

Nonclassicality in phase by breaking classical bounds on statistics

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We derive upper bounds on the statistics of phase and phase difference that are satisfied by all classical states. They are obtained by finding the maximum projection of classical states on phase states. For a single-mode phase, meaningful bounds are obtained conditioned to a fixed mean number of photons. We also derive classical bounds for the projection on phase-coherent states, discussing their relation with phase-state bounds within the context of analytic representations. We find states with nonclassical phase properties disclosed by the violation of these classical bounds. These are quadrature and SU(2) squeezed states and phase-coherent states.

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I. INTRODUCTION

Nonclassicality is a key concept supporting the necessity of the quantum theory [1]. In a recent work, we have derived simple and robust practical procedures to reveal the quantum nature of states [2]. These are upper bounds on observable probabilities which are satisfied when the Glauber-Sudarshan \mathcal{P} function of the state is compatible with classical physics [3–5]. The lack of compliance of these statistical bounds is thus a nonclassical signature. The main advantages of these nonclassical tests are extreme simplicity, robustness under practical imperfections, independence of other typical quantum signatures, and that they can be equally well applied to any observable [2].

In this work, we apply this approach to the phase. This is an irreplaceable variable in classical optics that allows us to understand a great variety of optical phenomena, from propagation to interference. In particular, nonclassicality is crucial in interferometry for improving resolution beyond classical interferometry [5]. In quantum optics, some difficulties arise in the quantum description of phase [6–8]. Although the definition of a phase operator may be problematic, there is a large consensus concerning proper phase states. This allows us to define proper phase statistics by projection on the phase states, considering the phase observable as a positive operator-valued measure.

The objective of this work is twofold: (i) we derive upper bounds on phase statistics that are satisfied by all classical states, and (ii) we find states with nonclassical phase properties disclosed by the violation of these bounds. We split the analysis into single-mode phase (Sec. II) and phase difference (Sec. III). This is appropriate since in the quantum domain these variables are not as simply related as might be expected from classical optics [7,8].

II. CLASSICAL BOUNDS ON SINGLE-MODE PHASE

We consider that the phase variable is represented in the quantum domain by phase states defined as the eigenstates of the Susskind-Glogower operator E . This operator enters in a polar decomposition of the complex amplitude operator a [6],

$$a = E\sqrt{a^\dagger a} = \sqrt{aa^\dagger}E, \quad (1)$$

so that E should represent the exponential of the phase. In the photon-number basis $|n\rangle$, we have

$$E|n\rangle = |n-1\rangle, \quad E|0\rangle = 0. \quad (2)$$

Difficulties arise since E is not unitary but one-sided unitary: $EE^\dagger = I$, $E^\dagger E \neq I$, where I is the identity.

There are two sets of eigenstates of E , the unnormalized, nonorthogonal states $|\phi\rangle$,

$$E|\phi\rangle = e^{i\phi}|\phi\rangle, \quad |\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\phi}|n\rangle, \quad (3)$$

providing the resolution of the identity

$$\int_{\phi_0}^{\phi_0+2\pi} d\phi |\phi\rangle\langle\phi| = I, \quad (4)$$

where ϕ_0 is any phase. This defines suitable phase statistics $P(\rho, \phi)$ for every state ρ as

$$P(\rho, \phi) = \langle\phi|\rho|\phi\rangle. \quad (5)$$

For completeness we also consider projection on the normalized, nonorthogonal eigenstates of E , referred to as phase-coherent states [9],

$$E|\xi\rangle = \xi|\xi\rangle, \quad |\xi\rangle = \sqrt{1-|\xi|^2} \sum_{n=0}^{\infty} \xi^n |n\rangle, \quad (6)$$

with $|\xi| < 1$ and mean number of photons

$$\bar{N} = \langle\xi|a^\dagger a|\xi\rangle = \frac{|\xi|^2}{1-|\xi|^2}. \quad (7)$$

These states have interesting properties [9–14]. In particular, they provide an analytic representation in the unit disk $Z(\psi; \xi)$ of states $|\psi\rangle$ as [13,14]

$$Z(\psi; \xi) = \frac{1}{\sqrt{1-|\xi|^2}} \langle\psi|\xi\rangle = \sum_{n=0}^{\infty} \psi_n^* \xi^n, \quad (8)$$

where $\psi_n = \langle n|\psi\rangle$. The projection of $|\psi\rangle$ on the phase states $|\phi\rangle$ is obtained by the unit-modulus limit

$$\Theta(\psi; \phi) = \lim_{|\xi| \rightarrow 1} Z(\psi; \xi) = \lim_{|\xi| \rightarrow 1} \langle\psi|\xi\rangle = \sum_{n=0}^{\infty} \psi_n^* e^{in\phi}, \quad (9)$$

where the representation on the unit circle $\Theta(\psi; \phi)$ is the *boundary function* of $Z(\psi; \xi)$. This allows us to express

