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Lineability, differentiable functions and special derivatives

J. Fernández-Sánchez ·
D.L. Rodríguez-Vidanes ·
J.B. Seoane-Sepúlveda ·
W. Trutschnig

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Abstract The present work either extends or improves several results on lineability of differentiable functions and derivatives enjoying certain special properties. Among many other results, we show that there exist *large* algebraic structures inside the following sets of *special* functions: (i.-) The class of differentiable functions with discontinuous derivative on a set of positive measure, (ii.-) the family of differentiable functions with a bounded, non-Riemann integrable derivative, (iii.-) the family of functions from $(0, 1)$ to $\mathbb{R}$ that are not derivatives, or (iv.-) the family of mappings that do not satisfy Rolle's Theorem on real infinite dimensional Banach spaces. Several examples and graphics illustrate the obtained results.

Keywords lineability · algebrability · Rolle's theorem · differentiable function · derivative

Mathematics Subject Classification (2010) 15A03 · 46B87 · 26A15 · 26A27 · 46J10

J. Fernández-Sánchez
Grupo de investigación "Teoría de cópulas y aplicaciones", Universidad de Almería. Carretera de Sacramento s/n, 04120 Almería (Spain)
E-mail: juanfernandez@ual.es

D.L. Rodríguez-Vidanes
Departamento de Análisis y Matemática Aplicada, Facultad de Ciencias Matemáticas. Plaza de Ciencias 3, Universidad Complutense de Madrid, Madrid, 28040 (Spain)
E-mail: dl.rodriguez.vidanes@ucm.es

J.B. Seoane-Sepúlveda
Instituto de Matemática Interdisciplinar (IMI), Departamento de Análisis y Matemática Aplicada, Facultad de Ciencias Matemáticas. Plaza de Ciencias 3, Universidad Complutense de Madrid, Madrid, 28040 (Spain)
E-mail: jseoane@ucm.es

W. Trutschnig
Department for Mathematics, University Salzburg. Hellbrunnerstrasse 34, 5020 Salzburg (Austria)
E-mail: wolfgang@trutschnig.net

1 Introduction and preliminaries

For the last two decades many mathematicians have shown interest in the search for large algebraic structures within nonlinear sets of a topological vector space. This area of research receives the name of lineability, a terminology dating back to the early 2000's, coined by V.I. Gurariy (1935–2005) and first introduced in [3,33]. There has been plenty of work in this direction since its appearance. As a matter of fact, this notion was (just recently) introduced by the American Mathematical Society under the MSC2020 15A03 and 46B87 classification references.

Research on lineability has shown to be extremely fruitful, a comprehensive description of theses concepts (as well as numerous examples and some general techniques) can be found in, e.g., [1, 2, 7–16, 18, 25, 32].

Definition 1.1 Assume that X is a vector space, that α is a cardinal number and that $A \subset X$. Then A is said to be:

- lineable if there is an infinite dimensional vector space M such that $M \setminus \{0\} \subset A$, and
- α-lineable if there exists a vector space M with $\dim(M) = \alpha$ and $M \setminus \{0\} \subset A$.

If, in addition, X is a topological vector space, then A is said to be spaceable whenever there is a closed infinite dimensional vector subspace M of X satisfying $M \setminus \{0\} \subset A$. Finally, when X is a topological vector space contained in some (linear) algebra then A is called:

- algebrable if there is an algebra M so that $M \setminus \{0\} \subset A$ and M is infinitely generated, that is, the cardinality of any system of generators of M is infinite.
- α-algebrable if there is an α-generated algebra M with $M \setminus \{0\} \subset A$.
- strongly α-algebrable if there exists an α-generated free algebra M with $M \setminus \{0\} \subset A$.

The rest of this paper is organized as follows. After the introduction, Section 2 focuses on improving and extending several results from [24, 26, 33] on differentiable functions and derivatives enjoying certain special properties. Let us use the following notations for families of functions from $\mathbb{R}$ to $\mathbb{R}$:

$\mathcal{D}_a$: The family of differentiable functions that do not have a continuous derivative at $a \in \mathbb{R}$.
$\mathcal{M}_a$: The family of differentiable functions having a local extreme value at a point $a \in \mathbb{R}$ where the derivative does not make a simple change in sign.
$\mathcal{UD}_a$: The family of differentiable functions that do not have bounded derivative on any neighborhood of $a \in \mathbb{R}$.

