin{aligned} \|x|\ell_p + \ell_p(2^{-m})\| &\sim \Big(\sum_{m=-\infty}^{\infty} \min\{1, 2^{-m}\}^p |x_m|^p \Big)^{1/p} \\ &\le \|(\min\{1,2^{-m}\})|\ell_{q^*}(2^{m\theta}\ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m))\| \cdot \|x|\ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))\| \\ &= \Big(\sum_{m=-\infty}^{0} [2^{m\theta}\ell^{-\alpha_0}(2^{m\theta})\ell\ell^{-\beta_0}(2^m)]^{q^*} + \sum_{m=0}^{\infty} [2^{-m(1-\theta)}\ell^{-\alpha_\infty}(2^m)\ell\ell^{-\beta_\infty}(2^m)]^{q*} \Big)^{1/q^*} \\ &\quad \times \|x|\ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))\|. \end{aligned}$$

The last sums are finite by (2.30). Therefore we obtain that

$$\ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)) \hookrightarrow \ell_p + \ell_p(2^{-m}).$$

Conversely, if (2.30) does not hold then

$$\begin{cases} \theta = 1, \quad \alpha_\infty < 0 \quad \text{or} \quad \alpha_\infty = 0 \text{ and } \beta_\infty < 0 & \text{if} \quad 0 < q \le p; \\ \theta = 1, \quad \alpha_\infty + \frac{1}{q} - \frac{1}{p} < 0 \quad \text{or} \quad \alpha_\infty + \frac{1}{q} - \frac{1}{p} = 0 \text{ and } \beta_\infty + \frac{1}{q} - \frac{1}{p} \le 0 & \text{if} \quad p < q \le \infty; \\ \theta = 0, \quad \alpha_0 < 0 \quad \text{or} \quad \alpha_0 = 0 \text{ and } \beta_0 < 0 & \text{if} \quad 0 < q \le p; \\ \theta = 0, \quad \alpha_0 + \frac{1}{q} - \frac{1}{p} < 0 \quad \text{or} \quad \alpha_0 + \frac{1}{q} - \frac{1}{p} = 0 \text{ and } \beta_0 + \frac{1}{q} - \frac{1}{p} \le 0 & \text{if} \quad p < q \le \infty. \end{cases}$$

In this case $\|(\min\{1,2^{-m}\})|\ell_{q^*}(2^{m\theta}\ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m))\| = \infty$. If follows that for any $N \in \mathbb{N}$, there is $L_N \in \mathbb{N}$ such that

$$N < \Big(\sum_{|m| \le L_N} [\min\{1, 2^{-m}\} 2^{m\theta}\ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m)]^{q^*} \Big)^{1/q^*}.$$

Assume now that $\theta = 1$ and $0 < p < q < \infty$. Let

$$y_m = \begin{cases} \min\{1, 2^{-m}\}^{\frac{p}{q-p}} 2^{m\frac{q}{q-p}} \ell^{-\frac{q}{q-p}\mathbb{A}}(2^m) \ell\ell^{-\frac{q}{q-p}\mathbb{B}}(2^m) & \text{if} \quad |m| \le L_N, \\ 0 & \text{otherwise,} \end{cases}$$

put $y = (y_m)$ and $x = \big(y_m / \|y|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))\|\big)$. Then x belongs to the unit ball of the sequence space $\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))$ but

$$\begin{aligned} \|x|\ell_p + \ell_p(2^{-m})\| &\sim \Big(\sum_{m=-\infty}^{\infty} [\min\{1, 2^{-m}\}|x_m|]^p \Big)^{1/p} \\ &= \frac{\Big(\sum_{|m| \le L_N} [\min\{1, 2^{-m}\} 2^m \ell^{-\mathbb{A}}(2^m) \ell\ell^{-\mathbb{B}}(2^m)]^{q^*} \Big)^{1/p}}{\Big(\sum_{|m| \le L_N} [\min\{1, 2^{-m}\} 2^m \ell^{-\mathbb{A}}(2^m) \ell\ell^{-\mathbb{B}}(2^m)]^{q^*} \Big)^{1/q}} \\ &= \Big(\sum_{|m| \le L_N} [\min\{1, 2^{-m}\} 2^m \ell^{-\mathbb{A}}(2^m) \ell\ell^{-\mathbb{B}}(2^m)]^{q^*} \Big)^{1/q^*} > N. \end{aligned}$$

Hence, $\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))$ is not continuously embedded in $\ell_p + \ell_p(2^{-m})$.

In the case $\theta = 1$ and $q = \infty$, where $q^* = p$, we put

$$y_m = \begin{cases} 2^m \ell^{-\mathbb{A}}(2^m) \ell\ell^{-\mathbb{B}}(2^m) & \text{if} \quad |m| \le L_N, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|(y_m)|\ell_\infty(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))\| = 1$ but

$$\|(y_m)|\ell_p + \ell_p(2^{-m})\| \sim \Big(\sum_{|m| \le L_N} [\min\{1, 2^{-m}\} 2^m \ell^{-\mathbb{A}}(2^m) \ell\ell^{-\mathbb{B}}(2^m)]^p \Big)^{1/p} > N.$$

Finally, if $\theta = 1$ and $0 < q < p$ we have $q^* = \infty$. We may assume that L_N satisfies that

$$\min\{1, 2^{-L_N}\} 2^{L_N} \ell^{-\mathbb{A}}(2^{L_N}) \ell\ell^{-\mathbb{B}}(2^{L_N}) > N.$$

Let $e_{L_N} = (\delta_m^{L_N})$ and $x = 2^{L_N} \ell^{-\mathbb{A}}(2^{L_N}) \ell\ell^{-\mathbb{B}}(2^{L_N}) e_{L_N}$. Then we have

$$\begin{aligned} &\|x|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))\| = 1 \text{ and} \\ &\|x|\ell_p + \ell_p(2^{-m})\| = \min\{1, 2^{-L_N}\} 2^{L_N} \ell^{-\mathbb{A}}(2^{L_N}) \ell\ell^{-\mathbb{B}}(2^{L_N}) > N. \end{aligned}$$

This completes the proof for $\theta = 1$. If $\theta = 0$ we can proceed analogously. □

If $\Gamma = \ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))$ with the parameters satisfying (2.30), then sometimes we will denote by $(A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}}$ (or simply $(A_0, A_1)^J_{\theta,q,\mathbb{A}}$ if $\mathbb{B} = (0,0)$) to the space $(A_0, A_1)_{\Gamma;J}$. If, in addition, $\bar{A} = (A_0, A_1)$ is a Banach couple and $q \ge 1$, then $(A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}}$ admits a continuous norm defined by:

$$\begin{aligned} &\|a|\bar{A}^J_{\theta,q,\mathbb{A},\mathbb{B}}\| \qquad (2.31) \\ &\sim \inf\Big\{ \Big(\int_0^\infty [t^{-\theta} J(t, u(t)) \ell^{\mathbb{A}}(t) \ell\ell^{\mathbb{B}}(t)]^q \frac{dt}{t} \Big)^{1/q} : a = \int_0^\infty u(t) \frac{dt}{t}, u(t) : (0,\infty) \longrightarrow A_0 \cap A_1 \Big\} \end{aligned}$$

understanding $a = \int_0^\infty u(t) \frac{dt}{t}$ as the Bochner integral with convergence in $A_0 + A_1$.

2.1.3 The complex method

The complex method was introduced by Calderón in [26] and, together with the real method, is one of the most celebrated interpolation methods. We are going to give here a very rough introduction, focusing only on the results we are going to need later. For a detailed study one could consult, for example, [11, 110].

Let $\bar{A} = (A_0, A_1)$ be a Banach couple. We put $F(\bar{A})$ for the space of all functions f from the closed strip $S = \{z \in \mathbb{C} : 0 \leq \Re ez \leq 1\}$ into $A_0 + A_1$, which are bounded on S, analytic on the open string $S_0 = \{z \in \mathbb{C} : 0 < \Re ez < 1\}$ and moreover, the functions $t \longrightarrow f(j + it)$ $(j = 0, 1)$ are continuous from the real line into A_j and tend to zero as $|t| \to \infty$. We endow $F(\bar{A})$ with the norm

$$\|f|F(\bar{A})\| = \max\{\sup \|f(it)|A_0\|, \sup \|f(1 + it)|A_1\|\}.$$

For $0 < \theta < 1$, the *complex interpolation space* $[A_0, A_1]_\theta$ consists of all $a \in A_0 + A_1$ such that $a = f(\theta)$ for some $f \in F(\bar{A})$. We consider on $[A_0, A_1]_\theta$ the following norm:

$$\|a|[A_0, A_1]_\theta\| = \inf\{\|f|F(\bar{A})\| : f(\theta) = a,\ f \in F(\bar{A})\}.$$

It turns out that $[\cdot, \cdot]_\theta$ is an interpolation method for Banach couples and it has the following property: If $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ are Banach couples, $T \in \mathcal{L}(\bar{A}, \bar{B})$ and $0 < \theta < 1$, then $T : [A_0, A_1]_\theta \longrightarrow [B_0, B_1]_\theta$ is continuous and

$$\|T|\mathcal{L}([A_0, A_1]_\theta, [B_0, B_1]_\theta)\| \leq \|T|\mathcal{L}(A_0, B_0)\|^{1-\theta} \|T|\mathcal{L}(A_1, B_1)\|^\theta.$$

The real and complex interpolation methods are connected through the following reiteration result (see [11, Theorem 4.7.2]):

Theorem 2.11. Let $\bar{A} = (A_0, A_1)$ be a Banach couple, $0 < \theta_0 \neq \theta_1 < 1$, $0 < \eta < 1$ and $\theta = (1 - \eta)\theta_0 + \eta\theta_1$.

(i) If $0 < q \leq \infty$ then

$$([A_0, A_1]_{\theta_0}, [A_0, A_1]_{\theta_1})_{\eta,q} = (A_0, A_1)_{\theta,q} \quad \text{with equivalent quasi-norms.}$$

(ii) If $1 \leq q_i \leq \infty$, $i = 0, 1$ and $\frac{1}{q} = \frac{1-\eta}{q_0} + \frac{\eta}{q_1}$ then

$$[(A_0, A_1)_{\theta_0,q_0}, (A_0, A_1)_{\theta_1,q_1}]_\eta = (A_0, A_1)_{\theta,q} \quad \text{with equivalent norms.}$$

As in the case of the real method, continuous bilinear operators can also be interpolated by the complex method. The result, proved by Calderón in [26], reads as follows:

Theorem 2.12. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be Banach couples and let $0 < \theta < 1$. Assume that T is a bilinear operator defined on $(A_0 \cap A_1) \times (B_0 \cap B_1)$ with values on $E_0 \cap E_1$ such that

$$\|T(a, b)|E_j\| \leq M_j \|a|A_j\| \|b|B_j\|, \quad a \in A_0 \cap A_1,\ b \in B_0 \cap B_1,\ j = 0, 1.$$

Then T might be uniquely extended to a bounded bilinear operator from $[A_0, A_1]_\theta \times [B_0, B_1]_\theta$ to $[E_0, E_1]_\theta$.

We close the study of the complex method with an example. Let (Ω, μ) be a σ-finite measure space, $1 \leq p_0, p_1 \leq \infty$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then

$$[L_{p_0}(\Omega), L_{p_1}(\Omega)]_\theta = L_p(\Omega) \quad \text{with equal norms.}$$

2.2 Besov and Triebel-Lizorkin spaces. Some basic properties.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. We write $\mathcal{S}'(\mathbb{R}^n)$ for the space of all tempered distributions. For $f \in \mathcal{S}'(\mathbb{R}^n)$, we put $\hat{f}$ for its Fourier transform and $\check{f}$ for its inverse Fourier transform.

Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\varphi_0(x) = 1 \text{ if } |x| \leq 1 \text{ and } \varphi_0(x) = 0 \text{ if } |x| \geq 3/2.$$
$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad k \in \mathbb{N}. \tag{2.32}$$

Here $|x|$ stands for the norm in $\mathbb{R}^n$. Since

$$\sum_{k=0}^{\infty} \varphi_k(x) = 1 \text{ for all } x \in \mathbb{R}^n,$$

the sequence $(\varphi_k)_{k=0}^{\infty}$ is a *smooth dyadic resolution of unity*.

Definition 2.13. For $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, then *Triebel-Lizorkin space* $F^s_{p,q}(\mathbb{R}^n)$ collects all $f \in \mathcal{S}'(\mathbb{R}^n)$ having finite quasi-norm

$$\|f|F^s_{p,q}(\mathbb{R}^n)\| = \Big\| \Big(\sum_{k=0}^{\infty} 2^{ksq} |(\varphi_k \hat{f})^\vee|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\|.$$

Triebel-Lizorkin spaces do not depend on the smooth dyadic resolution of unity and generalize other well-known spaces (see [111, Theorem 2.5.6]). For example, if $1 < p < \infty$ and $s \in \mathbb{R}$ then $F^s_{p,2}(\mathbb{R}^n)$ coincides with the *fractional Sobolev space* $H^s_p(\mathbb{R}^n)$ of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f|H^s_p(\mathbb{R}^n)\| = \|((1+|x|^2)^{s/2}\hat{f})^\vee|L_p(\mathbb{R}^n)\| < \infty. \tag{2.33}$$

If, in addition, $s = m \in \mathbb{N}$ then $F^m_{p,2}(\mathbb{R}^n) = H^m_p(\mathbb{R}^n) = W^m_p(\mathbb{R}^n)$ where this last space is the classical *Sobolev space* of all functions in $L_p(\mathbb{R}^n)$ whose distributional derivatives up to order m belong to $L_p(\mathbb{R}^n)$ as well.

Furthermore, according to [111, Theorem 2.5.8/1], for any $0 < p < \infty$,

$$F^0_{p,2}(\mathbb{R}^n) = h_p(\mathbb{R}^n), \tag{2.34}$$

with equivalent quasi-norms, where $h_p(\mathbb{R}^n)$ is the *local Hardy space* consisting of all $f \in \mathcal{S}'(\mathbb{R}^n)$ having finite quasi-norm

$$\|f|h_p(\mathbb{R}^n)\| = \Big\| \sup_{0<t<\infty} |(\psi(t\cdot)\hat{f})^\vee| |L_p(\mathbb{R}^n) \Big\|. \tag{2.35}$$

Here $\psi \in \mathcal{S}(\mathbb{R}^n)$ is compactly supported and $\psi(x) = 1$ if $|x| \leq 1$. The space $h_p(\mathbb{R}^n)$ does not depend on the choice of ψ. Finally, if $1 < p < \infty$, then

$$F^0_{p,2}(\mathbb{R}^n) = H^0_p(\mathbb{R}^n) = h_p(\mathbb{R}^n) = L_p(\mathbb{R}^n),$$

with equivalent quasi-norms.

Definition 2.14. For $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, then *Besov space* $B^s_{p,q}(\mathbb{R}^n)$ collects all $f \in \mathcal{S}'(\mathbb{R}^n)$ having finite quasi-norm

$$\|f|B^s_{p,q}(\mathbb{R}^n)\| = \Big(\sum_{k=0}^{\infty} 2^{ksq} \|(\varphi_k \hat{f})^\vee |L_p(\mathbb{R}^n)\|^q \Big)^{1/q}.$$

Again the definition of Besov spaces is independent from the resolution of unity taken (see [111, Proposition 2.3.2.1]).

It turns out that under certain conditions the interpolation of Triebel-Lizorkin spaces by the real method results in Besov spaces. In particular, if $0 < p < \infty$, $0 < \theta < 1$, $0 < q_0, q_1, q \leq \infty$, $-\infty < s_0 \neq s_1 < \infty$ and $s = (1-\theta)s_0 + \theta s_1$, then

$$(F^{s_0}_{p,q_0}(\mathbb{R}^n), F^{s_1}_{p,q_1}(\mathbb{R}^n))_{\theta,q} = B^s_{p,q}(\mathbb{R}^n).$$

See, for example, [111, Theorem 2.4.2]. According to [35, Theorem 5.3], under the same hypothesis as before if, in addition, $b \in \mathbb{R}$ then

$$(F^{s_0}_{p,q_0}(\mathbb{R}^n), F^{s_1}_{p,q_1}(\mathbb{R}^n))_{\theta,q,(b,b)} = B^{s,b}_{p,q}(\mathbb{R}^n), \tag{2.36}$$

where $B^{s,b}_{p,q}(\mathbb{R}^n)$ is a Besov space with additional logarithmic smoothness consisting of all $f \in \mathcal{S}'(\mathbb{R}^n)$ with finite quasi-norm

$$\|f|B^{s,b}_{p,q}(\mathbb{R}^n)\| = \Big(\sum_{k=0}^{\infty} [2^{ks}(1+k)^b \|(\varphi_k \hat{f})^\vee |L_p(\mathbb{R}^n)\|]^q \Big)^{1/q}. \tag{2.37}$$

Chapter 3

Duality for logarithmic interpolation spaces

The description of the dual space is an important and useful item on the list of properties of any interpolation method. The aim of this chapter is to determine the dual of limiting logarithmic interpolation spaces $(A_0, A_1)_{1,q,\mathbb{A}}$ and $(A_0, A_1)_{0,q,\mathbb{A}}$ in terms of the K-functional when $0 < q < 1$.

In Section 3.1, we review briefly some duality results in the literature for the real interpolation method, the interpolation method with a function parameter, logarithmic interpolation methods when $0 < \theta < 1$ and the limiting cases when $\theta = 0, 1$ and $1 \leq q < \infty$. We also introduce the concept of Banach envelope of a quasi-Banach space which is helpful when computing its dual and calculate the Banach envelope of logarithmic interpolation spaces defined by means of the J-functional when $0 < q < 1$.

In Section 3.2, we study equivalence theorems for logarithmic interpolation spaces defined by means of K and J-functionals when $0 < q \leq 1$ and $\theta = 0, 1$. These results will be the key for computing the dual of limiting logarithmic spaces under the previous hypothesis in Section 3.3 and they complement the already known properties for $1 \leq q \leq \infty$ (see [48]).

In the last section of this chapter we show applications of the duality results to function spaces and to operator spaces. In particular, among other things, we study the dual of the Besov spaces $\mathbf{B}^{0,b}_{p,q}$ defined by means of the modulus of smoothness associated to L_p $(1 < p < \infty)$, having classical smoothness 0 and logarithmic smoothness with exponent b. Finally we consider two scales of spaces of compact operators on a Hilbert space related to the Macaev ideals (see [90, 69]) and we describe the duality relationships between them.

The main results of this chapter have appeared in [13].

3.1 Dual of some interpolation methods

As we saw in Chapter 2 quasi-Banach spaces are defined in a similar way to Banach spaces. However, some of their properties differ in significant ways. For example, the dual of a quasi-Banach space can be $\{0\}$. This is the case of Lebesgue spaces $L_p([0,1])$ with the Lebesgue measure on $[0,1]$ (see [85, § 3.15.9, p.158]). In order to avoid this inconvenience, during the current chapter we are going to work with Banach couples.

When working with the dual of interpolation methods the first thing we wonder is under which hypothesis if $\bar{A} = (A_0, A_1)$ is a Banach couple then $\bar{A}^* = (A_0^*, A_1^*)$ is a Banach couple too. It turns out that if $\bar{A} = (A_0, A_1)$ is a *regular Banach couple*, that is to say, $A_0 \cap A_1$ is dense in A_0 and A_1 then

$$A_0^* \hookrightarrow (A_0 \cap A_1)^* \quad \text{and} \quad A_1^* \hookrightarrow (A_0 \cap A_1)^*,$$

and these embeddings are bounded. Therefore, $\bar{A}^* = (A_0^*, A_1^*)$ is a Banach couple.

The dual of the real interpolation method $(\cdot, \cdot)_{\theta,q}$ was already studied by Lions and Peetre in their foundational paper [89] when $0 < \theta < 1$ and $1 \leq q < \infty$. They proved that if $\bar{A} = (A_0, A_1)$ is a regular Banach couple then for $0 < \theta < 1$ and $1 \leq q < \infty$

$$(A_0, A_1)_{\theta,q}^* = (A_0^*, A_1^*)_{\theta,q'} \quad \text{where} \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

For the case of the real method with a function parameter $(\cdot, \cdot)_{\rho,q}$ Persson proved in [103, Theorem 2.4] that if $\bar{A} = (A_0, A_1)$ is a regular Banach couple, ρ is a function parameter and $1 \leq q < \infty$, then

$$(A_0, A_1)_{\rho,q}^* = (A_0^*, A_1^*)_{\rho_1,q'} \quad \text{where} \quad \rho_1(t) = \frac{1}{\rho(1/t)} \quad \text{and} \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

In particular, if $\rho(t) = t^\theta \ell^{-\mathbb{A}}(t)$ with $0 < \theta < 1$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ then we have $\rho_1(t) = t^\theta \ell^{(\alpha_\infty,\alpha_0)}(t)$. Thus

$$(A_0, A_1)_{\theta,q,\mathbb{A}}^* = (A_0^*, A_1^*)_{\theta,q',(-\alpha_\infty,-\alpha_0)}, \tag{3.1}$$

for any regular Banach couple $\bar{A}$ and $1 \leq q < \infty$.

As we saw in Theorem 2.8, for $1 \leq q < \infty$, the logarithmic interpolation space $(A_0, A_1)_{\theta,q,\mathbb{A}}$ still makes sense if

$$\begin{cases} \theta = 0, & \alpha_\infty + 1/q < 0; \\ \theta = 1, & \alpha_0 + 1/q < 0. \end{cases} \tag{3.2}$$

Cobos and Segurado in [48] studied the dual formula of the logarithmic methods in this limiting cases. They proved that if $1 \leq q < \infty$, $\theta = 1$ and $\alpha_0 + 1/q < 0$, then

$$(A_0, A_1)_{1,q,\mathbb{A}}^* = \begin{cases} (A_0^*, A_1^*)_{1,q',(-\alpha_\infty-1,-\alpha_0-1)} & \text{if} \quad \alpha_\infty + 1/q > 0; \\ (A_0^*, A_1^*)_{1,q',(-1/q',-\alpha_0-1),(-1,0)} & \text{if} \quad \alpha_\infty + 1/q = 0; \\ A_1^* \cap (A_0^*, A_1^*)_{1,q',(-1-1/q',-\alpha_0-1)} & \text{if} \quad \alpha_\infty + 1/q < 0. \end{cases} \tag{3.3}$$

for any regular Banach couple $\bar{A} = (A_0, A_1)$.

Noting that $(A_0, A_1)_{0,q,\mathbb{A}} = (A_1, A_0)_{1,q,(\alpha_\infty,\alpha_0)}$ for any quasi-Banach couple $\bar{A}$ (see (2.4)), we can also derive that if $1 \leq q < \infty$, $\theta = 0$ and $\alpha_\infty + 1/q < 0$, then

$$(A_0, A_1)_{0,q,\mathbb{A}}^* = \begin{cases} (A_0^*, A_1^*)_{0,q',(-\alpha_\infty-1,-\alpha_0-1)} & \text{if} \quad \alpha_0 + 1/q > 0; \\ (A_0^*, A_1^*)_{0,q',(-\alpha_\infty-1,-1/q'),(0,-1)} & \text{if} \quad \alpha_0 + 1/q = 0; \\ A_0^* \cap (A_0^*, A_1^*)_{0,q',(-\alpha_\infty-1,-1-1/q')} & \text{if} \quad \alpha_0 + 1/q < 0. \end{cases}$$

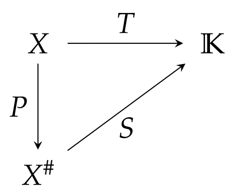
Observe that formulae in the limiting cases present a shift in the logarithmic exponents that does not appear in (3.1).

Now we wonder what happens when $0 < q < 1$. In this case the interpolation methods that we have seen before lead to quasi-Banach spaces even when applied to Banach couples. When working with the dual of a quasi-Banach space is convenient to consider its *Banach envelope* that we define next.

Let $(X, \| \cdot |X\|)$ be a quasi-Banach space. We consider

$$\|x|X\|^{\#} = \inf \left\{ \sum_{k=1}^{n} \|x_k|X\| : x = \sum_{k=1}^{n} x_k \right\}, \quad x \in X.$$

It turns out that $\| \cdot |X\|^{\#}$ is a semi-norm since it satisfies all the properties of a norm except that $\|x|X\|^{\#}$ equal 0 does not imply $x = 0$. We denote by N the space of all x in X such that $\|x|X\|^{\#} = 0$ and we designate as $X^{\#}$ the completion of the quotient space X/N. We call $X^{\#}$ the *Banach envelope* of X. It is a Banach space. Furthermore, for any $T \in X^*$ there exists a linear mapping $S \in (X^{\#})^*$ satisfying the following factorization:



where $P : X \longrightarrow X^{\#}$ is the canonical quotient map. In particular, for any quasi-Banach space X we have that

$$X^* = (X^{\#})^*. \tag{3.4}$$

For example, if X is a Banach space then $\| \cdot |X\|^{\#} = \| \cdot |X\|$ and therefore $X = X^{\#}$. If $0 < p < 1$, then according to [100], $(L_p([0,1]))^{\#} = \{0\}$ and $(\ell_p)^{\#} = \ell_1$. This implies that, $(\ell_p)^* = \ell_\infty$ and, as we said before, $(L_p([0,1]))^* = \{0\}$.

Peetre proved in [100] that if $0 < \theta < 1$ and $0 < q < 1$, for any Banach couple

$$[(A_0, A_1)_{\theta,q}]^{\#} = (A_0, A_1)_{\theta,1},$$

and, thereby, if $\bar{A}$ is regular, then $(A_0, A_1)^*_{\theta,q} = (A_0^*, A_1^*)_{\theta,\infty}$.

Following analogous ideas, it was showed by Cobos in [30, Theorem 3.1] and Mastyło in [92, Corollary 2] that if $0 < q < 1$ and ρ is a function parameter, then for any Banach couple $\bar{A} = (A_0, A_1)$

$$[(A_0, A_1)_{\rho,q}]^{\#} = (A_0, A_1)_{\rho,1}.$$

If, in addition, $\bar{A}$ is a regular Banach couple then we get that for $0 < q < 1$

$$(A_0, A_1)^*_{\rho,q} = (A_0^*, A_1^*)_{\rho_1,\infty} \quad \text{where} \quad \rho_1(t) = \frac{1}{\rho(1/t)}.$$

As we want to compute the dual of logarithmic interpolation methods when $0 < q < 1$, we should start by computing its Banach envelope. It turns out that it is easier to work in this case with J-spaces than with K-spaces.

Let $0 < q \leq 1, \theta \in \mathbb{R}, \mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. According to Theorem 2.10 the quasi-Banach sequence lattice $\Gamma = \ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))$ is $(1, J)$-non trivial if, and only if,

$$\begin{cases} 0 < \theta < 1; \\ \theta = 1, \quad \alpha_\infty > 0 \quad \text{or} \quad \alpha_\infty = 0 \text{ and } \beta_\infty \geq 0; \\ \theta = 0, \quad \alpha_0 > 0 \quad \text{or} \quad \alpha_0 = 0 \text{ and } \beta_0 \geq 0. \end{cases} \tag{3.5}$$

In this case we can calculate the Banach envelope as follows:

Theorem 3.1. Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Let $0 < q < 1, \theta \in \mathbb{R}, \mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ satisfying (3.5). We have that

$$[(A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}}]^\# = (A_0, A_1)^J_{\theta,1,\mathbb{A},\mathbb{B}}.$$

Proof. Since $\ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)) \hookrightarrow \ell_1(2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))$, then

$$(A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}} \hookrightarrow (A_0, A_1)^J_{\theta,1,\mathbb{A},\mathbb{B}}.$$

Hence, if $a \in (A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}}$ and $a = \sum_{k=1}^n a_k$, using that $(A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}}$ is a norm space, we obtain

$$\|a|(A_0, A_1)^J_{\theta,1,\mathbb{A},\mathbb{B}}\| \leq \sum_{k=1}^n \|a_k|(A_0, A_1)^J_{\theta,1,\mathbb{A},\mathbb{B}}\| \leq \sum_{k=1}^n \|a_k|(A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}}\|.$$

Taking the infimum over all finite decompositions $a = \sum_{k=1}^n a_k$ of a in $(A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}}$, it follows that

$$\|a|(A_0, A_1)^J_{\theta,1,\mathbb{A},\mathbb{B}}\| \leq \|a|(A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}}\|^\#.$$

In particular, we get that $\| \cdot |(A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}}\|^\#$ is not only a semi-norm but a norm and that

$$\left[(A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}}\right]^\# \hookrightarrow (A_0, A_1)^J_{\theta,1,\mathbb{A},\mathbb{B}}. \tag{3.6}$$

In order to establish the converse embedding, take any $u \in A_0 \cap A_1$ and let δ^k_m be the Kronecker delta. For any $k \in \mathbb{Z}$, using the J-representation $u = \sum_{m=-\infty}^\infty \delta^k_m u$, we obtain

$$\|u|(A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}}\| \leq 2^{-k\theta}\ell^{\mathbb{A}}(2^k)\ell\ell^{\mathbb{B}}(2^k)J(2^k, u), \quad \text{for any} \quad k \in \mathbb{Z}.$$

Let now $a \in (A_0, A_1)^J_{\theta,1,\mathbb{A},\mathbb{B}}$ and take any J-representation $a = \sum_{m=-\infty}^\infty u_m$ of a. The previous estimate yields that

$$\begin{aligned} \sum_{m=N}^M \|u_m|(A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}}\|^\# &\leq \sum_{m=N}^M \|u_m|(A_0, A_1)^J_{\theta,1,\mathbb{A},\mathbb{B}}\| \\ &\leq \sum_{m=N}^M 2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)J(2^m, u_m). \end{aligned}$$

It follows that the series $\sum_{m=-\infty}^{\infty} u_m$ is convergent in $\left[(A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}}\right]^\#$. The sum of the series should be also a because of the embedding (3.6). Therefore,

$$\begin{aligned}\|a|(A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}}\|^\# &\leq \sum_{m=-\infty}^{\infty} \|u_m|(A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}}\|^\# \\ &\leq \sum_{m=-\infty}^{\infty} 2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)J(2^m, u_m).\end{aligned}$$

This yields the embedding

$$(A_0, A_1)^J_{\theta,1,\mathbb{A},\mathbb{B}} \hookrightarrow [(A_0, A_1)^J_{\theta,q,\mathbb{A},\mathbb{B}}]^\#$$

and completes the proof. □

Corollary 3.2. Let $0 < q < 1$, $0 < \theta < 1$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. For any regular Banach couple $\bar{A} = (A_0, A_1)$ we have that

$$(A_0, A_1)^*_{\theta,q,\mathbb{A}} = (A_0^*, A_1^*)_{\theta,\infty,(-\alpha_\infty,-\alpha_0)}.$$

Proof. Under the above hypothesis, for any Banach couple

$$(A_0, A_1)_{\theta,q,\mathbb{A}} = (A_0, A_1)^J_{\theta,q,\mathbb{A}}.$$

This, together with Theorem 3.1 and (3.1), implies that

$$\begin{aligned}(A_0, A_1)^*_{\theta,q,\mathbb{A}} &= \left((A_0, A_1)^J_{\theta,q,\mathbb{A}}\right)^* = \left([(A_0, A_1)^J_{\theta,q,\mathbb{A}}]^\#\right)^* \\ &= \left((A_0, A_1)^J_{\theta,1,\mathbb{A}}\right)^* = (A_0, A_1)^*_{\theta,1,\mathbb{A}} \\ &= (A_0^*, A_1^*)_{\theta,\infty,(-\alpha_\infty,-\alpha_0)}.\end{aligned}$$

□

In order to apply the previous reasoning for the limiting cases $\theta = 0, 1$ and $\mathbb{A} = (\alpha_0, \alpha_\infty)$ satisfying (3.2), we need to study first the representation by means of the J-functional of logarithmic interpolation methods under these conditions. This is the main goal of next section.

3.2 Equivalence between K and J spaces for logarithmic methods and Banach couples

If $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, we put $\mathbb{A}\lambda = (\alpha_0\lambda, \alpha_\infty\lambda)$, $\mathbb{A} + \lambda = (\alpha_0 + \lambda, \alpha_\infty + \lambda)$ and $\widetilde{\mathbb{A}} = (\alpha_\infty, \alpha_0)$ for the reverse pair.

Let $\theta \in \mathbb{R}$, $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty)$. According to Theorem 2.8, the interpolation method $(\cdot,\cdot)_{\theta,q,\mathbb{A}}$ is K-non-trivial when the parameters satisfy

$$\begin{cases} 0 < \theta < 1; \\ \theta = 0, \quad 0 < q < \infty, \quad \alpha_\infty + 1/q < 0; \\ \theta = 0, \quad q = \infty, \quad \alpha_\infty \leq 0; \\ \theta = 1, \quad 0 < q < \infty, \quad \alpha_0 + 1/q < 0; \\ \theta = 1, \quad q = \infty, \quad \alpha_0 \leq 0. \end{cases} \tag{3.7}$$

As we already said, if $0 < \theta < 1$ then $\rho(t) = t^\theta \ell^{-\mathbb{A}}(t)$ is equivalent to a function parameter and, consequently, for any Banach couple $\bar{A} = (A_0, A_1)$ we have with equivalence of quasi-norms

$$(A_0, A_1)_{\theta,q,\mathbb{A}} = (A_0, A_1)^J_{\theta,q,\mathbb{A}}.$$

For the limiting cases when $\theta = 1$ and $1 \leq q \leq \infty$ Cobos and Segurado proved in [48] that for any Banach couple $\bar{A} = (A_0, A_1)$ we have the following equalities with equivalent norms:

$$(A_0, A_1)_{1,q,\mathbb{A}} = \begin{cases} (A_0, A_1)^J_{1,q,\mathbb{A}+1} & \text{if} \quad 1 \leq q \leq \infty, \quad \alpha_0 + 1/q < 0 < \alpha_\infty + 1/q; \\ (A_0, A_1)^J_{1,q,\mathbb{A}+1,(0,1)} & \text{if} \quad 1 \leq q < \infty, \quad \alpha_0 + 1/q < 0 = \alpha_\infty + 1/q; \\ (A_0 + A_1, A_1)^J_{1,q,(\alpha_0+1,1)} & \text{if} \quad 1 \leq q \leq \infty, \quad \alpha_0 + 1/q < 0, \quad \alpha_\infty + 1/q < 0; \\ & \text{or} \quad q = \infty, \quad \alpha_0 < 0, \quad \alpha_\infty = 0. \end{cases} \tag{3.8}$$

Analogous results can be derived for $\theta = 0$ noting that $(A_0, A_1)_{0,q,\mathbb{A}} = (A_1, A_0)_{1,q,(\alpha_\infty,\alpha_0)}$ for any Banach couple $\bar{A}$ (see 2.4). Observe that the case $\theta = 1$, $q = \infty$, $\alpha_0 = 0$ is included in (3.7) but not in (3.8). This special case will be consider later in Chapter 5.

Remark 3.3. Notice that in the case $1 \leq q \leq \infty$, $\alpha_0 + 1/q < 0$ and $\alpha_\infty + 1/q < 0$; or $q = \infty$, $\alpha_0 < 0$ and $\alpha_\infty = 0$, according to Theorem 2.9, there exists a Banach couple $\bar{A} = (A_0, A_1)$ such that $(A_0, A_1)_{1,q,\mathbb{A}}$ is not included in $(A_0 + A_1)^\circ$. This implies that $(A_0, A_1)_{1,q,\mathbb{A}}$ does not admits a J-representation in terms of (A_0, A_1). However, as we can see in (3.8), the spaces $(A_0, A_1)_{1,q,\mathbb{A}}$ admits a J-representation in terms of $(A_0 + A_1, A_1)$.

In what follows we are going to study the remaining limiting cases, that is to say, $\theta \in \{0, 1\}$, $0 < q \leq 1$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ verifying:

$$\begin{cases} \theta = 0, \quad \alpha_\infty + 1/q < 0; \\ \theta = 1, \quad \alpha_0 + 1/q < 0. \end{cases}$$

As before, we are going to focus on the case $\theta = 1$ since corresponding results for $\theta = 0$ can be derived straightforward later.

We start with an auxiliary result.

Lemma 3.4. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \leq 1$ such that $\alpha_0 + 1/q < 0$. Put

$$\mathrm{v}_{q,\mathbb{A}}(2^k) = \Big(\sum_{m=-\infty}^{\infty} [\min\{1, 2^{m-k}\} 2^{-m} \ell^{\mathbb{A}}(2^m)]^q \Big)^{1/q}, \quad k \in \mathbb{Z}.$$

Then we have:

a) If $\alpha_\infty + 1/q > 0$, $\mathrm{v}_{q,\mathbb{A}}(2^k) \sim 2^{-k}\ell^{\mathbb{A}+1/q}(2^k)$.

b) If $\alpha_\infty + 1/q < 0$, $\mathrm{v}_{q,\mathbb{A}}(2^k) \sim 2^{-k}\ell^{(\alpha_0+1/q,0)}(2^k)$.

c) If $\alpha_\infty + 1/q = 0$, $\mathrm{v}_{q,\mathbb{A}}(2^k) \sim \begin{cases} 2^{-k}\ell^{\alpha_0+1/q}(2^k) & \text{if } k \leq 0, \\ 2^{-k}\ell\ell^{1/q}(2^k) & \text{if } k > 0. \end{cases}$

Proof. We have

$$\begin{aligned} \mathrm{v}_{q,\mathbb{A}}(2^k)^q &= 2^{-kq}\sum_{m=-\infty}^{k}\ell^{\mathbb{A}q}(2^m) + \sum_{m=k+1}^{\infty}2^{-mq}\ell^{\mathbb{A}q}(2^m) \\ &\sim 2^{-kq}\int_0^{2^k}\ell^{\mathbb{A}q}(t)\frac{dt}{t} + 2^{-kq}\ell^{\mathbb{A}q}(2^k). \end{aligned}$$

To estimate the integral $I = \int_0^{2^k}\ell^{\mathbb{A}q}(t)\frac{dt}{t}$ we distinguish several cases. If $k \in \mathbb{Z}$ and $k \leq 0$, since $\alpha_0 + 1/q < 0$, we get

$$I = \int_0^{2^k}(1-\log t)^{\alpha_0 q}\frac{dt}{t} \sim (1-\log 2^k)^{\alpha_0 q+1} = \ell^{\mathbb{A}q+1}(2^k).$$

This yields that $\mathrm{v}_{q,\mathbb{A}}(2^k)^q \sim 2^{-kq}\ell^{\mathbb{A}q+1}(2^k)$ for $k \leq 0$ and $k \in \mathbb{Z}$. Assume now that $k > 0$, $k \in \mathbb{Z}$, then

$$\begin{aligned} I &= \int_0^1(1-\log t)^{\alpha_0 q}\frac{dt}{t} + \int_1^{2^k}(1+\log t)^{\alpha_\infty q}\frac{dt}{t} \\ &\sim \int_1^{2^k}(1+\log t)^{\alpha_\infty q}\frac{dt}{t}. \end{aligned}$$

In the case (a) we obtain $I \sim \ell^{\alpha_\infty q+1}(2^k)$, which implies that

$$\mathrm{v}_{q,\mathbb{A}}(2^k)^q \sim 2^{-kq}\ell^{\mathbb{A}q+1}(2^k), \quad \text{for } k > 0, \quad k \in \mathbb{Z}.$$

In the case (b) we get $I \sim 1$ and so

$$\mathrm{v}_{q,\mathbb{A}}(2^k)^q \sim 2^{-kq} \quad \text{for } k > 0, \quad k \in \mathbb{Z}.$$

Finally, in the case (c),

$$I \sim \int_1^{2^k}(1+\log t)^{-1}\frac{dt}{t} \sim \ell\ell(2^k),$$

which yields that

$$\mathrm{v}_{q,\mathbb{A}}(2^k)^q \sim 2^{-kq}\ell\ell(2^k) \quad \text{for } k > 0, \quad k \in \mathbb{Z}.$$

The proof is finished. □

For a Banach couple $\bar{A} = (A_0, A_1)$ the Gagliardo completions $A_0^\sim$ and $A_1^\sim$ defined in Section 2.1.1 will be useful now.

Theorem 3.5. Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \leq 1$ such that $\alpha_0 + 1/q < 0$. Then we have with equivalence of quasi-norms

(i) $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q}$ if $\alpha_\infty + 1/q > 0$,

(ii) $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q,(0,1/q)}$ if $\alpha_\infty + 1/q = 0$.

If, in addition, $\bar{A}$ is regular then

(iii) $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)^J_{1,q,(\alpha_0+1/q,0)}$ if $\alpha_\infty + 1/q < 0$.

Proof. It is enough to work with the couple $(A_0^\sim, A_1^\sim)$ because, by (2.8), we know that

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}}.$$

Let a be any element of the J-spaces of the statement and let $a = \sum_{k=-\infty}^{\infty} u_k$ be a J-representation of a. Since $0 < q \le 1$, we have

$$K(2^m, a)^q \le \Big(\sum_{k=-\infty}^{\infty} K(2^m, u_k)\Big)^q \le \sum_{k=-\infty}^{\infty} K(2^m, u_k)^q \le \sum_{k=-\infty}^{\infty} [\min\{1, 2^{m-k}\} J(2^k, u_k)]^q.$$

Hence

$$\begin{aligned} \|a|(A_0, A_1)_{1,q,\mathbb{A}}\| &= \Big(\sum_{m=-\infty}^{\infty} [2^{-m}\ell^{\mathbb{A}}(2^m) K(2^m, a)]^q\Big)^{1/q} \\ &\le \Big(\sum_{k=-\infty}^{\infty} J(2^k, u_k)^q \sum_{m=-\infty}^{\infty} [\min\{1, 2^{m-k}\} 2^{-m}\ell^{\mathbb{A}}(2^m)]^q\Big)^{1/q}. \end{aligned}$$

The interior sum can be estimated by using Lemma 3.4, with the outcome that in any of the cases (i), (ii), (iii), the J-space of the statement is continuously embedded in $(A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}}$.

Suppose now that $a \in (A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}}$. In the cases (i) and (ii), it follows from Theorem 2.9 that $a \in (A_0^\sim, A_1^\sim)^\circ$. In the case (iii), the additional assumption that $\bar{A}$ is regular and [22, Corollary 2.2.23] imply that $a \in (A_0^\sim + A_1^\sim)^\circ$ as well. According to [95, Theorem 3.2] there exists a representation $a = \sum_{k=-\infty}^{\infty} u_k$ (convergence in $A_0^\sim + A_1^\sim$) with $(u_k) \subseteq A_0^\sim \cap A_1^\sim$ and

$$\Big(\sum_{k=-\infty}^{\infty} [\min\{1, 2^{m-k}\} J(2^k, u_k; A_0^\sim, A_1^\sim)]^q\Big)^{1/q} \le cK(2^m, a), \quad m \in \mathbb{Z}, \tag{3.9}$$

where c is a constant depending on q only. Let

$$\Gamma = \begin{cases} \ell_q(2^{-m}\ell^{\mathbb{A}+1/q}(2^m)) & \text{if} \quad \alpha_\infty + 1/q > 0; \\ \ell_q(2^{-m}\ell^{\mathbb{A}+1/q}(2^m)\ell\ell^{(0,1/q)}(2^m)) & \text{if} \quad \alpha_\infty + 1/q = 0; \\ \ell_q(2^{-m}\ell^{(\alpha_0+1/q,0)}(2^m)) & \text{if} \quad \alpha_\infty + 1/q < 0. \end{cases}$$

By Lemma 3.4 the weight in front of $J(2^k, u_k; A_0^\sim, A_1^\sim)$ in the J-norm $\|a|(A_0^\sim, A_1^\sim)_{\Gamma;J}\|$ is equivalent to $\mathrm{v}_{q,\mathbb{A}}(2^k)$ in any of the cases (i), (ii) and (iii). Consequently,

$$\begin{aligned} \|a|(A_0^\sim, A_1^\sim)_{\Gamma;J}\| &\lesssim \Big(\sum_{k=-\infty}^{\infty} (\mathrm{v}_{q,\mathbb{A}}(2^k) J(2^k, u_k; A_0^\sim, A_1^\sim))^q\Big)^{1/q} \\ &= \Big(\sum_{m=-\infty}^{\infty} 2^{-mq}\ell^{\mathbb{A}q}(2^m) \sum_{k=-\infty}^{\infty} \min\{1, 2^{m-k}\}^q J(2^k, u_k; A_0^\sim, A_1^\sim)^q\Big)^{1/q} \\ &\lesssim \Big(\sum_{m=-\infty}^{\infty} [2^{-m}\ell^{\mathbb{A}}(2^m) K(2^m, a)]^q\Big)^{1/q} = \|a|(A_0, A_1)_{1,q,\mathbb{A}}\|, \end{aligned}$$

where the last inequality follows from (3.9). This completes the proof. □

Theorem 3.6. Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \leq 1$ such that $\alpha_\infty + 1/q < 0$. Then we have with equivalence of quasi-norms

(i) $(A_0, A_1)_{0,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)^J_{0,q,\mathbb{A}+1/q}$ if $\alpha_0 + 1/q > 0$,

(ii) $(A_0, A_1)_{0,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)^J_{0,q,\mathbb{A}+1/q,(1/q,0)}$ if $\alpha_0 + 1/q = 0$.

If, in addition, $\bar{A}$ is regular then

(iii) $(A_0, A_1)_{0,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)^J_{0,q,(0,\alpha_\infty+1/q)}$ if $\alpha_0 + 1/q < 0$.

Proof. From (2.4) it follows that $(A_0, A_1)_{0,q,\mathbb{A}} = (A_1, A_0)_{1,q,\tilde{\mathbb{A}}}$. Applying Theorem 3.5 we derive that

$$(A_0, A_1)_{0,q,\mathbb{A}} = (A_1, A_0)_{1,q,\tilde{\mathbb{A}}} = \begin{cases} (A_1^\sim, A_0^\sim)^J_{1,q,\tilde{\mathbb{A}}+1/q} & \text{if } \alpha_0 + 1/q > 0 \\ (A_1^\sim, A_0^\sim)^J_{1,q,\tilde{\mathbb{A}}+1/q,(0,1/q)} & \text{if } \alpha_0 + 1/q = 0, \\ (A_1^\sim, A_0^\sim)^J_{1,q,(\alpha_\infty+1/q,0)} & \text{if } \alpha_0 + 1/q < 0 \\ & \text{and } \bar{A} \text{ is regular.} \end{cases}$$

Now we get the result from (2.11). □

Remark 3.7. If $0 < q < 1$, $\alpha_0 + 1/q < 0$ and $\alpha_\infty + 1/q < 0$, Theorem 2.9 implies that there exists a Banach couple $\bar{A} = (A_0, A_1)$ such that $(A_0, A_1)_{1,q,\mathbb{A}}$ is not included in $(A_0 + A_1)^\circ$. This means that $(A_0, A_1)_{1,q,\mathbb{A}}$ does not admits a J-representation in terms of (A_0, A_1). However, Theorem 2.7/(iii) proves that if the Banach couple is regular then such representation does exist.

Now we show another equivalence result that will be useful later. It refers to the Banach case where $1 \leq q \leq \infty$, therefore we will use the continuous description for J-spaces given in (2.31).

Theorem 3.8. Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Let $1 \leq q \leq \infty$, $1/q + 1/q' = 1$ and $\alpha_\infty, \beta_0 \in \mathbb{R}$ such that

$$\begin{cases} \alpha_\infty + 1/q > 0 \text{ and } \beta_0 + 1/q < 0 & \text{if} \quad 1 \leq q < \infty, \\ \alpha_\infty > 0 \text{ and } \beta_0 < 0 & \text{if} \quad q = \infty. \end{cases} \tag{3.10}$$

Then we have with equivalent norms

$$(A_0, A_1)_{1,q,(-1/q,\alpha_\infty),(\beta_0,0)} = (A_0, A_1)^J_{1,q,(1/q',\alpha_\infty+1),(\beta_0+1,0)}.$$

Proof. Let us call for short $\bar{A}_K$ to the K-space and $\bar{A}_J$ to the J-space. Assume that $1 \leq q < \infty$. The case $q = \infty$ can be treated analogously. Take any $a \in \bar{A}_J$ and let $a = \int_0^\infty u(t)\frac{dt}{t}$ be a J-representation of a such that

$$\left(\int_0^1 \left[\frac{\ell^{1/q'}(t)\ell\ell^{\beta_0+1}(t)J(t,u(t))}{t} \right]^q \frac{dt}{t} + \int_1^\infty \left[\frac{\ell^{\alpha_\infty+1}(t)J(t,u(t))}{t} \right]^q \frac{dt}{t} \right)^{1/q} \leq 2\|a|\bar{A}_J\|.$$

Using that

$$\begin{aligned} \frac{1}{t}K(t,a) &\leq \frac{1}{t}\int_0^\infty K(t,u(s))\frac{ds}{s} \leq \frac{1}{t}\int_0^\infty \min\{1,t/s\}J(s,u(s))\frac{ds}{s} \\ &\leq \frac{1}{t}\int_0^t J(s,u(s))\frac{ds}{s} + \int_t^\infty \frac{1}{s}J(s,u(s))\frac{ds}{s}, \end{aligned}$$

we obtain

$$\begin{aligned}
\|a|\bar{A}_K\| &\le \Big(\int_0^1 \Big[\frac{(1-\log t)^{-1/q}(1+\log(1-\log t))^{\beta_0}}{t} \int_0^t J(s,u(s)) \frac{ds}{s} \Big]^q \frac{dt}{t} \Big)^{1/q} \\
&+ \Big(\int_0^1 \Big[(1-\log t)^{-1/q}(1+\log(1-\log t))^{\beta_0} \int_t^1 \frac{J(s,u(s))}{s} \frac{ds}{s} \Big]^q \frac{dt}{t} \Big)^{1/q} \\
&+ \Big(\int_0^1 \Big[(1-\log t)^{-1/q}(1+\log(1-\log t))^{\beta_0} \int_1^\infty \frac{J(s,u(s))}{s} \frac{ds}{s} \Big]^q \frac{dt}{t} \Big)^{1/q} \\
&+ \Big(\int_1^\infty \Big[\frac{(1+\log t)^{\alpha_\infty}}{t} \int_0^1 J(s,u(s)) \frac{ds}{s} \Big]^q \frac{dt}{t} \Big)^{1/q} \\
&+ \Big(\int_1^\infty \Big[\frac{(1+\log t)^{\alpha_\infty}}{t} \int_1^t J(s,u(s)) \frac{ds}{s} \Big]^q \frac{dt}{t} \Big)^{1/q} \\
&+ \Big(\int_1^\infty \Big[(1+\log t)^{\alpha_\infty} \int_t^\infty \frac{J(s,u(s))}{s} \frac{ds}{s} \Big]^q \frac{dt}{t} \Big)^{1/q} \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

Next we show that $I_j \lesssim \|a|\bar{A}_J\|$ for $j = 1, ..., 6$. We start with I_1 in the special case $q = 1$. Taking any $0 < \varepsilon < 1$ and using Fubini's theorem, we get

$$\begin{aligned}
I_1 &= \int_0^1 J(s,u(s)) \int_s^1 \frac{(1-\log t)^{-1}(1+\log(1-\log t))^{\beta_0}}{t} \frac{dt}{t} \frac{ds}{s} \\
&\lesssim \int_0^1 J(s,u(s)) \frac{(1-\log s)^{-1}(1+\log(1-\log s))^{\beta_0}}{s^\varepsilon} \int_s^1 t^{\varepsilon-1} \frac{dt}{t} \frac{ds}{s} \\
&\lesssim \int_0^1 J(s,u(s)) \frac{(1-\log s)^{-1}(1+\log(1-\log s))^{\beta_0}}{s} \frac{ds}{s} \\
&\le \int_0^1 J(s,u(s)) \frac{(1+\log(1-\log s))^{\beta_0+1}}{s} \frac{ds}{s} \lesssim \|a|\bar{A}_J\|.
\end{aligned}$$

If $1 < q < \infty$, by Hölder's inequality, for the interior integral in I_1 we get

$$\begin{aligned}
\int_0^t J(s,u(s)) \frac{ds}{s} &\le \Big(\int_0^t \Big[\frac{J(s,u(s))}{s} \ell^{1/q'}(s) \ell\ell^{\beta_0+1}(s) \Big]^q \frac{ds}{s} \Big)^{1/q} \\
&\quad \times \Big(\int_0^t \Big[s\ell^{-1/q'}(s) \ell\ell^{-(\beta_0+1)}(s) \Big]^{q'} \frac{ds}{s} \Big)^{1/q'} \\
&\lesssim t\ell^{-1/q'}(t) \ell\ell^{-\beta_0-1}(t) \Big(\int_0^t \Big[\frac{J(s,u(s))}{s} \ell^{1/q'}(s) \ell\ell^{\beta_0+1}(s) \Big]^q \frac{ds}{s} \Big)^{1/q}.
\end{aligned}$$

Whence, by Fubini's theorem,

$$\begin{aligned}
I_1 &\lesssim \Big(\int_0^1 \Big[\frac{J(s,u(s))}{s} \ell^{1/q'}(s) \ell\ell^{\beta_0+1}(s) \Big]^q \int_s^1 \Big[\ell^{-1}(t) \ell\ell^{-1}(t) \Big]^q \frac{dt}{t} \frac{ds}{s} \Big)^{1/q} \\
&\lesssim \Big(\int_0^1 \Big[\frac{J(s,u(s))}{s} \ell^{1/q'}(s) \ell\ell^{\beta_0+1}(s) \Big]^q \frac{ds}{s} \Big)^{1/q} \lesssim \|a|\bar{A}_J\|.
\end{aligned}$$

To estimate I_2 when $q = 1$, since $\beta_0 + 1 < 0$, using Fubini's theorem, we get

$$\begin{aligned}
I_2 &= \int_0^1 \frac{J(s,u(s))}{s} \int_0^s (1-\log t)^{-1}(1+\log(1-\log t))^{\beta_0} \frac{dt}{t} \frac{ds}{s} \\
&= \int_0^1 \frac{J(s,u(s))}{s} (1+\log(1-\log s))^{\beta_0+1} \frac{ds}{s} \lesssim \|a|\bar{A}_J\|.
\end{aligned}$$

If $1 < q < \infty$, taking $\varepsilon > 0$ such that $\beta_0 + 1 + 1/q < \varepsilon < 1$, we obtain

$$\begin{aligned}\int_t^1 \frac{J(s,u(s))}{s}\frac{ds}{s} &\leq \Big(\int_t^1 \Big[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\varepsilon-1/q}(s)\Big]^q\frac{ds}{s}\Big)^{1/q}\\ &\quad\times\Big(\int_t^1 \ell^{-1}(s)\ell\ell^{-\varepsilon q'+q'/q}(s)\frac{ds}{s}\Big)^{1/q'}\\ &\lesssim \ell\ell^{-\varepsilon+1}(t)\Big(\int_t^1 \Big[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\varepsilon-1/q}(s)\Big]^q\frac{ds}{s}\Big)^{1/q}.\end{aligned}$$

Hence, by Fubini's theorem,

$$\begin{aligned}I_2 &\lesssim \Big(\int_0^1 \Big[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\varepsilon-1/q}(s)\Big]^q\int_0^s \ell^{-1}(t)\ell\ell^{(\beta_0-\varepsilon+1)q}(t)\frac{dt}{t}\frac{ds}{s}\Big)^{1/q}\\ &\lesssim \Big(\int_0^1 \Big[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\beta_0+1}(s)\Big]^q\frac{ds}{s}\Big)^{1/q}\lesssim \|a|\bar{A}_J\|.\end{aligned}$$

As for I_3, since $\beta_0 + 1/q < 0$ and $\alpha_\infty + 1/q > 0$, for any $1 \leq q < \infty$ we derive

$$\begin{aligned}I_3 &= \int_1^\infty \frac{J(s,u(s))}{s}\frac{ds}{s}\Big(\int_0^1 (1-\log t)^{-1}(1+\log(1-\log t))^{\beta_0 q}\frac{dt}{t}\Big)^{1/q}\\ &\lesssim \int_1^\infty \frac{J(s,u(s))}{s}\frac{ds}{s}\\ &\lesssim \Big(\int_1^\infty \Big[\frac{J(s,u(s))}{s}\ell^{\alpha_\infty+1}(s)\Big]^q\frac{ds}{s}\Big)^{1/q}\Big(\int_1^\infty \ell^{-(\alpha_\infty+1)q'}(s)\frac{ds}{s}\Big)^{1/q'}\\ &\lesssim \Big(\int_1^\infty \Big[\frac{J(s,u(s))}{s}\ell^{\alpha_\infty+1}(s)\Big]^q\frac{ds}{s}\Big)^{1/q}\lesssim \|a|\bar{A}_J\|.\end{aligned}$$

Consider now I_4 for any $1 \leq q < \infty$. We obtain

$$\begin{aligned}I_4 &\lesssim \int_0^1 J(s,u(s))\frac{ds}{s}\\ &\leq \Big(\int_0^1 \Big[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\beta_0+1}(s)\Big]^q\frac{ds}{s}\Big)^{1/q}\\ &\quad\times\Big(\int_0^1 \Big[s\ell^{-1/q'}(s)\ell\ell^{-(\beta_0+1)}(s)\Big]^{q'}\frac{ds}{s}\Big)^{1/q'}\\ &\lesssim \Big(\int_0^1 \Big[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{\beta_0+1}(s)\Big]^q\frac{ds}{s}\Big)^{1/q}\lesssim \|a|\bar{A}_J\|.\end{aligned}$$

The integral I_5 (respectively, I_6) coincides with J_4 (respectively, I_4) in the proof of [48, Theorem 3.5]. Therefore,

$$I_5 \lesssim \Big(\int_1^\infty \Big[\frac{J(s,u(s))}{s}(1+\log s)^{\alpha_\infty}\Big]^q\frac{ds}{s}\Big)^{1/q}\lesssim \|a|\bar{A}_J\|$$

and

$$I_6 \lesssim \Big(\int_1^\infty \Big[\frac{J(s,u(s))}{s}(1+\log s)^{\alpha_\infty+1}\Big]^q\frac{ds}{s}\Big)^{1/q}\lesssim \|a|\bar{A}_J\|.$$

This shows the embedding $\bar{A}_J \hookrightarrow \bar{A}_K$.

Next we proceed with the converse embedding. Take any $a \in \bar{A}_K$. By Theorem 2.9, we have that

$$\min\{1, 1/t\}K(t,a) \to 0 \text{ as } t \to 0 \text{ and as } t \to \infty. \tag{3.11}$$

For $\nu \in \mathbb{Z}$, we put

$$\gamma_\nu = \begin{cases} 2^{-2^{2^{-\nu-1}}} & \text{if} \quad \nu < 0, \\ 1 & \text{if} \quad \nu = 0, \\ 2^{2^{\nu-1}} & \text{if} \quad \nu > 0. \end{cases}$$

We can decompose $a = a_{0,\nu} + a_{1,\nu}$ with $a_{j,\nu} \in A_j$, $j = 0,1$, such that

$$\|a_{0,\nu}|A_0\| + \gamma_{\nu-1}\|a_{1,\nu}|A_1\| \le 2K(\gamma_{\nu-1}, a), \quad \nu \in \mathbb{Z}. \tag{3.12}$$

Write

$$u_\nu = a_{0,\nu} - a_{0,\nu-1} = a_{1,\nu-1} - a_{1,\nu} \in A_0 \cap A_1.$$

According to (3.12) we have

$$\begin{aligned} \|a - \sum_{\nu=N}^{M} u_\nu|A_0 + A_1\| &\le \|a_{0,N-1}|A_0\| + \|a_{1,M}|A_1\| \\ &\lesssim K(\gamma_{N-2}, a) + \frac{K(\gamma_{M-1}, a)}{\gamma_{M-1}} \to 0 \text{ as } M \to \infty \text{ and } N \to -\infty \end{aligned}$$

by (3.11). So $a = \sum_{\nu=-\infty}^{\infty} u_\nu$ in $A_0 + A_1$.

Let $D_\nu = (\gamma_{\nu-1}, \gamma_\nu]$, $\nu \in \mathbb{Z}$. We have

$$\int_{D_\nu} \frac{dt}{t} = \begin{cases} \log 2 & \text{if} \quad \nu = 1, \\ 2^{\nu-2}\log 2 & \text{if} \quad \nu > 1. \end{cases}$$

For $\nu \le 0$ put

$$\delta_\nu = \int_{D_\nu} (1 - \log t)^{-1}(1 + \log(1 - \log t))^{-1}\frac{dt}{t} \sim 1.$$

Consider the function

$$w(t) = \begin{cases} \frac{u_\nu}{\delta_\nu \ell(t)\ell\ell(t)} & \text{if} \quad t \in D_\nu \text{ and } \nu \le 0, \\ \frac{u_1}{\log 2} & \text{if} \quad t \in D_1, \\ \frac{u_\nu}{2^{\nu-2}\log 2} & \text{if} \quad t \in D_\nu \text{ and } \nu > 1. \end{cases}$$

Then

$$\int_0^\infty w(t)\frac{dt}{t} = \sum_{\nu=-\infty}^{\infty} u_\nu = a \text{ in } A_0 + A_1.$$

If $\nu \le 0$ and $t \in D_\nu$, using (3.12), we get

$$\frac{J(t, w(t))}{t} \sim \ell^{-1}(t)\ell\ell^{-1}(t)\frac{J(t, u_\nu)}{t} \le \ell^{-1}(t)\ell\ell^{-1}(t)\frac{J(\gamma_{\nu-1}, u_\nu)}{\gamma_{\nu-1}} \lesssim \ell^{-1}(t)\ell\ell^{-1}(t)\frac{K(\gamma_{\nu-2}, a)}{\gamma_{\nu-2}}.$$

Whence,

$$
\begin{aligned}
\Big(\int_{D_\nu}\Big[\ell^{1/q'}(t)\ell\ell^{\beta_0+1}(t)\frac{J(t,w(t))}{t}\Big]^q\frac{dt}{t}\Big)^{1/q}
&\lesssim \frac{K(\gamma_{\nu-2},a)}{\gamma_{\nu-2}}\Big(\int_{D_\nu}\Big[\ell^{-1/q}(t)\ell\ell^{\beta_0}(t)\Big]^q\frac{dt}{t}\Big)^{1/q}\\
&\sim \frac{K(\gamma_{\nu-2},a)}{\gamma_{\nu-2}}2^{|\nu|(\beta_0+1/q)}\\
&\sim \frac{K(\gamma_{\nu-2},a)}{\gamma_{\nu-2}}\Big(\int_{D_{\nu-2}}\Big[\ell^{-1/q}(t)\ell\ell^{\beta_0}(t)\Big]^q\frac{dt}{t}\Big)^{1/q}\\
&\le \Big(\int_{D_{\nu-2}}\Big[\frac{K(t,a)}{t}\ell^{-1/q}(t)\ell\ell^{\beta_0}(t)\Big]^q\frac{dt}{t}\Big)^{1/q}
\end{aligned}
$$

because $t^{-1}K(t,a)$ is a non-increasing function. If $\nu > 2$, proceeding as in [48, Theorem 3.5] we obtain

$$
\Big(\int_{D_\nu}\Big[\ell^{\alpha_\infty+1}(t)\frac{J(t,w(t))}{t}\Big]^q\frac{dt}{t}\Big)^{1/q}\lesssim\Big(\int_{D_{\nu-2}}\Big[\ell^{\alpha_\infty}(t)\frac{K(t,a)}{t}\Big]^q\frac{dt}{t}\Big)^{1/q}.
$$

For $\nu = 1, 2$ we derive

$$
\Big(\int_{D_\nu}\Big[\ell^{\alpha_\infty+1}(t)\frac{J(t,w(t))}{t}\Big]^q\frac{dt}{t}\Big)^{1/q}\lesssim\Big(\int_{D_{\nu-2}}\Big[\ell^{-1/q}(t)\ell\ell^{\beta_0}(t)\frac{K(t,a)}{t}\Big]^q\frac{dt}{t}\Big)^{1/q}.
$$

Consequently,

$$
\begin{aligned}
\|a|\bar{A}_J\|^q &\le \sum_{\nu=-\infty}^{0}\int_{D_\nu}\Big[\ell^{1/q'}(t)\ell\ell^{\beta_0+1}(t)\frac{J(t,w(t))}{t}\Big]^q\frac{dt}{t}\\
&\quad+\sum_{\nu=1}^{\infty}\int_{D_\nu}\Big[\ell^{\alpha_\infty+1}(t)\frac{J(t,w(t))}{t}\Big]^q\frac{dt}{t}\\
&\lesssim \int_0^1\Big[\frac{K(t,a)}{t}\ell^{-1/q}(t)\ell\ell^{\beta_0}(t)\Big]^q\frac{dt}{t}+\int_1^\infty\Big[\frac{K(t,a)}{t}\ell^{\alpha_\infty}(t)\Big]^q\frac{dt}{t}\\
&= \|a|\bar{A}_K\|^q.
\end{aligned}
$$

This completes the proof. □

The following result can be establish by using similar arguments as in Theorem 3.8.

Theorem 3.9. Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Let $1 \le q \le \infty$, $1/q + 1/q' = 1$ and $\alpha_0, \beta_\infty \in \mathbb{R}$ such that

$$
\begin{cases}\alpha_0 + 1/q < 0 \text{ and } \beta_\infty + 1/q > 0 & \text{if} \quad 1 \le q < \infty,\\ \alpha_0 < 0 \text{ and } \beta_\infty > 0 & \text{if} \quad q = \infty.\end{cases}
$$

Then

$$
(A_0, A_1)_{1,q,(\alpha_0,-1/q),(0,\beta_\infty)} = (A_0, A_1)^J_{1,q,(\alpha_0+1,1/q'),(0,\beta_\infty+1)}.
$$

3.3 Duality of limiting logarithmic methods when 0 < q < 1

We start by computing the Banach envelope of limiting logarithmic interpolation spaces defined by means of the K-functional.

Theorem 3.10. Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q < 1$ such that $\alpha_0 + 1/q < 0$. Then we have:

(i) $(A_0, A_1)^{\#}_{1,q,\mathbb{A}} = (A_0, A_1)_{1,1,\mathbb{A}+1/q-1}$, if $\alpha_\infty + 1/q > 0$.

(ii) $(A_0, A_1)^{\#}_{1,q,\mathbb{A}} = (A_0, A_1)_{1,1,(\alpha_0+1/q-1,-1),(0,1/q-1)}$, if $\alpha_\infty + 1/q = 0$.

If, in addition, $\bar{A}$ is a regular Banach couple, then

(iii) $(A_0, A_1)^{\#}_{1,q,\mathbb{A}} = (A_0, A_1)_{1,1,(\alpha_0+1/q-1,\delta)}$ if $\alpha_\infty + 1/q < 0$ and $\delta < -1$.

Proof. For the case (i), according to Theorem 3.5, Theorem 3.1 and (2.8), we derive

$$\begin{aligned}(A_0, A_1)^{\#}_{1,q,\mathbb{A}} &= \left((A_0^{\sim}, A_1^{\sim})^J_{1,q,\mathbb{A}+1/q}\right)^{\#} \\ &= (A_0^{\sim}, A_1^{\sim})^J_{1,1,\mathbb{A}+1/q} \\ &= (A_0^{\sim}, A_1^{\sim})_{1,1,\mathbb{A}+1/q-1} = (A_0, A_1)_{1,1,\mathbb{A}+1/q-1}.\end{aligned}$$

For the case (ii), we proceed similarly but using also Theorem 3.9. We obtain

$$\begin{aligned}(A_0, A_1)^{\#}_{1,q,\mathbb{A}} &= \left((A_0^{\sim}, A_1^{\sim})^J_{1,q,(\alpha_0+1/q,0),(0,1/q)}\right)^{\#} \\ &= (A_0^{\sim}, A_1^{\sim})^J_{1,1,(\alpha_0+1/q,0),(0,1/q)} \\ &= (A_0, A_1)_{1,1,(\alpha_0+1/q-1,-1),(0,1/q-1)}.\end{aligned}$$

Finally, in the case (iii), we have

$$\begin{aligned}(A_0, A_1)^{\#}_{1,q,\mathbb{A}} &= \left((A_0^{\sim}, A_1^{\sim})^J_{1,q,(\alpha_0+1/q,0)}\right)^{\#} \\ &= (A_0^{\sim}, A_1^{\sim})^J_{1,1,(\alpha_0+1/q,0)} \\ &= (A_0, A_1)_{1,1,(\alpha_0+1/q-1,\delta)}.\end{aligned}$$

□

Next we proceed to determine the dual of $(A_0, A_1)_{1,q,\mathbb{A}}$ for $0 < q < 1$.

Theorem 3.11. Let $\bar{A} = (A_0, A_1)$ be a regular Banach couple. Let $\mathbb{A} = (\alpha_0, \alpha_\infty)$ and $0 < q < 1$ such that $\alpha_0 + 1/q < 0$.

(i) If $\alpha_\infty + 1/q > 0$, then $(A_0, A_1)^*_{1,q,\mathbb{A}} = (A_0^*, A_1^*)_{1,\infty,(-\alpha_\infty-1/q,-\alpha_0-1/q)}$.

(ii) If $\alpha_\infty + 1/q = 0$, then $(A_0, A_1)^*_{1,q,\mathbb{A}} = (A_0^*, A_1^*)_{1,\infty,(0,-\alpha_0-1/q),(-1/q,0)}$.

(iii) If $\alpha_\infty + 1/q < 0$, then $(A_0, A_1)^*_{1,q,\mathbb{A}} = A_1^* \cap (A_0^*, A_1^*)_{1,\infty,(-1,-\alpha_0-1/q)}$.

Proof. In the case (i), according to (3.4) and Theorem 3.10, we obtain

$$(A_0, A_1)^*_{1,q,\mathbb{A}} = \left((A_0, A_1)^{\#}_{1,q,\mathbb{A}}\right)^* = \left((A_0, A_1)_{1,1,\mathbb{A}+1/q-1}\right)^*.$$

As we saw in (3.3), the last dual space has been determined in [48, Theorem 5.6] with the result that

$$(A_0, A_1)^*_{1,q,\mathbb{A}} = (A_0^*, A_1^*)_{1,\infty,(-\alpha_\infty-1/q,-\alpha_0-1/q)}.$$

For the case (ii), applying Theorem 3.10 and [51, Theorem 3.1], we get

$$\begin{aligned}(A_0, A_1)^*_{1,q,\mathbb{A}} &= (A_0, A_1)^*_{1,1,(\alpha_0+1/q-1,-1),(0,1/q-1)} \\ &= (A_0^*, A_1^*)^J_{1,\infty,(1,-\alpha_0-1/q+1),(-1/q+1,0)}.\end{aligned}$$

We can describe the J-space in terms of the K-functional by using Theorem 3.8. We derive that

$$(A_0, A_1)^*_{1,q,\mathbb{A}} = (A_0^*, A_1^*)_{1,\infty,(0,-\alpha_0-1/q),(-1/q,0)}.$$

In the last case (iii), take any $\delta < -1$. Using Theorem 3.10 we obtain

$$(A_0, A_1)^*_{1,q,\mathbb{A}} = \Big((A_0, A_1)_{1,1,(\alpha_0+1/q-1,\delta)}\Big)^*$$

and, by [48, Theorem 5.10], we get

$$(A_0, A_1)^*_{1,q,\mathbb{A}} = A_1^* \cap (A_0^*, A_1^*)_{1,\infty,(-1,-\alpha_0-1/q)}.$$

□

We can derive straightforward the corresponding duality formulae for $\theta = 0$.

Theorem 3.12. Let $\bar{A} = (A_0, A_1)$ be a regular Banach couple. Let $\mathbb{A} = (\alpha_0, \alpha_\infty)$ and $0 < q < 1$ such that $\alpha_\infty + 1/q < 0$.

(i) If $\alpha_0 + 1/q > 0$, then $(A_0, A_1)^*_{0,q,\mathbb{A}} = (A_0^*, A_1^*)_{0,\infty,(-\alpha_\infty-1/q,-\alpha_0-1/q)}$.

(ii) If $\alpha_0 + 1/q = 0$, then $(A_0, A_1)^*_{0,q,\mathbb{A}} = (A_0^*, A_1^*)_{0,\infty,(-\alpha_\infty-1/q,0),(0,-1/q)}$.

(iii) If $\alpha_0 + 1/q < 0$, then $(A_0, A_1)^*_{0,q,\mathbb{A}} = A_0^* \cap (A_0^*, A_1^*)_{0,\infty,(-\alpha_\infty-1/q,-1)}$.

3.4 Some applications

3.4.1 Application to Besov spaces

Now we focus our attention on Besov spaces with logarithmic smoothness $B^{s,b}_{p,q}(\mathbb{R}^n)$ introduced in (2.37). It turns out that if $s > 0$ then these spaces can also be defined by means of the modulus of smoothness.

Let f be a (complex-valued) function on $\mathbb{R}^n$, $h \in \mathbb{R}^n$ and $k \in \mathbb{N}$. We put

$$(\Delta_h^1 f)(x) = f(x+h) - f(x) \quad \text{and} \quad (\Delta_h^{k+1} f)(x) = \Delta_h^1(\Delta_h^k f)(x).$$

The *k-th order modulus of smoothness* of a function $f \in L_p(\mathbb{R}^n)$ is defined by

$$\omega_k(f,t)_p = \sup\{\|\Delta_h^k f | L_p(\mathbb{R}^n)\| : |h| \le t\}, \quad t > 0.$$

If $k = 1$, we simply wirte $\omega(f,t)_p$ instead of $\omega_1(f,t)_p$.

For $1 \le p \le \infty$, $0 < q \le \infty$, $b \in \mathbb{R}$, $s \ge 0$ and $k \in \mathbb{N}$ with $k > s$, the space $\mathbf{B}^{s,b}_{p,q}(\mathbb{R}^n)$ consists of all $f \in L_p(\mathbb{R}^n)$ with finite quasi-norm

$$\|f|\mathbf{B}^{s,b}_{p,q}(\mathbb{R}^n)\| = \|f|L_p(\mathbb{R}^n)\| + \left(\int_0^1 [t^{-s}(1-\log t)^b \omega_k(f,t)_p]^q \frac{dt}{t}\right)^{1/q} < \infty.$$

As usual, the integral should be replaced by the supremum if $q = \infty$. If $s = 0$ and $b < -1/q$ then $\int_0^1 (1-\log t)^{bq} \frac{dt}{t} < \infty$ and so $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. Also $\mathbf{B}^{0,0}_{p,\infty} = L_p(\mathbb{R}^n)$.

If $s > 0$, according to [75, Theorem 2.5], it follows that $B^{s,b}_{p,q}(\mathbb{R}^n) = \mathbf{B}^{s,b}_{p,q}(\mathbb{R}^n)$. When $s = 0$ spaces $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^n)$ and $B^{0,b}_{p,q}(\mathbb{R}^n)$ are different but closely related. For $1 < p < \infty$, $0 < q \le \infty$ and $b > -1/q$ Cobos and Domínguez proved in [32, Theorem 3.3], the following embeddings:

$$B^{0,b+\frac{1}{\min\{2,p,q\}}}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathbf{B}^{0,b}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{0,b+\frac{1}{\max\{2,p,q\}}}_{p,q}(\mathbb{R}^n).$$

Duality of spaces $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^n)$ and $B^{0,b}_{p,q}(\mathbb{R}^n)$ has been studied in [32, Section 4] for the Banach case $1 \le q < \infty$. Next we want to study the quasi-Banach case $0 < q < 1$. We start with spaces $B^{0,b}_{p,q}(\mathbb{R}^n)$.

Theorem 3.13. Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $0 < q < 1$ and $b \in \mathbb{R}$. Then

$$(B^{0,b}_{p,q}(\mathbb{R}^n))^* = B^{0,-b}_{p',\infty}(\mathbb{R}^n).$$

Proof. It follows from (2.36) that $B^{0,b}_{p,q}(\mathbb{R}^n) = (H^{-1}_p(\mathbb{R}^n), H^1_p(\mathbb{R}^n))_{1/2,q,(b,b)}$ where $H^s_p(\mathbb{R}^n)$ is the fractional Sobolev space defined in (2.33). Note that $(H^{-1}_p(\mathbb{R}^n), H^1_p(\mathbb{R}^n))$ is a regular Banach couple since $\mathcal{S}(\mathbb{R}^n)$ is densely embedded in both spaces (see [11, Theorem 6.2.3]). Using that $(H^s_p(\mathbb{R}^n))^* = H^{-s}_{p'}(\mathbb{R}^n)$ (see [110, Theorem 2.6.1]) and Corollary 3.2 we obtain

$$\begin{aligned}(B^{0,b}_{p,q}(\mathbb{R}^n))^* &= (H^{-1}_p(\mathbb{R}^n), H^1_p(\mathbb{R}^n))^*_{1/2,q,(b,b)} \\ &= (H^1_{p'}(\mathbb{R}^n), H^{-1}_{p'}(\mathbb{R}^n))_{1/2,\infty,(-b,-b)} = B^{0,-b}_{p',\infty}(\mathbb{R}^n).\end{aligned}$$

□

Before computing the dual space of $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^n)$ we need to make the following remark regarding the logarithmic interpolation of ordered Banach couples.

Remark 3.14. Let $\bar{A} = (A_0, A_1)$ be a Banach couple such that $A_0 \hookrightarrow A_1$, with the embedding having norm less than or equal to 1, then it can be easily seen that $K(t,a) = t\|a|A_1\|$ for $0 < t \le 1$. In this case, if $0 \le \theta \le 1$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \le \infty$ satisfy (3.7), then

$$\|a|\bar{A}_{\theta,q,\mathbb{A}}\| \sim \left(\int_1^\infty [t^{-\theta} \ell^{\alpha_\infty}(t) K(t,a)]^q \frac{dt}{t}\right)^{1/q} \tag{3.13}$$

since

$$\begin{aligned}\left(\int_0^1 [t^{-\theta} K(t,a) \ell^{\alpha_0}(t)]^q \frac{dt}{t}\right)^{1/q} &= \|a|A_1\| \left(\int_0^1 t^{(1-\theta)q} \ell^{\alpha_0 q}(t) \frac{dt}{t}\right)^{1/q} \\ &\sim \|a|A_1\| \left(\int_1^\infty t^{-\theta q} \ell^{\alpha_\infty q}(t) \frac{dt}{t}\right)^{1/q} \\ &\le \left(\int_1^\infty [t^{-\theta} \ell^{\alpha_\infty}(t) K(t,a)]^q \frac{dt}{t}\right)^{1/q}.\end{aligned}$$

On the other hand, when $A_1 \hookrightarrow A_0$ with the embedding having norm less than or equal to 1, then $K(t,a) = \|a|A_0\|$ for every $t \geq 1$. If $0 \leq \theta \leq 1$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \leq \infty$ satisfy (3.7), we have

$$\begin{aligned}\Big(\int_1^\infty [t^{-\theta}\ell^{\alpha_\infty}(t)K(t,a)]^q \frac{dt}{t}\Big)^{1/q} &= \|a|A_0\| \Big(\int_1^\infty [t^{-\theta}\ell^{\alpha_\infty}(t)]^q \frac{dt}{t}\Big)^{1/q} \\ &\sim \|a|A_0\| \Big(\int_0^1 [t^{1-\theta}\ell^{\alpha_0}(t)]^q \frac{dt}{t}\Big)^{1/q} \\ &\leq \Big(\int_0^1 [t^{-\theta}\ell^{\alpha_0}(t)K(t,a)]^q \frac{dt}{t}\Big)^{1/q},\end{aligned}$$

where we have used that $K(t,a)/t$ is non-increasing and $K(1,a) = \|a|A_0\|$. Whence, we obtain

$$\|a|\bar{A}_{\theta,q,\mathbb{A}}\| \sim \Big(\int_0^1 [t^{-\theta}\ell^{\alpha_0}(t)K(t,a)]^q \frac{dt}{t}\Big)^{1/q}. \tag{3.14}$$

Next we determine the dual of spaces $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^n)$ with the help of *Lipschitz spaces* $\mathrm{Lip}^{(1,-\alpha)}_{p,\infty}(\mathbb{R}^n)$. For $1 < p < \infty$ and $\alpha \geq 0$, the space $\mathrm{Lip}^{(1,-\alpha)}_{p,\infty}(\mathbb{R}^n)$ consists of all $f \in L_p(\mathbb{R}^n)$ having a finite norm

$$\|f|\mathrm{Lip}^{(1,-\alpha)}_{p,\infty}(\mathbb{R}^n)\| = \|f|L_p(\mathbb{R}^n)\| + \sup_{0<t<1} \frac{\omega(f,t)_p}{t(1-\log t)^\alpha}.$$

These Lipschitz spaces have been studied by Haroske in [73, 74] (see also [55, p.149]). Note that the difference between spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,\infty}(\mathbb{R}^n)$ and $\mathbf{B}^{(1,-\alpha)}_{p,\infty}(\mathbb{R}^n)$ is that the modulus of smoothness in the definition of the first one is of order 1 while in the second one is of order strictly bigger than 1.