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the phase statistics as $P(|\psi\rangle, \phi) = |\Theta(\psi; \phi)|^2/(2\pi)$. It has been shown that the boundary function $\Theta(\psi; \phi)$ determines uniquely the analytic function $Z(\psi; \xi)$ [13,14]. In this context, an interesting set of states regarding phase properties are the *outer states*, which are the ones completely determined by their phase distribution $P(|\psi\rangle, \phi)$, that is, by the modulus of the boundary function $|\Theta(\psi; \phi)|$ [13,14].

A. Classical bound on phase statistics

In the quest of classical upper bounds let us begin with the phase statistics of coherent states $P(|\alpha\rangle, \phi) = |\langle\phi|\alpha\rangle|^2$, since they are the cornerstone of classical light, with

$$\langle\phi|\alpha\rangle = \frac{e^{-|\alpha|^2/2}}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{e^{-in\phi}\alpha^n}{\sqrt{n!}}. \quad (10)$$

In order to derive meaningful conclusions from analytical formulas let us consider $|\alpha| \gg 1$. In such a case the Poissonian number distribution of coherent states can be safely replaced by a Gaussian. Treating n as a continuous variable and replacing the sum in Eq. (10) by an integral, we get

$$\langle\phi|\alpha\rangle \simeq \frac{\int_{-\infty}^{\infty} dn \exp[in(\theta - \phi)] \exp\left[-\frac{(n-\bar{n})^2}{4(\Delta n)^2}\right]}{[(2\pi)^3(\Delta n)^2]^{1/4}}, \quad (11)$$

where \bar{n} is the mean number of photons in the coherent state,

$$\alpha = \sqrt{\bar{n}} \exp(i\theta), \quad \Delta n = \sqrt{\bar{n}} = |\alpha|, \quad (12)$$

and we have extended the lower limit of integration to $-\infty$. This leads to

$$\langle\phi|\alpha\rangle \simeq \left[\frac{2}{\pi}(\Delta n)^2\right]^{1/4} \exp[i\bar{n}(\theta - \phi)] \exp[-(\Delta n)^2(\phi - \theta)^2] \quad (13)$$

and

$$P(|\alpha\rangle, \phi) \simeq \sqrt{\frac{2}{\pi}} \Delta n \exp[-2(\Delta n)^2(\phi - \theta)^2]. \quad (14)$$

Within this approximation the phase distribution $P(|\alpha\rangle, \phi)$ is a Gaussian centered at θ . The phase uncertainty $\Delta\phi$ is inversely proportional to the number uncertainty Δn , which is $\Delta\phi = 1/(2\Delta n)$. For a fixed mean number of photons, \bar{n} , the maximum of $P(|\alpha\rangle, \phi)$ occurs for $\theta = \phi$,

$$P_{\max}(\bar{n}) \simeq \sqrt{\frac{2\bar{n}}{\pi}} = \sqrt{\frac{2}{\pi}} |\alpha|, \quad (15)$$

which is independent of ϕ . In Fig. 1 we have represented a numerical evaluation of the exact $P(|\alpha\rangle, \phi)$ for $\theta = \phi$ and the approximation $P_{\max}(\bar{n})$ in Eq. (15) as functions of \bar{n} , showing that the approximation works well even for rather low mean photon numbers. The exact calculus and the approximation differ by less than 1% for $\bar{n} \geq 14$.

It can be seen that there is no absolute upper bound for the phase statistics valid for all classical states at once, since $P_{\max}(\bar{n}) \rightarrow \infty$ when $\bar{n} \rightarrow \infty$ [note that $P(\rho, \phi)$ is probability density rather than probability]. This is reasonable since coherent states tend to have a well-defined phase as the mean number of photons increases, as illustrated in Fig. 2.