With these previous notations in hand, among other results, we shall prove that, for every $a \in \mathbb{R}$, the set $\mathcal{D}_a$ is $\mathfrak{c}$-lineable (Theorem 2.1) and the set $\mathcal{M}_a$ is strongly $\mathfrak{c}$-algebrable (Theorem 2.2). Also, completing some results from [24] we prove that the family of differentiable functions with discontinuous derivative on a set with positive (Lebesgue) measure is $\mathfrak{c}$-lineable (Theorem 2.3). Among other results of independent interest, we also show that $\mathcal{UD}_a$ is strongly $\mathfrak{c}$-algebrable for every $a \in \mathbb{R}$ (Theorem 2.5) and that (Theorem 2.6) the family of differentiable functions that have an unbounded derivative on every neighborhood of the points of a set with positive (Lebesgue) measure is strongly $\mathfrak{c}$-algebrable. All the previous assertions are either new or improve results appearing in [24, 26, 33]. At the end of this section we also provide results regarding real valued functions that are not derivatives (Theorem 2.7).

Next, Section 3 shall deal with lineability within the framework of Fréchet, α-Hölder and limiting subdifferentials (Theorems 3.1, 3.2, 3.3, and 3.4). The final section (Section 4) will deal with the lineability problem of mappings that do not satisfy Rolle's Theorem on real infinite dimensional Banach spaces. Examples and several remarks complement and illustrate the derived results.

2 Derivatives with special properties

This second section improves some results that can be found in [33, subsection 5.2.1 and 5.2.2]. The first result goes deeper into the study of [33, theorem 5.2.1], where it is shown that the family $\mathcal{D}_0$ is $\aleph_0$-lineable. To do so, we begin by recalling a theorem from number theory and dynamical systems known as Kronecker's density theorem (see, for instance, [20, 28, 30]).

Lemma 2.1 *Let $n \in \mathbb{N}$. If $\{h_i : i \in \{1, \ldots, n\}\}$ is a set of n real numbers linearly independent over $\mathbb{Q}$, then*

$$\{(kh_1, \ldots, kh_n) (\mathrm{mod}\ 1) : k \in \mathbb{N}\}$$

is dense in $[0, 1]^n$.

Theorem 2.1 *Let $a \in \mathbb{R}$. Then the family $\mathcal{D}_a$ is $\mathfrak{c}$-lineable for every $a \in \mathbb{R}$.*

Proof Fix $a \in \mathbb{R}$. Let $H \subset (0, 1)$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. For every $h \in H$, we define the function

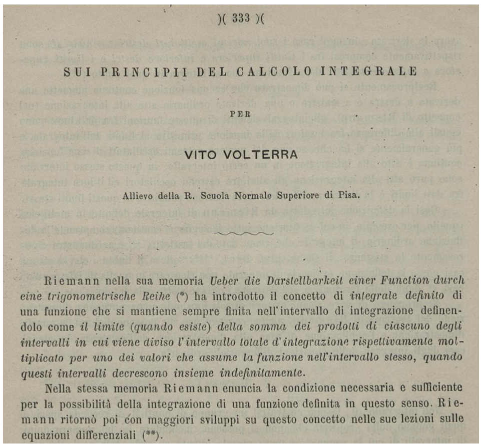
$$f_h(x) = \begin{cases} (x-a)^2 \sin\left(\dfrac{h}{x-a}\right) & \text{if } x \neq a, \\ 0 & \text{if } x = a. \end{cases} \tag{2.1}$$

It is well known that $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ is differentiable everywhere, $f'(0) = 0$ but the derivative is not continuous at 0. This function first appeared in a 1881 paper [36, p. 335] of Vito Volterra.

As an immediate consequence we obtain that f_h is differentiable everywhere, $f'(a) = 0$ but the derivative is not continuous at $a \in \mathbb{R}$ for any $h \in H$. Thus, $\mathcal{F} = \{f_h : h \in H\} \subset \mathcal{D}_a$.