For $s \in \mathbb{R}$, we denote by $I_s = ((1+|x|^2)^{s/2}\hat{f})^\vee$.

Theorem 3.15. Let $1 < p < \infty$, $1/p + 1/p' = 1$, $0 < q < 1$ and $b + 1/q > 0$. The space $(\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^n))^*$ is formed by all $f \in H^{-1}_{p'}(\mathbb{R}^n)$ such that $I_{-1}f \in \mathrm{Lip}^{1,-b-1/q}_{p',\infty}(\mathbb{R}^n)$. Moreover

$$\|f|(\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^n))^*\| \sim \|I_{-1}f|\mathrm{Lip}^{1,-b-1/q}_{p',\infty}(\mathbb{R}^n)\|.$$

Proof. Consider the Banach couple formed by the Sobolev space $W^1_p(\mathbb{R}^n)$ and $L_p(\mathbb{R}^n)$. By [6, Theorem V.4.12], we get

$$K(t,f;W^1_p(\mathbb{R}^n),L_p(\mathbb{R}^n)) = tK(t^{-1},f;L_p(\mathbb{R}^n),W^1_p(\mathbb{R}^n)) \sim \min\{1,t\}\|f|L_p(\mathbb{R}^n)\| + t\omega(f,t^{-1})_p.$$

Take any τ with $\tau + 1/q < 0$. Using (3.13), we obtain that

$$\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^n) = \Big(W^1_p(\mathbb{R}^n),L_p(\mathbb{R}^n)\Big)_{1,q,(\tau,b)}.$$

Applying Theorem 3.11 and having in mind that $(H^s_p(\mathbb{R}^n))^* = H^{-s}_{p'}(\mathbb{R}^n)$, we derive that

$$\Big(\mathbf{B}^{0,b}_{p,q}\Big)^* = \Big(H^{-1}_{p'}(\mathbb{R}^n),L_{p'}(\mathbb{R}^n)\Big)_{1,\infty,(-b-1/q,-\tau-1/q)}.$$

On the other hand, since $I_{-1}: H^s_{p'}(\mathbb{R}^n) \to H^{s+1}_{p'}(\mathbb{R}^n)$ is bijective and bounded, we have

$$\begin{aligned}K(t,f;H^{-1}_{p'}(\mathbb{R}^n),L_{p'}(\mathbb{R}^n)) &\sim K(t,I_{-1}f;L_{p'}(\mathbb{R}^n),W^1_{p'}(\mathbb{R}^n)) \\ &\sim \min\{1,t\}\|I_{-1}f|L_{p'}(\mathbb{R}^n)\| + \omega(I_{-1}f,t)_{p'}.\end{aligned}$$

Therefore, using (3.14), we conclude

$$\begin{aligned}\|f|(\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^n))^*\| &\sim \sup_{0<t<1}\left(\frac{K(t,f;H^{-1}_{p'}(\mathbb{R}^n),L_{p'}(\mathbb{R}^n))}{t(1-\log t)^{b+1/q}}\right)\\ &\sim \|I_{-1}f|L_{p'}(\mathbb{R}^n)\| + \sup_{0<t<1}\left(\frac{\omega(I_{-1}f,t)_{p'}}{t(1-\log t)^{b+1/q}}\right)\\ &= \|I_{-1}f|\mathrm{Lip}^{1,-b-1/q}_{p',\infty}\|.\end{aligned}$$

□

For the special case when $b=0$ we obtain the following.

Corollary 3.16. Let $1<p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$ and $0<q<1$. The space $(\mathbf{B}^0_{p,q}(\mathbb{R}^n))'$ is formed by all $f\in H^{-1}_{p'}(\mathbb{R}^n)$ such that $I_{-1}f\in \mathrm{Lip}^{(1,-1/q)}_{p',\infty}(\mathbb{R}^n)$. Moreover

$$\|f|(\mathbf{B}^0_{p,q}(\mathbb{R}^n))^*\| \sim \|I_{-1}f|\mathrm{Lip}^{(1,-1/q)}_{p',\infty}(\mathbb{R}^n)\|.$$

As follows from Theorem 3.13, the dual of $B^{0,b}_{p,q}(\mathbb{R}^n)$ does not depend on q for $0<q<1$. However this is not the case for $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^n)$. Moreover, Theorem 3.15 point out a remarkable difference between the dual of $\mathbf{B}^{0,b}_{p,q}$ in the quasi-Banach case $0<q<1$ and the dual in the Banach case $1\le q<\infty$ described in [32, Theorem 4.3]: $\|f|(\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^n))^*\| \sim \|I_{-1}f|\mathrm{Lip}^{1,-b-1}_{p',q'}(\mathbb{R}^n)\|$.

As one can see, the role of q when $0<q<1$ is in the exponent, $b-1/q$, of the logarithm in the associated Lipschitz space, while in the Banach case the exponent has the constant value $-b-1$ and q has a role in the second index q', of the Lipschitz space.

3.4.2 Application to operator spaces

Let H be a Hilbert space. For $T\in\mathcal{L}(H)$, *the singular numbers* of T are defined by

$$s_n(T)=\inf\{\|T-R|\mathcal{L}(H)\| : R\in\mathcal{L}(H) \text{ with rank } R<n\},\ n\in\mathbb{N}.$$

Let S_∞ be the subspace of $\mathcal{L}(H)$ formed by all compact operators and for $1\le p<\infty$ let S_p be *the Schatten p-class*, formed by all those $T\in\mathcal{L}(H)$ having a finite norm

$$\|T|S_p\| = \left(\sum_{n=1}^{\infty} s_n(T)^p\right)^{1/p}$$

(see [69, 84]). The so-called *Macaev ideals* $S_\Pi, S_{\infty,1}$ are defined as the collection of all $T\in\mathcal{L}(H)$ which have a finite norm

$$\begin{aligned}\|T|S_\Pi\| &= \sup\{(1+\log n)^{-1}\sum_{k=1}^{n} s_k(T) : n\in\mathbb{N}\},\\ \|T|S_{\infty,1}\| &= \sum_{n=1}^{\infty} s_n(T)n^{-1},\end{aligned}$$

respectively (see [90, 69, 4, 49]). We have the following continuous embeddings

$$S_1 \hookrightarrow S_\Pi^\circ \hookrightarrow S_\Pi \hookrightarrow S_q \hookrightarrow S_{\infty,1} \hookrightarrow S_\infty \hookrightarrow \mathcal{L}(H).$$

Here $1 < q < \infty$ and S° stands for the closure in the space S of the set of all finite rank operators in H. Furthermore, the following duality formulae hold

$$(S_1)^* = \mathcal{L}(H) \quad \text{and} \quad (S_p)^* = S_{p'} \quad \text{for } 1 < p \leq \infty, 1/p + 1/p' = 1, \tag{3.15}$$

$$(S_\Pi^\circ)^* = S_{\infty,1} \quad \text{and} \quad (S_{\infty,1})^* = S_\Pi, \tag{3.16}$$

(see [69, Theorems III.12.3 and III.15.2]).

The space $S_{\infty,1}$ is a member of the scale

$$S_{\infty,q,b} = \Big\{T \in \mathcal{L}(H) : \|T|S_{\infty,q,b}\| = \Big(\sum_{n=1}^{\infty}[(1+\log n)^b s_n(T)]^q n^{-1}\Big)^{1/q} < \infty\Big\}$$

where $0 < q \leq \infty$, $b + 1/q \geq 0$ if $q < \infty$ and $b > 0$ if $q = \infty$ because otherwise the space is equal to $\mathcal{L}(H)$. Theses spaces make sense even for operators between Banach spaces provided that we replace the singular numbers by the approximation numbers. They have been studied in [29, 46] among other papers.

Associated to S_Π we may consider the scale

$$S_{\Pi,q,b} = \Big\{T \in \mathcal{L}(H) : \|T|S_{\Pi,q,b}\| = \Big(\sum_{n=1}^{\infty}[(1+\log n)^b \sum_{k=1}^{n} s_k(T)]^q n^{-1}\Big)^{1/q} < \infty\Big\}$$

where $b + 1/q < 0$ if $q < \infty$ and $b \leq 0$ if $q = \infty$ because otherwise the space is just $\{0\}$.

All these spaces can be obtained by interpolation between S_1 and S_∞ by using logarithmic interpolation methods as we show next.

Lemma 3.17. Let $0 < q \leq \infty$ and $b, \eta \in \mathbb{R}$.

(i) If $b + 1/q < 0 \leq \eta + 1/q$ and $q < \infty$, or $b \leq 0 < \eta$ and $q = \infty$, then we have, with equivalent quasi-norms,

$$(S_\infty, S_1)_{1,q,(b,\eta)} = S_{\Pi,q,b}.$$

(ii) If $\eta + 1/q < 0 \leq b + 1/q$ and $q < \infty$, or $\eta \leq 0 < b$ and $q = \infty$, then we have, with equivalent quasi-norms,

$$(S_1, S_\infty)_{1,q,(\eta,b)} = S_{\infty,q,b}.$$

Proof. Since $S_1 \hookrightarrow S_\infty$ and

$$K(n^{-1}, T; S_\infty, S_1) = n^{-1}\sum_{k=1}^{n} s_k(T), \ n \in \mathbb{N} \tag{3.17}$$

(see [109, 94, 110]), using (3.14) we obtain

$$\begin{aligned}
\|T|(S_\infty, S_1)_{1,q,(b,\eta)}\| &\sim \Big(\int_0^1 [t^{-1}(1-\log t)^b K(t,T)]^q \frac{dt}{t} \Big)^{1/q} \\
&\sim \Big(\sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} [t^{-1}(1-\log t)^b K(t,T)]^q \frac{dt}{t} \Big)^{1/q} \\
&\sim \Big(\sum_{n=1}^{\infty} [(1+\log n)^b \sum_{k=1}^{n} s_k(T)]^q n^{-1} \Big)^{1/q} \\
&\sim \|T|S_{\Pi,q,b}\|.
\end{aligned}$$

To prove (ii) we use (3.13) obtaining that

$$\begin{aligned}
\|T|(S_1, S_\infty)_{1,q,(\eta,b)}\| &\sim \Big(\int_1^\infty [t^{-1}(1+\log t)^b K(t,T)]^q \frac{dt}{t} \Big)^{1/q} \\
&\sim \Big(\sum_{n=1}^{\infty} \int_{n}^{n+1} [t^{-1}(1+\log t)^b K(t,T)]^q \frac{dt}{t} \Big)^{1/q} \\
&\sim \Big(\sum_{n=1}^{\infty} [(1+\log n)^b \frac{1}{n} \sum_{k=1}^{n} s_k(T)]^q n^{-1} \Big)^{1/q} \\
&\sim \Big(\sum_{n=1}^{\infty} [(1+\log n)^b s_n(T)]^q n^{-1} \Big)^{1/q} \\
&= \|T|S_{\infty,q,b}\|
\end{aligned}$$

where we have used the generalized Hardy's inequality established in [47, Theorem 1.2] for the last equivalence. □

Remark 3.18. Under the assumptions of Lemma 3.17, according to Theorem 2.9, we have

$$(\mathcal{L}(H), S_1)_{1,q,(b,\eta)} \subseteq S_\infty, \quad (S_1, \mathcal{L}(H))_{1,q,(\eta,b)} \subseteq S_\infty.$$

Since $S_1 \hookrightarrow S_\infty \hookrightarrow \mathcal{L}(H)$, it is easy to check that

$$K(t, T; S_1, \mathcal{L}(H)) = K(t, T; S_1, S_\infty), \quad T \in S_\infty.$$

Therefore, under the assumptions on b, q, η as in Lemma 3.17, we also have

$$(\mathcal{L}(H), S_1)_{1,q,(b,\eta)} = S_{\Pi,q,b}, \quad (S_1, \mathcal{L}(H))_{1,q,(\eta,b)} = S_{\infty,q,b}.$$

We close the chapter by showing the duality relationships between these two scales of spaces.

Theorem 3.19. Let $0 < q \leq \infty$ and $b \in \mathbb{R}$ with $b + 1/q < 0$. Then we have

(i) $\left(S_{\Pi,q,b}\right)^* = S_{\infty,\infty,-b-1/q}$ if $0 < q < 1$,

(ii) $\left(S_{\Pi,q,b}\right)^* = S_{\infty,q',-b-1}$ if $1 \leq q < \infty$, $1/q + 1/q' = 1$,

(iii) $\left(S^{\circ}_{\Pi,\infty,b}\right)^* = S_{\infty,1,-b-1}$.

Proof. By Lemma 3.17, we know that $S_{\Pi,q,b} = (S_\infty, S_1)_{1,q,(b,0)}$. Moreover, (S_∞, S_1) is a regular Banach couple since finite rank operators are dense in both spaces. Hence, if $0 < q < 1$, according to

(3.15), Theorem 3.11/(i) and Remark 3.18, we obtain

$$(S_{\Pi,q,b})^* = (S_1, \mathcal{L}(H))_{1,\infty,(-1/q,-b-1/q)} = S_{\infty,\infty,-b-1/q}.$$

If $1 \le q < \infty$, we proceed similarly but using now [48, Theorem 5.6]. Finally, if $q = \infty$, we get by [48, Theorem 5.9]

$$\begin{aligned}(S^{\circ}_{\Pi,\infty,b})^* &= \Big((S_\infty, S_1)^{\circ}_{1,\infty,(b,1)}\Big)^* \\ &= (S_1, \mathcal{L}(H))_{1,1,(-2,-b-1)} = S_{\infty,1,-b-1}.\end{aligned}$$

□

Theorem 3.20. Let $0 < q \le \infty$ and $0 < b + 1/q$. Then we have

(i) $(S_{\infty,q,b})^* = S_{\Pi,\infty,-b-1/q}$ if $0 < q < 1$,

(ii) $(S_{\infty,q,b})^* = S_{\Pi,q',-b-1}$ if $1 \le q < \infty$, $1/q + 1/q' = 1$,

(iii) $(S^{\circ}_{\infty,\infty,b})^* = S_{\Pi,1,-b-1}$.

Proof. For $0 < q < 1$, according to Lemma 3.17, Theorem 3.11/(i) and Remark 3.18, we derive

$$\begin{aligned}(S_{\infty,q,b})^* &= \Big((S_1, S_\infty)_{1,q,(-2/q,b)}\Big)^* \\ &= (\mathcal{L}(H), S_1)_{1,\infty,(-b-1/q,1/q)} = S_{\Pi,\infty,-b-1/q}.\end{aligned}$$

The case $1 \le q < \infty$ is similar but using [48, Theorem 5.6]. If $q = \infty$, by Lemma 3.17 and [48, Theorem 5.9], we obtain

$$\begin{aligned}(S^{\circ}_{\infty,\infty,b})^* &= \Big((S_1, S_\infty)^{\circ}_{1,\infty,(-1,b)}\Big)^* \\ &= (\mathcal{L}(H), S_1)_{1,1,(-b-1,0)} = S_{\Pi,1,-b-1}.\end{aligned}$$

□

Since $S_\Pi = S_{\Pi,\infty,-1}$ and $S_{\infty,1} = S_{\infty,1,0}$, formulae (3.16) follows from Theorems 3.19 and 3.20.

Remark 3.21. Looking at the statements of Theorems 3.19 and 3.20 one can see that the exponent of the logarithm depends on q when $0 < q < 1$, while this is not the case for $1 \le q \le \infty$.

The dual of the space $S_{\infty,q,-1/q}$ does not belong to the scale of spaces $S_{\Pi,p,b}$. We compute it in the case $0 < q < 1$.

Theorem 3.22. Let $0 < q < 1$. Then $(S_{\infty,q,-1/q})^*$ coincides with the collection of all $T \in \mathcal{L}(H)$ which have a finite norm

$$\|T\| = \sup\{(1 + \log(1 + \log n))^{-1/q} \sum_{k=1}^{n} s_k(T) : k \in \mathbb{N}\}.$$

Proof. Since $S_{\infty,q,-1/q} = (S_1, S_\infty)_{1,q,(-2/q,-1/q)}$, it follows from Theorem 3.11/(ii) that

$$(S_{\infty,q,-1/q})^* = (\mathcal{L}(H), S_1)_{1,\infty,(0,1/q),(-1/q,0)}.$$

In order to identify this interpolation space, we observe that $K(t, T; \mathcal{L}(H), S_1) = \|T|\mathcal{L}(H)\|$ for $t \geq 1$. Hence,

$$\sup\{t^{-1}K(t, T)\ell^{1/q}(t) : t \geq 1\} \sim \|T|\mathcal{L}(H)\|.$$

Consequently, using (3.17), we derive

$$\begin{aligned}
\|T|(\mathcal{L}(H), S_1)_{1,\infty,(0,1/q),(-1/q,0)}\| &\sim \sup\{t^{-1}K(t, T; \mathcal{L}(H), S_1)\ell\ell^{-1/q}(t) : 0 < t < 1\} \\
&= \sup\{K(s, T; S_1, \mathcal{L}(H))\ell\ell^{-1/q}(s) : 1 < s < \infty\} \\
&\sim \sup\{(1 + \log(1 + \log n))^{-1/q} \sum_{k=1}^{n} s_k(T) : n \in \mathbb{N}\}.
\end{aligned}$$

□

Chapter 4

Associate spaces of logarithmic interpolation spaces

In Chapter 3 we have studied the dual spaces of logarithmic interpolation methods $(A_0, A_1)_{\theta,q,\mathbb{A}}$ when $\theta = 0, 1$ and $A_0 \cap A_1$ is dense in A_0 and A_1. Curiously, although the couple (L_1, L_∞) does not satisfy that $L_1 \cap L_\infty$ is dense in L_∞, writing down the duality results for (L_1, L_∞) the outcome coincides with the associate spaces of generalized Lorentz-Zygmund spaces $L_{(p,q,\mathbb{A})}$, determined by Opic and Pick in [97]. In this chapter we aim to clarify this coincidence.

We compute the associate space of $(X_0, X_1)_{\theta,q,\mathbb{A}}$ where X_j are Banach function spaces on a $\sigma-$finite measure space (Ω, μ). Here, again, we need the description of logarithmic spaces in terms of the J-functional. Since, for $\theta = 0$ and $\theta = 1$, there is no J-description in a certain range of the parameters (see Remarks 3.3 and 3.7), we show first in Section 4.1 that in such range the space $(A_0, A_1)_{j,q,\mathbb{A}}$ turns out to be equal to the sum of A_j and a modified logarithmic J-space. This result is of independent interest, it complements those stated in (3.8) and Theorem 3.5 and it is useful in Section 4.2. Finally, in Section 4.3, we show some applications of the abstract results. We first consider a non-atomic $\sigma-$finite measure space (Ω, μ) and applying the results to the couple $(L_1(\Omega), L_\infty(\Omega))$ we recover the results of Opic and Pick [97] on associate spaces of generalized Lorentz-Zygmund spaces $L_{(p,q,\mathbb{A})}(\Omega)$. Then we establish the corresponding results for the sequence spaces $\ell_{(p,q,\alpha)}$. For this aim, we work with the measure space $(\mathbb{N}, \#)$, where # is the counting measure, which is completely atomic. The sequence case has not been studied previously.

The main results of this chapter form the paper [18].

4.1 Further equivalence results for logarithmic interpolation methods

Let $0 < q \le \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying

$$\begin{cases} 0 < q < \infty, & \alpha_0 + 1/q < 0, \quad \alpha_\infty + 1/q < 0; \\ q = \infty, & \alpha_0 < 0, \quad \alpha_\infty \le 0. \end{cases} \tag{4.1}$$

According to Remarks 3.3 and 3.7, under the above hypothesis there is not a J-representation of $(A_0, A_1)_{1,q,\mathbb{A}}$ in terms of (A_0, A_1) for every Banach couple (A_0, A_1). However, we show next that in this range $(A_0, A_1)_{1,q,\mathbb{A}}$ is the sum of A_1 with a modified J-space that we define next.

Put $\mathbb{Z}^- = \{0, -1, -2, -3, ...\}$. If $0 < q \le \infty$ and $\alpha \in \mathbb{R}$, we write $[\bar{A}]^J_{1,q,\alpha} = [A_0, A_1]^J_{1,q,\alpha}$ for the collection of all $a \in A_0 + A_1$ such that there exists $(u_m)_{m\in\mathbb{Z}^-} \subseteq A_0 \cap A_1$ satisfying

$$a = \sum_{m=-\infty}^{0} u_m \text{ (convergence in } A_0 + A_1)$$

and

$$\Big(\sum_{m=-\infty}^{0} \big[2^{-m} \ell^{\alpha}(2^m) J(2^m, u_m)\big]^q \Big)^{1/q} < \infty.$$

We endow $[A_0, A_1]^J_{1,q,\alpha}$ with the quasi-norm

$$\|a|[A_0, A_1]^J_{1,q,\alpha}\| = \inf\Big\{ \Big(\sum_{m=-\infty}^{0} \big[2^{-m} \ell^{\alpha}(2^m) J(2^m, u_m)\big]^q \Big)^{1/q} : a = \sum_{m=-\infty}^{0} u_m \Big\}.$$

Proposition 4.1. Let $0 < q \le \infty$, $\alpha \in \mathbb{R}$ and $\bar{A} = (A_0, A_1)$ be a Banach couple. Then

$$A_0 \cap A_1 \hookrightarrow [A_0, A_1]^J_{1,q,\alpha} \hookrightarrow A_0 + A_1.$$

Proof. Indeed, take any $a \in A_0 \cap A_1$ and for any $m \in \mathbb{Z}^-$, put $u_m = \delta^0_m a$ where δ^k_m is the Kronecker delta. So, $a = \sum_{m=-\infty}^{0} u_m$ and $\|a|[A_0, A_1]^J_{1,q,\alpha}\| \le J(1, a) = \|a|A_0 \cap A_1\|$. On the other hand, take any $a \in [A_0, A_1]^J_{1,q,\alpha}$ and let $a = \sum_{m=-\infty}^{0} u_m$ be a representation with

$$\Big(\sum_{m=-\infty}^{0} [2^{-m} \ell^{\alpha}(2^m) J(2^m, u_m)]^q \Big)^{1/q} \le 2\|a|[A_0, A_1]^J_{1,q,\alpha}\|.$$

Then

$$\begin{aligned} \|a|A_0 + A_1\| = K(1, a) &\le \sum_{m=-\infty}^{0} K(1, u_m) \\ &\le \sum_{m=-\infty}^{0} \min\{1, 2^{-m}\} J(2^m, u_m) = \sum_{m=-\infty}^{0} J(2^m, u_m). \end{aligned}$$

If $1 \le q \le \infty$, applying Hölder's inequality, we obtain that

$$\begin{aligned} \|a|A_0 + A_1\| &\le \Big(\sum_{m=-\infty}^{0} [2^{-m} \ell^{\alpha}(2^m) J(2^m, u_m)]^q \Big)^{1/q} \Big(\sum_{m=-\infty}^{0} [2^m \ell^{-\alpha}(2^m)]^{q'} \Big)^{1/q'} \\ &\lesssim \Big(\sum_{m=-\infty}^{0} [2^{-m} \ell^{\alpha}(2^m) J(2^m, u_m)]^q \Big)^{1/q} \\ &\lesssim \|a|[A_0, A_1]^J_{1,q,\alpha}\|. \end{aligned}$$

If $0 < q \le 1$, we get

$$\begin{aligned} \|a|A_0 + A_1\| &\le \Big(\sum_{m=-\infty}^{0} J(2^m, u_m)^q \Big)^{1/q} \\ &\le \Big(\sum_{m=-\infty}^{0} [2^{-m} \ell^{\alpha}(2^m) J(2^m, u_m)]^q \Big)^{1/q} \sup_{m\in\mathbb{Z}^-} \Big(2^m \ell^{-\alpha}(2^m) \Big) \\ &\lesssim \|a|[A_0, A_1]^J_{1,q,\alpha}\|. \end{aligned}$$

This proves that $[A_0, A_1]^J_{1,q,\alpha} \hookrightarrow A_0 + A_1$. □

It is easy to check that $[\cdot,\cdot]^J_{1,q,\alpha}$ is an exact interpolation method (see Section 2.1).

Let $0 < q \le \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and let $\bar{A} = (A_0, A_1)$ be a Banach couple. Remember that, according to Theorem 2.10, it makes sense to consider $(A_0, A_1)^J_{1,q,\mathbb{A}}$ if, and only if,

$$\begin{cases} 0 < q \le 1, & \alpha_\infty \ge 0; \\ 1 < q \le \infty, & \alpha_\infty - 1/q' > 0; \end{cases} \tag{4.2}$$

where the parameter q' is given by the equality $\frac{1}{q} + \frac{1}{q'} = 1$.

Lemma 4.2. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \le \infty$ satisfying (4.2). Given any Banach couple $\bar{A} = (A_0, A_1)$, we have with equivalent quasi-norms

$$A_1 + (A_0, A_1)^J_{1,q,\mathbb{A}} = A_1 + [A_0, A_1]^J_{1,q,\alpha_0}.$$

Proof. Let $v = a_1 + a$ with $a_1 \in A_1$ and $a \in (A_0, A_1)^J_{1,q,\mathbb{A}}$. Find $(u_m)_{m\in\mathbb{Z}} \subseteq A_0 \cap A_1$ such that $a = \sum_{m=-\infty}^{\infty} u_m$ and

$$\left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbb{A}}(2^m) J(2^m, u_m) \right]^q \right)^{1/q} \le 2\|a|(A_0, A_1)^J_{1,q,\mathbb{A}}\|.$$

Then $w = \sum_{m=1}^{\infty} u_m$ belongs to A_1. Indeed, if $0 < q \le 1$, we have

$$\begin{aligned} \sum_{m=1}^{\infty} \|u_m|A_1\| &\le \sum_{m=1}^{\infty} 2^{-m} J(2^m, u_m) \le \left(\sum_{m=1}^{\infty} [2^{-m} J(2^m, u_m)]^q \right)^{1/q} \\ &\le \left(\sum_{m=1}^{\infty} [2^{-m} \ell^{\alpha_\infty}(2^m) J(2^m, u_m)]^q \right)^{1/q} \sup_{m\in\mathbb{N}} \ell^{-\alpha_\infty}(2^m) \\ &\lesssim \|a|(A_0, A_1)^J_{1,q,\mathbb{A}}\| \end{aligned}$$

where we have used that $\alpha_\infty \ge 0$ in the last inequality. If $1 < q \le \infty$, we proceed using Hölder's inequality. We get

$$\begin{aligned} \sum_{m=1}^{\infty} \|u_m|A_1\| &\le \sum_{m=1}^{\infty} 2^{-m} J(2^m, u_m) \\ &\le \left(\sum_{m=1}^{\infty} [2^{-m} \ell^{\alpha_\infty}(2^m) J(2^m, u_m)]^q \right)^{1/q} \left(\sum_{m=1}^{\infty} \ell^{-\alpha_\infty q'}(2^m) \right)^{1/q'} \\ &\lesssim \|a|(A_0, A_1)^J_{1,q,\mathbb{A}}\| \end{aligned}$$

because $\alpha_\infty - 1/q' > 0$.

Therefore, $v = (a_1 + w) + \sum_{m=-\infty}^{0} u_m$ belongs to $A_1 + [A_0, A_1]^J_{1,q,\alpha_0}$ with

$$\|v|A_1 + [A_0, A_1]^J_{1,q,\alpha_0}\| \le \|a_1 + w|A_1\| + \left(\sum_{m=-\infty}^{0} [2^{-m}\ell^{\alpha_0}(2^m)J(2^m, u_m)]^q\right)^{1/q}$$
$$\lesssim \|a_1|A_1\| + \|a|(A_0, A_1)^J_{1,q,\mathbb{A}}\|.$$

This yields that

$$A_1 + (A_0, A_1)^J_{1,q,\mathbb{A}} \hookrightarrow A_1 + [A_0, A_1]^J_{1,q,\alpha_0}.$$

The converse inclusion is clear. □

Now we are ready to show the announced result for the modified J-spaces.

Theorem 4.3. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \le \infty$ satisfying (4.1). Given any Banach couple $\bar{A} = (A_0, A_1)$ we have with equivalence of quasi-norms

$$(A_0, A_1)_{1,q,\mathbb{A}} = \begin{cases} A_1 + [A_0, A_1]^J_{1,q,\alpha_0+1} & \text{if} \quad 1 \le q \le \infty, \\ A_1^\sim + [A_0^\sim, A_1^\sim]^J_{1,q,\alpha_0+1/q} & \text{if} \quad 0 < q < 1. \end{cases}$$

Proof. The argument in the proof of [48, Lemma 2.3] for $1 \le q \le \infty$ is still valid for $0 < q \le \infty$ showing that in the assumption (4.1) we have

$$\|a|\bar{A}_{1,q,\mathbb{A}}\| \sim \left(\int_0^1 [t^{-1}K(t,a)\ell^{\alpha_0}(t)]^q \frac{dt}{t}\right)^{1/q}. \tag{4.3}$$

In particular, if $a \in A_1$, we obtain

$$\|a|\bar{A}_{1,q,\mathbb{A}}\| \lesssim \left(\int_0^1 \ell^{\alpha_0 q}(t)\frac{dt}{t}\right)^{1/q} \|a|A_1\| \lesssim \|a|A_1\|.$$

This yields that

$$A_1 \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}}. \tag{4.4}$$

Take any $\alpha \in \mathbb{R}$ with $\alpha + 1/q > 0$. We claim that

$$(A_0, A_1)_{1,q,\mathbb{A}} = A_1 + (A_0, A_1)_{1,q,(\alpha_0,\alpha)} \tag{4.5}$$

with equivalent quasi-norms. Indeed, by (4.3) and (4.4) we have that

$$(A_0, A_1)_{1,q,(\alpha_0,\alpha)} \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}} \quad \text{and} \quad A_1 \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}}.$$

So,

$$A_1 + (A_0, A_1)_{1,q,(\alpha_0,\alpha)} \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}}.$$

Conversely, if $a \in (A_0, A_1)_{1,q,\mathbb{A}}$, we can write $a = a_0 + a_1$ with $a_j \in A_j$ and

$$\|a_0|A_0\| + \|a_1|A_1\| \le 2\|a|A_0 + A_1\|. \tag{4.6}$$

Now we check that a_0 belongs to $(A_0, A_1)_{1,q,(\alpha_0,\alpha)}$. We have

$$\begin{aligned}\|a_0|\bar{A}_{1,q,(\alpha_0,\alpha)}\| &\lesssim \Big(\int_1^\infty \Big[t^{-1}K(t,a_0)\ell^\alpha(t)\Big]^q \frac{dt}{t}\Big)^{1/q} \\ &+ \Big(\int_0^1 \Big[t^{-1}K(t,a_1)\ell^{\alpha_0}(t)\Big]^q \frac{dt}{t}\Big)^{1/q} + \Big(\int_0^1 \Big[t^{-1}K(t,a)\ell^{\alpha_0}(t)\Big]^q \frac{dt}{t}\Big)^{1/q} \\ &\lesssim \Big(\int_1^\infty \Big[t^{-1}\ell^\alpha(t)\Big]^q \frac{dt}{t}\Big)^{1/q} \|a_0|A_0\| + \Big(\int_0^1 \ell^{\alpha_0 q}(t)\frac{dt}{t}\Big)^{1/q} \|a_1|A_1\| + \|a|\bar{A}_{1,q,\mathbb{A}}\| \\ &\lesssim \|a|A_0+A_1\| + \|a_1|A_1\| + \|a|\bar{A}_{1,q,\mathbb{A}}\|\end{aligned}$$

where we have used (4.6) in the last inequality. Hence,

$$\|a_0|\bar{A}_{1,q,(\alpha_0,\alpha)}\| \lesssim \|a_1|A_1\| + \|a|\bar{A}_{1,q,\mathbb{A}}\|.$$

This shows that $(A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow A_1 + (A_0, A_1)_{1,q,(\alpha_0,\alpha)}$ with

$$\begin{aligned}\|a|A_1 + (A_0, A_1)_{1,q,(\alpha_0,\alpha)}\| &\le \|a_1|A_1\| + \|a_0|\bar{A}_{1,q,(\alpha_0,\alpha)}\| \\ &\lesssim \|a|A_0+A_1\| + \|a|\bar{A}_{1,q,\mathbb{A}}\| \\ &\lesssim \|a|\bar{A}_{1,q,\mathbb{A}}\|\end{aligned}$$

which establishes (4.5).

Combining (4.5) with (3.8) and Lemma 4.2, we conclude for $1 \le q \le \infty$ that

$$\begin{aligned}(A_0, A_1)_{1,q,\mathbb{A}} &= A_1 + (A_0, A_1)_{1,q,(\alpha_0,\alpha)} \\ &= A_1 + (A_0, A_1)^J_{1,q,(\alpha_0+1,\alpha+1)} \\ &= A_1 + [A_0, A_1]^J_{1,q,\alpha_0+1}.\end{aligned}$$

If $0 < q < 1$, we use (2.8), (4.5), Theorem 3.5 and Lemma 4.2 to derive

$$\begin{aligned}(A_0, A_1)_{1,q,\mathbb{A}} &= (A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}} \\ &= A_1^\sim + (A_0^\sim, A_1^\sim)_{1,q,(\alpha_0,\alpha)} \\ &= A_1^\sim + (A_0^\sim, A_1^\sim)^J_{1,q,(\alpha_0+1/q,\alpha+1/q)} \\ &= A_1^\sim + [A_0^\sim, A_1^\sim]^J_{1,q,\alpha_0+1/q}.\end{aligned}$$

The proof is complete. □

We study now the corresponding result for $\theta = 0$. Put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $0 < q \le \infty$ and $\alpha \in \mathbb{R}$, we put $\langle \bar{A} \rangle^J_{0,q,\alpha} = \langle A_0, A_1 \rangle^J_{0,q,\alpha}$ for the collection of all $a \in A_0 + A_1$ such that there exists $(u_m)_{m\in\mathbb{N}_0} \subseteq A_0 \cap A_1$ satisfying

$$a = \sum_{m=0}^\infty u_m \text{ (convergence in } A_0 + A_1)$$

and

$$\Big(\sum_{m=0}^\infty [\ell^\alpha(2^m) J(2^m, u_m)]^q\Big)^{1/q} < \infty.$$

We endow $\langle A_0, A_1\rangle^J_{0,q,\alpha}$ with the quasi-norm

$$\|a|\langle A_0, A_1\rangle^J_{0,q,\alpha}\| = \inf\Big\{\Big(\sum_{m=0}^{\infty}[\ell^\alpha(2^m)J(2^m,u_m)]^q\Big)^{1/q} : a = \sum_{m=0}^{\infty} u_m\Big\}.$$

Notice that (2.11) implies that $\langle A_0, A_1\rangle^J_{0,q,\alpha} = [A_1, A_0]^J_{1,q,\alpha}$. Then $\langle\cdot,\cdot\rangle_{0,q,\alpha}$ is an exact interpolation method as well. Furthermore, the following equivalence result holds:

Theorem 4.4. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \leq \infty$ satisfying

$$\begin{cases} 0 < q < \infty, & \alpha_\infty + 1/q < 0, \quad \alpha_0 + 1/q < 0; \\ q = \infty, & \alpha_\infty < 0, \quad \alpha_0 \leq 0. \end{cases}$$

For any Banach couple $\bar{A} = (A_0, A_1)$ we have with equivalence of quasi-norms

$$(A_0, A_1)_{0,q,\mathbb{A}} = \begin{cases} A_0 + \langle A_0, A_1\rangle^J_{0,q,\alpha_\infty+1} & \text{if} \quad 1 \leq q \leq \infty, \\ A_0^\sim + \langle A_0^\sim, A_1^\sim\rangle^J_{0,q,\alpha_\infty+1/q} & \text{if} \quad 0 < q < 1. \end{cases}$$

4.2 Associate spaces

Let (Ω,μ) be a σ-finite measure space and let $\mathcal{M}$ denote the collection of all (equivalence classes of) scalar valued μ-measurable functions on Ω which are finite μ-almost everywhere. We endow $\mathcal{M}$ with the topology of convergence in measure on sets of finite measure.

Definition 4.5. By a *Banach function space* over (Ω,μ) we mean a Banach space $(X, \|\cdot|X\|)$ of functions in $\mathcal{M}$ satisfying the following three properties:

(i) Whenever $g \in \mathcal{M}$, $f \in X$ and $|g(x)| \leq |f(x)|$ μ-a.e., then $g \in X$ and $\|g|X\| \leq \|f|X\|$ (lattice property).

(ii) $\chi_E \in X$ for every $E \subseteq \Omega$ with $\mu(E) < \infty$.

(iii) For every $E \subseteq \Omega$ with $\mu(E) < \infty$ there is $c_E > 0$ such that $\int_E |f|d\mu \leq c_E\|f|X\|$ for every $f \in X$.

See, for example, the book by Zaanen [122], the book by Kreĭn, Petunin and Semenov [88] or the paper by Cobos and Fernández-Cabrera [36] for more properties of these spaces. One should be careful when consulting the existing literature on Banach function spaces since there is not an standard definition and some books, such as [6] and [55], consider that these spaces also satisfy the so-called Fatou property.

Clearly, simple functions are contained in any Banach function space X and $\||f||X\| = \|f|X\|$ for every $f \in X$. Moreover, the argument in [6, p. 4] based on (iii) can still be applied with the result that

$$X \hookrightarrow \mathcal{M}. \tag{4.7}$$

Generalized Lorentz-Zygmund spaces $L_{(p,q,\mathbb{A})}(\Omega)$ defined in (2.24) are Banach function spaces when $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. These spaces also make sense when $0 < q < 1$ but then they are quasi-Banach spaces.

The *associate space* X' of the Banach function space X consists of all $g \in \mathcal{M}$ such that

$$\int_\Omega |fg| d\mu < \infty, \qquad \text{for every } f \in X.$$

It is also a Banach function space over Ω endowed with the norm

$$\|g|X'\| = \sup \left\{ \int_\Omega |fg| d\mu : \|f|X\| \leq 1 \right\}.$$

Indeed, the arguments in the proof of [6, Theorem I.2.2] can be applied to show that $(X', \|\cdot|X'\|)$ is a normed space of functions in $\mathcal{M}$ which satisfies the corresponding versions of (i), (ii) and (iii). Moreover, using the definition of $\|\cdot|X'\|$, it is not hard to check that if $(g_n) \subseteq X'$ and $\sum_{n=1}^{\infty} \|g_n|X'\| < \infty$ then the function $g = \sum_{n=1}^{\infty} g_n$ belongs to X' and $\|g - \sum_{j=1}^{n} g_j|X'\| \to 0$ as $n \to \infty$. Hence, $(X', \|\cdot|X'\|)$ is a Banach function space.

Notice that $\int_\Omega |fg| d\mu \leq \|f|X\| \|g|X'\|$, for every $f \in X$ and $g \in X'$. Furthermore, if $(Y, \|\cdot|Y\|)$ is a quasi-Banach space of functions in $\mathcal{M}$ such that the corresponding versions of (i), (ii) and (iii) hold, then we can still define Y' as above.

It is known that $\ell_q' = \ell_{q'}$ for $1 \leq q \leq \infty$ where $\frac{1}{q} + \frac{1}{q'} = 1$. For later use we compute now the associate space of the quasi-Banach space ℓ_q when $0 < q < 1$.

Lemma 4.6. Let $0 < q < 1$. Then $\ell_q' = \ell_\infty$ with equality of norms.

Proof. Take any $\eta = (\eta_m)_{m\in\mathbb{Z}} \in \ell_\infty$ and $\xi = (\xi_m)_{m\in\mathbb{Z}} \in \ell_q$. It follows from

$$\sum_{m=-\infty}^{\infty} |\xi_m \eta_m| \leq \left(\sum_{m=-\infty}^{\infty} |\xi_m|^q |\eta_m|^q \right)^{1/q} \leq \|\xi|\ell_q\| \|\eta|\ell_\infty\|$$

that $\ell_\infty \hookrightarrow \ell_q'$ and that the embedding has norm less than or equal to 1. Conversely, take any $\eta = (\eta_m)_{m\in\mathbb{Z}} \in \ell_q'$ and for $n \in \mathbb{Z}$ let $e_n = (\delta_m^n)_{m\in\mathbb{Z}}$. We have

$$|\eta_n| = \sum_{m=-\infty}^{\infty} |\eta_m \delta_m^n| \leq \|\eta|\ell_q'\| \|e_n|\ell_q\| = \|\eta|\ell_q'\|.$$

Hence, η belongs to ℓ_∞ and $\|\eta|\ell_\infty\| \leq \|\eta|\ell_q'\|$. This completes the proof. □

Let X_0, X_1 be Banach function spaces over the measure space (Ω, μ). According to (4.7), we have that $X_j \hookrightarrow \mathcal{M}$. Hence $\bar{X} = (X_0, X_1)$ is a Banach couple.

Let $g \in \mathcal{M}$ and $f \in X_0 \cap X_1$ with $|g(x)| \leq |f(x)|$ μ-a.e., then it is clear that $g \in X_0 \cap X_1$ with $J(t, g; X_0, X_1) \leq J(t, f; X_0, X_1)$, $t > 0$. So, $X_0 \cap X_1$ is a Banach function space with the norm $J(t, \cdot; X_0, X_1)$. As for the K-functional, using that

$$K(t, f; X_0, X_1) = \inf\{\|f_0|X_0\| + t\|f_1|X_1\| : |f| \leq f_0 + f_1, f_j \geq 0, f_j \in X_j, \ j = 0, 1\}$$

(see, for example, [43, Lemma 3.1]), it follows that $K(t, g; X_0, X_1) \leq K(t, f; X_0, X_1)$ provided that $|g| \leq |f|$, $f \in X_0 + X_1$. Now it is not hard to check that $X_0 + X_1$ is also a Banach function space.

The above mentioned properties for the J- and K-functionals also yield that when we interpolate the Banach couple $\bar{X} = (X_0, X_1)$ by any J- or K-method the resulting space also satisfy properties (i), (ii) and (iii). To check (iii) one can rely on the exact interpolation property applied to the operator $f \longrightarrow \int_E f d\mu$.

A Banach function space X over Ω is said to have *absolutely continuous norm* if for any $f \in X$ and any decreasing sequence (E_n) of μ-measurable sets with empty intersection we have that $\|f\chi_{E_n}|X\| \downarrow 0$ as $n \to \infty$. If X has absolutely continuous norm then X' coincides with the dual space X^* of X (see [122, Theorem 15.72.5, p. 480]).

Lemma 4.7. Let $\bar{X} = (X_0, X_1)$ be a couple of Banach function spaces over a σ-finite measure space (Ω, μ). Then

$$J(t, g; X_0', X_1') = \sup_{f \in X_0 + X_1} \frac{\int_\Omega |fg| d\mu}{K(t^{-1}, f; X_0, X_1)}, \quad g \in X_0' \cap X_1', \quad t > 0. \tag{4.8}$$

If, in addition, X_0 or X_1 has absolutely continuous norm, then

$$K(t^{-1}, g; X_0', X_1') = \sup_{f \in X_0 \cap X_1} \frac{\int_\Omega |fg| d\mu}{J(t, f; X_0, X_1)}, \quad f \in X_0' + X_1', \quad t > 0. \tag{4.9}$$

Proof. Let $g \in X_0' \cap X_1'$ and $f \in X_0 + X_1$ with $f = f_0 + f_1$, $f_j \in X_j$, $j = 0, 1$ and $t > 0$. We have

$$\begin{aligned} \int_\Omega |fg| d\mu &\leq \int_\Omega |f_0 g| d\mu + \int_\Omega |f_1 g| d\mu \leq \|f_0|X_0\| \|g|X_0'\| + \|f_1|X_1\| \|g|X_1'\| \\ &\leq J(t, g; X_0', X_1')(\|f_0|X_0\| + t^{-1}\|f_1|X_1\|). \end{aligned}$$

This yields

$$\int_\Omega |fg| d\mu \leq J(t, g; X_0', X_1') K(t^{-1}, f; X_0, X_1). \tag{4.10}$$

To check the converse inequality we write λX for the space X normed by $\lambda \| \cdot |X\|$. Since the embeddings $X_0 \hookrightarrow (X_0 + X_1, K(t^{-1}, \cdot; X_0, X_1))$ and $t^{-1}X_1 \hookrightarrow (X_0 + X_1, K(t^{-1}, \cdot; X_0, X_1))$ have norm less than or equal to 1, we have $(X_0 + X_1, K(t^{-1}, \cdot; X_0, X_1))' \hookrightarrow X_0'$ and $(X_0 + X_1, K(t^{-1}, \cdot; X_0, X_1))' \hookrightarrow tX_1'$ with norm less than or equal to 1. Hence

$$(X_0 + X_1, K(t^{-1}, \cdot; X_0, X_1))' \hookrightarrow (X_0' \cap X_1', J(t, \cdot; X_0', X_1'))$$

with

$$J(t, g; X_0', X_1') \leq \sup_{f \in X_0 + X_1} \frac{\int_\Omega |fg| d\mu}{K(t^{-1}, f; X_0, X_1)}.$$

This establishes (4.8).

Inequality

$$\sup_{f \in X_0 \cap X_1} \frac{\int_\Omega |fg| d\mu}{J(t, f; X_0, X_1)} \leq K(t^{-1}, g; X_0', X_1'), \quad g \in X_0' + X_1', \quad t > 0,$$

can be proved proceeding as in (4.10). To check the reverse inequality, consider $X_0 \cap X_1$ with the norm $J(t, \cdot; X_0, X_1)$, endow $X_0' + X_1'$ with the norm $K(t^{-1}, \cdot; X_0', X_1')$ and take any $g \in (X_0 \cap X_1)'$. Then the functional T assigning to any $f \in X_0 \cap X_1$ the scalar $Tf = \int_\Omega fg d\mu$ belongs to $(X_0 \cap X_1)^*$ with

$$\|T|(X_0 \cap X_1)^*\| = \sup \left\{ \left| \int_\Omega fg d\mu \right| : J(t, f; X_0, X_1) \leq 1 \right\} = \|g|(X_0 \cap X_1)'\|.$$

Consider the space $X_0 \times X_1$ normed by

$$\|(f_0, f_1)|X_0 \times X_1\| = \max\{\|f_0|X_0\|, t\|f_1|X_1\|\}$$

and put $A = \{(f_0, f_1) \in X_0 \times X_1 : f_0 = f_1\}$. The linear functional $F : A \to \mathbb{K}$ defined by

$$F(f_0, f_1) = T\left(\frac{f_0 + f_1}{2}\right) = \int_\Omega f_0 g d\mu, \quad (f_0, f_1) \in A,$$

is bounded with

$$\|F|A^*\| = \sup\left\{\Big|\int_\Omega fgd\mu\Big| : f \in X_0 \cap X_1 \text{ with } J(t, f; X_0, X_1) \leq 1\right\} = \|g|(X_0 \cap X_1)'\|.$$

According to the Hahn-Banach theorem, we can extend F to a bounded linear functional $\hat{F} \in (X_0 \times X_1)^*$ with $\|\hat{F}|(X_0 \times X_1)^*\| = \|g|(X_0 \cap X_1)'\|$. Hence, there are $L_j \in X_j^*$, $j = 0, 1$, such that

$$L_0 f_0 = \hat{F}(f_0, 0), \;\; L_1 f_1 = \hat{F}(0, f_1) \;\text{ and }\; \hat{F}(f_0, f_1) = L_0 f_0 + L_1 f_1. \tag{4.11}$$

Assume that X_0 has absolutely continuous norm. Then $X_0^* = X_0'$ and so there is $g_0 \in X_0'$ such that $L_0 f_0 = \int_\Omega f_0 g_0 d\mu$ and $\|L_0|X_0^*\| = \|g_0|X_0'\|$. For any $f \in X_0 \cap X_1$, we have

$$\int_\Omega fgd\mu = \hat{F}(f, f) = L_0 f + L_1 f = \int_\Omega fg_0 d\mu + L_1 f.$$

Whence

$$L_1 f = \int_\Omega f(g - g_0)d\mu, \quad f \in X_0 \cap X_1.$$

We claim that $g - g_0 \in X_1'$. Indeed, take any $f \in X_1$. We can find an increasing sequence of simple functions (f_n) such that $0 \leq f_n \uparrow |f|$. Since $(f_n) \subseteq X_0 \cap X_1$, we get

$$\begin{aligned}\int_\Omega f_n|g - g_0|d\mu &= \Big|\int_\Omega (\text{sgn}(g - g_0)f_n)(g - g_0)d\mu\Big| \\ &= |L_1(\text{sgn}(g - g_0)f_n)| \\ &\leq \|L_1|X_1^*\|\|f_n|X_1\| \\ &\leq \|L_1|X_1^*\|\|f|X_1\|.\end{aligned}$$

Thus, using the monotone convergence theorem, we derive that

$$\int_\Omega |f||g - g_0|d\mu = \lim_{n\to\infty}\int_\Omega f_n|g - g_0|d\mu < \infty.$$

This shows that $g_1 = g - g_0$ belongs to X_1' with $\|g_1|X'\| \leq \|L_1|X_1^*\|$. So, $g = g_0 + g_1 \in X_0' + X_1'$. Moreover, given any $\varepsilon > 0$, we have

$$\begin{aligned}\|g|X_0' + X_1'\| - \varepsilon(1 + t^{-1}) &\leq \|g_0|X_0'\| - \varepsilon + t^{-1}(\|g_1|X_1'\| - \varepsilon) \\ &\leq \|L_0|X_0^*\| - \varepsilon + t^{-1}(\|L_1|X_1^*\| - \varepsilon) \\ &\leq |L_0 f_0| + t^{-1}|L_1 f_1|\end{aligned}$$

for some $f_j \in X_j$ with $\|f_j|X_j\| \leq 1$. Therefore, using (4.11), we get

$$\begin{aligned}\|g|X_0' + X_1'\| - \varepsilon(1+t^{-1}) &\leq L_0\left(\frac{|L_0 f_0|}{L_0 f_0} f_0\right) + L_1\left(t^{-1}\frac{|L_1 f_1|}{L_1 f_1} f_1\right)\\ &= \hat{F}\left(\frac{|L_0 f_0|}{L_0 f_0} f_0, t^{-1}\frac{|L_1 f_1|}{L_1 f_1} f_1\right)\\ &\leq \|\hat{F}|(X_0\times X_1)^*\| \max\left\{\left\|\frac{|L_0 f_0|}{L_0 f_0} f_0|X_0\right\|, tt^{-1}\left\|\frac{|L_1 f_1|}{L_1 f_1} f_1|X_1\right\|\right\}\\ &\leq \|\hat{F}|(X_0\times X_1)^*\| = \|g|(X_0\cap X_1)'\|.\end{aligned}$$

Letting $\varepsilon \to 0$, this yields that

$$K(t^{-1}, g; X_0', X_1') = \|g|X_0' + X_1'\| \leq \|g|(X_0\cap X_1)'\| = \sup_{f\in X_0\cap X_1} \frac{\int_\Omega |fg| d\mu}{J(t, f; X_0, X_1)}$$

and completes the proof.

The case when X_1 has absolutely continuous norm can be treated analogously. □

Our next objective is to compute the associate space of logarithmic interpolation methods when acting on couples of Banach function spaces. We start computing it for logarithmic J-spaces. Remember that, according to Theorem 2.10, for $0 < q \leq \infty$, $0 \leq \theta \leq 1$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$, the sequence lattice $\ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m))$ is $(1, J)$-non trivial if, and only if,

$$\begin{cases} 0 < \theta < 1; & \\ \theta = 1, \quad \alpha_\infty > 0 \quad \text{or} \quad \alpha_\infty = 0 \text{ and } \beta_\infty \geq 0 & \text{if} \quad 0 < q \leq 1;\\ \theta = 1, \quad \alpha_\infty + \frac{1}{q} - 1 > 0 \quad \text{or} \quad \alpha_\infty + \frac{1}{q} - 1 = 0 \text{ and } \beta_\infty + \frac{1}{q} - 1 > 0 & \text{if} \quad 1 < q \leq \infty;\\ \theta = 0, \quad \alpha_0 > 0 \quad \text{or} \quad \alpha_0 = 0 \text{ and } \beta_0 \geq 0 & \text{if} \quad 0 < q \leq 1;\\ \theta = 0, \quad \alpha_0 + \frac{1}{q} - 1 > 0 \quad \text{or} \quad \alpha_0 + \frac{1}{q} - 1 = 0 \text{ and } \beta_0 + \frac{1}{q} - 1 > 0 & \text{if} \quad 1 < q \leq \infty. \end{cases} \tag{4.12}$$

Remember that if $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, we use the notation $\tilde{\mathbb{A}} = (\alpha_\infty, \alpha_0)$, $\lambda\mathbb{A} = (\lambda\alpha_0, \lambda\alpha_\infty)$ and $\mathbb{A} + \lambda = (\alpha_0 + \lambda, \alpha_\infty + \lambda)$.

Theorem 4.8. Let $\bar{X} = (X_0, X_1)$ be a couple of Banach function spaces over Ω. Suppose that X_0 or X_1 has absolutely continuous norm. Let $0 < q \leq \infty$, $0 \leq \theta \leq 1$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ satisfying (4.12). Put

$$q^* = \begin{cases} q' & \text{if} \quad 1 \leq q \leq \infty,\\ \infty & \text{if} \quad 0 < q < 1.\end{cases}$$

Then $((X_0, X_1)^J_{\theta,q,\mathbb{A},\mathbb{B}})' = (X_0', X_1')_{\theta,q^*,-\tilde{\mathbb{A}},-\tilde{\mathbb{B}}}$.

Proof. We proceed following the lines of [63, Theorem 3.4]. Take any $g \in (\bar{X}^J_{\theta,q,\mathbb{A},\mathbb{B}})'$ and any $\varepsilon > 0$. According to (4.9), for any $m \in \mathbb{Z}$, there is $f_m \in X_0 \cap X_1$ such that

$$(1-\varepsilon)K(2^{-m}, |g|; X_0', X_1') \leq J(2^m, |f_m|; X_0, X_1)^{-1} \int_\Omega |f_m g| d\mu.$$

Take any sequence (δ_m) of non-negative scalars such that

$$\Big(\sum_{m=-\infty}^{\infty}[2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)\delta_m]^q\Big)^{1/q}\le 1.$$

Put $u_m = J(2^m,|f_m|;X_0,X_1)^{-1}\delta_m|f_m|$. Then the function $f=\sum_{m=-\infty}^{\infty}u_m$ belongs to $\bar{X}^J_{\theta,q,\mathbb{A},\mathbb{B}}$ and

$$\begin{aligned}\|f|\bar{X}^J_{\theta,q,\mathbb{A},\mathbb{B}}\| &\le \left(\sum_{m=-\infty}^{\infty}\Big[2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)J(2^m,u_m;X_0,X_1)\Big]^q\right)^{1/q}\\ &\le \left(\sum_{m=-\infty}^{\infty}\Big[2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)\delta_m\Big]^q\right)^{1/q}\le 1.\end{aligned}$$

Moreover,

$$\begin{aligned}&(1-\varepsilon)\sum_{m=-\infty}^{\infty}2^{m\theta}\ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m)K(2^{-m},|g|;X_0',X_1')2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)\delta_m\\ &\quad\le \sum_{m=-\infty}^{\infty}\delta_m J(2^m,|f_m|;X_0,X_1)^{-1}\int_\Omega|f_m g|d\mu\\ &\quad=\int_\Omega|fg|d\mu\le\|g|(\bar{X}^J_{\theta,q,\mathbb{A},\mathbb{B}})'\|.\end{aligned}$$

Using that $\ell_q'=\ell_{q^*}$, we derive that $g\in(X_0',X_1')_{\theta,q^*,-(\alpha_\infty,\alpha_0),-(\beta_\infty,\beta_0)}$ and that

$$\|g|(X_0',X_1')_{\theta,q^*,-(\alpha_\infty,\alpha_0),-(\beta_\infty,\beta_0)}\|\le\|g|(\bar{X}^J_{\theta,q,\mathbb{A},\mathbb{B}})'\|.$$

Conversely, take any $g\in(X_0',X_1')_{\theta,q^*,-\tilde{\mathbb{A}},-\tilde{\mathbb{B}}}$. Let $f\in\bar{X}^J_{\theta,q,\mathbb{A},\mathbb{B}}$ and take a J-representation $f=\sum_{m=-\infty}^{\infty}f_m$ of f. By (4.9), we have

$$\int_\Omega|f_m g|d\mu\le J(2^m,f_m;X_0,X_1)K(2^{-m},g;X_0',X_1'),\quad m\in\mathbb{Z}.$$

Hence, if $1\le q\le\infty$, it follows by using Hölder's inequality that

$$\begin{aligned}\int_\Omega|fg|d\mu &\le\sum_{m=-\infty}^{\infty}J(2^m,f_m;X_0,X_1)K(2^{-m},g;X_0',X_1')\\ &\le\Big(\sum_{m=-\infty}^{\infty}[2^{-m\theta}\ell^{\mathbb{A}}(2^m)\ell\ell^{\mathbb{B}}(2^m)J(2^m,f_m;X_0,X_1)]^q\Big)^{1/q}\\ &\quad\times\left(\sum_{m=-\infty}^{\infty}\Big[2^{m\theta}\ell^{-\mathbb{A}}(2^m)\ell\ell^{-\mathbb{B}}(2^m)K(2^{-m},g;X_0',X_1')\Big]^{q'}\right)^{1/q'}.\end{aligned}$$

If $0 < q < 1$, we obtain

$$\begin{aligned}\int_\Omega |fg| d\mu &\leq \sum_{m=-\infty}^{\infty} J(2^m, f_m; X_0, X_1) K(2^{-m}, g; X_0', X_1') \\ &\leq \left(\sum_{m=-\infty}^{\infty} [J(2^m, f_m; X_0, X_1) K(2^{-m}, g; X_0', X_1')]^q \right)^{1/q} \\ &\leq \left(\sum_{m=-\infty}^{\infty} \left[2^{-m\theta} \ell^{\mathbb{A}}(2^m) \ell\ell^{\mathbb{B}}(2^m) J(2^m, f_m; \bar{X})\right]^q \right)^{1/q} \\ &\quad \times \sup_{m \in \mathbb{Z}} \{2^{m\theta} \ell^{-\mathbb{A}}(2^m) \ell\ell^{-B}(2^m) K(2^{-m}, g; \bar{X})\}.\end{aligned}$$

Therefore, for any $0 < q \leq \infty$, we get that

$$\int_\Omega |fg| d\mu \leq \|f | \bar{X}^J_{\theta,q,\mathbb{A},\mathbb{B}}\| \left\| g | (X_0', X_1')_{\theta,q^*,-(\alpha_\infty,\alpha_0),-(\beta_\infty,\beta_0)} \right\|.$$

This shows that g belongs to $(\bar{X}^J_{\theta,q,\mathbb{A},\mathbb{B}})'$ with

$$\|g | (\bar{X}^J_{\theta,q,\mathbb{A},\mathbb{B}})'\| \leq \left\| g | (X_0', X_1')_{\theta,q^*,-(\alpha_\infty,\alpha_0),-(\beta_\infty,\beta_0)} \right\|.$$

The proof is complete. □

Remark 4.9. Notice that if $\ell_q(2^{-m\theta} \ell^{\mathbb{A}}(2^m) \ell\ell^{\mathbb{B}}(2^m))$ satisfies (4.12), then

$$\ell_{q*}(2^{-m\theta} \ell^{(-\alpha_\infty,-\alpha_0)}(2^m) \ell\ell^{(-\beta_\infty,-\beta_0)}(2^m))$$

is K-non trivial (see Theorem 2.8). Therefore the K-space in Theorem 4.8 is non-trivial.

The same techniques from Theorem 4.8 are useful to determine the associate space of the modified J-space $[X_0, X_1]^J_{1,q,\alpha}$ and $\langle X_0, X_1 \rangle^J_{0,q,\alpha}$. The relevant K-spaces are now $\langle X_0, X_1 \rangle_{1,q,\alpha}$ the space of all $f \in X_0 + X_1$ with finite quasi-norm

$$\|f | \langle X_0, X_1 \rangle_{1,q,\alpha}\| = \left(\sum_{m=0}^{\infty} [2^{-m} \ell^\alpha(2^m) K(2^m, f; X_0, X_1)]^q \right)^{1/q}; \tag{4.13}$$

and $[X_0, X_1]_{0,q,\alpha}$, the collection of all $f \in X_0 + X_1$ with finite quasi-norm

$$\|f | [X_0, X_1]_{0,q,\alpha}\| = \left(\sum_{m=-\infty}^{0} [\ell^\alpha(2^m) K(2^m, f; X_0, X_1)]^q \right)^{1/q}.$$

If $q = \infty$, as usual, the sum should be replaced by the supremum. For every $0 < q \leq \infty$ and $\alpha \in \mathbb{R}$, the previous constructions are exact interpolation methods. We have the following result.

Theorem 4.10. Let $\bar{X} = (X_0, X_1)$ be a couple of Banach function spaces over Ω. Suppose that X_0 or X_1 has absolutely continuous norm. Let $0 < q \leq \infty$, $0 \leq \theta \leq 1$, $\alpha \in \mathbb{R}$ and q^* as in Theorem 4.8. Then

$$\begin{aligned}([X_0, X_1]^J_{1,q,\alpha})' &= \langle X_0', X_1' \rangle_{1,q^*,-\alpha} \\ (\langle X_0, X_1 \rangle^J_{0,q,\alpha})' &= [X_0', X_1']_{0,q^*,-\alpha}.\end{aligned}$$

Remark 4.11. On the contrary to the duality formulae of Chapter 3 where it is essential that $A_0 \cap A_1$ is dense in A_0 and A_1, such assumption is not needed in Theorem 4.8. It suffices that X_0 or X_1 has absolutely continuous norm. Moreover, the parameter q can also take the value ∞ in Theorem 4.8.

We study now the associate space of logarithmic interpolation spaces defined by means of the K-functional. Let $0 \leq \theta \leq 1$, $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. Remember that, $\ell_q(2^{-m\theta}\ell^{\mathbb{A}}(2^m))$ is K-non trivial if, and only if, the parameters satisfy (3.7). However, we are going to leave out the cases $\theta = 0$, $q = \infty$, $\alpha_\infty = 0$ and $\theta = 1$, $q = \infty$, $\alpha_0 = 0$ that, because of its special properties, are going to be studied in Chapter 5. Therefore, we assume

$$\begin{cases} 0 < \theta < 1; \\ \theta = 0, \quad \alpha_\infty + 1/q < 0; \\ \theta = 1, \quad \alpha_0 + 1/q < 0. \end{cases} \tag{4.14}$$

Theorem 4.12. Let $\bar{X} = (X_0, X_1)$ be a couple of Banach function spaces over a σ-finite measure space (Ω, μ). Let $0 < q \leq \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ verifying (4.14) and put q^* as in Theorem 4.8. Suppose that X_0 or X_1 has absolutely continuous norm.

1. If $0 < \theta < 1$, then $(\bar{X}_{\theta,q,\mathbb{A}})' = (X_0', X_1')_{\theta,q^*,-\tilde{\mathbb{A}}}$.

Now, when $0 < q \leq 1$ we assume, in addition, that (X_0, X_1) is mutually closed, that is to say, $X_j^\sim = X_j$.

2. If $\theta = 1$, then

$$(\bar{X}_{1,q,\mathbb{A}})' = \begin{cases} (X_0', X_1')_{1,q^*,-\tilde{\mathbb{A}}-1/\min\{1,q\}} & \text{if} \quad \alpha_\infty + 1/q > 0, \\ (X_0', X_1')_{1,q^*,-\tilde{\mathbb{A}}-1/\min\{1,q\},(-1/\min\{1,q\},0)} & \text{if} \quad q < \infty \quad \text{and} \quad \alpha_\infty + 1/q = 0, \\ X_1' \cap \langle X_0', X_1' \rangle_{1,q^*,-\alpha_0-1/\min\{1,q\}} & \text{if} \quad \alpha_\infty + 1/q < 0 \quad \text{and} \quad q < \infty, \\ & \quad\ \text{or} \quad \alpha_\infty \leq 0 \quad \text{and} \quad q = \infty. \end{cases}$$

3. If $\theta = 0$, then

$$(\bar{X}_{0,q,\mathbb{A}})' = \begin{cases} (X_0', X_1')_{0,q^*,-\tilde{\mathbb{A}}-1/\min\{1,q\}} & \text{if} \quad \alpha_0 + 1/q > 0, \\ (X_0', X_1')_{0,q^*,-\tilde{\mathbb{A}}-1/\min\{1,q\},(0,-1/\min\{1,q\})} & \text{if} \quad q < \infty \quad \text{and} \quad \alpha_0 + 1/q = 0, \\ X_0' \cap [X_0', X_1']_{0,q^*,-\alpha_\infty-1/\min\{1,q\}} & \text{if} \quad \alpha_0 + 1/q < 0 \quad \text{and} \quad q < \infty, \\ & \quad\ \text{or} \quad \alpha_0 \leq 0 \quad \text{and} \quad q = \infty. \end{cases}$$

Proof. Statement (1) follows from the equivalence result

$$(X_0, X_1)_{\theta,q,\mathbb{A}} = (X_0, X_1)^J_{\theta,q,\mathbb{A}}$$

and Theorem 4.8.

For the proof of (2) we should distinguish multiple cases:

If $1 \leq q \leq \infty$ and $\alpha_\infty + 1/q \geq 0$, then we use (3.8) and Theorem 4.8. For $1 \leq q \leq \infty$ and $\alpha_\infty + 1/q < 0$, the proof follows from Theorem 4.3, (4.8) and Theorem 4.10.

For $0 < q \leq 1$ we have assumed that $X_0^\sim = X_0$ and $X_1^\sim = X_1$. Then using Theorems 3.5 and 4.8 we get the result for $\alpha_\infty + 1/q \geq 0$. Finally, if $\alpha_\infty + 1/q < 0$ we use Theorem 4.3, (4.8) and Theorem 4.10.

The proof of (3) follows analogously, but using the corresponding results for $\theta = 0$. $\square$

4.3 Applications to generalized Lorentz-Zygmund spaces

First we assume that (Ω, μ) is a non-atomic σ-finite measure space and we deal with the spaces $L_{(p,q,\mathbb{A})}(\Omega)$ (see (2.24)). Their associate spaces have been determined by Opic and Pick [97, Theorems 6.2, 6.6, 6.7 and 6.9] by means of direct calculations. In this section we derive them from the abstract results obtained in Section 4.2 as an specific example.

Consider the Banach couple $(L_1(\Omega), L_\infty(\Omega))$. It is well-known (see, for example, [11, Theorem 5.2.1]) that

$$K(t, f; L_1(\Omega), L_\infty(\Omega)) = \int_0^t f^*(s)ds, \quad t > 0, \tag{4.15}$$

here f^* is the non-increasing rearrangement defined in (2.15). From this equality it is not hard to check that $(L_1(\Omega), L_\infty(\Omega))$ is a mutually closed couple (see the comment after [6, Theorem V.1.6]). Besides, the norm of $L_1(\Omega)$ is absolutely continuous.

It follows straightforward from (4.15) and (2.24) that if $1 \leq p \leq \infty$, $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfy (2.26), then

$$(L_1(\Omega), L_\infty(\Omega))_{1-1/p,q,\mathbb{A}} = L_{(p,q,\mathbb{A})}(\Omega), \tag{4.16}$$

with equivalent quasi-norms.

For $1 \leq r \leq \infty$ we put $r' = \frac{r}{r-1}$ and for $0 < r \leq \infty$ we put $r^* = \begin{cases} r' & \text{if} \quad 1 < r \leq \infty, \\ \infty & \text{if} \quad 0 < r \leq 1. \end{cases}$

Theorem 4.13. Let $1 < p < \infty$, $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. Then

$$(L_{(p,q,\mathbb{A})}(\Omega))' = L_{(p',q^*,-\mathbb{A})}(\Omega).$$

Proof. Using that $L_1(\Omega)' = L_\infty(\Omega)$, $L_\infty(\Omega)' = L_1(\Omega)$, (4.16) and Theorem 4.12/1 we have that

$$\begin{aligned} L_{(p,q,\mathbb{A})}(\Omega) &= (L_1(\Omega), L_\infty(\Omega))'_{1-1/p,q,\mathbb{A}} = (L_\infty(\Omega), L_1(\Omega))_{1-1/p,q^*,-\tilde{\mathbb{A}}} \\ &= (L_1(\Omega), L_\infty(\Omega))_{1/p,q^*,-\mathbb{A}} = L_{(p',q^*,-\mathbb{A})}(\Omega). \end{aligned}$$

$\square$

Theorem 4.14. Let $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ such that $\alpha_0 + 1/q < 0$. We have with equivalence of quasi-norms:

1. If $\alpha_\infty + 1/q > 0$, then $(L_{(\infty,q,\mathbb{A})}(\Omega))' = L_{(1,q^*,-\mathbb{A}-1/\min\{1,q\})}(\Omega)$.
2. If $\alpha_\infty + 1/q = 0$ and $q < \infty$, then $(L_{(\infty,q;\mathbb{A})}(\Omega))' =$

$$\left\{ g \in \mathcal{M} : \|g\| = \left(\int_0^\infty \left[\ell^{(-\mathbb{A}-1/\min\{1,q\})}(t) \ell\ell^{(0,-1/\min\{1,q\})}(t) \int_0^t g^*(s)ds \right]^{q^*} \frac{dt}{t} \right)^{1/q^*} < \infty \right\}.$$

3. If $\alpha_\infty + 1/q < 0$ and $q < \infty$ or $\alpha_\infty \le 0$ and $q = \infty$, then

$$(L_{(\infty,q;\mathbb{A})}(\Omega))' = \Big\{ g \in \mathcal{M} : \|g\| = \int_0^\infty g^*(t)dt + \|g|L_{(1,q^*,-\alpha_0-1/\min\{1,q\})}(0,1)\| < \infty \Big\},$$

where $\|\cdot|L_{(1,q',-\alpha_0-1/\min\{1,q\})}(0,1)\|$ means that the integral in (2.24) is taken only on the interval $(0,1)$ instead of $(0,\infty)$.

Proof. Using (4.16) and Theorem 4.12/2 we get that:

If $\alpha_\infty + 1/q > 0$, then

$$\begin{aligned}(L_{(\infty,q,\mathbb{A})}(\Omega))' &= (L_1(\Omega), L_\infty(\Omega))'_{1,q,\mathbb{A}} = (L_\infty(\Omega), L_1(\Omega))_{1,q^*,-\tilde{\mathbb{A}}-1/\min\{1,q\}} \\ &= (L_1(\Omega), L_\infty(\Omega))_{0,q^*,-\mathbb{A}-1/\min\{1,q\}} = L_{(1,q^*,-\mathbb{A}-1/\min\{1,q\})}(\Omega).\end{aligned}$$

If $\alpha_\infty + 1/q = 0$ and $q < \infty$, then

$$\begin{aligned}(L_{(\infty,q,\mathbb{A})}(\Omega))' &= (L_1(\Omega), L_\infty(\Omega))'_{1,q,\mathbb{A}} = (L_\infty(\Omega), L_1(\Omega))_{1,q^*,-\tilde{\mathbb{A}}-1/\min\{1,q\},(-1/\min\{1,q\},0)} \\ &= (L_1(\Omega), L_\infty(\Omega))_{0,q^*,-\mathbb{A}-1/\min\{1,q\},(0,-1/\min\{1,q\})}.\end{aligned}$$

Thus

$$\|f|(L_{(\infty,q,\mathbb{A})}(\Omega))'\| \sim \left(\int_0^\infty \left[\ell^{(-\mathbb{A}-1/\min\{1,q\})}(t)\ell\ell^{(0,-1/\min\{1,q\})}(t) \int_0^t f^*(s)ds \right]^{q^*} \frac{dt}{t} \right)^{1/q^*}.$$

Finally, if $\alpha_\infty + 1/q < 0$ and $q < \infty$, or $\alpha_\infty \le 0$ and $q = \infty$, then

$$(L_{(\infty,q,\mathbb{A})}(\Omega))' = (L_1(\Omega), L_\infty(\Omega))'_{1,q,\mathbb{A}} = L_1(\Omega) \cap \langle L_\infty(\Omega), L_1(\Omega)\rangle_{1,q^*,-\alpha_0-1/\min\{1,q\}}.$$

Hence,

$$\|f|(L_{(\infty,q,\mathbb{A})}(\Omega))'\| \sim \int_0^\infty f^*(t)dt + \|f|\langle L_\infty(\Omega), L_1(\Omega)\rangle_{1,q^*,-\alpha_0-1/\min\{1,q\}}\|,$$

and

$$\begin{aligned}&\|f|\langle L_\infty(\Omega), L_1(\Omega)\rangle_{1,q^*,-\alpha_0-1/\min\{1,q\}}\| \\ &= \left(\sum_{m=0}^{\infty} \left[2^{-m}\ell^{-\alpha_0-1/\min\{1,q\}}(2^m)K(2^m, g; L_\infty, L_1) \right]^{q^*} \right)^{1/q^*} \\ &= \left(\sum_{m=0}^{\infty} \left[\ell^{-\alpha_0-1/\min\{1,q\}}(2^m)K(2^{-m}, g; L_1, L_\infty) \right]^{q^*} \right)^{1/q^*} \\ &\sim \left(\int_0^1 \left[\ell^{-\alpha_0-1/\min\{1,q\}}(t)K(t, g; L_1, L_\infty) \right]^{q^*} \frac{dt}{t} \right)^{1/q^*} \\ &= \left(\int_0^1 \left[\ell^{-\alpha_0-1/\min\{1,q\}}(t) \int_0^t g^*(s)ds \right]^{q^*} \frac{dt}{t} \right)^{1/q^*} \\ &= \|g|L_{(1,q^*;-\alpha_0-1/\min\{1,q\})}(0,1)\|.\end{aligned}$$

□

Notice that the case $q = \infty$ and $\alpha_0 = 0$ is not included in the previous theorem. This special case will be treated in Chapter 5.

Theorem 4.15. Let $0 < q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ such that $\alpha_\infty + 1/q < 0$. We have with equivalence of quasi-norms:

1. If $\alpha_0 + 1/q > 0$, then $(L_{(1,q,\mathbb{A})}(\Omega))' = L_{(\infty,q^*,-\mathbb{A}-1/\min\{1,q\})}(\Omega)$.
2. If $\alpha_0 + 1/q = 0$ and $q < \infty$, then $(L_{(1,q;\mathbb{A})}(\Omega))' =$

$$\left\{ g \in \mathcal{M} : \|g\| = \left(\int_0^\infty \left[\ell^{(-\mathbb{A}-1/\min\{1,q\})}(t) \ell\ell^{(-1/\min\{1,q\},0)}(t) \int_0^t g^*(s)ds \right]^{q^*} \frac{dt}{t} \right)^{1/q^*} < \infty \right\}.$$

3. If $\alpha_0 + 1/q < 0$ and $q < \infty$ or $\alpha_0 \le 0$ and $q = \infty$, then $(L_{(1,q,\mathbb{A})}(\Omega))' =$

$$\left\{ g \in \mathcal{M} : \|g\| = \|g|L_\infty(\Omega)\| + \|g|L_{(\infty,q^*,-\alpha_\infty-1/\min\{1,q\})}(1,\infty)\| < \infty \right\},$$

   where $\| \cdot |L_{(\infty,q^*,-\alpha_\infty-1/\min\{1,q\})}(1,\infty)\|$ means that the integral in (2.24) is taken only on the interval $(1,\infty)$ instead of $(0,\infty)$.

Proof. One can proceed as in Theorem 4.14 but replacing Theorem 4.12/2 by Theorem 4.12/3. □

We close the chapter by establishing the corresponding results to Theorems 4.13, 4.14 and 4.15 for sequence spaces. This question has not been discussed in [97], but it can be treated as another specific application of the abstract results of Section 4.2.

Let $\Omega = \mathbb{N}$ and $\mu = \#$ the counting measure. Given any bounded sequence of scalars $\xi = (\xi_n)$, we put

$$\xi_n^* = \inf\{\tau > 0 : \text{card}\{j \in \mathbb{N} : |\xi_j| \ge \tau\} < n\}.$$

The sequence (ξ_n^*) is the decreasing rearrangement of (ξ_n) by magnitude of modulus. If $\xi = (\xi_n)$ converges to zero, then

$$\xi_1^* = \max\{|\xi_n| : n \in \mathbb{N}\} = |\xi_{n_1}|, \quad \xi_2^* = \max\left\{|\xi_n| : n \in \mathbb{N} \setminus \{n_1\}\right\} \quad \text{and so on.}$$

For $\alpha \in \mathbb{R}$, $1 \le p \le \infty$ and $0 < q \le \infty$, let $\ell_{(p,q,\alpha)}$ be the collection of all bounded sequences $\xi = (\xi_n)$ such that

$$\|\xi|\ell_{(p,q,\alpha)}\| = \left(\sum_{n=1}^\infty \left[\ell^\alpha(n) n^{1/p-1} \sum_{j=1}^n \xi_j^*\right]^q n^{-1} \right)^{1/q} < \infty.$$

Replacing the weight $\ell^\alpha(n)$ by $\ell^\alpha(n)\ell\ell^\beta(n)$, where $\beta \in \mathbb{R}$, we obtain the spaces $\ell_{(p,q,\alpha,\beta)}$.

To determine the associate space of $\ell_{(p,q,\alpha)}$, we work with the Banach couple $(\ell_1(\mathbb{N}), \ell_\infty(\mathbb{N}))$. The K-functional for this couple is

$$K(n, \xi; \ell_1(\mathbb{N}), \ell_\infty(\mathbb{N})) = \sum_{j=1}^n \xi_j^*, \quad n \in \mathbb{N} \tag{4.17}$$

(see [110, p. 126]). Observe that

$$\|\xi|\ell_1(\mathbb{N})^{\sim}\| = \sup_{0<t<\infty} K(t,\xi) = \lim_{n\to\infty} \sum_{j=1}^{n} \xi_j^* = \|\xi|\ell_1(\mathbb{N})\|.$$

Therefore $(\ell_1(\mathbb{N}), \ell_\infty(\mathbb{N}))$ is a mutually closed couple. Moreover, $\ell_1(\mathbb{N})$ has absolutely continuous norm.

Some of the spaces $\ell_{(p,q,\alpha)}$ are very well-known. Indeed, if $\alpha + 1/q < 0$ then the space $\ell_{(\infty,q,\alpha)} = \ell_\infty$, because

$$\begin{aligned}\|\xi|\ell_\infty(\mathbb{N})\| \leq \|\xi|\ell_{(\infty,q,\alpha)}\| &= \Big(\sum_{n=1}^{\infty} [\ell^\alpha(n) n^{-1} \sum_{j=1}^{n} \xi_j^*]^q n^{-1} \Big)^{1/q} \\ &\leq \|\xi|\ell_\infty(\mathbb{N})\| \Big(\sum_{n=1}^{\infty} \ell^{\alpha q}(n) n^{-1} \Big)^{1/q} \sim \|\xi|\ell_\infty(\mathbb{N})\|.\end{aligned}$$

On the other hand, if $\alpha + 1/q \geq 0$ and $q < \infty$ or $\alpha > 0$ and $q = \infty$, then $\ell_{(1,q,\alpha)} = \{0\}$. Indeed, if $\xi \in \ell_{(1,q,\alpha)}$ we have

$$\infty > \|\xi|\ell_{(1,q,\alpha)}\| \geq \|\xi|\ell_\infty(\mathbb{N})\| \Big(\sum_{n=1}^{\infty} \ell^{\alpha q}(n) n^{-1} \Big)^{1/q}.$$

Since $\Big(\sum_{n=1}^{\infty} \ell^{\alpha q}(n) n^{-1} \Big)^{1/q} = \infty$, we derive that $\|\xi|\ell_\infty(\mathbb{N})\| = 0$. Thus, $\xi = 0$.

Furthermore, $\|\xi|\ell_{(1,\infty,0)}\| = \sup_{n\in\mathbb{N}} \sum_{j=1}^{n} \xi_j^* = \|\xi|\ell_1(\mathbb{N})\|$. Hence, $\ell_{(1,\infty,0)} = \ell_1$.

We compute next the associate space of all non-trivial $\ell_{(p,q,\alpha)}$.

Theorem 4.16. Let $1 \leq p \leq \infty$, $0 < q \leq \infty$ and $\alpha \in \mathbb{R}$. We have the following equalities with equivalence of quasi-norms.

1. If $1 < p < \infty$, then $\ell'_{(p,q,\alpha)} = \ell_{(p',q^*,-\alpha)}$.
2. If $p = \infty$, then $\ell'_{(\infty,q,\alpha)} = \begin{cases} \ell_{(1,q^*,-\alpha-1/\min\{1,q\})} & \text{if} \quad \alpha + 1/q > 0, \\ \ell_{(1,q^*,-\alpha-1/\min\{1,q\},-1/\min\{1,q\})} & \text{if} \quad \alpha + 1/q = 0. \end{cases}$
3. If $p = 1$ and $\alpha + 1/q < 0$, then $\ell'_{(1,q,\alpha)} = \ell_{(\infty,q^*,-\alpha-1/\min\{1,q\})}$.