Therefore, nonclassical phase properties of a single mode must be elucidated by a classical upper bound conditioned to

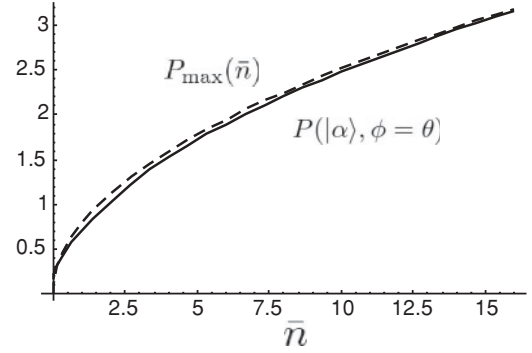


FIG. 1. Numerical evaluation of the exact $P(|\alpha\rangle, \phi = \theta)$ (solid line) and its approximation $P_{\max}(\bar{n}) \simeq \sqrt{2\bar{n}/\pi}$ in Eq. (15) (dashed line) as functions of \bar{n} . They are almost indistinguishable even for small photon numbers.

the mean number of photons of the state whose classicality is being tested. Next we derive such classical bounds. Classical states are all incoherent superpositions of coherent states,

$$\rho_{\text{class}} = \int d^2\alpha \mathcal{P}(\alpha) |\alpha\rangle\langle\alpha|, \quad \mathcal{P}(\alpha) \geq 0, \quad (16)$$

where $\mathcal{P}(\alpha)$ is any classical probability distribution (this is real, normalized, non-negative, and no more singular than a δ function). Thus, admitting the approximate result (15), we get the following inequality:

$$P(\rho_{\text{class}}, \phi) = \int d^2\alpha \mathcal{P}(\alpha) P(|\alpha\rangle, \phi) \leq \sqrt{\frac{2}{\pi}} \int d^2\alpha \mathcal{P}(\alpha) |\alpha| \leq \sqrt{\frac{2\bar{n}}{\pi}}, \quad (17)$$

where \bar{n} represents the mean number of photons in the state, ρ_{class} , and we have used that for classical $\mathcal{P}(\alpha)$ distributions:

$$\left(\int d^2\alpha \mathcal{P}(\alpha) |\alpha|\right)^2 \leq \int d^2\alpha \mathcal{P}(\alpha) |\alpha|^2 = \bar{n}. \quad (18)$$

This defines the following nonclassical-phase criterion for states ρ with any mean number of photons \bar{n} :

$$P(\rho, \phi) > P_{\max}(\bar{n}) \simeq \sqrt{\frac{2\bar{n}}{\pi}} \rightarrow \rho \text{ is nonclassical}. \quad (19)$$

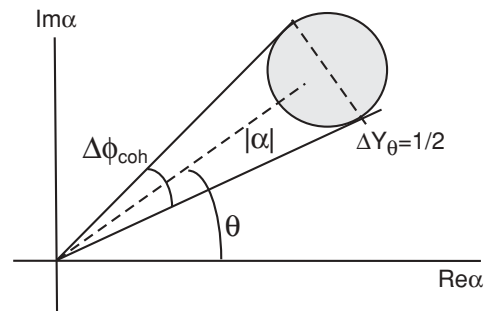


FIG. 2. Illustration representing a coherent state as an uncertainty disk of diameter $1/2$ in the complex-amplitude plane. The diameter of the disk is given by the uncertainty of the out-of-phase quadrature $Y_\theta = i(a^\dagger e^{i\theta} - a e^{-i\theta})/2$. The phase uncertainty $\Delta\phi_{\text{coh}} \simeq 1/(2|\alpha|)$ tends to zero as $|\alpha| \rightarrow \infty$.

For Gaussian phase distributions, after Eq. (14) this criterion admits a simple interpretation: phase uncertainty $\Delta\phi$ below the coherent-state value $\Delta\phi_{\text{coh}} = 1/(2\sqrt{\bar{n}})$ is a nonclassical property. From a practical perspective, phase fluctuations degrade visibility and interferometric resolution. In this regard, the above result shows that states providing larger interferometric resolution than coherent states are nonclassical, in agreement with other approaches [5].

B. Classical bounds on the projection on phase-coherent states

For completeness, next we derive an absolute bound independent on the mean number of photons by considering the projection on the normalizable phase-coherent states $|\xi\rangle$ in Eq. (6), $P(\rho, \xi) = \langle \xi | \rho | \xi \rangle$. Resorting again to the Gaussian and continuous number approximations, and assuming that $|\alpha| \gg 1$ and $|\xi| \rightarrow 1$, we get

$$\begin{aligned} \langle \xi | \alpha \rangle &\simeq \frac{\sqrt{1 - |\xi|^2}}{[2\pi(\Delta n)^2]^{1/4}} \int_{-\infty}^{\infty} dn |\xi|^n \exp[in(\theta - \phi)] \\ &\times \exp\left[-\frac{(n - \bar{n})^2}{4(\Delta n)^2}\right], \end{aligned} \quad (20)$$

where $\theta = \arg \alpha$ and $\phi = \arg \xi$, leading to, for $\theta = \phi$,