It suffices to show that $\mathcal{F}$ is a family of linearly independent functions and that $\mathrm{span}\{\mathcal{F}\} \setminus \{0\} \subset \mathcal{D}_a$. Let $f = \sum_{i=1}^{n} \alpha_i f_{h_i}$, where $f_{h_1}, \ldots, f_{h_n} \in \mathcal{F}$ are distinct, $\alpha_1, \ldots, \alpha_n \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$. Let $\beta_i = \alpha_i h_i$ for every $i \in \{1, \ldots, n\}$. Notice that f is differentiable everywhere and $\beta_i \neq 0$ for every $i \in \{1, \ldots, n\}$. Moreover, if $x \neq a$, we have

$$f'(x) = \sum_{i=1}^{n} \alpha_i \left(2(x-a)\sin\left(\frac{h_i}{x-a}\right) - h_i \cos\left(\frac{h_i}{x-a}\right)\right),$$



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SUI PRINCIPII DEL CALCOLO INTEGRALE

PER

VITO VOLTERRA

Allievo della R. Scuola Normale Superiore di Pisa.

Riemann nella sua memoria *Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe* (*) ha introdotto il concetto di *integrale definito* di una funzione che si mantiene sempre finita nell'intervallo di integrazione definendolo come *il limite (quando esiste) della somma dei prodotti di ciascuno degli intervalli in cui viene diviso l'intervallo totale d'integrazione rispettivamente moltiplicato per uno dei valori che assume la funzione nell'intervallo stesso, quando questi intervalli decrescono insieme indefinitamente.*

Nella stessa memoria Riemann enuncia la condizione necessaria e sufficiente per la possibilità della integrazione di una funzione definita in questo senso. Riemann ritornò poi con maggiori sviluppi su questo concetto nelle sue lezioni sulle equazioni differenziali (**).

Fig: 1. Vito Volterra (1860–1940) and an excerpt of the first page of Volterra's paper where the example given in equation (2.1) first appeared.

and $f'(a) = 0$. Since $\lim_{x \to a} \sum_{i=1}^{n} 2\alpha_i (x-a) \sin\left(\frac{h_i}{x-a}\right) = 0$, it is enough to prove that $\lim_{x \to a} \sum_{i=1}^{n} \beta_i \cos\left(\frac{h_i}{x-a}\right)$ does not exist in order to show that f' is not continuous at a. This is equivalent to proving that $\lim_{x \to +\infty} \sum_{i=1}^{n} \beta_i \cos(h_i(x-a))$ does not exist. Due to the fact that $\{(kh_1, \cdots, kh_n) (\text{mod } 1) : k \in \mathbb{N}\}$ is dense in $[0,1]^n$ it follows that for infinitely many $k_1 \in \mathbb{N}$ we have that $\sum_{i=1}^{n} \beta_i \cos(h_i k_1) \geq \sum_{i=1}^{n} \frac{|\beta_i|}{2}$ and for infinitely many $k_2 \in \mathbb{N}$ we have $\sum_{i=1}^{n} \beta_i \cos(h_i k_2) \leq -\sum_{i=1}^{n} \frac{|\beta_i|}{2}$. Thus, the limit $\lim_{x \to +\infty} \sum_{i=1}^{n} \beta_i \cos(h_i(x-a))$ does not exist. Notice that using the same reasoning as above f cannot be identically zero. Hence $\mathcal{F}$ is linearly independent. □

Theorem 2.2 *Let $a \in \mathbb{R}$. Then the family $\mathcal{M}_a$ is strongly $\mathfrak{c}$-algebrable.*

Proof Fix $a \in \mathbb{R}$. Let $H \subset (4,5)$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. For every $h \in H$, we define

$$f_h(x) = \begin{cases} |x-a|^h \left(2 + \sin\left(\frac{1}{x-a}\right)\right) & \text{if } x \neq a, \\ 0 & \text{if } x = a. \end{cases}$$

By the definition of the derivative, it is easy to see that f_h is differentiable at a and $f_h'(a) = 0$ for any $h \in H$. Moreover, f_h' is continuous everywhere for any $h \in H$. We will prove that $\mathcal{F} = \{f_h : h \in H\}$ is a system of generators of an algebra and that $\mathcal{F}$ is a family of algebraically independent functions such that $\mathcal{A}(\mathcal{F})$ (the algebra generated by $\mathcal{F}$) is contained in $\mathcal{M}_a \cup \{0\}$.