Proof. Choose $\beta \in \mathbb{R}$ such that $\beta + 1/q < 0$. As $\ell_1 \hookrightarrow \ell_\infty$, according to (3.13) under the above hypothesis we have that

$$\begin{aligned}\|\xi|(\ell_1(\mathbb{N}), \ell_\infty(\mathbb{N}))_{1-1/p,q,(\beta,\alpha)}\| &\sim \Big(\int_1^\infty [t^{1/p-1} \ell^\alpha(t) K(t,a)]^q \frac{dt}{t} \Big)^{1/q} \\ &\sim \Big(\sum_{n=1}^{\infty} [n^{1/p-1} \ell^\alpha(n) \sum_{j=1}^{n} \xi_j^*]^q n^{-1} \Big)^{1/q} = \|\xi|\ell_{(p,q,\alpha)}\|.\end{aligned}$$

Using the previous equivalence and Theorem 4.12 we get the result. □

Chapter 5

Structure of spaces $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$

According to Theorem 2.8, for any Banach couple $\bar{A} = (A_0, A_1)$ and $\alpha_0,\ \alpha_\infty \in \mathbb{R}$, it makes sense to consider the logarithmic interpolation spaces $(A_0, A_1)_{0,\infty,(\alpha_0,0)}$ and $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$. However we have left out these cases when studying duality results for logarithmic interpolation methods in Chapter 3 and its associate spaces in Chapter 4. The reason is that both kind of results are based on the J-representation of the spaces and, although Cobos and Segurado [48, Theorem 3.5] intended to cover also spaces $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ in their study of equivalence theorems, unfortunately the arguments given there do not work in this limit case.

In this chapter we focus on the study of $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$. As we know, the symmetry of logarithmic methods implies that $(A_0, A_1)_{0,\infty,(\alpha_0,0)} = (A_1, A_0)_{1,\infty,(0,\alpha_0)}$ and, therefore, the results given here cover also the case $\theta = 0$, $q = \infty$ and $\alpha_\infty = 0$. We start this chapter by looking into some basic properties of $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$. In Section 5.1, we show a representation of this space in terms of the J-functional, completing (3.8). It turns out that it is of a different type than the other descriptions we have seen in Sections 3.2 and 4.1. Then using this description we compute the associate space of $(X_0, X_1)_{1,\infty,(0,\alpha_\infty)}$ when X_0 and X_1 are Banach function spaces on a σ-finite measure space (Ω, μ) (see Section 4.2). As an application, we describe the associate space of the generalized Lorentz-Zygmund space $L_{(\infty,\infty,(0,\alpha_\infty))}(\Omega)$ defined in (2.24). In Section 5.3, we determine the dual of $(A_0, A_1)^\circ_{1,\infty,(0,\alpha_\infty)}$ for (A_0, A_1) a regular Banach couple. Here the circle indicates the closure of $A_0 \cap A_1$ in $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$.

The results include in this chapter form the paper [19].

5.1 Description in terms of the J-functional

Let $\bar{A} = (A_0, A_1)$ be a Banach couple and $\alpha_\infty \in \mathbb{R}$, remember that $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ is the space of all $a \in A_0 + A_1$ with finite norm

$$\begin{aligned}\|a|\bar{A}_{1,\infty,(0,\alpha_\infty)}\| &= \sup_{m\in\mathbb{Z}} 2^{-m}\ell^{(0,\alpha_\infty)}(2^m)K(2^m,a) \sim \sup_{0<t<\infty} \frac{K(t,a)\ell^{(0,\alpha_\infty)}(t)}{t} \\ &= \max\left\{ \sup_{0<t\le 1} \frac{K(t,a)}{t},\ \sup_{1<t<\infty} \frac{K(t,a)\ell^{\alpha_\infty}(t)}{t} \right\}.\end{aligned}$$

Note that if $\alpha_\infty \le 0$, as $K(t,a)/t$ is decreasing, then

$$\|a|\bar{A}_{1,\infty,(0,\alpha_\infty)}\| = \sup_{0<t\le 1} \frac{K(t,a)}{t} = \|a|A_1^\sim\|.$$

That is to say, if $\alpha_\infty \le 0$ then $\bar{A}_{1,\infty,(0,\alpha_\infty)}$ coincides with the Gagliardo completion $A_1^{\sim}$ of A_1.

Observe also that

$$(A_0, A_1)_{1,\infty,(0,\alpha_\infty)} = A_1^{\sim} \cap \langle A_0, A_1\rangle_{1,\infty,\alpha_\infty}, \tag{5.1}$$

being

$$\langle A_0, A_1\rangle_{1,\infty,\alpha_\infty} = \left\{ a \in A_0 + A_1 : \|a|\langle A_0, A_1\rangle_{1,\infty,\alpha_\infty}\| = \sup_{1<t<\infty} \frac{K(t,a)\ell^{\alpha}(t)}{t} < \infty \right\}$$

a special case of spaces defined in (4.13).

Next, we investigate different descriptions of $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ for $\alpha_\infty > 0$. We start with an auxiliary lemma.

Lemma 5.1. Let (A_0, A_1) be a Banach couple. For any $a \in A_1$ we have

$$K(t,a;A_0 \cap A_1, A_1) \sim \begin{cases} t\|a|A_1\| & \text{if} \quad 0 < t \le 1, \\ \|a|A_1\| + K(t,a;A_0,A_1) & \text{if} \quad 1 < t < \infty. \end{cases}$$

Proof. Take any $a \in A_1$ and $0 < t \le 1$. Clearly, $K(t,a;A_0 \cap A_1, A_1) \le t\|a|A_1\|$. On the other hand, if $a = a_0 + a_1$ with $a_0 \in A_0 \cap A_1$, $a_1 \in A_1$, then we have

$$t\|a|A_1\| \le t\left(\|a_0|A_1\| + \|a_1|A_1\|\right) \le \|a_0|A_0 \cap A_1\| + t\|a_1|A_1\|.$$

Hence, $K(t,a;A_0 \cap A_1, A_1) = t\|a|A_1\|$.

Assume now that $1 < t < \infty$ and $a \in A_1$. If $a = a_0 + a_1$ with $a_j \in A_j$, then $a_0 = a - a_1$ belongs to A_1 and we get

$$\begin{aligned} K(t,a;A_0 \cap A_1, A_1) &\le \|a_0|A_0 \cap A_1\| + t\|a_1|A_1\| \\ &\le \max\left(\|a_0|A_0\|, \|a - a_1|A_1\|\right) + t\|a_1|A_1\| \\ &\le 2\left(\|a_0|A_0\| + t\|a_1|A_1\|\right) + \|a|A_1\|. \end{aligned}$$

Hence, $K(t,a;A_0 \cap A_1, A_1) \le 2K(t,a;A_0,A_1) + \|a|A_1\|$. On the other hand,

$$\|a|A_1\| = K(1,a;A_0 \cap A_1, A_1) \le K(t,a;A_0 \cap A_1, A_1)$$

and, if $a = a_0 + a_1$ with $a_j \in A_j$, then

$$K(t,a;A_0,A_1) \le \|a_0|A_0\| + t\|a_1|A_1\| \le \|a_0|A_0 \cap A_1\| + t\|a_1|A_1\|.$$

Therefore,

$$\|a|A_1\| + K(t,a;A_0,A_1) \le 2K(t,a;A_0 \cap A_1, A_1).$$

□

Proposition 5.2. Let (A_0, A_1) be a Banach couple and $-\infty < \alpha_0 < 0 < \alpha_\infty < \infty$. Then we have with equivalent norms

$$A_1 \cap \langle A_0, A_1\rangle_{1,\infty,\alpha_\infty} = (A_0 \cap A_1, A_1)_{1,\infty,(\alpha_0,\alpha_\infty)}.$$

Proof. Take any $a \in (A_0 \cap A_1, A_1)_{1,\infty,(\alpha_0,\alpha_\infty)}$. Then $a \in A_1$. Using Lemma 5.1 we obtain

$$\begin{aligned} \|a|\langle A_0, A_1\rangle_{1,\infty,\alpha_\infty}\| &= \sup_{1<t<\infty} \frac{\ell^{\alpha_\infty}(t)K(t,a;A_0,A_1)}{t} \\ &\lesssim \sup_{1<t<\infty} \frac{\ell^{\alpha_\infty}(t)K(t,a;A_0\cap A_1,A_1)}{t} \\ &\leq \|a|(A_0\cap A_1,A_1)_{1,\infty,(\alpha_0,\alpha_\infty)}\|. \end{aligned}$$

Moreover,

$$\|a|A_1\| = K(1,a;A_0\cap A_1,A_1) \leq \|a|(A_0\cap A_1,A_1)_{1,\infty,(\alpha_0,\alpha_\infty)}\|.$$

Therefore, $(A_0 \cap A_1, A_1)_{1,\infty,(\alpha_0,\alpha_\infty)} \hookrightarrow A_1 \cap \langle A_0, A_1\rangle_{1,\infty,\alpha_\infty}$.

Conversely, if $a \in A_1 \cap \langle A_0, A_1\rangle_{1,\infty,\alpha_\infty}$, then

$$\begin{aligned} \|a|(A_0\cap A_1,A_1)_{1,\infty,(\alpha_0,\alpha_\infty)}\| &\sim \sup_{0<t\leq 1} \frac{\ell^{\alpha_0}(t)K(t,a;A_0\cap A_1,A_1)}{t} \\ &+ \sup_{1<t<\infty} \frac{\ell^{\alpha_\infty}(t)K(t,a;A_0\cap A_1,A_1)}{t} = S_1+S_2. \end{aligned}$$

By Lemma 5.1 and using that $\alpha_0 < 0$, we obtain

$$\begin{aligned} S_1 &\sim \sup_{0<t\leq 1} \left(\ell^{\alpha_0}(t)\right)\|a|A_1\| \\ &\sim \sup_{1<t<\infty} \left(\frac{\ell^{\alpha_\infty}(t)}{t}\right) K(1,a;A_0\cap A_1,A_1) \\ &\leq \sup_{1<t<\infty} \left(\frac{\ell^{\alpha_\infty}(t)K(t,a;A_0\cap A_1,A_1)}{t}\right) = S_2. \end{aligned}$$

Furthermore,

$$\begin{aligned} S_2 &\sim \sup_{1<t<\infty} \frac{\ell^{\alpha_\infty}(t)(\|a|A_1\| + K(t,a;A_0,A_1))}{t} \\ &\leq \sup_{1<t<\infty} \left(\frac{\ell^{\alpha_\infty}(t)}{t}\right)\|a|A_1\| + \sup_{1<t<\infty} \frac{\ell^{\alpha_\infty}(t)K(t,a;A_0,A_1)}{t} \\ &\sim \|a|A_1\| + \|a|\langle A_0,A_1\rangle_{1,\infty,\alpha_\infty}\| \sim \|a|A_1\cap\langle A_0,A_1\rangle_{1,\infty,\alpha_\infty}\|. \end{aligned}$$

□

Since $A_j \hookrightarrow A_j^{\sim} \hookrightarrow A_0 + A_1$, it follows from (2.8) that for any $a \in A_0 + A_1$ and $t > 0$ we have

$$\begin{aligned} K(t,a;A_0,A_1) &= K(t,a;A_0^{\sim},A_1^{\sim}) \\ &\leq K(t,a;A_0,A_1^{\sim}) \leq K(t,a;A_0,A_1). \end{aligned}$$

This fact and (5.1) yield that

$$(A_0,A_1)_{1,\infty,(0,\alpha_\infty)} = A_1^{\sim} \cap \langle A_0,A_1\rangle_{1,\infty,\alpha_\infty} = A_1^{\sim} \cap \langle A_0,A_1^{\sim}\rangle_{1,\infty,\alpha_\infty}. \tag{5.2}$$

As a direct consequence of (5.2), Proposition 5.2 and (3.8), we derive the following result.

Corollary 5.3. Let (A_0, A_1) be a Banach couple and $0 < \alpha_\infty < \infty$. Then for any $-\infty < \alpha < 0$ we have with equivalent norms

$$\begin{aligned}(A_0, A_1)_{1,\infty,(0,\alpha_\infty)} &= (A_0 \cap A_1^{\sim}, A_1^{\sim})_{1,\infty,(\alpha,\alpha_\infty)} \\ &= (A_0 \cap A_1^{\sim}, A_1^{\sim})^J_{1,\infty,(\alpha+1,\alpha_\infty+1)}.\end{aligned}$$

Now we show that if the couple (A_0, A_1) is ordered by inclusion then $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ coincides with an usual J-space modelled on (A_0, A_1).

Corollary 5.4. Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$ and let $0 < \alpha_\infty < \infty$. Then we have with equivalence of norms

$$(A_0, A_1)_{1,\infty,(0,\alpha_\infty)} = (A_0, A_1)^J_{1,\infty,(1,\alpha_\infty+1)}.$$

Proof. Take any $\alpha < 0$. Then

$$(A_0, A_1)^J_{1,\infty,(1,\alpha_\infty+1)} \hookrightarrow (A_0, A_1)^J_{1,\infty,(\alpha+1,\alpha_\infty+1)}.$$

Since $A_0 \hookrightarrow A_1$, the Gagliardo completion of A_1 is A_1. Moreover, $A_0 \cap A_1^{\sim} = A_0$. It follows from Corollary 5.3 that

$$(A_0, A_1)^J_{1,\infty,(1,\alpha_\infty+1)} \hookrightarrow (A_0, A_1)^J_{1,\infty,(\alpha+1,\alpha_\infty+1)} = (A_0, A_1)_{1,\infty,(0,\alpha_\infty)}.$$

To complete the proof it suffices to check that

$$(A_0, A_1)^J_{1,\infty,(\alpha+1,\alpha_\infty+1)} \hookrightarrow (A_0, A_1)^J_{1,\infty,(1,\alpha_\infty+1)}. \tag{5.3}$$

We work here with the continuous definition of J-spaces (see (2.31)). Take any $a \in \bar{A}^J_{1,\infty,(\alpha+1,\alpha_\infty+1)}$ and let $a = \int_0^\infty u(t)\frac{dt}{t}$ be a representation such that

$$\begin{aligned}\max\Big\{ \sup_{0<t\le 1} \frac{\ell^{\alpha+1}(t)J(t,u(t))}{t}, \sup_{1<t<\infty} \frac{\ell^{\alpha_\infty+1}(t)J(t,u(t))}{t}\Big\} \\ \le 2\|a|(A_0, A_1)^J_{1,\infty,(\alpha+1,\alpha_\infty+1)}\|.\end{aligned}$$

Then $\int_0^1 u(t)\frac{dt}{t}$ is convergent in A_0 because

$$\begin{aligned}\int_0^1 \|u(t)|A_0\|\frac{dt}{t} &\le \int_0^1 J(t,u(t))\frac{dt}{t} = \int_0^1 \frac{\ell^{\alpha}(t)J(t,u(t))}{t} t\ell^{-\alpha}(t)\frac{dt}{t} \\ &\le \left(\int_0^1 t\ell^{-\alpha}(t)\frac{dt}{t}\right) \sup_{0<t\le 1}\left(\frac{\ell^{\alpha}(t)J(t,u(t))}{t}\right) \\ &\lesssim \|a|(A_0, A_1)^J_{1,\infty,(\alpha+1,\alpha_\infty+1)}\|.\end{aligned}$$

Put $w = \int_0^1 u(t)\frac{dt}{t}$ and let

$$v(t) = \begin{cases} 0 & \text{if} \quad 0 < t \le 1, \\ u(t) + \frac{w}{\log 2} & \text{if} \quad 1 < t < 2, \\ u(t) & \text{if} \quad 2 \le t < \infty. \end{cases}$$

Then $a = \int_0^\infty v(t)\frac{dt}{t}$ (convergence in $A_1 = A_0 + A_1$), and for $1 < t < 2$ we have

$$\begin{aligned}\frac{\ell^{\alpha_\infty+1}(t)J(t,v(t))}{t} &\leq \frac{\ell^{\alpha_\infty+1}(t)J(t,u(t))}{t} + \frac{\ell^{\alpha_\infty+1}(2)}{\log 2}J(1,w)\\ &\lesssim \|a|(A_0,A_1)^J_{1,\infty,(\alpha+1,\alpha_\infty+1)}\| + \int_0^1 \|u(t)|A_0\|\frac{dt}{t}\\ &\lesssim \|a|(A_0,A_1)^J_{1,\infty,(\alpha+1,\alpha_\infty+1)}\|.\end{aligned}$$

This yields that a belongs to $(A_0,A_1)^J_{1,\infty,(1,\alpha_\infty+1)}$ with

$$\|a|(A_0,A_1)^J_{1,\infty,(1,\alpha_\infty+1)}\| \leq \sup_{1<t<\infty}\frac{\ell^{\alpha_\infty+1}(t)J(t,v(t))}{t} \lesssim \|a|(A_0,A_1)^J_{1,\infty,(\alpha+1,\alpha_\infty+1)}\|.$$

This establish (5.3) and finishes the proof. □

Now we return to work with arbitrary Banach couples. We are going to give another description of $(A_0,A_1)_{1,\infty,(0,\alpha_\infty)}$ in terms of the J-functional. For this aim we need to introduce the following mixed-type J-space: for $\alpha_\infty > 0$, $(A_0,A_1)^J_{\alpha_\infty}$ consists of all $a \in A_0 + A_1$ for which there is a strongly measurable function $u(t)$ with values in $A_0 \cap A_1$ such that

$$a = \int_0^\infty u(t)\frac{dt}{t} \quad \text{(convergence in } A_0 + A_1\text{)}$$

and

$$\int_0^1 \frac{J(t,u(t))}{t}\frac{dt}{t} + \sup_{1<t<\infty}\frac{\ell^{\alpha_\infty+1}(t)J(t,u(t))}{t} < \infty.$$

The norm in $(A_0,A_1)^J_{\alpha_\infty}$ is

$$\begin{aligned}&\|a|(A_0,A_1)^J_{\alpha_\infty}\|\\ &\quad = \inf\left\{\int_0^1 \frac{J(t,u(t))}{t}\frac{dt}{t} + \sup_{1<t<\infty}\frac{\ell^{\alpha_\infty+1}(t)J(t,u(t))}{t} : a = \int_0^\infty u(t)\frac{dt}{t}\right\}.\end{aligned}$$

Observe that

$$A_0 \cap A_1 \hookrightarrow (A_0,A_1)^J_{\alpha_\infty} \hookrightarrow A_0 + A_1$$

because if $a = \int_0^\infty u(t)\frac{dt}{t}$ is a representation of $a \in (A_0,A_1)^J_{\alpha_\infty}$ then

$$\begin{aligned}\|a|A_0+A_1\| &\leq \int_0^\infty \|u(t)|A_0+A_1\|\frac{dt}{t} \leq \int_0^1 J(t,u(t))\frac{dt}{t} + \int_1^\infty \frac{J(t,u(t))}{t}\frac{dt}{t}\\ &\leq \int_0^1 \frac{J(t,u(t))}{t}\frac{dt}{t} + \left(\int_1^\infty \ell^{-\alpha_\infty-1}(t)\frac{dt}{t}\right)\sup_{1<t<\infty}\frac{\ell^{\alpha_\infty+1}(t)J(t,u(t))}{t}.\end{aligned}$$

Furthermore, any $a \in A_0 \cap A_1$ can be written as $a = \int_0^\infty u(t)\frac{dt}{t}$ with $u(t) = \begin{cases}\frac{1}{\log 2}a & \text{if } \frac{1}{2} < t < 1,\\ 0 & \text{otherwise}\end{cases}.$ Hence

$$\|a|(A_0,A_1)^J_{\alpha_\infty}\| \leq \int_{1/2}^1 \frac{J(t,a)}{t\log 2}\frac{dt}{t} \leq 2J(1,a) = 2\|a|A_0\cap A_1\|.$$

Theorem 5.5. Let (A_0, A_1) be a Banach couple and let $0 < \alpha_\infty < \infty$. Then we have with equivalence of norms

$$(A_0, A_1)_{1,\infty,(0,\alpha_\infty)} = (A_0, A_1)^J_{\alpha_\infty}.$$

Proof. We follow a direct approach based on ideas developed in [37]. Let $a \in (A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$. Since

$$\sup_{0<t\leq 1} \frac{K(t,a)}{t} < \infty \quad \text{and} \quad \sup_{1<t<\infty} \frac{\ell^{\alpha_\infty}(t)K(t,a)}{t} < \infty,$$

we have that

$$\lim_{t\to 0} K(t,a) = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{K(t,a)}{t} = 0. \tag{5.4}$$

Given $\nu \in \mathbb{Z}$, let

$$\gamma_\nu = \begin{cases} 2^{2^{\nu-1}} & \text{if } \nu > 0, \\ 1 & \text{if } \nu = 0, \\ 2^{-2^{-\nu-1}} & \text{if } \nu < 0. \end{cases}$$

We can decompose $a = a_{0,\nu} + a_{1,\nu}$ with $a_{j,\nu} \in A_j$ such that

$$\|a_{0,\nu}|A_0\| + \gamma_{\nu-1}\|a_{1,\nu}|A_1\| \leq 2K(\gamma_{\nu-1}, a),\ \nu \in \mathbb{Z}. \tag{5.5}$$

Write

$$u_\nu = a_{0,\nu} - a_{0,\nu-1} = a_{1,\nu-1} - a_{1,\nu},\ \nu \in \mathbb{Z}.$$

Then $(u_\nu) \subseteq A_0 \cap A_1$. Moreover, by (5.4), we have that

$$\begin{aligned} \|a - \sum_{\nu=N}^{M} u_\nu|A_0 + A_1\| &= \|a - a_{0,M} + a_{0,N-1}|A_0 + A_1\| \\ &\leq \|a_{0,N-1}|A_0\| + \|a_{1,M}|A_1\| \\ &\leq 2\left[K(\gamma_{N-2}, a) + \frac{K(\gamma_{M-1}, a)}{\gamma_{M-1}}\right] \to 0 \end{aligned}$$

as $M \to \infty$ and $N \to -\infty$. Therefore $a = \sum_{\nu=-\infty}^{\infty} u_\nu$ in $A_0 + A_1$.

For $\nu \in \mathbb{Z}$, let $D_\nu = (\gamma_{\nu-1}, \gamma_\nu]$. We have

$$\int_{D_1} \frac{dt}{t} = \log 2 = \int_{D_0} \frac{dt}{t}.$$

If $\nu > 1$ then

$$\int_{D_\nu} \frac{dt}{t} = \int_{2^{2^{\nu-2}}}^{2^{2^{\nu-1}}} \frac{dt}{t} = 2^{\nu-2}\log 2,$$

and if $\nu < 0$ we get

$$\int_{D_\nu} \frac{dt}{t} = \int_{2^{-2^{-\nu}}}^{2^{-2^{-\nu-1}}} \frac{dt}{t} = 2^{-\nu-1}\log 2.$$

Put

$$u(t) = \begin{cases} \frac{u_\nu}{\log 2} & \text{if } t \in D_\nu \text{ and } \nu = 0, 1, \\ \frac{u_\nu}{2^{\nu-2}\log 2} & \text{if } t \in D_\nu \text{ and } \nu > 1, \\ \frac{u_\nu}{2^{-\nu-1}\log 2} & \text{if } t \in D_\nu \text{ and } \nu < 0. \end{cases}$$

Then

$$\int_0^\infty u(t)\frac{dt}{t} = \sum_{\nu=-\infty}^{\infty} u_\nu = a \text{ in } A_0 + A_1.$$

The functions $\frac{J(t,u)}{t}$ and $\frac{K(t,a)}{t}$ are decreasing. Hence, using (5.5), for $\nu \geq 2$ and $t \in D_\nu$ we obtain

$$\begin{aligned}\frac{J(t,u(t))}{t} &\sim \frac{1}{2^\nu}\frac{J(t,u_\nu)}{t} \leq \frac{1}{2^\nu}\frac{J(\gamma_{\nu-1},u_\nu)}{\gamma_{\nu-1}} \\ &\lesssim \frac{1}{2^\nu}\left[\frac{K(\gamma_{\nu-1},a)}{\gamma_{\nu-1}} + \frac{K(\gamma_{\nu-2},a)}{\gamma_{\nu-2}}\right] \\ &\lesssim \frac{1}{2^\nu}\frac{K(\gamma_{\nu-2},a)}{\gamma_{\nu-2}}.\end{aligned}$$

Note also that

$$2^{\nu-2}\log 2 < \log t \leq 2^{\nu-1}\log 2 \quad \text{if } t \in D_\nu.$$

Whence,

$$\begin{aligned}\sup_{t\in D_\nu}\frac{\ell^{\alpha_\infty+1}(t)J(t,u(t))}{t} &\lesssim \frac{2^{\nu(\alpha_\infty+1)}K(\gamma_{\nu-2},a)}{2^\nu\gamma_{\nu-2}} \\ &\lesssim \sup_{t\in D_{\nu-2}}\frac{\ell^{\alpha_\infty}(t)K(t,a)}{t}.\end{aligned}$$

Proceeding similarly, if $\nu < 0$ and $t \in D_\nu$ we get

$$\frac{J(t,u(t))}{t} \sim \frac{J(t,u_\nu)}{2^{-\nu}t} \leq \frac{J(\gamma_{\nu-1},u_\nu)}{2^{-\nu}\gamma_{\nu-1}} \lesssim \frac{1}{2^{-\nu}}\frac{K(\gamma_{\nu-2},a)}{\gamma_{\nu-2}}.$$

Therefore,

$$\begin{aligned}\int_{D_\nu}\frac{J(t,u(t))}{t}\frac{dt}{t} &\lesssim \frac{1}{2^{-\nu}}\frac{K(\gamma_{\nu-2},a)}{\gamma_{\nu-2}}\int_{D_\nu}\frac{dt}{t} \sim \frac{K(\gamma_{\nu-2},a)}{\gamma_{\nu-2}} \\ &\leq \sup_{t\in D_{\nu-2}}\frac{K(t,a)}{t}.\end{aligned}$$

Furthermore, if $\nu = 1$ and $t \in D_1 = (1,2]$, we get

$$\frac{\ell^{\alpha_\infty+1}(t)J(t,u(t))}{t} \lesssim J(2,u_1) \lesssim K(1,a) \leq \sup_{0<t\leq 1}\frac{K(t,a)}{t},$$

and for $\nu = 0$ and $t \in D_0 = (2^{-1},1]$ we derive

$$\int_{D_0}\frac{J(t,u(t))}{t}\frac{dt}{t} \lesssim J(1,u_0) \lesssim K(1,a) \leq \sup_{0<t\leq 1}\frac{K(t,a)}{t}.$$

Consequently,

$$(A_0,A_1)_{1,\infty,(0,\alpha_\infty)} \hookrightarrow (A_0,A_1)^J_{\alpha_\infty}.$$

Conversely, take any $a \in (A_0,A_1)^J_{\alpha_\infty}$ and choose a representation $a = \int_0^\infty u(t)\frac{dt}{t}$ with

$$\int_0^1\frac{J(t,u(t))}{t}\frac{dt}{t} + \sup_{1<t<\infty}\frac{\ell^{\alpha_\infty+1}(t)J(t,u(t))}{t} \leq 2\|a|(A_0,A_1)^J_{\alpha_\infty}\|.$$

We have

$$K(t,a) \le \int_0^\infty K(t,u(s))\frac{ds}{s} \le \int_0^\infty \min\left(1,\frac{t}{s}\right) J(s,u(s))\frac{ds}{s}$$
$$= \int_0^t J(s,u(s))\frac{ds}{s} + t\int_t^\infty \frac{J(s,u(s))}{s}\frac{ds}{s}.$$

Therefore,

$$\|a|(A_0,A_1)_{1,\infty,(0,\alpha_\infty)}\| \le \sup_{0<t\le 1}\frac{K(t,a)}{t} + \sup_{1<t<\infty}\frac{\ell^{\alpha_\infty}(t)K(t,a)}{t}$$
$$\le \sup_{0<t\le 1}\frac{1}{t}\int_0^t J(s,u(s))\frac{ds}{s} + \sup_{0<t\le 1}\int_t^1 \frac{J(s,u(s))}{s}\frac{ds}{s}$$
$$+ \sup_{0<t\le 1}\int_1^\infty \frac{J(s,u(s))}{s}\frac{ds}{s} + \sup_{1<t<\infty}\frac{\ell^{\alpha_\infty}(t)}{t}\int_0^1 J(s,u(s))\frac{ds}{s}$$
$$+ \sup_{1<t<\infty}\frac{\ell^{\alpha_\infty}(t)}{t}\int_1^t J(s,u(s))\frac{ds}{s} + \sup_{1<t<\infty}\ell^{\alpha_\infty}(t)\int_t^\infty \frac{J(s,u(s))}{s}\frac{ds}{s}$$
$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

We proceed to estimate each of these six terms. We have

$$I_1 = \sup_{0<t\le 1}\frac{1}{t}\int_0^t s\frac{J(s,u(s))}{s}\frac{ds}{s} \le \int_0^1 \frac{J(s,u(s))}{s}\frac{ds}{s} \le 2\|a|(A_0,A_1)^J_{\alpha_\infty}\|.$$

For I_2, we obtain

$$I_2 \le \int_0^1 \frac{J(s,u(s))}{s}\frac{ds}{s} \le 2\|a|(A_0,A_1)^J_{\alpha_\infty}\|.$$

As for I_3, we have

$$I_3 = \int_1^\infty \frac{J(s,u(s))}{s}\frac{ds}{s} \le \left(\int_1^\infty \ell^{-\alpha_\infty-1}(s)\frac{ds}{s}\right)\sup_{1<t<\infty}\frac{\ell^{\alpha_\infty+1}(t)J(t,u(t))}{t}$$
$$\lesssim \|a|(A_0,A_1)^J_{\alpha_\infty}\|.$$

For I_4 we have

$$I_4 \le \sup_{1<t<\infty}\left(t^{-1}\ell^{\alpha_\infty}(t)\right)\int_0^1 \frac{J(s,u(s))}{s}\frac{ds}{s} \lesssim \|a|(A_0,A_1)^J_{\alpha_\infty}\|.$$

To estimate I_5 we use that $s\ell^{-\alpha_\infty-1}(s)$ is equivalent to an increasing function. We obtain

$$I_5 = \sup_{1<t<\infty}\frac{\ell^{\alpha_\infty}(t)}{t}\int_1^t \frac{\ell^{\alpha_\infty+1}(s)J(s,u(s))}{s}s\ell^{-\alpha_\infty-1}(s)\frac{ds}{s}$$
$$\le \sup_{1<t<\infty}\left[\frac{\ell^{\alpha_\infty}(t)}{t}\sup_{1<s<t}\left(\frac{\ell^{\alpha_\infty+1}(s)J(s,u(s))}{s}\right)\int_1^t s\ell^{-\alpha_\infty-1}(s)\frac{ds}{s}\right]$$
$$\lesssim \sup_{1<t<\infty}\left[\ell^{-1}(t)\sup_{1<s<t}\left(\frac{\ell^{\alpha_\infty+1}(s)J(s,u(s))}{s}\right)\int_1^t \frac{ds}{s}\right]$$
$$= \sup_{1<s<\infty}\frac{\ell^{\alpha_\infty+1}(s)J(s,u(s))}{s} \le 2\|a|(A_0,A_1)^J_{\alpha_\infty}\|.$$

Finally, for I_6, we get

$$
\begin{aligned}
I_6 &\le \sup_{1<t<\infty} \left[\ell^{\alpha_\infty}(t) \sup_{t<s<\infty} \left(\frac{\ell^{\alpha_\infty+1}(s) J(s,u(s))}{s} \right) \int_t^\infty \ell^{-\alpha_\infty-1}(s) \frac{ds}{s} \right] \\
&\lesssim \sup_{1<t<\infty} \left[\sup_{t<s<\infty} \frac{\ell^{\alpha_\infty+1}(s) J(s,u(s))}{s} \right] \lesssim \sup_{1<s<\infty} \frac{\ell^{\alpha_\infty+1}(s) J(s,u(s))}{s} \\
&\le 2\|a|(A_0,A_1)^J_{\alpha_\infty}\|.
\end{aligned}
$$

This completes the proof. □

Using the J-description given in Theorem 5.5 we can have a deeper understanding of the structure of $(A_0,A_1)_{1,\infty,(0,\alpha_\infty)}$.

Lemma 5.6. Let (A_0,A_1) be a Banach couple and let $0<\alpha_\infty<\infty$. Then

$$
(A_0,A_1)_{1,\infty,(0,\alpha_\infty)} \hookrightarrow A_1.
$$

Proof. Take any $a \in (A_0,A_1)_{1,\infty,(0,\alpha_\infty)} = (A_0,A_1)^J_{\alpha_\infty}$ and let $a = \int_0^\infty u(t)\frac{dt}{t}$ be a J-representation with

$$
\int_0^1 \frac{J(t,u(t))}{t}\frac{dt}{t} + \sup_{1<t<\infty} \frac{\ell^{\alpha_\infty+1}(t) J(t,u(t))}{t} \le 2\|a|(A_0,A_1)^J_{\alpha_\infty}\|.
$$

We have

$$
\begin{aligned}
\int_0^\infty \|u(t)|A_1\|\frac{dt}{t} &\le \int_0^\infty \frac{J(t,u(t))}{t}\frac{dt}{t} \\
&\le \int_0^1 \frac{J(t,u(t))}{t}\frac{dt}{t} + \sup_{1<t<\infty} \left(\frac{\ell^{\alpha_\infty+1}(t) J(t,u(t))}{t} \right) \int_1^\infty \ell^{-\alpha_\infty-1}(t)\frac{dt}{t} \\
&\lesssim \|a|(A_0,A_1)^J_{\alpha_\infty}\|.
\end{aligned}
$$

Therefore, a belongs to A_1 and $(A_0,A_1)_{1,\infty,(0,\alpha_\infty)} \hookrightarrow A_1$. □

Now we can get rid of the Gagliardo closure in Corollary 5.3.

Corollary 5.7. Let (A_0,A_1) be a Banach couple and let $0<\alpha_\infty<\infty$. Then for any $-\infty<\alpha<0$ we have with equivalent norms

$$
(A_0,A_1)_{1,\infty,(0,\alpha_\infty)} = (A_0\cap A_1, A_1)_{1,\infty,(\alpha,\alpha_\infty)} = (A_0\cap A_1, A_1)^J_{1,\infty,(\alpha+1,\alpha_\infty+1)}.
$$

Proof. Since $A_1 \hookrightarrow A_1^\sim$, using Corollary 5.3 we have

$$
(A_0\cap A_1, A_1)_{1,\infty,(\alpha,\alpha_\infty)} \hookrightarrow (A_0\cap A_1^\sim, A_1^\sim)_{1,\infty,(\alpha,\alpha_\infty)} = (A_0,A_1)_{1,\infty,(0,\alpha_\infty)}.
$$

On the other hand, by Lemma 5.6, $(A_0,A_1)_{1,\infty,(0,\alpha_\infty)} \hookrightarrow A_1$, and clearly $(A_0,A_1)_{1,\infty,(0,\alpha_\infty)} \hookrightarrow \langle A_0,A_1\rangle_{1,\infty,\alpha_\infty}$. So,

$$
\begin{aligned}
(A_0,A_1)_{1,\infty,(0,\alpha_\infty)} \hookrightarrow A_1 \cap \langle A_0,A_1\rangle_{1,\infty,\alpha_\infty} &= (A_0\cap A_1, A_1)_{1,\infty,(\alpha,\alpha_\infty)} \\
&= (A_0\cap A_1, A_1)^J_{1,\infty,(\alpha+1,\alpha_\infty+1)}
\end{aligned}
$$

where the last two equalities follow from Proposition 5.2 and [48, Theorem 3.5], respectively. □

5.2 Associate space

Let (Ω, μ) be a σ-finite measure space and let X_0 and X_1 be a Banach function space over (Ω, μ) (see Definition 4.5). Then, as it was pointed out in Section 4.2, $X_0 \cap X_1$, $X_0 + X_1$ and $(X_0, X_1)_{1,\infty,(0,\alpha_\infty)}$ are also Banach function spaces.

We prove now an auxiliary result that complements Lemma 4.7.

Lemma 5.8. Let (X_0, X_1) be a couple of Banach function spaces over a σ-finite space (Ω, μ). Assume that X_0 or X_1 has absolutely continuous norm. Then for any $g \in X_0' + X_1'$ and $t > 0$, we have that

$$K(t^{-1}, g; X_0' + X_1', X_1') = \sup_{f \in X_0 \cap X_1} \frac{\int_\Omega |fg| d\mu}{J(t, f; X_0 \cap X_1, X_1)}.$$

Proof. Take any $g \in X_0' + X_1'$ and $t > 0$. It follows from Lemma 4.7 that

$$K(t^{-1}, g; X_0', X_1') = \sup_{f \in X_0 \cap X_1} \frac{\int_\Omega |fg| d\mu}{J(t, f; X_0, X_1)}.$$

On the other hand, if $f \in X_0 \cap X_1$, then we have

$$\begin{aligned} J(t, f; X_0 \cap X_1, X_1) &= \max\left(\|f|X_0\|, \|f|X_1\|, t\|f|X_1\|\right) \\ &= \begin{cases} J(1, f; X_0, X_1) & \text{if} \quad 0 < t \le 1, \\ J(t, f; X_0, X_1) & \text{if} \quad 1 < t < \infty. \end{cases} \end{aligned}$$

Whence, for $0 < t \le 1$, we get

$$K(1, g; X_0', X_1') = \sup_{f \in X_0 \cap X_1} \frac{\int_\Omega |fg| d\mu}{J(t, f; X_0 \cap X_1, X_1)}$$

and, if $1 < t < \infty$, then

$$K(t^{-1}, g; X_0', X_1') = \sup_{f \in X_0 \cap X_1} \frac{\int_\Omega |fg| d\mu}{J(t, f; X_0 \cap X_1, X_1)}.$$

Using now that

$$K(\min\{1, t^{-1}\}, g; X_0', X_1') = K(t^{-1}, g; X_0' + X_1', X_1')$$

(see [91, Theorem 2]), the wanted equality follows. □

Theorem 5.9. Let (X_0, X_1) be a couple of Banach function spaces over a σ-finite measure space (Ω, μ). Assume that X_0 or X_1 has absolutely continuous norm. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ such that $\alpha_0 < 1 < \alpha_\infty$. Then

$$\left((X_0 \cap X_1, X_1)^J_{1,\infty,(\alpha_0,\alpha_\infty)}\right)' = (X_0' + X_1', X_1')_{1,1,(-\alpha_\infty,-\alpha_0)}.$$

Proof. We can proceed following the lines of Theorem 4.8 and using Lemma 5.8. □

The next results complete Theorem 4.12.

Theorem 5.10. Let (X_0, X_1) be a couple of Banach function spaces over a σ-finite measure space (Ω, μ). Assume that X_0 or X_1 has absolutely continuous norm and let $0 < \alpha_\infty < \infty$. Then for any

$\alpha < -1$ we have that

$$\Big((X_0, X_1)_{1,\infty,(0,\alpha_\infty)}\Big)' = (X_0', X_1')_{1,1,(-\alpha_\infty-1,\alpha)}.$$

Proof. Take any $\alpha_0 < 0$. According to Corollary 5.7 and Theorem 5.9, we derive

$$\begin{aligned}\Big((X_0, X_1)_{1,\infty,(0,\alpha_\infty)}\Big)' &= \Big((X_0 \cap X_1, X_1)^J_{1,\infty,(\alpha_0+1,\alpha_\infty+1)}\Big)' \\ &= (X_0' + X_1', X_1')_{1,1,(-\alpha_\infty-1,-\alpha_0-1)}.\end{aligned}$$

Now, applying [48, Corollary 2.5], we conclude that for any $\alpha < -1$ we have

$$\Big((X_0, X_1)_{1,\infty,(0,\alpha_\infty)}\Big)' = (X_0', X_1')_{1,1,(-\alpha_\infty-1,\alpha)}.$$

□

Corollary 5.11. Let (X_0, X_1) be a couple of Banach function spaces over a σ-finite measure space (Ω, μ). Assume that X_0 or X_1 has absolutely continuous norm and let $0 < \alpha_0 < \infty$. Then for any $\alpha < -1$ we have that

$$\Big((X_0, X_1)_{0,\infty,(\alpha_0,0)}\Big)' = (X_0', X_1')_{0,1,(\alpha,-\alpha_0-1)}.$$

Assume now that (Ω, μ) is a non-atomic measure space. Opic and Pick have shown by means of direct calculations that if $\alpha_\infty > 0$ and $\alpha_0 < -1$ then

$$\Big(L_{(\infty,\infty,(0,\alpha_\infty))}(\Omega)\Big)' = \left\{f \in \mathcal{M} : \|f\| = \int_0^\infty \ell^{(\alpha_0,-\alpha_\infty-1)}(t)\left(\int_0^t f^*(s)ds\right)\frac{dt}{t} < \infty\right\} \tag{5.6}$$

(see [97, Theorems 3.8, 6.2/(iv) and 3.8/(ii)]). We close this section by deriving (5.6) as an application of our abstract results and, therefore, completing Theorem 4.14.

It follows from (4.16) that

$$L_{(\infty,\infty,(0,\alpha_\infty))}(\Omega) = (L_1(\Omega), L_\infty(\Omega))_{1,\infty,(0,\alpha_\infty)}.$$

Thus, according to Theorem 5.10, we obtain

$$\begin{aligned}\Big(L_{(\infty,\infty,(0,\alpha_\infty))}(\Omega)\Big)' &= (L_\infty(\Omega), L_1(\Omega))_{1,1,(-\alpha_\infty-1,\alpha_0)} \\ &= \left\{f \in \mathcal{M} : \int_0^\infty t^{-1}\ell^{(-\alpha_\infty-1,\alpha_0)}(t)K(t,f;L_\infty(\Omega),L_1(\Omega))\frac{dt}{t} < \infty\right\} \\ &= \left\{f \in \mathcal{M} : \int_0^\infty \ell^{(-\alpha_\infty-1,\alpha_0)}(t)\left(\int_0^{t^{-1}} f^*(s)ds\right)\frac{dt}{t} < \infty\right\} \\ &= \left\{f \in \mathcal{M} : \int_0^\infty \ell^{(\alpha_0,-\alpha_\infty-1)}(t)\left(\int_0^t f^*(s)ds\right)\frac{dt}{t} < \infty\right\}.\end{aligned}$$

5.3 Duality

In what follows we work with regular Banach couples (A_0, A_1), that is, we assume that $A_0 \cap A_1$ is dense in A_j for $j = 0, 1$ (as we saw in Chapter 3, this is the appropriate assumption when studying the dual of an interpolation method). If A is a Banach space satisfying that $A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1$ then, as in Chapter 2, we write A° for the closure of $A_0 \cap A_1$ in A.

Theorem 5.12. Let $\bar{A} = (A_0, A_1)$ be a regular Banach couple and let $0 < \alpha_\infty < \infty$. Take any $\beta < -1$. Then we have with equivalent norms

$$\left((A_0, A_1)^{\circ}_{1,\infty,(0,\alpha_\infty)}\right)^* = (A_0^*, A_1^*)_{1,1,(-\alpha_\infty-1,\beta)}.$$

Proof. Let $\alpha < 0$. By Corollary 5.7 we have that

$$(A_0, A_1)^{\circ}_{1,\infty,(0,\alpha_\infty)} = (A_0 \cap A_1, A_1)^{\circ}_{1,\infty,(\alpha,\alpha_\infty)}.$$

Then, according to [48, Theorems 5.9 and 2.5] we obtain

$$\begin{aligned}\left((A_0, A_1)^{\circ}_{1,\infty,(0,\alpha_\infty)}\right)^* &= ((A_0 \cap A_1)^*, A_1^*)_{1,1,(-\alpha_\infty-1,-\alpha-1)} \\ &= (A_0^* + A_1^*, A_1^*)_{1,1,(-\alpha_\infty-1,-\alpha-1)} \\ &= (A_0^*, A_1^*)_{1,1,(-\alpha_\infty-1,\beta)}.\end{aligned}$$

□

We finish the chapter by characterizing $(A_0, A_1)^{\circ}_{1,\infty,(0,\alpha_\infty)}$.

Proposition 5.13. Let $\bar{A} = (A_0, A_1)$ be a regular Banach couple and let $0 < \alpha_\infty < \infty$. Then $(A_0, A_1)^{\circ}_{1,\infty,(0,\alpha_\infty)}$ consists of all those $a \in (A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ such that

$$t^{-1}\ell^{\alpha_\infty}(t)K(t,a) \xrightarrow{t\to\infty} 0. \tag{5.7}$$

Proof. Take any $\alpha < 0$. By Corollary 5.7 we know that $(A_0, A_1)^{\circ}_{1,\infty,(0,\alpha_\infty)} = (A_0 \cap A_1, A_1)^{\circ}_{1,\infty,(\alpha,\alpha_\infty)}$. Furthermore, Lemma 5.1 yields that condition (5.7) is equivalent to

$$\begin{cases} t^{-1}\ell^{\alpha}(t)K(t,a;A_0\cap A_1,A_1) \to 0 & \text{as} \quad t \to 0, \\ t^{-1}\ell^{\alpha_\infty}(t)K(t,a;A_0\cap A_1,A_1) \to 0 & \text{as} \quad t \to \infty. \end{cases}$$

Consequently, the result follows by applying [48, Proposition 3.10] to

$$(A_0 \cap A_1, A_1)_{1,\infty,(\alpha,\alpha_\infty)}.$$

□

Chapter 6

Interpolation of the measure of non-compactness of logarithmic spaces

In 1960 Kranosel'skiĭ proved in [86] that if T is a linear operator, continuous from L_{p_j} into L_{q_j} for $1 \le p_0, q_0, p_1, q_1 \le \infty$ and, in addition, $q_0 < \infty$ and $T : L_{p_0} \longrightarrow L_{q_0}$ is compact, then $T : L_p \longrightarrow L_q$ compactly provided that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and $0 < \theta < 1$. This result motivated the investigation of interpolation properties of compact operators under abstract interpolation methods. The first result already appeared in the seminal paper by Lions and Peetre [89] on the real method. The final result was proved by Cobos, Kühn and Schonbek [42] and Cwikel [50]: Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be Banach couples, $0 < \theta < 1$, $1 \le q \le \infty$ and $T \in \mathcal{L}(\bar{A}, \bar{B})$. If, in addition, $T : A_j \longrightarrow B_j$ is compact for $j = 0$ or $j = 1$, then $T : \bar{A}_{\theta,q} \longrightarrow \bar{B}_{\theta,q}$ is also compact. Later, this result was extended to quasi-Banach couples and $0 < q \le \infty$ by Cobos and Persson in [45].

The next natural step was to search for quantitative results. There are different ways for measuring how far a linear operator T is from being compact. One of the most popular is the measure of non-compactness $\beta(T)$ (see (6.6)). Cobos, Fernández-Martínez and Martínez proved in [41] that for $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ Banach couples, $0 < \theta < 1$, $1 \le q \le \infty$ and $T \in \mathcal{L}(\bar{A}, \bar{B})$ we have the following estimate for the measure of non-compactness of the interpolated operator:

$$\beta(T : \bar{A}_{\theta,q} \longrightarrow \bar{B}_{\theta,q}) \le C\beta(T : A_0 \longrightarrow B_0)^{1-\theta}\beta(T : A_1 \longrightarrow B_1)^{\theta}.$$

Again, this result was later extended to quasi-Banach couples and $0 < q \le \infty$ (see [66]).

In 2014 Edmunds and Opic [56] established a limiting variant of Kranosel'skiĭ's theorem in the Banach setting and for finite measure spaces (R, μ) and (S, ν) with the outcome that if $T : L_{p_0}(R) \to L_{q_0}(S)$ is compact and $T : L_{p_1}(R) \to L_{q_1}(S)$ is continuous, then T is also compact when acting between Lorentz-Zygmund spaces which are very close to $L_{p_0}(R)$ and $L_{q_0}(S)$. Abstract versions of the result by Edmunds and Opic in terms of logarithmic methods for $\theta = 0, 1$ have been obtained by Cobos, Fernández-Cabrera and Martínez [38] and Cobos and Segurado [48]. In the last paper it is shown that the limit variant of Kranosel'skiĭ's theorem also holds when the function spaces are defined on $\sigma-$finite measure spaces. One of the goals of this chapter is to show that the result is also valid if the function spaces which are the target of T, are quasi-Banach spaces.

The other aim of this chapter is to complete the results of [38, 48] by estimating the measure of non-compactness of the operator T acting between the Lorentz-Zygmund spaces in terms of the measure of non-compactness of the restriction of T from L_{p_0} into L_{q_0}. These results can be found in Section 6.2. But first, we give in Section 6.1 an estimate for the measure of non-compactness of the interpolated operator by logarithmic interpolation methods when working with quasi-Banach couples. This type of results had been proved before for Banach couples by Cobos, Fernández-Cabrera and Martínez [38, 39], with the help of the connection between the measure of non-compactness and another ideal measure. Since we work here with quasi-Banach couples where those connections are not available, we follow a more direct approach based on ideas originated in the papers by Cobos and Peetre [44] and Cobos, Kühn and Schonbek [42]. We split the operator with the help of certain projections associated to the vector-valued sequence spaces that arise with the construction of the logarithmic spaces, and then we proceed to estimate the measure of non-compactness of each piece.

The main results of this chapter form the papers [12, 15].

6.1 Measure of non-compactness and logarithmic interpolation methods

Subsequently, we work with limiting logarithmic interpolation spaces ($\theta = 0$ or $\theta = 1$). In fact, we are going to develop the theory for $\theta = 1$ and derive from here the relevant results for $\theta = 0$. Let $\bar{A} = (A_0, A_1)$ be a (p-normed) quasi-Banach couple, $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. Remember that, according to Theorem 2.8, $(A_0, A_1)_{1,q,\mathbb{A}}$ is non-trivial if, and only if,

$$\begin{cases} 0 < q < \infty, & \alpha_0 + 1/q < 0; \\ q = \infty, & \alpha_0 \leq 0. \end{cases} \tag{6.1}$$

As we have seen in Chapters 3, 4 and 5, if $\bar{A} = (A_0, A_1)$ is a Banach couple the study of equivalent representations of $(A_0, A_1)_{1,q,\mathbb{A}}$ is a quite complicate matter. We need here a J-representation for $(A_0, A_1)_{1,q,\mathbb{A}}$ when $\bar{A} = (A_0, A_1)$ is a (p-normed) quasi-Banach couple. We recall that a necessary condition for this to happen is that $(A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow (A_0 + A_1)^\circ$. According to Theorem 2.9, the last inclusion holds if, and only, if the parameters satisfy (6.1) and

$$\begin{cases} 0 < q < \infty, & \alpha_\infty + 1/q \geq 0; \\ q = \infty, & \alpha_\infty > 0; \end{cases} \tag{6.2}$$

The following abstract equivalence result is a consequence of [95, Theorem 3.19].

Theorem 6.1. Let $\overline{A} = (A_0, A_1)$ be a mutually closed (p-normed) quasi-Banach couple ($0 < p \leq 1$). Let $0 < q \leq \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying (6.1) and (6.2) and put $\Lambda = (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}$. Then we have with equivalent quasi-norms

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0, A_1)_{\Lambda;J}.$$

Proof. Take any $a \in (A_0, A_1)_{\Lambda;J}$ and let $a = \sum_{m=-\infty}^{\infty} u_m$ be a J-representation of a with

$$\|(J(2^m, u_m))|\Lambda\| \leq 2\|a|(A_0, A_1)_{\Lambda;J}\|.$$

For any $m \in \mathbb{Z}$, we have

$$K_p(2^k, a) \leq \Big(\sum_{m=-\infty}^{\infty} \min\{1, 2^{k-m}\}^p J(2^m, u_m)^p \Big)^{1/p}.$$

Hence

$$\begin{aligned} \|a|(A_0, A_1)_{1,q,\mathbb{A}}\| &\leq \Big(\sum_{k=-\infty}^{\infty} \Big[2^{-k} \ell^{\mathbb{A}}(2^k) \Big(\sum_{m=-\infty}^{\infty} \min\{1, 2^{k-m}\}^p J(2^m, u_m)^p \Big)^{1/p} \Big]^q \Big)^{1/q} \\ &\leq 2^{1/p} \|(J(2^m, u_m))|(\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}\| \leq 2^{1+1/p} \|a|(A_0, A_1)_{\Lambda;J}\|. \end{aligned}$$

Conversely, let $a \in (A_0, A_1)_{1,q,\mathbb{A}}$. Using (2.8) and [95, Theorem 3.2], we can find $(u_m) \subseteq A_0 \cap A_1$ with $a = \sum_{m=-\infty}^{\infty} u_m$ (in $A_0 + A_1$) and such that

$$\Big(\sum_{m=-\infty}^{\infty} \min\{1, 2^{k-m}\}^p J(2^m, u_m)^p \Big)^{1/p} \leq cK(2^k, a),\ k \in \mathbb{Z},$$

where $c = c_p$ is a constant independent of a and k. Therefore

$$\begin{aligned} \|a|(A_0, A_1)_{\Lambda;J}\| &\leq \|(J(2^m, u_m))|(\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}\| \\ &\leq \Big(\sum_{k=-\infty}^{\infty} \Big[2^{-k}\ell^{\mathbb{A}}(2^k) \Big(\sum_{m=-\infty}^{\infty} \min\{1, 2^{k-m}\}^p J(2^m, u_m)^p \Big)^{1/p} \Big]^q \Big)^{1/q} \\ &\leq c \Big(\sum_{k=-\infty}^{\infty} [2^{-k}\ell^{\mathbb{A}}(2^k) K(2^k, a)]^q \Big)^{1/q} = c\|a|(A_0, A_1)_{1,q,\mathbb{A}}\|. \end{aligned}$$

□

Notice that under the hypothesis in Theorem 6.1, the J-space $(A_0, A_1)_{\Lambda;J}$ is well-defined for p-normed quasi-Banach couples since $\Lambda \hookrightarrow \ell_p + \ell_p(2^{-m})$ and, therefore, is (p, J)-non trivial (see Subsection 2.1.2). Although the involved J-space can look more complex that logarithmic J-spaces that we have used in the previous chapters it has the advantage that we do not need to distinguish cases for the parameters and that it is valid for quasi-Banach spaces (in Chapter 7 we will see that under certain hypothesis, logarithmic interpolation methods acting on quasi-Banach couples do not admit a representation as a logarithmic J-space).

For later use, we prove now some results on the *shift operator* τ_k defined by $\tau_k \xi = (\xi_{m+k})$ for $\xi = (\xi_m)$. Here $k \in \mathbb{Z}$.

Subsequently, given $\lambda \in \mathbb{R}$ we put $\lambda^+ = \max\{0, \lambda\}$.

The following estimate is useful in what follows. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $\mathbb{B} = (\alpha_0^+ + (-\alpha_\infty)^+, \alpha_\infty^+ + (-\alpha_0)^+)$. Then for any $u > 0$ we have

$$\sup_{s>0} \frac{\ell^{\mathbb{A}}(su)}{\ell^{\mathbb{A}}(s)} \leq \ell^{\mathbb{B}}(u). \tag{6.3}$$

Inequality (6.3) was established in [39, Lemma 2.1] for $u \geq 1$. If $0 < u < 1$ it follows by applying the same result for $1/u > 1$ since

$$\sup_{s>0} \frac{\ell^{\mathbb{A}}(su)}{\ell^{\mathbb{A}}(s)} = \sup_{s>0} \frac{\ell^{(\alpha_\infty, \alpha_0)}(\frac{1}{s}\frac{1}{u})}{\ell^{(\alpha_\infty, \alpha_0)}(\frac{1}{s})} \leq \ell^{\alpha_0^+ + (-\alpha_\infty)^+}(1/u) = \ell^{\alpha_0^+ + (-\alpha_\infty)^+}(u).$$

Lemma 6.2. Let $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying (6.1). Put $\mathbb{B} = ((-\alpha_\infty)^+, \alpha_\infty^+ - \alpha_0)$. Then

$$\|\tau_k | \mathcal{L}(\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)), \ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)))\| \leq 2^k \ell^{\mathbb{B}}(2^{-k}), \quad k \in \mathbb{Z}.$$

Proof. For $\xi = (\xi_m)$ and $k \in \mathbb{Z}$, we have

$$\begin{aligned}\|\tau_k\xi|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\| &= \Big(\sum_{m=-\infty}^{\infty}(2^{-m}\ell^{\mathbb{A}}(2^m)|\xi_{m+k}|)^q\Big)^{1/q}\\ &\le 2^k\sup_{m\in\mathbb{Z}}\left\{\frac{\ell^{\mathbb{A}}(2^m)}{\ell^{\mathbb{A}}(2^{m+k})}\right\}\|\xi|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\|\\ &\le 2^k\ell^{\mathbb{B}}(2^{-k})\|\xi|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\|\end{aligned}$$

where we have used (6.3) in the last inequality. □

Lemma 6.3. Let $0 < q \le \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying (6.1), $0 < p \le 1$ and $\Lambda = (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}$. Then

$$\|\tau_k|\mathcal{L}(\Lambda,\Lambda)\| \le 2^{1/p-1}2^k\ell^{\mathbb{B}}(2^{-k}),\quad k\in\mathbb{Z}.$$

Proof. Let $\xi = (\xi_m) \in \Lambda$. For any $r \in \mathbb{Z}$, we have

$$\begin{aligned}K(2^r,\xi;\ell_p,\ell_p(2^{-m})) &\le K_p(2^r,\xi;\ell_p,\ell_p(2^{-m}))\\ &= \Big(\sum_{m=-\infty}^{\infty}[\min\{1,2^{r-m}\}|\xi_m|]^p\Big)^{1/p} \le 2^{1/p-1}K(2^r,\xi;\ell_p,\ell_p(2^{-m})).\end{aligned}$$

Hence, given any $k \in \mathbb{Z}$, we derive

$$\begin{aligned}\|\tau_k\xi|\Lambda\| &\le \Big(\sum_{r=-\infty}^{\infty}\Big[2^{-r}\ell^{\mathbb{A}}(2^r)\Big(\sum_{m=-\infty}^{\infty}\min\{1,2^{r-m}\}^p|\xi_{m+k}|^p\Big)^{1/p}\Big]^q\Big)^{1/q}\\ &= \Big(\sum_{r=-\infty}^{\infty}\Big[2^{-r}\ell^{\mathbb{A}}(2^r)\Big(\sum_{m=-\infty}^{\infty}\min\{1,2^{r+k-m}\}^p|\xi_m|^p\Big)^{1/p}\Big]^q\Big)^{1/q}\\ &\le 2^{1/p-1}2^k\sup_{r\in\mathbb{Z}}\left\{\frac{\ell^{\mathbb{A}}(2^r)}{\ell^{\mathbb{A}}(2^{r+k})}\right\}\Big(\sum_{r=-\infty}^{\infty}[2^{-(r+k)}\ell^{\mathbb{A}}(2^{r+k})K(2^{r+k},\xi;\ell_p,\ell_p(2^{-m}))]^q\Big)^{1/q}\\ &\le 2^{1/p-1}2^k\ell^{\mathbb{B}}(2^{-k})\|\xi|\Lambda\|,\end{aligned}$$

where the last inequality follows from (6.3). □

Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be quasi-Banach couples. If $0 < q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfy (6.1), then it follows from Theorem 2.6 that if $T \in \mathcal{L}(\bar{A}, \bar{B})$, then $T \in \mathcal{L}(\bar{A}_{1,q,\mathbb{A}}, \bar{B}_{1,q,\mathbb{A}})$ and

$$\|T|\mathcal{L}(\bar{A}_{1,q,\mathbb{A}},\bar{B}_{1,q,\mathbb{A}})\| \le \max\{\|T|\mathcal{L}(A_0,B_0)\|,\|T|\mathcal{L}(A_1,B_1)\|\}. \tag{6.4}$$

We give now based on [39, Theorem 2.2], a better estimation for the quasi-norm of the interpolated operator.

Theorem 6.4. Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two quasi-Banach couples, $0 < q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying (6.1). If $T \in \mathcal{L}(\bar{A}, \bar{B})$ and $M_j > \|T|\mathcal{L}(A_j, B_j)\|$, $j = 0, 1$, then

$$\|T|\mathcal{L}(\bar{A}_{1,q,\mathbb{A}},\bar{B}_{1,q,\mathbb{A}})\| \le 2M_1\Big(1+\Big(\log\frac{M_0}{M_1}\Big)^+\Big)^{\alpha_\infty^+-\alpha_0}. \tag{6.5}$$

Proof. If $M_0 \le M_1$, the result follows from (6.4).

Assume now that $M_1 < M_0$ and take $r \in \mathbb{Z}$ such that $2^{r-1} \leq M_1/M_0 < 2^r$. Let $a \in A_0 + A_1$ and take any decomposition of a as $a = a_0 + a_1$ with $a_j \in A_j$, $j = 0, 1$. Then

$$\begin{aligned} K(2^k, Ta; B_0, B_1) &\leq \|Ta_0|B_0\| + 2^k\|Ta_1|B_1\| \leq M_0\|a_0|A_0\| + 2^k M_1\|a_1|A_1\| \\ &\leq M_0\Big(\|a_0|A_0\| + 2^k\frac{M_1}{M_0}\|a_1|A_1\|\Big) \\ &\leq M_0\Big(\|a_0|A_0\| + 2^{k+r}\|a_1|A_1\|\Big), \quad k \in \mathbb{Z}. \end{aligned}$$

Therefore, $K(2^k, Ta; B_0, B_1) \leq M_0 K(2^{k+r}, a; A_0, A_1)$ for any $k \in \mathbb{Z}$. From here, using Lemma 6.2 we derive that

$$\begin{aligned} \|Ta|\bar{B}_{1,q,\mathbb{A}}\| &= \|K(2^m, Ta; B_0, B_1)|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\| \leq M_0\|K(2^{m+r}, a; A_0, A_1)|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\| \\ &= M_0\|\tau_r K(2^m, a; A_0, A_1)|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\| \\ &\leq M_0\|\tau_r|\mathcal{L}(\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)), \ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)))\|\|K(2^m, a; A_0, A_1)|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\| \\ &\leq M_0 2^r \ell^{\mathbb{B}}(2^{-r})\|K(2^m, a; A_0, A_1)|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\| \\ &\leq 2M_1\ell^{\alpha_\infty^+ - \alpha_0}(M_0/M_1). \end{aligned}$$

□

Observe that, in particular, if $\|T|\mathcal{L}(A_1, B_1)\| = 0$ then $\|T|\mathcal{L}(\bar{A}_{1,q,\mathbb{A}}, \bar{B}_{1,q,\mathbb{A}})\| = 0$. Moreover, if in addition to (6.1) we also have that (6.2) and $\|T|\mathcal{L}(A_0, B_0)\| = 0$, then it follows from Theorem 6.1 that $\|T|\mathcal{L}(\bar{A}_{1,q,\mathbb{A}}, \bar{B}_{1,q,\mathbb{A}})\| = 0$.

Let A and B be quasi-Banach spaces and $T \in \mathcal{L}(A, B)$. The *measure of non-compactness* $\beta(T) = \beta(T : A \to B)$ is defined to be the infimum of all the numbers $\sigma > 0$ for which there exists a finite subset $\{b_1, ..., b_s\} \subseteq B$ such that

$$T(U_A) \subseteq \bigcup_{k=1}^{s}\{b_k + \sigma U_B\}. \tag{6.6}$$

Here U_A (respectively, U_B) is the closed unit ball of A (respectively, B). See [54, 27].

Clearly, $\beta(T) \leq \|T|\mathcal{L}(A, B)\|$. Moreover, T is compact if, and only if, $\beta(T) = 0$. The next two properties are easy to check and will be used freely in what follows:

- If E is another quasi-Banach space and $S \in \mathcal{L}(B, E)$, then for $ST = S \circ T$ we have

$$\beta(ST : A \to E) \leq \|S|\mathcal{L}(B, E)\|\beta(T : A \to B). \tag{6.7}$$

  Moreover, if $\|Sb|E\| = \|b|B\|$ for all $b \in B$, then $\beta(T : A \to B) \leq 2c_E\beta(ST : A \to E)$.

- If X is another quasi-Banach space and $R \in \mathcal{L}(X, A)$, then

$$\beta(TR : X \to B) \leq \|R|\mathcal{L}(X, A)\|\beta(T : A \to B). \tag{6.8}$$

  Moreover, if for any $a \in A$ with $\|a|A\| < 1$, there is $x \in X$ with $\|x|X\| < 1$ and $Rx = a$, then $\beta(T : A \to B) \leq \beta(TR : X \to B)$.

Lemma 6.5. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let $T \in \mathcal{L}(\bar{A}, \bar{B})$. Then

$$\beta(T : A_j^\sim \to B_j^\sim) \leq \max\{c_{B_0}, c_{B_1}\}\beta(T : A_j \to B_j), \quad j = 0, 1.$$

Proof. Suppose $j = 0$ and let $\beta = \beta(T : A_0 \to B_0)$. Given any $\varepsilon > 0$ there is a finite set $\{b_1, ..., b_s\} \subseteq B_0$ such that

$$T(U_{A_0}) \subseteq \bigcup_{k=1}^{s} \{b_k + (\beta + \varepsilon) U_{B_0}\}.$$

Take any $a \in U_{A_0^{\sim}}$. According to (2.7), there exists $(a_n) \subseteq A_0$ such that $\|a - a_n|A_0 + A_1\| \xrightarrow{n\to\infty} 0$ and $\|a_n|A_0\| \leq 1 + \varepsilon$. Since

$$(Ta_n) \subseteq \bigcup_{k=1}^{s} \{(1+\varepsilon) b_k + (\beta + \varepsilon)(1+\varepsilon) U_{B_0}\},$$

we can find $k_0 \in \{1, ..., s\}$ and a subsequence $(Ta_{n'})$ of (Ta_n) such that

$$(Ta_{n'}) \subseteq \{(1+\varepsilon) b_{k_0} + (\beta + \varepsilon)(1+\varepsilon) U_{B_0}\}.$$

It follows that

$$\|Ta_{n'} - (1+\varepsilon) b_{k_0}|B_0\| \leq (\beta + \varepsilon)(1+\varepsilon), \quad n' \in \mathbb{N}$$

and

$$\begin{aligned} &\|Ta - (1+\varepsilon) b_{k_0} - (Ta_{n'} - (1+\varepsilon) b_{k_0})|B_0 + B_1\| \\ &\quad \leq \|T|\mathcal{L}(A_0 + A_1, B_0 + B_1)\| \|a - a_{n'}|A_0 + A_1\| \xrightarrow{n'\to\infty} 0. \end{aligned}$$

Hence,

$$\begin{aligned} \|Ta - (1+\varepsilon) b_{k_0}|B_0^{\sim}\| &\leq \max\{c_{B_0}, c_{B_1}\} \sup_{n'} \{\|Ta_{n'} - (1+\varepsilon) b_{k_0}|B_0\|\} \\ &\leq \max\{c_{B_0}, c_{B_1}\}(\beta + \varepsilon)(1+\varepsilon). \end{aligned}$$

This yields that

$$T(U_{A_0^{\sim}}) \subseteq \bigcup_{k=1}^{s} \{(1+\varepsilon) b_k + (\beta + \varepsilon)(1+\varepsilon) \max\{c_{B_0}, c_{B_1}\} U_{B_0^{\sim}}\}$$

and therefore $\beta(T : A_0^{\sim} \to B_0^{\sim}) \leq \max\{c_{B_0}, c_{B_1}\}\beta$.

The case $j = 1$ can be treated similarly. □

Subsequently we are going to work with vector-valued sequence spaces. Let (W_m) be a sequence of quasi-Banach spaces with the same constant $c \geq 1$ in the quasi-triangle inequality for any $m \in \mathbb{Z}$, which is the case if W_m is $p-$Banach for any $m \in \mathbb{Z}$. Let (λ_m) be a sequence of positive numbers and $0 < q \leq \infty$. We write

$$\ell_q(\lambda_m W_m) = \{w = (w_m) : w_m \in W_m \quad \text{and} \quad \|w|\ell_q(\lambda_m W_m)\| = \|(\lambda_m \|w_m|W_m\|)|\ell_q\| < \infty\}.$$

If Γ is a quasi-Banach sequence lattice, we define $\Gamma(\lambda_m W_m)$ in a similar way.

Lemma 6.6. Let $0 < p \leq 1$ and let (W_m) be a sequence of $p-$Banach spaces. Assume that $0 < q \leq \infty$ and let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying (6.1). Let $\Lambda = (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}$. We have with equivalence of quasi-norms

$$(\ell_p(W_m), \ell_p(2^{-m} W_m))_{1,q,\mathbb{A}} = \Lambda(W_m).$$

Proof. Let $w = (w_m) \in (\ell_p(W_m), \ell_p(2^{-m}W_m))_{1,q,\mathbb{A}}$. For any $k \in \mathbb{Z}$ and $\varepsilon > 0$, we can find $w^j = (w_{j,m}) \in \ell_p(2^{-jm}W_m)$ such that $w = w^0 + w^1$ and $\|w^0|\ell_p(W_m)\| + 2^k\|w^1|\ell_p(2^{-m}W_m)\| \leq (1+\varepsilon)K(2^k, w; \ell_p(W_m), \ell_p(2^{-m}W_m))$. Hence,

$$\Big(\sum_{m=-\infty}^{\infty} \min\{1, 2^{k-m}\}^p \|w_m|W_m\|^p\Big)^{1/p} \leq \Big(\|w^0|\ell_p(W_m)\|^p + 2^{kp}\|w^1|\ell_p(2^{-m}W_m)\|^p\Big)^{1/p}$$
$$\leq 2^{1/p}(1+\varepsilon)K(2^k, w; \ell_p(W_m), \ell_p(2^{-m}W_m)).$$

It follows that

$$\begin{aligned}
\|w|\Lambda(W_m)\| &= \Big(\sum_{k=-\infty}^{\infty} [2^{-k}\ell^{\mathbb{A}}(2^k)K(2^k, (\|w_m|W_m\|); \ell_p, \ell_p(2^{-m}))]^q\Big)^{1/q} \\
&\leq \Big(\sum_{k=-\infty}^{\infty} \Big[2^{-k}\ell^{\mathbb{A}}(2^k)\Big(\sum_{m=-\infty}^{\infty} \min\{1, 2^{k-m}\}^p \|w_m|W_m\|^p\Big)^{1/p}\Big]^q\Big)^{1/q} \\
&\leq 2^{1/p}\Big(\sum_{k=-\infty}^{\infty} [2^{-k}\ell^{\mathbb{A}}(2^k)K(2^k, w; \ell_p(W_m), \ell_p(2^{-m}W_m))]^q\Big)^{1/q} \\
&= 2^{1/p}\|w|(\ell_p(W_m), \ell_p(2^{-m}W_m))_{1,q,\mathbb{A}}\|.
\end{aligned}$$

Conversely, if $w = (w_m) \in \Lambda(W_m)$, given $k \in \mathbb{Z}$, put $w^j = (w_{j,m})$ with

$$w_{0,m} = \begin{cases} w_m & \text{if} \quad m \leq k, \\ 0 & \text{if} \quad m > k, \end{cases} \qquad w_{1,m} = \begin{cases} 0 & \text{if} \quad m \leq k, \\ w_m & \text{if} \quad m > k. \end{cases}$$

Then $w = w^0 + w^1$ and

$$K(2^k, w; \ell_p(W_m), \ell_p(2^{-m}W_m)) \leq \Big(\sum_{m=-\infty}^{k} \|w_m|W_m\|^p\Big)^{1/p} + 2^k\Big(\sum_{m=k+1}^{\infty} (2^{-m}\|w_m|W_m\|)^p\Big)^{1/p}$$
$$\leq 2\Big(\sum_{m=-\infty}^{\infty} \min\{1, 2^{k-m}\}^p \|w_m|W_m\|^p\Big)^{1/p}.$$

Therefore,

$$\begin{aligned}
\|w|(\ell_p(W_m), \ell_p(2^{-m}W_m))_{1,q,\mathbb{A}}\| &= \Big(\sum_{k=-\infty}^{\infty} [2^{-k}\ell^{\mathbb{A}}(2^k)K(2^k, w; \ell_p(W_m), \ell_p(2^{-m}W_m))]^q\Big)^{1/q} \\
&\leq 2\Big(\sum_{k=-\infty}^{\infty} \Big[2^{-k}\ell^{\mathbb{A}}(2^k)\Big(\sum_{m=-\infty}^{\infty} \min\{1, 2^{k-m}\}^p \|w_m|W_m\|^p\Big)^{1/p}\Big]^q\Big)^{1/q} \\
&\leq 2^{1+1/p}\|w|\Lambda(W_m)\|.
\end{aligned}$$

□

Lemma 6.7. Let (W_m) be a sequence of quasi-Banach spaces with the same constant $c \geq 1$ in the quasi-triangle inequality. Let $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying (6.1). Then

$$(\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))_{1,q,\mathbb{A}} \hookrightarrow \ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)W_m).$$

Proof. Let $w = (w_m) \in (\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))_{1,q,\mathbb{A}}$. Given any decomposition $w = w^0 + w^1$ with $w^j = (w_{j,m}) \in \ell_\infty(2^{-mj}W_m)$, $j = 0, 1$, we have

$$\begin{aligned}\|w_k|W_k\| &\le c(\|w_{0,k}|W_k\| + \|w_{1,k}|W_k\|)\\ &\le c(\|w^0|\ell_\infty(W_m)\| + 2^k\|w^1|\ell_\infty(2^{-m}W_m)\|), \quad k \in \mathbb{Z}.\end{aligned}$$

Hence, $\|w_k|W_k\| \le cK(2^k, w; \ell_\infty(W_m), \ell_\infty(2^{-m}W_m))$ for every $k \in \mathbb{Z}$ and so

$$\|w|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)W_m)\| \le c\|w|(\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))_{1,q,\mathbb{A}}\|.$$

□

Given $0 < q \le \infty$, $\mathbb{A} \in \mathbb{R}^2$ satisfying (6.1) and any quasi-Banach couple $\overline{B} = (B_0, B_1)$, there are natural vector valued sequence spaces that arise with the construction of $(B_0, B_1)_{1,q,\mathbb{A}}$. Indeed, let $F_m = (B_0 + B_1, K(2^m, \cdot))$, $m \in \mathbb{Z}$. The sequence (F_m) is formed by quasi-Banach spaces with the same constant $c = \max\{c_{B_0}, c_{B_1}\}$ in the quasi-triangle inequality. Consider the spaces $\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)F_m)$, $\ell_\infty(F_m)$ and $\ell_\infty(2^{-m}F_m)$. The operator $\iota b = (..., b, b, b, ...)$ is a metric injection from $(B_0, B_1)_{1,q,\mathbb{A}}$ into $\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)F_m)$, that is to say,

$$\|\iota b|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m)F_m)\| = \|b|(B_0, B_1)_{1,q,\mathbb{A}}\|, \quad b \in (B_0, B_1)_{1,q,\mathbb{A}}.$$

Note also that $\iota : B_j \to \ell_\infty(2^{-mj}F_m)$, $j = 0, 1$, is continuous with norm less than or equal to 1.

We use this notation to state next lemma.

Lemma 6.8. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let $T \in \mathcal{L}(\bar{A}, \bar{B})$. Fix $j \in \{0, 1\}$ and put $\beta_j = \beta(T : A_j \to B_j)$. Assume that there is a quasi-Banach space X and continuous linear operators $R_n \in \mathcal{L}(X, A_j)$ such that $\|R_n|\mathcal{L}(X, A_j)\| \le 1$ and $\lim_{n\to\infty} \|TR_n|\mathcal{L}(X, B_0 + B_1)\| = 0$. Then the following holds.

a) If $\beta_j = 0$, then there is a subsequence (n') such that $\lim_{n'\to\infty} \|\iota TR_{n'}|\mathcal{L}(X, \ell_\infty(2^{-mj}F_m))\| = 0$.

b) If $\beta_j > 0$, then there is a constant $C > 0$ independent of T and a subsequence (n') such that $\lim_{n'\to\infty} \|\iota TR_{n'}|\mathcal{L}(X, \ell_\infty(2^{-mj}F_m))\| \le C\beta_j$.

Proof. Since $\sup_{n\in\mathbb{N}} \|\iota TR_n|\mathcal{L}(X, \ell_\infty(2^{-mj}F_m))\| \le \|T|\mathcal{L}(A_j, B_j)\| < \infty$, there exists a subsequence (n') such that

$$\lim_{n'\to\infty} \|\iota TR_{n'}|\mathcal{L}(X, \ell_\infty(2^{-mj}F_m))\| = \lambda \ge 0.$$

Let $(x_{n'}) \subseteq U_X$ such that

$$\|\iota TR_{n'}x_{n'}|\ell_\infty(2^{-mj}F_m)\| \xrightarrow{n'\to\infty} \lambda.$$

Take $\sigma > \beta_j$. There exists a finite set $\{z_1, ..., z_s\} \subseteq B_j$ such that

$$T(U_{A_j}) \subseteq \bigcup_{k=1}^{s} \{z_k + \sigma U_{B_j}\}.$$

Passing to another subsequence, if necessary, that we continue to denote by (n'), we may find $k \in \{1, ..., s\}$ such that

$$TR_{n'}x_{n'} \in \{z_k + \sigma U_{B_j}\}, \quad \text{for all } n'. \tag{6.9}$$

Now we estimate the quasi-norm of ιz_k in $\ell_\infty(2^{-mj}F_m)$. Using that $\lim_{n\to\infty}\|TR_n|\mathcal{L}(X,B_0+B_1)\| = 0$, we can find n' belonging to the previous subsequence such that

$$2^{-jm}\max\{1,2^m\}\|TR_{n'}|\mathcal{L}(X,B_0+B_1)\| \le \sigma.$$

Whence,

$$\begin{aligned}
&2^{-mj}K(2^m,z_k;B_0,B_1)\\
&\quad\le \max\{c_{B_0},c_{B_1}\}(2^{-mj}K(2^m,z_k-TR_{n'}x_{n'};B_0,B_1)+2^{-mj}K(2^m,TR_{n'}x_{n'};B_0,B_1))\\
&\quad\le \max\{c_{B_0},c_{B_1}\}(\|z_k-TR_{n'}x_{n'}|B_j\|+2^{-mj}\max\{1,2^m\}\|TR_{n'}|\mathcal{L}(X,B_0+B_1)\|)\\
&\quad\le 2\max\{c_{B_0},c_{B_1}\}\sigma.
\end{aligned}$$

This yields that $\|\iota z_k|\ell_\infty(2^{-mj}F_m)\| \le 2\max\{c_{B_0},c_{B_1}\} < \sigma$. Consequently, using that

$$\|i|\mathcal{L}(B_j,\ell_\infty(2^{-mj}F_m))\| \le 1$$

and (6.9) we obtain with $C = 2\max\{c_{B_0},c_{B_1}\}(1+2\max\{c_{B_0},c_{B_1}\})$ that

$$\begin{aligned}
&\lim_{n'\to\infty}\|\iota TR_{n'}|\mathcal{L}(X,\ell_\infty(2^{-mjF_m}))\|\\
&\qquad\le \max\{c_{B_0},c_{B_1}\}(\|\iota TR_{n'}x_{n'}-\iota z_k|\ell_\infty(2^{-mj}F_m)\|+\|\iota z_k|\ell_\infty(2^{-mj}F_m)\|) \le C\sigma/2.
\end{aligned}$$

If $\beta_j = 0$, it follows that $\lim_{n'\to\infty}\|\iota TR_{n'}|\mathcal{L}(X,\ell_\infty(2^{-mj}F_m))\| = 0$. If $\beta_j > 0$, then taking $\sigma = 2\beta_j$ we conclude that

$$\lim_{n'\to\infty}\|\iota TR_{n'}|\mathcal{L}(X,\ell_\infty(2^{-mj}F_m))\| \le C\beta_j.$$

□

We shall also need later the following lemma.

Lemma 6.9. Let A,B,Z be quasi-Banach spaces and let D be a dense subspace of A. Assume that $T\in\mathcal{L}(A,B)$ and let $(S_n)\subseteq\mathcal{L}(B,Z)$ such that $M=\sup\{\|S_n|\mathcal{L}(B,Z)\| : n\in\mathbb{N}\}<\infty$ and $\lim_{n\to\infty}\|S_nTu|Z\| = 0$ for all $u\in D$. Let $\beta=\beta(T:A\to B)$. Then the following holds.

a) If $\beta = 0$ then $\lim_{n\to\infty}\|S_nT|\mathcal{L}(A,Z)\| = 0$.

b) If $\beta > 0$ then there is a constant $C > 0$ independent of T and there is $L\in\mathbb{N}$ such that $\|S_nT|\mathcal{L}(A,Z)\| \le C\beta$ for all $n \ge L$.