$$|\langle \xi | \alpha \rangle|^2 \simeq \frac{2\sqrt{2\pi\bar{n}}}{\bar{N} + 1} \left(\frac{\bar{N}}{\bar{N} + 1}\right)^{\bar{n}} \exp\left(\frac{\bar{n}}{2} \ln^2 \frac{\bar{N}}{\bar{N} + 1}\right), \quad (21)$$

where \bar{N} is the mean number of photons of the state $|\xi\rangle$, \bar{n} is the mean number of photons of $|\alpha\rangle$, and $\Delta n = \sqrt{\bar{n}}$ is the photon-number uncertainty in $|\alpha\rangle$. In Fig. 3 we have represented a numerical evaluation of the exact $P(|\alpha\rangle, \xi)$ for $\theta = \phi$ along with its approximation (21) as functions of \bar{n} for $\bar{N} = 50$, showing that the approximation works well even for rather low mean photon numbers. The exact calculus and the approximation differ by less than 2% for $\bar{n} \geq 15$.

Finally, we look for the maximum $P_{\text{max}}(\bar{N})$ of $|\langle \xi | \alpha \rangle|^2$ when \bar{n} is varied, which holds for

$$\bar{n} = \frac{1}{2 \ln \frac{\bar{N}+1}{\bar{N}} \left(1 + \frac{1}{2} \ln \frac{\bar{N}}{1+\bar{N}}\right)}. \quad (22)$$

In the limit $\bar{N} \gg 1$, we can further approximate $P_{\text{max}}(\bar{N})$ as

$$P_{\text{max}}(\bar{N}) \simeq 2\sqrt{\frac{\pi}{e\bar{N}}}. \quad (23)$$

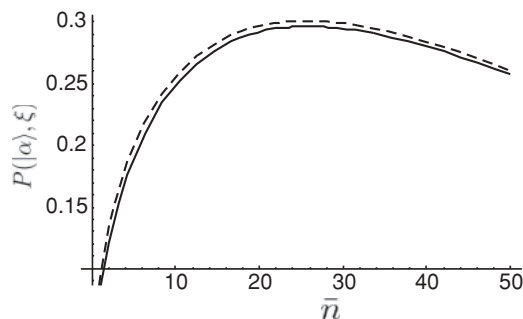


FIG. 3. Numerical evaluation of $P(|\alpha\rangle, \xi)$ for $\theta = \phi$ (solid line) and its approximation (21) (dashed line) as functions of \bar{n} for $\bar{N} = 50$.

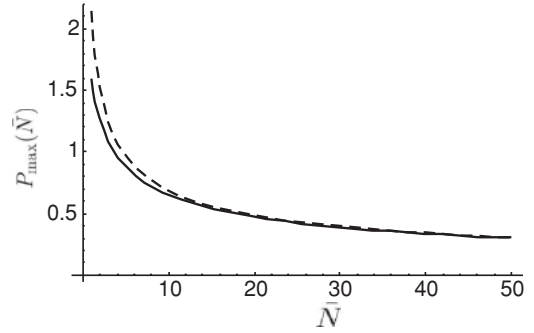


FIG. 4. Plot of $P_{\text{max}}(\bar{N})$ after Eqs. (21) and (22) (solid line) and its approximation (23) (dashed line) as functions of \bar{N} .

In Fig. 4 we have represented $P_{\text{max}}(\bar{N})$ in Eqs. (21) and (22) and its approximation (23). They differ by less than 1% for $\bar{N} \geq 50$.

C. Phase-state versus phase-coherent-state bounds

It is natural to compare the two nonclassicality criteria (19) and (23) obtained above. Their differences stem from the different normalization of $|\phi\rangle$ and $|\xi\rangle$. The lack of normalization of $|\phi\rangle$ forces that any meaningful classical bound must be conditioned to the mean number of photons, \bar{n} , of the state being tested, since otherwise $\langle \phi | \rho | \phi \rangle$ has no absolute upper bound for classical states ρ . On the other hand, since $\langle \xi | \rho | \xi \rangle \leq 1$ for all ρ , an absolute maximum of $\langle \xi | \rho | \xi \rangle$ can be found when ρ is varied among classical states. Such a maximum holds for all \bar{n} and only depends on the mean number of photons \bar{N} of $|\xi\rangle$.