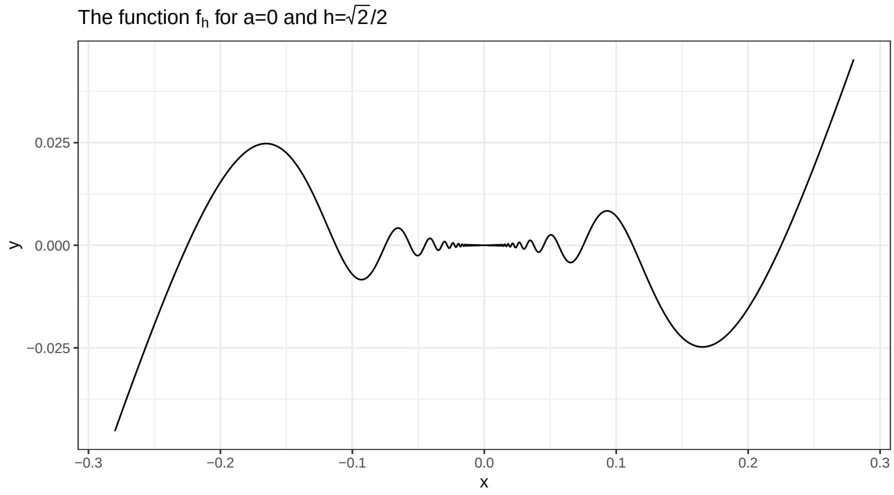

Fig: 2. Approximate representation of the functions f_h defined in the proof of Theorem 2.1 when $a = 0$.

Let $f_{h_1}, \ldots, f_{h_n} \in \mathcal{F}$ (not all necessarily distinct) and let P be a polynomial of positive degree in n variables and without independent term. Then $P(f_{h_1}, \ldots, f_{h_n})$ is of the form

$$f(x) = \begin{cases} \sum_{i=1}^{k} \alpha_i f_{\beta_i, n_i} & \text{if } x \neq a, \\ 0 & \text{if } x = a, \end{cases}$$

where $\alpha_i \in \mathbb{R} \setminus \{0\}$, $4 < \beta_1 < \cdots < \beta_k$, $n_i \in \mathbb{N}$ and, if $x \neq a$, $f_{\beta_i, n_i}(x) = |x-a|^{\beta_i} \left(2 + \sin\left(\frac{1}{x-a}\right)\right)^{n_i}$ for every $i \in \{1, \ldots, k\}$. Notice that f is differentiable everywhere, $f'(a) = 0$ and f' is continuous everywhere. Furthermore, f cannot be identically zero since by construction $\lim_{x \to +\infty} |f(x)| = +\infty$. Thus, the family $\mathcal{F}$ is a system of generators of an algebra and the functions are algebraically independent. Assume now that $x > a$, then

$$\begin{aligned} f'(x) &= \sum_{i=1}^{k} \alpha_i f'_{\beta_i, n_i}(x) \\ &= \sum_{i=1}^{k} \alpha_i \left(2 + \sin\left(\frac{1}{x-a}\right)\right)^{n_i - 1} \\ &\quad \cdot \left[\beta_i (x-a)^{\beta_i - 1} \left(2 + \sin\left(\frac{1}{x-a}\right)\right) - n_i (x-a)^{\beta_i - 2} \cos\left(\frac{1}{x-a}\right)\right] \\ &= (x-a)^{\beta_1 - 2} \sum_{i=1}^{k} \alpha_i \left(2 + \sin\left(\frac{1}{x-a}\right)\right)^{n_i - 1} \\ &\quad \cdot \left[\beta_i (x-a)^{\beta_i - \beta_1 + 1} \left(2 + \sin\left(\frac{1}{x-a}\right)\right) - n_i (x-a)^{\beta_i - \beta_1} \cos\left(\frac{1}{x-a}\right)\right] \end{aligned}$$

In the latter equality, since $\beta_1 < \beta_i$ for every $i \in \{2, \ldots, n\}$, all the terms in the sum go to 0 when $x \to a^+$ except $-n_1 \cos\left(\dfrac{1}{x-a}\right)$. Hence, we have $f'(x) =$

$$-\alpha_1 n_1 (x-a)^{\beta_1-2}\left(2+\sin\left(\frac{1}{x-a}\right)\right)^{n_1-1}\cos\left(\frac{1}{x-a}\right)+o(x-a) \text{ as } x\to a^+.$$