Proof. Take $\sigma > \beta$. There exists a finite set $\{w_1,\ldots,w_s\}\subseteq B$ such that

$$T(U_A)\subseteq\bigcup_{k=1}^{s}\{w_k+\sigma U_B\}.$$

If $\{w_k+\sigma U_B\}\cap T(U_A)\ne\emptyset$, choose $a_k\in U_A$ such that $Ta_k\in w_k+\sigma U_B$. Then

$$T(U_A)\subseteq\bigcup_{k=1}^{s}\{Ta_k+2c_B\sigma U_B\}.$$

From the density assumption, we derive that there is $u_k\in D$ such that

$$\|a_k-u_k|A\| \le \frac{\sigma}{\|T|\mathcal{L}(A,B)\|}.$$

Hence, $\|Ta_k - Tu_k|B\| \leq \sigma$. It follows that

$$T(U_A) \subseteq \bigcup_{k=1}^{s} \{Tu_k + c_B(2c_B+1)\sigma U_B\}.$$

Let $C = 2c_Z(Mc_B(2c_B+1)+1)$ and let $L \in \mathbb{N}$ such that for any $n \geq L$ we have $\|S_nTu_k|Z\| \leq \sigma$ for any $1 \leq k \leq s$. Given any $a \in U_A$ we can find k such that $\|Ta - Tu_k|B\| \leq c_B(2c_B+1)\sigma$. Therefore, we obtain

$$\begin{aligned}\|S_nTa|Z\| &\leq c_Z(\|S_n(Ta - Tu_k)|Z\| + \|S_nTu_k|Z\|)\\ &\leq c_Z(Mc_B(2c_B+1)\sigma + \sigma) = c\sigma/2.\end{aligned}$$

This yields that $\|S_nT|\mathcal{L}(A,Z)\| \leq C\sigma/2$, for $n \geq L$.

If $\beta = 0$, we derive that $\lim_{n\to\infty} \|S_nT|\mathcal{L}(A,Z)\| = 0$. If $\beta > 0$, the choice $\sigma = 2\beta$ gives that $\|S_nT|\mathcal{L}(A,Z)\| \leq C\beta$ for any $n \geq L$. □

To complete the preparation for the central result of this section, note that if Γ is a (p-normed) quasi-Banach sequence lattice then it induces a p-norm $\|\cdot|\tilde{\Gamma}\|$ in $\mathbb{R}^{2n+1}$, $n \in \mathbb{N}$. Indeed, if $x = (x_k)_{k=-n}^{n} \in \mathbb{R}^{2n+1}$, put $\|x|\tilde{\Gamma}\| = \|\tilde{x}|\Gamma\|$ where $\tilde{x} = \sum_{k=-n}^{n} x_k e_k$ and $e_k = (\delta_m^k)$ with δ_m^k being the Kronecker delta. The unit ball $U = U_{(\mathbb{R}^{2n+1}, \|\cdot|\tilde{\Gamma}\|)}$ is compact in $(\mathbb{R}^{2n+1}, \|\cdot|\tilde{\Gamma}\|)$ and therefore for any $\varepsilon > 0$ we can find an $\varepsilon-$net for U.

Subsequently, in order to make the arguments more transparent, constants are denoted by c_1, c_2,... In this way we emphasize the new constants that show up and a change in an existing constant.

Theorem 6.10. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be mutually closed quasi-Banach couples and let $T \in \mathcal{L}(\bar{A}, \bar{B})$. Let $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying that

$$\begin{cases} \alpha_0 + 1/q < 0 \leq \alpha_\infty + 1/q & \text{if} \quad 0 < q < \infty, \\ \alpha_0 \leq 0 < \alpha_\infty & \text{if} \quad q = \infty. \end{cases}$$

Then we have:

a) $\beta(T : (A_0, A_1)_{1,q,\mathbb{A}} \to (B_0, B_1)_{1,q,\mathbb{A}}) = 0$ if $\beta(T : A_1 \to B_1) = 0$,

b) $\beta(T : (A_0, A_1)_{1,q,\mathbb{A}} \to (B_0, B_1)_{1,q,\mathbb{A}}) \leq C\beta(T : A_1 \to B_1)$ if $\beta(T : A_0 \to B_0) = 0$,

c) $\beta(T : (A_0, A_1)_{1,q,\mathbb{A}} \to (B_0, B_1)_{1,q,\mathbb{A}}) \leq C\beta(T : A_1 \to B_1) \left(1 + \left(\log \frac{\beta(T:A_0\to B_0)}{\beta(T:A_1\to B_1)}\right)^+\right)^{\alpha_\infty^+ - \alpha_0}$ if $\beta(T : A_j \to B_j) > 0$ for $j = 0, 1$.

Here $C > 0$ is a constant independent of $\bar{A}$, $\bar{B}$ and T.

Proof. Without loss of generality we can suppose that the spaces A_0, A_1, B_0, B_1 are p-normed for some $0 < p \leq 1$. Furthermore, since $\bar{A}$ is mutually closed, by Theorem 6.1, we have that

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0, A_1)_{\Lambda;J} \quad \text{with} \quad \Lambda = (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}.$$

For each $m \in \mathbb{Z}$ let

$$G_m = (A_0 \cap A_1, J(2^m, \cdot; A_0, A_1)), \quad F_m = (B_0 + B_1, K(2^m, \cdot; B_0, B_1)).$$

The operator $\pi(u_m) = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$) is surjective from $\Lambda(G_m)$ into $(A_0, A_1)_{\Lambda;J}$ and it induces the quasi-norm of $(A_0, A_1)_{\Lambda;J}$. On the other hand, as we pointed out before Lemma 6.8, if $\Delta = \ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))$ and $\iota b = (\dots, b, b, b, b, \dots)$, then ι is a metric injection from $(B_0, B_1)_{1,q,\mathbb{A}}$ into $\Delta(F_m)$. Let $\hat{T} = \iota T \pi : \Lambda(G_m) \to \Delta(F_m)$. According to (6.7) and (6.8), we get

$$\beta(T : \bar{A}_{1,q,\mathbb{A}} \to \bar{B}_{1,q,\mathbb{A}}) \leq c_1 \beta(\iota T : \bar{A}_{1,q,\mathbb{A}} \to \Delta(F_m)) \leq c_2 \beta(\hat{T} : \Lambda(G_m) \to \Delta(F_m)). \tag{6.10}$$

With the aim of estimating the last measure of non-compactness, consider the couples

$$\overline{\ell_p(G)} = (\ell_p(G_m), \ell_p(2^{-m}G_m)) \quad \text{and} \quad \overline{\ell_\infty(F)} = (\ell_\infty(F_m), \ell_\infty(2^{-m}F_m))$$

and note that $\pi \in \mathcal{L}\left(\ell_p(2^{-mj}G_m), A_j\right)$, $j = 0, 1$, with norm less than or equal to 1. On the other hand, $\iota \in \mathcal{L}\left(B_j, \ell_\infty(2^{-mj}F_m)\right)$, $j = 0, 1$, also with norm less than or equal to 1. The relevant picture to keep in mind is

$$\begin{array}{ccccccc}
\ell_p(G_m) & \xrightarrow{\pi} & A_0 & \xrightarrow{T} & B_0 & \xrightarrow{\iota} & \ell_\infty(F_m) \\
\ell_p(2^{-m}G_m) & \xrightarrow{\pi} & A_1 & \xrightarrow{T} & B_1 & \xrightarrow{\iota} & \ell_\infty(2^{-m}F_m) \\
\hline\hline
\overline{\ell_p(G)}_{1,q,\mathbb{A}} & \xrightarrow{\pi} & \bar{A}_{1,q,\mathbb{A}} & \xrightarrow{T} & \bar{B}_{1,q,\mathbb{A}} & \xrightarrow{\iota} & \overline{\ell_\infty(F)}_{1,q,\mathbb{A}}.
\end{array}$$

Moreover, by Lemmata 6.6 and 6.7, we have

$$\Lambda(G_m) \hookrightarrow \overline{\ell_p(G)}_{1,q,\mathbb{A}} \quad \text{and} \quad \overline{\ell_\infty(F)}_{1,q,\mathbb{A}} \hookrightarrow \Delta(F_m).$$

On $\overline{\ell_p(G)}$ we can consider the following projections. For $n \in \mathbb{N}$ and $u = (u_m)$, let

$$\begin{aligned}
P_n u &= (\dots, 0, 0, u_{-n}, u_{-n+1}, \dots, u_{n-1}, u_n, 0, 0, \dots), \\
P_n^+ u &= (\dots, 0, 0, u_{n+1}, u_{n+2}, u_{n+3}, \dots), \\
P_n^- u &= (\dots, u_{-n-3}, u_{-n-2}, u_{-n-1}, 0, 0, \dots).
\end{aligned}$$

These mappings have norm 1 when acting from $\ell_p(2^{-jm}G_m)$ into $\ell_p(2^{-jm}G_m)$, $j = 0, 1$, or from $\Lambda(G_m)$ into $\Lambda(G_m)$. The identity operator I on $\ell_p(G_m) + \ell_p(2^{-m}G_m)$ can be split as $I = P_n + P_n^+ + P_n^-$, $n \in \mathbb{N}$. Moreover

$$\|P_n^+ | \mathcal{L}(\ell_p(G_m), \ell_p(2^{-m}G_m))\| = 2^{-(n+1)} = \|P_n^- | \mathcal{L}(\ell_p(2^{-m}G_m), \ell_p(G_m))\|. \tag{6.11}$$

On the couple $\overline{\ell_\infty(F)}$ we can consider similar sequences of projections with analogous properties. We denote them by (Q_n), (Q_n^+), (Q_n^-).

Since

$$\hat{T} = \hat{T}P_n + Q_n\hat{T}(P_n^+ + P_n^-) + Q_n^+\hat{T}P_n^- + Q_n^-\hat{T}P_n^+ + Q_n^-\hat{T}P_n^- + Q_n^+\hat{T}P_n^+, \tag{6.12}$$

we proceed to estimate the measure of non-compactness of each one of these six operators acting from $\Lambda(G_m)$ into $\Delta(F_m)$. Take any $\sigma_j > \beta(T : A_j \to B_j)$ and let $N \in \mathbb{N} \cup \{0\}$ such that $2^N \leq \sigma_0/\sigma_1 < 2^{N+1}$ if $\sigma_1 \leq \sigma_0$ and $N = 0$ if $\sigma_0 < \sigma_1$.

For $\hat{T}P_n$ we have

$$\beta(\hat{T}P_n : \Lambda(G_m) \to \Delta(F_m)) \leq c\beta(T\pi P_n : \Lambda(G_m) \to (B_0, B_1)_{\Lambda;J}).$$

Let $\eta = \big\| \sum_{k=-n}^{n} \frac{e_k}{\|e_k|\Lambda\|} |\Lambda\big\|^{-1}$ and consider the quasi-norm $\|\cdot|\tilde{\Lambda}\|$ in $\mathbb{R}^{2n+1}$. We can find $\Upsilon = \{\lambda^1, ..., \lambda^s\} \subseteq U_{(\mathbb{R}^{2n+1}, \|\cdot|\tilde{\Lambda}\|)}$ such that

$$U_{(\mathbb{R}^{2n+1}, \|\cdot|\tilde{\Lambda}\|)} \subseteq \bigcup_{d=1}^{s} \{\lambda^d + \eta U_{(\mathbb{R}^{2n+1}, \|\cdot|\tilde{\Lambda}\|)}\}.$$

For each $\lambda^d = (\lambda_k^d)_{k=-n}^{n}$, let

$$\varphi_k^j = \varphi_{k,\lambda^d}^j = \left(\frac{\eta}{\|e_k|\Lambda\|} + |\lambda_k^d| \right) 2^{-kj}, \quad j = 0, 1.$$

Besides, there are finite sets $\Sigma_j = \{h_1^j, ..., h_{L_j}^j\} \subseteq B_j$ such that

$$T(U_{A_j}) \subseteq \bigcup_{\ell=1}^{L_j} \{h_\ell^j + \sigma_j U_{B_j}\}, \quad j = 0, 1.$$

Given any $\lambda^d \in \Upsilon$, $h_\ell^0 \in \Sigma_0$ and $h_y^1 \in \Sigma_1$, for any $-n \le k \le n$, let g_k be an element of $(\varphi_k^0 h_\ell^0 + \varphi_k^0 \sigma_0 U_{B_0}) \cap (\varphi_k^1 h_y^1 + \varphi_k^1 \sigma_1 U_{B_1})$ provided the intersection is non-empty and let $g_k = 0$ otherwise. Denote by Φ the collection of all sums $\sum_{k=-n}^{n} g_k$ of the elements g_k. Note that Φ is a finite set. Now we show that Φ is a suitable net for $T\pi P_n$.

Given any $u = (u_m) \in U_{\Lambda(G_m)}$, there is $\lambda^d \in \Upsilon$ such that

$$|J(2^k, u_k) - \lambda_k^d| \|e_k|\Lambda\| \le \|(J(2^k, u_k) - \lambda_k^d)|\tilde{\Lambda}\| \le \eta, \quad -n \le k \le n.$$

Hence, $|J(2^k, u_k)| \le \frac{\eta}{\|e_k|\Lambda\|} + |\lambda_k^d| = \varphi_k^0$ and so $\|u_k|A_j\| \le \varphi_k^j$, $j = 0, 1$, $-n \le k \le n$. We can find $h_\ell^0 \in \Sigma_0$ and $h_y^1 \in \Sigma_1$ such that

$$\|Tu_k - \varphi_k^0 h_\ell^0 |B_0\| \le \varphi_k^0 \sigma_0, \quad \|Tu_k - \varphi_k^1 h_y^1 |B_1\| \le \varphi_k^1 \sigma_1.$$

Thus,

$$(\varphi_k^0 h_\ell^0 + \varphi_k^0 \sigma_0 U_{B_0}) \cap (\varphi_k^1 h_y^1 + \varphi_k^1 \sigma_1 U_{B_1}) \neq \emptyset$$

and for the corresponding g_k we obtain that

$$\begin{aligned} J(2^{k+N}, Tu_k - g_k) &\le \max\{\|Tu_k - \varphi_k^0 h_\ell^0 |B_0\|^p + \|\varphi_k^0 h_\ell^0 - g_k|B_0\|^p, \\ &\qquad 2^{(k+N)p}(\|Tu_k - \varphi_k^1 h_y^1 |B_1\|^p + \|\varphi_k^1 h_y^1 - g_k|B_1\|^p)\}^{1/p} \\ &\le 2^{1/p-1} \max\{\sigma_0, 2^N \sigma_1\} \varphi_k^0. \end{aligned}$$

For $g = \sum_{k=-n}^{n} g_k \in \Phi$, it follows that

$$\begin{aligned} \|T\pi P_n u - g|(B_0, B_1)_{\Lambda;J}\| &= \| \sum_{k=-n}^{n} (Tu_k - g_k)|(B_0, B_1)_{\Lambda;J}\| \\ &\le \|\tau_{-N}(..., 0, 0, J(2^{-n+N}, Tu_{-n} - g_{-n}), ..., J(2^{n+N}, Tu_n - g_n), 0, 0, ...)|\Lambda\| \\ &\le 2^{1/p-1} \|\tau_{-N}|\mathcal{L}(\Lambda, \Lambda)\| \max\{\sigma_0, 2^N \sigma_1\} \|(..., 0, 0, \varphi_{-n}^0, ..., \varphi_n^0, 0, 0, ...)|\Lambda\| \\ &\le c_3 \|\tau_{-N}|\mathcal{L}(\Lambda, \Lambda)\| \max\{\sigma_0, 2^N \sigma_1\}. \end{aligned}$$

By our choice of N and Lemma 6.3 we conclude that

$$\beta(\hat{T}P_n : \Lambda(G_m) \to \Delta(F_m)) \leq 2^{1/p} c_3 \sigma_1 \Big(1 + \Big(\log \frac{\sigma_0}{\sigma_1}\Big)^+\Big)^{\alpha_\infty^+ - \alpha_0}.$$

As for $Q_n \hat{T}(P_n^+ + P_n^-)$ we get

$$\beta(Q_n \hat{T}(P_n^+ + P_n^-) : \Lambda(G_m) \to \Delta(F_m)) \leq c_4 \beta(Q_n \iota T : \bar{A}_{1,q,\mathbb{A}} \to \Delta(F_m)).$$

To estimate the last measure of non-compactness let σ_0, σ_1 and N be as before. Put

$$\eta = \Big\| \sum_{k=-n}^{n} \frac{e_k}{\|e_k|\Delta\|} |\Delta\Big\|^{-1}$$

and choose $\Psi = \{\mu^1, ..., \mu^s\} \subseteq U_{(\mathbb{R}^{2n+1}, \|\cdot|\tilde{\Delta}\|)}$ such that for any $x \in U_{(\mathbb{R}^{2n+1}, \|\cdot|\tilde{\Delta}\|)}$ there is μ^d such that $\|x - \mu^d|\tilde{\Delta}\| < \eta$. Let associate to each $\mu^d = (\mu_k^d)_{k=-n}^n$ the numbers

$$\psi_k^j = \psi_{k,\mu^d}^j = \|\tau_{-N}|\mathcal{L}(\Delta, \Delta)\| \left(\frac{\eta}{\|e_k|\Delta\|} + |\mu_k^d| \right) 2^{-(k-N)j}, \quad j = 0, 1.$$

Choose a σ_j−net $\Sigma_j = \{h_1^j, ..., h_{L_j}^j\}$ for $T(U_{A_j})$ in B_j, $j = 0, 1$, and given any $\mu^d \in \Psi$, $h_\ell^0 \in \Sigma_0$ and $h_y^1 \in \Sigma_1$, let $z^{d,\ell,y} = (z_m^{d,\ell,y})_{m \in \mathbb{Z}}$ the sequence of vectors defined by

$$z_m^{d,\ell,y} = \begin{cases} 0 & \text{if} \quad m \notin [-n, n], \\ \psi_m^0 h_\ell^0 + \psi_m^1 h_y^1 & \text{if} \quad -n \leq m \leq n. \end{cases}$$

Clearly, the collection of these $z^{d,\ell,y}$ forms a finite subset Ξ of $\Delta(F_m)$. We are going to show that Ξ is a suitable net for $Q_n \iota T$.

Take any $a \in U_{\bar{A}_{1,q,\mathbb{A}}}$. Since

$$\|(K(2^{m-N}, a))|\Delta\| \leq \|\tau_{-N}|\mathcal{L}(\Delta, \Delta)\| \|a|\bar{A}_{1,q,\mathbb{A}}\| \leq \|\tau_{-N}|\mathcal{L}(\Delta, \Delta)\|,$$

there is $\mu^d \in \Psi$ such that

$$\|(K(2^{m-N}, a) - \|\tau_{-N}|\mathcal{L}(\Delta, \Delta)\| \mu_m^d)_{m=-n}^n |\tilde{\Delta}\| < \eta \|\tau_{-N}|\mathcal{L}(\Delta, \Delta)\|.$$

So,

$$|(K(2^{m-N}, a) - \|\tau_{-N}|\mathcal{L}(\Delta, \Delta)\| \mu_m^d)| \cdot \|e_m|\Delta\| < \eta \|\tau_{-N}|\mathcal{L}(\Delta, \Delta)\|, \quad -n \leq m \leq n.$$

This yields that $K(2^{m-N}, a) < \psi_m^0$, $-n \leq m \leq n$. We can decompose $a = a_{0,m} + a_{1,m}$ with $a_{j,m} \in A_j$, $j = 0, 1$, and

$$\|a_{0,m}|A_0\| + 2^{m-N} \|a_{1,m}|A_1\| < \psi_m^0.$$

It follows that for some $h_\ell^0 \in \Sigma_0$, $h_y^1 \in \Sigma_1$

$$\|Ta_{0,m} - \psi_m^0 h_\ell^0 |B_0\| \leq \psi_m^0 \sigma_0 \quad \text{and} \quad \|Ta_{1,m} - \psi_m^1 h_y^1 |B_1\| \leq \psi_m^1 \sigma_1, \quad -n \leq m \leq n.$$

Let $z = z^{d,\ell,y}$ be the element of Ξ associated to μ^d, h^0_ℓ and h^1_y. Then,

$$\begin{aligned}\|Q_n\iota Ta - z|\Delta(F_m)\| &= \|(K(2^m, Ta - z_m))_{m=-n}^n|\tilde{\Delta}\| \\ &\le \|(\|Ta_{0,m} - \psi^0_m h^0_\ell|B_0\| + 2^m\|Ta_{1,m} - \psi^1_m h^1_y|B_1\|)_{m=-n}^n|\tilde{\Delta}\| \\ &\le \|(\psi^0_m\sigma_0 + 2^m\psi^1_m\sigma_1)_{m=-n}^n|\tilde{\Delta}\| \le 2c_\Delta\|\tau_{-N}|\mathcal{L}(\Delta,\Delta)\|(\sigma_0 + 2^N\sigma_1).\end{aligned}$$

By the choice of N and Lemma 6.2, we derive that

$$\beta(Q_n\iota T : \bar{A}_{1,q,\mathbb{A}} \to \Delta(F_m)) \le 8c_\Delta\sigma_1\Big(1 + \Big(\log\frac{\sigma_0}{\sigma_1}\Big)^+\Big)^{\alpha^+_\infty - \alpha_0}.$$

As for $Q^+_n\hat{T}P^-_n$ we are going to estimate the measure of non-compactness by the norm of the operator and then to show that it tends to 0 as $n \to \infty$. We have

$$\|Q^+_n\hat{T}P^-_n|\mathcal{L}(\ell_p(G_m), \ell_\infty(F_m))\| \le \|T|\mathcal{L}(A_0, B_0)\|.$$

On the other hand, the factorization

$$\begin{array}{ccc} \ell_p(2^{-m}G_m) & \xrightarrow{\;Q^+_n\hat{T}P^-_n\;} & \ell_\infty(2^{-m}F_m) \\ {\scriptstyle P^-_n}\big\downarrow & & \big\uparrow{\scriptstyle Q^+_n} \\ \ell_p(G_m) & \xrightarrow{\quad\hat{T}\quad} & \ell_\infty(F_m) \end{array}$$

and (6.11) give that

$$\begin{aligned}\|Q^+_n\hat{T}P^-_n|\mathcal{L}(\ell_p(2^{-m}G_m), \ell_\infty(2^{-m}G_m))\| &\le 2^{-(n+1)}\|\hat{T}|\mathcal{L}(\ell_p(G_m), \ell_\infty(F_m))\|2^{-(n+1)} \\ &\le 2^{-2(n+1)}\|T|\mathcal{L}(A_0, B_0)\|.\end{aligned}$$

Hence, according to Lemmata 6.6, 6.7 and estimate (6.5), we get

$$\begin{aligned}\beta(Q^+_n\hat{T}P^-_n : \Lambda(G_m) \to \Delta(F_m)) &\le c\|Q^+_n\hat{T}P^-_n|\mathcal{L}(\overline{\ell_p(G)}_{1,q,\mathbb{A}}, \overline{\ell_\infty(F)}_{1,q,\mathbb{A}})\| \\ &\le c2^{-2(n+1)}\|T|\mathcal{L}(A_0, B_0)\|(1 + \log 2^{2(n+1)})^{\alpha^+_\infty - \alpha_0} \xrightarrow{n\to\infty} 0.\end{aligned}$$

As for $Q^-_n\hat{T}P^+_n$, we observe that

$$\|P^+_n|\mathcal{L}(\ell_p(G_m), \ell_p(2^{-m}G_m))\| = 2^{-(n+1)} \quad \text{and} \quad \|P^+_n|\mathcal{L}(\ell_p(2^{-m}G_m), \ell_p(2^{-m}G_m))\| = 1,$$

hence $\|P^+_n|\mathcal{L}(\Lambda(G_m), \ell_p(2^{-m}G_m))\| \le c_5$. Similarly, $\|Q^-_n|\mathcal{L}(\ell_\infty(2^{-m}F_m), \Delta(F_m))\| \le c_6$. Using the diagram

$$\begin{array}{ccc} \Lambda(G_m) & \xrightarrow{\;Q^-_n\hat{T}P^+_n\;} & \Delta(F_m) \\ {\scriptstyle P^+_n}\big\downarrow & & \big\uparrow{\scriptstyle Q^-_n} \\ \ell_p(2^{-m}G_m) & \xrightarrow{\quad\hat{T}\quad} & \ell_\infty(2^{-m}F_m) \end{array}$$

we derive that

$$\begin{aligned}\beta(Q_n^- \hat{T} P_n^+ : \Lambda(G_m) \to \Delta(F_m)) &\leq c_5 c_6 \beta(\hat{T} : \ell_p(2^{-m} G_m) \to \ell_\infty(2^{-m} F_m)) \\ &\leq c_5 c_6 \sigma_1 \leq c_5 c_6 \sigma_1 \Big(1 + \Big(\log \frac{\sigma_0}{\sigma_1}\Big)^+\Big)^{\alpha_\infty^+ - \alpha_0}.\end{aligned}$$

The remaining two operators can be estimated with the help of Lemmata 6.8 and 6.9. Using the factorization

$$\begin{array}{ccc} \ell_p(2^{-m}G_m) & \xrightarrow{\;T\pi P_n^-\;} & B_0 + B_1 \\ {\scriptstyle P_n^-}\Big\downarrow & & \Big\uparrow{\scriptstyle T} \\ \ell_p(G_m) & \xrightarrow{\;\;\pi\;\;} & A_0 \end{array}$$

we get

$$\|T\pi P_n^- | \mathcal{L}(\ell_p(2^{-m}G_m), B_0 + B_1)\| \leq 2^{-(n+1)} \|T | \mathcal{L}(A_0, B_0 + B_1)\| \xrightarrow{n\to\infty} 0.$$

Whence, Lemma 6.8 yields that there is $c_7 > 0$ and a subsequence (n') such that

$$\lim_{n' \to \infty} \|\hat{T} P_{n'}^- | \mathcal{L}(\ell_p(2^{-m}G_m), \ell_\infty(2^{-m}F_m))\| \leq c_7 \beta(T\pi : \ell_p(2^{-m}G_m) \to B_1) \leq c_7 \sigma_1.$$

On the other hand, let D be a subset of $\ell_p(G_m)$ formed by all sequences having only a finite number of coordinates different from 0. Clearly, D is dense in $\ell_p(G_m)$ and if $u \in D$ then

$$\|Q_n^- \hat{T} u | \ell_\infty(F_m)\| \leq 2^{-(n+1)} \|\hat{T} u | \ell_\infty(2^{-m}F_m)\| \xrightarrow{n\to\infty} 0.$$

According to Lemma 6.9, there is $c_8 > 0$ and $L \in \mathbb{N}$ such that for all $n \geq L$

$$\|Q_n^- \hat{T} | \mathcal{L}(\ell_p(G_m), \ell_\infty(F_m))\| \leq c_8 \beta(\hat{T} : \ell_p(G_m) \to \ell_\infty(F_m)) \leq c_8 \sigma_0.$$

Using now Lemmata 6.6, 6.7 and (6.5), if $n' > L$ we derive that

$$\begin{aligned}\beta(Q_{n'}^- \hat{T} P_{n'}^- : \Lambda(G_m) \to \Delta(F_m)) &\leq c_9 \|Q_{n'}^- \hat{T} P_{n'}^- | \mathcal{L}(\overline{\ell_p(G)}_{1,q,\mathbb{A}}, \overline{\ell_\infty(F)}_{1,q,\mathbb{A}})\| \\ &\leq c\sigma_1 \Big(1 + \Big(\log \frac{\sigma_0}{\sigma_1}\Big)^+\Big)^{\alpha_\infty^+ - \alpha_0}.\end{aligned}$$

With a similar argument one can show that there is another subsequence that we also denote by (n') and another positive integer that we also call L such that if $n' > L$ we have that

$$\beta(Q_{n'}^+ \hat{T} P_{n'}^+ : \Lambda(G_m) \to \Delta(F_m)) \leq c\sigma_1 \Big(1 + \Big(\log \frac{\sigma_0}{\sigma_1}\Big)^+\Big)^{\alpha_\infty^+ - \alpha_0}.$$

Collecting all these estimates we derive that there is a constant $C > 0$ independent of $\bar{A}$, $\bar{B}$ and T such that if we split the operator as in (6.12) with a suitable n then

$$\beta(\hat{T} : \Lambda(G_m) \to \Delta(F_m)) \leq C\sigma_1 \Big(1 + \Big(\log \frac{\sigma_0}{\sigma_1}\Big)^+\Big)^{\alpha_\infty^+ - \alpha_0}.$$

Now take $\sigma_j = (1+\varepsilon)\beta(T : A_j \to B_j)$ if $\beta(T : A_j \to B_j) > 0$ and $\sigma_j = \varepsilon$ otherwise. Letting $\varepsilon \to 0$ and using (6.10) the result follows. □

Now we remove the assumption of mutually closed couples from last theorem.

Theorem 6.11. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let $T \in \mathcal{L}(\bar{A}, \bar{B})$. If q and $\mathbb{A}$ are as in Theorem 6.10, then the conclusion of that theorem is still valid.

Proof. The quasi-Banach couples $\overline{A^\sim} = (A_0^\sim, A_1^\sim)$, $\overline{B^\sim} = (B_0^\sim, B_1^\sim)$ are mutually closed and, by (2.8), $\bar{A}_{1,q,\mathbb{A}} = \overline{A^\sim}_{1,q,\mathbb{A}}$ and $\bar{B}_{1,q,\mathbb{A}} = \overline{B^\sim}_{1,q,\mathbb{A}}$. Furthermore, T belongs to $\mathcal{L}(\overline{A^\sim}, \overline{B^\sim})$ and, according to Lemma 6.5,

$$\beta(T : A_j^\sim \to B_j^\sim) \leq \max\{c_{B_0}, c_{B_1}\}\beta(T : A_j \to B_j),\ j = 0, 1.$$

Consequently, Theorem 6.10 yields the result. □

Now from Theorem 6.11 we can deduce straightforward the following result for logarithmic methods with $\theta = 0$.

Theorem 6.12. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let $T \in \mathcal{L}(\bar{A}, \bar{B})$. Let $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying that

$$\begin{cases} \alpha_\infty + 1/q < 0 \leq \alpha_0 + 1/q & \text{if} \quad 0 < q < \infty, \\ \alpha_\infty \leq 0 < \alpha_0 & \text{if} \quad q = \infty. \end{cases}$$

Then we have:

a) $\beta(T : (A_0, A_1)_{0,q,\mathbb{A}} \to (B_0, B_1)_{0,q,\mathbb{A}}) = 0$ if $\beta(T : A_0 \to B_0) = 0$,

b) $\beta(T : (A_0, A_1)_{0,q,\mathbb{A}} \to (B_0, B_1)_{0,q,\mathbb{A}}) \leq C\beta(T : A_0 \to B_0)$ if $\beta(T : A_1 \to B_1) = 0$,

c) $\beta(T : (A_0, A_1)_{0,q,\mathbb{A}} \to (B_0, B_1)_{0,q,\mathbb{A}}) \leq C\beta(T : A_0 \to B_0)\Big(1 + \Big(\log \frac{\beta(T:A_1\to B_1)}{\beta(T:A_0\to B_0)}\Big)^+\Big)^{\alpha_0^+ - \alpha_\infty}$ if $\beta(T : A_j \to B_j) > 0$ for $j = 0, 1$.

Here $C > 0$ is a constant independent of $\bar{A}$, $\bar{B}$ and T.

Another consequence of Theorem 6.11 is the following compactness result. The novelty with respect to the results in [48] is that the source and target couples are quasi-Banach and, moreover, q can take any value bigger than 0. Note also that the assumption on the parameter $\mathbb{A}$ is weaker than in Theorem 6.10. It is valid for all non-trivial logarithmic methods with $\theta = 1$ (see (6.1)).

Theorem 6.13. Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be quasi-Banach couples. Assume that $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfy that

$$\begin{cases} \alpha_0 + 1/q < 0 & \text{if} \quad 0 < q < \infty, \\ \alpha_0 \leq 0 & \text{if} \quad q = \infty. \end{cases}$$

If $T \in \mathcal{L}(\bar{A}, \bar{B})$ and $T : A_1 \to B_1$ is compact, then $T : (A_0, A_1)_{1,q,\mathbb{A}} \to (B_0, B_1)_{1,q,\mathbb{A}}$ is also compact.

Proof. Step 1. If $0 \leq \alpha_\infty + 1/q$ and $0 < q < \infty$, or $0 < \alpha_\infty$ and $q = \infty$, then the result follows from Theorem 6.11.

Step 2. Suppose now that $\alpha_\infty + 1/q < 0$ and $0 < q < \infty$, or $\alpha_\infty \leq 0$ and $q = \infty$. Take any $\alpha > -1/q$. The argument in [48, Corollary 2.5] for Banach couples still work in the quasi-Banach context showing that

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0 + A_1, A_1)_{1,q,(\alpha_0,\alpha)} \quad \text{and} \quad (B_0, B_1)_{1,q,\mathbb{A}} = (B_0 + B_1, B_1)_{1,q,(\alpha_0,\alpha)}.$$

Since $T \in \mathcal{L}((A_0 + A_1, A_1), (B_0 + B_1, B_1))$, the result established in the Step 1 yields that

$$T : (A_0, A_1)_{1,q,\mathbb{A}} \to (B_0, B_1)_{1,q,\mathbb{A}} \text{ compactly.}$$

□

The compactness theorem for $\theta = 0$ reads as follows.

Theorem 6.14. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples. Assume that $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfy that

$$\begin{cases} \alpha_\infty + 1/q < 0 & \text{if} \quad 0 < q < \infty, \\ \alpha_\infty \leq 0 & \text{if} \quad q = \infty. \end{cases}$$

If $T \in \mathcal{L}(\bar{A}, \bar{B})$ and $T : A_0 \to B_0$ is compact, then $T : (A_0, A_1)_{0,q,\mathbb{A}} \to (B_0, B_1)_{0,q,\mathbb{A}}$ compactly.

Remark 6.15. As we said at the beginning of this chapter, the compactness result of Cwikel [50] and Cobos, Kühn and Schonbek [42] shows that if any restriction of the operator is compact, then the interpolated operator by the classical real method is also compact. However, this is not the case for logarithmic methods with $\theta = 0$ or 1. Theorem 6.14 does not hold if we move the compactness assumption from $T : A_0 \to B_0$ to $T : A_1 \to B_1$. The following counterexample was given by Cobos, Fernández-Cabrera and Martínez in [38, Remark 2.4]: let $A_0 = A_1 = B_0 = c_0$ the sequence of all $(\xi_n)_{n=1}^\infty$ such that $\xi_n \xrightarrow{n\to\infty} 0$ with $\|\xi_n|c_0\| = \sup_{n\in\mathbb{N}} |\xi_n|$, let $B_1 = \{(\xi_n)_{n=1}^\infty : \|\xi|B_1\| = \sup_{n\in\mathbb{N}} 2^{-n}|\xi_n| < \infty\}$ and let T be the identity. Clearly, $T : A_0 \longrightarrow B_0$ is continuous and $T : A_1 \longrightarrow B_1$ is compact, since it is the limit of a sequence of finite rank operators. Indeed, let $P_m(\xi) = (\xi_1, ..., \xi_m, 0, 0, ...)$ then

$$\|(T - P_m)(\xi)|B_1\| = \sup_{n\geq m} 2^{-n}|\xi_n| \leq 2^{-m}\|\xi|c_0\|, \quad \xi \in c_0, \ m \in \mathbb{N},$$

and therefore, $\|T - P_m|\mathcal{L}(A_1, B_1)\| \leq 2^{-m} \xrightarrow{m\to\infty} 0$. However, as

$$K(t, \xi; B_0, B_1) \sim \sup_{n\in\mathbb{N}} \min\{1, t2^{-n}\}|\xi_n|, \quad \text{for every } t > 0,$$

it is not difficult to check that the Gagliardo completion of $B_0 = c_0$ is $B_0 + B_1 = B_1$ and $T : (A_0, A_1)_{0,\infty,(0,0)} = A_0^\sim \longrightarrow (B_0, B_1)_{0,\infty,(0,0)} = B_0^\sim$ is not compact, because it is the inclusion of c_0 in $\ell_\infty(\mathbb{N})$.

6.2 Applications to generalized Lorentz-Zygmund spaces

Next we apply the abstract results obtained in the previous section to derive quantitative versions of the limiting variant of Kranosels'kiĭ compactness result for generalized Lorentz-Zygmund spaces $L_{p,q,\mathbb{A}}(\Omega)$ (see (2.23)) proved by Edmunds and Opic in [56, Corollary 4 and Theorem 6]. Instead of the finite measure spaces used in [56], we work here with function spaces on σ-finite measure spaces. Moreover, the range for parameters q_0, q_1, q is broader than in [56].

Theorem 6.16. Let (R, μ), (S, ν) be σ-finite measure spaces. Take $1 < p_0 < p_1 \leq \infty$, $0 < q_0 < q_1 \leq \infty$, $0 < q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying that $\alpha_\infty + 1/q < 0 < \alpha_0 + 1/q$. Let $T \in \mathcal{L}((L_{p_0}(R), L_{p_1}(R)), (L_{q_0}(S), L_{q_1}(S)))$, put $\beta_j = \beta(T : L_{p_j}(R) \to L_{q_j}(S))$, $j = 0, 1$, and

$$\beta = \beta(T : L_{p_0,q,\mathbb{A}+\frac{1}{\min\{p_0,q\}}}(R) \to L_{q_0,q,\mathbb{A}+\frac{1}{\max\{q_0,q\}}}(S)).$$

Then there exists a constant $C > 0$ independent of T such that

a) $\beta \leq C\beta_0\Big(1 + \Big(\log \frac{\beta_1}{\beta_0}\Big)^+\Big)^{\alpha_0^+ - \alpha_\infty}$ if $\beta_j > 0$, for $j = 0, 1$;

b) $\beta \leq C\beta_0$ if $\beta_1 = 0$;

c) $\beta = 0$ if $\beta_0 = 0$.

Proof. According to (2.17) for any $r < q_0$ we have

$$\begin{aligned} L_{p_0}(R) &= (L_1(R), L_\infty(R))_{1-1/p_0,p_0}, \\ L_{p_1}(R) &= (L_1(R), L_\infty(R))_{1-1/p_1,p_1}, \\ L_{q_0}(S) &= (L_r(S), L_\infty(S))_{1-r/q_0,q_0}, \\ L_{q_1}(S) &= (L_r(S), L_\infty(S))_{1-r/q_1,q_1}. \end{aligned}$$

It follows from [58, Theorem 4.7 and Theorem 5.9]

$$(L_1(R), L_\infty(R))_{1-1/p_0,q,\mathbb{A}+\frac{1}{\min\{p_0,q\}}} \hookrightarrow (L_{p_0}(R), L_{p_1}(R))_{0,q,\mathbb{A}} \quad \text{and}$$
$$(L_{q_0}(S), L_{q_1}(S))_{0,q,\mathbb{A}} \hookrightarrow (L_r(S), L_\infty(S))_{1-r/q_0,q,\mathbb{A}+\frac{1}{\max\{q,q_0\}}}.$$

Besides by (2.27) we have

$$\begin{aligned} L_{p_0,q,\mathbb{A}+\frac{1}{\min\{p_0,q\}}} &= (L_1(R), L_\infty(R))_{1-1/p_0,q,\mathbb{A}+\frac{1}{\min\{p_0,q\}}}, \\ L_{q_0,q,\mathbb{A}+\frac{1}{\max\{q_0,q\}}} &= (L_r(S), L_\infty(S))_{1-r/q_0,q,\mathbb{A}+\frac{1}{\max\{q,q_0\}}}. \end{aligned}$$

Whence the result follows from Theorem 6.12. □

Corollary 6.17. Let (R, μ) and (S, ν) be σ-finite measure spaces. Take $1 < p_0 < p_1 \leq \infty$, $0 < q_0 < q_1 \leq \infty$, $0 < q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ with $\alpha_\infty + 1/q < 0 < \alpha_0 + 1/q$. Let T be a linear operator such that

$$\begin{aligned} T : L_{p_0}(R) &\longrightarrow L_{q_0}(S) \quad \text{is compact and ,} \\ T : L_{p_1}(R) &\longrightarrow L_{q_1}(S) \quad \text{is continuous.} \end{aligned}$$

Then $T : L_{p_0,q,\mathbb{A}+\frac{1}{\min\{p_0,q\}}}(R) \to L_{q_0,q,\mathbb{A}+\frac{1}{\max\{q_0,q\}}}(S)$ is also compact.

Theorem 6.18. Let (R, μ) and (S, ν) be σ-finite measure spaces. Take $1 \leq p_0 < p_1 < \infty$, $0 < q_0 < q_1 < \infty$, $0 < q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying that $\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q$. Let $T \in \mathcal{L}((L_{p_0}(R), L_{p_1}(R)), (L_{q_0}(S), L_{q_1}(S)))$, put $\beta_j = \beta(T : L_{p_j}(R) \to L_{q_j}(S))$, $j = 0, 1$, and

$$\beta = \beta(T : L_{p_1,q,\mathbb{A}+\frac{1}{\min\{p_1,q\}}}(R) \to L_{q_1,q,\mathbb{A}+\frac{1}{\max\{q_1,q\}}}(S)).$$

Then there is a constant $C > 0$ independent of T such that

a) $\beta \leq C\beta_1\Big(1 + \Big(\log \frac{\beta_0}{\beta_1}\Big)^+\Big)^{\alpha_\infty^+ - \alpha_0}$ if $\beta_j > 0$ for $j = 0, 1$;

b) $\beta \leq C\beta_1$ if $\beta_0 = 0$;

c) $\beta = 0$ if $\beta_1 = 0$.

Proof. We can proceed as in the previous theorem but using now Theorem 6.11. □

Corollary 6.19. Let (R,μ) and (S,ν) be σ-finite measure spaces. Take $1 \leq p_0 < p_1 < \infty$, $0 < q_0 < q_1 < \infty$, $0 < q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ with $\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q$. Let T be a linear operator such that

$$\begin{aligned} T : L_{p_0}(R) &\longrightarrow L_{q_0}(S) \quad \text{is continuous and,} \\ T : L_{p_1}(R) &\longrightarrow L_{q_1}(S) \quad \text{is compact.} \end{aligned}$$

Then $T : L_{p_1,q,\mathbb{A}+\frac{1}{\min\{p_1,q\}}}(R) \to L_{q_1,q,\mathbb{A}+\frac{1}{\max\{q_1,q\}}}(S)$ compactly.

Chapter 7

Equivalence theorems for logarithmic interpolation methods on quasi-Banach couples

In this chapter we study the representation of $(A_0, A_1)_{\theta,q,\mathbb{A}}$ by means of the J-functional when $\bar{A} = (A_0, A_1)$ is a (p-normed) quasi-Banach couple. If $0 < \theta < 1$, as we discussed in Section 2.1.2, $\rho(t) = t^{\theta}\ell^{-\mathbb{A}}(t)$ is equivalent to a function parameter and $(A_0, A_1)_{\theta,q,\mathbb{A}} = (A_0, A_1)^J_{\theta,q,\mathbb{A}}$. Therefore, the interesting cases are $\theta = 0$ and $\theta = 1$. For simplicity, we are going to focus here on the case $\theta = 1$, but analogous procedures could be applied for the case $\theta = 0$.

For $\theta = 1$, $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, according to Theorem 2.8, the interpolation method $(\cdot,\cdot)_{1,q,\mathbb{A}}$ is non-trivial if, and only if,

$$\begin{cases} 0 < q < \infty, & \alpha_0 + 1/q < 0; \\ q = \infty, & \alpha_0 \leq 0. \end{cases} \tag{7.1}$$

Furthermore, according to Theorem 2.9, $(A_0, A_1)_{1,q,\mathbb{A}} \subseteq (A_0 + A_1)^\circ$ for every (p-normed) quasi-Banach couple if, and only if,

$$\begin{cases} 0 < q < \infty, & \alpha_\infty + 1/q \geq 0; \\ q = \infty, & \alpha_\infty > 0; \end{cases} \tag{7.2}$$

and this condition is necessary for $(A_0, A_1)_{\theta,q,\mathbb{A}}$ to admit a representation as a J-space $(A_0, A_1)_{\Gamma;J}$ for every quasi-Banach couple $\bar{A} = (A_0, A_1)$. If $\bar{A} = (A_0, A_1)$ is a Banach couple, from (3.8), Theorem 3.5 and Theorem 5.5, we have the following identity with equivalent quasi-norms:

$$(A_0, A_1)_{1,q,\mathbb{A}} = \begin{cases} (A_0, A_1)^J_{1,q,\mathbb{A}+1} & \text{if} \quad 1 \leq q \leq \infty, \quad \alpha_0 + 1/q < 0 < \alpha_\infty + 1/q; \\ (A_0, A_1)_{1,q,\mathbb{A}+1,(0,1)} & \text{if} \quad 1 \leq q < \infty, \quad \alpha_0 + 1/q < 0 = \alpha_\infty + 1/q; \\ (A_0, A_1)^J_{\alpha_\infty} & \text{if} \quad q = \infty, \quad \alpha_0 = 0 < \alpha_\infty; \\ (A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q} & \text{if} \quad 0 < q < 1, \quad \alpha_0 + 1/q < 0 < \alpha_\infty + 1/q; \\ (A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q,(0,1/q)} & \text{if} \quad 0 < q < 1, \quad \alpha_0 + 1/q < 0 = \alpha_\infty + 1/q. \end{cases}$$

In general, for any (p-normed) quasi-Banach couple $\bar{A} = (A_0, A_1)$, we already saw in Theorem 6.1 that if $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty)$ satisfy (7.1) and (7.2), then

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)_{\Lambda;J},$$

for $\Lambda = (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}$. We wonder if under these hypothesis $(A_0, A_1)_{1,q,\mathbb{A}}$ admits a J-representation $(A_0, A_1)_{\Gamma;J}$ with Γ a logarithmic sequence lattice of the kind $\ell_q(2^{-m}\ell^{\mathbb{M}}(2^m))$. Sometimes, such a representation does not exists and then we determine the best $\mathbb{M}$ and $\mathbb{B}$ such that

$$(A_0, A_1)^J_{1,q,\mathbb{B}} \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow (A_0, A_1)^J_{1,q,\mathbb{M}}$$

for every p-normed quasi-Banach couple.

In Section 7.1, we start by computing $(A, 2^{-k}A)_{1,q,\mathbb{A}}$ for $k \in \mathbb{Z}$ and A a p-normed quasi-Banach space. We also compute it for logarithmic interpolation J-spaces. With the help of these results, in Section 7.2 we establish the equivalence and embedding theorems.

The most important results of this chapter form the paper [16].

7.1 Characteristic function for logarithmic interpolation methods

If $(A, \|\cdot|A\|)$ is a quasi-Banach space and $s > 0$, we write sA for the space A endowed with the quasi-norm $s\|\cdot|A\|$.

The *characteristic function* $\Phi_{\mathcal{F}}$ of an interpolation functor $\mathcal{F}$ is the function defined by

$$\mathcal{F}(\mathbb{R}, (1/t)\mathbb{R}) = (1/\Phi_{\mathcal{F}}(t))\mathbb{R}, \quad t > 0$$

(see [22, 80, 98]). Next we describe the characteristic function $\Phi_{q,\mathbb{A}}$ of $(\cdot,\cdot)_{1,q,\mathbb{A}}$.

The following auxiliary result is useful.

Lemma 7.1. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \le \infty$ satisfying (7.1). Put

$$\mathrm{v}_{q,\mathbb{A}}(2^k) = \left(\sum_{m=-\infty}^{\infty} [\min\{1, 2^{m-k}\} 2^{-m}\ell^{\mathbb{A}}(2^m)]^q\right)^{1/q}, \quad k \in \mathbb{Z}.$$

Then we have for $0 < q < \infty$

$$\mathrm{v}_{q,\mathbb{A}}(2^k) \sim \begin{cases} 2^{-k}\ell^{\mathbb{A}+1/q}(2^k) & \text{if} \quad \alpha_\infty + 1/q > 0, \\ 2^{-k}\ell^{\mathbb{A}+1/q}(2^k)\ell\ell^{(0,1/q)}(2^k) & \text{if} \quad \alpha_\infty + 1/q = 0, \\ 2^{-k}\ell^{(\alpha_0+1/q,0)}(2^k) & \text{if} \quad \alpha_\infty + 1/q < 0, \end{cases}$$

and for $q = \infty$

$$\mathrm{v}_{\infty,q}(2^k) \sim \begin{cases} 2^{-k}\ell^{\mathbb{A}}(2^k) & \text{if} \quad \alpha_\infty \ge 0, \\ 2^{-k}\ell^{(\alpha_0,0)}(2^k) & \text{if} \quad \alpha_\infty < 0. \end{cases}$$

Proof. If $0 < q < \infty$ then the result follows proceeding as in Lemma 3.4. Suppose $q = \infty$. We have

$$\begin{aligned} \mathrm{v}_{\infty,q}(2^k) &= \sup_{m\in\mathbb{Z}}\{\min\{1, 2^{m-k}\}2^{-m}\ell^{\mathbb{A}}(2^m)\} \\ &= \max\{\sup_{m\le k}\{2^{-k}\ell^{\mathbb{A}}(2^m)\}, \sup_{m\ge k}\{2^{-m}\ell^{\mathbb{A}}(2^m)\}\} \\ &\sim 2^{-k}\sup_{m\le k}\{\ell^{\mathbb{A}}(2^m)\}. \end{aligned}$$

If $k \leq 0$, since $\alpha_0 \leq 0$, we get

$$\mathrm{v}_{\infty,q}(2^k) \sim 2^{-k} \sup_{m\leq k}\{(1-\log 2^m)^{\alpha_0}\} = 2^{-k}\ell^{\alpha_0}(2^k).$$

If $k > 0$, we have

$$\begin{aligned} \mathrm{v}_{\infty,\mathbb{A}}(2^k) &\sim 2^{-k}\max\Big\{\sup_{m\leq 0}\{(1-\log t)^{\alpha_0}\},\ \sup_{0\leq m\leq k}\{(1+\log t)^{\alpha_\infty}\}\Big\} \\ &= 2^{-k}\max\{1,\ \sup_{0\leq m\leq k}\{(1+\log 2^m)^{\alpha_\infty}\}\} \\ &= \begin{cases} 2^{-k}\ell^{\alpha_\infty}(2^k) & \text{if} \quad \alpha_\infty \geq 0, \\ 2^{-k} & \text{if} \quad \alpha_\infty < 0. \end{cases} \end{aligned}$$

This completes the proof. □

Proposition 7.2. Let $(A, \|\cdot|A\|)$ be a p-normed quasi-Banach space and let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \leq \infty$ satisfying (7.1). Then, with equivalence of quasi-norms, we have

$$(A, 2^{-k}A)_{1,q,\mathbb{A}} = \mathrm{v}_{q,\mathbb{A}}(2^k)A,$$

where the constants in the equivalence depend on p but they are independent of $k \in \mathbb{Z}$ and of the concrete p-normed space A.

Proof. Let $0 < p \leq 1$ such that A is p-normed. Take any $a \in A$ and $m \in \mathbb{Z}$. It is not hard to check that $K_p(2^m, a; A, 2^{-k}A) = \min\{1, 2^{m-k}\}\|a|A\|$. Therefore

$$\begin{aligned} \|a|(A,2^{-k}A)_{1,q,\mathbb{A}}\| &\sim \left(\sum_{m=-\infty}^{\infty}[\min\{1,2^{m-k}\}2^{-m}\ell^{\mathbb{A}}(2^m)]^q\right)^{1/q}\|a|A\| \\ &= \mathrm{v}_{q,\mathbb{A}}(2^k)\|a|A\|. \end{aligned}$$

□

Corollary 7.3. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \leq \infty$ satisfying (7.1). Then $\Phi_{q,\mathbb{A}}(2^k) \sim \mathrm{v}_{q,\mathbb{A}}(2^k)^{-1}$.

Now we compute $(A, 2^{-k}A)^J_{1,q,\mathbb{A}}$ for $k \in \mathbb{Z}$ and A a p-normed quasi-Banach space. Remember that, according to (2.10), the interpolation method $(\cdot,\cdot)^J_{1,q,\mathbb{A}}$ is (p, J)-non trivial if, and only if,

$$\begin{cases} 0 < q \leq p, & \alpha_\infty \geq 0; \\ p < q \leq \infty, & \alpha_\infty + \frac{1}{q} - \frac{1}{p} > 0. \end{cases} \tag{7.3}$$

In what follows, given $0 < p \leq 1$ and $0 < q \leq \infty$, we put

$$q^* = \begin{cases} \infty & \text{if} \quad 0 < q \leq p; \\ \frac{pq}{q-p} & \text{if} \quad 0 < p < q < \infty; \\ p & \text{if} \quad q = \infty. \end{cases}$$

Lemma 7.4. Let $0 < p \leq 1$, $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ verifying (7.3). For $k \in \mathbb{Z}$, put

$$\mathrm{u}_{q,\mathbb{A},p}(2^k) = \sup\Big\{\Big(\sum_{m=-\infty}^{\infty}[\min\{1,2^{k-m}\}|x_m|]^p\Big)^{1/p} : \|(x_m)|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\| \leq 1\Big\}.$$

Then we have

$$\mathrm{u}_{q,\mathbb{A},p}(2^k) = \|(\min\{1,2^{k-m}\})|\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))\|.$$

Proof. Let $x = (x_m) \in \ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))$ with $\|x|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\| \le 1$. Applying Hölder's inequality we get

$$\Big(\sum_{m=-\infty}^{\infty}[\min\{1,2^{k-m}\}|x_m|]^p\Big)^{1/p} \le \|(\min\{1,2^{k-m}\})|\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))\| \cdot \|x|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\|.$$

Therefore, $\mathrm{u}_{q,\mathbb{A},p}(2^k) \le \|(\min\{1,2^{k-m}\})|\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))\|$.

Conversely, if $0 < q \le p$, so $q^* = \infty$, given any $\varepsilon > 0$ we can find $n \in \mathbb{Z}$ such that

$$\min\{1,2^{k-n}\}2^n\ell^{-\mathbb{A}}(2^n) \ge \|(\min\{1,2^{k-m}\})|\ell_\infty(2^m\ell^{-\mathbb{A}}(2^m))\| - \varepsilon.$$

Take $x = 2^n\ell^{-\mathbb{A}}(2^n)e_n$. Clearly $\|x|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\| = 1$. Moreover

$$\begin{aligned}\left(\sum_{m=-\infty}^{\infty}[\min\{1,2^{k-m}\}|x_m|]^p\right)^{1/p} &= \min\{1,2^{k-n}\}2^n\ell^{-\mathbb{A}}(2^n)\\ &> \|(\min\{1,2^{k-m}\})|\ell_\infty(2^m\ell^{-\mathbb{A}}(2^m))\| - \varepsilon.\end{aligned}$$

Whence $\mathrm{u}_{q,\mathbb{A},p}(2^k) = \|(\min\{1,2^{k-m}\})|\ell_\infty(2^m\ell^{-\mathbb{A}}(2^m))\|$.

Suppose now $0 < p < q < \infty$. Let

$$y_m = \min\{1,2^{k-m}\}^{\frac{p}{q-p}}2^{m\frac{q}{q-p}}\ell^{-\frac{q}{q-p}\mathbb{A}}(2^m), \quad m \in \mathbb{Z},$$

and $y = (y_m)$. Then

$$\|y|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\| = \|(\min\{1,2^{k-m}\})|\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))\|^{p/(q-p)}.$$

Put

$$x_m = \frac{y_m}{\|(\min\{1,2^{k-m}\})|\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))\|^{p/(q-p)}} \quad \text{and} \quad x = (x_m).$$

Then, $\|x|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\| = 1$ and

$$\begin{aligned}\Big(\sum_{m=-\infty}^{\infty}[\min\{1,2^{k-m}\}|x_m|]^p\Big)^{1/p} &= \frac{\Big(\sum_{m=-\infty}^{\infty}[\min\{1,2^{k-m}\}2^m\ell^{-\mathbb{A}}(2^m)]^{q^*}\Big)^{1/p}}{\Big(\sum_{m=-\infty}^{\infty}[\min\{1,2^{k-m}\}2^m\ell^{-\mathbb{A}}(2^m)]^{q^*}\Big)^{1/q}}\\ &= \|(\min\{1,2^{k-m}\})|\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))\|.\end{aligned}$$

Consequently,

$$\mathrm{u}_{q,\mathbb{A},p}(2^k) = \|(\min\{1,2^{k-m}\})|\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))\|.$$

Finally, assume that $q = \infty$ so $q^* = p$. Let $x_m = 2^m\ell^{-\mathbb{A}}(2^m)$, $m \in \mathbb{Z}$. We have

$$\|(x_m)|\ell_\infty(2^{-m}\ell^{\mathbb{A}}(2^m))\| = 1$$

and

$$\Big(\sum_{m=-\infty}^{\infty}[\min\{1,2^{k-m}\}|x_m|]^p\Big)^{1/p} = \Big(\sum_{m=-\infty}^{\infty}[\min\{1,2^{k-m}\}2^m\ell^{-\mathbb{A}}(2^m)]^p\Big)^{1/p}.$$

This yields that

$$\mathrm{u}_{q,\mathbb{A},p}(2^k) = \|(\min\{1,2^{k-m}\})|\ell_p(2^m\ell^{-\mathbb{A}}(2^m))\|.$$

□

Proposition 7.5. Let A be a p-normed quasi-Banach space, let $0 < q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying (7.3). For any $k \in \mathbb{Z}$, we have

$$(A, 2^{-k}A)^J_{1,q,\mathbb{A}} \hookrightarrow \mathrm{u}_{q,\mathbb{A},p}(2^k)^{-1}A,$$

being the norm of the embedding less than or equal to 1. Moreover, if in addition we suppose that $p = 1$ or $0 < q \le p$, then we have with equal quasi-norms

$$(A, 2^{-k}A)^J_{1,q,\mathbb{A}} = \mathrm{u}_{q,\mathbb{A},p}(2^k)^{-1}A.$$

Proof. Observe that $J(2^m, u; A, 2^{-k}A) = \max\{1, 2^{m-k}\}\|u|A\|$. If $(u_m) \subseteq A$ and $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in A), we obtain

$$\begin{aligned}
\|a|A\| &\le \Big(\sum_{m=-\infty}^{\infty} \|u_m|A\|^p\Big)^{1/p} = \left(\sum_{m=-\infty}^{\infty} [\min\{1,2^{k-m}\}J(2^m,u_m)]^p\right)^{1/p} \\
&\le \left(\sum_{m=-\infty}^{\infty} \left[\min\{1,2^{k-m}\}\frac{J(2^m,u_m)}{\|(J(2^m,u_m))|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\|}\right]^p\right)^{1/p} \\
&\qquad \times \|(J(2^m,u_m))|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\| \\
&\le \mathrm{u}_{q,\mathbb{A},p}(2^k)\|(J(2^m,u_m))|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\|.
\end{aligned}$$

This yields that the embedding $(A, 2^{-k}A)^J_{1,q,\mathbb{A}} \hookrightarrow \mathrm{u}_{q,\mathbb{A},p}(2^k)^{-1}A$ has a norm less than or equal to 1.

Conversely, if $p = 1$, given any $\varepsilon > 0$ we can find $x = (x_m)$ such that $\|x|\ell_q(2^{-m}\ell^{\mathbb{A}}(2^m))\| \le 1$ and

$$\mathrm{u}_{q,\mathbb{A},1}(2^k) - \varepsilon < \sum_{m=-\infty}^{\infty} \min\{1,2^{k-m}\}|x_m|.$$

We can represent any $a \in A$ as $a = \sum_{m=-\infty}^{\infty} u_m$ with

$$u_m = \frac{\min\{1,2^{k-m}\}|x_m|}{\sum_{m=-\infty}^{\infty}\min\{1,2^{k-m}\}|x_m|}a, \quad m \in \mathbb{Z}.$$

Then

$$J(2^m,u_m) = \frac{|x_m|}{\sum_{m=-\infty}^{\infty}\min\{1,2^{k-m}\}|x_m|}\|a|A\| \le \frac{|x_m|}{\mathrm{u}_{q,\mathbb{A},1}(2^k)-\varepsilon}\|a|A\|.$$

Whence

$$\|a|(A,2^{-k}A)^J_{1,q,\mathbb{A}}\| \le (\mathrm{u}_{q,\mathbb{A},1}(2^k)-\varepsilon)^{-1}\|a|A\|.$$

This yields that the embedding $\mathrm{u}_{q,\mathbb{A},1}(2^k)^{-1}A \hookrightarrow (A, 2^{-k}A)^J_{1,q,\mathbb{A}}$ has norm less than or equal to 1.

Suppose now that $0 < q \le p$. Then $q^* = \infty$. Given any $\varepsilon > 0$ we can find $n \in \mathbb{Z}$ such that

$$\mathrm{u}_{q,\mathbb{A},p}(2^k) - \varepsilon < 2^n\ell^{-\mathbb{A}}(2^n)\min\{1,2^{k-n}\}.$$

For any $a \in A$ we obtain

$$J(2^n, a) = \frac{1}{\min\{1, 2^{k-n}\}}\|a|A\| < \left(\mathrm{u}_{q,\mathbb{A},p}(2^k) - \varepsilon\right)^{-1} 2^n \ell^{-\mathbb{A}}(2^n)\|a|A\|.$$

Consider the representation $a = \sum_{m=-\infty}^{\infty} \delta_m^n a$. We have

$$\|a|(A, 2^{-k}A)^J_{1,q,\mathbb{A}}\| \leq 2^{-n}\ell^{\mathbb{A}}(2^n)J(2^n, a) < \left(\mathrm{u}_{q,\mathbb{A},p}(2^k) - \varepsilon\right)^{-1}\|a|A\|.$$

This completes the proof. □

Corollary 7.6. Let A be a p-normed quasi-Banach space, let $k \in \mathbb{Z}, 0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying (7.3). The following equalities hold with equivalence of quasi-norms:

For $p = 1$,

$$(A, 2^{-k}A)^J_{1,q,\mathbb{A}} = \begin{cases} 2^{-k}\ell^{\mathbb{A}}(2^k)A & \text{if} \quad 0 < q \leq 1 \quad \text{and} \quad \alpha_0 \leq 0, \\ 2^{-k}\ell^{(0,\alpha_\infty)}(2^k)A & \text{if} \quad 0 < q \leq 1 \quad \text{and} \quad \alpha_0 > 0, \\ 2^{-k}\ell^{\mathbb{A}-1/q^*}(2^k)A & \text{if} \quad 1 < q \leq \infty \quad \text{and} \quad \alpha_0 < \frac{1}{q^*}, \\ 2^{-k}\ell^{\mathbb{A}-1/q^*}(2^k)\ell\ell^{(-1/q^*,0)}(2^k)A & \text{if} \quad 1 < q \leq \infty \quad \text{and} \quad \alpha_0 = \frac{1}{q^*}, \\ 2^{-k}\ell^{(0,\alpha_\infty-1/q^*)}(2^k)A & \text{if} \quad 1 < q \leq \infty \quad \text{and} \quad \alpha_0 > \frac{1}{q^*}. \end{cases}$$

For $0 < q \leq p$,

$$(A, 2^{-k}A)^J_{1,q,\mathbb{A}} = \begin{cases} 2^{-k}\ell^{\mathbb{A}}(2^k)A & \text{if} \quad \alpha_0 \leq 0, \\ 2^{-k}\ell^{(0,\alpha_\infty)}(2^k)A & \text{if} \quad \alpha_0 > 0. \end{cases}$$

The constants in the equivalence depend on p but they are independent of $k \in \mathbb{Z}$ and of the concrete p-normed space A.

Proof. Let $\tilde{\mathbb{A}} = (\alpha_\infty, \alpha_0)$. Using Lemma 7.4 and a change of variables we can relate $\mathrm{u}_{q,\mathbb{A},p}$ with the function $\mathrm{v}_{q^*,-\tilde{\mathbb{A}}}$ defined in Lemma 7.1. We have

$$\begin{aligned} \mathrm{u}_{q,\mathbb{A},p}(2^k) &= \|(\min\{1, 2^{k-m}\})|\ell_{q^*}(2^m\ell^{-\mathbb{A}}(2^m))\| \\ &= \|(\min\{1, 2^{k+m}\})|\ell_{q^*}(2^{-m}\ell^{-\tilde{\mathbb{A}}}(2^m))\| = \mathrm{v}_{q^*,-\tilde{\mathbb{A}}}(2^{-k}). \end{aligned}$$

Now the result follows by applying Proposition 7.5 and Lemma 7.1. □

7.2 The equivalence results

As we recalled in the introduction of this chapter, if $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfy (7.1) and (7.2), then for any p-normed quasi-Banach couple we have that

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)_\Lambda,$$

where $\Lambda = (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}$ and $A_j^\sim$ is the Gagliardo completion of A_j (see Theorem 6.1). The next result gives a more detailed description of the quasi-Banach sequence lattice Λ.

Lemma 7.7. Let $0 < p \leq 1$, $0 < q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying (7.1) and (7.2). If $x = (x_m)_{m\in\mathbb{Z}}$ then

$$\|x|(\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}\| \sim \Big(\sum_{m=-\infty}^{0} \big[\ell^{\alpha_0}(2^m)(\sum_{k=m}^{0} 2^{-kp}|x_k|^p)^{1/p}\big]^q\Big)^{1/q}$$
$$+ \Big(\sum_{m=0}^{\infty} \big[\ell^{\alpha_\infty}(2^m)(\sum_{k=m}^{\infty} 2^{-kp}|x_k|^p)^{1/p}\big]^q\Big)^{1/q}.$$

Proof. Since $K_p(2^k, x; \ell_p, \ell_p(2^{-m})) = (\sum_{m=-\infty}^{\infty}[\min\{1, 2^{k-m}\}|x_m|]^p)^{1/p}$ we have

$$\|x|(\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}\| \sim \Big(\sum_{m=-\infty}^{\infty} \big[2^{-m}\ell^{\mathbb{A}}(2^m)(\sum_{k=-\infty}^{\infty} \min\{1, 2^{m-k}\}^p|x_k|^p)^{1/p}\big]^q\Big)^{1/q}$$
$$\sim \Big(\sum_{m=-\infty}^{0} \big[2^{-m}\ell^{\alpha_0}(2^m)(\sum_{k=-\infty}^{m} |x_k|^p)^{1/p}\big]^q\Big)^{1/q}$$
$$+ \Big(\sum_{m=-\infty}^{0} \big[\ell^{\alpha_0}(2^m)(\sum_{k=m}^{0} 2^{-kp}|x_k|^p)^{1/p}\big]^q\Big)^{1/q}$$
$$+ \Big(\sum_{m=-\infty}^{0} \big[\ell^{\alpha_0}(2^m)(\sum_{k=0}^{\infty} 2^{-kp}|x_k|^p)^{1/p}\big]^q\Big)^{1/q}$$
$$+ \Big(\sum_{m=0}^{\infty} \big[2^{-m}\ell^{\alpha_\infty}(2^m)(\sum_{k=-\infty}^{0} |x_k|^p)^{1/p}\big]^q\Big)^{1/q}$$
$$+ \Big(\sum_{m=0}^{\infty} \big[2^{-m}\ell^{\alpha_\infty}(2^m)(\sum_{k=0}^{m} |x_k|^p)^{1/p}\big]^q\Big)^{1/q}$$
$$+ \Big(\sum_{m=0}^{\infty} \big[\ell^{\alpha_\infty}(2^m)(\sum_{k=m}^{\infty} 2^{-kp}|x_k|^p)^{1/p}\big]^q\Big)^{1/q}$$
$$= S_1 + S_2 + S_3 + S_4 + S_5 + S_6.$$

Having in mind (7.1), we get $S_3 \sim (\sum_{k=0}^{\infty} 2^{-kp}|x_k|^p)^{1/p}$. Hence, the term in S_6 with $m = 0$ shows that $S_3 \lesssim S_6$. On the other hand,

$$S_4 \sim (\sum_{k=-\infty}^{0} |x_k|^p)^{1/p} \leq S_1.$$

Next we check that $S_1 \lesssim S_2$ and $S_5 \lesssim S_6$. Assume first that $0 < q \leq p$. Then

$$S_1 \leq \Big(\sum_{m=-\infty}^{0} 2^{-mq}\ell^{\alpha_0 q}(2^m) \sum_{k=-\infty}^{m} |x_k|^q\Big)^{1/q} = \Big(\sum_{k=-\infty}^{0} |x_k|^q \sum_{m=k}^{0} 2^{-mq}\ell^{\alpha_0 q}(2^m)\Big)^{1/q}$$
$$\lesssim \Big(\sum_{k=-\infty}^{0} |x_k|^q 2^{-kq}\ell^{\alpha_0 q}(2^k)\Big)^{1/q} \leq S_2.$$

Similarly,

$$S_5 \leq \Big(\sum_{m=0}^{\infty} 2^{-mq}\ell^{\alpha_\infty q}(2^m) \sum_{k=0}^{m} |x_k|^q\Big)^{1/q} = \Big(\sum_{k=0}^{\infty} |x_k|^q \sum_{m=k}^{\infty} 2^{-mq}\ell^{\alpha_0 q}(2^m)\Big)^{1/q}$$
$$\lesssim \Big(\sum_{k=0}^{\infty} |x_k|^q 2^{-kq}\ell^{\alpha_\infty q}(2^k)\Big)^{1/q} \leq S_6.$$

Suppose now that $0 < p < q < \infty$. Let $0 < \varepsilon < 1$. Using Hölder's inequality we get

$$\Big(\sum_{k=-\infty}^{m} |x_k|^p\Big)^{q/p} \le \Big(\sum_{k=-\infty}^{m} 2^{-k(1-\varepsilon)q}|x_k|^q\Big)\Big(\sum_{k=-\infty}^{m} 2^{k(1-\varepsilon)pq/(q-p)}\Big)^{(q-p)/p}$$
$$\sim 2^{m(1-\varepsilon)q} \sum_{k=-\infty}^{m} 2^{-k(1-\varepsilon)q}|x_k|^q.$$

Hence,

$$\begin{aligned} S_1 &\lesssim \Big(\sum_{m=-\infty}^{0} 2^{-m\varepsilon q}\ell^{\alpha_0 q}(2^m) \sum_{k=-\infty}^{m} 2^{-k(1-\varepsilon)q}|x_k|^q\Big)^{1/q} \\ &= \Big(\sum_{k=-\infty}^{0} 2^{-k(1-\varepsilon)q}|x_k|^q \sum_{m=k}^{0} 2^{-m\varepsilon q}\ell^{\alpha_0 q}(2^m)\Big)^{1/q} \\ &\lesssim \Big(\sum_{k=-\infty}^{0} 2^{-kq}|x_k|^q \ell^{\alpha_0 q}(2^k)\Big)^{1/q} \le S_2. \end{aligned}$$

As for S_5, for the interior sum we obtain

$$\Big(\sum_{k=0}^{m} |x_k|^p\Big)^{q/p} \le \Big(\sum_{k=0}^{m} 2^{-kq\varepsilon}|x_k|^q\Big)\Big(\sum_{k=0}^{m} 2^{k\varepsilon pq/(q-p)}\Big)^{(q-p)/p}$$
$$\sim 2^{m\varepsilon q} \sum_{k=0}^{m} 2^{-kq\varepsilon}|x_k|^q.$$

Therefore,

$$\begin{aligned} S_5 &\lesssim \Big(\sum_{m=0}^{\infty} 2^{-mq(1-\varepsilon)}\ell^{\alpha_\infty q}(2^m) \sum_{k=0}^{m} 2^{-kq\varepsilon}|x_k|^q\Big)^{1/q} \\ &= \Big(\sum_{k=0}^{\infty} 2^{-kq\varepsilon}|x_k|^q \sum_{m=k}^{\infty} 2^{-mq(1-\varepsilon)}\ell^{\alpha_\infty q}(2^m)\Big)^{1/q} \\ &\lesssim \Big(\sum_{k=0}^{\infty} 2^{-kq}|x_k|^q \ell^{\alpha_\infty q}(2^k)\Big)^{1/q} \le S_6. \end{aligned}$$

The case $q = \infty$ can be treated analogously. This completes the proof. □

Next we show that if $p = q$, then Λ is a weighted ℓ_q-space.

Lemma 7.8. Let $0 < q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying $\alpha_0 + 1/q < 0 \le \alpha_\infty + 1/q$. Then we have, with equivalence of quasi-norms

$$(\ell_q, \ell_q(2^{-m}))_{1,q,\mathbb{A}} = \begin{cases} \ell_q(2^{-m}\ell^{\mathbb{A}+1/q}(2^m)) & \text{if} \quad 0 < \alpha_\infty + 1/q, \\ \ell_q(2^{-m}\ell^{\mathbb{A}+1/q}(2^m)\ell\ell^{(0,1/q)}(2^m)) & \text{if} \quad 0 = \alpha_\infty + 1/q. \end{cases}$$

Proof. According to Lemma 7.7, we obtain

$$\|x|(\ell_q, \ell_q(2^{-m}))_{1,q,\mathbb{A}}\| \sim \Big(\sum_{m=-\infty}^{0} \ell^{\alpha_0 q}(2^m) \sum_{k=m}^{0} 2^{-kq}|x_k|^q\Big)^{1/q} + \Big(\sum_{m=0}^{\infty} \ell^{\alpha_\infty q}(2^m) \sum_{k=m}^{\infty} 2^{-kq}|x_k|^q\Big)^{1/q}.$$

Changing the order of summation, we derive

$$\|x|(\ell_q,\ell_q(2^{-m}))_{1,q,\mathbb{A}}\| \sim \left(\sum_{k=-\infty}^{0} 2^{-kq}|x_k|^q \sum_{m=-\infty}^{k} \ell^{\alpha_0 q}(2^m)\right)^{1/q} + \left(\sum_{k=0}^{\infty} 2^{-kq}|x_k|^q \sum_{m=0}^{k} \ell^{\alpha_\infty q}(2^m)\right)^{1/q}.$$

Since

$$\sum_{m=-\infty}^{k} \ell^{\alpha_0 q}(2^m) \sim \ell^{\alpha_0 q+1}(2^k) \quad \text{if} \quad \alpha_0 + 1/q < 0$$

and

$$\sum_{m=0}^{k} \ell^{\alpha_\infty q}(2^m) \sim \begin{cases} \ell^{\alpha_\infty q+1}(2^k) & \text{if} \quad 0 < \alpha_\infty + 1/q, \\ \ell\ell(2^k) & \text{if} \quad 0 = \alpha_\infty + 1/q, \end{cases}$$

the result follows. □

Now we are ready to focus on the main problem of this chapter consisting in the investigation of descriptions of $(A_0,A_1)_{1,q,\mathbb{A}}$ as a J-space $(A_0,A_1)_{\Gamma;J}$, for Γ a quasi-Banach sequence lattice of logarithmic type $\ell_q(2^{-m}\ell^{\mathbb{M}}(2^m))$ and (A_0,A_1) a p-normed quasi-Banach couple. This means that $\mathbb{A}=(\alpha_0,\alpha_\infty)$ and q should satisfy (7.1) and (7.2); and $\mathbb{M}=(\mu_0,\mu_\infty)$, q and p should satisfy (7.3). A more detailed outline of the problem is the following:

Given $0<p\le 1$, $\mathbb{A}=(\alpha_0,\alpha_\infty)\in\mathbb{R}^2$ and $0<q\le\infty$ such that

$$\begin{cases} \alpha_0+1/q<0\le\alpha_\infty+1/q & \text{if} \quad 0<q<\infty, \\ \alpha_0\le 0<\alpha_\infty & \text{if} \quad q=\infty, \end{cases} \tag{7.4}$$

find $\mathbb{M}=(\mu_0,\mu_\infty)\in\mathbb{R}^2$ with

$$\begin{cases} \mu_\infty \ge 0 & \text{if} \quad 0<q\le p, \\ \mu_\infty > \frac{1}{p}-\frac{1}{q} & \text{if} \quad 0<p<q\le\infty, \end{cases} \tag{7.5}$$

and such that for any p-normed quasi-Banach couple $\bar{A}=(A_0,A_1)$ we have

$$(A_0,A_1)_{1,q,\mathbb{A}} = (A_0,A_1)^J_{1,q,\mathbb{M}} \tag{7.6}$$

with equivalence of quasi-norms where the constants are independent of $\bar{A}$.

Unfortunately, for some values of parameters p, q and $\mathbb{A}$ there is no $\mathbb{M}$ satisfying (7.6). Take, for example, $0<p<q\le 1$ and $\alpha_0+1/q<0<\alpha_\infty+1/q\le 1/p-1/q$. Let $\mathbb{M}=(\mu_0,\mu_\infty)$ with $\mu_\infty>1/p-1/q$. If (7.6) holds then Proposition 7.2 and Corollary 7.6 with $A=\ell_1$ would yield

$$\mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1} \sim \mathrm{v}_{q,\mathbb{A}}(2^k)$$

with constant in the equivalence independent of k. By Lemma 7.1 and Corollary 7.6 we get for $k\in\mathbb{Z}$, k positive, that $\mathrm{v}_{q,\mathbb{A}}(2^k)\sim 2^{-k}\ell^{\alpha_\infty+1/q}(2^k)$ and $\mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1}\sim 2^{-k}\ell^{\mu_\infty}(2^k)$. It follows that $\alpha_\infty+1/q=\mu_\infty$. Hence $\alpha_\infty+1/q>1/p-1/q$ which contradicts that $\alpha_\infty+1/q\le 1/p-1/q$.

This example leads us to investigate the weaker questions of finding the best $\mathbb{M}$ and $\mathbb{B}$ such that

$$(A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow (A_0, A_1)^J_{1,q,\mathbb{M}} \tag{7.7}$$

or

$$(A_0, A_1)^J_{1,q,\mathbb{B}} \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}} \tag{7.8}$$

for any p-normed quasi-Banach couple (A_0, A_1) and with constants in the embeddings independent of (A_0, A_1).

In the case $0 < p < q \leq 1$ and $\alpha_0 + 1/q < 0$, if we assume that (7.7) holds for some $\mathbb{M} = (\mu_0, \mu_\infty)$ with $\mu_\infty > 1/p - 1/q$, then proceeding as above we obtain that $\mathrm{v}_{q,\mathbb{A}}(2^k)\ell_1 \hookrightarrow \mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1}\ell_1$. The values of $\mathrm{v}_{q,\mathbb{A}}(2^k)$ and $\mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1}$ have been pointed out above. Since the embedding is valid for any positive k, we get $\mu_\infty \leq \alpha_\infty + 1/q$. Notice this inequality is only possible when $\alpha_\infty + 1/q > 1/p - 1/q$, otherwise (7.7) does not happen for any $\mathbb{M} = (\mu_0, \mu_\infty)$ with $\mu_\infty > 1/p - 1/q$. Let now $k \in \mathbb{Z}$, k negative, then $\mathrm{v}_{q,\mathbb{A}}(2^k) \sim 2^{-k}\ell^{\alpha_0+1/q}(2^k)$ while

$$\mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1} \sim \begin{cases} 2^{-k}\ell^{\mu_0}(2^k) & \text{if} \quad \mu_0 \leq 0, \\ 2^{-k} & \text{if} \quad \mu_0 > 0. \end{cases}$$

The option 2^{-k} is not possible since $\alpha_0 + 1/q < 0$. Hence, we should have $\mu_0 \leq \alpha_0 + 1/q$. In other words, for these p, q and $\mathbb{A}$, the best possible $\mathbb{M}$ would be $(\alpha_0 + 1/q, \alpha_\infty + 1/q)$. Next we show that (7.7) holds for this choice of $\mathbb{M}$. We shall use that $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^{\sim}, A_1^{\sim})_{1,q,\mathbb{A}}$ (see (2.8)).

Theorem 7.9. Let $0 < p < q \leq 1$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying

$$\alpha_0 + 1/q < 0, \quad \alpha_\infty + 1/q > 1/p - 1/q. \tag{7.9}$$

Then, for any p-normed quasi-Banach couple $\bar{A} = (A_0, A_1)$, we have

$$(A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow (A_0^{\sim}, A_1^{\sim})^J_{1,q,\mathbb{A}+1/q}.$$

Proof. According to Theorem 6.1, we get

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^{\sim}, A_1^{\sim})_{1,q,\mathbb{A}} = (A_0^{\sim}, A_1^{\sim})_{(\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}};J}.$$

Besides, since $p < q$, we have $\ell_p \hookrightarrow \ell_q$ and $\ell_p(2^{-m}) \hookrightarrow \ell_q(2^{-m})$. Therefore,

$$(\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}} \hookrightarrow (\ell_q, \ell_q(2^{-m}))_{1,q,\mathbb{A}} = \ell_q(2^{-m}\ell^{\mathbb{A}+1/q}(2^m))$$

where the last equality follows from Lemma 7.8. Consequently,

$$(A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow (A_0^{\sim}, A_1^{\sim})^J_{1,q,\mathbb{A}+1/q}.$$

□

Next we consider the case $1 \leq q \leq \infty$. This time our arguments are based on decompositions of the type considered in [37] and [48].