Otherwise, if we remove normalization factors, we get the boundary limit

$$\lim_{|\xi| \rightarrow 1} \frac{|\langle \xi | \alpha \rangle|^2}{1 - |\xi|^2} = 2\pi |\langle \phi | \alpha \rangle|^2, \quad (24)$$

which is fully consistent with the boundary relation between analytic representations recalled in Eq. (9). Moreover, for $|\alpha| \gg 1$, this boundary limit allows us to get explicitly the phase-state criteria (19) from the phase-coherent-state bound in Eqs. (21) and (22) as

$$\begin{aligned} \lim_{|\xi| \rightarrow 1} \frac{|\langle \xi | \alpha \rangle|^2}{1 - |\xi|^2} &= \lim_{\bar{N} \rightarrow \infty} 2\sqrt{2\pi\bar{n}} \left(\frac{\bar{N}}{\bar{N} + 1}\right)^{\bar{n}} \\ &\times \exp\left(\frac{\bar{n}}{2} \ln^2 \frac{\bar{N}}{\bar{N} + 1}\right) = 2\pi \sqrt{\frac{2\bar{n}}{\pi}}. \end{aligned} \quad (25)$$

D. Examples

Next we provide some examples of nonclassical phase statistics illustrated by quadrature squeezed states and phase states. For some other field states, such as number, Schrödinger cat states, and superpositions of number states of the form $\alpha|0\rangle + \beta|n\rangle$, we have found no nonclassical phase behavior.

1. Number states

For number states $\rho = |n\rangle\langle n|$ we have

$$P(|n\rangle, \phi) = \frac{1}{2\pi}, \quad |\langle \xi | n \rangle|^2 = \frac{1}{\bar{N} + 1} \left(\frac{\bar{N}}{\bar{N} + 1} \right)^n. \quad (26)$$

The phase distribution $P(|n\rangle, \phi)$ is uniform and always below the classical upper bound $P_{\max}(\bar{n} = n)$ for all integers n since (see Fig. 1)

$$P_{\max}(n \geq 1) \geq P_{\max}(n = 1) = 0.705 > P(|n\rangle, \phi) = \frac{1}{2\pi} = 0.159. \quad (27)$$

It can be easily seen numerically that the same conclusion holds for the projection on phase-coherent states $|\xi\rangle$. In particular, for $\bar{N} \gg 1$, we get

$$|\langle \xi | n \rangle|^2 = \frac{1}{\bar{N} + 1} \left(\frac{\bar{N}}{\bar{N} + 1} \right)^n \simeq \frac{1}{\bar{N}} \left(1 - \frac{n}{\bar{N}} \right), \quad (28)$$

which is below the classical upper bound (23).

These results are consistent since complementarity demands that states with well-defined number should have fully random phase. Then, the uniform phase distribution is fully compatible with classical physics.

2. Quadrature coherent squeezed states

For quadrature squeezed states $|\psi\rangle$ in a pure state with moderate squeezing and large mean photon numbers, the continuous number and Gaussian approximations above work quite well (for large squeezing, other approximations may be used [15]), so that the phase distribution is given by Eq. (14) with $\Delta n \neq \sqrt{\bar{n}}$. Thus, the phase distribution is Gaussian so the classical bound is surpassed when the phase fluctuations are below the coherent-state level, caused by reduced fluctuations of the out-of-phase quadrature $\Delta Y_\theta < 1/2$, as illustrated in Fig. 5. In turn, this implies super-Poissonian number statistics because of the enlarged fluctuations of the in-phase quadrature.

We can relate this nonclassical phase behavior with the usefulness of squeezed states in quantum metrology [16]. A convenient measure of the metrological usefulness is provided by the quantum Fisher information F_Q [17], where larger F_Q means larger resolution. For standard interferometry with light in a pure state, it holds that $F_Q = 4(\Delta n)^2$. Thus, improved resolution beyond coherent states implies $\Delta n > \sqrt{\bar{n}}$, which we

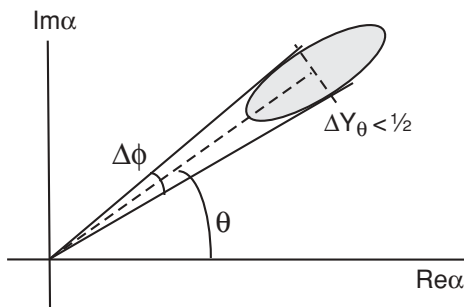


FIG. 5. Illustration representing a phase-squeezed state as an uncertainty disk in the complex-amplitude plane.

have just shown is equivalent to nonclassical phase statistics $\Delta\phi < \Delta\phi_{\text{coh}}$.

Some other previous works on squeezed states can also be well framed within this approach. In Ref. [18] we have studied the optimum combination of coherent displacement and squeezing so that $P(|\psi\rangle, \phi)$ and $P(|\psi\rangle, \xi)$ are maximums for fixed mean photon number. This was intended to approximate phase states by experimentally realizable states. From the perspective of this work this corresponds to maximizing phase nonclassicality. Some other practical approximations of phase states can be found in Ref. [9].