Thus, f' changes sign in every neighborhood of a (this can be seen simply by taking the sequence $x_n = a + \frac{1}{n\pi}$). Furthemore, the function f satisfies that there exists $\delta > 0$ sufficiently small such that

(i) either $f > 0$ in $(a-\delta, a+\delta)$ if $\alpha_1 > 0$, which implies that f has a local minimum at a;
(ii) or $f < 0$ in $(a-\delta, a+\delta)$ if $\alpha_1 < 0$, which implies that f has a local maximum at a.

This proves the desired result. □

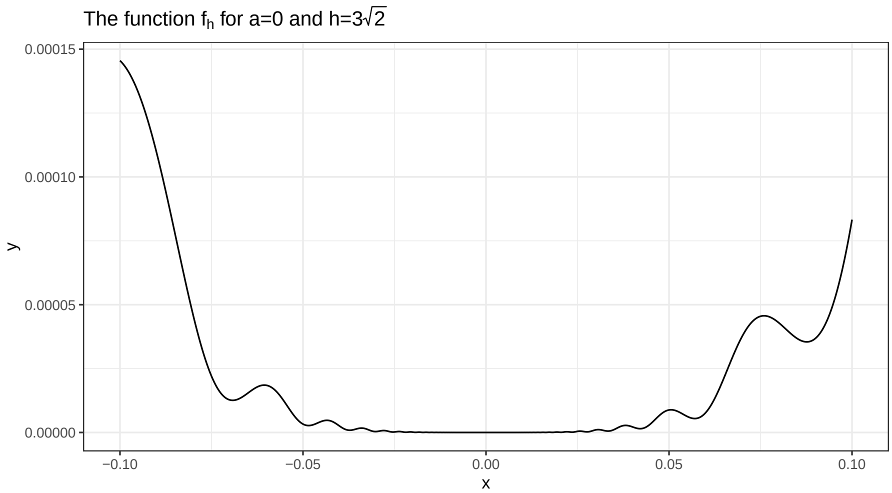

Fig: 3. Approximate representation of the functions f_h defined in the proof of Theorem 2.2 when $a = 0$.

The following two theorems of this section are stronger versions of Theorems 2.1 and 2.2. In order to prove them we recall for completeness what is known as a Smith-Cantor-Volterra set or a fat Cantor set, named after the examples given by H. Smith in 1875 (see [35]) and V. Volterra in 1881. The latter example was used by V. Volterra to construct a non-constant function with a bounded non-Riemann integrable derivative (see, for instance, [31]).

Consider the interval $[0, 1]$ and in the first step remove the middle open interval $\left(\frac{3}{8}, \frac{5}{8}\right)$ that has measure $\frac{1}{4}$. We are left with $\left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right]$. In the next step, remove the middle open intervals of $\left[0, \frac{3}{8}\right]$ and $\left[\frac{5}{8}, 1\right]$ that have measure $\frac{1}{16}$. Thus, for every $n \in \mathbb{N}$, in the n-th step remove the middle open intervals that have length $\frac{1}{2^{2n}}$ in each one of the closed intervals that are left. Notice that we have removed 2^{n-1} open intervals of length $\frac{1}{2^{2n}}$. Hence, the set that is left is known as a fat Cantor set C which satisfies the following properties:

(i) C is a Lebesgue measurable set such that

$$\lambda(C) = 1 - \sum_{n=1}^{\infty} \frac{2^{n-1}}{2^{2n}} = \frac{1}{2},$$

where λ denotes the Lebesgue measure.
(ii) C is perfect and totally disconnected.

For every $0 < \varepsilon < 1$, the above construction can be modified in order to obtain a perfect and totally disconnected set C on the interval $[0, 1]$ such that $\lambda(C) = \varepsilon$.

Definition 2.1 Let $a < b$. We say that $C \subset [a, b]$ is a fat Cantor set in $[a, b]$ if C is perfect, totally disconnected and $0 < \lambda(C) < b - a$.