Theorem 7.10. Let $0 < p \leq 1 \leq q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying

$$\alpha_0 + 1/q < 0 \quad \text{and} \quad \alpha_\infty + 1/q > 1/p - 1. \tag{7.10}$$

Then, for any p-Banach couple $\bar{A} = (A_0, A_1)$, we have

$$(A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow (A_0, A_1)^J_{1,q,\mathbb{A}+1}.$$

Proof. Take any $a \in (A_0, A_1)_{1,q,\mathbb{A}}$. It follows from (7.10) (see (7.2)) that $a \in (A_0 + A_1)^\circ$. Hence,

$$\min\{1, t^{-1}\}K(t, a) \to 0 \text{ as } t \to 0 \text{ and as } t \to \infty.$$

For $\nu \in \mathbb{Z}$, let

$$\gamma_\nu = \begin{cases} 2^{-2^{-\nu-1}} & \text{if} \quad \nu < 0, \\ 1 & \text{if} \quad \nu = 0, \\ 2^{2^{\nu-1}} & \text{if} \quad \nu > 0. \end{cases}$$

Choose $a_{0,\nu} \in A_0$, $a_{1,\nu} \in A_1$ such that $a = a_{0,\nu} + a_{1,\nu}$ and

$$\|a_{0,\nu}|A_0\|^p + \gamma^p_{\nu-1}\|a_{1,\nu}|A_1\|^p \le 2^p K_p(\gamma_{\nu-1}, a)^p.$$

Let $u_\nu = a_{0,\nu} - a_{0,\nu-1} = a_{1,\nu-1} - a_{1,\nu} \in A_0 \cap A_1$. We have

$$\begin{aligned} \|a - \sum_{\nu=N}^{M} u_\nu|A_0 + A_1\|^p &= \|a + a_{0,N-1} - a_{0,M}|A_0 + A_1\|^p \\ &= \|a_{1,M} + a_{0,N-1}|A_0 + A_1\|^p \\ &\le 2^p\left(\gamma^{-p}_{M-1}K_p(\gamma_{M-1}, a)^p + K_p(\gamma_{N-2}, a)^p\right) \to 0 \end{aligned}$$

as $M \to \infty$ and $N \to -\infty$. So $a = \sum_{\nu=-\infty}^{\infty} u_\nu$. Moreover, $\|u_\nu|A_0 + A_1\| \to 0$ as $\nu \to -\infty$ and as $\nu \to \infty$. Besides,

$$\frac{J(\gamma_{\nu-1}, u_\nu)}{\gamma_{\nu-1}} \lesssim \frac{K_p(\gamma_{\nu-2}, a)}{\gamma_{\nu-2}}, \quad \nu \in \mathbb{Z}. \tag{7.11}$$

Indeed,

$$\begin{aligned} \frac{J(\gamma_{\nu-1}, u_\nu)^p}{\gamma^p_{\nu-1}} &\le \gamma^{-p}_{\nu-1}\left(\|u_\nu|A_0\|^p + \gamma^p_{\nu-1}\|u_\nu|A_1\|^p\right) \\ &\le \gamma^{-p}_{\nu-1}\left(\|a_{0,\nu}|A_0\|^p + \|a_{0,\nu-1}|A_0\|^p + \gamma^p_{\nu-1}\|a_{1,\nu}|A_1\|^p + \gamma^p_{\nu-1}\|a_{1,\nu-1}|A_1\|^p\right) \\ &\le 2^p\gamma^{-p}_{\nu-1}\left(K_p(\gamma_{\nu-1}, a)^p + \frac{\gamma^p_{\nu-1}}{\gamma^p_{\nu-2}}K_p(\gamma_{\nu-2}, a)^p\right) \lesssim \frac{K_p(\gamma_{\nu-2}, a)^p}{\gamma^p_{\nu-2}} \end{aligned}$$

where we have used in the last inequality that the function $t^{-1}K(t, a)$ is non-increasing.

For $\nu \in \mathbb{Z}$, put $I_\nu = [\gamma_{\nu-1}, \gamma_\nu)$. If $2^m \in I_\nu$, let w_m be u_ν divided by the number of m such that $2^m \in I_\nu$. That is

$$w_m = \begin{cases} \frac{u_\nu}{2^{-\nu-1}} & \text{if} \quad 2^m \in I_\nu \quad \text{and} \quad \nu < 0, \\ u_\nu & \text{if} \quad 2^m \in I_\nu \quad \text{and} \quad \nu = 0, 1, \\ \frac{u_\nu}{2^{\nu-2}} & \text{if} \quad 2^m \in I_\nu \quad \text{and} \quad \nu > 1. \end{cases}$$

This sequence also satisfies that $(w_m) \subseteq A_0 \cap A_1$ with $a = \sum_{m=-\infty}^{\infty} w_m$ because for some $0 \leq d, f < 1$ we have

$$\begin{aligned}\|a - \sum_{m=N}^{M} w_m | A_0 + A_1\|^p &= \|a - \sum_{\nu=P}^{Q} u_\nu - d u_{P-1} - f u_{Q+1} | A_0 + A_1\|^p \\ &\leq \|a - \sum_{\nu=P}^{Q} u_\nu | A_0 + A_1\|^p + \|u_{P-1}|A_0 + A_1\|^p + \|u_{Q+1}|A_0 + A_1\|^p \to 0\end{aligned}$$

as $N \to -\infty$ and $M \to \infty$. Consequently, using (7.11), we derive if $1 \leq q < \infty$

$$\begin{aligned}\|a|(A_0, A_1)^J_{1,q,\mathbb{A}+1}\| &\leq \Big(\sum_{m=-\infty}^{\infty} 2^{-mq} \ell^{\mathbb{A}q+q}(2^m) J(2^m, w_m)^q \Big)^{1/q} \\ &\sim \Big(\sum_{\nu=-\infty}^{\infty} \sum_{2^m \in I_\nu} 2^{-|\nu| q} \frac{J(2^m, u_\nu)^q}{2^{mq}} \ell^{\mathbb{A}q+q}(2^m) \Big)^{1/q} \lesssim \Big(\sum_{\nu=-\infty}^{\infty} \sum_{2^m \in I_\nu} 2^{-|\nu| q} \frac{J(\gamma_{\nu-1}, u_\nu)^q}{\gamma_{\nu-1}^q} 2^{|\nu|(\mathbb{A}q+q)} \Big)^{1/q} \\ &\lesssim \Big(\sum_{\nu=-\infty}^{\infty} \sum_{2^m \in I_\nu} \frac{K_p(\gamma_{\nu-2}, a)^q}{\gamma_{\nu-2}^q} 2^{|\nu| \mathbb{A}q} \Big)^{1/q} \sim \Big(\sum_{\nu=-\infty}^{\infty} \sum_{2^m \in I_{\nu-2}} \frac{K_p(\gamma_{\nu-2}, a)^q}{\gamma_{\nu-2}^q} 2^{|\nu| \mathbb{A}q} \Big)^{1/q} \\ &\lesssim \Big(\sum_{\nu=-\infty}^{\infty} \sum_{2^m \in I_{\nu-2}} \frac{K_p(2^m, a)^q}{2^{mq}} \ell^{\mathbb{A}q}(2^m) \Big)^{1/q} = \|a|(A_0, A_1)_{1,q,\mathbb{A}}\|.\end{aligned}$$

The case $q = \infty$ can be treated similarly. □

Now we prove that the embedding in Theorem 7.10 is the best possible. We rely on the characteristic functions computed in Section 7.1.

Let $0 < p < 1 < q < \infty$ and $\mathbb{A}, \mathbb{M} \in \mathbb{R}^2$ satisfying (7.4) and (7.5). If (7.7) holds then Proposition 7.2 and Corollary 7.6 with $A = \ell_1$ yield that

$$\mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1} \lesssim \mathrm{v}_{q,\mathbb{A}}(2^k). \tag{7.12}$$

Take any $k \in \mathbb{Z}$, k negative. Using Lemma 7.1 we have that $\mathrm{v}_{q,\mathbb{A}}(2^k) = 2^{-k} \ell^{\alpha_0 + 1/q}(2^k)$ and by Corollary 7.6 we get

$$\mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1} \sim \begin{cases} 2^{-k} \ell^{\mu_0 - 1 + 1/q}(2^k) & \text{if} \quad \mu_0 < 1 - 1/q, \\ 2^{-k} \ell \ell^{-1+1/q}(2^k) & \text{if} \quad \mu_0 = 1 - 1/q, \\ 2^{-k} & \text{if} \quad \mu_0 > 1 - 1/q. \end{cases}$$

Hence, (7.12) is only possible if $\mu_0 \leq \alpha_0 + 1$. Consider now k positive. Then

$$\mathrm{v}_{q,\mathbb{A}}(2^k) \sim \begin{cases} 2^{-k} \ell^{\alpha_\infty + 1/q}(2^k) & \text{if} \quad \alpha_\infty + 1/q > 0, \\ 2^{-k} \ell \ell^{1/q}(2^k) & \text{if} \quad \alpha_\infty + 1/q = 0 \end{cases}$$

and $\mathrm{u}_{q,\mathbb{M},1}(2^k)^{-1} \sim 2^{-k} \ell^{\mu_\infty - 1 + 1/q}(2^k)$. Since $\mu_\infty > 1/p - 1/q > 1 - 1/q$, we derive that if (7.12) holds, then $\mu_\infty \leq \alpha_\infty + 1$ and that there is no $\mathbb{M}$ if $0 \leq \alpha_\infty + 1/q \leq 1/p - 1$. Consequently, the embedding in Theorem 7.10 is optima. Note that these arguments also explain the assumption on α_∞ in the statement of the theorem. The cases $p = 1 < q \leq \infty$, $0 < p < 1$ with $q = \infty$ and $0 < p < 1 = q$ can be treated analogously.

Now we turn our attention to reciprocal embeddings of the type (7.8).

Theorem 7.11. Let $0 < p \le 1$, $0 < p < q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying

$$\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q. \tag{7.13}$$

Then, for any p-Banach couple $\bar{A} = (A_0, A_1)$, we have

$$(A_0, A_1)^J_{1,q,\mathbb{A}+1/p} \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}}.$$

Proof. Take any $a \in (A_0, A_1)^J_{1,q,\mathbb{A}+1/p}$ and let $(u_m) \subseteq A_0 \cap A_1$ such that $a = \sum_{m=-\infty}^{\infty} u_m$ and $\|(J(2^m, u_m))|\ell_q(2^{-m}\ell^{\mathbb{A}+1/p}(2^m))\| \le 2\|a|(A_0, A_1)^J_{1,q,\mathbb{A}+1/p}\|$. Then

$$\begin{aligned}
\|a|(A_0, A_1)_{1,q,\mathbb{A}}\| &\le \Big(\sum_{k=-\infty}^{\infty} \big[2^{-k}\ell^{\mathbb{A}}(2^k)\big(\sum_{m=-\infty}^{\infty} \min\{1, 2^{k-m}\}^p J(2^m, u_m)^p\big)^{1/p}\big]^q \Big)^{1/q} \\
&\sim \|(J(2^m, u_m))|(\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}\| \\
&\sim \Big(\sum_{k=-\infty}^{0} \big[\ell^{\alpha_0}(2^k)\big(\sum_{m=k}^{0} 2^{-mp} J(2^m, u_m)^p\big)^{1/p}\big]^q \Big)^{1/q} \\
&\quad + \Big(\sum_{k=0}^{\infty} \big[\ell^{\alpha_\infty}(2^k)\big(\sum_{m=k}^{\infty} 2^{-mp} J(2^m, u_m)^p\big)^{1/p}\big]^q \Big)^{1/q}
\end{aligned}$$

where we have used Lemma 7.7 in the last equivalence. Suppose now that $q < \infty$. To estimate the first term take any $0 < \varepsilon < -(\alpha_0 + 1/q)$. Using Hölder's inequality, for the interior sum of the first term, we get

$$\begin{aligned}
&\Big(\sum_{m=k}^{0} 2^{-mp} J(2^m, u_m)^p \Big)^{q/p} \\
&\quad \le \Big(\sum_{m=k}^{0} 2^{-mq} J(2^m, u_m)^q \ell^{\alpha_0 q + q/p + \varepsilon q}(2^m) \Big) \times \Big(\sum_{m=k}^{0} \ell^{-qp(\alpha_0 + 1/p + \varepsilon)/(q-p)}(2^m) \Big)^{(q-p)/p} \\
&\quad \lesssim \ell^{-\alpha_0 q - \varepsilon q - 1}(2^k) \Big(\sum_{m=k}^{0} 2^{-mq} J(2^m, u_m)^q \ell^{\alpha_0 q + q/p + \varepsilon q}(2^m) \Big)
\end{aligned}$$

where we have used the condition on ε in the last estimate. Whence, changing the order of summation, we derive

$$\begin{aligned}
&\Big(\sum_{k=-\infty}^{0} \big[\ell^{\alpha_0}(2^k)\big(\sum_{m=k}^{0} 2^{-mp} J(2^m, u_m)^p\big)^{1/p}\big]^q \Big)^{1/q} \\
&\quad \le \Big(\sum_{k=-\infty}^{0} \ell^{-\varepsilon q - 1}(2^k)\big(\sum_{m=k}^{0} 2^{-mq} J(2^m, u_m)^q \ell^{\alpha_0 q + q/p + \varepsilon q}(2^m)\big) \Big)^{1/q} \\
&\quad = \Big(\sum_{m=-\infty}^{0} 2^{-mq} J(2^m, u_m)^q \ell^{\alpha_0 q + q/p + \varepsilon q}(2^m) \sum_{k=-\infty}^{m} \ell^{-\varepsilon q - 1}(2^k) \Big)^{1/q} \\
&\quad \sim \Big(\sum_{m=-\infty}^{0} 2^{-mq} J(2^m, u_m)^q \ell^{\alpha_0 q + q/p}(2^m) \Big)^{1/q} \\
&\quad \lesssim \|a|(A_0, A_1)^J_{1,q,\mathbb{A}+1/p}\|.
\end{aligned}$$

For the second term, we choose now any $0 < \varepsilon < \alpha_\infty + 1/q$ and we proceed similarly. We get

$$\Big(\sum_{m=k}^{\infty} 2^{-mp} J(2^m, u_m)^p \Big)^{q/p} \leq \Big(\sum_{m=k}^{\infty} 2^{-mq} J(2^m, u_m)^q \ell^{\alpha_\infty q + q/p - \varepsilon q}(2^m) \Big)$$
$$\times \Big(\sum_{m=k}^{\infty} \ell^{-pq(\alpha_\infty + 1/p - \varepsilon)/(q-p)}(2^m) \Big)^{(q-p)/p}$$
$$\sim \ell^{-\alpha_\infty q - 1 + \varepsilon q}(2^k) \Big(\sum_{m=k}^{\infty} 2^{-mq} J(2^m, u_m)^q \ell^{\alpha_\infty q + q/p - \varepsilon q}(2^m) \Big).$$

Therefore

$$\Big(\sum_{k=0}^{\infty} \big[\ell^{\alpha_\infty}(2^k) \big(\sum_{m=k}^{\infty} 2^{-mp} J(2^m, u_m)^p \big)^{1/p} \big]^q \Big)^{1/q}$$
$$\lesssim \Big(\sum_{k=0}^{\infty} \ell^{-1+\varepsilon q}(2^k) \big(\sum_{m=k}^{\infty} 2^{-mq} J(2^m, u_m)^q \ell^{\alpha_\infty q + q/p - \varepsilon q}(2^m) \big) \Big)^{1/q}$$
$$= \Big(\sum_{m=0}^{\infty} 2^{-mq} J(2^m, u_m)^q \ell^{\alpha_\infty q + q/p - \varepsilon q}(2^m) \Big(\sum_{k=0}^{m} \ell^{-1+\varepsilon q}(2^k) \Big) \Big)^{1/q}$$
$$\lesssim \Big(\sum_{m=0}^{\infty} 2^{-mq} J(2^m, u_m)^q \ell^{\alpha_\infty q + q/p}(2^m) \Big)^{1/q}$$
$$\lesssim \|a|(A_0, A_1)^J_{1,q,\mathbb{A}+1/p}\|.$$

Consequently,

$$(A_0, A_1)^J_{1,q,\mathbb{A}+1/p} \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}}.$$

The case $q = \infty$ can be treated analogously. □

Now we aim to show that the embedding of Theorem 7.11 is the best possible. For this purpose we need some preparation.

Lemma 7.12. Let $0 < p \leq 1$, $0 < q \leq \infty$ and take any $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ with

$$\begin{cases} \beta_\infty \geq 0 & \text{if} \quad 0 < q \leq p, \\ \beta_\infty > \frac{1}{p} - \frac{1}{q} & \text{if} \quad 0 < p < q \leq \infty. \end{cases} \tag{7.14}$$

Then

$$\ell_q(2^{-m} \ell^{\mathbb{B}}(2^m)) \hookrightarrow (\ell_p, \ell_p(2^{-m}))^J_{1,q,\mathbb{B}}.$$

Proof. Take any $x = (x_m) \in \ell_q(2^{-m} \ell^{\mathbb{B}}(2^m))$ and $e_k = (\delta^k_m)$, $k \in \mathbb{Z}$. Then

$$J(2^k, x_k e_k; \ell_p, \ell_p(2^{-m})) = |x_k| \quad \text{and} \quad x = \sum_{m=-\infty}^{\infty} x_m e_m.$$

Whence

$$\|x|(\ell_p, \ell_p(2^{-m}))^J_{1,q,\mathbb{B}}\| \leq \|(J(2^k, x_k e_k; \ell_p, \ell_p(2^{-m})))|\ell_q(2^{-m} \ell^{\mathbb{B}}(2^m))\| = \|x|\ell_q(2^{-m} \ell^{\mathbb{B}}(2^m))\|.$$

□

Proposition 7.13. Let $0 < p \le 1$, $0 < q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ satisfying (7.1) and (7.14). Then the embedding

$$\ell_q(2^{-m}\ell^{\mathbb{B}}(2^m)) \hookrightarrow (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}$$

is a necessary and sufficient condition for $(A_0, A_1)^J_{1,q,\mathbb{B}} \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}}$ for any p-Banach couple (A_0, A_1).

Proof. The condition is necessary by Lemma 7.12. Let us check that the condition is sufficient. Take any p-Banach couple (A_0, A_1). For any $a \in (A_0, A_1)^J_{1,q,\mathbb{B}}$ we can find a representation $a = \sum_{m=-\infty}^{\infty} u_m$ with $(u_m) \subseteq A_0 \cap A_1$ satisfying that

$$\|(J(2^m, u_m))|\ell_q(2^{-m}\ell^{\mathbb{B}}(2^m))\| \le 2\|a|(A_0, A_1)^J_{1,q,\mathbb{B}}\|.$$

Since

$$\begin{aligned} K(2^k, a) \le K_p(2^k, a) &\le \Big(\sum_{m=-\infty}^{\infty} K_p(2^k, u_m)^p \Big)^{1/p} \\ &\le \Big(\sum_{m=-\infty}^{\infty} \min\{1, 2^{k-m}\}^p J(2^m, u_m)^p \Big)^{1/p}, \end{aligned}$$

we obtain

$$\begin{aligned} \|a|(A_0, A_1)_{1,q,\mathbb{A}}\| &\le \|(K(2^k, a))|\ell_q(2^{-k}\ell^{\mathbb{A}}(2^k))\| \\ &\le \Big\| \Big(\sum_{m=-\infty}^{\infty} \min\{1, 2^{k-m}\}^p J(2^m, u_m)^p \Big)^{1/p} |\ell_q(2^{-k}\ell^{\mathbb{A}}(2^k)) \Big\| \\ &\sim \|(K(2^k, (J(2^m, u_m)); \ell_p, \ell_p(2^{-m}))|\ell_q(2^{-k}\ell^{\mathbb{A}}(2^k))\| \\ &= \|(J(2^m, u_m))|(\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}\|. \end{aligned}$$

Now using the condition we derive that

$$\begin{aligned} \|a|(A_0, A_1)_{1,q,\mathbb{A}}\| &\lesssim \|(J(2^m, u_m))|\ell_q(2^{-m}\ell^{\mathbb{B}}(2^m))\| \\ &\le 2\|a|(A_0, A_1)^J_{1,q,\mathbb{B}}\|. \end{aligned}$$

□

The next result refers to the operators

$$\begin{aligned} H_1 x &= \Big(\sum_{n=1}^{k} (1 + \log 2^k)^{\alpha_0 p} (1 + \log 2^n)^{-\beta_0 p} x_n \Big)_{k \in \mathbb{N}}, \\ H_2 x &= \Big(\sum_{n=k}^{\infty} (1 + \log 2^k)^{\alpha_\infty p} (1 + \log 2^n)^{-\beta_\infty p} x_n \Big)_{k \in \mathbb{N}}. \end{aligned}$$

Here $x = (x_n)_{n \in \mathbb{N}}$. We consider H_1 and H_2 acting on the space $\ell_r(\mathbb{N})$ of r-summable sequences with $\mathbb{N}$ as index set.

Lemma 7.14. Let $0 < p \le 1$, $0 < q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ satisfying (7.1) and (7.14). Assume also that

$$\ell_q(2^{-m}\ell^{\mathbb{B}}(2^m)) \hookrightarrow (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}.$$

Then H_1 and H_2 are continuous on $\ell_{q/p}(\mathbb{N})$.

Proof. Take any $x = (x_n) \in \ell_{q/p}(\mathbb{N})$. For $k \in \mathbb{Z}$, let

$$y_k = \begin{cases} |x_{-k}|^{1/p} 2^k (1 - \log 2^k)^{-\beta_0} & \text{if} \quad k \leq -1, \\ 0 & \text{if} \quad k \geq 0, \end{cases}$$

and put $y = (y_k)_{k\in\mathbb{Z}}$. Then $\|y|\ell_q(2^{-k}\ell^{\mathbb{B}}(2^k))\| = \|x|\ell_{q/p}(\mathbb{N})\|^{1/p} < \infty$. By the assumption and applying Lemma 7.7, we obtain

$$\begin{aligned}
\|x|\ell_{q/p}(\mathbb{N})\|^{1/p} = \|y|\ell_q(2^{-k}\ell^{\mathbb{B}}(2^k))\| &\gtrsim \|y|(\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}\| \\
&\sim \Big(\sum_{k=-\infty}^{0} \big[\ell^{\alpha_0}(2^k)\big(\sum_{m=k}^{0} 2^{-mp}|y_m|^p\big)^{1/p}\big]^q \Big)^{1/q} \\
&\quad + \Big(\sum_{k=0}^{\infty} \big[\ell^{\alpha_\infty}(2^k)\big(\sum_{m=k}^{\infty} 2^{-mp}|y_m|^p\big)^{1/p}\big]^q \Big)^{1/q} \\
&= \Big(\sum_{k=-\infty}^{-1} \big[(1-\log 2^k)^{\alpha_0}\big(\sum_{m=k}^{-1} (1-\log 2^m)^{-\beta_0 p}|x_{-m}|\big)^{1/p}\big]^q \Big)^{1/q} \\
&= \Big(\sum_{k=1}^{\infty} \big[(1+\log 2^k)^{\alpha_0}\big(\sum_{m=1}^{k} (1+\log 2^m)^{-\beta_0 p}|x_m|\big)^{1/p}\big]^q \Big)^{1/q} \\
&\geq \|H_1 x|\ell_{q/p}(\mathbb{N})\|^{1/p}.
\end{aligned}$$

Now we return our attention to H_2. Let $z = (z_k)_{k\in\mathbb{Z}}$ where

$$z_k = \begin{cases} 0 & \text{if} \quad k \leq 0, \\ |x_k|^{1/p} 2^k (1 + \log 2^k)^{-\beta_\infty} & \text{if} \quad k \geq 1. \end{cases}$$

Using again the assumption and Lemma 7.7, we derive

$$\begin{aligned}
\|x|\ell_{q/p}(\mathbb{N})\|^{1/p} = \|z|\ell_q(2^{-k}\ell^{\mathbb{B}}(2^k))\| &\gtrsim \|z|(\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}\| \\
&\gtrsim \Big(\sum_{k=1}^{\infty} \big[(1+\log 2^k)^{\alpha_\infty}\big(\sum_{m=k}^{\infty} 2^{-mp}|z_m|^p\big)^{1/p}\big]^q \Big)^{1/q} \\
&= \Big(\sum_{k=1}^{\infty} \big[(1+\log 2^k)^{\alpha_\infty}\big(\sum_{m=k}^{\infty} (1+\log 2^m)^{-\beta_\infty p}|x_m|\big)^{1/p}\big]^q \Big)^{1/q} \\
&\geq \|H_2 x|\ell_{q/p}(\mathbb{N})\|^{1/p}.
\end{aligned}$$

□

We shall also need some results on matrix transformations of $\ell_r(\mathbb{N})$-spaces established by G. Bennett [7, 8]. Let (a_n), (b_n) be sequences of non-negative numbers and let $\mathbf{M} = (a_{nk})$ with

$$a_{nk} = \begin{cases} a_n b_k & \text{if} \quad 1 \leq k \leq n, \\ 0 & \text{if} \quad k > n. \end{cases}$$

Put $Tx = \left(\sum_{k=1}^{\infty} a_{nk} x_k \right)_{n\in\mathbb{N}} = \left(a_n \sum_{k=1}^{\infty} b_k x_k \right)_{n\in\mathbb{N}}$ for the operator defined by $\mathbf{M}$. Let $1 \leq r \leq \infty$ and $1/r + 1/r' = 1$. According to [7, Theorem 2] and [8, Theorem 1], the operator $T : \ell_r(\mathbb{N}) \to$

$\ell_r(\mathbb{N})$ is continuous if, and only if,

$$\begin{cases} \sup_{N\in\mathbb{N}} \left(\sum_{n=N}^{\infty} a_n^r\right)^{1/r} \left(\sum_{k=1}^{N} b_k^{r'}\right)^{1/r'} < \infty & \text{for} \quad 1 < r < \infty, \\ \sup_{N\in\mathbb{N}} \left(\sum_{n=N}^{\infty} a_n^r\right)^{1/r} b_N < \infty & \text{for} \quad r = 1, \\ \sup_{N\in\mathbb{N}} a_N \sum_{k=1}^{N} b_k < \infty & \text{for} \quad r = \infty. \end{cases} \tag{7.15}$$

Put $\widehat{\mathbf{M}} = (\widehat{a}_{nk})$ with

$$\widehat{a}_{nk} = \begin{cases} 0 & \text{if} \quad k < n, \\ a_k b_n & \text{if} \quad k \geq n, \end{cases}$$

and let $\widehat{T}x = \left(\sum_{k=1}^{\infty} \widehat{a}_{nk} x_k\right)_{n\in\mathbb{N}} = \left(b_n \sum_{k=n}^{\infty} a_k x_k\right)_{n\in\mathbb{N}}$. Since $\widehat{T}$ is the adjoint of the operator T, it follows from (7.15) that $\widehat{T} : \ell_r(\mathbb{N}) \to \ell_r(\mathbb{N})$ is continuous if, and only if,

$$\sup_{N\in\mathbb{N}} \left(\sum_{n=N}^{\infty} a_n^{r'}\right)^{1/r'} \left(\sum_{k=1}^{N} b_k^{r}\right)^{1/r} < \infty \quad \text{for} \quad 1 < r < \infty. \tag{7.16}$$

Furthermore, a direct computation shows that $\widehat{T} : \ell_\infty(\mathbb{N}) \to \ell_\infty(\mathbb{N})$ is continuous if, and only if,

$$\sup_{N\in\mathbb{N}} b_N \sum_{k=N}^{\infty} a_k < \infty. \tag{7.17}$$

Observe that H_1 is the transformation defined by the matrix $\mathbf{M} = (a_{nk})$ given by the sequences

$$a_n = (1 + \log 2^n)^{\alpha_0 p} \quad \text{and} \quad b_k = (1 + \log 2^k)^{-\beta_0 p}$$

while H_2 is the transformation defined by $\widehat{\mathbf{M}}$ with

$$a_k = (1 + \log 2^k)^{-\beta_\infty p} \quad \text{and} \quad b_n = (1 + \log 2^n)^{\alpha_\infty p}.$$

Now we can show that the embedding of Theorem 7.11 is the best possible.

Proposition 7.15. Let $0 < p \leq 1$, $0 < p < q \leq \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ satisfying (7.13) and (7.14) and consider the p-Banach couple $(\ell_p, \ell_p(2^{-m}))$. If $(\ell_p, \ell_p(2^{-m}))^J_{1,q,\mathbb{B}} \hookrightarrow (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}$ then $\beta_0 \geq \alpha_0 + 1/p$ and $\beta_\infty \geq \alpha_\infty + 1/p$.

Proof. By the assumptions on parameters and Lemma 7.12 we have that

$$\ell_q(2^{-m}\ell^{\mathbb{B}}(2^m)) \hookrightarrow (\ell_p, \ell_p(2^{-m}))_{1,q,\mathbb{A}}.$$

Hence, according to Lemma 7.14, operators H_1 and H_2 are continuous on $\ell_{q/p}(\mathbb{N})$. If $q < \infty$, applying (7.15) to H_1 we obtain that

$$\sup_{N\in\mathbb{N}} \left(\sum_{n=N}^{\infty} (1 + \log 2^n)^{\alpha_0 q}\right)^{p/q} \left(\sum_{k=1}^{N} (1 + \log 2^k)^{-\beta_0 \frac{pq}{q-p}}\right)^{\frac{q-p}{p}} < \infty.$$

Using that $\alpha_0 + 1/q < 0$, it follows that

$$\left(\sum_{k=1}^{N} (1 + \log 2^k)^{-\beta_0 \frac{pq}{q-p}}\right)^{(q-p)/p} \lesssim (1 + \log 2^n)^{-\alpha_0 - 1/q}.$$

Whence

$$\left.\begin{cases}(1+\log 2^N)^{-\beta_0+1/p-1/q} & \text{if} \quad \beta_0 < 1/p-1/q \\ (1+\log(1+\log 2^N)) & \text{if} \quad \beta_0 = 1/p-1/q \\ 1 & \text{if} \quad \beta_0 > 1/p-1/q.\end{cases}\right\} \lesssim (1+\log 2^n)^{-\alpha_0-1/q}.$$

Since $-\alpha_0 - 1/q > 0$, it follows that

$$\beta_0 \geq 1/p - 1/q, \text{ or } \beta_0 < 1/p - 1/q \text{ and } -\beta_0 + 1/p - 1/q \leq -\alpha_0 - 1/q.$$

Consequently, in both cases $\beta_0 \geq \alpha_0 + 1/p$.

On the other hand, since H_2 is also continuous on $\ell_{q/p}(\mathbb{N})$, according to (7.16), we obtain

$$\sup_{N\in\mathbb{N}} \Big(\sum_{n=N}^{\infty} (1+\log 2^n)^{-\beta_\infty \frac{pq}{q-p}} \Big)^{(q-p)/q} \Big(\sum_{k=1}^{N} (1+\log 2^k)^{\alpha_\infty q} \Big)^{p/q} < \infty.$$

Having in mind that $\beta_\infty > 1/p - 1/q$ and $\alpha_\infty + 1/q > 0$, we get

$$(1+\log 2^N)^{\alpha_\infty+1/q} \lesssim (1+\log 2^N)^{\beta_\infty-1/p+1/q}.$$

This yields that $\beta_\infty \geq \alpha_\infty + 1/p$.

The proof for $q = \infty$ is similar but using now (7.17). □

The next result complements Theorem 7.11. It corresponds to the limit case $\alpha_\infty + 1/q = 0$. One can establish it by using similar arguments to those of Theorem 7.11.

Theorem 7.16. Let $0 < p \leq 1$, $0 < p < q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying

$$\alpha_0 + 1/q < 0 = \alpha_\infty + 1/q.$$

Then, for any p-Banach couple $\bar{A} = (A_0, A_1)$, we have

$$(A_0, A_1)^J_{1,q,\mathbb{A}+1/p,(0,1/p)} \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}}.$$

Using the previous ideas based on matrix transformations of $\ell_r(\mathbb{N})$-spaces, one can also show that the embedding in Theorem 7.16 is the best possible.

Writing down Theorems 7.10 and 7.11 for $1 \leq q \leq \infty$ and $\bar{A}$ a Banach couple, so $p = 1$, we obtain the following equivalence result of Cobos and Segurado [48, Theorem 3.5] already mentioned in Chapter 3 (see (3.8)).

Corollary 7.17. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $1 \leq q \leq \infty$ such that $\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q$. Then, for any Banach couple (A_0, A_1), we have with equivalence of norms

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0, A_1)^J_{1,q,\mathbb{A}+1}.$$

Finally, we study the case $0 < q \leq p \leq 1$ where we have also equality.

Theorem 7.18. Let $0 < q \leq p \leq 1$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying $\alpha_0 + 1/q < 0 \leq \alpha_\infty + 1/q$. Then, for any p-Banach couple $\bar{A} = (A_0, A_1)$, we have with equivalence of quasi-norms

i) $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^{\sim}, A_1^{\sim})^J_{1,q,\mathbb{A}+1/q}$ if $\alpha_\infty + 1/q > 0$,

ii) $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q,(0,1/q)}$ if $\alpha_\infty + 1/q = 0$.

Proof. Recall that $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}}$. From Theorem 6.1, using that $q \leq p$ and Lemma 7.8 we get that

$$(A_0, A_1)_{1,q,\mathbb{A}} \hookleftarrow \begin{cases} (A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q} & \text{if} \quad \alpha_\infty + 1/q > 0, \\ (A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q,(0,1/q)} & \text{if} \quad \alpha_\infty + 1/q = 0. \end{cases}$$

In order to check the converse embedding, consider the function $\mathrm{v}_{q,\mathbb{A}}(\cdot)$. By Lemma 7.1, we have

$$\mathrm{v}_{q,\mathbb{A}}(2^k) \sim \begin{cases} 2^{-k}\ell^{\mathbb{A}+1/q}(2^k) & \text{if} \quad \alpha_\infty + 1/q > 0, \\ 2^{-k}\ell^{\mathbb{A}+1/q}(2^k)\ell\ell^{(0,1/q)}(2^k) & \text{if} \quad \alpha_\infty + 1/q = 0. \end{cases} \tag{7.18}$$

Take any $a \in (A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}}$. Since (7.2) holds, we know that $(A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}} \subseteq (A_0^\sim + A_1^\sim)^\circ$. Using [95, Theorem 3.2], we can find $(u_m) \subseteq A_0^\sim \cap A_1^\sim$ with $a = \sum_{m=-\infty}^{\infty} u_m$ in $A_0^\sim + A_1^\sim$ and

$$\Big(\sum_{m=-\infty}^{\infty} [\min\{1, 2^{k-m}\} J(2^m, u_m)]^q \Big)^{1/q} \leq cK(2^k, a), \quad k \in \mathbb{Z}. \tag{7.19}$$

Therefore, if $\alpha_\infty + 1/q > 0$, according to (7.18) and (7.19), we have

$$\begin{aligned} \|a|(A_0^\sim, A_1^\sim)^J_{1,q,\mathbb{A}+1/q}\| &\leq \|(J(2^m, u_m))|\ell_q(2^{-m}\ell^{\mathbb{A}+1/q}(2^m))\| \\ &\sim \Big(\sum_{m=-\infty}^{\infty} J(2^m, u_m)^q \sum_{k=-\infty}^{\infty} \big[\min\{1, 2^{k-m}\} 2^{-k}\ell^{\mathbb{A}}(2^k)\big]^q \Big)^{1/q} \\ &\lesssim \Big(\sum_{k=-\infty}^{\infty} 2^{-kq}\ell^{\mathbb{A}q}(2^k) K(2^k, a)^q \Big)^{1/q} \\ &= \|a|(A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}}\|. \end{aligned}$$

The case $\alpha_\infty + 1/q = 0$ can be treated analogously. □

Writing down Theorem 7.18 for $\bar{A}$ a Banach couple, we recover some cases from Theorem 3.5.

We close this chapter with two diagrams that collect the identities and optimal embeddings obtained for $0 < p < 1$ and $0 < q < \infty$.

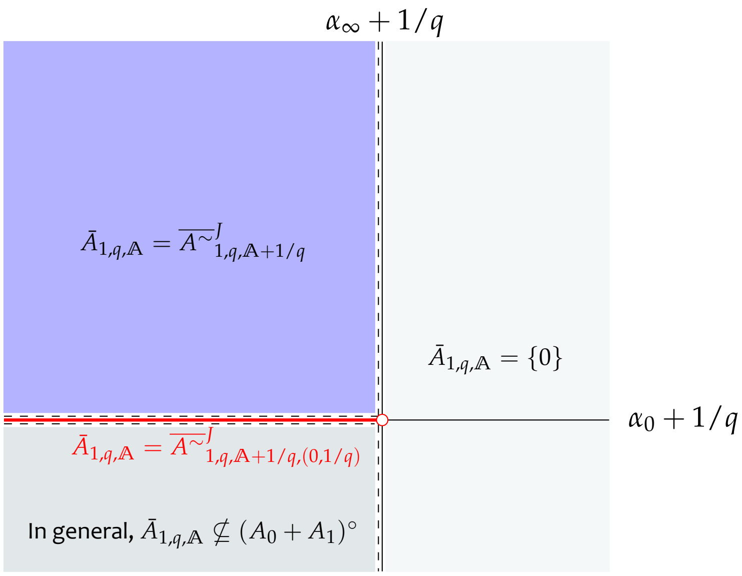


FIGURE 7.1: Case $0 < q \leq p < 1$

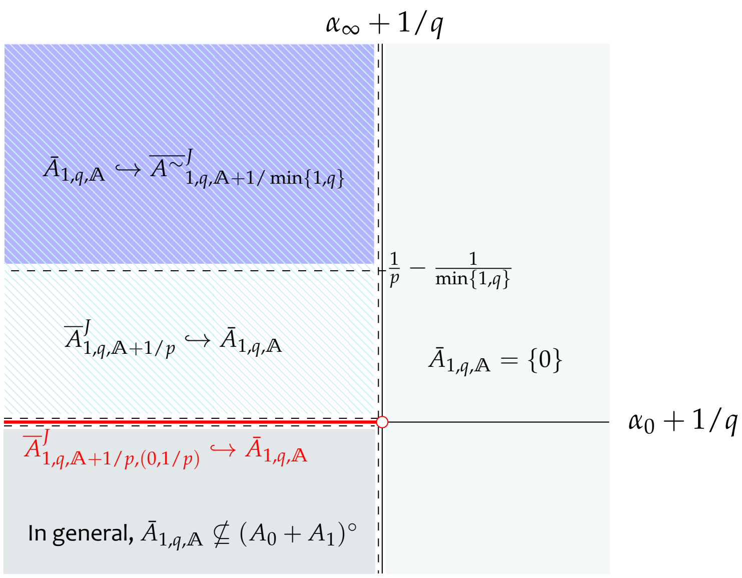


FIGURE 7.2: Case $0 < p < 1, 0 < p < q < \infty$

Chapter 8

Interpolation of the measure of non-compactness of bilinear operators

As we saw in Theorem 2.3, continuous bilinear operators can be interpolated by the real method. Besides boundedness, another useful property that a bilinear operator may have is compactness. Recently, it was shown that compact bilinear operators occur rather naturally in harmonic analysis (see, for example, the papers by Bényi and Torres [10], Bényi and Oh [9] and Hu [79]). This fact motivated the research on interpolation properties of compact bilinear operators, a problem already considered by Calderón [26] for the complex method. For the real method, the question has been investigated more recently by Fernández and Silva [62], Fernández-Cabrera and Martínez [64, 65] and Cobos, Fernández-Cabrera and Martínez [40], among other authors. One of the main results corresponds to [40, Theorem 4.9] and reads as follows (we use notation introduced in page 29):

Theorem. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach spaces and let $\bar{E} = (E_0, E_1)$ be an r-normed quasi-Banach couple. Let $0 < \theta < 1$, $0 < q_0, q_1 \leq \infty$ and let $0 < q \leq \infty$ satisfying

$$\frac{1}{q} = \begin{cases} \frac{1}{q_0} + \frac{1}{q_1} - \frac{1}{r} & \text{if} \quad q_0, q_1 \geq r, \\ \frac{1}{\max\{q_0, q_1\}} & \text{if} \quad q_0 < r \quad \text{or} \quad q_1 < r. \end{cases}$$

If $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ and $T : A_j \times B_j \longrightarrow E_j$ is compact for $j = 0$ or $j = 1$, then

$$T : (A_0, A_1)_{\theta,q_0} \times (B_0, B_1)_{\theta,q_1} \longrightarrow (E_0, E_1)_{\theta,q} \text{ is also compact.}$$

In fact, this type of results can be extended to the generalized real method that we introduced in Section 2.1.2. The next natural step was to inquire about quantitative results involving the measure of non-compactness, similar to the one introduced in Chapter 6 but for bilinear operators. Mastyło and Silva [93] proved, among other things, that

$$\beta(T : \bar{A}_{\theta,q_0} \times \bar{B}_{\theta,q_1} \to \bar{E}_{\theta,q}) \leq C\beta(T : A_0 \times B_0 \to E_0)^{1-\theta}\beta(T : A_1 \times B_1 \to E_1)^{\theta} \tag{8.1}$$

provided that $\bar{A}$, $\bar{B}$ and $\bar{E}$ are Banach couples, $1 \leq q_0, q_1 < \infty$, $1 < q < \infty$ and $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1} - 1$. Their proof is based on duality and therefore it cannot be extended to the quasi-Banach setting. The aim of this chapter is to study the interpolation by the general real method of the measure of non-compactness of bilinear operators acting between quasi-Banach couples. In particular, we want to establish a quantitative version of the theorem above and an extension of (8.1) to quasi-Banach couples. The techniques used here are similar to the ones of Chapter 6 but now for bilinear operators and the general real method.

The results in this chapter form the paper [14].

8.1 Some interpolation properties of bilinear operators

For $k \in \mathbb{Z}$, we use here again the *shift operator* τ_k given by $\tau_k \xi = (\xi_{m+k})_{m\in\mathbb{Z}}$ for $\xi = (\xi_m)_{m\in\mathbb{Z}}$. Let Γ be a quasi-Banach sequence lattice verifying that τ_k is continuous from Γ to Γ for all $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} 2^{-n}\|\tau_n|\mathcal{L}(\Gamma,\Gamma)\| = 0 = \lim_{n\to\infty}\|\tau_{-n}|\mathcal{L}(\Gamma,\Gamma)\|. \tag{8.2}$$

We put

$$f_\Gamma(t) = \|\tau_{[\log_2 t]}|\mathcal{L}(\Gamma,\Gamma)\|, \quad t > 0,$$

where the logarithm is taken in base 2 and $[\cdot]$ is the greatest integer function. Let

$$\begin{aligned} M_1 &= \max\{1, \|\tau_1|\mathcal{L}(\Gamma,\Gamma)\|\}, \\ M_2 &= \sup\{f_\Gamma(t) : 0 < t \le 1\} = \sup\{\|\tau_{-n}|\mathcal{L}(\Gamma,\Gamma)\| : n \ge 0\}, \\ M_3 &= \sup\{f_\Gamma(t)/t : 1 \le t < \infty\} = \sup\{2^{-n}\|\tau_n|\mathcal{L}(\Gamma,\Gamma)\| : n \ge 0\}. \end{aligned}$$

The following properties hold for the function f_Γ:

$$\lim_{t\to 0} f_\Gamma(t) = \lim_{t\to\infty} f_\Gamma(t)/t = 0. \tag{8.3}$$

For any $0 < s, t < \infty$, $f_\Gamma(st) \le M_1 f_\Gamma(s) f_\Gamma(t)$. Hence, if $0 < s < t < \infty$ then

$$f_\Gamma(s) \le M_1 M_2 f_\Gamma(t) \quad \text{and} \quad f_\Gamma(t)/t \le M_1 M_3 f_\Gamma(s)/s. \tag{8.4}$$

If $\xi = (\xi_m)_{m\in\mathbb{Z}}$ and $\eta = (\eta_m)_{m\in\mathbb{Z}}$ are non-negative scalar sequences, we write

$$\xi \star \eta = \left(\sum_{k=-\infty}^{\infty} \xi_k \eta_{m-k}\right)_{m\in\mathbb{Z}}$$

for their convolution. If $r > 0$, we put $\xi^r = (\xi_m^r)_{m\in\mathbb{Z}}$.

Remember now the definition of K-non trivial and (p, J)-non trivial quasi-Banach sequence lattices Γ (see (2.21) and (2.22)) and the result proved in Theorem 2.7 showing that if

$$\Gamma \text{ is } K\text{-non trivial, } (p, J)\text{-non trivial and } \Gamma \hookrightarrow (\ell_p, \ell_p(2^{-m}))_{\Gamma;K}, \tag{8.5}$$

then $(A_0, A_1)_{\Gamma;K} = (A_0, A_1)_{\Gamma;J}$ for every p-normed quasi-Banach couple (A_0, A_1). In this case, sometimes we just write $(A_0, A_1)_\Gamma$ for the interpolated space.

Recall also that if $\bar{A}$, $\bar{B}$ and $\bar{E}$ are quasi-Banach couples, we put $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ if $T \in \mathcal{B}((A_0 + A_1) \times (B_0 + B_1), E_0 + E_1)$ and the restriction of T to $A_j \times B_j$ belongs to $\mathcal{B}(A_j \times B_j, E_j)$, for $j = 0$ and $j = 1$.

The following result shows the interpolation properties of continuous bilinear operators by the general real method (see [40, Theorem 3.1]). For the sake of completeness, we include the proof.

Theorem 8.1. Let $\bar{A} = (A_0, A_1)$ be a quasi-Banach couple, let $\bar{B} = (B_0, B_1)$ be a p-normed quasi-Banach couple and let $\bar{E} = (E_0, E_1)$ be an r-normed quasi-Banach couple $(0 < p, r \le 1)$. Assume that Γ_0 and Γ_2 are K-non-trivial quasi-Banach sequence lattices and Γ_1 is a (p, J)-non-trivial quasi-Banach sequence lattice satisfying (8.2). Assume that, in addition, there is a constant $M > 0$ such that for all non-negative scalar sequences $\xi \in \Gamma_0$ and $\eta \in \Gamma_1$ we have

$$\|(\xi^r \star \eta^r)^{1/r}|\Gamma_2\| \le M\|\xi|\Gamma_0\|\|\eta|\Gamma_1\|. \tag{8.6}$$

Then, for each $T : \bar{A} \times \bar{B} \to \bar{E}$ the restriction of T to $\bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;J}$ defines a continuous bilinear operator $T : \bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;J} \to \bar{E}_{\Gamma_2;K}$ with

$$\|T|\mathcal{B}(\bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;J}, \bar{E}_{\Gamma_2;K})\| \leq \begin{cases} 0 & \text{if} \quad \|T\|_j = 0 \quad \text{for} \quad j = 0 \text{ or } j = 1, \\ C\|T\|_0 f_{\Gamma_1}(\|T\|_1/\|T\|_0) & \text{otherwise.} \end{cases}$$

Here $\|T\|_j = \|T|\mathcal{B}(A_j \times B_j, E_j)\|$ and C is a constant independent of T.

Proof. Let $\sigma_j > \|T\|_j$, $j = 0, 1$ and choose $n \in \mathbb{Z}$ such that $2^n \leq \sigma_1/\sigma_0 < 2^{n+1}$. Take any $a \in \bar{A}_{\Gamma_0;K}$, any $u \in B_0 \cap B_1$ and $m, k \in \mathbb{Z}$. If $a = a_0 + a_1$ with $a_j \in A_j$, we get

$$\begin{aligned} K(2^m, T(a,u)) &\leq \|T(a_0,u)|E_0\| + 2^m\|T(a_1,u)|E_1\| \\ &\leq \sigma_0\|a_0|A_0\|\|u|B_0\| + 2^{m-k-n}2^{k+n}\sigma_1\|a_1|A_1\|\|u|B_1\| \\ &\leq \max\{\sigma_0, 2^{-n}\sigma_1\}(\|a_0|A_0\| + 2^{m-k}\|a_1|A_1\|)J(2^{k+n}, u). \end{aligned}$$

Taking the infimum among all possible decompositions $a = a_0 + a_1$ with $a_j \in A_j$ and noting that $2^{-n}\sigma_1 < 2\sigma_0$, we get

$$K(2^m, T(a,u)) \leq 2\sigma_0 K(2^{m-k}, a) J(2^{k+n}, u). \tag{8.7}$$

Take $b \in \bar{B}_{\Gamma_1;J}$ and let $b = \sum_{k=-\infty}^{\infty} u_k$ any J-representation of b. Then we also have that $b = \sum_{k=-\infty}^{\infty} u_{k+n}$. Moreover, since $K_r(t, \cdot; E_0, E_1)$ is an r-norm on $E_0 + E_1$ equivalent to $K(t, \cdot; E_0, E_1)$, we obtain that $K(2^m, T(a,b)) \leq C_1\Big(\sum_{k=-\infty}^{\infty} K(2^m, T(a, u_{k+n}))^r\Big)^{1/r}$. Using (8.6) and (8.7), we obtain that

$$\begin{aligned} \|T(a,b)|\bar{E}_{\Gamma_2;K}\| &\leq C_1\Big\|\Big(\sum_{k=-\infty}^{\infty} K(2^m, T(a, u_{k+n}))^r\Big)^{1/r}|\Gamma_2\Big\| \\ &\leq 2C_1\sigma_0\Big\|\Big(\sum_{k=-\infty}^{\infty} K(2^{m-k}, a)^r J(2^{k+n}, u_{k+n})^r\Big)^{1/r}|\Gamma_2\Big\| \\ &= 2C_1\sigma_0\Big\|\Big(\sum_{j=-\infty}^{\infty} K(2^j, a)^r J(2^{m+n-j}, u_{m+n-j})^r\Big)^{1/r}|\Gamma_2\Big\| \\ &\leq 2C_1\sigma_0 M\|(K(2^m, a))|\Gamma_0\|\|(J(2^{m+n}, u_{m+n}))|\Gamma_1\| \\ &\leq 2C_1 M\sigma_0\|\tau_n|\mathcal{L}(\Gamma_1, \Gamma_1)\|\|a|\bar{A}_{\Gamma_0;K}\|\|(J(2^m, u_m))|\Gamma_1\|. \end{aligned}$$

Noting that $\|\tau_n|\mathcal{L}(\Gamma_1, \Gamma_1)\| = f_{\Gamma_1}(\sigma_1/\sigma_0)$, we get that

$$\|T|\mathcal{B}(\bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;J}, \bar{E}_{\Gamma_2;K})\| \leq C\sigma_0 f_{\Gamma_1}(\sigma_1/\sigma_0).$$

Finally, if $\|T\|_j = 0$ for $j = 0$ or $j = 1$, letting $\sigma_j \to 0$ and using (8.3) we have that $\|T|\mathcal{B}(\bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;J}, \bar{E}_{\Gamma_2;K})\| = 0$. If $\|T\|_j \neq 0$ for $j = 0, 1$, taking $\sigma_j = (1+\varepsilon)\|T\|_j$ and letting $\varepsilon \to 0$, we conclude $\|T|\mathcal{B}(\bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;J}, \bar{E}_{\Gamma_2;K})\| \leq C\|T\|_0 f_{\Gamma_1}(\|T\|_1/\|T\|_0)$. ☐

Example 8.2. Let $\rho : (0,\infty) \longrightarrow (0,\infty)$ be a function parameter (see Section 2.1.2). Then we already know that for any $0 < q \leq \infty$ the quasi-Banach sequence lattice $\Gamma = \ell_q(1/\rho(2^m))$ is K-non trivial and (p, J)-non trivial for every $0 < p \leq 1$. Furthermore, for every $k \in \mathbb{Z}$

$$\|\tau_k|\mathcal{L}(\ell_q(1/\rho(2^m)), \ell_q(1/\rho(2^m)))\| \leq s_\rho(2^k),$$

where $s_\rho(t) = \sup\{\rho(ts)/\rho(s) : s > 0\}$. It follows from the definition of function parameter that τ_k satisfies (8.2). The above inequality allows to replace $f_{\ell_q(1/\rho(2^m))}$ by s_ρ in Theorem 8.1. Furthermore,

s_ρ is multiplicative, non-decreasing and $s_\rho(t)/t$ is non-decreasing. Hence, s_ρ satisfies (8.4) with $M_1 = M_2 = M_3 = 1$.

We apply now Theorem 8.1 to interpolation spaces with a function parameter and the real method.

Corollary 8.3. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let $\bar{E} = (E_0, E_1)$ be an r-normed quasi-Banach couple $(0 < r \le 1)$. Suppose that ρ_0, ρ_1, ρ_2 are function parameters such that for some constant $L > 0$ we have

$$\rho_0(t)\rho_1(s) \le L\rho_2(ts), \quad t, s > 0. \tag{8.8}$$

Let $0 < q_0, q_1 \le \infty$ and put

$$\frac{1}{q} = \begin{cases} \frac{1}{q_0} + \frac{1}{q_1} - \frac{1}{r} & \text{if} \quad q_0, q_1 \ge r, \\ \frac{1}{\max\{q_0,q_1\}} & \text{if} \quad q_0 < r \quad \text{or} \quad q_1 < r. \end{cases} \tag{8.9}$$

Then, for each $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ the restriction of T to $\bar{A}_{\rho_0,q_0} \times \bar{B}_{\rho_1,q_1}$ defines a continuous bilinear operator $T : \bar{A}_{\rho_0,q_0} \times \bar{B}_{\rho_1,q_1} \longrightarrow \bar{E}_{\rho_2,q}$ with

$$\|T|\mathcal{B}(\bar{A}_{\rho_0,q_0} \times \bar{B}_{\rho_1,q_1}, \bar{E}_{\rho_2,q})\| \le \begin{cases} 0 & \text{if} \quad \|T\|_j = 0 \quad \text{for} \quad j = 0 \text{ or } j = 1, \\ C\|T\|_0 s_{\rho_1}(\|T\|_1/\|T\|_0) & \text{otherwise.} \end{cases}$$

Here $\|T\|_j = \|T|\mathcal{B}(A_j \times B_j, E_j)\|$ and C is a constant independent of T.

Proof. Let $\Gamma_0 = \ell_{q_0}(1/\rho_0(2^m))$, $\Gamma_1 = \ell_{q_1}(1/\rho_1(2^m))$ and $\Gamma_2 = \ell_q(1/\rho_2(2^m))$. Then Γ_0 and Γ_2 are K-non trivial and Γ_1 is (p, J)-non trivial. Let us show that these quasi-Banach sequence lattices satisfy (8.6). Assume first that $q_0, q_1 \ge r$, so $\frac{r}{q} + 1 = \frac{r}{q_0} + \frac{r}{q_1}$. If $\xi = (\xi_m) \in \ell_{q_0}(1/\rho_0(2^m))$ and $\eta = (\eta_m) \in \ell_{q_1}(1/\rho_1(2^m))$ are non-negative scalar sequences, then according to (8.8) and Young's convolution inequality we obtain

$$\begin{aligned} \|(\xi^r * \eta^r)^{1/r}|\ell_q(1/\rho_2(2^m))\| &= \Big(\sum_{m=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{\infty} \xi_k^r \eta_{m-k}^r/\rho_2(2^m)^r\Big)^{q/r}\Big)^{1/q} \\ &\le L\Big(\sum_{m=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{\infty} (\xi_k/\rho_0(2^k))^r(\eta_{m-k}/\rho_1(2^{m-k}))^r\Big)^{q/r}\Big)^{1/q} \\ &\le C\|(\xi_m/\rho_0(2^m))^r|\ell_{q_0/r}\|^{1/r}\|(\eta_m/\rho_1(2^m))|\ell_{q_1/r}\|^{1/r} \\ &= C\|\xi|\ell_{q_0}(1/\rho_0(2^m))\|\|\eta|\ell_{q_1}(1/\rho_1(2^m))\|. \end{aligned}$$

Suppose now that $q_0 < r$ or $q_1 < r$. If $q = q_1 = \max\{q_0, q_1\}$, then $q_0 = \min\{q_0, q_1\} < r$. Thus $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1} - \frac{1}{q_0}$ and the couple (E_0, E_1) is q_0-normed because $q_0 < r$. For every $\xi = (\xi_m) \in \ell_{q_0}(1/\rho_0(2^m))$ and $\eta = (\eta_m) \in \ell_{q_1}(1/\rho_1(2^m))$ non-negative scalar sequences, we get

$$\|(\eta^{q_0} * \xi^{q_0})^{1/q_0}|\ell_q(1/\rho_2(2^m))\| \le C\|\xi|\ell_{q_0}(1/\rho_0(2^m))\|\|\eta|\ell_{q_1}(1/\rho_1(2^m))\|.$$

The case $q = q_0 = \max\{q_0, q_1\}$ is analogous.

Now the result follows by applying Theorem 8.1 and having in mind what we have pointed out in Example 8.2. □

Taking $\rho_0(t) = \rho_1(t) = \rho_2(t) = t^\theta$ in the previous theorem, we obtain Theorem 2.3.

Corollary 8.4. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let $\bar{E} = (E_0, E_1)$ be an r-normed quasi-Banach couple $(0 < r \leq 1)$. Let $0 < \theta < 1$, $0 < q_0, q_1 \leq \infty$ and take q verifying (8.9). Then, for each $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ the restriction of T to $\bar{A}_{\theta,q_0} \times \bar{B}_{\theta,q_1}$ defines a continuous bilinear operator $T : \bar{A}_{\theta,q_0} \times \bar{B}_{\theta,q_1} \longrightarrow \bar{E}_{\theta,q}$ with

$$\|T|\mathcal{B}(\bar{A}_{\theta,q_0} \times \bar{B}_{\theta,q_1}, \bar{E}_{\theta,q})\| \leq \begin{cases} 0 & \text{if} \quad \|T\|_j = 0 \quad \text{for} \quad j = 0 \text{ or } j = 1, \\ C\|T\|_0^{1-\theta}\|T\|_1^{\theta} & \text{otherwise.} \end{cases}$$

Here $\|T\|_j = \|T|\mathcal{B}(A_j \times B_j, E_j)\|$ and C is a constant independent of T.

8.2 Interpolation of the measure of non-compactness

Let $A,\ B,\ E$ be quasi-Banach spaces and let $T \in \mathcal{B}(A \times B, E)$ a continuous bilinear operator from $A \times B$ into E. The operator T is said to be *compact* if for any bounded sets $V \subseteq A$, $W \subseteq B$ we have that the closure of $T(V, W) = \{T(a, b) : a \in V, b \in W\}$ is compact in E. This condition is equivalent to the fact that $T(U_A, U_B)$ is precompact in E.

In Chapter 6 we introduced the concept and properties of the measure of non-compactness for continuous linear operators. We shall need the corresponding notion for bilinear operators.

The *measure of non-compactness* $\beta(T) = \beta(T : A \times B \to E)$ of $T \in \mathcal{B}(A \times B,\ E)$ is defined to be the infimum of the set of all $\sigma > 0$ for which there exists a finite subset $\{w_1, ..., w_s\} \subseteq E$ such that

$$T(U_A, U_B) \subseteq \bigcup_{j=1}^{s} \{w_j + \sigma U_E\}.$$

The measure of non-compactness for bilinear operators satisfies analogous properties to the ones we saw for linear operators (see (6.7) and (6.8)):

- If $T \in \mathcal{B}(A \times B,\ E)$, then

$$\beta(T : A \times B \to E) \leq \|T|\mathcal{B}(A \times B, E)\|. \tag{8.10}$$

- T is compact if, and only if,

$$\beta(T : A \times B \to E) = 0. \tag{8.11}$$

- If F is another quasi-Banach space and R is a bounded linear operator $R \in \mathcal{L}(E, F)$, then for $RT = R \circ T$ we have

$$\beta(RT : A \times B \to F) \leq \|R|\mathcal{L}(E, F)\|\beta(T : A \times B \to E). \tag{8.12}$$

  Moreover, if $\|Rv|F\| = \|v|E\|$ for any $v \in E$, then

$$\beta(T : A \times B \to E) \leq 2c_F\beta(RT : A \times B \to F).$$

- If X, Y are quasi-Banach spaces and R_1, R_2 are bounded linear operators $R_1 \in \mathcal{L}(X, A)$, $R_2 \in \mathcal{L}(Y, B)$, then the operator $T \circ (R_1, R_2)(x, y) = T(R_1, R_2)(x, y) = T(R_1 x, R_2 y)$ belongs to $\mathcal{B}(X \times Y, E)$ and

$$\beta(T(R_1, R_2) : X \times Y \to E) \leq \|R_1|\mathcal{L}(X, A)\|\|R_2|\mathcal{L}(Y, B)\|\beta(T : A \times B \to E). \tag{8.13}$$

Moreover, if for any $a \in A, b \in B$ with $\|a|A\| < 1$, $\|b|B\| < 1$ there exists $x \in X, y \in Y$ with $\|x|X\| < 1$, $\|y|Y\| < 1$ and $(R_1, R_2)(x,y) = (a,b)$, then

$$\beta(T : A \times B \to E) \leq \beta(T(R_1, R_2) : X \times Y \to E).$$

- If $S \in \mathcal{B}(A \times B, E)$ then

$$\beta(S + T : A \times B \to E) \leq c_E(\beta(S : A \times B \to E) + \beta(T : A \times B \to E)). \tag{8.14}$$

We prove now two auxiliary results that are the bilinear version of Lemma 6.8 and Lemma 6.9. Here, we use vector-valued sequence spaces and the operator ι such that given a quasi-Banach couple (E_0, E_1), for any $w \in E_0 + E_1$ assigns $\iota w = (..., w, w, w, w, ...)$. Remember (see the explanation just before Lemma 6.8), that $\iota : E_j \longrightarrow \ell_\infty(2^{-mj} W_m)$ is continuous with norm less than or equal to 1, being $W_m = (E_0 + E_1, K(2^m, \cdot))$.

From now on, if $\bar{A} = (A_0, A_1)$ is a quasi-Banach couple with constants c_{A_0}, c_{A_1} in the quasi-triangle inequality, respectively, we put $c_{\bar{A}} = \max\{c_{A_0}, c_{A_1}\}$.

Lemma 8.5. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be quasi-Banach couples and let $T : \bar{A} \times \bar{B} \to \bar{E}$. Fix $j \in \{0,1\}$and put $\beta_j = \beta(T : A_j \times B_j \to E_j)$. Assume that there are quasi-Banach spaces X, Y and continuous linear operators $R_n \in \mathcal{L}(X, A_j)$, $S_n \in \mathcal{L}(Y, B_j)$ such that $\|R_n|\mathcal{L}(X, A_j)\| \leq 1$, $\|S_n|\mathcal{L}(Y, B_j)\| \leq 1$ and $\lim_{n\to\infty} \|T(R_n, S_n)|\mathcal{B}(X \times Y, E_0 + E_1)\| = 0$. Then the following holds.

a) If $\beta_j = 0$, then there is a subsequence (n') such that

$$\lim_{n'\to\infty} \|\iota T(R_{n'}, S_{n'})|\mathcal{B}(X \times Y, \ell_\infty(2^{-mj} W_m))\| = 0.$$

b) If $\beta_j > 0$, then there is a constant C independent of T and a subsequence (n') such that

$$\lim_{n'\to\infty} \|\iota T(R_{n'}, S_{n'})|\mathcal{B}(X \times Y, \ell_\infty(2^{-mj} W_m))\| \leq C\beta_j.$$

Proof. Since $\sup_{n\in\mathbb{N}} \|\iota T(R_n, S_n)|\mathcal{B}(X \times Y, \ell_\infty(2^{-mj} W_m))\| \leq \|T|\mathcal{B}(A_j \times B_j, E_j)\| < \infty$, there exists a subsequence (n') such that

$$\lim_{n'\to\infty} \|\iota T(R_{n'}, S_{n'})|\mathcal{B}(X \times Y, \ell_\infty(2^{-mj} W_m))\| = \lambda \geq 0.$$

Let $(x_{n'}) \subseteq U_X$, $(y_{n'}) \subseteq U_Y$ so that

$$\|\iota T(R_{n'} x_{n'}, S_{n'} y_{n'})|\ell_\infty(2^{-mj} W_m)\| \xrightarrow{n'\to\infty} \lambda.$$

Take any $\sigma > \beta_j$. There exists a finite set $\{z_1, ..., z_s\} \subseteq E_j$ such that

$$T(U_{A_j}, U_{B_j}) \subseteq \bigcup_{k=1}^{s} \{z_k + \sigma U_{E_j}\}.$$

Passing to another subsequence if necessary that we continue denoting by (n'), we may find $k \in \{1, ..., s\}$ such that

$$T(R_{n'} x_{n'}, S_{n'} y_{n'}) \in z_k + \sigma U_{E_j} \quad \text{for all } n'. \tag{8.15}$$

Now we estimate the quasi-norm of $\iota(z_k)$ in $\ell_\infty(2^{-mj}W_m)$. Take any $m \in \mathbb{Z}$. Using that

$$\lim_{n\to\infty} \|T(R_n, S_n)|\mathcal{B}(X \times Y, E_0 + E_1)\| = 0,$$

we can find n' belonging to the subsequence and sufficiently large so that

$$2^{-jm} \max\{1, 2^m\}\|T(R_{n'}, S_{n'})|\mathcal{B}(X \times Y, E_0 + E_1)\| \le \sigma.$$

Whence,

$$\begin{aligned}
2^{-jm}K(2^m, z_k; E_0, E_1) &\le c_{\bar{E}}\Big(2^{-jm}K(2^m, z_k - T(R_{n'}x_{n'}, S_{n'}y_{n'}); E_0, E_1) \\
&\quad + 2^{-mj}K(2^m, T(R_{n'}x_{n'}, S_{n'}y_{n'}); E_0, E_1)\Big) \\
&\le c_{\bar{E}}\Big(\|z_k - T(R_{n'}x_{n'}, S_{n'}y_{n'})|E_j\| \\
&\quad + 2^{-jm} \max\{1, 2^m\}\|T(R_{n'}, S_{n'})|\mathcal{B}(X \times Y, E_0 + E_1)\|\Big) \\
&\le 2c_{\bar{E}}\sigma.
\end{aligned}$$

This yields that $\|\iota z_k|\ell_\infty(2^{-mj}W_m)\| \le 2c_{\bar{E}}\sigma$. Consequently, using that $\|\iota|\mathcal{L}(E_j, \ell_\infty(2^{-mj}W_m))\| \le 1$ and (8.15), we obtain with $C = 2c_{\bar{E}}(1 + 2c_{\bar{E}})$ that

$$\begin{aligned}
&\lim_{n'\to\infty} \|\iota T(R_{n'}, S_{n'})|\mathcal{B}(X \times Y, \ell_\infty(2^{-mj}W_m))\| \\
&\le c_{\bar{E}}\Big(\|\iota T(R_{n'}x_{n'}, S_{n'}y_{n'}) - \iota z_k|\ell_\infty(2^{-mj}W_m)\| + \|\iota z_k|\ell_\infty(2^{-mj}W_m)\|\Big) \\
&\le c_{\bar{E}}(\sigma + 2c_{\bar{E}}\sigma) \le C\sigma/2.
\end{aligned}$$

If $\beta_j = 0$, it follows that $\lim_{n'\to\infty} \|\iota T(R_{n'}, S_{n'})|\mathcal{B}(X \times Y, \ell_\infty(2^{-mj}W_m))\| = 0$. If $\beta_j > 0$, then taking $\sigma = 2\beta_j$ we conclude that

$$\lim_{n'\to\infty} \|\iota T(R_{n'}, S_{n'})|\mathcal{B}(X \times Y, \ell_\infty(2^{-mj}W_m))\| \le C\beta_j.$$

□

Lemma 8.6. Let A, B, E, Z be quasi-Banach spaces, let D be a dense subspace of A and let V be a dense subspace of B. Let $T \in \mathcal{B}(A \times B, E)$, put $\beta = \beta(T : A \times B \to E)$ and assume that there exists $(S_n) \subseteq \mathcal{L}(E, Z)$ with $\sup_{n\in\mathbb{N}} \|S_n|\mathcal{L}(E, Z)\| = M < \infty$ and $\lim_{n\to\infty} \|S_n T(u, v)|Z\| = 0$ for all $(u, v) \in D \times V$. Then the following holds.

a) If $\beta = 0$, then $\lim_{n\to\infty} \|S_n T|\mathcal{B}(A \times B, Z)\| = 0$.

b) If $\beta > 0$, then there is a constant C independent of T and there is $N \in \mathbb{N}$ such that

$$\|S_n T|\mathcal{B}(A \times B, Z)\| \le C\beta \quad \text{for all } n \ge N.$$

Proof. Take $\sigma > \beta$. There exists a finite set $\{w_1, ..., w_s\} \subseteq E$ such that

$$T(U_A, U_B) \subseteq \bigcup_{k=1}^{s} \{w_k + \sigma U_E\}.$$

If $\{w_k + \sigma U_E\} \cap T(U_A, U_B) \neq \emptyset$, choose $a_k \in U_A$, $b_k \in U_B$ such that $T(a_k, b_k) \in w_k + \sigma U_E$. Then

$$T(U_A, U_B) \subseteq \bigcup_{k=1}^{s} \{T(a_k, b_k) + 2c_E \sigma U_E\}.$$

By the density assumption, there are $u_k \in D$, $v_k \in V$ such that

$$\|a_k - u_k|A\| \leq \frac{\sigma}{2c_E\|T|\mathcal{B}(A \times B, E)\|} \quad \text{and}$$
$$\|b_k - v_k|B\| \leq \frac{\sigma}{2c_E c_A\|T|\mathcal{B}(A \times B, E)\| \left(1 + \frac{\sigma}{2c_E\|T|\mathcal{B}(A\times B,E)\|}\right)}.$$

Hence

$$\|u_k|A\| \leq c_A(\|u_k - a_k|A\| + \|a_k|A\|) \leq c_A \left(\frac{\sigma}{2c_E\|T|\mathcal{B}(A \times B, E)\|} + 1\right)$$

and so

$$\begin{aligned}\|T(a_k, b_k) - T(u_k, v_k)|E\| &= \|T(a_k - u_k, b_k) + T(u_k, b_k - v_k)|E\| \\ &\leq c_E\|T|\mathcal{B}(A \times B, E)\| \left(\|a_k - u_k|A\|\|b_k|B\| + \|u_k|A\|\|b_k - v_k|B\|\right) \\ &\leq \sigma.\end{aligned}$$

It follows that

$$T(U_A, U_B) \subseteq \bigcup_{k=1}^{s} \{T(u_k, v_k) + c_E(2c_E + 1)\sigma U_E\}.$$

Let $C = 2c_Z(Mc_E(2c_E + 1) + 1)$ and let $N \in \mathbb{N}$ such that for any $n \geq N$ we have

$$\|S_n T(u_k, v_k)|Z\| \leq \sigma, \quad \text{for any } 1 \leq k \leq s.$$

Given any $(a, b) \in U_A \times U_B$, we can find k such that $\|T(a, b) - T(u_k, v_k)|E\| \leq c_E(2c_E + 1)\sigma$. Therefore, we obtain

$$\begin{aligned}\|S_n T(a, b)|Z\| &\leq c_Z \left(\|S_n(T(a, b) - T(u_k, v_k))|Z\| + \|S_n T(u_k, v_k)|Z\|\right) \\ &\leq c_Z \left(Mc_E(2c_E + 1) + 1\right)\sigma \\ &= C\sigma/2.\end{aligned}$$

This yields that $\|S_n T|\mathcal{B}(A \times B, Z)\| \leq C\sigma/2$ for $n \geq N$.

If $\beta = 0$, we derive that $\lim_{n\to\infty} \|S_n T|\mathcal{B}(A \times B, E)\| = 0$. If $\beta > 0$, the choice $\sigma = 2\beta$ gives that $\|S_n T|\mathcal{B}(A \times B, Z)\| \leq C\beta$ for any $n \geq N$. □

Given $n \in \mathbb{N}$, if $x = (x_k)_{k=-n}^{n} \in \mathbb{R}^{2n+1}$ we write $\tilde{x} = \sum_{k=-n}^{n} x_k e_k$. If Γ is a quasi-Banach sequence lattice and $\|\cdot|\Gamma\|$ is a p-norm, then the functional $\|x|\tilde{\Gamma}\| = \|\tilde{x}|\Gamma\|$ defines a p-norm on $\mathbb{R}^{2n+1}$. It is not hard to check that $\|\cdot|\tilde{\Gamma}\|$ is equivalent to $\|x\|_p = \left(\sum_{k=-n}^{n} |x_k|^p\right)^{1/p}$ on $\mathbb{R}^{2n+1}$ and that $U_{\left(\mathbb{R}^{2n+1},\ \|\cdot|\tilde{\Gamma}\|\right)}$ is compact in $\left(\mathbb{R}^{2n+1},\ \|\cdot|\tilde{\Gamma}\|\right)$. This yields that for any quasi-Banach sequence lattice Γ and for any $\varepsilon > 0$, there exists an $\varepsilon-$*net* for $U_{\left(\mathbb{R}^{2n+1},\ \|\cdot|\tilde{\Gamma}\|\right)}$. That is to say, there is a finite set $\{v_1, ..., v_s\} \subseteq \mathbb{R}^{2n+1}$ such that for any $x \in U_{\left(\mathbb{R}^{2n+1},\ \|\cdot|\tilde{\Gamma}\|\right)}$ we have

$$\min_{1\leq k\leq s} \|x - v_k|\tilde{\Gamma}\| \leq \varepsilon. \tag{8.16}$$

This remark will be useful in the proof of next theorem, which is the main result of the chapter. The proof uses some bilinear versions of techniques that we already saw in Theorem 6.10.

Theorem 8.7. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be p-normed quasi-Banach couples $(0 < p \leq 1)$, let $\bar{E} = (E_0, E_1)$ be an r-normed quasi-Banach couple $(0 < r \leq 1)$ and let $\Gamma_0, \Gamma_1, \Gamma_2$ be quasi-Banach sequence lattices. We assume that Γ_0, Γ_1 satisfy (8.5) and (8.2) and that Γ_2 satisfies (8.5) with parameter r. Suppose also that the sequence spaces satisfy the condition (8.6) on convolutions. Let $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ and put $\beta_j = \beta\left(T : A_j \times B_j \longrightarrow E_j\right)$, $j = 0, 1$. Then

$$\beta\left(T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \longrightarrow \bar{E}_{\Gamma_2}\right) \leq \begin{cases} 0 & \text{if } \quad \beta_j = 0 \text{ for } j = 0 \text{ or } 1, \\ C\beta_0 f_{\Gamma_1}(\beta_1/\beta_0) & \text{otherwise.} \end{cases} \tag{8.17}$$

Here C is a constant independent of T.

Proof. Step 1. Since $\bar{A}$ and $\bar{B}$ are p-normed, the spaces $F_m = (A_0 \cap A_1, J(2^m, \cdot; A_0, A_1))$ and $G_m = (B_0 \cap B_1, J(2^m, \cdot; B_0, B_1))$ are also p-normed for each $m \in \mathbb{Z}$. Consider the couples

$$\bar{F}_p = (\ell_p(F_m), \ell_p(2^{-m}F_m)), \quad \bar{G}_p = (\ell_p(G_m), \ell_p(2^{-m}G_m)).$$

According to [40, Lemma 2.4], we have with equivalence of quasi-norms

$$(\ell_p(F_m), \ell_p(2^{-m}F_m))_{\Gamma_0} = \Gamma_0(F_m), \quad (\ell_p(G_m), \ell_p(2^{-m}G_m))_{\Gamma_1} = \Gamma_1(G_m). \tag{8.18}$$

Let $\pi(u_m) = \sum_{m=-\infty}^{\infty} u_m$ be the linear operator assigning to any sequence (u_m) its sum in $A_0 + A_1$. Realizing $\bar{A}_{\Gamma_0}$ by means of the J-functional, the map $\pi : \Gamma_0(F_m) \to \bar{A}_{\Gamma_0}$ is bounded and for any $a \in \bar{A}_{\Gamma_0}$ with $\|a|\bar{A}_{\Gamma_0;J}\| < 1$ there is $(u_m) \in \Gamma_0(F_m)$ with $\|(u_m)|\Gamma_0(F_m)\| < 1$ such that $\pi(u_m) = a$. Moreover, $\pi : \ell_p(2^{-mj}F_m) \to A_j$ is bounded with norm less than or equal 1 for $j = 0, 1$. Similar properties hold for $\pi : \Gamma_1(G_m) \to \bar{B}_{\Gamma_1}$ and $\pi : \ell_p(2^{-mj}G_m) \to B_j$.

As for the r-normed couple (E_0, E_1), put $W_m = (E_0 + E_1, K(2^m, \cdot; E_0, E_1))$, consider the couple $\overline{W}_\infty = (\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))$ and the linear operator $\iota w = (..., w, w, w, ...)$ that already appeared in Lemma 8.5. If we realize $\bar{E}_{\Gamma_2}$ by means of the K-functional, then the operator $\iota : \bar{E}_{\Gamma_2} \to \Gamma_2(W_m)$ is bounded with $\|\iota w|\Gamma_2(W_m)\| = \|w|\bar{E}_{\Gamma_2;K}\|$. Moreover, if we consider $\iota : E_j \to \ell_\infty(2^{-mj}W_m)$ then its norm is less than or equal to 1 for $j = 0, 1$, and the following interpolation formula holds $(\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))_{\Gamma_2} = \Gamma_2(W_m)$ (see [40, Lemma 2.4]).

The following diagram illustrates the situation

$$\begin{array}{ccccccc} \ell_p(F_m) \times \ell_p(G_m) & \xrightarrow{(\pi,\pi)} & A_0 \times B_0 & \xrightarrow{T} & E_0 & \xrightarrow{\iota} & \ell_\infty(W_m) \\ \ell_p(2^{-m}F_m) \times \ell_p(2^{-m}G_m) & \xrightarrow{(\pi,\pi)} & A_1 \times B_1 & \xrightarrow{T} & E_1 & \xrightarrow{\iota} & \ell_\infty(2^{-m}W_m) \\ \hline \Gamma_0(F_m) \times \Gamma_1(G_m) & \xrightarrow{(\pi,\pi)} & \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} & \xrightarrow{T} & \bar{E}_{\Gamma_2} & \xrightarrow{\iota} & \Gamma_2(W_m). \end{array}$$

Put $\widehat{T} = \iota T(\pi, \pi)$. Then $\widehat{T} : \overline{F}_p \times \overline{G}_p \to \overline{W}_\infty$.

According to (8.12) and (8.13) and properties of π and ι, we get

$$\begin{aligned} \beta\left(T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \to \bar{E}_{\Gamma_2}\right) &\leq 2c_{\bar{E}}c_{\Gamma_2}\beta\left(\iota T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \to \Gamma_2(W_m)\right) \\ &\leq 2c_{\bar{E}}c_{\Gamma_2}\beta\left(\widehat{T} : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right). \end{aligned} \tag{8.19}$$

It is easier to estimate $\beta(\widehat{T})$ than $\beta(T)$ because on the couples $\overline{F}_p$, $\overline{G}_p$, $\overline{W}_\infty$ we can use the following families of projections: For $n \in \mathbb{N}$, let

$$R_n(u_m) = (..., 0, 0, u_{-n}, u_{-n+1}, ..., u_{n-1}, u_n, 0, 0, ...),$$
$$R_n^+(u_m) = (..., 0, 0, u_{n+1}, u_{n+2}, u_{n+3}, ...),$$
$$R_n^-(u_m) = (..., u_{-n-3}, u_{-n-2}, u_{-n-1}, 0, 0, ...).$$

It is clear that the identity operator I on $\ell_p(F_m) + \ell_p(2^{-m}F_m)$ can be decomposed as $I = R_n + R_n^+ + R_n^-$, $n \in \mathbb{N}$. These projections are bounded from $\ell_p(2^{-mj}F_m)$ into $\ell_p(2^{-mj}F_m)$ with norm less than or equal to 1 for $j = 0, 1$, and the same happens on $\Gamma_0(F_m)$. Moreover, the restrictions $R_n : \ell_p(F_m) + \ell_p(2^{-m}F_m) \to \ell_p(F_m) \cap \ell_p(2^{-m}F_m)$, $R_n^+ : \ell_p(F_m) \to \ell_p(2^{-m}F_m)$ and $R_n^- : \ell_p(2^{-m}F_m) \to \ell_p(F_m)$ are bounded with

$$\|R_n|\mathcal{L}(\ell_p(F_m) + \ell_p(2^{-m}F_m), \ell_p(F_m) \cap \ell_p(2^{-m}F_m))\| \le c_{\bar{A}} 2^{1/p} 2^n,$$
$$\|R_n^+|\mathcal{L}(\ell_p(F_m), \ell_p(2^{-m}F_m))\| = 2^{-(n+1)} = \|R_n^-|\mathcal{L}(\ell_p(2^{-m}F_m), \ell_p(F_m))\|. \tag{8.20}$$

Let S_n, S_n^+, S_n^- and P_n, P_n^+ and P_n^- similar sequences of projections defined on the couples $\overline{G}_p$, $\overline{W}_\infty$, respectively. They satisfy the corresponding version of (8.20).