3. Squeezed vacuum

As a further example let us consider the squeezed vacuum, also with application in quantum metrology [19]. In the photon-number basis this is

$$|r\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} \tanh^n r |2n\rangle, \quad (29)$$

with mean number of photons $\bar{n} = \sinh^2 r$. This does not admit the continuous number and Gaussian approximations.

It can be easily seen that the phase distribution has two symmetrical peaks at $\phi = 0, \pi$. In Fig. 6 we have represented numerical evaluations of $P(|r\rangle, \phi = 0)$ and $P_{\max}(\bar{n})$ for $\bar{n} = \sinh^2 r$ as functions of \bar{n} . The nonclassical behavior holds for all $\bar{n} > 1.75$ and increases as the mean number of photons \bar{n} grows.

4. Phase-coherent states

Let us consider the phase distribution of the phase-coherent states $|\xi\rangle$ [11,14],

$$P(|\xi\rangle, \phi) = |\langle \phi | \xi \rangle|^2 = \frac{1}{2\pi} \frac{1 - |\xi|^2}{1 + |\xi|^2 - 2|\xi| \cos(\phi - \theta)}, \quad (30)$$

where $\theta = \arg \xi$. The maximum holds for $\phi = \theta$, being

$$P(|\xi\rangle, \phi = \theta) = \frac{1 + |\xi|}{2\pi(1 - |\xi|)} \simeq \frac{2\bar{N}}{\pi}, \quad (31)$$

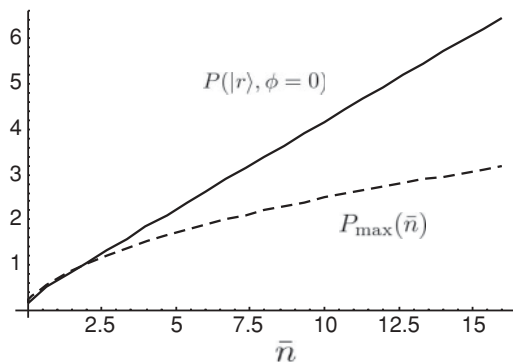


FIG. 6. Numerical evaluations of $P(|r\rangle, \phi = 0)$ (solid line) and $P_{\max}(\bar{n})$ (dashed line) for $\bar{n} = \sinh^2 r$ as functions of \bar{n} .

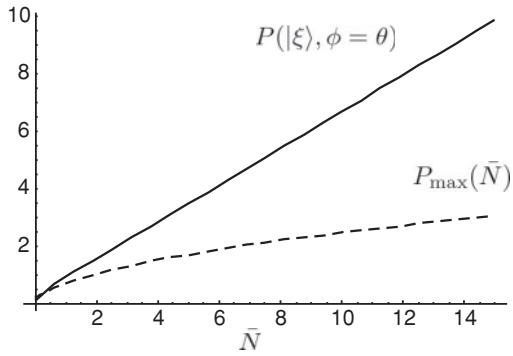


FIG. 7. Numerical evaluation of $P(|\xi\rangle, \phi = \theta)$ (solid line) and $P_{\max}(\bar{N})$ (dashed line) as functions of \bar{N} .

where \bar{N} is the mean number of photons of the state $|\xi\rangle$, and the last approximation holds for $\bar{N} \gg 1$. The approximated condition for nonclassical phase behavior (19),

$$P(|\xi\rangle, \phi = \theta) \simeq \frac{2\bar{N}}{\pi} > P_{\max}(\bar{N}) \simeq \sqrt{2\frac{\bar{N}}{\pi}}, \quad (32)$$

is satisfied for $\bar{N} > \pi/2$. Actually, the numerical evaluation represented in Fig. 7 shows that the nonclassical phase behavior holds for every \bar{N} .

5. Outer states

As mentioned above, for the outer states all quantum-statistical properties are determined by their phase distribution. Therefore, it is interesting to inquire whether these states have special properties concerning phase nonclassicality. In particular, we examine whether for these states phase nonclassicality extends to arbitrary field observables, according to the general approach in Ref. [2].