Notice that every fat Cantor set C in $[0, 1]$ has the following property:

- For every $x \in [0, 1] \setminus C$, there exists an open non-degenerate interval $(a, b) \subset [0, 1] \setminus C$ with endpoints in C such that $x \in (a, b)$.

Theorem 2.3 *The family of differentiable functions on the open interval* $(0, 1)$ *with discontinuous derivative on a set with positive measure is* $\mathfrak{c}$*-lineable.*

Proof Let $H \subset (0, 1)$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ and $C \subset [0, 1]$ be a fat Cantor set of the form constructed above. The following construction is an adaptation from the one given by V. Volterra which will be used in other proofs of this work. For every $h \in H$, we define $f_h : [0, 1] \to \mathbb{R}$ as follows:

For every $I = [a, b] \subset \mathbb{R}$ non-degenerate interval, we define an auxiliar function $f_{h,I} : I \to \mathbb{R}$ by:

(i) $f_{h,I}(a) = f_{h,I}(b) = 0$;
(ii) $f_{h,I}(x) = (x - a)^2 \sin\left(\dfrac{h}{x - a}\right)$ on $(a, x_{I,1})$, where $x_{I,1}$ is the largest number less than or equal to $\frac{a+b}{2}$ for which $(x - a)^2 \sin\left(\frac{h}{x-a}\right)$ has a local maximum value;
(iii) $f_{h,I}(x) = (b - x)^2 \sin\left(\dfrac{h}{b - x}\right)$ on $(x_{I,2}, b)$, where $x_{I,2}$ is the smallest number greater than or equal to $\frac{a+b}{2}$ for which $(b - x)^2 \sin\left(\frac{h}{b-x}\right)$ has a local maximum value;
(iv) $f_{h,I}(x) = (x_{I,1} - a)^2 \sin\left(\dfrac{h}{x_{I,1} - a}\right)$ on $[x_{I,1}, x_{I,2}]$.

Notice that $f_{h,I}(x) = (b - x_{I,2})^2 \sin\left(\dfrac{h}{b - x_{I,2}}\right)$ on $[x_{I,1}, x_{I,2}]$ since $x_{I,1}$ is symmetric to $x_{I,2}$ with respect to $\frac{a+b}{2}$. Using the same arguments as in the proof of Theorem 2.1, note that the functions $f_{h,I}$ are differentiable on (a, b) and have continuous derivative but $\lim_{x\to a^+} f'_{h,I}(x)$ and $\lim_{x\to b^-} f'_{h,I}(x)$ do not exist. Moreover,

$$|f_{h,I}(x)| \leq \begin{cases} (x - a)^2 \\ (b - x)^2 \end{cases} \tag{2.2}$$

for every $x \in I$. Now set

$$f_h(x) = \begin{cases} f_{h,I}(x) & \text{if } x \in \text{int}(I), \\ 0 & \text{if } x \in C, \end{cases}$$

where I is some interval $[a, b]$ such that $(a, b) \subset [0, 1] \setminus C$ with $a, b \in C$ and $\text{int}(I)$ denotes the interior of I. This can be done since $[0, 1] \setminus C$ is the union of intervals (a, b) where $a, b \in C$.

It is enough to show that the family $\mathcal{F} = \{f_h : h \in H\}$ is a family of linearly independent functions such that $\text{span}\{\mathcal{F}\} \setminus \{0\}$ is a family of functions from $[0, 1]$ to $\mathbb{R}$ that are differentiable on $(0, 1)$ with discontinuous derivative on $C \setminus \{0, 1\}$.