Having in mind (8.19), in order to prove (8.17) it suffices to show that if $\beta_j > 0$ for $j = 0$ and $j = 1$, then there is a constant C independent of T such that for any $\varepsilon > 0$ we have

$$\beta\left(\widehat{T} : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) \le C\beta_0 f_{\Gamma_1}(\beta_1/\beta_0) + \varepsilon,$$

and if $\beta_j = 0$ for $j = 0$ or $j = 1$, then

$$\beta\left(\widehat{T} : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) = 0.$$

With this aim, for $n \in \mathbb{N}$ we decompose $\widehat{T}$ as

$$\begin{aligned}\widehat{T} &= P_{3n}\widehat{T} + P_{3n}^+\widehat{T} + P_{3n}^-\widehat{T} = P_{3n}\widehat{T}(R_{4n}, S_{4n}) \\ &\quad + P_{3n}\widehat{T}(R_{4n}, S_{4n}^+) + P_{3n}\widehat{T}(R_{4n}, S_{4n}^-) + P_{3n}\widehat{T}(R_{4n}^+, S_{4n} + S_{4n}^+) \\ &\quad + P_{3n}\widehat{T}(R_{4n}^-, S_{4n} + S_{4n}^-) + P_{3n}\widehat{T}(R_{4n}^+, S_{4n}^-) \\ &\quad + P_{3n}\widehat{T}(R_{4n}^-, S_{4n}^+) + P_{3n}^+\widehat{T} + P_{3n}^-\widehat{T}.\end{aligned} \tag{8.21}$$

Step 2. Now we proceed to give a direct estimate for the measure of non-compactness of the operator $P_{3n}\widehat{T}(R_{4n}, S_{4n})$. First note that we have by (8.12) that

$$\begin{aligned}&\beta\left(P_{3n}\widehat{T}(R_{4n}, S_{4n}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) \\ &\quad \le \beta\left(T(\pi R_{4n}, \pi S_{4n}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \bar{E}_{\Gamma_2}\right) \\ &\quad \le c\beta\left(T(\pi R_{4n}, \pi S_{4n}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \bar{E}_{\Gamma_2;J}\right)\end{aligned}$$

where the last target space is provided with the J-quasi-norm.

Consider on $\mathbb{R}^{8n+1}$ the quasi-norm

$$\|x|\tilde{\Gamma}_j\| = \|(..., 0, 0, x_{-4n}, ..., x_{4n}, 0, 0, ...)|\Gamma_j\|, \quad j = 0, 1,$$

for $x = (x_k)_{k=-4n}^{4n}$. Let

$$\eta = \left(\max_{j=0,1} \Big\| \sum_{k=-4n}^{4n} \frac{e_k}{\|e_k|\Gamma_j\|} |\Gamma_j \Big\| \right)^{-1}.$$

By (8.16), there exists a finite η-net for $U_{(\mathbb{R}^{8n+1}, \|\cdot|\tilde{\Gamma}_0\|)}$. That is, there is a finite set $\Lambda_0 = \{\lambda^1, ..., \lambda^s\} \subseteq U_{(\mathbb{R}^{8n+1}, \|\cdot|\tilde{\Gamma}_0\|)}$ such that for any $x \in \mathbb{R}^{8n+1}$ with $\|x|\tilde{\Gamma}_0\| \le 1$ we can find $\lambda^d \in \Lambda_0$ with $\|x - \lambda^d|\tilde{\Gamma}_0\| \le \eta$. Similarly, let $\Lambda_1 = \{\mu^1, ..., \mu^t\} \subseteq U_{(\mathbb{R}^{8n+1}, \|\cdot|\tilde{\Gamma}_1\|)}$ be an η-net for $U_{(\mathbb{R}^{8n+1}, \|\cdot|\tilde{\Gamma}_1\|)}$. We can associate to each $\lambda^d = (\lambda_k^d)_{k=-4n}^{4n}$ the positive numbers

$$\varphi_k^j = \varphi_{k,\lambda^d}^j = \left(\frac{\eta}{\|e_k|\Gamma_0\|} + |\lambda_k^d| \right) 2^{-kj}, \quad j = 0, 1.$$

In a parallel way, we associate to each $\mu^z = (\mu_k^z)_{k=-4n}^{4n} \in \Lambda_1$ the positive numbers

$$\psi_k^j = \psi_{k,\mu^z}^j = \left(\frac{\eta}{\|e_k|\Gamma_1\|} + |\mu_k^z| \right) 2^{-kj}, \quad j = 0, 1.$$

Let $\sigma_0 > \beta_0, \sigma_1 > \beta_1$ and choose $N \in \mathbb{N}$ such that $2^N \le \sigma_1/\sigma_0 < 2^{N+1}$. There are finite sets

$$\Delta_0 = \{h_l : l = 1, ..., L_0\} \subseteq E_0, \quad \Delta_1 = \{f_y : y = 1, ..., L_1\} \subseteq E_1$$

such that

$$T(U_{A_0}, U_{B_0}) \subseteq \bigcup_{l=1}^{L_0} \{h_l + \sigma_0 U_{E_0}\}, \quad T(U_{A_1}, U_{B_1}) \subseteq \bigcup_{y=1}^{L_1} \{f_y + \sigma_1 U_{E_1}\}.$$

Take any $\lambda^d \in \Lambda_0$, $\mu^z \in \Lambda_1$, $h_l \in \Delta_0$ and $f_y \in \Delta_1$. For any $-4n \le k, s \le 4n$, take an element $g_{k,s} = g_{k,s,\lambda^d,\mu^z,h_l,f_y}$ belonging to

$$(\varphi_k^0 \psi_s^0 \{h_l + \sigma_0 U_{E_0}\}) \cap \left(\varphi_k^1 \psi_s^1 \{f_y + \sigma_1 U_{E_1}\} \right) \tag{8.22}$$

provided the intersection is non-empty. Put $g_{k,s} = 0$ if (8.22) is empty. Let

$$\bar{g}_{k,s} = \begin{cases} g_{k,s} & \text{if} \quad k \in [-4n, 4n] \text{ and } s \in [-4n, 4n] \\ 0 & \text{otherwise.} \end{cases}$$

For $m \in \mathbb{Z}$, put $\xi_m = \sum_{k=-\infty}^{\infty} \bar{g}_{k,m+N-k}$. This series is convergent, with $\xi_m \in E_0 \cap E_1$ and $\xi_m = 0$ if $m \notin [-8n - N, 8n - N]$. Put $\xi = \sum_{m=-\infty}^{\infty} \xi_m$. Then $\xi \in E_0 \cap E_1 \subseteq \bar{E}_{\Gamma_2;J}$. Let Y be the collection of all elements ξ as constructed above. The set Y is finite because $\Lambda_0, \Lambda_1, \Delta_0$ and Δ_1 are finite. Next we show that there is a constant L independent of T such that Y is an $L\sigma_0 f_{\Gamma_1}(\sigma_1/\sigma_0)$-net for $T(U_{\Gamma_0(F_m)}, U_{\Gamma_1(G_m)})$ in $\bar{E}_{\Gamma_2}$.

Given any $u = (u_m) \in U_{\Gamma_0(F_m)}$, $v = (v_m) \in U_{\Gamma_1}(G_m)$, there exists $\lambda^d = (\lambda_k) \in \Lambda_0$, $\mu^z = (\mu_k) \in \Lambda_1$ such that for $k = -4n, ..., 4n$ we have

$$|J(2^k, u_k) - \lambda_k| \|e_k|\Gamma_0\| \le \|(J(2^m, u_m) - \lambda_m)|\tilde{\Gamma}_0\| \le \eta,$$
$$|J(2^k, v_k) - \mu_k| \|e_k|\Gamma_1\| \le \|(J(2^m, v_m) - \mu_m)|\tilde{\Gamma}_1\| \le \eta.$$

Hence,

$$J(2^k, u_k) \le \frac{\eta}{\|e_k|\Gamma_0\|} + |\lambda_k|, \quad J(2^k, v_k) \le \frac{\eta}{\|e_k|\Gamma_1\|} + |\mu_k|.$$

This yields that

$$\|u_k|A_j\| \leq \varphi_k^j, \quad \|v_k|B_j\| \leq \psi_k^j, \quad j=0,1,\ -4n \leq k \leq 4n.$$

Therefore,

$$u_k \in \varphi_k^0 U_{A_0} \cap \varphi_k^1 U_{A_1}, \quad v_s \in \psi_s^0 U_{B_0} \cap \psi_s^1 U_{B_1}, \quad -4n \leq k,s \leq 4n.$$

We can find $h_l \in \Delta_0$, $f_y \in \Delta_1$ such that

$$\begin{aligned} \|T(u_k,v_s) - \varphi_k^0\psi_s^0 h_l|E_0\| &\leq \varphi_k^0\psi_s^0\sigma_0, \\ \|T(u_k,v_s) - \varphi_k^1\psi_s^1 f_y|E_1\| &\leq \varphi_k^1\psi_s^1\sigma_1, \end{aligned} \tag{8.23}$$

and so the intersection (8.22) is non-empty. Let $\xi \in Y$ the vector associated to λ^d, μ^z, h_l and f_y. Put

$$\bar{u}_m = \begin{cases} u_m & \text{if} \quad m \in [-4n,4n], \\ 0 & \text{if} \quad m \notin [-4n,4n], \end{cases}$$

define $\bar{v}_m$, $\bar{\varphi}_m^j$, $\bar{\psi}_m^j$ similarly, and write

$$T_k(u,v) = \sum_{m=-\infty}^{\infty} T(\bar{u}_m, \bar{v}_{k+N-m}) \in E_0 \cap E_1, \quad k \in \mathbb{Z}.$$

We have

$$T(\pi R_{4n}u, \pi S_{4n}v) = \sum_{k=-\infty}^{\infty} T_k(u,v).$$

Since $\bar{E}$ is r-normed, using (8.23) we get

$$\begin{aligned} J\left(2^k, T_k(u,v) - \xi_k\right) &= J\left(2^k, \sum_{m=-\infty}^{\infty} \left(T(\bar{u}_m, \bar{v}_{k+N-m}) - \bar{g}_{m,k+N-m}\right)\right) \\ &\leq \left(\sum_{m=-\infty}^{\infty} J\left(2^k, T(\bar{u}_m, \bar{v}_{k+N-m}) - \bar{g}_{m,k+N-m}\right)^r\right)^{1/r} \\ &\leq \left(\sum_{m=-\infty}^{\infty} \max\left\{2\bar{\varphi}_m^0\bar{\psi}_{k+N-m}^0\sigma_0, 22^k\bar{\varphi}_m^1\bar{\psi}_{k+N-m}^1\sigma_1\right\}^r\right)^{1/r} \\ &\leq 4\sigma_0\left(\sum_{m=-\infty}^{\infty} \left(\bar{\varphi}_m^0\bar{\psi}_{k+N-m}^0\right)^r\right)^{1/r}, \end{aligned}$$

where in the last inequality we have used that $\bar{\varphi}_m^1 = 2^{-m}\bar{\varphi}_m^0$, $\bar{\psi}_{k+N-m}^1 = 2^{-k-N+m}\bar{\psi}_{k+N-m}^0$ and $2^{-N}\sigma_1 < 2\sigma_0$. Consequently, by condition (8.6) on convolutions and definition of f_{Γ_1}, we obtain

$$
\begin{aligned}
\|T(\pi R_{4n}u, \pi S_{4n}v) - \xi|\bar{E}_{\Gamma_2;J}\| &\leq \|(J(2^k, T_k(u,v) - \xi_k))|\Gamma_2\| \\
&\leq 4\sigma_0 \Big\| \Big(\sum_{m=-\infty}^{\infty} (\bar{\varphi}_m^0 \bar{\psi}_{k+N-m}^0)^r \Big)^{1/r} |\Gamma_2 \Big\| \\
&\leq 4M\sigma_0 \|(\bar{\varphi}_m^0)|\Gamma_0\| \|(\bar{\psi}_{m+N}^0)|\Gamma_1\| \\
&\leq 4M\sigma_0 \|\tau_N|\mathcal{L}(\Gamma_1,\Gamma_1)\| c_{\Gamma_0} \left(1 + \eta \Big\| \sum_{k=-4n}^{4n} \frac{e_k}{\|e_k|\Gamma_0\|} |\Gamma_0 \Big\| \right) \\
&\quad \times c_{\Gamma_1} \left(1 + \eta \Big\| \sum_{k=-4n}^{4n} \frac{e_k}{\|e_k|\Gamma_1\|} |\Gamma_1 \Big\| \right) \\
&\leq L\sigma_0 f_{\Gamma_1}(\sigma_1/\sigma_0)
\end{aligned}
$$

where $L = c16Mc_{\Gamma_0}c_{\Gamma_1}$.

It follows that

$$\beta\left(P_{3n}\widehat{T}(R_{4n}, S_{4n}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) \leq L\sigma_0 f_{\Gamma_1}(\sigma_1/\sigma_0).$$

If $\beta_0 = 0$ or $\beta_1 = 0$, then (8.3) implies that

$$\beta\left(P_{3n}\widehat{T}(R_{4n}, S_{4n}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) = 0.$$

Otherwise, the choice $\sigma_j = (1+\varepsilon)\beta_j$ with $\varepsilon > 0$ yields that

$$\beta\left(P_{3n}\widehat{T}(R_{4n}, S_{4n}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) \leq L(1+\varepsilon)\beta_0 f_{\Gamma_1}(\beta_1/\beta_0).$$

Letting $\varepsilon \to 0$ we conclude that

$$\beta\left(P_{3n}\widehat{T}(R_{4n}, S_{4n}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) \leq L\beta_0 f_{\Gamma_1}(\beta_1/\beta_0).$$

Step 3. Now we show that each one of the other six operators involving P_{3n} in the decomposition (8.21) has norm which tends to 0 as $n \to \infty$. To establish it we will use the norm estimate given by Theorem 8.1 and also the fact that $T : (A_0 + A_1) \times (B_0 + B_1) \to E_0 + E_1$ is continuous. Hence, $T : A_i \times B_j \to E_0 + E_1$ is also continuous for $i = 0, 1$, $j = 0, 1$.

Consider, for example, $P_{3n}\widehat{T}(R_{4n}^+, S_{4n} + S_{4n}^+)$. The following commutative diagram holds:

$$
\begin{array}{ccc}
\ell_p(F_m) \times \ell_p(G_m) & \xrightarrow{\;P_{3n}\widehat{T}(R_{4n}^+, S_{4n} + S_{4n}^+)\;} & \ell_\infty(W_m) \\
\Big\downarrow (R_{4n}^+, S_{4n} + S_{4n}^+) & & \Big\uparrow P_{3n} \\
\ell_p(2^{-m}F_m) \times \ell_p(G_m) & \xrightarrow{\;\widehat{T}\;} & \ell_\infty(W_m) + \ell_\infty(2^{-m}W_m).
\end{array}
$$

Moreover, by (8.20), we know that

$$\|R^+_{4n}|\mathcal{L}(\ell_p(F_m),\ell_p(2^{-m}F_m))\| \le 2^{-4n}, \quad \|S_{4n}+S^+_{4n}|\mathcal{L}(\ell_p(G_m),\ell_p(G_m))\| \le 1 \quad \text{and}$$
$$\|P_{3n}|\mathcal{L}(\ell_\infty(W_m)+\ell_\infty(2^{-m}W_m),\ell_\infty(W_m))\| \le c_{\bar{E}}2^{3n}.$$

Hence,

$$\|P_{3n}\widehat{T}(R^+_{4n},S_{4n}+S^+_{4n})|\mathcal{B}(\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m))\| \le c_{\bar{E}}2^{-n}\|T|\mathcal{B}(A_1\times B_0,E_0+E_1)\| \xrightarrow{n\to\infty} 0.$$

On the other hand,

$$\|P_{3n}\widehat{T}(R^+_{4n},S_{4n}+S^+_{4n})|\mathcal{B}(\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m),\ell_\infty(2^{-m}W_m))\| \le \|T|\mathcal{B}(A_1\times B_1,E_1)\|.$$

Using the interpolation formulae (8.18), the corresponding formula for $\Gamma_2(W_m)$ and Theorem 8.1, we conclude that

$$\begin{aligned}
&\beta\left(P_{3n}\widehat{T}(R^+_{4n},S_{4n}+S^+_{4n}):\Gamma_0(F_m)\times\Gamma_1(G_m)\to\Gamma_2(W_m)\right)\\
&\quad\le \|P_{3n}\widehat{T}(R^+_{4n},S_{4n}+S^+_{4n})|\mathcal{B}(\Gamma_0(F_m)\times\Gamma_1(G_m),\Gamma_2(W_m))\|\\
&\quad\le C2^{-n}f_{\Gamma_1}(2^n)\to 0 \text{ as } n\to\infty.
\end{aligned}$$

The following operators can be treated similarly:

$$P_{3n}\widehat{T}(R_{4n},S^+_{4n}),\ P_{3n}\widehat{T}(R_{4n},S^-_{4n}),\ P_{3n}\widehat{T}(R^-_{4n},S_{4n}+S^-_{4n}),\ P_{3n}\widehat{T}(R^+_{4n},S^-_{4n}),\ P_{3n}\widehat{T}(R^-_{4n},S^+_{4n}).$$

Step 4. Next we work with the other two operators $P^+_{3n}\widehat{T}$, $P^-_{3n}\widehat{T}$ in the decomposition (8.21). It is convenient to split them as follows

$$\begin{aligned}
P^+_{3n}\widehat{T}+P^-_{3n}\widehat{T} &= P^+_{3n}\widehat{T}(R_n+R^-_n,S_n+S^-_n)+P^-_{3n}\widehat{T}(R_n+R^+_n,S_n+S^+_n)\\
&\quad+P^+_{3n}\widehat{T}(R_n+R^-_n,S^+_n)+P^-_{3n}\widehat{T}(R_n+R^+_n,S^-_n)\\
&\quad+P^+_{3n}\widehat{T}(R^+_n,I)+P^-_{3n}\widehat{T}(R^-_n,I).
\end{aligned}$$

Factorization

$$\begin{array}{ccc}
\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m) & \xrightarrow{\;P^+_{3n}\widehat{T}(R_n+R^-_n,S_n+S^-_n)\;} & \ell_\infty(2^{-m}W_m)\\
\Big\downarrow{\scriptstyle (R_n+R^-_n,\,S_n+S^-_n)} & & \Big\uparrow{\scriptstyle P^+_{3n}}\\
\ell_p(F_m)\times\ell_p(G_m) & \xrightarrow{\;\widehat{T}\;} & \ell_\infty(W_m)
\end{array}$$

and the fact that $\|P^+_{3n}|\mathcal{L}(\ell_\infty(W_m),\ell_\infty(2^{-m}W_m))\| \le 2^{-3n}$, $\|R_n+R^-_n|\mathcal{L}(\ell_p(2^{-m}F_m),\ell_p(F_m))\| \le 2^n$ and $\|S_n+S^-_n|\mathcal{L}(\ell_p(2^{-m}G_m),\ell_p(G_m))\| \le 2^n$ yields that

$$\begin{aligned}
&\|P^+_{3n}\widehat{T}(R_n+R^-_n,S_n+S^-_n)|\mathcal{B}(\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m),\ell_\infty(2^{-m}W_m))\|\\
&\quad\le 2^{2n}2^{-3n}\|T|\mathcal{B}(A_0\times B_0,E_0)\| \xrightarrow{n\to\infty} 0.
\end{aligned}$$

Since

$$\|P_{3n}^{+}\widehat{T}(R_n+R_n^-,S_n+S_n^-)|\mathcal{B}(\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m))\|\le\|T|\mathcal{B}(A_0\times B_0,E_0)\|,$$

it follows from Theorem 8.1 and properties of f_{Γ_1} that

$$\begin{aligned}&\beta\left(P_{3n}^{+}\widehat{T}(R_n+R_n^-,S_n+S_n^-):\Gamma_0(F_m)\times\Gamma_1(G_m)\to\Gamma_2(W_m)\right)\\&\quad\le\|P_{3n}^{+}\widehat{T}(R_n+R_n^-,S_n+S_n^-)|\mathcal{B}(\Gamma_0(F_m)\times\Gamma_1(G_m),\Gamma_2(W_m))\|\xrightarrow{n\to\infty}0.\end{aligned}$$

With the operator $P_{3n}^{-}\widehat{T}(R_n+R_n^+,S_n+S_n^+)$ we can proceed in a similar way.

For the four remaining operators we shall need Lemmata 8.6 and 8.5. In applications of Lemma 8.6, as dense subspace of $\ell_p(2^{-mj}F_m)$ (respectively, $\ell_p(2^{-mj}G_m)$) for $j=0,1$, we take the subspace of all sequences having only a finite number of co-ordinates different from 0. Besides, if $S:\bar{F}_p\times\bar{G}_p\to\bar{W}_\infty$, we put

$$\|S\|_j=\|S|\mathcal{B}(\ell_p(2^{-mj}F_m)\times\ell_p(2^{-mj}G_m),\ell_\infty(2^{-mj}W_m))\|,\quad j=0,1.$$

Consider $P_{3n}^{+}\widehat{T}(R_n+R_n^-,S_n^+)$. Factorization

$$\begin{array}{ccc}\ell_p(F_m)\times\ell_p(G_m) & \xrightarrow{\quad T(\pi(R_n+R_n^-),\,\pi S_n^+)\quad} & E_0+E_1\\ \Big\downarrow{\scriptstyle (R_n+R_n^-,\,S_n^+)} & & \Big\uparrow{\scriptstyle T}\\ \ell_p(F_m)\times\ell_p(2^{-m}G_m) & \xrightarrow{\quad(\pi,\pi)\quad} & A_0\times B_1\end{array}$$

shows that

$$\|T(\pi(R_n+R_n^-),\pi S_n^+)|\mathcal{B}(\ell_p(F_m)\times\ell_p(G_m),E_0+E_1)\|\le 2^{-n}\|T|\mathcal{B}(A_0\times B_1,E_0+E_1)\|\xrightarrow{n\to\infty}0.$$

Since

$$\|P_{3n}^{+}\widehat{T}(R_n+R_n^-,S_n^+)\|_0\le\|\widehat{T}(R_n+R_n^-,S_n^+)\|_0,$$

it follows from Lemma 8.5 that there are a constant C_1 independent of T, a subsequence (n') and $N_1\in\mathbb{N}$ such that for any $n'\ge N_1$ we have

$$\|P_{3n'}^{+}\widehat{T}(R_{n'}+R_{n'}^-,S_{n'}^+)\|_0\le C_1\beta_0 \tag{8.24}$$

provided that $\beta_0>0$. If $\beta_0=0$, we obtain that

$$\lim_{n'\to\infty}\|P_{3n'}^{+}\widehat{T}(R_{n'}+R_{n'}^-,S_{n'}^+)\|_0=0.$$

On the other hand, if $u\in\ell_p(2^{-m}F_m)$ and $v\in\ell_p(2^{-m}G_m)$ are sequence with only a finite number of co-ordinates different from 0, we have

$$\|P_{3n}^{+}\widehat{T}(u,v)|\ell_\infty(2^{-m}W_m)\|\le 2^{-3n}\|\widehat{T}(u,v)|\ell_\infty(W_m)\|\xrightarrow{n\to\infty}0.$$

Moreover,

$$\|P_{3n}^{+}\widehat{T}(R_n+R_n^-,S_n^+)\|_1\le\|P_{3n}^{+}\widehat{T}\|_1.$$

Hence, applying Lemma 8.6, we obtain that there is a constant C_2 independent of T and $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$

$$\|P^+_{3n}\widehat{T}(R_n + R^-_n, S^+_n)\|_1 \leq C_2\beta_1 \tag{8.25}$$

provided that $\beta_1 > 0$. If $\beta_1 = 0$, then we get that

$$\lim_{n\to\infty} \|P^+_{3n}\widehat{T}(R_n + R^-_n, S^+_n)\|_1 = 0.$$

Put $L = \max\{C_1, C_2\}$ and take any n' from the subsequence with $n' \geq \max\{N_1, N_2\}$. If $\beta_j > 0$ for $j = 0, 1$, it follows from Theorem 8.1 and estimates (8.24), (8.25) that

$$\begin{aligned}
\beta &\left(P^+_{3n'}\widehat{T}(R_{n'} + R^-_{n'}, S^+_{n'}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) \\
&\leq \|P^+_{3n'}\widehat{T}(R_{n'} + R^-_{n'}, S^+_{n'})|\mathcal{B}(\Gamma_0(F_m) \times \Gamma_1(G_m), \Gamma_2(W_m))\| \\
&\leq C\|P^+_{3n'}\widehat{T}(R_{n'} + R^-_{n'}, S^+_{n'})\|_0 f_{\Gamma_1}\left(\frac{\|P^+_{3n'}\widehat{T}(R_{n'} + R^-_{n'}, S^+_{n'})\|_1}{\|P^+_{3n'}\widehat{T}(R_{n'} + R^-_{n'}, S^+_{n'})\|_0}\right) \\
&\leq CL\beta_0 f_{\Gamma_1}(\beta_1/\beta_0).
\end{aligned}$$

If $\beta_j = 0$ for $j = 0$ or $j = 1$, then we obtain

$$\beta\left(P^+_{3n'}\widehat{T}(R_{n'} + R^-_{n'}, S^+_{n'}) : \Gamma_0(F_m) \times \Gamma_1(G_m) \to \Gamma_2(W_m)\right) \xrightarrow{n'\to\infty} 0.$$

Proceeding similarly, an analogous conclusion holds for each one of the operators $P^-_{3n}\widehat{T}(R_n + R^+_n, S^-_n)$, $P^+_{3n}\widehat{T}(R^+_n, I)$ and $P^-_{3n}\widehat{T}(R^-_n, I)$.

Step 5. Having in mind (8.19), (8.21) and collecting the estimates in the previous steps, if $\beta_j > 0$ for $j = 0, 1$, then we conclude that there is a constant $C > 0$ independent of T such that for any $\varepsilon > 0$ we can decompose the operator by (8.21) with $n = n'$ belonging to the subsequence appeared in Step 4 and being sufficiently large, with the result that

$$\beta\left(T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \to \bar{E}_{\Gamma_2}\right) \leq C\beta_0 f_{\Gamma_1}(\beta_1/\beta_0) + \varepsilon.$$

Consequently,

$$\beta\left(T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \to \bar{E}_{\Gamma_2}\right) \leq C\beta_0 f_{\Gamma_1}(\beta_1/\beta_0).$$

If $\beta_j = 0$ for $j = 0$ or $j = 1$, then we derive that

$$\beta\left(T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \to \bar{E}_{\Gamma_2}\right) = 0.$$

This finishes the proof. □

In particular, for the case of the real method with a function parameter we obtain the following result.

Theorem 8.8. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let $\bar{E} = (E_0, E_1)$ be an r-normed quasi-Banach couple ($0 < r \leq 1$). Suppose that ρ_0, ρ_1, ρ_2 are function parameters such that for some constant L we have

$$\rho_0(t)\rho_1(s) \leq L\rho_2(ts), \quad t, s > 0. \tag{8.26}$$

Let $0 < q_0, q_1 \leq \infty$ and write

$$\frac{1}{q} = \begin{cases} \frac{1}{q_0} + \frac{1}{q_1} - \frac{1}{r} & \text{if} \quad q_0, q_1 \geq r, \\ \frac{1}{\max\{q_0,q_1\}} & \text{if} \quad q_0 < r \quad \text{or} \quad q_1 < r. \end{cases}$$

If $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ and $\beta_j = \beta\left(T : A_j \times B_j \longrightarrow E_j\right)$, $j = 0, 1$, then we have:

a) $\beta\left(T : (A_0, A_1)_{\rho_0,q_0} \times (B_0, B_1)_{\rho_1,q_1} \longrightarrow (E_0, E_1)_{\rho_2,q}\right) = 0$, if $\beta_0 = 0$ or $\beta_1 = 0$.

b) $\beta\left(T : (A_0, A_1)_{\rho_0,q_0} \times (B_0, B_1)_{\rho_1,q_1} \longrightarrow (E_0, E_1)_{\rho_2,q}\right) \leq C\beta_0 s_{\rho_1}(\beta_1/\beta_0)$ if $\beta_0 > 0$ and $\beta_1 > 0$.

Here C is a constant independent of T.

Proof. Proceeding as in Corollary 8.3 one can check that assumptions of Theorem 8.7 hold. Having in mind that we can replace $f_{\ell_q(1/\rho_1(2^m))}$ by s_{ρ_1}, the result follows from (8.17). ☐

For the case of the real method, we get the following result.

Theorem 8.9. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let $\bar{E} = (E_0, E_1)$ be an r-normed quasi-Banach couple ($0 < r \leq 1$). Let $0 < \theta < 1$, $0 < q_0, q_1 \leq \infty$ and put

$$\frac{1}{q} = \begin{cases} \frac{1}{q_0} + \frac{1}{q_1} - \frac{1}{r} & \text{if} \quad q_0, q_1 \geq r, \\ \frac{1}{\max\{q_0,q_1\}} & \text{if} \quad q_0 < r \text{ or } q_1 < r. \end{cases}$$

If $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ and $\beta_j = \beta\left(T : A_j \times B_j \longrightarrow E_j\right)$, $j = 0, 1$, then we have:

$$\beta(T : (A_0, A_1)_{\theta,q_0} \times (B_0, B_1)_{\theta,q_1} \longrightarrow (E_0, E_1)_{\theta,q}) \leq C\beta_0^{1-\theta}\beta_1^{\theta}.$$

Here C is a constant independent of T.

Theorems 8.7, 8.8 and 8.9 refine the compactness results for bilinear operators established in [40, Theorems 4.7, 4.8 and 4.9].

When $\bar{A}$, $\bar{B}$ and $\bar{E}$ are Banach couples, so $r = 1$, and $1 \leq q_0, q_1, q \leq \infty$ with $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1} - 1$, Theorem 8.9 includes [93, Theorem 3.2] and shows that the estimate for the measure of non-compactness holds in any of the cases $q_0 = \infty$, $q_1 = \infty$, $q = \infty$ or $q = 1$, cases that had not been studied in [93].

We close the chapter with a final remark.

Remark 8.10. In the assumptions of Theorem 8.7, if $\beta_j > 0$ for $j = 0, 1$, then it also holds

$$\beta\left(T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \longrightarrow \bar{E}_{\Gamma_2}\right) \leq C\beta_0 f_{\Gamma_0}(\beta_1/\beta_0). \tag{8.27}$$

This follows by applying Theorem 8.7 to the operator

$$\tilde{T} : (B_0 + B_1) \times (A_0 + A_1) \longrightarrow (E_0 + E_1), \quad \tilde{T}(b, a) = T(a, b),$$

exchanging the roles of $\bar{A}$, $\bar{B}$, and of Γ_0 and Γ_1.

Estimates (8.18) and (8.27) are not comparable as we show with the following example.

Let $\bar{A}, \bar{B}, \bar{E}, r, q_0, q_1, q$ as in the statement of Theorem 8.8. Assume that $0 < \theta < 1$, $-\infty < \alpha_2 < 0 < \alpha_0, \alpha_1 < \infty$ and put $\rho_k(t) = t^{\theta}(1 + |\log t|)^{-\alpha_k}$ for $k = 0, 1, 2$. Then (8.26) is satisfied. Using the estimation (6.3) that we saw in Chapter 6, it is straightforward that $s_{\rho_k}(t) = t^{\theta}(1 + |\log t|)^{|\alpha_k|}$. Then Theorem 8.8 yields

$$\beta\left(T : \bar{A}_{\rho_0,q_0} \times \bar{B}_{\rho_1,q_1} \longrightarrow \bar{E}_{\rho_2,q}\right) \leq C\beta_0^{1-\theta}\beta_1^{\theta}(1 + |\log(\beta_1/\beta_0)|)^{\alpha_1}, \tag{8.28}$$

while it follows from (8.27) that

$$\beta\left(T : \bar{A}_{\rho_0,q_0} \times \bar{B}_{\rho_1,q_1} \longrightarrow \bar{E}_{\rho_2,q}\right) \leq C\beta_0^{1-\theta}\beta_1^{\theta}(1 + |\log(\beta_1/\beta_0)|)^{\alpha_0}. \tag{8.29}$$

Therefore, if $\alpha_1 < \alpha_0$ then it is clear that (8.28) is a better estimate that (8.29), while if $\alpha_0 < \alpha_1$ then (8.29) is better that (8.28).

In the special case when we have equality (8.26), i.e. $\rho_0(t)\rho_1(s) = L\rho_2(ts)$, $t, s > 0$, then we have that $s_{\rho_0} = s_{\rho_1}$ and the estimates coincide. This is the case in Theorem 8.9.

Chapter 9

Function spaces of Lorentz-Sobolev type

Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ play a main role in the theory of function spaces and, as we saw in Section 2.2, they contain many other important scales of functions and distributions. However, in general, the interpolation by the real method of spaces $B^s_{p,q}(\mathbb{R}^n)$ or $F^s_{p,q}(\mathbb{R}^n)$ does not belong to these classical scales (see [110, Section 2.4]). This fact led to introduce spaces $B^s_q L_{p,r}(\mathbb{R}^n)$ and $F^s_q L_{p,r}(\mathbb{R}^n)$ where the role of the Lebesgue space $L_p(\mathbb{R}^n)$ in the Fourier analytical definition is replaced by the Lorentz space $L_{p,r}(\mathbb{R}^n)$. These spaces have been receiving growing attention in the last years. See, for example, the papers by Fefferman, Riviere and Sagher [60], Caetano [25], Cianchi and Pick [28], Xiang and Yan [119, 120], Almeida and Caetano [3, 2] and more recently, the articles by Grafakos and Slavíková [71], Seeger and Trebels [107] and Hobus and Saal [78].

The aim of this chapter is to study some basic properties of spaces $B^s_q L_{p,r}(\mathbb{R}^n)$ and $F^s_q L_{p,r}(\mathbb{R}^n)$. In particular, we characterize them by means of wavelets (see Section 9.2). This is a well-known property in the case of classical Besov and Triebel-Lizorkin spaces (see, for example, the books by Triebel [114, 113]). For Lorentz-smoothness spaces some previous results are already known. The characterization of $F^s_q L_{p,r}(\mathbb{R}^n)$ was proved by Yang, Cheng and Peng [121, Theorems 3 and 4] in a paper written in Chinese. For the case of Besov-Lorentz spaces, Almeida in [1, Corollary 3.2] gave the wavelet decomposition of spaces $B^s_q L_{p,q}(\mathbb{R}^n)$. We eliminate here the restriction $r = q$ of the result of Almeida, establishing the characterization in terms of wavelets for general Besov-Lorentz spaces $B^s_q L_{p,r}(\mathbb{R}^n)$. The approach we follow is based on a recent paper by Haroske, Skandera and Triebel [77], needing to establish first, in Section 9.1, the decomposition of Lorentz smoothness spaces in terms of atoms. Finally, in Section 9.3, by using wavelets decompositions we obtain interpolation formulae for spaces $F^s_q L_{p,r}(\mathbb{R}^n)$ and also for spaces $B^s_q L_{p,r}(\mathbb{R}^n)$. A description of $B^s_q L_{p,r}(\mathbb{R}^n)$ as an approximation space is also given.

The results in this chapter are taken from the preprint [20] and the work in preparation [17].

9.1 Atomic decomposition

We start by giving the definition of Besov-Lorentz and Triebel-Lizorkin-Lorentz spaces in terms of dyadic resolutions of unity as we did in Section 2.2 for spaces $F^s_{p,q}(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$. We also show some basic properties of theses spaces and then we prove that they admit an atomic decomposition.

As usual, we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 9.1. Let $0 < q, r \leq \infty$, $0 < p < \infty$, $s \in \mathbb{R}$ and $(\varphi_k)_{k\in\mathbb{N}_0}$ be a smooth dyadic resolution of unity (see (2.32)).

1. The *Besov-Lorentz space* $B^s_q L_{p,r}(\mathbb{R}^n)$ is defined as the space of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that
$$\|f|B^s_q L_{p,r}(\mathbb{R}^n)\| = \Big(\sum_{k=0}^{\infty} 2^{ksq}\|(\varphi_k \hat{f})^{\vee}|L_{p,r}(\mathbb{R}^n)\|^q\Big)^{1/q}$$
is finite.

2. The *Triebel-Lizorkin-Lorentz space* $F^s_q L_{p,r}(\mathbb{R}^n)$ is defined as the space of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that
$$\|f|F^s_q L_{p,r}(\mathbb{R}^n)\| = \Big\|\Big(\sum_{k=0}^{\infty} 2^{ksq}|(\varphi_k \hat{f})^{\vee}|^q\Big)^{1/q}|L_{p,r}(\mathbb{R}^n)\Big\|$$
is finite.

Observe that for $p = r$, the space $B^s_q L_{p,p}(\mathbb{R}^n)$ (respectively, $F^s_q L_{p,p}(\mathbb{R}^n)$) coincides with the classical Besov (respectively, Triebel-Lizorkin) space $B^s_{p,q}(\mathbb{R}^n)$ (respectively, $F^s_{p,q}(\mathbb{R}^n)$).

Proposition 9.2. Let $0 < q, r \le \infty$, $0 < p < \infty$ and $s \in \mathbb{R}$, then
$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \quad \text{and}$$
$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Proof. For $p = r$, the result corresponds to [111, Theorem 2.3.3]. This together with [107, Theorem 1.5] implies that
$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{s'}_{r,q}(\mathbb{R}^n) \hookrightarrow B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n), \quad \text{if} \quad r \le p,\ s' > s + n(1/r - 1/p),$$
$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow B^{s''}_{r,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n), \quad \text{if} \quad p \le r,\ s'' < s - n(1/p - 1/r).$$
As for Triebel-Lizorkin-Lorentz spaces, let $s'' < s < s'$, then it follows from [107, Theorem 1.1 and 1.2] that
$$B^{s'}_q L_{p,r}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow B^{s''}_q L_{p,r}(\mathbb{R}^n),$$
and the proof is reduced to the first case. □

Let $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $m = (m_1, ..., m_n) \in \mathbb{Z}^n$. Consider the *dyadic cubes*
$$Q_{jm} = \prod_{\ell=1}^{n}(2^{-j}m_\ell - 2^{-j-1}, 2^{-j}m_\ell + 2^{-j-1}).$$
with center in $x_{jm} = (2^{-j}m_1, ..., 2^{-j}m_n)$ and sides of length 2^{-j}. Let $\chi_{j,m}$ be the characteristic function of Q_{jm}.

For $L \in \mathbb{N}$, we designate by $C^L(\mathbb{R}^n, \mathbb{C})$ the space of functions from $\mathbb{R}^n$ to $\mathbb{C}$ with continuous derivatives up to order L (included).

Definition 9.3. Let $L \in \mathbb{N}$ and $d > 1$. If $j \in \mathbb{N}$ and $m \in \mathbb{Z}^n$ the function $a_{jm} \in C^L(\mathbb{R}^n, \mathbb{C})$ is said to be an (L, d)*-atom* (or simply L-atom) if

1. supp $a_{jm} \subset dQ_{jm} := \prod_{\ell=1}^{n}(2^{-j}m_\ell - d2^{-j-1}, 2^{-j}m_\ell + d2^{-j-1})$.
2. $|\partial^\alpha a_{jm}(x)| \le 2^{j|\alpha|}$ for $x \in \mathbb{R}^n$ and $|\alpha| \le L$.
3. $\int_{\mathbb{R}^n} x^\beta a_{jm}(x)dx = 0$ for $|\beta| \le L - 1$.

When $j=0$ we simply ask that a_{0m} verifies (1) and (2), no moment conditions are required.

Classical Besov and Triebel-Lizorkin spaces admit atomic representations (see, for example, [114, Theorem 1.7]). To study the case of spaces $B^s_q L_{p,r}(\mathbb{R}^n)$ and $F^s_q L_{p,r}(\mathbb{R}^n)$, we start with some auxiliary results.

Lemma 9.4. Let $(\varphi_k)_{k\in\mathbb{N}_0}$ be a smooth dyadic resolution of unity (2.32), let $(a_{jm})_{j\in\mathbb{N}_0,\, m\in\mathbb{Z}^n}$ be a sequence of (L,d)-atoms on $\mathbb{R}^n$ and $\lambda \geq L$. Then

$$|(\varphi_k \hat{a}_{jm})^\vee(x)| \leq C\frac{2^{-L|k-j|}2^{n\min\{k-j,0\}}}{(1+2^{\min\{k,j\}}|x-x_{jm}|)^\lambda},$$

for every $x\in\mathbb{R}^n$ with a constant C independent of j, k and m.

Proof. Let $N>0$, then for any multi-index α with $|\alpha|\leq L$ we have

$$\sup_{y\in\mathbb{R}^n} |\partial^\alpha a_{jm}(y)|(1+2^j|y-x_{jm}|)^N \leq C(d,n,N)2^{j|\alpha|}. \tag{9.1}$$

Indeed, from properties (1) and (2) in Definition 9.3, we obtain

$$|\partial^\alpha a_{jm}(y)| \leq 2^{j|\alpha|}\frac{(1+2^j|y-x_{jm}|)^N}{(1+2^j|y-x_{jm}|)^N}\chi_{dQ_{jm}}(y) \leq 2^{j|\alpha|}\left(1+\frac{d}{2}\sqrt{n}\right)^N(1+2^j|y-x_{jm}|)^{-N}.$$

Besides, by construction of the partition of unity, if $k\in\mathbb{N}$, we have

$$\varphi_k^\vee(x) = \varphi_0(2^{-k}\cdot)^\vee(x) - \varphi_0(2^{-k+1}\cdot)^\vee(x) = 2^{kn}\varphi_0^\vee(2^k x) - 2^{(k-1)n}\varphi_0^\vee(2^{k-1}x)\,.$$

Hence,

$$\partial^\gamma\varphi_k^\vee(x) = 2^{k(n+|\gamma|)}\partial^\gamma\varphi_0^\vee(2^k x) - 2^{(k-1)(n+|\gamma|)}\partial^\gamma\varphi_0^\vee(2^{k-1}x)\,.$$

Let $M>0$, for any multi-index γ with $|\gamma|\leq M$ we have

$$\begin{aligned}
&\sup_{y\in\mathbb{R}^n} |\partial^\gamma\varphi_k^\vee(x-y)|(1+2^k|x-y|)^M \\
&\quad = 2^{k(n+|\gamma|)}\sup_{y\in\mathbb{R}^n}|\partial^\gamma\varphi_0^\vee(2^k(x-y)) - 2^{-(n+|\gamma|)}\partial^\gamma\varphi_0^\vee(2^{k-1}(x-y))|(1+2^k|x-y|)^M \\
&\quad \leq 2^{k(n+|\gamma|)}C(M,n,\varphi_0).
\end{aligned} \tag{9.2}$$

Assume first that $k\leq j$ and $(j,k)\neq(0,0)$, then a_{jm} has vanishing moments up to order $L-1$ (included). Put $\alpha=0$, $N=L+\lambda+n+1$, $M=\lambda$ and $|\gamma|\leq L$. Then (9.1) and (9.2) allow us to use the results of [70, Appendix B.2, p. 596] based on Taylor's theorem, obtaining that

$$|(\varphi_k\hat{a}_{jm})^\vee(x)| = \left|\int_{\mathbb{R}^n}\varphi_k^\vee(x-y)a_{jm}(y)dy\right| \leq C(L,d,n,\lambda,\varphi_0)\frac{2^{-L(j-k)}2^{-n(j-k)}}{(1+2^k|x-x_{jm}|)^\lambda}, \quad x\in\mathbb{R}^n.$$

Let now $j\leq k$ and $(j,k)\neq(0,0)$. We claim that for any $x\in\mathbb{R}^n$ the Schwartz function $y\to\varphi_k^\vee(x-y)$ has vanishing moment conditions of any order. Indeed, let $\beta\in\mathbb{N}_0^n$ be any multi-index, for any $k\in\mathbb{N}$, we get $\partial^\beta\varphi_k(0) = 2^{-k|\beta|}\partial^\beta\varphi_0(0) - 2^{(-k+1)|\beta|}\partial^\beta\varphi_0(0) = 0$ because $\varphi_0\equiv 1$ in a neighborhood of 0. This implies that $\int_{\mathbb{R}^n} y^\beta \varphi_k^\vee(y)dy = 0$ for any $\beta \in \mathbb{N}_0^n$ and so

$$\int_{\mathbb{R}^n} y^\beta \varphi_k^\vee(x-y)dy = \int_{\mathbb{R}^n} (y-x+x)^\beta \varphi_k^\vee(y)dy = \sum_{\eta\leq\beta} \binom{\beta}{\eta} x^\eta \int_{\mathbb{R}^n} (y-x)^{\beta-\eta} \varphi_k^\vee(x-y)dy = 0. \tag{9.3}$$

For $|\alpha| \leq L$, $N = \lambda$, $M = L+\lambda+n+1$ and $\gamma = 0$, using the results in [70, Appendix B.2] we get that

$$|(\varphi_k \hat{a}_{jm})^\vee(x)| = \left|\int_{\mathbb{R}^n} \varphi_k^\vee(x-y)a_{jm}(y)dy\right| \leq C(L,d,n,\lambda,\varphi_0)\frac{2^{-L(k-j)}}{(1+2^j|x-x_{jm}|)^\lambda}, \quad x \in \mathbb{R}^n.$$

Finally we deal with the case $k = j = 0$. From (9.1) with $\alpha = 0$, $N = \lambda$ and (9.2) with $\gamma = 0$ and $M = L+\lambda+n+1$ we derive that

$$|a_{0m}(y)| \leq C(d,n,\lambda)(1+|y-x_{0m}|)^{-\lambda}, \quad y \in \mathbb{R}^n$$
$$|\varphi_0^\vee(x-y)| \leq C(L,n,\lambda,\varphi_0)(1+|x-y|)^{-L-\lambda-n-1}, \quad x,y \in \mathbb{R}^n.$$

Thus,

$$\begin{aligned}
|(\varphi_0\hat{a}_{0m})^\vee(x)| &\leq \int_{\mathbb{R}^n} |\varphi_0^\vee(x-y)||a_{0m}(y)|dy \\
&\leq C(L,d,n,\lambda,\varphi_0)\int_{\mathbb{R}^n} (1+|x-y|)^{-L-\lambda-n-1}(1+|y-x_{0m}|)^{-\lambda}dy \\
&\leq C(L,d,n,\lambda,\varphi_0)(1+|x-x_{0m}|)^{-\lambda}\int_{\mathbb{R}^n} (1+|x-y|)^{-n-L-1}dy \\
&= C(L,d,n,\lambda,\varphi_0)(1+|x-x_{0m}|)^{-\lambda}.
\end{aligned}$$

□

Let $(A, \|\cdot|A\|)$ be a quasi-Banach space, $0 < p < \infty$ and $0 < r \leq \infty$. The space $L_{p,r}(A)$ consists of all (equivalence classes of) strongly Lebesgue measurable functions $f : \mathbb{R}^n \longrightarrow A$ which have finite quasi-norm

$$\|f|L_{p,r}(A)\| = \|\|f(\cdot)|A\||L_{p,r}(\mathbb{R}^n)\|.$$

If $p = r$ we just write $L_p(A)$ for $L_{p,p}(A)$. See, for example, [87, 110, 11]. According to [110, Theorem 1.18.6.2], if $1 < p_0 \neq p_1 < \infty$, $0 < r \leq \infty$ and $0 < \theta < 1$, then

$$(L_{p_0}(A), L_{p_1}(A))_{\theta,r} = L_{p,r}(A), \quad \text{with} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \tag{9.4}$$

with equivalent quasi-norms.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be a Lebesgue measurable function. Then

$$Mf(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y)|dy, \quad x \in \mathbb{R}^n,$$

is the *Hardy-Littlewood maximal operator*. Here Q is a cube containing x and $|Q|$ stands for its Lebesgue measure.

According to Fefferman-Stein maximal inequality for vector valued Lebesgue spaces (see [61]) we know that for any sequence of Lebesgue measurable functions $(f_j)_{j\in\mathbb{N}_0}$, $1 < p < \infty$ and $0 <$

$q \le \infty$, the following inequality holds:

$$\Big\| \Big(\sum_{j=0}^{\infty} (Mf_j)^q \Big)^{1/q} | L_p(\mathbb{R}^n) \Big\| \le C \Big\| \Big(\sum_{j=0}^{\infty} |f_j|^q \Big)^{1/q} | L_p(\mathbb{R}^n) \Big\|. \tag{9.5}$$

Using now interpolation formula (9.4) with $A = \ell_q(\mathbb{N}_0)$ and suitable parameters, and Theorem 2.1, we get that for any sequence of Lebesgue measurable functions $(f_j)_{j\in\mathbb{N}_0}$, $1 < p < \infty$, $0 < q \le \infty$ and $0 < r \le \infty$, we have that

$$\Big\| \Big(\sum_{j=0}^{\infty} (Mf_j)^q \Big)^{1/q} | L_{p,r}(\mathbb{R}^n) \Big\| \le C \Big\| \Big(\sum_{j=0}^{\infty} |f_j|^q \Big)^{1/q} | L_{p,r}(\mathbb{R}^n) \Big\|. \tag{9.6}$$

Notice that the Hardy-Littlewood maximal operator is not linear but sublinear. However, Theorem 2.1 is still valid for sublinear operators if the couples are formed by quasi-Banach spaces of measurable functions with the lattice property (that is, if $|f| < |g|$ almost everywhere, then the quasi-norm of f is smaller that the quasi-norm of g).

Lemma 9.5. Let $(\eta_{jm})_{\substack{j\in\mathbb{N}_0 \\ m\in\mathbb{Z}^n}} \subset \mathbb{C}$, $0 < b \le 1$ and $\lambda > \frac{n}{b}$. Then

$$\sum_{m\in\mathbb{Z}^n} \frac{|\eta_{jm}|}{(1 + 2^{\min\{j,k\}}|x - x_{jm}|)^\lambda} \le C 2^{-\frac{n}{b}\min\{k-j,0\}} \Big\{ M\Big(\sum_{m\in\mathbb{Z}^n} |\eta_{jm}|^b \chi_{j,m} \Big)(x) \Big\}^{1/b},$$

for every $x \in \mathbb{R}^n$ with a constant C independent of $j, k \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$.

Proof. For any $j, k \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$, set

$$F_0 = \{ m \in \mathbb{Z}^n : |x_{jm} - x| 2^{\min\{k,j\}} \le 1 \},$$
$$F_u = \{ m \in \mathbb{Z}^n : 2^{u-1} \le |x_{jm} - x| 2^{\min\{k,j\}} \le 2^u \}, \quad u \in \mathbb{N}.$$

We have

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n} \frac{|\eta_{jm}|}{(1 + 2^{\min\{j,k\}}|x - x_{jm}|)^\lambda} &= \sum_{u=0}^{\infty} \sum_{F_u} \frac{|\eta_{jm}|}{(1 + 2^{\min\{j,k\}}|x - x_{jm}|)^\lambda} \le \sum_{u=0}^{\infty} 2^{-\lambda(u-1)} \sum_{F_u} |\eta_{jm}| \\ &\le \sum_{u=0}^{\infty} 2^{-\lambda(u-1)} \Big(\sum_{F_u} |\eta_{jm}|^b \Big)^{1/b}.\end{aligned}$$

Note that $\big(\sum_{F_u} |\eta_{jm}|^b \big)^{1/b} = 2^{jn/b} \Big(\int_{\mathbb{R}^n} \sum_{F_u} |\eta_{jm}|^b \chi_{j,m}(y) dy \Big)^{1/b}$. Furthermore,

$$\bigcup_{F_u} Q_{jm} \subset Q := \prod_{\ell=1}^{n} [x_\ell - 2^{u-\min\{k,j\}+1} x_\ell + 2^{u-\min\{k,j\}+1}].$$

Indeed, if $y \in Q_{jm}$ and $m \in F_u$ then

$$|y_\ell - x_\ell| \le |y_\ell - (x_{jm})_\ell| + |(x_{jm})_\ell - x_\ell| \le 2^{-j-1} + 2^{u-\min\{j,k\}} \le 2^{u-\min\{j,k\}+1}, \quad \ell = 1, ..., n.$$

Collecting the previous estimates we have that

$$\begin{aligned}
\sum_{m\in\mathbb{Z}^n} \frac{|\eta_{jm}|}{(1+2^{\min\{j,k\}}|x-x_{jm}|)^{\lambda}} &\le 2^{jn/b}\sum_{u=0}^{\infty} 2^{-\lambda(u-1)}\Big(\int_{\mathbb{R}^n}\sum_{F_u}|\eta_{jm}|^b\chi_{j,m}(y)dy\Big)^{1/b}\\
&= 2^{jn/b}\sum_{u=0}^{\infty} 2^{-\lambda(u-1)}\Big(\int_{Q}\sum_{F_u}|\eta_{jm}|^b\chi_{j,m}(y)dy\Big)^{1/b}\\
&\le C(n)2^{jn/b-\min\{j,k\}n/b}\sum_{u=0}^{\infty} 2^{-\lambda(u-1)}2^{un/b}\Big\{M\Big(\sum_{F_u}|\eta_{jm}|^b\chi_{j,m}\Big)(x)\Big\}^{1/b}\\
&\le C(n)2^{jn/b-\min\{j,k\}n/b}\Big\{M\Big(\sum_{m\in\mathbb{Z}^n}|\eta_{jm}|^b\chi_{j,m}\Big)(x)\Big\}^{1/b}\sum_{u=0}^{\infty} 2^{-\lambda(u-1)}2^{un/b}\\
&\le C(n,\lambda)2^{-\frac{n}{b}\min\{k-j,0\}}\Big\{M\Big(\sum_{m\in\mathbb{Z}^n}|\eta_{jm}|^b\chi_{j,m}\Big)(x)\Big\}^{1/b},
\end{aligned}$$

where we have used that $\lambda > n/b$ in the last inequality. □

Definition 9.6. Let $a > 0$ and $(f_k)_{k=0}^{\infty}$ be a sequence of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{C}$. We define the *Peetre's maximal function* by

$$(f_k)_a^* = \sup_{y\in\mathbb{R}^n}\frac{|f_k(x-y)|}{(1+2^k|y|)^a} = \sup_{y\in\mathbb{R}^n}\frac{|f_k(y)|}{(1+2^k|x-y|)^a}.$$

We recall here an important property of Peetre's maximal function whose proof can be found in [121, Lemma 6] or [70, Lemma 2.2.3].

Proposition 9.7. Let $(\varphi_k)_{k\in\mathbb{N}_0}$ be a smooth dyadic resolution of unity, $f\in\mathcal{S}'(\mathbb{R}^n)$, $0<r\le 1$ and $a=n/r$. Then

$$((\varphi_k\hat{f})^{\vee})_a^*(x) \le C(n,\varphi_0,r)(M|(\varphi_k\hat{f})^{\vee}|^r)^{1/r}(x), \quad x\in\mathbb{R}^n.$$

Now we introduce some sequence spaces.

Definition 9.8. Let $0<q,r\le\infty$, $0<p<\infty$ and $s\in\mathbb{R}$.

1. The space $\mathbf{b^s_q L_{p,r}}$ is the collection of all sequences $(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}\subset\mathbb{C}$ having a finite quasi-norm

$$\|(\mu_{jm})|\mathbf{b^s_q L_{p,r}}\| = \Big(\sum_{j=0}^{\infty}2^{jsq}\Big\|\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|\chi_{j,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q}.$$

   If $p=r$ then we just write $\mathbf{b^s_{p,q}}$ instead of $\mathbf{b^s_q L_{p,p}}$.

2. The space $\mathbf{f^s_q L_{p,r}}$ is formed by all sequences $(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}\subset\mathbb{C}$ having a finite quasi-norm

$$\|(\mu_{jm})|\mathbf{f^s_q L_{p,r}}\| = \Big\|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}2^{jsq}|\mu_{jm}|^q\chi_{j,m}(\cdot)\Big)^{1/q}|L_{p,r}(\mathbb{R}^n)\Big\|.$$

   If $p=r$ then we just write $\mathbf{f^s_{p,q}}$ instead of $\mathbf{f^s_q L_{p,p}}$.

Spaces $\mathbf{b^s_q L_{p,r}}$ can be also defined by using the Lorentz sequence spaces:

From now on we will use the notation $\ell_{p,r}(\mathbb{Z}^n)$ for Lorentz space $L_{p,r}(\mathbb{Z}^n)$ with the counting measure. It can be easily shown that $\ell_{p,r}(\mathbb{Z}^n)$ admits the following equivalent quasi-norm

$$\|(x_m)_{m\in\mathbb{Z}^n}|\ell_{p,r}(\mathbb{Z}^n)\| \sim \Big(\sum_{k=0}^{\infty}|x_k^*|^r(k+1)^{r/p-1}\Big)^{1/r},$$

where (x_k^*) is the decreasing rearrangement of $(x_m)_{m\in\mathbb{Z}^n}$. As usual, if $p=r$ we just write $\ell_p(\mathbb{Z}^n)$.

Lemma 9.9. For any $0<q,r\le\infty$, $0<p<\infty$ and $s\in\mathbb{R}$, let $\mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\ell_{\mathbf{p},\mathbf{r}}$ be the collection of all sequences $(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}\subset\mathbb{C}$ having a finite quasi-norm

$$\|(\mu_{jm})|\mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\ell_{\mathbf{p},\mathbf{r}}\| = \Big(\sum_{j=0}^{\infty}2^{j(s-n/p)q}\|(\mu_{jm})_{m\in\mathbb{Z}^n}|\ell_{p,r}(\mathbb{Z}^n)\|^q\Big)^{1/q}.$$

Then we have with equivalent quasi-norms

$$\mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}} = \mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\ell_{\mathbf{p},\mathbf{r}}\,.$$

Proof. Take any sequence $(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}\subset\mathbb{C}$. For any $j\in\mathbb{N}_0$ put $f_j(x)=\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|\chi_{j,m}(x)$, $x\in\mathbb{R}^n$. If $\mu^j:=(\mu_{jm})_{m\in\mathbb{Z}^n}$, since $|Q_{jm}|=2^{-jn}$, we have that

$$f_j^*(t)=\sum_{k=0}^{\infty}(\mu^j)_k^*\chi_{[2^{-jn}k,2^{-jn}(k+1))}(t)\,.$$

Therefore

$$\begin{aligned}\Big\|\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|\chi_{j,m}(x)|L_{p,r}(\mathbb{R}^n)\Big\| &= \Big(\int_0^{\infty}[t^{1/p}f_j^*(t)]^r\frac{dt}{t}\Big)^{1/r} = \Big(\sum_{k=0}^{\infty}[(\mu^j)_k^*]^r\int_{2^{-jn}k}^{2^{-jn}(k+1)}t^{r/p}\frac{dt}{t}\Big)^{1/r}\\ &= \Big(\frac{p}{r}2^{-jnr/p}\sum_{k=0}^{\infty}[(\mu^j)_k^*]^r\big((k+1)^{r/p}-k^{r/p}\big)\Big)^{1/r}\\ &\sim 2^{-jn/p}\|(\mu_{jm})_{m\in\mathbb{Z}^n}|\ell_{p,r}(\mathbb{Z}^n)\|. \qquad (9.7)\end{aligned}$$

This yields the result. □

We prove now an interpolation formula for $\mathbf{f}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}$ spaces that will be useful later on.

Theorem 9.10. Let $s\in\mathbb{R}$, $0<q,r\le\infty$, $0<p_0\ne p_1<\infty$, $0<\theta<1$ and $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$. Then we have with equivalent quasi-norms

$$(\mathbf{f}^{\mathbf{s}}_{\mathbf{p_0},\mathbf{q}},\mathbf{f}^{\mathbf{s}}_{\mathbf{p_1},\mathbf{q}})_{\theta,r}=\mathbf{f}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}.$$

Proof. Let A be the quasi-Banach space consisting of all $\mu=(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}\subset\mathbb{C}$ with finite quasi-norm

$$\|(\mu_{jm})|A\|=\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}2^{jsq}|\mu_{jm}|^q\Big)^{1/q}.$$

For every $\mu=(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}\in\mathbf{f}^{\mathbf{s}}_{\mathbf{p_k},\mathbf{q}}$, $k=0,1$, put

$$R(\mu)(x)=(\mu_{jm}\chi_{j,m}(x))_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n}.$$

It is clear that $R : \mathbf{f}^{\mathbf{s}}_{\mathbf{p_k,q}} \longrightarrow L_{p_k}(A)$ is continuous and $\|R(\mu)|L_{p_k}(A)\| = \|\mu|\mathbf{f}^{\mathbf{s}}_{\mathbf{p_k,q}}\|$.

Take any $0 < b < \min\{p_0, p_1, q\}$ and consider the operator P which assigns to any $g = (g_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ the sequence

$$P(g_{jm}) = \Big(\Big(\frac{1}{|Q_{j,m}|}\int_{Q_{j,m}} |g_{jm}(y)|^b dy\Big)^{1/b}\Big)_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}.$$

Applying (9.5) we derive

$$\begin{aligned}\|P(g_{jm})|\mathbf{f}^{\mathbf{s}}_{\mathbf{p_k,q}}\| &= \Big\|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\Big[2^{js}\Big(\frac{1}{|Q_{j,m}|}\int_{Q_{j,m}}|g_{jm}(y)|^b dy\Big)^{1/b}\chi_{j,m}(\cdot)\Big]^q\Big)^{b/q}|L_{p_k/b}(\mathbb{R}^n)\Big\|^{1/b}\\ &\le \Big\|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}[2^{jsb}M|g_{jm}|^b(\cdot)]^{q/b}\Big)^{b/q}|L_{p_k/b}(\mathbb{R}^n)\Big\|\\ &\lesssim \|g_{jm}|L_{p_k}(A)\|.\end{aligned}$$

Whence, $P : L_{p_k}(A) \longrightarrow \mathbf{f}^{\mathbf{s}}_{\mathbf{p_k,q}}$ is continuous. Note also that

$$(P\circ R)(\mu) = (|\mu_{jm}|)_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}.$$

The linearity and continuity of R imply that for every $\mu \in \mathbf{f}^{\mathbf{s}}_{\mathbf{p_0,q}} + \mathbf{f}^{\mathbf{s}}_{\mathbf{p_1,q}}$ we get

$$K(t, R(\mu); L_{p_0}(A), L_{p_1}(A)) \le K(t, \mu; \mathbf{f}^{\mathbf{s}}_{\mathbf{p_0,q}}, \mathbf{f}^{\mathbf{s}}_{\mathbf{p_1,q}}). \tag{9.8}$$

On the other hand, due to the lattice properties of $\mathbf{f}^{\mathbf{s}}_{\mathbf{p_k,q}}$, the last K-functional is

$$K(t,\mu;\mathbf{f}^{\mathbf{s}}_{\mathbf{p_0,q}},\mathbf{f}^{\mathbf{s}}_{\mathbf{p_1,q}}) = \inf\{\|(\mu^0_{jm})|\mathbf{f}^{\mathbf{s}}_{\mathbf{p_0,q}}\| + t\|(\mu^1_{jm})|\mathbf{f}^{\mathbf{s}}_{\mathbf{p_1,q}}\| : |\mu_{jm}| \le \mu^0_{jm} + \mu^1_{jm},\ \mu^0_{jm}, \mu^1_{jm} \ge 0\}$$

(see, for example, [43, Lemma 3.1]). This together with the fact that $|Pf| \le C(b)(P|f_0| + P|f_1|)$ if $f = f_0 + f_1$, imply that

$$\begin{aligned}K(t, Pf; \mathbf{f}^{\mathbf{s}}_{\mathbf{p_0,q}}, \mathbf{f}^{\mathbf{s}}_{\mathbf{p_1,q}}) &= K(t, |Pf|; \mathbf{f}^{\mathbf{s}}_{\mathbf{p_0,q}}, \mathbf{f}^{\mathbf{s}}_{\mathbf{p_1,q}}) \le C(b)K(t, P|f_0| + P|f_1|; \mathbf{f}^{\mathbf{s}}_{\mathbf{p_0,q}}, \mathbf{f}^{\mathbf{s}}_{\mathbf{p_1,q}})\\ &\lesssim \|P|f_0||\mathbf{f}^{\mathbf{s}}_{\mathbf{p_0,q}}\| + t\|P|f_1||\mathbf{f}^{\mathbf{s}}_{\mathbf{p_1,q}}\| \lesssim \|f_0|L_{p_0}(A)\| + t\|f_1|L_{p_1}(A)\|.\end{aligned}$$

Taking the infimum over all possible decompositions we obtain that

$$K(t, Pf; \mathbf{f}^{\mathbf{s}}_{\mathbf{p_0,q}}, \mathbf{f}^{\mathbf{s}}_{\mathbf{p_1,q}}) \lesssim K(t, f; L_{p_0}(A), L_{p_1}(A)), \quad \text{for every } f \in L_{p_0}(A) + L_{p_1}(A).$$

Thus,

$$\begin{aligned}K(t, \mu; \mathbf{f}^{\mathbf{s}}_{\mathbf{p_0,q}}, \mathbf{f}^{\mathbf{s}}_{\mathbf{p_1,q}}) &= K(t, |\mu|; \mathbf{f}^{\mathbf{s}}_{\mathbf{p_0,q}}, \mathbf{f}^{\mathbf{s}}_{\mathbf{p_1,q}})\\ &\lesssim K(t, R(\mu); L_{p_0}(A), L_{p_1}(A)).\end{aligned} \tag{9.9}$$

From (9.8), (9.9) and (9.4), we conclude that

$$\|\mu|(\mathbf{f}^{\mathbf{s}}_{\mathbf{p_0,q}}, \mathbf{f}^{\mathbf{s}}_{\mathbf{p_1,q}})_{\theta,r}\| \sim \|R(\mu)|(L_{p_0}(A), L_{p_1}(A))_{\theta,r}\| \sim \|R(\mu)|L_{p,r}(A)\| = \|\mu|\mathbf{f}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p,r}}\|.$$

□

Next we establish the atomic decomposition for function spaces with Lorentz smoothness.

Theorem 9.11. Let $0 < q, r \le \infty$, $0 < p < \infty$, $s \in \mathbb{R}$, $n, L \in \mathbb{N}$.

1. Let $L > \max\{n(\frac{1}{\min\{p,r\}} - 1)_+ - s, s\}$. A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^s_q L_{p,r}(\mathbb{R}^n)$ if, and only if, there exists a sequence of L-atoms $(a_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ and a sequence $(\mu_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ in $\mathbf{b^s_q L_{p,r}}$ such that
$$f(x) = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \mu_{jm} a_{jm}(x) \quad (\text{convergence in } \mathcal{S}'(\mathbb{R}^n)).$$
Moreover, $\|f|B^s_q L_{p,r}(\mathbb{R}^n)\| \sim \inf\{\|(\mu_{jm})|\mathbf{b^s_q L_{p,r}}\|\}$.

2. Let $L > \max\{n(\frac{1}{\min\{p,q\}} - 1)_+ - s, s\}$. A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $F^s_q L_{p,r}(\mathbb{R}^n)$ if, and only if, there exists a sequence of L-atoms $(a_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ and a sequence $(\mu_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ in $\mathbf{f^s_q L_{p,r}}$ such that
$$f(x) = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \mu_{jm} a_{jm}(x) \quad (\text{convergence in } \mathcal{S}'(\mathbb{R}^n)).$$
Moreover, $\|f|F^s_q L_{p,r}(\mathbb{R}^n)\| \sim \inf\{\|(\mu_{jm})|\mathbf{f^s_q L_{p,r}}\|\}$.

Proof. Assume that $f(x) = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \mu_{jm} a_{jm}(x)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$) with $(a_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ and $(\mu_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ satisfying the assumptions of (1). Let $0 < b < \min\{p, r, 1\}$ such that $L > n(\frac{1}{b} - 1) - s > n(\frac{1}{\min\{p,r\}} - 1)_+ - s$. Take $\lambda > \max\{L, n/b\}$. It follows from Lemmata 9.4 and 9.5 that for any $k \in \mathbb{N}_0$,

$$\begin{aligned} |(\varphi_k \hat{f})^\vee(x)| &\le \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} |\mu_{jm}| |(\varphi_k \hat{a}_{jm})^\vee|(x) \\ &\lesssim \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} |\mu_{jm}| \frac{2^{-L|k-j|} 2^{n\min\{k-j,0\}}}{(1 + 2^{\min\{k,j\}}|x - x_{jm}|)^\lambda} \\ &\lesssim \sum_{j=0}^{\infty} \Big\{ M\Big(\sum_{m\in\mathbb{Z}^n} 2^{-L|k-j|b} 2^{-n\min\{k-j,0\}(1-b)} |\mu_{jm}|^b \chi_{j,m} \Big)(x) \Big\}^{1/b}. \end{aligned} \tag{9.10}$$

Set $g_j(x) = \sum_{m\in\mathbb{Z}^n} 2^{-L|k-j|b} 2^{-n\min\{k-j,0\}(1-b)} |\mu_{jm}|^b \chi_{j,m}(x)$, $j \in \mathbb{N}_0$. Using (9.6) and the fact that $L_{p,r}(\mathbb{R}^n)$ is b-normed quasi-Banach space, we derive that

$$\begin{aligned} \|(\varphi_k \hat{f})^\vee(x)|L_{p,r}(\mathbb{R}^n)\| &\lesssim \Big\| \Big(\sum_{j=0}^{\infty} (M g_j)^{1/b} \Big)^b |L_{p/b,r/b}(\mathbb{R}^n) \Big\|^{1/b} \lesssim \Big\| \Big(\sum_{j=0}^{\infty} |g_j|^{1/b} \Big)^b |L_{p/b,r/b}(\mathbb{R}^n) \Big\|^{1/b} \\ &= \Big\| \sum_{j=0}^{\infty} |g_j|^{1/b} |L_{p,r}(\mathbb{R}^n) \Big\| \lesssim \Big(\sum_{j=0}^{\infty} \Big\| |g_j|^{1/b} |L_{p,r}(\mathbb{R}^n) \Big\|^b \Big)^{1/b} \\ &= \Big(\sum_{j=0}^{\infty} 2^{-L|k-j|b} 2^{-n\min\{k-j,0\}(1-b)} \Big\| \sum_{m\in\mathbb{Z}^n} |\mu_{jm}| \chi_{j,m}(\cdot) |L_{p,r}(\mathbb{R}^n) \Big\|^b \Big)^{1/b}. \end{aligned} \tag{9.11}$$

Consequently, if $q^* = \min\{q/b, 1\}$ we obtain that

$$\begin{aligned}
\|f|B^s_q L_{p,r}(\mathbb{R}^n)\| &= \Big(\sum_{k=0}^{\infty} 2^{ksq}\|(\varphi_k\hat{f})^\vee|L_{p,r}(\mathbb{R}^n)\|^q\Big)^{1/q} \\
&\lesssim \Big(\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{\infty} 2^{(k-j)sb}2^{-L|k-j|b}2^{-n\min\{k-j,0\}(1-b)}2^{jsb}\Big\|\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|\chi_{j,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^b\Big)^{q/b}\Big)^{1/q} \\
&\lesssim \Big(\sum_{j=-\infty}^{\infty} 2^{jsbq*}2^{-L|j|bq^*}2^{-n\min\{j,0\}(1-b)q^*}\Big)^{1/bq^*}\Big(\sum_{j=0}^{\infty}2^{jsq}\Big\|\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|\chi_{j,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q}
\end{aligned} \tag{9.12}$$

where we have used Young's inequality for convolution in the last inequality. Taking into consideration that $L > s$ and $L > n(\frac{1}{b}-1)-s$, it follows that

$$\|f|B^s_q L_{p,r}(\mathbb{R}^n)\| \lesssim \Big(\sum_{j=0}^{\infty}2^{jsq}\Big\|\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|\chi_{j,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q} = \|(\mu_{jm})|\mathbf{b^s_q L_{p,r}}\|.$$

Assume now that $f \in B^s_q L_{p,r}(\mathbb{R}^n)$. According to [68, Lemma 5.12], [67, p. 783] and [76, Lemma 3.11], there exist Θ_0, Θ, Φ_0, $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with the following properties:

- $\operatorname{supp}\Theta_0, \Theta \subset \{x \in \mathbb{R}^n : |x| \le 1\}$ and $\int_{\mathbb{R}^n} x^\beta\Theta(x)dx = 0$, for every $|\beta| \le L-1$.
- $\operatorname{supp}\Phi_0 \subset \{x \in \mathbb{R}^n : |x| \le 2\}$ and $\operatorname{supp}\Phi \subset \{x \in \mathbb{R}^n : 1/2 \le |x| \le 2\}$.
- $\hat{\Theta}_0(x)\Phi_0(x) + \sum_{j=1}^{\infty}\hat{\Theta}(2^{-j}x)\Phi(2^{-j}x) = 1$ for all $x \in \mathbb{R}^n$.

Then, it follows from [70, Proposition 1.1.6/(b)] that

$$\begin{aligned}
f(x) &= (\hat{\Theta}_0\Phi_0\hat{f})^\vee(x) + \sum_{j=1}^{\infty}(\hat{\Theta}(2^{-j}\cdot)\Phi(2^{-j}\cdot)\hat{f})^\vee(x) \\
&= \Theta_0 * (\Phi_0\hat{f})^\vee(x) + \sum_{j=1}^{\infty}2^{jn}\Theta(2^j\cdot) * (\Phi(2^{-j}\cdot)\hat{f})^\vee(x),
\end{aligned}$$

being the convergence in $\mathcal{S}'(\mathbb{R}^n)$. For every $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, set

$$\mu_{jm} = C\sup_{y\in Q_{jm}}|(\Phi(2^{-j}\cdot)\hat{f})^\vee(y)| \quad \text{with} \quad C = \max\Big\{\sup_{|\gamma|\le L}\sup_{|x|\le 1}|\partial^\gamma\Theta(x)|, \sup_{|\gamma|\le L}\sup_{|x|\le 1}|\partial^\gamma\Theta_0(x)|\Big\} \tag{9.13}$$

and

$$a_{jm}(x) = \begin{cases} \frac{1}{\mu_{0m}}\int_{Q_{0m}}\Theta_0(x-y)(\Phi_0\hat{f})^\vee(y)dy & \text{if } j = 0, \\ \frac{2^{jn}}{\mu_{jm}}\int_{Q_{jm}}\Theta(2^j(x-y))(\Phi(2^{-j}\cdot)\hat{f})^\vee(y)dy & \text{if } j \in \mathbb{N}. \end{cases} \tag{9.14}$$

Notice that

$$\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\mu_{jm}a_{jm}(x) = \sum_{m\in\mathbb{Z}^n}\int_{Q_{0m}}\Theta_0(x-y)(\Phi_0\hat{f})^\vee(y)dy$$
$$+\sum_{j=1}^{\infty}\sum_{m\in\mathbb{Z}^n}2^{jn}\int_{Q_{jm}}\Theta(2^j(x-y))(\Phi(2^{-j}\cdot)\hat{f})^\vee(y)dy$$
$$=\Theta_0*(\Phi_0\hat{f})^\vee(x)+\sum_{j=1}^{\infty}2^{jn}\Theta(2^j\cdot)*(\Phi(2^{-j}\cdot)\hat{f})^\vee(x)=f(x),$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$. Moreover, $(a_{jm})_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ is a sequence of L-atoms. Indeed, the support of a_{jm} satisfies

$$\operatorname{supp} a_{jm}\subset Q_{jm}+\operatorname{supp}\Theta(2^j\cdot)=Q_{jm}+\{x\in\mathbb{R}^n:|x|\le 2^{-j}\}\subset 3Q_{jm}\quad j\in\mathbb{N}_0,\ m\in\mathbb{Z}^n.$$

Furthermore, if $j\in\mathbb{N}$ we have that for any multi-index α with $|\alpha|\le L$

$$|\partial^\alpha a_{jm}(x)|\le\frac{2^{jn}}{\mu_{jm}}\int_{Q_{jm}}2^{j|\alpha|}|\partial^\alpha\Theta(2^j(x-y))||(\Phi(2^{-j}\cdot)\hat{f})^\vee(y)|dy$$
$$\le\frac{2^{j(n+|\alpha|)}}{\mu_{jm}}\sup_{y\in Q_{jm}}|(\Phi(2^{-j}\cdot)\hat{f})^\vee(y)|\sup_{x\in\mathbb{R}^n}|\partial^\alpha\Theta(x)||Q_{jm}|\le 2^{j|\alpha|}.$$

Analogously for $j=0$. Finally, for any multi-index β with $|\beta|\le L-1$ and $j\in\mathbb{N}$ it holds that

$$\int_{\mathbb{R}^n}x^\beta a_{jm}(x)dx=\frac{2^{jn}}{\mu_{jm}}\int_{\mathbb{R}^n}x^\beta\Big(\int_{Q_{jm}}\Theta(2^j(x-y))(\Phi(2^{-j}\cdot)\hat{f})^\vee(y)dy\Big)dx=0,$$

due to the moment conditions on Θ.

Let now $(\varphi_j)_{j\in\mathbb{N}_0}$ be the dyadic resolution of unity in (2.32). For any $x,y\in\mathbb{R}^n$, $j\in\mathbb{N}_0$ and $a>\frac{n}{\min\{1,p\}}$ we have that

$$|(\Phi(2^{-j}\cdot)\hat{f})^\vee(y)|=\Big|\sum_{\ell=-1}^{1}(\varphi_{j+\ell}\Phi(2^{-j}\cdot)\hat{f})^\vee(y)\Big|\le\sum_{\ell=-1}^{1}2^{jn}\int_{\mathbb{R}^n}|\Phi^\vee(2^j(y-z))||(\varphi_{j+\ell}\hat{f})^\vee(z)|dz$$
$$\lesssim 2^{jn}(1+2^j|y-x|)^a\sum_{\ell=-1}^{1}\int_{\mathbb{R}^n}(1+2^j|y-z|)^{-n-1}(1+2^j|z-x|)^{-a}|(\varphi_{j+\ell}\hat{f})^\vee(z)|dz$$
$$\le 2^{jn}\Big(\int_{\mathbb{R}^n}(1+2^j|y-z|)^{-n-1}dz\Big)(1+2^j|y-x|)^a\sum_{\ell=-1}^{1}((\varphi_{j+\ell}\hat{f})^\vee)^*_a(x)$$
$$\lesssim(1+2^j|y-x|)^a\sum_{\ell=-1}^{1}((\varphi_{j+\ell}\hat{f})^\vee)^*_a(x).$$

Whence, for $b=n/a<\min\{1,p\}$, using Proposition 9.7 we obtain

$$\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|\chi_{j,m}(x)\lesssim\sum_{m\in\mathbb{Z}^n}\sup_{y\in Q_{jm}}(1+2^j|y-x|)^a\sum_{\ell=-1}^{1}((\varphi_{j+\ell}\hat{f})^\vee)^*_a(x)\chi_{j,m}(x)$$
$$\lesssim\sum_{\ell=-1}^{1}((\varphi_{j+\ell}\hat{f})^\vee)^*_a(x)\lesssim\sum_{\ell=-1}^{1}(M|(\varphi_{j+\ell}\hat{f})^\vee|^b(x))^{1/b}.\qquad(9.15)$$

Thus, from (9.15) and (9.6)

$$\|(\mu_{jm})|\mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}\| \lesssim \Big(\sum_{j=0}^{\infty} 2^{jsq}\Big\|\sum_{\ell=-1}^{1}(M|(\varphi_{j+\ell}\hat{f})^{\vee}|^{b}(x))^{1/b}|L_{p,r}(\mathbb{R}^n)\Big\|\Big)^{1/q}$$
$$\lesssim \sum_{\ell=-1}^{1}\Big(\sum_{j=0}^{\infty} 2^{jsq}\|(\varphi_{j+\ell}\hat{f})^{\vee}|L_{p,r}(\mathbb{R}^n)\|^{q}\Big)^{1/q} \lesssim \|f|B^s_qL_{p,r}(\mathbb{R}^n)\|.$$

The proof of (2) follows the same pattern. Assume that $f(x) = \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\mu_{jm}a_{jm}(x)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$) with $(a_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ and $(\mu_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ satisfying the assumptions in (2). Let $0 < b < \min\{1,p,q\}$ such that $L > n(1/b-1) - s > n(\frac{1}{\min\{p,q\}}-1)_+ - s$. Following the same ideas as in (9.10), (9.11) and (9.12), now with $q^* = \min\{1,q\}$, we derive that for any $x\in\mathbb{R}^n$

$$\Big(\sum_{k=0}^{\infty}[2^{ks}|(\varphi_k\hat{f})^{\vee}|(x)]^q\Big)^{1/q}$$
$$\lesssim \Big(\sum_{k=0}^{\infty}\Big[\sum_{j=0}^{\infty}2^{(k-j)s}2^{-L|k-j|}2^{-n\min\{k-j,0\}(1/b-1)}\Big\{M\Big(\sum_{m\in\mathbb{Z}^n}2^{jsb}|\mu_{jm}|^b\chi_{j,m}\Big)(x)\Big\}^{1/b}\Big]^q\Big)^{1/q}$$
$$\lesssim \Big(\sum_{j=-\infty}^{\infty}[2^{js}2^{-L|j|}2^{-n\min\{j,0\}(1/b-1)}]^{q^*}\Big)^{1/q^*}\Big(\sum_{j=0}^{\infty}\Big\{M\Big(\sum_{m\in\mathbb{Z}^n}2^{jsb}|\mu_{jm}|^b\chi_{j,m}\Big)(x)\Big\}^{q/b}\Big)^{1/q}.$$

The first sum in the last inequality is finite since $L > s$ and $L > n(1/b-1) - s$. Consequently using (9.6), we finally get that

$$\|f|F^s_qL_{p,r}(\mathbb{R}^n)\| \lesssim \Big\|\Big(\sum_{j=0}^{\infty}\Big\{M\Big(\sum_{m\in\mathbb{Z}^n}2^{jsb}|\mu_{jm}|^b\chi_{j,m}\Big)(x)\Big\}^{q/b}\Big)^{b/q}|L_{p/b,r/b}(\mathbb{R}^n)\Big\|^{1/b}$$
$$\lesssim \Big\|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}2^{jsq}|\mu_{jm}|^q\chi_{j,m}(x)\Big)^{1/q}|L_{p,r}(\mathbb{R}^n)\Big\| = \|(\mu_{jm})|\mathbf{f}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}\|.$$

Conversely, assume that $f\in F^s_qL_{p,r}(\mathbb{R}^n)$. Take $(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}$ and $(a_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}$ as in (9.13) and (9.14), then (a_{jm}) is a sequence of L-atoms and $f(x) = \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\mu_{jm}a_{jm}(x)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$). Take any $0 < b < \min\{1,p,q\}$. Proceeding as in (9.15), we have that

$$\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|^q\chi_{j,m}(x) \lesssim \sum_{\ell=-1}^{1}(M|(\varphi_{j+\ell}\hat{f})^{\vee}|^b(x))^{q/b}, \quad x\in\mathbb{R}^n.$$

Consequently,

$$\|(\mu_{jm})|\mathbf{f}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}\| \lesssim \sum_{\ell=-1}^{1}\Big\|\Big(\sum_{j=0}^{\infty}\Big(2^{jsb}M|(\varphi_{j+\ell}\hat{f})^{\vee}|^b(x)\Big)^{q/b}\Big)^{b/q}|L_{p/b,r/b}(\mathbb{R}^n)\Big\|$$
$$\lesssim \sum_{\ell=-1}^{1}\Big\|\Big(\sum_{j=0}^{\infty}\Big(2^{js}|(\varphi_{j+\ell}\hat{f})^{\vee}|(x)\Big)^{q}\Big)^{1/q}|L_{p,r}(\mathbb{R}^n)\Big\| \lesssim \|f|F^s_qL_{p,r}(\mathbb{R}^n)\|.$$

This completes the proof. □

We close this section by showing that in the statement of Theorem 9.11 we may change convergence in $\mathcal{S}'(\mathbb{R}^n)$ by unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$. As usual, for any $1\le r\le\infty$, we put

$1/r' = 1 - 1/r$.