There are outer states that are classical in phase while some others are nonclassical with regard to phase. For example, the phase-coherent states $|\xi\rangle$ are outer states [13] and nonclassical with regard to phase (see above). On the other hand, the coherent superpositions of vacuum and number state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |n\rangle), \quad (33)$$

are also outer states [13] and classical with regard to phase. This is because the maximum of the phase statistics $P(|\psi\rangle, \phi = 0) = 1/\pi$ is always below the classical upper bound for the corresponding mean number of photons $P_{\max}(\bar{n} = n/2)$ for all integers n , since

$$\begin{aligned} P_{\max}(\bar{n} \geq 1/2) &\geq P_{\max}(\bar{n} = 1/2) = 0.501 \\ &> P(|\psi\rangle, \phi = 0) = \frac{1}{\pi} = 0.318. \end{aligned} \quad (34)$$

Phase nonclassicality of outer states does not extend to arbitrary field observables. A suitable counterexample is provided by the number observable for phase-coherent states, since their photon-number statistics $|\langle n|\xi\rangle|^2$ coincides exactly

with the number statistics of thermal-chaotic states, which are classical states [9,12,14].

III. CLASSICAL BOUNDS ON PHASE DIFFERENCE

In practical terms, phase usually manifests as a phase difference between two field modes (sometimes one of them merely acting as a phase reference). Interferometry is a typical example.

Maybe, surprisingly, in the quantum domain this is not a trivial remark, since there are definite properties of quantum phase difference that cannot be derived from single-mode phases [7,8]. In particular, there is a bona fide unitary operator exponential of the phase difference satisfying a polar decomposition of the product of complex amplitudes, $a_1 a_2^\dagger$ [7]. In any case, single-mode phase and phase difference require different analyses.

A. Phase difference and classical states

A convenient set of observables suitable to express two-mode properties including the phase difference is [20]

$$\begin{aligned} j_0 &= \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2), & j_1 &= \frac{1}{2}(a_2^\dagger a_1 + a_1^\dagger a_2), \\ j_2 &= \frac{i}{2}(a_2^\dagger a_1 - a_1^\dagger a_2), & j_3 &= \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2), \end{aligned} \quad (35)$$

which satisfy the commutation relations of an angular momentum or spin,

$$[j_k, j_\ell] = i \sum_{n=1}^3 \epsilon_{k,\ell,n} j_n, \quad [j_0, \mathbf{j}] = \mathbf{0}, \quad (36)$$

with

$$\mathbf{j}^2 = j_0(j_0 + 1), \quad (37)$$

where $\epsilon_{k,\ell,n}$ is the fully antisymmetric tensor with $\epsilon_{1,2,3} = 1$. There is a useful correspondence between the angular momentum basis $|j, m\rangle_3$, with $j_3|j, m\rangle_3 = m|j, m\rangle_3$ and $j_0|j, m\rangle = j|j, m\rangle_3$, and the two-mode photon-number basis $|n_1\rangle|n_2\rangle$ in the form

$$|j, m\rangle_3 = |n_1 = j + m\rangle|n_2 = j - m\rangle. \quad (38)$$

For definiteness, throughout we focus on states with definite total photon number $N = 2j$.

Phase difference is the variable conjugated to number difference j_3 , so it is formally equivalent to the azimuthal angle. It can be properly represented by the phase-difference states [7,8,21]

$$|j, \phi\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^j e^{im\phi} |j, m\rangle_3, \quad (39)$$

defining phase-difference statistics

$$P_j(\rho, \phi) = \langle j, \phi | \rho | j, \phi \rangle. \quad (40)$$

In the spin context, the classical states are [22]

$$\rho_{\text{class}} = \int d\Omega \mathcal{P}_{\text{class}}(\Omega) |j, \Omega\rangle \langle j, \Omega|, \quad \mathcal{P}_{\text{class}}(\Omega) \geq 0, \quad (41)$$

where $|j, \Omega\rangle$ are the SU(2) coherent states,

$$|j, \Omega\rangle = \sum_{m=-j}^j \sqrt{p_j(\vartheta, m)} e^{im\varphi} |j, m\rangle_3, \quad (42)$$

with

$$p_j(\vartheta, m) = \binom{2j}{j+m} \left(\sin \frac{\vartheta}{2}\right)^{2j+2m} \left(\cos \frac{\vartheta}{2}\right)^{2j-2m}, \quad (43)$$

$\Omega = (\vartheta, \varphi)$ are state parameters, and $d\Omega = \sin \vartheta d\vartheta d\varphi$.

B. Classical upper bound

We can derive an upper bound for the phase-difference statistics of classical states as

$$P_j(\rho_{\text{class}}, \phi) \leq P_{j, \text{max}}, \quad P_{j, \text{max}} = |\langle j, \phi | j, \Omega_{\text{max}} \rangle|^2, \quad (44)$$

with

$$\langle j, \phi | j, \Omega \rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^j \sqrt{p_j(\vartheta, m)} e^{im(\varphi-\phi)}, \quad (45)$$

where $|j, \Omega_{\text{max}}\rangle$ is the SU(2) coherent state that maximizes $|\langle j, \phi | j, \Omega \rangle|$. This holds for $\vartheta = \pi/2$ and $\varphi = \phi$. Actually the coherent states $|j, \Omega_{\text{max}}\rangle$ have been regarded as feasible counterparts of phase-difference states [23].