Recall that by the proof of Theorem 2.1, it follows that $\mathcal{F} = \{f_h : h \in H\}$ is linearly independent and every linear combination of the form $f = \sum_{i=1}^{k} \alpha_i f_{h_i}$, where $f_{h_1}, \dots, f_{h_k} \in \mathcal{F}$ are distinct, $\alpha_1, \dots, \alpha_k \in \mathbb{R}\setminus\{0\}$ and $k \in \mathbb{N}$, is differentiable on the intervals (a, b). Moreover, in a similar fashion as the arguments employed in Theorem 2.1 we have that $\lim\limits_{x \to a^+} f'(x)$ and $\lim\limits_{x \to b^-} f'(x)$ do not exist. Also, if $x \in C$ and is not of the form a, b considered previously, then there exists a sequence of non-degenerate intervals $\{[a_n, b_n]\}_{n \in \mathbb{N}}$ such that $(a_n, b_n) \cap C = \varnothing$ for every $n \in \mathbb{N}$ and $\{a_n\}_{n \in \mathbb{N}}$ converges to x. For every open set U that contains x, there are infinitely many intervals $[a_n, b_n]$ that are contained in U. As we have seen in the proof of Theorem 2.1, in U there are infinitely many values such that the derivative of f is greater than or equal to $\sum_{i=1}^{k} \frac{|\beta_i|}{2}$, and there are also infinitely many values such that the derivative of f is less than or equal to $-\sum_{i=1}^{k} \frac{|\beta_i|}{2}$. Thus, f' is not continuous at x.

Next, it suffices to show that f is differentiable at every $x \in C \setminus \{0, 1\}$, in fact, we will prove that $f'(x) = 0$ for every $x \in C \setminus \{0, 1\}$.

Fix $x \in C \setminus \{0, 1\}$. Let $\varepsilon > 0$. If $y \in (0, 1) \setminus \{x\}$ is such that $|x - y| < \varepsilon$, then there are two possible cases:

(i) If $y \in C$, then $\frac{f(y) - f(x)}{y - x} = 0$.

(ii) If $y \notin C$, then there exists $(a, b) \subset [0, 1] \setminus C$ with $a, b \in C$ such that $y \in (a, b)$. We can assume, without loss of generality, that a is closer than b to x (this implies that $x < y$). Hence, by (2.2), we have

$$\left| \frac{f(y) - f(x)}{y - x} \right| = \left| \frac{f(y)}{y - x} \right| \leq \left| \frac{f(y)}{y - a} \right| \leq \frac{(y - a)^2}{|y - a|} = |y - a| < |x - y| < \varepsilon.$$

Thus, $f'(x) = 0$ for every $x \in C \setminus \{0, 1\}$. □

Remark 2.1 Let $\mathcal{F} = \{f_h : h \in H\}$ be the family of functions defined in the proof of Theorem 2.3. For every $I = (a, b) \subset [0, 1] \setminus C$ with $a, b \in C$, we have

$$|f_h'(x)| = \left| 2(x - a) \sin\left(\frac{h}{x - a}\right) - h \cos\left(\frac{h}{x - a}\right) \right| \leq 2(x - a) + h \leq 2 + h$$

provided $x \in (a, x_{\overline{I},1})$, where $\overline{I}$ denotes the closure of I, and

$$|f_h'(x)| = \left| 2(b - x) \sin\left(\frac{h}{b - x}\right) - h \cos\left(\frac{h}{b - x}\right) \right| \leq 2(b_n - x) + h \leq 2 + h$$

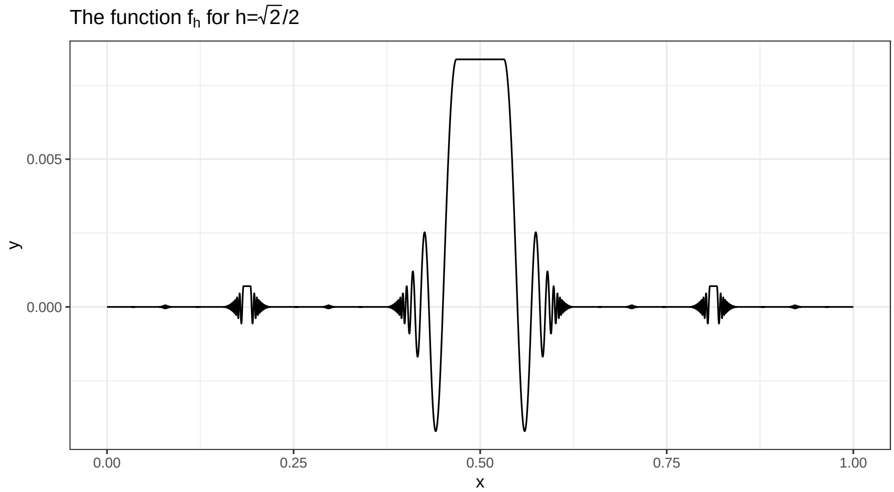

Fig: 4. Sketch of the functions f_h defined in the proof of Theorem 2.3.