Proposition 9.12. Let $0 < q, r \le \infty$, $0 < p < \infty$, $s \in \mathbb{R}^n$, $n, L \in \mathbb{N}$.

1. Let $L > \max\{n(\frac{1}{\min\{p,r\}} - 1)_+ - s, s\}$, let $(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}} \in \mathbf{b^s_q L_{p,r}}$ and assume that $(a_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}$ is a sequence of L-atoms. Put

$$f(x) = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \mu_{jm} a_{jm}(x) \quad (\text{convergence in } \mathcal{S}'(\mathbb{R}^n)).$$

Then the convergence of the series is unconditional on $\mathcal{S}'(\mathbb{R}^n)$.

2. Let $L > \max\{n(\frac{1}{\min\{p,q\}} - 1)_+ - s, s\}$, let $(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}} \in \mathbf{f^s_q L_{p,r}}$ and assume that $(a_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}$ is a sequence of L-atoms. Put

$$f(x) = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \mu_{jm} a_{jm}(x) \quad (\text{convergence in } \mathcal{S}'(\mathbb{R}^n)),$$

Then the convergence of the series is unconditional on $\mathcal{S}'(\mathbb{R}^n)$.

Proof. It suffices to show that the series $\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}(\mu_{jm}a_{jm}, \varphi)$ converges absolutely for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Assume first that $1 < p < \infty$ and let $1 < p_1 < p < p_2 < \infty$, $M > n/p_2'$, $N > L + n$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\sup_{|\beta|=L} \sup_{x\in\mathbb{R}^n} |\partial^\beta \varphi(x)|(1+|x|)^M < \infty,$$

and according to (9.1) $\sup_{x\in\mathbb{R}^n} |a_{jm}(x)|(1+2^j|x-x_{jm}|)^N$ is uniformly bounded on j and m. Since a_{jm} has vanishing moments up to order $L-1$, using Taylor's theorem (see [70, Appendix B.2, p. 596]), we derive that

$$\sum_{m\in\mathbb{Z}^n} |(\mu_{jm}a_{jm}, \varphi)| = \sum_{m\in\mathbb{Z}^n} |\mu_{jm}| \left| \int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)dx \right| \lesssim \sum_{m\in\mathbb{Z}^n} |\mu_{jm}| \frac{2^{-j(L+n)}}{(1+|x_{jm}|)^M}, \tag{9.16}$$

where the constant in the last inequality is independent of $j \in \mathbb{N}_0$.

Let $T_j((\mu_{jm})_{m\in\mathbb{Z}^n}) = \left(2^{-j(L+n)}\frac{\mu_{jm}}{(1+|x_{jm}|)^M}\right)_{m\in\mathbb{Z}^n}$. Then $T_j : \ell_{p_k}(\mathbb{Z}^n) \longrightarrow \ell_1(\mathbb{Z}^n)$ is bounded for $k = 1, 2$, because

$$\begin{aligned}
\|T_j(\mu_{jm})|\ell_1(\mathbb{Z}^n)\| &= 2^{-j(L+n)} \sum_{m\in\mathbb{Z}^n} \frac{|\mu_{jm}|}{(1+|x_{jm}|)^M} \\
&\le 2^{-j(L+n)} \Big(\sum_{m\in\mathbb{Z}^n} |\mu_{jm}|^{p_k}\Big)^{1/p_k} \Big(\sum_{m\in\mathbb{Z}^n} (1+|x_{jm}|)^{-Mp_k'}\Big)^{1/p_k'} \\
&\lesssim 2^{-j(L+n/p_k)} \Big(\sum_{m\in\mathbb{Z}^n} |\mu_{jm}|^{p_k}\Big)^{1/p_k} \Big(\int_{\mathbb{R}^n} (1+|x|)^{-Mp_k'}dx\Big)^{1/p_k'}.
\end{aligned}$$

The last expression is finite since $M > n/p_2' > n/p_1'$. By the estimates for the norm of the interpolated operator given in Theorem 2.1 and interpolation formula (2.18) for Lorentz spaces, we obtain that T_j is continuous from $\ell_{p,r}(\mathbb{Z}^n)$ to $\ell_1(\mathbb{Z}^n)$ and

$$\sum_{m\in\mathbb{Z}^n} |\mu_{jm}| \frac{2^{-j(L+n)}}{(1+|x_{jm}|)^M} \lesssim 2^{-j(L+n/p)} \|(\mu_{jm})_{m\in\mathbb{Z}^n} | \ell_{p,r}(\mathbb{Z}^n)\|, \tag{9.17}$$

where the constant in the last inequality is independent of $j \in \mathbb{N}_0$. Now (9.16), (9.17) together with (9.7) imply that

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|(\mu_{jm}a_{jm},\varphi)| &\lesssim \sum_{j=0}^{\infty}2^{-j(L+n/p)}\|(\mu_{jm})_{m\in\mathbb{Z}^n}|\ell_{p,r}(\mathbb{Z}^n)\| \\ &\lesssim \sum_{j=0}^{\infty}2^{-j(L+s)}\|(\mu_{jm})|\mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}\| < \infty,\end{aligned}$$

since $(\mu_{jm}) \in \mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}$ and $L > \max\{n(\frac{1}{\min\{p,r\}}-1)_+ - s, s\} > -s$.

If $0 < p \leq 1$, as $L > n(1/p-1)-s$, there exists $P > 1$ such that $L > n(1/p-1/P)-s$. Proceeding as before and noting that $\ell_{p,r}(\mathbb{Z}^n) \hookrightarrow \ell_{P,r}(\mathbb{Z}^n)$ we have that

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|(\mu_{jm}a_{jm},\varphi)| &\lesssim \sum_{j=0}^{\infty}2^{-j(L+n/P)}\|(\mu_{jm})_{m\in\mathbb{Z}^n}|\ell_{P,r}(\mathbb{Z}^n)\| \\ &\lesssim \sum_{j=0}^{\infty}2^{-j(L+n/P)}\|(\mu_{jm})_{m\in\mathbb{Z}^n}|\ell_{p,r}(\mathbb{Z}^n)\| \\ &\lesssim \sum_{j=0}^{\infty}2^{-j(L+n(1/P-1/p)+s)}\|(\mu_{jm})|\mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}\| < \infty,\end{aligned}$$

since $(\mu_{jm}) \in \mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}$ and $L > n(\frac{1}{p}-\frac{1}{P})-s$. This establishes (1).

In order to prove (2) first observe that

$$\Big\|\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|\chi_{jm}|L_{p,r}(\mathbb{R}^n)\Big\| = \Big\|\Big(\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|^q\chi_{jm}\Big)^{1/q}|L_{p,r}(\mathbb{R}^n)\Big\| \tag{9.18}$$

since for any fixed $j \in \mathbb{N}_0$ the dyadic cubes $(Q_{jm})_{m\in\mathbb{Z}^n}$ are pairwise disjoints. If $1 < p < \infty$, proceeding as for the part (1) we obtain

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|(\mu_{jm}a_{jm},\varphi)| &\lesssim \sum_{j=0}^{\infty}2^{-j(L+n/p)}\|(\mu_{jm})_{m\in\mathbb{Z}^n}|\ell_{p,r}(\mathbb{Z}^n)\| \\ &\sim \sum_{j=0}^{\infty}2^{-jL}\Big\|\Big(\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|^q\chi_{j,m}\Big)^{1/q}|L_{p,r}(\mathbb{R}^n)\Big\| \\ &\sim \sum_{j=0}^{\infty}2^{-j(L+s)}\Big\|\Big(\sum_{m\in\mathbb{Z}^n}2^{jsq}|\mu_{jm}|^q\chi_{j,m}\Big)^{1/q}|L_{p,r}(\mathbb{R}^n)\Big\| \\ &\leq \sum_{j=0}^{\infty}2^{-j(L+s)}\|(\mu_{jm})|\mathbf{f}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}\| < \infty,\end{aligned}$$

because now $(\mu_{jm}) \in \mathbf{f}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}$ and $L > \max\{n(\frac{1}{\min\{p,q\}}-1)_+ - s, s\} > -s$. For the case $0 < p \leq 1$, taking into consideration (9.18), we can proceed again as in (1). $\square$

9.2 Wavelet characterization

For $L \in \mathbb{N}$, let $\psi_F, \psi_M \in C^L(\mathbb{R})$ be real-valued compactly supported functions with

$$\int_{\mathbb{R}} \psi_F(t)^2 dt = 1, \quad \int_{\mathbb{R}} \psi_M(t)^2 dt = 1 \quad \text{and} \quad \int_{\mathbb{R}} \psi_M(t) t^\ell dt = 0, \quad \ell < L.$$

Let $G = (G_1, ..., G_n) \in G^0 = \{F, M\}^n$ which means that G_ℓ is either F or M. For $j \in \mathbb{N}$, let $G = (G_1, ..., G_n) \in G^j = \{F, M\}^{n*}$ which means that G_ℓ is either F or M but at least one of the components of G must be M. Put

$$\psi^j_{G,m}(x) = 2^{jn/2} \prod_{\ell=1}^{n} \psi_{G_\ell}(2^j x_\ell - m_\ell), \quad G \in G^j,\ m \in \mathbb{Z}^n,\ j \in \mathbb{N}_0. \tag{9.19}$$

Then $\{\psi^j_{G,m} : G \in G^j, m \in \mathbb{Z}^n, j \in \mathbb{N}_0\}$ is called a *wavelet system.*

We start by introducing the relevant sequence spaces.

Definition 9.13. Let $0 < q, r \le \infty$, $0 < p < \infty$ and $s \in \mathbb{R}$.

1. The space $b^s_q L_{p,r}$ is the collection of all sequences $(\mu^{j,G}_m) \subset \mathbb{C}$ having a finite quasi-norm

$$\|(\mu^{j,G}_m)|b^s_q L_{p,r}\| = \Big(\sum_{j=0}^{\infty} 2^{jsq} \sum_{G \in G^j} \Big\| \sum_{m \in \mathbb{Z}^n} |\mu^{j,G}_m| \chi_{j,m}(\cdot) | L_{p,r}(\mathbb{R}^n) \Big\|^q \Big)^{1/q}.$$

   If $p = r$ we write $b^s_{p,q}$ for $b^s_q L_{p,p}$.

2. The space $f^s_q L_{p,r}$ is formed by all sequences $(\mu^{j,G}_m) \subset \mathbb{C}$ having a finite quasi-norm

$$\|(\mu^{j,G}_m)|f^s_q L_{p,r}\| = \Big\| \Big(\sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} 2^{jsq} |\mu^{j,G}_m|^q \chi_{j,m}(\cdot) \Big)^{1/q} | L_{p,r}(\mathbb{R}^n) \Big\|.$$

   If $p = r$ we write $f^s_{p,q}$ for $f^s_q L_{p,p}$.

The space $b^s_q L_{p,r}$ can be also defined by using Lorentz sequence space $\ell_{p,r}$: Consider the space $b^s_q \ell_{p,r}$ formed by all sequences $(\mu^{j,G}_m) \subset \mathbb{C}$ having a finite quasi-norm

$$\|(\mu^{j,G}_m)|b^s_q \ell_{p,r}\| = \Big(\sum_{j=0}^{\infty} 2^{j(s-n/p)q} \sum_{G \in G^j} \|\mu^{j,G}|\ell_{p,r}(\mathbb{Z}^n)\|^q \Big)^{1/q}$$

where $\mu^{j,G} = (\mu^{j,G}_m)_{m \in \mathbb{Z}^n}$. With the same argument as in Lemma 9.9 but putting now $f_{j,G} = \sum_{m \in \mathbb{Z}} |\mu^{j,G}_m| \chi_{j,m}(x)$, we derive that

$$b^s_q L_{p,r} = b^s_q \ell_{p,r} \quad \text{(equivalent quasi-norms).} \tag{9.20}$$

Remark 9.14. There is an useful connection between these sequence spaces and those of Definition 9.8. To describe it, let $G^* = \{F, M\}^{n*}$ and $G_0 = \{F\}^n$. If $(\mu_m^{j,G}) \in b_q^s L_{p,r}$ we have

$$\begin{aligned}
&\|(\mu_m^{j,G})|b_q^s L_{p,r}\| \\
&= \Big(\Big\| \sum_{m\in\mathbb{Z}^n} |\mu_m^{0,G_0}|\chi_{0,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^q + \sum_{j=0}^{\infty} 2^{jsq} \sum_{G\in G^*} \Big\| \sum_{m\in\mathbb{Z}^n} |\mu_m^{j,G}|\chi_{j,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^q \Big)^{1/q} \\
&\sim \Big\| \sum_{m\in\mathbb{Z}^n} |\mu_m^{0,G_0}|\chi_{0,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\| + \sum_{G\in G^*} \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big\| \sum_{m\in\mathbb{Z}^n} |\mu_m^{j,G}|\chi_{j,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^q \Big)^{1/q} \\
&= \Big\| \sum_{m\in\mathbb{Z}^n} |\mu_m^{0,G_0}|\chi_{0,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\| + \sum_{G\in G^*} \|(\mu_m^{j,G})|\mathbf{b_q^s L_{p,r}}\|.
\end{aligned}$$

Analogously, if $(\mu_m^{j,G}) \in f_q^s L_{p,r}$ then

$$\|(\mu_m^{j,G})|f_q^s L_{p,r}\| \sim \Big\| \sum_{m\in\mathbb{Z}^n} |\mu_m^{0,G_0}|\chi_{0,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\| + \sum_{G\in G^*} \|(\mu_m^{j,G})|\mathbf{f_q^s L_{p,r}}\|.$$

Note that the above computations give equivalent quasi-norms in $b_q^s L_{p,r}$ and $f_q^s L_{p,r}$, respectively.

Let $m \in \mathbb{Z}^n$, $j, J \in \mathbb{N}_0$, $d > 1$ and $C_1 > 0$. From now on put

$$I_J^j(m) = \{M \in \mathbb{Z}^n : dQ_{JM} \cap C_1 Q_{jm} \neq \emptyset\}.$$

The following result shows that spaces $\mathbf{b_q^s L_{p,r}}$ and $\mathbf{f_q^s L_{p,r}}$ are κ-sequence spaces in the sense of [77, Definition 4.1].

Lemma 9.15. Let $0 < p < \infty$, $0 < q, r \le \infty$, $s \in \mathbb{R}$, $d > 1$ and $C_1 > 0$.

1. If $\kappa > \max\{s, n(\frac{1}{\min\{p,r\}} - 1)_+ - s, \frac{n}{p} - s\}$, then $\mathbf{b_q^s L_{p,r}}$ has the following properties.

   a) If $(\mu_{jm}) \in \mathbf{b_q^s L_{p,r}}$ and $(\lambda_{jm}) \subset \mathbb{C}$ satisfy

   $$|\lambda_{j,m}| \le \sum_{J\in\mathbb{N}_0} 2^{-\kappa|J-j|} \sum_{M\in I_J^j(m)} 2^{-n(J-j)_+}|\mu_{J,M}|, \quad j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{9.21}$$

   then $(\lambda_{jm}) \in \mathbf{b_q^s L_{p,r}}$ and $\|(\lambda_{jm})|\mathbf{b_q^s L_{p,r}}\| \lesssim \|(\mu_{jm})|\mathbf{b_q^s L_{p,r}}\|$.

   b) For any cube Q there exists a constant $C_Q > 0$ such that for any $(\mu_{jm}) \in \mathbf{b_q^s L_{p,r}}(\mathbb{R}^n)$, we have that

   $$|\mu_{JM}| \le C_Q 2^{J\kappa}\|(\mu_{jm})|\mathbf{b_q^s L_{p,r}}\| \quad \text{for every } J \in \mathbb{N}_0 \text{ and } M \in \mathbb{Z}^n \text{ such that } Q_{J,M} \subset Q.$$

2. If $\kappa > \max\{s, n(\frac{1}{\min\{p,q\}} - 1)_+ - s, \frac{n}{p} - s\}$, then $\mathbf{f_q^s L_{p,r}}$ has the following properties.

   a) If $(\mu_{jm}) \in \mathbf{f_q^s L_{p,r}}$ and $(\lambda_{jm}) \subset \mathbb{C}$ satisfy (9.21), then $(\lambda_{jm}) \in \mathbf{f_q^s L_{p,r}}$ and

   $$\|(\lambda_{jm})|\mathbf{f_q^s L_{p,r}}\| \lesssim \|(\mu_{jm})|\mathbf{f_q^s L_{p,r}}\|.$$

   b) For any cube Q there exists a constant $C_Q > 0$ such that for any $(\mu_{jm}) \in \mathbf{f_q^s L_{p,r}}$, we have that

   $$|\mu_{JM}| \le C_Q 2^{J\kappa}\|(\mu_{jm})|\mathbf{f_q^s L_{p,r}}\| \quad \text{for every } J \in \mathbb{N}_0 \text{ and } M \in \mathbb{Z}^n \text{ such that } Q_{J,M} \subset Q.$$

Proof. Let $(\mu_{jm}) \in \mathbf{b^s_q L_{p,r}}$ and $(\lambda_{jm}) \subset \mathbb{C}$ satisfying (9.21). Then for any $x \in \mathbb{R}^n$

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|\chi_{j,m}(x) &\lesssim \sum_{m\in\mathbb{Z}^n}\sum_{J=0}^{\infty} 2^{-\kappa|J-j|}2^{-n(J-j)_+} \sum_{M\in I^j_J(m)} |\mu_{JM}|\chi_{j,m}(x)\\ &= \sum_{J=0}^{\infty} 2^{-\kappa|J-j|}2^{-n(J-j)_+} \sum_{m\in\mathbb{Z}^n}\sum_{M\in I^j_J(m)} |\mu_{JM}|\chi_{j,m}(x)\\ &= \sum_{J=0}^{\infty} 2^{-\kappa|J-j|}2^{-n(J-j)_+} \sum_{M\in\mathbb{Z}^n} |\mu_{JM}| \sum_{\substack{m\in\mathbb{Z}^n\\ dQ_{JM}\cap C_1Q_{jm}\neq\emptyset}} \chi_{j,m}(x).\end{aligned}$$

It is straightforward that if $dQ_{JM}\cap C_1Q_{jm}\neq\emptyset$, then $Q_{jm}\subset Q_{j,J,M}$ where

$$Q_{j,J,M} = \prod_{\ell=1}^{n}[2^{-J}M_\ell - D2^{-\min\{j,J\}-1}, 2^{-J}M_\ell + D2^{-\min\{j,J\}-1}]$$

and $D = 1+d+C_1$. We have

$$M\chi_{J,M}(x) \geq \frac{1}{|Q_{j,J,M}|}\int_{Q_{j,J,M}}\chi_{J,M}(y)dy = \frac{|Q_{JM}\cap Q_{j,J,M}|}{|Q_{j,J,M}|} = \frac{|Q_{JM}|}{|Q_{j,J,M}|} = \frac{1}{D^n}2^{-n(J-j)_+},$$

for any $x \in Q_{j,J,M}$. This implies that

$$\sum_{\substack{m\in\mathbb{Z}^n\\ dQ_{JM}\cap C_1Q_{jm}\neq\emptyset}} \chi_{j,m}(x) \leq \chi_{Q_{j,J,M}}(x) \lesssim 2^{n(J-j)_+}M\chi_{J,M}(x), \quad x\in\mathbb{R}^n.$$

Let now $0<b<\min\{1,p,r\}$ such that $\kappa > n(1/b-1)-s > n(\frac{1}{\min\{p,r\}}-1)_+ - s$. Then

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|\chi_{j,m}(x) &\lesssim \sum_{J=0}^{\infty} 2^{-\kappa|J-j|}2^{-n(J-j)_+} \sum_{M\in\mathbb{Z}^n} |\mu_{JM}|\chi_{Q_{j,J,M}}(x)\\ &\lesssim \sum_{J=0}^{\infty}\sum_{M\in\mathbb{Z}^n}\Big(|\mu_{JM}|^b 2^{-\kappa|J-j|b}2^{-n(J-j)_+(b-1)}M\chi_{J,M}(x)\Big)^{1/b}\\ &= \sum_{J=0}^{\infty}\sum_{M\in\mathbb{Z}^n}\Big\{M(|\mu_{JM}|^b 2^{-\kappa|J-j|b}2^{-n(J-j)_+(b-1)}\chi_{J,M})(x)\Big\}^{1/b}.\end{aligned}$$

Using now (9.6), as we did in (9.11), we derive that

$$\begin{aligned}\Big\|\sum_{m\in\mathbb{Z}^n}|\lambda_{jm}|\chi_{j,m}(x)|L_{p,r}\Big\| &\lesssim \Big\|\Big(\sum_{J=0}^{\infty}2^{-\kappa|J-j|}2^{-n(J-j)_+(1-1/b)}\sum_{M\in\mathbb{Z}^n}|\mu_{JM}|\chi_{J,M})(x)\Big|L_{p,r}(\mathbb{R}^n)\Big\|\\ &\lesssim \Big(\sum_{J=0}^{\infty}2^{-\kappa|J-j|b}2^{-n(J-j)_+(b-1)}\Big\|\sum_{M\in\mathbb{Z}^n}|\mu_{JM}|\chi_{J,M}(\cdot)\Big|L_{p,r}(\mathbb{R}^n)\Big\|^b\Big)^{1/b},\end{aligned}$$

where in the last inequality we have used that $L_{p,r}$ is a b-normed quasi-Banach space. Now proceeding as in (9.12) we have that for $q^* = \min\{q/b, 1\}$

$$\|(\lambda_{jm})|\mathbf{b^s_q L_{p,r}}\| = \Big(\sum_{j=0}^{\infty} 2^{jsq}\Big\|\sum_{m\in\mathbb{Z}^n}|\lambda_{jm}|\chi_{j,m}|L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q}$$
$$\lesssim \Big(\sum_{j=0}^{\infty}\Big(\sum_{J=0}^{\infty} 2^{(j-J)sb}2^{-\kappa|j-J|b}2^{-n\min\{j-J,0\}(1-b)}2^{Jsb}\Big\|\sum_{M\in\mathbb{Z}^n}|\mu_{JM}|\chi_{J,M}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^b\Big)^{q/b}\Big)^{1/q}$$
$$\lesssim \Big(\sum_{j=-\infty}^{\infty} 2^{[js-\kappa|j|-n\min\{j,0\}(1/b-1)]bq^*}\Big)^{1/(q^*b)}\Big(\sum_{J=0}^{\infty}2^{Jsq}\Big\|\sum_{M\in\mathbb{Z}^n}|\mu_{JM}|\chi_{J,M}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q}.$$

Since $\kappa > n(1/b-1) - s$ and $\kappa > s$, the first series is finite and we get that $\|(\lambda_{jm})|\mathbf{b^s_q L_{p,r}}\| \lesssim \|(\mu_{jm})|\mathbf{b^s_q L_{p,r}}\|$.

On the other hand, we have

$$\|(\mu_{jm})|\mathbf{b^s_q L_{p,r}}\| = \Big(\sum_{j=0}^{\infty}2^{jsq}\Big\|\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|\chi_{j,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|\Big)^{1/q} \geq 2^{Js}|\mu_{JM}|2^{-Jn/p},$$

for every $J\in\mathbb{N}_0$ and $M\in\mathbb{Z}^n$. As $\kappa > n/p - s$, we have that for every $J\in\mathbb{N}_0$ and $M\in\mathbb{Z}^n$

$$|\mu_{JM}| \leq 2^{\kappa J}\|(\mu_{jm})|\mathbf{b^s_q L_{p,r}}\|.$$

For the sequence space $\mathbf{f^s_q L_{p,r}}$, the second property can be proved in the same way.

$$\|(\mu_{jm})|\mathbf{f^s_q L_{p,r}}\| = \Big\|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}2^{jsq}|\mu_{jm}|^q\chi_{j,m}\Big)^{1/q}|L_{p,r}(\mathbb{R}^n)\Big\|$$
$$\geq 2^{Js}|\mu_{JM}|2^{-Jn/p} = 2^{J(s-n/p)}|\mu_{JM}|,$$

for every $J\in\mathbb{N}_0$ and $M\in\mathbb{Z}^n$. As $\kappa > n/p - s$, we get that for every $J\in\mathbb{N}_0$ and $M\in\mathbb{Z}^n$

$$|\mu_{JM}| \leq 2^{\kappa J}\|(\mu_{jm})|\mathbf{f^s_q L_{p,r}}\|.$$

We check the first property for $\mathbf{f^s_q L_{p,r}}$ with the help of interpolation. Let $0 < p_0 < p < p_1$ such that $\kappa > \max\{s, n(\frac{1}{\min\{p_0,q\}}-1)_+ - s\}$. According to Theorem 9.10, we have that $(\mathbf{f^s_{p_0,q}}, \mathbf{f^s_{p_1,q}})_{\theta,r} = \mathbf{f^s_q L_{p,r}}$ for $\theta = \frac{p_1(p-p_0)}{p(p_1-p_0)}$. Consider the operator

$$T(\mu_{jm}) = \Big(\sum_{J\in\mathbb{N}_0}2^{-\kappa|J-j|}\sum_{M\in I^j_J(m)}2^{-n(J-j)_+}|\mu_{JM}|\Big)_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}.$$

T is sublineal and $T:\mathbf{f^s_{p_k,q}}\longrightarrow\mathbf{f^s_{p_k,q}}$ is continuous for $k=0,1$ (see [77, Proposition 6.5]). Then $T:\mathbf{f^s_q L_{p,r}}\longrightarrow\mathbf{f^s_q L_{p,r}}$ is continuous, which means that if $(\mu_{jm})\in\mathbf{f^s_q L_{p,r}}$ and $(\lambda_{jm})\subset\mathbb{C}$ satisfy (9.21), then

$$\|(\lambda_{jm})|\mathbf{f^s_q L_{p,r}}\| \lesssim \|T(\mu_{jm})|\mathbf{f^s_q L_{p,r}}\| \lesssim \|(\mu_{jm})|\mathbf{f^s_q L_{p,r}}\|.$$

□

Now we are ready to establish the characterization by means of wavelets for spaces with Lorentz smoothness.

Theorem 9.16. Let $0<p<\infty$, $0<q,r\leq\infty$, $s\in\mathbb{R}$ and let $\psi^j_{G,m}$ be the wavelets in (9.19).

1. Assume that
$$L > \max\{s, n(\frac{1}{\min\{p,r\}} - 1)_+ - s, \frac{n}{p} - s\}.$$
Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^s_q L_{p,r}(\mathbb{R}^n)$ if, and only if, it can be represented as
$$f(x) = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_m^{j,G} 2^{-jn/2} \psi^j_{G,m}(x), \quad (\lambda_m^{j,G}) \in b^s_q L_{p,r} \tag{9.22}$$
unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$. The representation (9.22) is unique,
$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2}(f, \psi^j_{G,m})$$
and
$$I : f \longrightarrow (2^{jn/2}(f, \psi^j_{G,m}))_{j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n}$$
is an isomorphism from $B^s_q L_{p,r}(\mathbb{R}^n)$ onto $b^s_q L_{p,r}$.

2. Assume that
$$L > \max\{s, n(\frac{1}{\min\{p,q\}} - 1)_+ - s, \frac{n}{p} - s\}.$$
Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $F^s_q L_{p,r}(\mathbb{R}^n)$ if, and only if, it can be represented as
$$f(x) = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_m^{j,G} 2^{-jn/2} \psi^j_{G,m}(x), \quad (\lambda_m^{j,G}) \in f^s_q L_{p,r} \tag{9.23}$$
unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$. The representation (9.23) is unique,
$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2}(f, \psi^j_{G,m})$$
and
$$I : f \longrightarrow (2^{jn/2}(f, \psi^j_{G,m}))_{j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n}$$
is an isomorphism from $F^s_q L_{p,r}(\mathbb{R}^n)$ onto $f^s_q L_{p,r}$.

Proof. Since we have Proposition 9.2, Theorem 9.11, Proposition 9.12 and Lemma 9.15, we can apply [77, Theorem 5.1] to get the wanted characterizations. □

9.3 Interpolation formulae for Lorentz-Sobolev spaces

Now we study some interpolation results for spaces $B^s_q L_{p,r}(\mathbb{R}^n)$ and $F^s_q L_{p,r}(\mathbb{R}^n)$ with the help of the wavelet representation obtained in Theorem 9.16.

Theorem 9.17. Let $s \in \mathbb{R}$ and $0 < q, r_0, r_1, r \leq \infty$. Assume that $0 < \theta < 1$, $0 < p_0 \neq p_1 < \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then we have with equivalent quasi-norms

1. $(F^s_{p_0,q}(\mathbb{R}^n), F^s_{p_1,q}(\mathbb{R}^n))_{\theta,r} = F^s_q L_{p,r}(\mathbb{R}^n)$.
2. $(F^s_q L_{p_0,r_0}(\mathbb{R}^n), F^s_q L_{p_1,r_1}(\mathbb{R}^n))_{\theta,r} = F^s_q L_{p,r}(\mathbb{R}^n)$.

Proof. The operator I in Theorem 9.16 is an isomorphism of $F^s_{p_k,q}(\mathbb{R}^n)$ onto $f^s_{p_k,q}$, $k = 0, 1$. Hence the interpolation theorem yields that

$$I : (F^s_{p_0,q}(\mathbb{R}^n), F^s_{p_1,q}(\mathbb{R}^n))_{\theta,r} \longrightarrow (f^s_{p_0,q}, f^s_{p_1,q})_{\theta,r}$$

is an isomorphism. Proceeding as in Theorem 9.10 is easy to prove that

$$(f^s_{p_0,q}, f^s_{p_1,q})_{\theta,r} = f^s_q L_{p,r}.$$

Therefore, using again Theorem 9.16 we get that I is an isomorphism of $F^s_q L_{p,r}(\mathbb{R}^n)$ into $f^s_q L_{p,r}$ and consequently

$$F^s_q L_{p,r}(\mathbb{R}^n) = (F^s_{p_0,q}(\mathbb{R}^n), F^s_{p_1,q}(\mathbb{R}^n))_{\theta,r},$$

with equivalent quasi-norms.

We derive (2) from (1) by applying the reiteration formula (2.13). □

Using the previous result, spaces $F^0_2 L_{p,r}(\mathbb{R}^n)$ can be characterized as *local Hardy spaces of Lorentz smoothness* ($h_{p,r}(\mathbb{R}^n)$) extending the result we saw for classical spaces in (2.34). For $0 < p, r < \infty$ we define $h_{p,r}(\mathbb{R}^n)$ as the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ having finite quasi-norm

$$\|f|h_{p,r}(\mathbb{R}^n)\| = \Big\| \sup_{0<t<1} |(\psi(t\cdot)\hat{f})^\vee| \Big| L_{p,r}(\mathbb{R}^n)\Big\|,$$

where $\psi \in \mathcal{S}(\mathbb{R}^n)$ has compact support and $\psi(x) = 1$ if $|x| \leq 1$. The space $h_{p,r}(\mathbb{R}^n)$ does not depend on the particular choice of ψ (see [2, Theorem 3.1]). Notice that if $p = r$, then $h_{p,p}(\mathbb{R}^n)$ coincides with the local Hardy space $h_p(\mathbb{R}^n)$ introduced in (2.35).

According to [2, Theorems 4.3 and 4.7], spaces $h_{p,r}(\mathbb{R}^n)$ can be obtained interpolating classical local Hardy spaces. In particular, if $0 < p_0 \neq p_1 < \infty$, $0 < \theta < 1$ and $0 < r < \infty$, we have that

$$(h_{p_0}(\mathbb{R}^n), h_{p_1}(\mathbb{R}^n))_{\theta,r} = h_{p,r}(\mathbb{R}^n), \quad \text{with} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

This together with Theorem 9.17 and (2.34), imply that for $0 < p_0 < p < p_1 < \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $0 < r < \infty$, we have

$$h_{p,r}(\mathbb{R}^n) = (h_{p_0}(\mathbb{R}^n), h_{p_1}(\mathbb{R}^n))_{\theta,r} = (F^0_{p_0,2}(\mathbb{R}^n), F^0_{p_1,2}(\mathbb{R}^n))_{\theta,r} = F^0_2 L_{p,r}(\mathbb{R}^n). \tag{9.24}$$

By means of the interpolation formulae of Theorem 9.17 we can transfer to spaces $F^s_q L_{p,r}(\mathbb{R}^n)$ many properties of the usual Triebel-Lizorkin spaces. For example, consider the Gauss-Weierstrass semi-group

$$W_t f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)dy, \quad f \in \mathcal{S}'(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

The following caloric smoothing holds.

Theorem 9.18. Let $d \geq 0$, $0 < p < \infty$, $s \in \mathbb{R}$ and $0 < q, r \leq \infty$. Then there is a constant $c > 0$ such that for all t with $0 < t \leq 1$ and all $f \in F^s_q L_{p,r}(\mathbb{R}^n)$,

$$t^{d/2}\|W_t f|F^{s+d}_q L_{p,r}(\mathbb{R}^n)\| \leq c\|f|F^s_q L_{p,r}(\mathbb{R}^n)\|.$$

Proof. Let $0 < p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Consider the operator $T_t f = t^{d/2} W_t f$. By [118, Theorem 3.35], there is a constant $c_1 > 0$ such that for all t with $0 <$

$t \le 1$ and all $f \in F^s_{p_j,q}$ we have $\|T_t f|F^{s+d}_{p_j,q}(\mathbb{R}^n)\| \le c_1 \|f|F^s_{p_j,q}(\mathbb{R}^n)\|$. Whence, the result follows by interpolating the operator T_t by the real method with parameter (θ, r) and using Theorem 9.17/(1). □

We can also derive the following result on Fourier multipliers.

Theorem 9.19. Let $1 < p < \infty$, $s \in \mathbb{R}$ and $0 < q, r \le \infty$. Then there are numbers $N \in \mathbb{N}$ and $c > 0$ such that

$$\|(m\hat{f})^\vee|F^s_q L_{p,r}(\mathbb{R}^n)\| \le C \sup_{\substack{|\alpha|\le N,\\ x\in\mathbb{R}^n}} |x|^{|\alpha|}|D^\alpha m(x)| \cdot \|f|F^s_q L_{p,r}(\mathbb{R}^n)\|.$$

Proof. Let $\sigma > n\Big(\max(\frac{1}{p}, 1) - 1\Big)$, $0 < p < \infty$ and $0 < q \le \infty$. Then

$$\|f|F^\sigma_{p,q}(\mathbb{R}^n)\| \sim \|f|L_p(\mathbb{R}^n)\| + \|f|\dot{F}^\sigma_{p,q}(\mathbb{R}^n)\|$$

(see [112, Section 2.3.3]), where $\dot{F}^\sigma_{p,q}(\mathbb{R}^n)$ is the related homogeneous space. Let $Tf = (m\hat{f})^\vee$. Using the corresponding multiplier assertion for the homogeneous spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$ [111, Theorem 5.2.2, p. 241] and the well-known Milchin-Hörmander Fourier multiplier theorem for $L_p(\mathbb{R}^n)$, $1 < p < \infty$, we derive for $s > 0$ that there are numbers N and $c_1 > 0$ such that

$$\|Tf|F^s_{p,q}(\mathbb{R}^n)\| \le c_1 \sup_{\substack{|\alpha|\le N,\\ x\in\mathbb{R}^n}} |x|^{|\alpha|}|D^\alpha m(x)| \cdot \|f|F^s_{p,q}(\mathbb{R}^n)\|.$$

We can extend this assertion from $s > 0$ to $s \in \mathbb{R}$ with the help of the lift operator

$$I_\delta f = \big((1 + |x|^2)^{-\delta/2}\hat{f}\big)^\vee, \quad \delta \in \mathbb{R}$$

which maps $F^s_{p,q}(\mathbb{R}^n)$ isomorphically onto $F^{s+\delta}_{p,q}(\mathbb{R}^n)$ (see [111, Section 2.3.8]) and satisfies that $T = I_{-\delta} T I_\delta$. Finally, interpolating T and using Theorem 9.17/(1) we conclude the wanted result. □

Before studying interpolation of $B^s_q L_{p,r}(\mathbb{R}^n)$ spaces, we need the following auxiliary result concerning the real interpolation of product spaces.

Let $k \in \mathbb{N}$ and let $(A_\ell)^k_{\ell=1}$ be a finite sequence of quasi-Banach spaces. We consider the space $\prod^k_{\ell=1} A_\ell = A_1 \times \cdots \times A_k$ quasi-normed by

$$\Big\|(a_1, \ldots, a_k)|\prod_{\ell=1}^k A_\ell\Big\| = \sum_{\ell=1}^k \|a_\ell|A_\ell\|, \quad a_\ell \in A_\ell,\ \ell = 1, \ldots, k.$$

Proposition 9.20. Let $((A^0_\ell, A^1_\ell))^k_{\ell=1}$ be a finite sequence of quasi-Banach couples and put $B_i = \prod^k_{\ell=1} A^i_\ell$, $i = 0, 1$. Then (B_0, B_1) is a quasi-Banach couple and for any $0 < \theta < 1$ and $0 < r \le \infty$,

$$(B_0, B_1)_{\theta,r} = \prod_{\ell=1}^k (A^0_\ell, A^1_\ell)_{\theta,r},$$

with equivalent quasi-norms.

Proof. First note that

$$B_i \hookrightarrow \prod_{\ell=1}^k (A^0_\ell + A^1_\ell), \quad \text{for } i = 0, 1.$$

Indeed, let $a = (a_1, \dots, a_k) \in B_i$, then

$$\|a| \prod_{\ell=1}^{k}(A_\ell^0 + A_\ell^1)\| = \sum_{\ell=1}^{k} \|a_\ell|A_\ell^0 + A_\ell^1\| \le \sum_{\ell=1}^{k} \|a_\ell|A_\ell^i\| = \|a|B_i\|.$$

Now we prove that $(B_0, B_1)_{\theta,r} \hookrightarrow \prod_{\ell=1}^{k}(A_\ell^0, A_\ell^1)_{\theta,r}$. Fix $t > 0$ and let $a = (a_1, \dots, a_k) \in B_0 + B_1$. For every $\varepsilon > 0$ we can decompose $a = a^0 + a^1$ with $a^i = (a_1^i, \dots, a_k^i) \in B_i$, $i = 0, 1$ and such that

$$\|a^0|B_0\| + t\|a^1|B_1\| = \sum_{\ell=1}^{k} \Big(\|a_\ell^0|A_\ell^0\| + t\|a_\ell^1|A_\ell^1\| \Big) \le K(t, a; B_0, B_1) + \varepsilon.$$

Then, $\sum_{\ell=1}^{k} K(t, a_\ell; A_\ell^0, A_\ell^1) \le \sum_{\ell=1}^{k}(\|a_\ell^0|A_\ell^0\| + t\|a_\ell^1|A_\ell^1\|) \le K(t, a; B_0, B_1) + \varepsilon$, for every $\varepsilon > 0$. From here we deduce that

$$\Big\|a| \prod_{\ell=1}^{k}(A_\ell^0, A_\ell^1)_{\theta,r}\Big\| = \sum_{\ell=1}^{k} \|a_\ell|(A_\ell^0, A_\ell^1)_{\theta,r}\| \sim \Big(\int_0^\infty t^{-\theta r}\Big(\sum_{\ell=1}^{k} K(t, a_\ell; A_\ell^0, A_\ell^1)\Big)^r \frac{dt}{t}\Big)^{1/r}$$

$$\le \left(\int_0^\infty t^{-\theta r} K(t, a; B_0, B_1)^r \frac{dt}{t}\right)^{1/r} = \|a|(B_0, B_1)_{\theta,r}\|.$$

An analogous argument works for the other direction. □

Let $s \in \mathbb{R}$, $0 < q \le \infty$ and X a quasi-Banach space. We write $\ell_q^s(X)$ for the collection of all sequences $(x_j)_{j\in\mathbb{N}_0} \subset X$ having finite quasi-norm

$$\|(x_j)|\ell_q^s(X)\| = \Big(\sum_{j=0}^{\infty} [2^{js}\|x_j|X\|]^q \Big)^{1/q}$$

(the sum should be replaced by the supremum if $q = \infty$). If $0 < q_0, q_1, q \le \infty$, $-\infty < s_0 \ne s_1 < \infty$, $0 < \theta < 1$ and $s = (1-\theta)s_0 + \theta s_1$ we have with equivalent quasi-norms

$$(\ell_{q_0}^{s_0}(X), \ell_{q_1}^{s_1}(X))_{\theta,q} = \ell_q^s(X), \tag{9.25}$$

for every quasi-Banach space X. This is a consequence of [110, Theorem 1.18.2] and [11, Theorem 5.6.1]. If (X, Y) is a quasi-Banach couple, $0 < q_0, q_1 < \infty$, $0 < \theta < 1$, $-\infty < s_0 \ne s_1 < \infty$, $s = (1-\theta)s_0 + \theta s_1$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{1}{q_1}$. Then

$$(\ell_{q_0}^{s_0}(X), \ell_{q_1}^{s_1}(Y))_{\theta,q} = \ell_q^s((X, Y)_{\theta,q}), \tag{9.26}$$

(see [110, Theorem 1.18.1 and Remark 1.18.1/4, pp. 120-123] and [11, Theorem 5.6.2]).

Theorem 9.21. Let $0 < \theta < 1$, $-\infty < s_0, s_1 < \infty$, $s = (1-\theta)s_0 + \theta s_1$, $0 < p_0 \ne p_1 < \infty$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $0 < q_0, q_1 < \infty$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and $0 < r_0, r_1 \le \infty$. Then we have with equivalent quasi-norms

$$(B_{q_0}^{s_0} L_{p_0,r_0}(\mathbb{R}^n), B_{q_1}^{s_1} L_{p_1,r_1}(\mathbb{R}^n))_{\theta,q} = B_q^s L_{p,q}(\mathbb{R}^n).$$

Proof. The operator I in Theorem 9.16 is an isomorphism from $B_{q_k}^{s_k} L_{p_k,r_k}(\mathbb{R}^n)$ onto $b_{q_k}^{s_k} L_{p_k,r_k}$. Remark 9.14 and Lemma 9.9 imply that

$$b_{q_k}^{s_k} L_{p_k,r_k} = \ell_{p_k,r_k}(\mathbb{Z}^n) \times \prod_{G\in G^*} \mathbf{b}_{\mathbf{q_k}}^{\mathbf{s_k}} \ell_{\mathbf{p_k,r_k}} = \ell_{p_k,r_k}(\mathbb{Z}^n) \times \prod_{G\in G^*} \ell_{q_k}^{s_k - n/p_k}(\ell_{p_k,r_k}(\mathbb{Z}^n)), \quad k = 0, 1. \tag{9.27}$$

Therefore, according to Proposition 9.20, (2.18) and (9.26), we have that

$$\begin{aligned}(b^{s_0}_{q_0}L_{p_0,r_0}, b^{s_1}_{q_1}L_{p_1,q_1})_{\theta,q} &= (\ell_{p_0,r_0}(\mathbb{Z}^n), \ell_{p_1,q_1}(\mathbb{Z}^n))_{\theta,q} \\ &\quad \times \prod_{G\in G^*} (\ell^{s_0-n/p_0}_{q_0}(\ell_{p_0,r_0}(\mathbb{Z}^n)), \ell^{s_1-n/p_1}_{q_1}(\ell_{p_1,r_1}(\mathbb{Z}^n)))_{\theta,q} \\ &= \ell_{p,q}(\mathbb{Z}^n) \times \prod_{G\in G^*} \ell^{s-n/p}_{q}(\ell_{p,q}(\mathbb{Z}^n)) = b^s_q L_{p,q}.\end{aligned}$$

Consequently, interpolating the operator I and using Theorem 9.16, we derive that

$$(B^{s_1}_{q_1}L_{p_1,r_1}(\mathbb{R}^n), B^{s_2}_{q_2}L_{p_2,r_2}(\mathbb{R}^n))_{\theta,q} = B^s_q L_{p,q}(\mathbb{R}^n).$$

□

Theorem 9.22. Let $-\infty < s_0 \neq s_1 < \infty$, $0 < p, r < \infty$ and $0 < q_0, q_1, q \leq \infty$. Let $0 < \theta < 1$ and $s = (1-\theta)s_0 + \theta s_1$. Then we have with equivalent quasi-norms

$$(B^{s_0}_{q_0}L_{p,r}(\mathbb{R}^n), B^{s_1}_{q_1}L_{p,r}(\mathbb{R}^n))_{\theta,q} = B^s_q L_{p,r}(\mathbb{R}^n).$$

Proof. From Theorem 9.16, we know that there is an isomorphism $I : B^{s_k}_{q_k}L_{p,r}(\mathbb{R}^n) \longrightarrow b^{s_k}_{q_k}L_{p,r}(\mathbb{R}^n)$, $k = 0, 1$. Proceeding as in Theorem 9.21, we now get that

$$b^{s_k}_{q_k}L_{p,r} = \ell_{p,r}(\mathbb{Z}^n) \times \prod_{G\in G^*} \mathbf{b^{s_k}_{q_k}}\ell_{\mathbf{p,r}} = \ell_{p,r}(\mathbb{Z}^n) \times \prod_{G\in G^*} \ell^{s_k-n/p}_{q_k}(\ell_{p,r}(\mathbb{Z}^n)), \quad k = 0, 1.$$

Therefore, according to Proposition 9.20, (2.18) and (9.25), we have that

$$(b^{s_0}_{q_0}L_{p_0,r_0}, b^{s_1}_{q_1}L_{p_1,q_1})_{\theta,q} = \ell_{p,q}(\mathbb{Z}^n) \times \prod_{G\in G^*} \ell^s_q(2^{-jn/p}\ell_{p,q}(\mathbb{Z}^n)) = b^s_q L_{p,r}.$$

Using again Theorem 9.16 we derive the result. □

Next we are going to characterize $B^s_q L_{p,r}(\mathbb{R}^n)$ as an interpolation space between Triebel-Lizorkin-Lorentz spaces.

Theorem 9.23. Let $-\infty < s_0 \neq s_1 < \infty$, $0 < p, r < \infty$ and $0 < q_0, q_1, q \leq \infty$. Let $0 < \theta < 1$ and $s = (1-\theta)s_0 + \theta s_1$. Then we have with equivalent quasi-norms

$$(F^{s_0}_{q_0}L_{p,r}(\mathbb{R}^n), F^{s_1}_{q_1}L_{p,r}(\mathbb{R}^n))_{\theta,q} = B^s_q L_{p,r}(\mathbb{R}^n).$$

Proof. For $k = 0, 1$, let $0 < \tau_k, \nu_k < \infty$ such that

$$\begin{cases} \tau_k \leq \min\{p, q_k, r\} & \text{for } p \neq q_k \text{ and for } p = q_k \geq r, \\ \tau_k < p & \text{for } p = q_k < r, \end{cases}$$

and

$$\begin{cases} \nu_k \geq \max\{p, q_k, r\} & \text{for } p \neq q_k \text{ and for } p = q_k \leq r, \\ \nu_k > p & \text{for } p = q_k > r. \end{cases}$$

According to the sharp embeddings obtained by Seeger and Trebels [107, Theorems 1.1 and 1.2] we have

$$B^{s_k}_{\tau_k}L_{p,r}(\mathbb{R}^n) \hookrightarrow F^{s_k}_{q_k}L_{p,r}(\mathbb{R}^n) \hookrightarrow B^{s_k}_{\nu_k}L_{p,r}(\mathbb{R}^n), \quad k = 0, 1.$$

Consequently, applying Theorem 9.22, we derive

$$\begin{aligned} B^s_q L_{p,r}(\mathbb{R}^n) &= (B^{s_0}_{\tau_0} L_{p,r}(\mathbb{R}^n), B^{s_1}_{\tau_1} L_{p,r}(\mathbb{R}^n))_{\theta,q} \\ &\hookrightarrow (F^{s_0}_{q_0} L_{p,r}(\mathbb{R}^n), F^{s_1}_{q_1} L_{p,r}(\mathbb{R}^n))_{\theta,q} \\ &\hookrightarrow (B^{s_0}_{\nu_0} L_{p,r}(\mathbb{R}^n), B^{s_1}_{\nu_1} L_{p,r}(\mathbb{R}^n))_{\theta,q} = B^s_q L_{p,r}(\mathbb{R}^n). \end{aligned}$$

□

Now we characterize Besov-Lorentz spaces $B^s_q L_{p,r}(\mathbb{R}^n)$ as approximation spaces (see (2.19)) by means of wavelets and $F^0_2 L_{p,r}(\mathbb{R}^n)$. Although this result is of independent interest, it will allow us to obtain another interpolation result.

We choose $X = F^0_2 L_{p,r}(\mathbb{R}^n)$. Remember that according to (9.24) (and [116, Theorem 3.15] for the case $1 < p < \infty$)

$$F^0_2 L_{p,r}(\mathbb{R}^n) = \begin{cases} L_{p,r}(\mathbb{R}^n) & \text{if} \quad 1 < p < \infty \quad \text{and} \quad 0 < r \le \infty, \\ h_{p,r}(\mathbb{R}^n) & \text{if} \quad 0 < p < \infty \quad \text{and} \quad 0 < r < \infty, \end{cases} \tag{9.28}$$

and by [107, Theorem 1.1] we have $B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow F^0_2 L_{p,r}(\mathbb{R}^n)$.

As the sequence of subsets $(A_u)_{u\in\mathbb{N}_0}$, following an idea of [34, Lemma 5.4] for the case of logarithmic Besov spaces, we choose $A_0 = \{0\}$ and for $u \in \mathbb{N}$ with $2^k \le u \le 2^{k+1}$, $k \in \mathbb{N}_0$,

$$A_u = \{g \in F^0_2 L_{p,r}(\mathbb{R}^n) : g = \sum_{\nu=0}^{k} \sum_{G\in G^\nu} \sum_{m\in\mathbb{Z}^n} c^{\nu,G}_m 2^{-\nu n/2} \psi^\nu_{G,m} \ , \ c^{\nu,G}_m \in \mathbb{C}\},$$

where $(\psi^\nu_{G,m})$ are the wavelets defined in (9.19). Consider also the operators $P_{2^k} : F^0_2 L_{p,r}(\mathbb{R}^n) \longrightarrow F^0_2 L_{p,r}(\mathbb{R}^n)$ defined by

$$P_{2^k} f = \sum_{\nu=0}^{k} \sum_{G\in G^\nu} \sum_{m\in\mathbb{Z}^n} \lambda^{\nu,G}_m(f) 2^{-\nu n/2} \psi^\nu_{G,m}.$$

It is clear that

$$A_u = P_{2^k}(F^0_2 L_{p,r}(\mathbb{R}^n)), \quad 2^k \le u < 2^{k+1}, \quad k \in \mathbb{N}_0\,. \tag{9.29}$$

By the characterization in terms of wavelets for $F^0_2 L_{p,r}(\mathbb{R}^n)$ (see [121] or Theorem 9.16/2) and the Banach-Steinhaus theorem [85, p. 169], we obtain that

$$\sup\{\|P_{2^k} | \mathcal{L}(F^0_2 L_{p,r}(\mathbb{R}^n), F^0_2 L_{p,r}(\mathbb{R}^n))\| : k \in \mathbb{N}\} < \infty. \tag{9.30}$$

It follows from (9.29) and (9.30) that

$$E_{2^k}(f) = E^A_{2^k}(f)_{F^0_2 L_{p,r}(\mathbb{R}^n)} \sim \|f - P_{2^k} f | F^0_2 L_{p,r}(\mathbb{R}^n)\|, \quad k \in \mathbb{N}_0. \tag{9.31}$$

Indeed, given any $g \in A_{2^k}$ we have

$$\begin{aligned} \|f - P_{2^k} f | F^0_2 L_{p,r}(\mathbb{R}^n)\| &\lesssim \|f - g | F^0_2 L_{p,r}(\mathbb{R}^n)\| + \|g - P_{2^k} f | F^0_2 L_{p,r}(\mathbb{R}^n)\| \\ &= \|f - g | F^0_2 L_{p,r}(\mathbb{R}^n)\| + \|P_{2^k}(g - f) | F^0_2 L_{p,r}(\mathbb{R}^n)\| \\ &\lesssim \|f - g | F^0_2 L_{p,r}(\mathbb{R}^n)\|. \end{aligned}$$

Theorem 9.24. Let $s > 0, 0 < p < \infty$ and $0 < q, r \le \infty$. Then we have with equivalence of quasi-norms

$$B^s_q L_{p,r}(\mathbb{R}^n) = (F^0_2 L_{p,r}(\mathbb{R}^n); A_k)^s_q\,.$$

Proof. Using (9.31), the quasi-norm for the approximation space (2.20) and the description of $F^0_2 L_{p,r}(\mathbb{R}^n)$ in terms of wavelets we obtain

$$\begin{aligned}
&\|f|(F^0_2 L_{p,r}(\mathbb{R}^n))^s_q\|^q \sim \|f|F^0_2 L_{p,r}(\mathbb{R}^n)\|^q + \sum_{k=1}^{\infty} 2^{ksq} E_{2^k}(f)^q \\
&\sim \|f|F^0_2 L_{p,r}(\mathbb{R}^n)\|^q + \sum_{k=1}^{\infty} 2^{ksq} \Big\| \sum_{\nu=k+1}^{\infty} \sum_{G\in G^\nu} \sum_{m\in\mathbb{Z}^n} \lambda^{\nu,G}_m(f) 2^{-\nu n/2} \psi^\nu_{G,m} | F^0_2 L_{p,r}(\mathbb{R}^n) \Big\|^q \\
&\sim \|f|F^0_2 L_{p,r}(\mathbb{R}^n)\|^q + \sum_{k=1}^{\infty} 2^{ksq} \Big\| \Big(\sum_{\nu=k+1}^{\infty} \sum_{G\in G^\nu} \sum_{m\in\mathbb{Z}^n} |\lambda^{\nu,G}_m(f) \chi_{\nu,m}(\cdot)|^2 \Big)^{1/2} | L_{p,r}(\mathbb{R}^n) \Big\|^q \\
&= \|f|F^0_2 L_{p,r}(\mathbb{R}^n)\|^q + \sum_{k=1}^{\infty} 2^{ksq} \Big\| \Big(\sum_{\nu=1}^{\infty} \sum_{G\in G^\nu} \sum_{m\in\mathbb{Z}^n} |\lambda^{k+\nu,G}_m(f) \chi_{k+\nu,m}(\cdot)|^2 \Big)^{1/2} | L_{p,r}(\mathbb{R}^n) \Big\|^q. \quad (9.32)
\end{aligned}$$

Take $0 < \rho < \min\{1, q, p, r\}$. Since $\ell_1 \hookrightarrow \ell_2$ and $L_{p,r}$ is a ρ-normed quasi-Banach space, we derive

$$\begin{aligned}
&\Big(\sum_{k=1}^{\infty} 2^{ksq} \Big\| \Big(\sum_{\nu=1}^{\infty} \sum_{G\in G^\nu} \sum_{m\in\mathbb{Z}^n} |\lambda^{k+\nu,G}_m(f) \chi_{k+\nu,m}(\cdot)|^2 \Big)^{1/2} | L_{p,r}(\mathbb{R}^n) \Big\|^q \Big)^{1/q} \\
&\quad \le \Big\{ \Big(\sum_{k=1}^{\infty} \Big(2^{ks\rho} \Big\| \sum_{\nu=1}^{\infty} \sum_{G\in G^\nu} \sum_{m\in\mathbb{Z}^n} |\lambda^{k+\nu,G}_m(f) \chi_{k+\nu,m}(\cdot)| | L_{p,r}(\mathbb{R}^n) \Big\|^\rho \Big)^{q/\rho} \Big)^{\rho/q} \Big\}^{1/\rho} \\
&\quad \le \Big\{ \sum_{\nu=1}^{\infty} \sum_{G\in G^\nu} \Big(\sum_{k=1}^{\infty} \Big(2^{ks\rho} \Big\| \sum_{m\in\mathbb{Z}^n} |\lambda^{k+\nu,G}_m(f) \chi_{k+\nu,m}(\cdot)| | L_{p,r}(\mathbb{R}^n) \Big\|^\rho \Big)^{q/\rho} \Big)^{\rho/q} \Big\}^{1/\rho} \\
&\quad = \Big\{ \sum_{\nu=1}^{\infty} 2^{-\nu s\rho} \sum_{G\in G^\nu} \Big(\sum_{k=1}^{\infty} 2^{(k+\nu)sq} \Big\| \sum_{m\in\mathbb{Z}^n} |\lambda^{k+\nu,G}_m(f) \chi_{k+\nu,m}(\cdot)| | L_{p,r}(\mathbb{R}^n) \Big\|^q \Big)^{\rho/q} \Big\}^{1/\rho} \\
&\quad \lesssim \Big\{ \sum_{\nu=1}^{\infty} 2^{-\nu s\rho} \Big\}^{1/\rho} \|f|B^s_q L_{p,r}(\mathbb{R}^n)\|
\end{aligned}$$

where the last inequality follows from Theorem 9.16/1. Since, as we already said (see [107, Theorem 1.1]), $B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow F^0_2 L_{p,r}(\mathbb{R}^n)$, we conclude that $B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow (F^0_2 L_{p,r}(\mathbb{R}^n))^s_q$.

In order to establish the converse embedding we start from

$$\|f|B^s_q L_{p,r}(\mathbb{R}^n)\| \sim \Big(\sum_{k=0}^{\infty} \sum_{G\in G^k} 2^{ksq} \Big\| \sum_{m\in\mathbb{Z}^n} \lambda^{k,G}_m(f) \chi_{k,m} | L_{p,r}(\mathbb{R}^n) \Big\|^q \Big)^{1/q} \quad (9.33)$$

(see Theorem 9.16). In this sum, the part with $k = 0$ and $k = 1$ can be bounded by

$$\|f|F^0_2 L_{p,r}(\mathbb{R}^n)\| \sim \Big\| \Big(\sum_{k=0}^{\infty} \sum_{G\in G^k} \sum_{m\in\mathbb{Z}^n} |\lambda^{k,G}_m(f) \chi_{k,m}(\cdot)|^2 \Big)^{1/2} | L_{p,r}(\mathbb{R}^n) \Big\|$$

since

$$\begin{aligned}
\Big(\sum_{k=0}^{1}\sum_{G\in G^k} 2^{ksq}\Big\|\sum_{m\in\mathbb{Z}^n}\lambda_m^{k,G}(f)\chi_{k,m}|L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q}
&\lesssim \sum_{k=0}^{1}\sum_{G\in G^k}\Big\|\sum_{m\in\mathbb{Z}^n}\lambda_m^{k,G}(f)\chi_{k,m}|L_{p,r}(\mathbb{R}^n)\Big\|\\
&= \sum_{k=0}^{1}\sum_{G\in G^k}\Big\|\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_m^{k,G}(f)\chi_{k,m}(\cdot)|^2\Big)^{1/2}|L_{p,r}(\mathbb{R}^n)\Big\|\\
&\lesssim \Big\|\Big(\sum_{k=0}^{1}\sum_{G\in G^k}\sum_{m\in\mathbb{Z}^n}|\lambda_m^{k,G}(f)\chi_{k,m}(\cdot)|^2\Big)^{1/2}|L_{p,r}(\mathbb{R}^n)\Big\|\\
&\lesssim \|f|F_2^0L_{p,r}(\mathbb{R}^n)\|.
\end{aligned}$$

For the remaining part of (9.33) we have

$$\begin{aligned}
\sum_{k=2}^{\infty}\sum_{G\in G^k} 2^{ksq}\Big\|\sum_{m\in\mathbb{Z}^n}\lambda_m^{k,G}(f)\chi_{k,m}|L_{p,r}(\mathbb{R}^n)\Big\|^q
&\lesssim \sum_{k=1}^{\infty}2^{(k+1)sq}\Big\|\sum_{G\in G^k}\sum_{m\in\mathbb{Z}^n}|\lambda_m^{k+1,G}(f)\chi_{k+1,m}||L_{p,r}(\mathbb{R}^n)\Big\|^q\\
&\lesssim \sum_{k=1}^{\infty}2^{ksq}\Big\|\Big(\sum_{\nu=1}^{\infty}\sum_{G\in G^\nu}\sum_{m\in\mathbb{Z}^n}|\lambda_m^{k+\nu,G}(f)\chi_{k+\nu,m}(\cdot)|^2\Big)^{1/2}|L_{p,r}(\mathbb{R}^n)\Big\|^q\\
&\lesssim \|f|(F_2^0L_{p,r}(\mathbb{R}^n))_q^s\|^q
\end{aligned}$$

where we have used (9.32) for the last inequality. Consequently,

$$\|f|B_q^sL_{p,r}(\mathbb{R}^n)\| \lesssim \|f|(F_2^0L_{p,r}(\mathbb{R}^n))_q^s\|.$$

This completes de proof. □

We finish the chapter with the following interpolation formula.

Theorem 9.25. Let $0<\theta<1$, $0<s,p<\infty$ and $0<q,u\le\infty$. Then we have with equivalence of quasi-norms

$$(F_2^0L_{p,r}(\mathbb{R}^n), B_q^sL_{p,r}(\mathbb{R}^n))_{\theta,u} = B_u^{\theta s}L_{p,r}(\mathbb{R}^n)\,.$$

Proof. Theorem 9.24 and the interpolation result for approximation spaces stated in Theorem 2.5 yield

$$(F_2^0L_{p,r}(\mathbb{R}^n), B_q^sL_{p,r}(\mathbb{R}^n))_{\theta,u} = (F_2^0L_{p,r}(\mathbb{R}^n), (F_2^0L_{p,r}(\mathbb{R}^n))_q^s)_{\theta,u} = (F_2^0L_{p,r}(\mathbb{R}^n))_u^{\theta s} = B_u^{\theta s}L_{p,r}(\mathbb{R}^n)\,.$$

□

Chapter 10

Key problems for Lorentz-Sobolev spaces

In this chapter we continue the research on Besov-Lorentz and Triebel-Lizorkin-Lorentz spaces ($B^s_q L_{p,r}(\mathbb{R}^n)$ and $F^s_q L_{p,r}(\mathbb{R}^n)$, respectively) introduced in Definition 9.1. The aim now is to extend some well known properties of classical Besov and Triebel-Lizorkin spaces to our setting. In particular, we focus on the key problems stated by Triebel in [112, Chapter 4] for the classical spaces and that are significant for their applications to PDEs. These problems are: Invariance of $A^s_q L_{p,r}(\mathbb{R}^n)$, $A \in \{B, F\}$ with respect to diffeomorphisms of $\mathbb{R}^n$ onto itself, the existence of linear extension operators of corresponding spaces $A^s_q L_{p,r}(\mathbb{R}^n_+)$ on $\mathbb{R}^n_+$ to $A^s_q L_{p,r}(\mathbb{R}^n)$, pointwise multipliers, multiplication properties of $A^s_q L_{p,r}(\mathbb{R}^n)$ and traces of $A^s_q L_{p,r}(\mathbb{R}^n)$ on hyperplanes.

In Section 10.1, we discuss the first three problems stated above. It turns out that for Triebel-Lizorkin spaces they follow straightforward from interpolation formulae in Theorem 9.17 and the already known results for classical $F^s_{p,q}(\mathbb{R}^n)$ spaces. In the case of Besov-Lorentz spaces, the outcome is a consequence of interpolation formula in Theorem 9.23 and the results for Triebel-Lizorkin-Lorentz spaces.

In Section 10.2, we study multiplication properties for Triebel-Lizorkin-Lorentz spaces and Besov-Lorentz spaces. We give sufficient conditions for them to be multiplication algebras. The results given here are also based on interpolation techniques, but this time we need more refined arguments involving sometimes not only the real method, but also complex interpolation and approximation results.

Finally, in Section 3, we focus on the problem of traces on hyperplanes. When studying traces for $F^s_q L_{p,r}(\mathbb{R}^n)$ we need to deal with the interpolation space

$$(B^{s-1/p_0}_{p_0,p_0}(\mathbb{R}^{n-1}), B^{s-1/p_1}_{p_1,p_1}(\mathbb{R}^{n-1}))_{\theta,r}.$$

In Theorem 9.21, it is shown that if $r = p$ with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ then

$$(B^{s-1/p_0}_{p_0,p_0}(\mathbb{R}^{n-1}), B^{s-1/p_1}_{p_1,p_1}(\mathbb{R}^{n-1}))_{\theta,r} = B^{s-1/p}_{p,p}(\mathbb{R}^{n-1}).$$

However, for if $r \neq p$ the characterization of this interpolation space is an open problem already stated by Peetre in [101, p.110]. Here we give a characterization of the interpolation space in terms of wavelets allowing us to describe the trace of Triebel-Lizorkin-Lorentz spaces. The section finishes with the computation of traces for $B^s_q L_{p,q}(\mathbb{R}^n)$ using interpolation formula in Theorem 9.21.

The results in this chapter are taken from the preprints [20, 21] and the work in preparation [17].

10.1 Invariance with respect to diffeomorphisms, existence of linear extension operators and pointwise multipliers

We say that a continuous bijective map of $\mathbb{R}^n$ onto itself

$$y = \psi(x) = (\psi_1(x), \psi_2(x), ..., \psi_n(x)), \quad x \in \mathbb{R}^n,$$
$$x = \psi^{-1}(y) = (\psi_1^{-1}(y), ..., \psi_n^{-1}(y)), \quad y \in \mathbb{R}^n$$

is a *differomorphism* if all components $\psi_j(x)$ and $\psi_j^{-1}(y)$ are real C^∞ functions on $\mathbb{R}^n$ and for $j = 1, 2, ..., n$

$$\sup_{x\in\mathbb{R}^n} (|\partial^\alpha \psi_j(x)| + |\partial^\alpha \psi_j^{-1}(x)|) < \infty, \quad \text{for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| > 0.$$

If ψ is a diffeomorphism, it is straightforward that

$$\begin{aligned} \mathcal{S}(\mathbb{R}^n) &\longrightarrow \mathcal{S}(\mathbb{R}^n) \\ \varphi &\longrightarrow (\varphi \circ \psi)(x) = \varphi(\psi(x)), \end{aligned}$$

is a one-to-one map from $\mathcal{S}(\mathbb{R}^n)$ onto itself. It can be extended to a one-to-one map from $\mathcal{S}'(\mathbb{R}^n)$ onto itself

$$\begin{aligned} \mathcal{S}'(\mathbb{R}^n) &\longrightarrow \mathcal{S}'(\mathbb{R}^n) \\ f &\longrightarrow (f \circ \psi)(\varphi) := (f, |\det J_{\psi^{-1}}| \varphi \circ \psi^{-1}), \end{aligned}$$

where $J_{\psi^{-1}}$ is the Jacobian of ψ^{-1}.

It turns out (see [118, Section 2.3]) that if ψ is a diffeomorphism, for every $s \in \mathbb{R}$ and $0 < p, q \leq \infty$,

$$\begin{aligned} D_\psi : A^s_{p,q} &\longrightarrow A^s_{p,q}, \quad A \in \{B, F\} \\ f &\longrightarrow D_\psi f := f \circ \psi, \end{aligned} \tag{10.1}$$

is a linear isomorphic map.

Theorem 10.1. Let ψ be a diffeomorphism.

(1) If $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q, r \leq \infty$, then

$$\begin{aligned} D_\psi : F^s_q L_{p,r}(\mathbb{R}^n) &\longrightarrow F^s_q L_{p,r}(\mathbb{R}^n) \\ f &\longrightarrow D_\psi f = f \circ \psi, \end{aligned}$$

is an isomorphic map in $F^s_q L_{p,r}(\mathbb{R}^n)$.

(2) If $s \in \mathbb{R}$, $0 < p, r < \infty$ and $0 < q \leq \infty$, then

$$\begin{aligned} D_\psi : B^s_q L_{p,r}(\mathbb{R}^n) &\longrightarrow B^s_q L_{p,r}(\mathbb{R}^n) \\ f &\longrightarrow D_\psi f = f \circ \psi, \end{aligned}$$

is an isomorphic map in $B^s_q L_{p,r}(\mathbb{R}^n)$.

Proof. The proof of (1) is a direct consequence of (10.1) and the interpolation result obtained in Theorem 9.17. On the other hand, (2) follows from (1) and Theorem 9.23. □

For $\rho \in \mathbb{R}$, consider now the Hölder-Zygmund space $C^\rho(\mathbb{R}^n) = B^\rho_{\infty,\infty}(\mathbb{R}^n)$. According to [118, Theorem 2.30], if $s \in \mathbb{R}$, $0 < p, q \leq \infty$ and $g \in C^\rho(\mathbb{R}^n)$ with $\rho > \max\left\{s, n\left(\max\{\frac{1}{p}, 1\} - 1\right) - s\right\}$ is a *pointwise multiplier* for $F^s_{p,q}(\mathbb{R}^n)$. That is to say, the operator induced by pointwise multiplication by g

$$\begin{aligned} T_g : F^s_{p,q}(\mathbb{R}^n) &\longrightarrow F^s_{p,q}(\mathbb{R}^n) \\ f &\longrightarrow T_g f = fg, \end{aligned}$$

is continuous in $F^s_{p,q}(\mathbb{R}^n)$. Furthermore, there is a constant $C > 0$ such that

$$\|gf|F^s_{p,q}(\mathbb{R}^n)\| \leq C\|g|C^\rho(\mathbb{R}^n)\| \cdot \|f|F^s_{p,q}(\mathbb{R}^n)\|, \quad \text{for every } g \in C^\rho(\mathbb{R}^n) \text{ and } f \in F^s_{p,q}(\mathbb{R}^n). \tag{10.2}$$

Now we extend this result to Lorentz smoothness spaces.

Theorem 10.2. Let $0 < q, r \leq \infty$, $0 < p < \infty$, $s \in \mathbb{R}$ and $\rho > \max\left\{s, n\left(\max\{\frac{1}{p}, 1\} - 1\right) - s\right\}$. Then $g \in C^\rho(\mathbb{R}^n)$ is a pointwise multiplier for $F^s_q L_{p,r}(\mathbb{R}^n)$. Furthermore, there is a constant $C > 0$ such that

$$\|gf|F^s_q L_{p,r}(\mathbb{R}^n)\| \leq C\|g|C^\rho(\mathbb{R}^n)\| \cdot \|f|F^s_q L_{p,r}(\mathbb{R}^n)\|, \quad \text{for all } g \in C^\rho(\mathbb{R}^n) \text{ and } f \in F^s_q L_{p,r}(\mathbb{R}^n). \tag{10.3}$$

If, in addition, $0 < r < \infty$, then $g \in C^\rho(\mathbb{R}^n)$ is a pointwise multiplier for $B^s_q L_{p,r}(\mathbb{R}^n)$. Furthermore, there is a constant $C > 0$ such that

$$\|gf|B^s_q L_{p,r}(\mathbb{R}^n)\| \leq C\|g|C^\rho(\mathbb{R}^n)\| \cdot \|f|B^s_q L_{p,r}(\mathbb{R}^n)\|, \quad \text{for all } g \in C^\rho(\mathbb{R}^n) \text{ and } f \in B^s_q L_{p,r}(\mathbb{R}^n). \tag{10.4}$$

Proof. As in the proof of the previous theorem, (10.3) follows from (10.2) and the interpolation result obtained in Theorem 9.17. Conversely, (10.4) follows from (10.3) and Theorem 9.23. □

Now let $\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$. As usual, we define $A^s_q L_{p,r}(\mathbb{R}^n_+)$, $A \in \{B, F\}$, by restriction of $A^s_q L_{p,r}(\mathbb{R}^n)$ with

$$\|f|A^s_q L_{p,r}(\mathbb{R}^n_+)\| = \inf\{\|g|A^s_q L_{p,r}(\mathbb{R}^n)\| : g|_{\mathbb{R}^n_+} = f\}, \quad A \in \{B, F\}.$$

The restriction operator $\operatorname{re} f = f|_{\mathbb{R}^n_+}$ defines a continuous linear map from $A^s_q L_{p,r}(\mathbb{R}^n)$ onto $A^s_q L_{p,r}(\mathbb{R}^n_+)$, $A \in \{B, F\}$. We wonder if there is a continuous linear operator

$$\operatorname{ext} : A^s_q L_{p,r}(\mathbb{R}^n_+) \longrightarrow A^s_q L_{p,r}(\mathbb{R}^n),$$

such that $\operatorname{re} \circ \operatorname{ext} = \operatorname{id}$, where id is the identity map in $A^s_q L_{p,r}(\mathbb{R}^n_+)$. This fact is known for classical spaces (see [118, Section 2.5.2]). For Lorentz smoothness spaces we have the following:

Theorem 10.3. Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q, r \leq \infty$. There is a continuous linear (and even universal) extension operator

$$\operatorname{ext} : F^s_q L_{p,r}(\mathbb{R}^n_+) \longrightarrow F^s_q L_{p,r}(\mathbb{R}^n),$$

such that $\operatorname{re} \circ \operatorname{ext} = \operatorname{id}$ in $F^s_q L_{p,r}(\mathbb{R}^n_+)$.

If, in addition, $0 < r < \infty$, then there is also a continuous linear (and even universal) extension operator from $B^s_q L_{p,r}(\mathbb{R}^n_+)$ to $B^s_q L_{p,r}(\mathbb{R}^n)$.

Proof. Let $0 < p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Interpolating $\mathrm{re} : F^s_{p_j,q}(\mathbb{R}^n) \longrightarrow F^s_{p_j,q}(\mathbb{R}^n_+)$ by the real method with parameters (θ, r) and using Theorem 9.17, we get that

$$\mathrm{re} : F^s_q L_{p,r}(\mathbb{R}^n) \longrightarrow (F^s_{p_0,q}(\mathbb{R}^n_+), F^s_{p_1,q}(\mathbb{R}^n_+))_{\theta,r}$$

is continuous. This yields that

$$F^s_q L_{p,r}(\mathbb{R}^n_+) \hookrightarrow (F^s_{p_1,q}(\mathbb{R}^n_+), F^s_{p_2,q}(\mathbb{R}^n_+))_{\theta,r}.$$

Besides, by the assertions on extensions for the spaces $F^s_{p,q}(\mathbb{R}^n)$ (see [118, Section 2.5.2] and the references given there), there is a continuous linear operator $\mathrm{ext} : F^s_{p_j,q}(\mathbb{R}^n_+) \longrightarrow F^s_{p_j,q}(\mathbb{R}^n)$, $j = 0, 1$, such that $\mathrm{re} \circ \mathrm{ext} = \mathrm{id}$, the identity map in $F^s_{p_j,q}(\mathbb{R}^n_+)$. Using again the interpolation theorem and Theorem 9.17, we obtain that

$$\mathrm{ext} : (F^s_{p_0,q}(\mathbb{R}^n_+), F^s_{p_1,q}(\mathbb{R}^n_+))_{\theta,r} \longrightarrow F^s_q L_{p,r}(\mathbb{R}^n)$$

is bounded. Take any $f \in (F^s_{p_0,q}(\mathbb{R}^n_+), F^s_{p_1,q}(\mathbb{R}^n_+))_{\theta,r}$. Then $\mathrm{ext}\, f \in F^s_q L_{p,r}(\mathbb{R}^n)$ and so $f = \mathrm{re}(\mathrm{ext}\, f) \in F^s_q L_{p,r}(\mathbb{R}^n_+)$. Furthermore,

$$\begin{aligned}\|f|F^s_q L_{p,r}(\mathbb{R}^n_+)\| &\le \|\mathrm{ext}\, f|F^s_q L_{p,r}(\mathbb{R}^n)\| \\ &\le c\|f|(F^s_{p_0,q}(\mathbb{R}^n_+), F^s_{p_1,q}(\mathbb{R}^n_+))_{\theta,r}\|.\end{aligned}$$

Consequently, $F^s_q L_{p,r}(\mathbb{R}^n_+) = (F^s_{p_0,q}(\mathbb{R}^n_+), F^s_{p_1,q}(\mathbb{R}^n_+))_{\theta,r}$ with equivalence of quasi-norms and

$$\mathrm{ext} : F^s_q L_{p,r}(\mathbb{R}^n_+) \longrightarrow F^s_q L_{p,r}(\mathbb{R}^n)$$

is continuous and satisfies that $\mathrm{re} \circ \mathrm{ext} = \mathrm{id}$ in $F^s_q L_{p,r}(\mathbb{R}^n_+)$.

Proceeding in a similar way but using Theorem 9.23 and the result we just proved for Triebel-Lizorkin spaces with Lorentz smoothness $F^s_q L_{p,r}(\mathbb{R}^n)$, we obtain the proof for spaces $B^s_q L_{p,r}(\mathbb{R}^n)$. □

Similarly, let Ω be a domain (a non-empty open set) in $\mathbb{R}^n$ and define $A^s_q L_{p,r}(\Omega)$ by restriction, $A \in \{B, F\}$. Proceeding as in Theorem 10.3 but using now the corresponding results on extensions for the spaces $A^s_{p,q}(\Omega)$ (see [118, Section 2.6.4]) one can derive the following.