No simple expression for $P_{j, \text{max}}$ is available. A suitable approximation valid for $j \gg 1$ can be carried out within the continuous number and Gaussian approximations [24]

$$p_j(\vartheta = \pi/2, m) \simeq \frac{1}{\sqrt{2\pi} \Delta j_3} \exp\left[-\frac{m^2}{2(\Delta j_3)^2}\right], \quad (46)$$

with $\Delta j_3 = \sqrt{j/2}$. Approximating the sum in Eq. (45) by an integral, we get

$$P_{j, \text{max}} \simeq \sqrt{\frac{\pi}{j}}. \quad (47)$$

In Fig. 8 we have represented a numerical evaluation of the exact $P_{j, \text{max}}$ and its approximation $P_{j, \text{max}} \simeq \sqrt{\pi/j}$ as functions of j . They differ by less than 2% for $j \geq 32$.

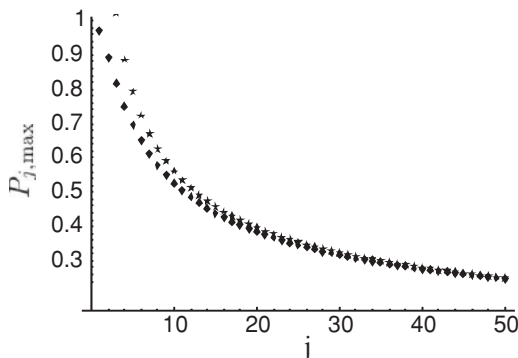


FIG. 8. Numerical evaluation of $P_{j, \text{max}}$ (diamonds) and its approximation $P_{j, \text{max}} \simeq \sqrt{\pi/j}$ (stars) as functions of j .

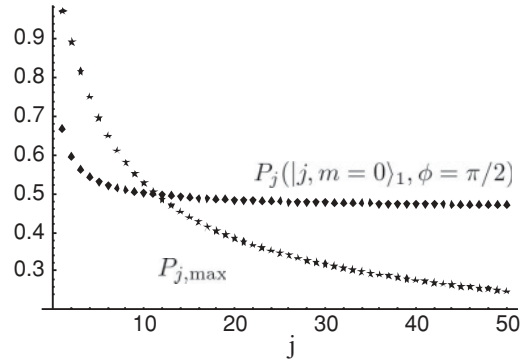


FIG. 9. Numerical evaluations of $P_j(|j, m=0\rangle_1, \phi = \pi/2)$ (diamonds) and $P_{j, \text{max}}$ (stars) as functions of j .

C. SU(2) squeezed states

SU(2) squeezed states are paradigmatic nonclassical states. There are several definitions of SU(2) or spin squeezing [25]. The most meaningful criteria are based on the fluctuations of the angular momentum components j_{\perp} orthogonal to the average vector $\langle j \rangle$, so by definition $\langle j_{\perp} \rangle = 0$. The general idea is that SU(2) squeezing holds when a suitable function of Δj_{\perp} and $|\langle j \rangle|$ is below its value for SU(2) coherent states with $\langle j_{\perp} \rangle = 0$. These states allow the most precise interferometric measurements in linear interferometers [25].

As a simple extreme case of SU(2) squeezing we can consider the eigenstate of j_1 with vanishing eigenvalue $|j, m=0\rangle_1$ for integer j , also referred to as twin-photon states since it corresponds to the product of identical number states in some suitable modes [26]. In the $|j, m\rangle_3$ basis, it reads

$$|j, m=0\rangle_1 = \frac{\sqrt{(2j)!}}{2^j j!} \sum_{k=0}^j (-1)^k \binom{j}{k} \binom{2j}{2k}^{-1/2} |j, j-2k\rangle_3, \quad (48)$$

which does not admit a simple Gaussian approximation. The maximum phase probability occurs for $\phi = \pm\pi/2$. In Fig. 9 we have represented numerical evaluations of $P_j(|j, m=0\rangle_1, \phi = \pi/2)$ and $P_{j, \text{max}}$ as functions of j showing nonclassical behavior for the phase difference for $j \geq 12$.

IV. CONCLUSIONS

We have derived very simple criteria disclosing nonclassical phase properties. These are obtained from upper bounds on the phase statistics that are satisfied by all classical light states, so that their infringement reveals quantum behavior. We have shown that phase-coherent states and both quadrature and SU(2) squeezed states display nonclassical phase properties.

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