Theorem 10.4. Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q, r \le \infty$ and let Ω be a bounded C^∞ domain in $\mathbb{R}^n$. There is a continuous linear (and even universal) extension operator from $F^s_q L_{p,r}(\Omega)$ to $F^s_q L_{p,r}(\mathbb{R}^n)$. If, in addition, $0 < r < \infty$, then there is also a continuous linear (and even universal) extension operator from $B^s_q L_{p,r}(\Omega)$ to $B^s_q L_{p,r}(\mathbb{R}^n)$

10.2 Multiplication algebras

Now we consider some properties of the multiplication of functions. If the quasi-Banach spaces X and Y are formed by regular distributions on $\mathbb{R}^n$, then for $f \in X$ and $g \in Y$ the product $f \cdot g$ makes sense almost everywhere. This is the case of spaces $A^s_{p,q}(\mathbb{R}^n)$, $A \in \{B, F\}$ when $0 < p < \infty$, $0 < q \le \infty$ and $s > n\left(\frac{1}{p} - 1\right)_+$, since $A^s_{p,q}(\mathbb{R}^n) \hookrightarrow L^{\mathrm{loc}}_1(\mathbb{R}^n)$ (see [106, Theorem 2.2.4/2, p.33] or [118, Theorem 2.4]). Using Theorems 9.21 and 9.17, it follows that $A^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow L^{\mathrm{loc}}_1(\mathbb{R}^n)$ for $0 < p < \infty$, $0 < q, r \le \infty$, $s > n\left(\frac{1}{p} - 1\right)_+$ and $A \in \{B, F\}$. Indeed, let $u = \max\{r, q\}$ and take

$0 < p_0 < p < p_1 < \infty$ such that $s > n\left(\frac{1}{p_0} - 1\right)_+$ and $0 < \theta < 1$ verifying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{1}{p_1}$. Then,

$$B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow B^s_u L_{p,u}(\mathbb{R}^n) = (B^s_{p_0,u}(\mathbb{R}^n), B^s_{p_1,u}(\mathbb{R}^n))_{\theta,u} \hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^n),$$
$$F^s_q L_{p,r}(\mathbb{R}^n) = (F^s_{p_0,q}(\mathbb{R}^n), F^s_{p_1,q}(\mathbb{R}^n))_{\theta,r} \hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^n).$$

If the spaces X and Y are formed by regular distributions, we put $X \cdot Y \hookrightarrow Z$ if there is a constant $C > 0$ such that $\|fg|Z\| \le C\|f|X\| \cdot \|g|Y\|$, for any $f \in X$ and $g \in Y$.

Classical Hölder inequality says that

$$L_{p_0}(\mathbb{R}^n) \cdot L_{p_1}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$$

provided that $0 < p_0, p_1 \le \infty$ and $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$.

For Lorentz spaces, the corresponding result reads

$$L_{p_0,q_0}(\mathbb{R}^n) \cdot L_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow L_{p,q}(\mathbb{R}^n) \tag{10.5}$$

provided that $0 < p_0, p_1 < \infty$, $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$, $0 < q_0, q_1 \le \infty$ and $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1}$. See [96] for the Banach case. In general, it can be derived from the interpolation result for multilinear operators given by Janson in [81].

10.2.1 Triebel-Lizorkin-Lorentz spaces

Triebel shows in his books [115, 117] that multiplication properties of Triebel-Lizorkin spaces are important for their applications to Navier-Stokes and nonlinear heat equations. Now we study the case of spaces $F^s_q L_{p,r}(\mathbb{R}^n)$.

We start with a preliminary result when $s = 0$ which extends Hölder inequality for Lorentz spaces (10.5).

Proposition 10.5. Let $1 < p_0, p_1, p < \infty$ with $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$, let $0 < q_0, q_1 \le 2 \le q \le \infty$ and $0 < r_0, r_1, r \le \infty$ with $\frac{1}{r} = \frac{1}{r_0} + \frac{1}{r_1}$. Then we have

$$F^0_{q_0} L_{p_0,r_0}(\mathbb{R}^n) \cdot F^0_{q_1} L_{p_1,r_1}(\mathbb{R}^n) \hookrightarrow F^0_q L_{p,r}(\mathbb{R}^n).$$

Proof. Since $q_j \le 2 \le q$, it is clear from Definition 9.1 that

$$F^s_{q_j} L_{p,r}(\mathbb{R}^n) \hookrightarrow F^s_2 L_{p,r}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p,r}(\mathbb{R}^n),\ j = 0, 1,\ s \in \mathbb{R}.$$

Hence, by (9.28) and Hölder inequality for Lorentz spaces (10.5), we get

$$F^0_{q_0} L_{p_0,r_0}(\mathbb{R}^n) \cdot F^0_{q_1} L_{p_1,r_1}(\mathbb{R}^n) \hookrightarrow F^0_2 L_{p_0,r_0}(\mathbb{R}^n) \cdot F^0_2 L_{p_1,r_1}(\mathbb{R}^n)$$
$$= L_{p_0,r_0}(\mathbb{R}^n) \cdot L_{p_1,r_1}(\mathbb{R}^n) \hookrightarrow L_{p,r}(\mathbb{R}^n) \hookrightarrow F^0_q L_{p,r}(\mathbb{R}^n).$$

□

We study now the case of spaces with positive smoothness s and with differential dimension $(s - n/p)$ satisfying the same condition as in Proposition 10.5, that is, the sum of the differential dimension of the spaces in the source equal to the differential dimension of the space in the target. We put $\frac{1}{p^s} = \frac{1}{p} + \frac{s}{n}$ and we base our arguments on the following result for classical Triebel-Lizorkin spaces (see [57, Theorem 2.4.3, p. 52] or [106, Theorem 4.8.2/1, p. 238]): If $s > 0$, $1 < p_0, p_1, p < \infty$ with $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$ and $0 < q \leq \infty$, then

$$F^s_{p^s_0,q}(\mathbb{R}^n) \cdot F^s_{p^s_1,q}(\mathbb{R}^n) \hookrightarrow F^s_{p^s,q}(\mathbb{R}^n). \tag{10.6}$$

Theorem 10.6. Let $s > 0$, $0 < q, r_0, r_1 \leq \infty$ and $1 < p_0, p_1, p < \infty$ with $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$. Then

$$F^s_q L_{p^s_0,r_0}(\mathbb{R}^n) \cdot F^s_q L_{p^s_1,r_1}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p^s,r}(\mathbb{R}^n)$$

where $\frac{1}{p^s_0} = \frac{1}{p_0} + \frac{s}{n}$, $\frac{1}{p^s_1} = \frac{1}{p_1} + \frac{s}{n}$, $\frac{1}{p^s} = \frac{1}{p} + \frac{s}{n}$ and $r \geq \min\{r_0, r_1\}$.

Proof. Fix p_1. Given any $0 < \varepsilon < \min\{1, p_0 - 1, p_1 - 1\}$, let $\delta = \varepsilon p_1/(p_0 + p_1)$. Then

$$\frac{1+\varepsilon}{p_0} + \frac{1}{p_1} = \frac{1+\delta}{p} \quad \text{and} \quad \frac{1-\varepsilon}{p_0} + \frac{1}{p_1} = \frac{1-\delta}{p}.$$

Write

$$u_1 = \frac{p_0}{1+\varepsilon}, \quad u_2 = \frac{p_0}{1-\varepsilon}, \quad v_1 = \frac{p}{1+\delta}, \quad v_2 = \frac{p}{1-\delta}.$$

According to (10.6), there is a constant $c > 0$ such that for any $g \in F^s_{p^s_1,q}(\mathbb{R}^n)$, the operator $Tf = fg$ is bounded from $F^s_{u^s_j,q}(\mathbb{R}^n)$ into $F^s_{v^s_j,q}(\mathbb{R}^n)$ with norm less than or equal to c for $j = 1, 2$. Since

$$\frac{1/2}{u^s_1} + \frac{1/2}{u^s_2} = \frac{1}{p^s_0} \quad \text{and} \quad \frac{1/2}{v^s_1} + \frac{1/2}{v^s_2} = \frac{1}{p^s},$$

using the interpolation theorem (Theorem 2.1) with parameters $(1/2, r_0)$ and Theorem 9.17/(1), we derive that

$$F^s_q L_{p^s_0,r_1}(\mathbb{R}^n) \cdot F^s_{p^s_1,q}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p^s,r_1}(\mathbb{R}^n). \tag{10.7}$$

To complete the proof we keep now p_0 fixed and apply the same argument but using this time Theorem 9.17/(1) to identify the resulting spaces. This produces the target space $F^s_q L_{p^s,r_1}(\mathbb{R}^n)$. The last index (r_1) can be improved to $r \geq \min\{r_0, r_1\}$ because we can start by fixing p_0 and then continue fixing p_1. □

We focus now on cases where the target space has positive differential dimension. For the next theorem we cannot guarantee that the involved function spaces consist only of regular distributions. Therefore, here we will understand the multiplication of general distributions in the sense of [106, Definition 4.2.1/1].

Theorem 10.7. Assume that $0 < p_0, p_1, p < \infty$ with $\max\{\frac{1}{p_0}, \frac{1}{p_1}\} < \frac{1}{p} < \frac{1}{p_0} + \frac{1}{p_1}$. Let $s > n\left(\frac{1}{p_0} + \frac{1}{p_1} - \frac{1}{p}\right)$ and $2s > n \max\{\frac{1}{p_0} + \frac{1}{p_1} - 1, 0\}$. Let $0 < q, r_0, r_1 \leq \infty$ and $r \geq \min\{r_0, r_1\}$. Then

$$F^s_q L_{p_0,r_0}(\mathbb{R}^n) \cdot F^s_q L_{p_1,r_1}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p,r}(\mathbb{R}^n).$$

Proof. According to [106, Theorem 4.8.2/2, p. 239], we have that

$$F^s_{p_0,q}(\mathbb{R}^n) \cdot F^s_{p_1,q}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n).$$

Then the result follows proceeding similarly as in the proof of the previous theorem and using Theorem 9.17. □

Hobus and Saal proved on their paper on Triebel-Lizorkin-Lorentz spaces and the Navier-Stokes equations (see [78, Lemma 6.4 and Remark 1.2]), the following multiplication result:

Let $s > 0$, $1 < p, q < \infty$ and $1 \leq r \leq \infty$. Then there exists $\bar{\varepsilon} > 0$ such that for all $0 < \varepsilon < \bar{\varepsilon}$ we have

$$F^s_q L_{2p-\varepsilon,r}(\mathbb{R}^n) \cdot F^s_q L_{2p-\varepsilon,r}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p,r}(\mathbb{R}^n). \tag{10.8}$$

Note that Theorem 10.7 allows to extend (10.8). Indeed, given any $s > 0$ we may allow $n/(2s + n) < p < \infty$ and $0 < q, r \leq \infty$, because if

$$\bar{\varepsilon} = \min\Big\{p, \frac{2p^2 s}{n + ps}, \frac{4sp + 2n(p-1)}{2s + n}\Big\},$$

then for any $0 < \varepsilon < \bar{\varepsilon}$, embedding (10.8) follows from Theorem 10.7.

According to [118, Section 2.7] if $0 < p_0 \leq p_1 < \infty$, $s > n/p_0$ and $0 < q \leq \infty$, one has

$$F^s_{p_1,q}(\mathbb{R}^n) \cdot F^s_{p_0,q}(\mathbb{R}^n) \hookrightarrow F^s_{p_0,q}(\mathbb{R}^n). \tag{10.9}$$

Note that this implies that any element in $F^s_{p_1,q}(\mathbb{R}^n)$ is a pointwise multiplier for $F^s_{p_0,q}(\mathbb{R}^n)$ (in the sense we saw in last section). Furthermore, when $p_0 = p_1$ formula (10.9) shows that $F^s_{p_0,q}(\mathbb{R}^n)$ is a multiplication algebra. With the help of Theorem 9.17 and proceeding in the same way we did in Theorem 10.6 we can derive the following result for Triebel-Lizorkin spaces with Lorentz smoothness.

Theorem 10.8. Let $0 < p_0 < p_1 < \infty$, $0 < q \leq \infty$, $s > n/p_0$ and $r \geq \min\{r_0, r_1\}$. Then

$$F^s_q L_{p_1,r_1}(\mathbb{R}^n) \cdot F^s_q L_{p_0,r_0}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p_0,r}(\mathbb{R}^n).$$

The last result shows that any element of $F^s_q L_{p_1,r}(\mathbb{R}^n)$ is a pointwise multiplier for $F^s_q L_{p_0,r}(\mathbb{R}^n)$ with $0 < p_0 < p_1 < \infty$, $s > n/p_0$ and $0 < q, r \leq \infty$. However the preceding argument does not cover the case $p_0 = p_1$, which corresponds to multiplication algebra. Nevertheless, using the bilinear interpolation theorem (Theorem 2.4), we can establish the following result.

Theorem 10.9. Let $0 < p < \infty$, $s > n/p$, $0 < q \leq \infty$, $0 < r \leq \min\{1, q\}$ and $r < p$. Then $F^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra.

Proof. Take $r < p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $s > n/p_0$. By (10.9), the bilinear operator $T(f, g) = fg$ is bounded acting from $F^s_{p_j,q}(\mathbb{R}^n) \times F^s_{p_j,q}(\mathbb{R}^n)$ into $F^s_{p_j,q}(\mathbb{R}^n)$, $j = 0, 1$. The space $F^s_{p_j,q}(\mathbb{R}^n)$ is a $\min\{1, p_j, q\}$-normed quasi-Banach space and so it is r-normed. Since, by Theorem 9.17/(1), we have that $(F^s_{p_0,q}(\mathbb{R}^n), F^s_{p_1,q}(\mathbb{R}^n))_{\theta,r} = F^s_q L_{p,r}(\mathbb{R}^n)$, applying Theorem 2.4 we derive that

$$T : F^s_q L_{p,r}(\mathbb{R}^n) \times F^s_q L_{p,r}(\mathbb{R}^n) \longrightarrow F^s_q L_{p,r}(\mathbb{R}^n) \text{ continuously.}$$

In other words, $F^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplicative algebra. □

In particular, $F^s_q L_{p,1}(\mathbb{R}^n)$ is both a Banach space and a multiplicative algebra for $1 < p < \infty$, $s > n/p$ and $1 \leq q \leq \infty$. Otherwise, for these p, s and q, if $1 \leq r_0 \leq 2$ and $\frac{1}{r_1} = \frac{2}{r_0} - 1$, then proceeding as in Theorem 10.9 we derive from the interpolation theorem for bilinear operators (Theorem 2.4) that

$$F^s_q L_{p,r_0}(\mathbb{R}^n) \cdot F^s_q L_{p,r_0}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p,r_1}(\mathbb{R}^n). \tag{10.10}$$

In Theorem 10.9 we need that $0 < r \le 1$. Next we use the complex interpolation theorem for bilinear operators to show some spaces $F^s_q L_{p,r}(\mathbb{R}^n)$ with $r > 1$ which are also multiplication algebras.

Theorem 10.10. Let $1 < r < p < \infty$, $s > n/p$ and $1 \le q \le \infty$. Then $F^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra.

Proof. We take $0 < \delta < 1$ with

$$1 - (1/r - 1/p) < \delta \tag{10.11}$$

and such that δ is so close to 1 that $u = \delta p$ satisfies that $1 < u$, $s > n/u$ and

$$1/u < 1/r. \tag{10.12}$$

Put

$$\theta = \frac{1 - 1/r}{1 - 1/u}\,.$$

Then we have $0 < \theta < 1$, where the second inequality follows from (10.12). Moreover, we have

$$1/r = (1-\theta) + \theta/u. \tag{10.13}$$

Write

$$1/p_1 = (1-\theta)^{-1}(1/p - \theta/u).$$

By (10.11), it follows that $0 < 1/p_1 < 1/p$ and $s > n/p_1$. Furthermore,

$$1/p = (1-\theta)/p_1 + \theta/u. \tag{10.14}$$

Consider the operators $Jf = \left((\varphi_k \hat{f})^\vee\right)$ and $R(f_k) = \sum_{k=0}^{\infty} (\tilde{\varphi}_k \hat{f}_k)^\vee$ where $(\varphi_k)_{k\in\mathbb{N}_0}$ is a smooth resolution of unity (2.32) and $\tilde{\varphi}_k = \varphi_{k-1} + \varphi_k + \varphi_{k+1}$ and $\varphi_{-1} = 0$. The restrictions

$$J : F^s_q L_{p_1,1}(\mathbb{R}^n) \longrightarrow L_{p_1,1}(\ell^s_q)\,,$$

$$J : F^s_{u,q}(\mathbb{R}^n) \longrightarrow L_u(\ell^s_q)\,,$$

$$R : L_{p_1,1}(\ell^s_q) \longrightarrow F^s_q L_{p_1,1}(\mathbb{R}^n)\,,$$

$$R : L_u(\ell^s_q) \longrightarrow F^s_{u,q}(\mathbb{R}^n)$$

are continuous. Moreover, by the reiteration formula for the complex method (Theorem 2.11) and interpolation formula (9.4), we have

$$[L_{p_1,1}(\ell^s_q), L_u(\ell^s_q)]_\theta = L_{p,r}(\ell^s_q) \text{ (equivalent norms)}.$$

Since $R(Jf) = f$, it follows from [110, Theorem 1.2.4] that

$$[F^s_q L_{p_1,1}(\mathbb{R}^n), F^s_{u,q}(\mathbb{R}^n)]_\theta = F^s_q L_{p,r}(\mathbb{R}^n) \text{ (equivalent norms)}. \tag{10.15}$$

The spaces $F^s_q L_{p_1,1}(\mathbb{R}^n)$ and $F^s_{u,q}(\mathbb{R}^n)$ are multiplication algebras by Theorem 10.9 and [106, Theorem 4.6.4/1]. Therefore, applying Theorem 2.12 to the bilinear operator $T(f,g) = fg$ and using (10.15) we obtain that $F^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra. □

Our next and last result concerning multiplication of Triebel-Lizorkin spaces with Lorentz smoothness fixes all parameters except for the s and it reads as follows:

Theorem 10.11. Let $0 < p < \infty$, $s > \max\{\sigma, n/p\}$ and $0 < q, r \leq \infty$. Then

$$F_q^s L_{p,r}(\mathbb{R}^n) \cdot F_q^s L_{p,r}(\mathbb{R}^n) \hookrightarrow F_q^\sigma L_{p,r}(\mathbb{R}^n).$$

Proof. Take $p < p_1 < \infty$ such that $s - \frac{n}{p} = \sigma - \frac{n}{p_1}$. By Jawerth-Franke type embedding proved in [107, Theorem 1.6], we have that $F_q^s L_{p,r}(\mathbb{R}^n) \hookrightarrow F_q^\sigma L_{p_1,r}(\mathbb{R}^n)$. Moreover, as a consequence of Definition 9.1, $F_q^s L_{p,r}(\mathbb{R}^n) \hookrightarrow F_q^\sigma L_{p,r}(\mathbb{R}^n)$. Consequently, according to Theorem 10.8, we obtain

$$F_q^s L_{p,r}(\mathbb{R}^n) \cdot F_q^s L_{p,r}(\mathbb{R}^n) \hookrightarrow F_q^\sigma L_{p_1,r}(\mathbb{R}^n) \cdot F_q^\sigma L_{p,r}(\mathbb{R}^n) \hookrightarrow F_q^\sigma L_{p,r}(\mathbb{R}^n).$$

□

10.2.2 Besov-Lorentz spaces

Now we study under which hypothesis Besov spaces with Lorentz smoothness are multiplication algebras. We start with a result that is a consequence of the interpolation formula obtained in Theorem 9.23 and the multiplication properties we have proved for $F_q^s L_{p,r}(\mathbb{R}^n)$ spaces.

Theorem 10.12. Let $0 < p < \infty$, $s > n/p$, $0 < q \leq r \leq 1$ and $r < p$. Then $B_q^s L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra.

Proof. Take $r < p_0 < p < p_1 < \infty$, $\frac{n}{p_0} < s_0 < s < s_1 < \infty$ and $0 < \tau, \theta < 1$ with $\frac{1}{p} = \frac{1-\tau}{p_0} + \frac{\tau}{p_1}$ and $s = (1-\theta)s_0 + \theta s_1$. According to Theorem 9.17 we have that

$$F_r^{s_j} L_{p,r}(\mathbb{R}^n) = (F_{p_0,r}^{s_j}(\mathbb{R}^n), F_{p_1,r}^{s_j}(\mathbb{R}^n))_{\tau,r}.$$

Since spaces $F_{p_j,r}^{s_j}(\mathbb{R}^n)$ are r-normed quasi-Banach spaces, it follows from Proposition 2.2 that $F_r^{s_j} L_{p,r}(\mathbb{R}^n)$ is also r-Banach. Consider now the bilinear operator $T(f,g) = fg$. By Theorem 10.9, the restrictions

$$T : F_r^{s_j} L_{p,r}(\mathbb{R}^n) \times F_r^{s_j} L_{p,r}(\mathbb{R}^n) \longrightarrow F_r^{s_j} L_{p,r}(\mathbb{R}^n), \quad j = 0, 1,$$

are bounded. Moreover, by Theorem 9.23, we have

$$(F_r^{s_0} L_{p,r}(\mathbb{R}^n), F_r^{s_1} L_{p,r}(\mathbb{R}^n))_{\theta,q} = B_q^s L_{p,r}(\mathbb{R}^n).$$

Consequently, applying the real interpolation theorem for bilinear operators (Theorem 2.4), we conclude that

$$B_q^s L_{p,r}(\mathbb{R}^n) \cdot B_q^s L_{p,r}(\mathbb{R}^n) \hookrightarrow B_q^s L_{p,r}(\mathbb{R}^n).$$

□

For the purpose of improving the previous result, we first show that one can replace $L_{p,r}(\mathbb{R}^n)$ by the local Hardy space with Lorentz smoothness $h_{p,r}(\mathbb{R}^n)$ in the definition of spaces $B_q^s L_{p,r}(\mathbb{R}^n)$ (see Definition 9.1). Note that if $1 < p < \infty$, according to (9.28) and (9.24), $h_{p,r}(\mathbb{R}^n) = L_{p,r}(\mathbb{R}^n)$ and therefore the result is trivial. In order to show that they also coincide when $0 < p \leq 1$, we first extend to Lorentz spaces several results of [111] for Lebesgue spaces.

For Ω a compact subset of $\mathbb{R}^n$, we put

$$\mathcal{S}^\Omega(\mathbb{R}^n) = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \text{supp } \hat{\varphi} \subseteq \Omega\}$$

and for $0 < p < \infty$ and $0 < r \leq \infty$, we write

$$L^{\Omega}_{p,r}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \text{supp } \hat{f} \subseteq \Omega \text{ and } \|f|L_{p,r}(\mathbb{R}^n)\| < \infty\}.$$

As usual, if $1 \leq p \leq \infty$ we put $\frac{1}{p} + \frac{1}{p'} = 1$.

Lemma 10.13. Let Ω be a compact subset of $\mathbb{R}^n$ and let $0 < p < \infty$ and $0 < r \leq \infty$. There is a constant $C > 0$ such that

$$\|\varphi|L_\infty(\mathbb{R}^n)\| \leq C\|\varphi|L_{p,r}(\mathbb{R}^n)\| \quad \text{for any } \varphi \in \mathcal{S}^{\Omega}(\mathbb{R}^n). \tag{10.16}$$

Furthermore,

$$L^{\Omega}_{p,r}(\mathbb{R}^n) \hookrightarrow L^{\Omega}_{\infty}(\mathbb{R}^n). \tag{10.17}$$

Proof. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\psi}(x) = 1$ if $x \in \Omega$. Then

$$|\varphi(x)| = |(\hat{\varphi})^\vee(x)| = |(\hat{\psi}\hat{\varphi})^\vee(x)| = C|\psi * \varphi(x)| = C\Big|\int_{\mathbb{R}^n} \psi(x-y)\varphi(y)dy\Big|.$$

Take any $0 < s < \min\{p, r, 1\}$. Using Hölder inequality for Lorentz spaces (10.5), we obtain

$$\begin{aligned}
|\varphi(x)| &\leq C\int_{\mathbb{R}^n} |\varphi(y)|^{1-s}|\varphi(y)|^s|\psi(x-y)|dy \\
&\leq C\sup_{y\in\mathbb{R}^n} |\varphi(y)|^{1-s}\int_{\mathbb{R}^n} |\varphi(y)|^s|\psi(x-y)|dy \\
&\leq C\sup_{y\in\mathbb{R}^n} |\varphi(y)|^{1-s}\||\varphi|^s|L_{p/s,r/s}(\mathbb{R}^n)\| \cdot \||\psi||L_{(p/s)',(r/s)'}(\mathbb{R}^n)\| \\
&= C(\Omega)\sup_{y\in\mathbb{R}^n} |\varphi(y)|^{1-s}\|\varphi|L_{p,r}(\mathbb{R}^n)\|^s.
\end{aligned}$$

This yields (10.16).

To establish (10.17) we proceed as in [111, Theorem 1.4.1/Step 1]. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi(0) = 1$ and supp $\hat{\varphi} \subseteq \{y : |y| \leq 1\}$. For $f \in L^{\Omega}_{p,r}(\mathbb{R}^n)$ and $0 < \delta < 1$, we write $f_\delta(x) = \varphi(\delta x)f(x)$. Then $f_\delta \in \mathcal{S}^B(\mathbb{R}^n)$ where B is a closed ball, centered at the origin, such that

$$\{y : \text{ there is } x \in \Omega \text{ with } |x-y| \leq 1\} \subseteq B,$$

and f_δ converges to f in $L_\infty(\mathbb{R}^n)$ as $\delta \downarrow 0$ (see [111, pp.22-23]). By (10.16) we get

$$\begin{aligned}
\|f_\delta|L_\infty(\mathbb{R}^n)\| &\leq C\|f_\delta|L_{p,r}(\mathbb{R}^n)\| = C\|\varphi(\delta\cdot)f|L_{p,r}(\mathbb{R}^n)\| \\
&\leq C\|\varphi|L_\infty(\mathbb{R}^n)\| \cdot \|f|L_{p,r}(\mathbb{R}^n)\| \leq C_1\|f|L_{p,r}(\mathbb{R}^n)\|.
\end{aligned}$$

Therefore, letting $\delta \downarrow 0$ we conclude that

$$\|f|L_\infty(\mathbb{R}^n)\| \leq C_1\|f|L_{p,r}(\mathbb{R}^n)\|.$$

□

Next we establish a Fourier multiplier result for Lorentz spaces.

Theorem 10.14. Let Ω, Γ be compact subsets of $\mathbb{R}^n$. Let $0 < p < \infty$, $0 < r \le \infty$ and put

$$\sigma = \begin{cases} 1 & \text{if} \quad \min\{p,r\} > 1, \\ \min\{p,r\}/2 & \text{if} \quad \min\{p,r\} \le 1. \end{cases}$$

Then there is a constant $C = C(\Omega, \Gamma, p, r) > 0$ such that for any $f \in L^{\Omega}_{p,r}(\mathbb{R}^n)$ and $\check{M} \in L^{\Gamma}_{\sigma}(\mathbb{R}^n)$ the inequality

$$\|(M\hat{f})^{\vee}|L_{p,r}(\mathbb{R}^n)\| \le C\|\check{M}|L_{\sigma}(\mathbb{R}^n)\| \cdot \|f|L_{p,r}(\mathbb{R}^n)\|$$

holds.

Proof. We know by Lemma 10.13 that $f \in L^{\Omega}_{\infty}(\mathbb{R}^n)$. If $\min\{p,r\} > 1$, since

$$(M\hat{f})^{\vee}(x) = C_1 \int_{\mathbb{R}^n} \check{M}(y) f(x-y) dy = C_1 \check{M} * f(x), \; x \in \mathbb{R}^n,$$

using the properties of the convolution in Lorentz spaces (see [96, Theorem 2.6] or [81, p.301]), we get that

$$\|(M\hat{f})^{\vee}|L_{p,r}(\mathbb{R}^n)\| = C_1\|\check{M} * f|L_{p,r}(\mathbb{R}^n)\| \le C\|\check{M}|L_1(\mathbb{R}^n)\| \cdot \|f|L_{p,r}(\mathbb{R}^n)\|.$$

Suppose now that $\min\{p,r\} \le 1$. This time the assumption is $|\check{M}|^{\sigma} \in L_1(\mathbb{R}^n)$ and so for each $x \in \mathbb{R}^n$ we have $\check{M}(\cdot)f(x-\cdot) \in L_{\sigma}(\mathbb{R}^n)$. Moreover, supp $(\check{M}(\cdot)f(x-\cdot))^{\wedge} \subseteq \Gamma - \Omega$ (see [111, pp. 25-26]). It follows from [111, 1.4.1/(3)] that $\check{M}(\cdot)f(x-\cdot) \in L_1(\mathbb{R}^n)$ and

$$\begin{aligned} |(M\hat{f})^{\vee}(x)|^{\sigma} &= C_2 \Big| \int_{\mathbb{R}^n} \check{M}(y) f(x-y) dy \Big|^{\sigma} \\ &\le C_3 \int_{\mathbb{R}^n} |\check{M}(y) f(x-y)|^{\sigma} dy \\ &= C_3 |\check{M}|^{\sigma} * |f|^{\sigma}(x). \end{aligned}$$

Consequently, using again convolution estimates for Lorentz spaces ([96, Theorem 2.6] or [81, p.301]), we conclude that

$$\begin{aligned} \|(M\hat{f})^{\vee}|L_{p,r}(\mathbb{R}^n)\| &= \||(M\hat{f})^{\vee}|^{\sigma}|L_{p/\sigma,r/\sigma}(\mathbb{R}^n)\|^{1/\sigma} \\ &\le C_4\||\check{M}|^{\sigma} * |f|^{\sigma}|L_{p/\sigma,r/\sigma}(\mathbb{R}^n)\|^{1/\sigma} \\ &\le C\||\check{M}|^{\sigma}|L_1(\mathbb{R}^n)\|^{1/\sigma}\||f|^{\sigma}|L_{p/\sigma,r/\sigma}(\mathbb{R}^n)\|^{1/\sigma} \\ &= C\|\check{M}|L_{\sigma}(\mathbb{R}^n)\| \cdot \|f|L_{p,r}(\mathbb{R}^n)\|. \end{aligned}$$

This completes the proof. □

Let $0 < q \le \infty$, $0 < p, r < \infty$, $s \in \mathbb{R}$ and $(\varphi_k)_{k \in \mathbb{N}_0}$ a smooth dyadic resolution of unity. We define $B^s_q h_{p,r}(\mathbb{R}^n)$ as the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ with finite quasi-norm

$$\|f|B^s_q h_{p,r}(\mathbb{R}^n)\| = \Big(\sum_{k=0}^{\infty} 2^{ksq} \|(\varphi_k \hat{f})^{\vee}|h_{p,r}(\mathbb{R}^n)\|^q \Big)^{1/q}.$$

Now we are ready to establish that $B^s_q L_{p,r}(\mathbb{R}^n)$ and $B^s_q h_{p,r}(\mathbb{R}^n)$ coincide.

Theorem 10.15. Let $-\infty < s < \infty$, $0 < q \le \infty$ and $0 < p, r < \infty$. Then we have with equivalence of quasi-norms

$$B^s_q L_{p,r}(\mathbb{R}^n) = B^s_q h_{p,r}(\mathbb{R}^n).$$

Proof. Let $(\varphi_k)_{k=0}^{\infty}$ be a smooth dyadic resolution of unity. It is enough to show that for any $f \in \mathcal{S}'(\mathbb{R}^n)$ we have

$$\|(\varphi_k \hat{f})^\vee | L_{p,r}(\mathbb{R}^n)\| \sim \|(\varphi_k \hat{f})^\vee | h_{p,r}(\mathbb{R}^n)\|.$$

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ compactly supported and with $\psi(x) = 1$ if $|x| \le 1$. For $t \le 2^{-(k+1)}$ we have that $\psi(t\cdot)\varphi_k = \varphi_k$ because supp $\varphi_k \subseteq \{x : 2^{k-1} \le |x| \le 2^{k+1}\}$. Therefore,

$$\begin{aligned}\|(\varphi_k \hat{f})^\vee | L_{p,r}(\mathbb{R}^n)\| &= \|(\psi(2^{-(k+1)}\cdot)\varphi_k \hat{f})^\vee | L_{p,r}(\mathbb{R}^n)\| \\ &\le \|\sup_{0<t<1} |(\psi(t\cdot)\varphi_k \hat{f})^\vee|\, | L_{p,r}(\mathbb{R}^n)\| = \|(\varphi_k \hat{f})^\vee | h_{p,r}(\mathbb{R}^n)\|.\end{aligned}$$

To check the converse inequality note that

$$\text{supp } \varphi_0 \subseteq \{x \in \mathbb{R}^n : |x| \le 2\},\ \varphi_1(x) = \varphi_0(2^{-1}x) - \varphi_0(x)$$

and

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x) = \varphi_1(2^{-k+1}x),\ k = 2, 3, \ldots$$

Let $(\psi_k)_{k=0}^{\infty}$ be another smooth dyadic resolution of unity. Then analogous relationships hold among ψ_0, ψ_1 and ψ_k. According to (9.24) for $k = 2, 3, \ldots$ we get

$$\begin{aligned}\|(\varphi_k \hat{f})^\vee | h_{p,r}\| &\sim \|(\varphi_k \hat{f})^\vee | F_2^0 L_{p,r}(\mathbb{R}^n)\| \\ &= \Big\|\Big(\sum_{m=0}^{\infty} |\psi_m \varphi_k \hat{f}(x)|^2\Big)^{1/2} | L_{p,r}(\mathbb{R}^n)\Big\| \\ &= \Big\|\Big(\sum_{j=-1}^{1} |(\psi_{k+j}\varphi_k \hat{f})^\vee(x)|^2\Big)^{1/2} | L_{p,r}(\mathbb{R}^n)\Big\| \\ &\lesssim \sum_{j=-1}^{1} \|(\psi_{k+j}\varphi_k \hat{f})^\vee | L_{p,r}(\mathbb{R}^n)\|.\end{aligned}$$

Moreover,

$$\begin{aligned}(\psi_{k+j}\varphi_k \hat{f})^\vee(\xi) &= (\psi_1(2^{-k-j+1}\cdot)\varphi_1(2^{-k+1}\cdot)\hat{f})^\vee(\xi) \\ &= 2^{(k-1)n}(\psi_1(2^{-j}\cdot)\varphi_1 \hat{f}(2^{k-1}\cdot))^\vee(2^{k-1}\xi).\end{aligned}$$

Hence

$$\begin{aligned}\|(\psi_{k+j}\varphi_k \hat{f})^\vee | L_{p,r}(\mathbb{R}^n)\| &= 2^{(k-1)n}\|(\psi_1(2^{-j}\cdot)\varphi_1 \hat{f}(2^{k-1}\cdot))^\vee(2^{k-1}\cdot) | L_{p,r}(\mathbb{R}^n)\| \\ &\le 2^{(k-1)n(1-(1/p))}\|(\psi_1(2^{-j}\cdot)\varphi_1 \hat{f}(2^{k-1}\cdot))^\vee | L_{p,r}(\mathbb{R}^n)\| \\ &= 2^{(k-1)n(1-(1/p))}\|(\psi_1(2^{-j}\cdot)\hat{h})^\vee | L_{p,r}(\mathbb{R}^n)\|\end{aligned}$$

where $h = (\varphi_1 \hat{f}(2^{k-1}\cdot))^\vee$ and we have used that

$$\|f(\lambda\cdot) | L_{p,r}(\mathbb{R}^n)\| = \lambda^{-n/p}\|f | L_{p,r}(\mathbb{R}^n)\| \quad \text{for } \lambda > 0.$$

Since $j = -1, 0, 1$, we have

$$\text{supp } \psi_1(2^{-j}\cdot) \subseteq \{x : |x| \le 8\}.$$

Furthermore,

$$\text{supp } \hat{h} = \text{supp } \varphi_1 \hat{f}(2^{k+1}\cdot) \subseteq \text{supp } \varphi_1 \subseteq \{x : |x| \le 4\}.$$

So, the supports of $\psi_1(2^{-j}\cdot)$ and $\hat{h}$ are contained in compact sets which are independent of j and k. Applying Theorem 10.14, we obtain that

$$\begin{aligned}\|(\psi_{k+j}\varphi_k\hat{f})^\vee|L_{p,r}(\mathbb{R}^n)\| &\le 2^{(k-1)n(1-(1/p))}\|(\psi_1(2^{-j}\cdot)\hat{h})^\vee|L_{p,r}(\mathbb{R}^n)\| \\ &\lesssim 2^{(k-1)n(1-(1/p))}\|\psi_1(2^{-j}\cdot)^\vee|L_\sigma(\mathbb{R}^n)\|\cdot\|(\varphi_1\hat{f}(2^{k-1}\cdot))^\vee|L_{p,r}(\mathbb{R}^n)\|.\end{aligned}$$

We have $\psi_1(2^{-j}\cdot)^\vee = 2^{jn}\check{\psi}_1(2^j\cdot)$ and

$$\begin{aligned}(\varphi_1\hat{f}(2^{k-1}\cdot))^\vee &= 2^{-(k-1)n}(\varphi_1(2^{-(k-1)}\cdot)\hat{f})^\vee(2^{-(k-1)}\cdot) \\ &= 2^{-(k-1)n}(\varphi_k\hat{f})^\vee(2^{-(k-1)}\cdot).\end{aligned}$$

Therefore

$$\begin{aligned}\|(\psi_{k+j}\varphi_k\hat{f})^\vee|L_{p,r}(\mathbb{R}^n)\| &\lesssim 2^{(k-1)n(1-(1/p))}\|\check{\psi}_1|L_\sigma(\mathbb{R}^n)\|2^{-(k-1)n}\|(\varphi_k\hat{f})^\vee(2^{-(k-1)}\cdot)|L_{p,r}(\mathbb{R}^n)\| \\ &\lesssim \|(\varphi_k\hat{f})^\vee|L_{p,r}(\mathbb{R}^n)\|.\end{aligned}$$

This yields that

$$\|(\varphi_k\hat{f})^\vee|h_{p,r}(\mathbb{R}^n)\| \lesssim \|(\varphi_k\hat{f})^\vee|L_{p,r}(\mathbb{R}^n)\|.$$

The cases $k = 0, 1$ can be treated similarly. The proof is complete. □

Now we are ready to give a description of $B^s_qL_{p,r}(\mathbb{R}^n)$ as approximation spaces based on entire functions of exponential type (see [110, Section 2.5.4]).

By the construction of the resolution of unity $(\varphi_k)_{k\in\mathbb{N}_0}$, for $N \in \mathbb{N}$, there exists $M < \infty$ independent of k, such that

$$\sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^n}(1+|x|^2)^{|\alpha|/2}|D^\alpha\varphi_k(x)| \le M, \quad k \in \mathbb{N}_0.$$

It follows from [111, Theorem 2.3.7] that for any $0 < p < \infty$ and $0 < r \le \infty$ there is $C > 0$ such that

$$\|(\varphi_k\hat{f})^\vee|F^0_{p,r}(\mathbb{R}^n)\| \le C\|f|F^0_{p,r}(\mathbb{R}^n)\|, \quad f \in F^0_{p,r}(\mathbb{R}^n), \quad k \in \mathbb{N}_0.$$

Whence, applying Theorem 9.17, we derive that for any $0 < p < \infty$ and $0 < r \le \infty$ there exists $C > 0$ such that

$$\|(\varphi_k\hat{f})^\vee|F^0_2L_{p,r}(\mathbb{R}^n)\| \le C\|f|F^0_2L_{p,r}(\mathbb{R}^n)\|, \quad f \in F^0_2L_{p,r}(\mathbb{R}^n), \quad k \in \mathbb{N}_0. \tag{10.18}$$

This estimate will be useful later.

Let $A_0 = \{0\}$ and for $k \in \mathbb{N}$ put

$$A_k = \{g \in F^0_2L_{p,r}(\mathbb{R}^n) : \operatorname{supp}\hat{g} \subseteq \{x : |x| \le k\}\}.$$

In what follows we work with approximation spaces $(X; A_k)^s_q$ generated by the sequence of subset (A_k) with X being $F^0_2L_{p,r}(\mathbb{R}^n)$ and $B^s_qL_{p,r}(\mathbb{R}^n)$.

Theorem 10.16. Let $0 < q \le \infty$, $s > 0$, $0 < p < \infty$ and $0 < r < \infty$, allowing also $r = \infty$ if $1 < p < \infty$. Then we have with equivalence of quasi-norms

$$(F^0_2L_{p,r}(\mathbb{R}^n); A_k)^s_q = B^s_qL_{p,r}(\mathbb{R}^n)\,.$$

Proof. Take any $f \in B^s_q L_{p,r}(\mathbb{R}^n)$. Using that, according to [107, Theorem 1.1], $B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow F^0_2 L_{p,r}(\mathbb{R}^n)$, we get

$$E_2(f) \leq E_1(f) \leq E_0(f) = \|f|F^0_2 L_{p,r}(\mathbb{R}^n)\| \leq C\|f|B^s_q L_{p,r}(\mathbb{R}^n)\|. \tag{10.19}$$

Moreover, since

$$\operatorname{supp}\Big(\sum_{m=0}^{k} \varphi_m \hat{f}\Big) \subseteq \{x : |x| \leq 2^{k+2}\}, \quad k = 0, 1, 2, \dots,$$

we obtain for $k \geq 2$ that

$$E_{2^k}(f) \leq \Big\| f - \big(\sum_{m=0}^{k-2} \varphi_m \hat{f}\big)^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\Big\| = \Big\| \sum_{m=k-1}^{\infty} (\varphi_m \hat{f})^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\Big\|.$$

Using the quasi-norm (2.20) for approximation spaces, we have

$$\begin{aligned} \|f|(F^0_2 L_{p,r}(\mathbb{R}^n); A_k)^s_q\|^\diamond &= \Big(\|f|F^0_2 L_{p,r}(\mathbb{R}^n)\|^q + \sum_{m=1}^{\infty} 2^{msq} E_{2^m}(f)^q\Big)^{1/q} \\ &\lesssim \Big(\|f|F^0_2 L_{p,r}(\mathbb{R}^n)\|^q + \sum_{m=2}^{\infty} 2^{msq} \Big\| \sum_{k=m-1}^{\infty} (\varphi_k \hat{f})^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q}. \end{aligned}$$

Take $0 < \rho < \min\{1, q\}$ such that $F^0_2 L_{p,r}(\mathbb{R}^n)$ is a ρ-normed quasi-Banach space. For the second term in the last expression, we get

$$\begin{aligned} &\Big(\sum_{m=2}^{\infty} \Big(2^{ms\rho} \Big\| \sum_{k=m-1}^{\infty} (\varphi_k \hat{f})^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\Big\|^\rho\Big)^{q/\rho}\Big)^{1/q} \\ &= \Big[\Big(\sum_{m=2}^{\infty} \Big(2^{ms\rho} \Big\| \sum_{k=0}^{\infty} (\varphi_{m-1+k} \hat{f})^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\Big\|^\rho\Big)^{q/\rho}\Big)^{\rho/q}\Big]^{1/\rho} \\ &\leq \Big[\sum_{k=0}^{\infty} 2^{(1-k)s\rho} \Big(\sum_{m=1}^{\infty} 2^{(m-1+k)sq} \|(\varphi_{m-1+k} \hat{f})^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\|^q\Big)^{\rho/q}\Big]^{1/\rho} \\ &\lesssim \Big[\sum_{k=0}^{\infty} 2^{(1-k)s\rho}\Big]^{1/\rho} \|f|B^s_q L_{p,r}(\mathbb{R}^n)\| \end{aligned}$$

where in the last inequality we have used (9.28) and the fact that $B^s_q L_{p,r}(\mathbb{R}^n) = B^s_q h_{p,r}(\mathbb{R}^n)$ (see Theorem 10.15). This together with (10.19) yield that $B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow (F^0_2 L_{p,r}(\mathbb{R}^n); A_k)^s_q$.

In order to establish the converse embedding, first note that given any $f \in F^0_2 L_{p,r}(\mathbb{R}^n)$, $k \in \mathbb{N}$ and $g \in D_{2^k}$, since $\operatorname{supp} \hat{g} \cap \operatorname{supp} \varphi_{k+1} = \emptyset$, it follows from (10.18) that

$$\|(\varphi_{k+1} \hat{f})^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\| = \|(\varphi_{k+1}(\hat{f} - \hat{g}))^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\| \leq C\|f - g|F^0_2 L_{p,r}(\mathbb{R}^n)\|.$$

This yields that

$$\|(\varphi_{k+1} \hat{f})^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\| \lesssim E_{2^k}(f), \quad k = 1, 2, \dots$$

Moreover, using again (10.18), we get

$$\|(\varphi_m \hat{f})^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\| \leq C\|f|F^0_2 L_{p,r}(\mathbb{R}^n)\| \quad \text{for} \quad m = 0, 1.$$

Consequently,

$$\begin{aligned}\|f|B^s_q L_{p,r}(\mathbb{R}^n)\| &\sim \Big(\sum_{k=0}^{\infty} 2^{ksq}\|(\varphi_k \hat{f})^\vee|F^0_2 L_{p,r}\|^q\Big)^{1/q} \\ &\lesssim \Big(\|f|F^0_2 L_{p,r}(\mathbb{R}^n)\|^q + \sum_{k=1}^{\infty} 2^{ksq} E_{2^k}(f)^q\Big)^{1/q} \\ &\sim \|f|(F^0_2 L_{p,r}(\mathbb{R}^n); A_k)^s_q\| .\end{aligned}$$

□

Next we study multiplication properties of Besov-Lorentz spaces.

For $m, k \in \mathbb{N}$, $f \in A_m$ and $g \in A_k$, since $fg = (\hat{f} * \hat{g})^\vee$, we have that supp $\widehat{fg} \subseteq \{x : |x| \leq m + k\}$. Let $T(f,g) = fg$. According to the previous observation, we get

$$T(A_{2^m}, A_{2^k}) \subseteq A_{2^m+2^k}, \quad m,k = 1,2,\ldots \tag{10.20}$$

Lemma 10.17. Let X, Y, Z be quasi-Banach function spaces on $\mathbb{R}^n$ containing the subsets (A_k). We assume that X and Y are formed by regular distributions and we put $T(f,g) = fg$. If the bilinear operator $T : X \times Y \longrightarrow Z$ is bounded then there is a constant $C > 0$ such that for any $k \in \mathbb{N}$ and any $f \in X$ and $g \in Y$ we have

$$E_{2^{k+1}}(fg)_Z \leq C\Big(E_{2^k}(f)_X\|g|Y\| + \|f|X\|E_{2^k}(g)_Y\Big) .$$

Proof. Let M be the norm of $T : X \times Y \longrightarrow Z$. Given any $f \in X$, $g \in Y$, $k \in \mathbb{N}$ and $\varepsilon > 0$, there are $f_0, g_0 \in A_{2^k}$ such that

$$\|f - f_0|X\| \leq E_{2^k}(f)_X + \varepsilon, \; \|g - g_0|Y\| \leq E_{2^k}(g)_Y + \varepsilon .$$

Having in mind (10.20), we obtain

$$\begin{aligned}E_{2^{k+1}}(fg)_Z &\leq \|fg - f_0 g_0|Z\| \\ &\leq c_Z(\|fg - f_0 g|Z\| + \|f_0 g - f_0 g_0|Z\|) \\ &\leq c_Z M(\|f - f_0|X\|\|g|Y\| + \|f_0 - f + f|X\|\|g - g_0|Y\|) \\ &\leq c_Z M\big[(E_{2^k}(f)_X + \varepsilon)\|g|Y\| + c_X\big((E_{2^k}(f)_X + \varepsilon + \|f|X\|)(E_{2^k}(g)_Y + \varepsilon)\big)\big] \\ &\leq c_Z M\big[(E_{2^k}(f)_X + \varepsilon)\|g|Y\| + c_X(2\|f|X\| + \varepsilon)(E_{2^k}(g)_Y + \varepsilon)\big] .\end{aligned}$$

Passing to the limit when $\varepsilon \to 0$ the wanted result follows with $C = 2c_X c_Z M$. □

As a first consequence of Theorem 10.16 and Lemma 10.17 we extend Hölder inequality to Besov-Lorentz spaces.

Theorem 10.18. Let $s > 0$, $1 < p_0, p_1, p < \infty$ with $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$, $0 < q \leq \infty$ and $0 < r_0, r_1, r \leq \infty$ with $\frac{1}{r} = \frac{1}{r_0} + \frac{1}{r_1}$. Then we have

$$B^s_q L_{p_0,r_0}(\mathbb{R}^n) \cdot B^s_q L_{p_1,r_1}(\mathbb{R}^n) \hookrightarrow B^s_q L_{p,r}(\mathbb{R}^n) .$$

Proof. Since $1 < p_j < \infty$, according to (9.28), $F_2^0 L_{p_j,r_j}(\mathbb{R}^n) = L_{p_j,r_j}(\mathbb{R}^n)$, $j = 0,1$. Moreover, Hölder inequality for Lorentz spaces (10.5) yields that

$$F_2^0 L_{p_0,r_0}(\mathbb{R}^n) \cdot F_2^0 L_{p_1,r_1}(\mathbb{R}^n) \hookrightarrow F_2^0 L_{p,r}(\mathbb{R}^n) .$$

Let again $T(f,g) = fg$ and write for simplicity $X = F_2^0 L_{p_0,r_0}(\mathbb{R}^n)$, $Y = F_2^0 L_{p_1,r_1}(\mathbb{R}^n)$ and $Z = F_2^0 L_{p,r}(\mathbb{R}^n)$. So, $T : X \times Y \longrightarrow Z$ is bounded. Using Lemma 10.17 we derive

$$\begin{aligned}
\|T(f,g)|Z_q^s\| &\sim \left[\|T(f,g)|Z\|^q + \sum_{k=2}^{\infty} 2^{ksq} E_{2^k}(T(f,g))_Z^q\right]^{1/q} \\
&\lesssim \left[\|f|X\|^q \|g|Y\|^q + \sum_{k=1}^{\infty} 2^{ksq} \left(E_{2^k}(f)_X \|g|Y\| + \|f|X\| E_{2^k}(g)_Y\right)^q\right]^{1/q} \\
&\leq \left[\|f|X\|^q + \sum_{k=1}^{\infty} 2^{ksq} E_{2^k}(f)_X^q\right]^{1/q} \left[\|g|Y\|^q + \sum_{k=1}^{\infty} 2^{ksq} E_{2^k}(g)_Y^q\right]^{1/q} \\
&\lesssim \|f|X_q^s\| \cdot \|g|Y_q^s\|.
\end{aligned}$$

This yields the result having in mind that, by Theorem 10.16, we have $(F_2^0 L_{p,r}(\mathbb{R}^n); A_k)_q^s = B_q^s L_{p,r}(\mathbb{R}^n)$ and $(F_2^0 L_{p_j,r_j}(\mathbb{R}^n); A_k)_q^s = B_q^s L_{p_j,r_j}(\mathbb{R}^n)$, $j = 0,1$. □

In what follows we focus on multiplication algebras. In Theorem 10.12 we proved that $B_q^s L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra provided that $0 < p < \infty$, $s > n/p$, $0 < q \leq r \leq 1$ and $r < p$. Now, we continue this research with the help of Theorem 10.16. Our first aim is to eliminate the restrictions on q.

Theorem 10.19. Let $0 < p < \infty$, $s > n/p$ and $0 < r < \infty$, allowing also $r = \infty$ if $1 < p < \infty$. If there are $0 < q_1 < \infty$ and $n/p < s_1 < s$ such that $B_{q_1}^{s_1} L_{p,r}(\mathbb{R}^n)$ is an algebra for multiplication, then for any $0 < q \leq \infty$ the space $B_q^s L_{p,r}(\mathbb{R}^n)$ is an algebra for multiplication.

Proof. Let $X = B_{q_1}^{s_1} L_{p,r}(\mathbb{R}^n)$, $T(f,g) = fg$ and $\tau = s - s_1$. Then $T : X \times X \longrightarrow X$ is bounded. Moreover, by Theorem 10.16 and the reiteration formula for approximation spaces [105, p.123], we obtain

$$X_q^\tau = \left((F_2^0 L_{p,r}(\mathbb{R}^n); A_k)_{q_1}^{s_1}; A_k\right)_q^\tau = (F_2^0 L_{p,r}(\mathbb{R}^n); A_k)_q^s = B_q^s L_{p,r}(\mathbb{R}^n) .$$

To complete the proof, it suffices to show that $T : X_q^\tau \times X_q^\tau \longrightarrow X_q^\tau$ is bounded and this follows by using Lemma 10.17 and proceeding as in the proof of Theorem 10.18. □

Now we are ready to get rid of the restrictions on q in Theorem 10.12.

Theorem 10.20. Let $0 < p < \infty$, $s > n/p$, $0 < q \leq \infty$, $0 < r \leq 1$ and $r < p$. Then $B_q^s L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra.

Proof. Take $n/p < s_1 < s$ and $0 < q_1 \leq r$. By Theorem 10.12, the space $B_{q_1}^{s_1} L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra. Then, Theorem 10.19 yields that $B_q^s L_{p,r}(\mathbb{R}^n)$ is also a multiplication algebra. □

Next we consider the case $1 < r < p < \infty$. As we did in Theorem 10.10, this time our arguments rely on complex interpolation.

Theorem 10.21. Let $1 < r < p < \infty$, $s > n/p$ and $0 < q \leq \infty$. Then $B_q^s L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra.

Proof. Assume first $1 \leq q < \infty$. Choose δ, u, θ and p_1 as in the proof of Theorem 10.10. So

$$1 < p_1, u < \infty, \quad s > \max\{n/p_1, n/u\}, \quad \frac{1}{r} = (1-\theta) + \frac{\theta}{u}, \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{u}.$$

Now using Theorem 2.11, we obtain that

$$[L_{p_1,1}(\mathbb{R}^n), L_u(\mathbb{R}^n)]_\theta = L_{p,r}(\mathbb{R}^n).$$

Hence, applying [110, Theorem 1.18.1], we derive that

$$[\ell_q(2^{ks}L_{p_1,1}(\mathbb{R}^n)), \ell_q(2^{ks}L_u(\mathbb{R}^n))]_\theta = \ell_q(2^{ks}L_{p,r}(\mathbb{R}^n)). \tag{10.21}$$

Next consider again the operators $Jf = ((\varphi_k \hat{f})^\vee)$ and $R(f_k) = \sum_{k=0}^{\infty}(\tilde{\varphi}_k \hat{f}_k)^\vee$ where $\tilde{\varphi}_k = \varphi_{k-1} + \varphi_k + \varphi_{k+1}$ and $\varphi_{-1} = 0$. Since restrictions

$$J : B^s_q L_{p_1,1}(\mathbb{R}^n) \longrightarrow \ell_q(2^{ks}L_{p_1,1}(\mathbb{R}^n)),$$

$$J : B^s_{u,q}(\mathbb{R}^n) \longrightarrow \ell_q(2^{ks}L_u(\mathbb{R}^n)),$$

$$R : \ell_q(2^{ks}L_{p_1,1}(\mathbb{R}^n)) \longrightarrow B^s_q L_{p_1,1}(\mathbb{R}^n),$$

$$R : \ell_q(2^{ks}L_u(\mathbb{R}^n)) \longrightarrow B^s_{u,q}(\mathbb{R}^n)$$

are bounded and $R(Jf) = f$, applying [110, Theorem 1.2.4] and using (10.21), we get

$$[B^s_q L_{p_1,1}(\mathbb{R}^n), B^s_{u,q}(\mathbb{R}^n)]_\theta = B^s_q L_{p,r}(\mathbb{R}^n). \tag{10.22}$$

By Theorem 10.20, the space $B^s_q L_{p_1,1}(\mathbb{R}^n)$ is a multiplication algebra and, according to [118, Theorem 2.41] or [106, Theorem 4.6.4/1], $B^s_{u,q}(\mathbb{R}^n)$ is also a multiplication algebra. Hence, applying the bilinear interpolation theorem for the complex method (Theorem 2.12) to the operator $T(f,g) = fg$ and using (10.22) we conclude that $B^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra.

Finally, to cover the whole range for q it suffices to apply Theorem 10.19. □

10.3 Traces

Let $n \in \mathbb{N}$, $n \geq 2$. If $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the *trace operator* is pointwise defined as

$$\text{tr} : \varphi(x) \longrightarrow \varphi(x_1, ..., x_{n-1}, 0).$$

Let A and B be two quasi-Banach spaces such that

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow A \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \quad \text{and} \quad \mathcal{S}(\mathbb{R}^{n-1}) \hookrightarrow B \hookrightarrow \mathcal{S}'(\mathbb{R}^{n-1}).$$

Suppose that there exists $C > 0$ such that

$$\|\text{tr}\varphi|B\| \leq C\|\varphi|A\|, \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Although the pointwise definition of the trace operator might not make sense in general, if $\mathcal{S}(\mathbb{R}^n)$ is dense in A, the trace operator can be uniquely extended to the whole space satisfying that tr : $A \longrightarrow B$ and

$$\|\text{tr}f|B\| \leq C\|f|A\|, \quad \text{for every } f \in A.$$

According to [111, Theorem 2.3.3/(ii)], if $0 < p, q < \infty$ and $s \in \mathbb{R}$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $F^s_{p,q}(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$. Moreover, we have the following result for the trace operator (see, for example, [118, Theorem 2.13]): Let $0 < p, q < \infty$ and $s - \frac{1}{p} > (n-1)(\frac{1}{p} - 1)_+$, then

(1) $\mathrm{tr} : F^s_{p,q}(\mathbb{R}^n) \longrightarrow B^{s-1/p}_{p,p}(\mathbb{R}^{n-1})$ and $\mathrm{tr} : B^s_{p,q}(\mathbb{R}^n) \longrightarrow B^{s-1/p}_{p,q}(\mathbb{R}^{n-1})$ are linear and continuous, and

(2) there exists linear and continuous extension operator

$$\mathrm{ext} : B^{s-1/p}_{p,p}(\mathbb{R}^{n-1}) \longrightarrow F^s_{p,q}(\mathbb{R}^n) \quad \text{with} \quad \mathrm{tr} \circ \mathrm{ext} = \mathrm{id} : B^{s-1/p}_{p,p}(\mathbb{R}^{n-1}) \longrightarrow B^{s-1/p}_{p,p}(\mathbb{R}^{n-1})$$
$$\mathrm{ext} : B^{s-1/p}_{p,q}(\mathbb{R}^{n-1}) \longrightarrow B^s_{p,q}(\mathbb{R}^n) \quad \text{with} \quad \mathrm{tr} \circ \mathrm{ext} = \mathrm{id} : B^{s-1/p}_{p,q}(\mathbb{R}^{n-1}) \longrightarrow B^{s-1/p}_{p,q}(\mathbb{R}^{n-1}).$$

The aim of this section is to study the trace operator when acting on Triebel-Lizorkin and Besov spaces with Lorentz smoothness. We start studying the density of the Schwartz class in these spaces.

Proposition 10.22. Let $0 < p, q, r < \infty$ and $s \in \mathbb{R}$. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is dense in $F^s_q L_{p,r}(\mathbb{R}^n)$ and $B^s_q L_{p,r}(\mathbb{R}^n)$.

Proof. According to Theorem 9.17, $F^s_q L_{p,r}(\mathbb{R}^n) = (F^s_{p_0,q}(\mathbb{R}^n), F^s_{p_1,q}(\mathbb{R}^n))_{\theta,r}$, for $0 < \theta < 1$, $0 < p_0 \neq p_1 < \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. It turns out that $\mathcal{S}(\mathbb{R}^n)$ is dense in $F^s_{p_0,q}(\mathbb{R}^n) \cap F^s_{p_1,q}(\mathbb{R}^n)$. Indeed, it follows from [111, Theorem 2.3.3/Step 5, p.49], that $\mathcal{S}(\mathbb{R}^n)$ is not only dense in both $F^s_{p_0,q}(\mathbb{R}^n)$ and $F^s_{p_1,q}(\mathbb{R}^n)$ but, in addition, for any $f \in F^s_{p_0,q}(\mathbb{R}^n) \cap F^s_{p_1,q}(\mathbb{R}^n)$ we can take the same sequence $(\varphi_n)_{n=1}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ such that $\|f - \varphi_n | F^s_{p_k,q}(\mathbb{R}^n)\| \xrightarrow{n\to\infty} 0$ for $j = 0, 1$. And therefore,

$$\|f - \varphi_n | F^s_{p_0,q}(\mathbb{R}^n) \cap F^s_{p_1,q}(\mathbb{R}^n)\| \le \|f - \varphi_n | F^s_{p_0,q}(\mathbb{R}^n)\| + \|f - \varphi_n | F^s_{p_1,q}(\mathbb{R}^n)\| \xrightarrow{n\to\infty} 0.$$

Furthermore, due to the representation of the real method by means of the J-functional (see (2.12)), $F^s_{p_0,q}(\mathbb{R}^n) \cap F^s_{p_1,q}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p,r}(\mathbb{R}^n)$ densely (see [11, Theorem 3.4.2]). Thus $\mathcal{S}(\mathbb{R}^n)$ is dense in $F^s_q L_{p,r}(\mathbb{R}^n)$.

Now we prove that $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^s_q L_{p,r}(\mathbb{R}^n)$. According to Theorem 9.23,

$$B^s_q L_{p,r}(\mathbb{R}^n) = (F^{s_0}_q L_{p,r}(\mathbb{R}^n), F^{s_1}_q L_{p,r}(\mathbb{R}^n))_{\theta,q},$$

with $0 < \theta < 1$, $-\infty < s_0 < s < s_1 < \infty$ and $s = (1-\theta)s_0 + \theta s_1$. Observe that $F^{s_1}_q L_{p,r}(\mathbb{R}^n) \hookrightarrow F^{s_0}_q L_{p,r}(\mathbb{R}^n)$. From the first part of the proof we know that $\mathcal{S}(\mathbb{R}^n)$ is dense in $F^{s_1}_q L_{p,r}(\mathbb{R}^n)$ and, as before, $F^{s_1}_q L_{p,r}(\mathbb{R}^n) = F^{s_0}_q L_{p,r}(\mathbb{R}^n) \cap F^{s_1}_q L_{p,r}(\mathbb{R}^n)$ is dense in $B^s_q L_{p,r}(\mathbb{R}^n)$ by properties of the real interpolation method. Now it is clear that $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^s_q L_{p,r}(\mathbb{R}^n)$. □

Theorem 10.23. Let $0 < p, q, r < \infty$ and $s - \frac{1}{p} > (n-1)(\frac{1}{p} - 1)_+$. For $0 < \theta < 1$ and $0 < p_0 < p < p_1 < \infty$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $s - \frac{1}{p_j} > (n-1)(\frac{1}{p_j} - 1)_+$, $j = 0, 1$. Then

(1) $\mathrm{tr} : F^s_q L_{p,r}(\mathbb{R}^n) \longrightarrow (B^{s-1/p_0}_{p_0,p_0}(\mathbb{R}^{n-1}), B^{s-1/p_1}_{p_1,p_1}(\mathbb{R}^{n-1}))_{\theta,r}$ is linear and continuous, and

(2) there exists a linear and continuous extension operator

$$\mathrm{ext} : (B^{s-1/p_0}_{p_0,p_0}(\mathbb{R}^{n-1}), B^{s-1/p_1}_{p_1,p_1}(\mathbb{R}^{n-1}))_{\theta,r} \longrightarrow F^s_q L_{p,r}(\mathbb{R}^n)$$

such that

$$\mathrm{tr} \circ \mathrm{ext} = \mathrm{id} : (B^{s-1/p_0}_{p_0,p_0}(\mathbb{R}^{n-1}), B^{s-1/p_1}_{p_1,p_1}(\mathbb{R}^{n-1}))_{\theta,r} \longrightarrow (B^{s-1/p_0}_{p_0,p_0}(\mathbb{R}^{n-1}), B^{s-1/p_1}_{p_1,p_1}(\mathbb{R}^{n-1}))_{\theta,r}.$$

Proof. The proof follows straightforward from the results we have seen for traces on classical Triebel-Lizorkin spaces and the interpolation formula in Theorem 9.17. □

In the previous theorem, if we take $r = p$, then according to Theorem 9.21,

$$(B^{s-1/p_0}_{p_0,p_0}(\mathbb{R}^{n-1}), B^{s-1/p_1}_{p_1,p_1}(\mathbb{R}^{n-1}))_{\theta,r} = B^{s-1/p}_{p,p}(\mathbb{R}^{n-1}),$$

and we obtain the already known result concerning the traces of classical Triebel-Lizorkin spaces. However if $p \neq r$, the characterization of this interpolation space is an open problem already stated by Peetre in his monograph on Besov spaces [101, p.110]. Next, we are going to characterize these spaces in terms of wavelets using the wavelet representation for Besov spaces.

Theorem 10.24. Let $0 < p_0 < p_1 < \infty$, $s \in \mathbb{R}$ and $0 < r \leq \infty$. Then

$$(B^{s-\frac{1}{p_0}}_{p_0,p_0}(\mathbb{R}^{n-1}), B^{s-\frac{1}{p_1}}_{p_1,p_1}(\mathbb{R}^{n-1}))_{\theta,r}$$

is the set of all $f \in \mathcal{S}'(\mathbb{R}^{n-1})$ that admit a wavelet representation on $\mathbb{R}^{n-1}$ as

$$f(x) = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^{n-1}} \lambda^{j,G}_m 2^{-j(n-1)/2} \psi^j_{G,m}(x), \quad x \in \mathbb{R}^{n-1} \tag{10.23}$$

convergence in $\mathcal{S}'(\mathbb{R}^{n-1})$ and

$$\|f|(B^{s-\frac{1}{p_0}}_{p_0,p_0}(\mathbb{R}^{n-1}), B^{s-\frac{1}{p_1}}_{p_1,p_1}(\mathbb{R}^{n-1}))_{\theta,r}\| \sim \|\Lambda^s((\lambda^{j,G}_m))|L_{p,r}(\mathbb{R}^n)\| < \infty,$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$,

$$\Lambda^s((\lambda^{j,G}_m)) = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^{n-1}} 2^{js} |\lambda^{j,G}_m| \chi_{E_{jm}}(x), \quad x \in \mathbb{R}^n$$

and $E_{jm} = Q^{(n-1)}_{jm} \times (2^{-j-2}, 2^{-j-1}) = \prod_{\ell=1}^{n-1}(2^{-j}m_\ell - 2^{-j-1}, 2^{-j}m_\ell + 2^{-j-1}) \times (2^{-j-2}, 2^{-j-1}) \subset \mathbb{R}^n$ for $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^{n-1}$.

Proof. According to wavelet decomposition results for Besov spaces (see [114, Theorem 1.20] or Theorem 9.16), for $k = 0, 1$, there is an isomorphism

$$\begin{aligned} I : B^{s-\frac{1}{p_k}}_{p_k,p_k}(\mathbb{R}^{n-1}) &\longrightarrow b^{s-\frac{1}{p_k}}_{p_k,p_k}(\mathbb{R}^{n-1}) \\ f &\longrightarrow (\lambda^{j,G}_m)_{j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^{n-1}}, \end{aligned}$$

with $\lambda^{j,G}_m = 2^{j(n-1)/2}(f, \psi^j_{G,m})$ and $\psi^j_{G,m}$ being the wavelet defined in (9.19) but on $\mathbb{R}^{n-1}$. For any $s \in \mathbb{R}$, the operator

$$\begin{aligned} T_s : b^{s-\frac{1}{p_k}}_{p_k,p_k}(\mathbb{R}^{n-1}) &\longrightarrow b^{-\frac{1}{p_k}}_{p_k,p_k}(\mathbb{R}^{n-1}) \\ (\lambda^{j,G}_m) &\longrightarrow (2^{js}\lambda^{j,G}_m) \end{aligned}$$

is an isometry for $k = 0, 1$ and, according to (9.27),

$$b^{-1/p_k}_{p_k,p_k}(\mathbb{R}^{n-1}) = \ell_{p_k}(\mathbb{Z}^{n-1}) \times \prod_{G \in G^*} \ell^{-n/p_k}_{p_k}(\ell_{p_k}(\mathbb{Z}^{n-1})), \quad k = 0, 1,$$

where $G^* = \{F, M\}^{n-1} \setminus \{(F, F, F, ..., F)\}$. Therefore, if $G_0 = \{F\}^{n-1}$, according to Proposition 9.20, we get

$$
\begin{aligned}
&\|f|(B^{s-\frac{1}{p_0}}_{p_0,p_0}(\mathbb{R}^{n-1}), B^{s-\frac{1}{p_1}}_{p_1,p_1}(\mathbb{R}^{n-1}))_{\theta,r}\| \\
&\quad\sim \|(\lambda^{j,G}_m)|(b^{s-\frac{1}{p_0}}_{p_0,p_0}(\mathbb{R}^{n-1}), b^{s-\frac{1}{p_1}}_{p_1,p_1}(\mathbb{R}^{n-1}))_{\theta,r}\| \\
&\quad\sim \|(2^{js}\lambda^{j,G}_m)|(b^{-\frac{1}{p_0}}_{p_0,p_0}(\mathbb{R}^{n-1}), b^{-\frac{1}{p_1}}_{p_1,p_1}(\mathbb{R}^{n-1}))_{\theta,r}\| \\
&\quad\sim \|(\lambda^{0,G_0}_m)|\ell_{p,r}(\mathbb{Z}^{n-1})\| \\
&\qquad + \sum_{G\in G^*} \|(2^{js}\lambda^{j,G}_m)|(\ell^{-n/p_0}_{p_0}(\ell_{p_0}(\mathbb{Z}^{n-1})), \ell^{-n/p_1}_{p_1}(\ell_{p_1}(\mathbb{Z}^{n-1})))_{\theta,r}\|.
\end{aligned}
$$

Note that for $k = 0, 1$, $\ell^{-n/p_k}_{p_k}(\ell_{p_k}(\mathbb{Z}^{n-1})) = L_{p_k}(\mathbb{N}_0 \times \mathbb{Z}^{n-1}, \mu)$ with μ the weighted discrete measure

$$\mu = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^{n-1}} 2^{-jn}\delta_{\{(j,m)\}}.$$

Then, it follows from the interpolation formula for Lebesgue spaces (2.18), that

$$
\begin{aligned}
(\ell^{-n/p_0}_{p_0}(\ell_{p_0}(\mathbb{Z}^{n-1})), \ell^{-n/p_1}_{p_1}(\ell_{p_1}(\mathbb{Z}^{n-1})))_{\theta,r} &= (L_{p_0}(\mathbb{N}_0 \times \mathbb{Z}^{n-1}, \mu), L_{p_1}(\mathbb{N}_0 \times \mathbb{Z}^{n-1}, \mu))_{\theta,r} \\
&= L_{p,r}(\mathbb{N}_0 \times \mathbb{Z}^{n-1}, \mu).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|f|(B^{s-\frac{1}{p_0}}_{p_0,p_0}(\mathbb{R}^{n-1}), B^{s-\frac{1}{p_1}}_{p_1,p_1}(\mathbb{R}^{n-1}))_{\theta,r}\| \\
&\quad\sim \|(\lambda^{0,G_0}_m)|\ell_{p,r}(\mathbb{Z}^{n-1})\| + \sum_{G\in G^*} \|(2^{js}\lambda^{j,G}_m)|L_{p,r}(\mathbb{N}_0 \times \mathbb{Z}^{n-1}, \mu)\| \\
&\quad\sim \|t^{1-1/r}\text{card}\{m \in \mathbb{Z}^{n-1} : |\lambda^{0,G_0}_m| > t\}^{1/p}|L_r(0,\infty)\| \\
&\qquad + \sum_{G\in G^*} \|t^{1-1/r}\Big(\sum_{j=0}^{\infty} 2^{-jn}\text{card}\{m \in \mathbb{Z}^{n-1} : 2^{js}|\lambda^{j,G}_m| > t\}\Big)^{1/p}|L_r(0,\infty)\|.
\end{aligned}
\tag{10.24}
$$

Finally, we compute $\|\Lambda^s(\lambda^{j,G}_m)|L_{p,r}(\mathbb{R}^n)\|$ to see that it is equivalent to (10.24). Indeed, note that $(E_{jm})_{j\in\mathbb{N}_0,m\in\mathbb{Z}^{n-1}}$ are disjoint sets and $\Lambda^s(\lambda) = \Lambda^s_{G_0}(\lambda) + \sum_{G\in G^*}\Lambda^s_G(\lambda)$ where

$$
\begin{aligned}
\Lambda^s_{G_0}(\lambda) &= \sum_{m\in\mathbb{Z}^{n-1}} |\lambda^{0,G_0}_m|\chi_{E_{0m}}(x), \\
\Lambda^s_G(\lambda) &= \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^{n-1}} |\lambda^{j,G}_m|\chi_{E_{jm}}(x), \quad G \in G^*.
\end{aligned}
$$

Therefore, $\|\Lambda^s(\lambda^{j,G}_m)|L_{p,r}(\mathbb{R}^n)\| \sim \|\Lambda^s_{G_0}(\lambda^{0,G_0}_m)|L_{p,r}(\mathbb{R}^n)\| + \sum_{G\in G^*}\|\Lambda^s_G(\lambda^{j,G}_m)|L_{p,r}(\mathbb{R}^n)\|$. Taking into account that $|E_{jm}| \sim 2^{-jn}$ being $|\cdot|$ the Lebesgue measure on $\mathbb{R}^n$, it is straightforward that

$$
\begin{aligned}
|\{x \in \mathbb{R}^n : \Lambda^s_{G_0}(\lambda^{0,G_0}_m) > t\}| &= \text{card}\{m \in \mathbb{Z}^{n-1} : |\lambda^{0,G_0}_m| > t\}, \\
|\{x \in \mathbb{R}^n : \Lambda^s_G(\lambda^{j,G_0}_m) > t\}| &= \sum_{j=0}^{\infty} 2^{-jn}\text{card}\{m \in \mathbb{Z}^{n-1} : |2^{js}\lambda^{j,G}_m| > t\}, \quad G \in G^*.
\end{aligned}
$$

And this completes the proof. $\square$

Observe that using Theorems 10.23 and 10.24 we deduce that if $0 < p, q, r < \infty$ and $s - \frac{1}{p} > (n-1)(\frac{1}{p} - 1)_+$, $\mathrm{tr} F^s_q L_{p,r}(\mathbb{R}^n)$ is the space of all $f \in \mathcal{S}'(\mathbb{R}^{n-1})$ that admit a wavelet representation on $\mathbb{R}^{n-1}$ as in (10.23) such that $\|\Lambda^s((\lambda_m^{j,m}))|L_{p,r}(\mathbb{R}^n)\|$ is finite, being Λ^s as in Theorem 10.24. This is a characterization depending only on s, p and r, but not on p_0 and p_1 as happened in Theorem 10.23.

Following some of the ideas in the proof of Theorem 10.24, we can obtain a more general interpolation formula for Besov spaces, but this time we need to work with a special type of weighted Lorentz spaces: Let $\alpha \in \mathbb{R}$, for $x = (x_1, ..., x_n) \in \mathbb{R}^n$ we define

$$w_\alpha(x) = \begin{cases} |x_n|^\alpha & \text{if} \quad |x_n| \le 1, \\ 1, & \text{if} \quad |x_n| > 1. \end{cases}$$

For $0 < p < \infty$ and $0 < r \le \infty$ we put $L_{p,r}(\mathbb{R}^n, w_\alpha)$ for the corresponding Lorentz space on $(\mathbb{R}^n, \mu)$ with $\mu = w_\alpha(x)dx$.

Theorem 10.25. Let $0 < p_0 < p_1 < \infty$, $-\infty < s, \alpha < \infty$ and $0 < r \le \infty$. Then

$$(B^{s-\frac{\alpha}{p_0}}_{p_0,p_0}(\mathbb{R}^n), B^{s-\frac{\alpha}{p_1}}_{p_1,p_1}(\mathbb{R}^n))_{\theta,r}$$

is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ that admit a wavelet representation on $\mathbb{R}^n$ as

$$f(x) = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^{n-1}} \lambda_m^{j,G} 2^{-j(n-1)/2} \psi^j_{G,m}(x), \quad x \in \mathbb{R}^n \tag{10.25}$$

convergence in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\|f|(B^{s-\frac{\alpha}{p_0}}_{p_0,p_0}(\mathbb{R}^n), B^{s-\frac{\alpha}{p_1}}_{p_1,p_1}(\mathbb{R}^n))_{\theta,r}\| \sim \|\Lambda^s((\lambda_m^{j,G}))|L_{p,r}(\mathbb{R}^{n+1}, w_\alpha)\| < \infty,$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$,

$$\Lambda^s((\lambda_m^{j,G})) = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} 2^{js} |\lambda_m^{j,G}| \chi_{E_{jm}}(x), \quad x \in \mathbb{R}^{n+1}$$

and $E_{jm} = Q^{(n)}_{jm} \times (2^{-j-2}, 2^{-j-1}) = \prod_{\ell=1}^n (2^{-j}m_\ell - 2^{-j-1}, 2^{-j}m_\ell + 2^{-j-1}) \times (2^{-j-2}, 2^{-j-1}) \subset \mathbb{R}^{n+1}$ for $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$.

Proof. Proceeding as in Theorem 10.24, we derive that

$$\begin{aligned} &\|f|(B^{s-\frac{\alpha}{p_0}}_{p_0,p_0}(\mathbb{R}^n), B^{s-\frac{\alpha}{p_1}}_{p_1,p_1}(\mathbb{R}^n))_{\theta,r}\| \\ &\quad \sim \|(\lambda_m^{0,G_0})|\ell_{p,r}(\mathbb{Z}^n)\| \\ &\qquad + \sum_{G \in G^*} \|(2^{js}\lambda_m^{j,G})|(\ell_{p_0}^{-(n+\alpha)/p_0}(\ell_{p_0}(\mathbb{Z}^n)), \ell_{p_1}^{-(n+\alpha)/p_1}(\ell_{p_1}(\mathbb{Z}^{n-1})))_{\theta,r}\| \\ &\quad \sim \|t^{1-1/r}\mathrm{card}\{m \in \mathbb{Z}^n : |\lambda_m^{0,G_0}| > t\}^{1/p}|L_r(0,\infty)\| \\ &\qquad + \sum_{G \in G^*} \|t^{1-1/r}\Big(\sum_{j=0}^{\infty} 2^{-j(n+\alpha)}\mathrm{card}\{m \in \mathbb{Z}^n : 2^{js}|\lambda_m^{j,G}| > t\}\Big)^{1/p}|L_r(0,\infty)\|, \end{aligned}$$

here $G_0 = \{F\}^n$ and $G^* = \{F, M\}^n \setminus \{(F, F, ..., F)\}$.

On the other hand, $(E_{jm})_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ are disjoint sets and $\int_{\mathbb{R}^{n+1}}\chi_{E_{jm}}(x)w_\alpha(x)dx \sim 2^{-j(\alpha+n)}$. Moreover, analogous to Theorem 10.24,

$$\|\Lambda^s(\lambda_m^{j,G})|L_{p,r}(\mathbb{R}^n)\| \sim \|\Lambda^s_{G_0}(\lambda_m^{0,G_0})|L_{p,r}(\mathbb{R}^n)\| + \sum_{G\in G^*}\|\Lambda^s_G(\lambda_m^{j,G})|L_{p,r}(\mathbb{R}^n)\|,$$

and for $\mu = w_\alpha(x)dx$ on $\mathbb{R}^{n+1}$ we derive that

$$\mu(\{x\in\mathbb{R}^n : \Lambda^s_{G_0}(\lambda_m^{0,G_0}) > t\}) = \text{card}\{m\in\mathbb{Z}^n : |\lambda_m^{0,G_0}| > t\},$$
$$\mu(\{x\in\mathbb{R}^n : \Lambda^s_{G}(\lambda_m^{j,G_0}) > t\}) = \sum_{j=0}^{\infty} 2^{-j(n+\alpha)}\text{card}\{m\in\mathbb{Z}^n : |2^{js}\lambda_m^{j,G}| > t\}, \quad G\in G^*.$$

Thus, we conclude that $\|f|(B^{s-\frac{\alpha}{p_0}}_{p_0,p_0}(\mathbb{R}^n), B^{s-\frac{\alpha}{p_1}}_{p_1,p_1}(\mathbb{R}^n))_{\theta,r}\| \sim \|\Lambda^s((\lambda_m^{j,G}))|L_{p,r}(\mathbb{R}^{n+1}, w_\alpha)\|$. □

We close the section with the study of the trace operator acting on Besov-Lorentz spaces $B^s_qL_{p,q}(\mathbb{R}^n)$. The following result is a direct consequence of the trace theorem for classical Besov spaces and the interpolation formula proved in Theorem 9.21.

Theorem 10.26. Let $0 < p,q < \infty$ and $s - \frac{1}{p} > (n-1)(\frac{1}{p}-1)_+$. Then

(1) $\text{tr} : B^s_qL_{p,q}(\mathbb{R}^n) \longrightarrow B^{s-1/p}_qL_{p,q}(\mathbb{R}^{n-1})$ is linear and continuous, and

(2) there exists a linear and continuous extension operator

$$\text{ext} : B^{s-1/p}_qL_{p,q}(\mathbb{R}^{n-1}) \longrightarrow B^s_qL_{p,q}(\mathbb{R}^n)$$

such that

$$\text{tr}\circ\text{ext} = \text{id} : B^{s-1/p}_qL_{p,q}(\mathbb{R}^{n-1}) \longrightarrow B^{s-1/p}_qL_{p,q}(\mathbb{R}^{n-1}).$$

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