provided $x \in (x_{\overline{I},2}, b)$. Since $f'(x) = 0$ for every $x \in C \cup \bigcup_{I \subset [0,1] \setminus C} [x_{\overline{I},1}, x_{\overline{I},2}]$, we have $|f_h'| \leq 2 + h$. Thus, for every linear combination $f = \sum_{i=1}^{k} \alpha_i f_{h_i}$ we have

$$|f'| \leq k \max \{|\alpha_i| : i \in \{1, \dots, k\}\} \max\{2 + h_i : i \in \{1, \dots, k\}\}.$$

Applying the Lebesgue criterion for Riemann integrability we have also proven the following result which already appeared in [23, remark 4.2] but with a different prove.

Corollary 2.1 *The family of differentiable functions on the open interval* $(0, 1)$ *with a bounded, non-Riemann integrable derivative is* $\mathfrak{c}$*-lineable.*

The following result is a weaker version from [19] but it can be obtained by using Theorem 2.2 and Volterra's construction.

Theorem 2.4 *The family of differentiable functions on the open interval* $(0, 1)$ *having an extreme value at every point* x *of a set with positive measure such that the derivative does not make a simple change of sign in a neighborhood of* x *is strongly* $\mathfrak{c}$*-algebrable.*

Proof Let $H \subset (4, 5)$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ and $C \subset [0, 1]$ be a fat Cantor set of the form constructed above. The following construction is similar to the one in Theorem 2.3. For every $h \in H$, let $f_h : [0, 1] \to \mathbb{R}$ be defined as follows:

For every $I = [a, b] \subset \mathbb{R}$ non-degenerate interval, we define first the function $f_{h,I} : I \to \mathbb{R}$ as:

(i) $f_{h,I}(a) = f_{h,I}(b) = 0$;

(ii) $f_{h,I}(x) = (x - a)^h \left(2 + \sin\left(\dfrac{1}{x - a}\right)\right)$ on $(a, x_{I,1})$, where $x_{I,1}$ is the largest number less than or equal to $\frac{a+b}{2}$ for which $(x - a)^h \left(2 + \sin\left(\frac{1}{x-a}\right)\right)$ has a maximum value;

(iii) $f_{h,I}(x) = (b-x)^h \left(2 + \sin\left(\frac{1}{b-x}\right)\right)$ on $(x_{I,2}, b)$, where $x_{I,2}$ is the smallest number greater than or equal to $\frac{a+b}{2}$ for which $(b-x)^h \left(2 + \sin\left(\frac{1}{b-x}\right)\right)$ has a maximum value;

(iv)

$$\begin{aligned} f_{h,I}(x) &= (x_{I,1}-a)^h \left(2 + \sin\left(\frac{1}{x_{I,1}-a}\right)\right) \\ &= (b-x_{I,2})^h \left(2 + \sin\left(\frac{1}{b-x_{I,2}}\right)\right) \end{aligned}$$

on $[x_{I,1}, x_{I,2}]$.

Note that

$$|f_{h,I}(x)| \leq \begin{cases} 3(x-a)^h \\ 3(b-x)^h \end{cases}$$

for any $x \in I$. Consider

$$f_h(x) = \begin{cases} f_{h,I}(x) & \text{if } x \in \operatorname{int}(I), \\ 0 & \text{if } x \in C, \end{cases}$$

where I is some interval $[a, b]$ such that $(a, b) \subset [0, 1] \setminus C$ with $a, b \in C$.

The family $\mathcal{F} = \{f_h : h \in H\}$ is a system of generators algebraically independent such that $\mathcal{A}(\mathcal{F})$ is a family of differentiable functions on $(0, 1)$ having an extreme value at every $x \in C$ such that the derivative does not make a simple change of sign in any neighborhood of x on $[0, 1]$.

Indeed, the functions are a system of generators algebraically independent (simply consider the functions f_h periodically on $\mathbb{R}$ and follow the proof of Theorem 2.2).

Notice that each function f_h is differentiable on the intervals $(a, b) \subset [0, 1] \setminus C$ with $a, b \in C$. Moreover, $f'_h(x) = 0$ for every $x \in C \setminus \{0, 1\}$. The latter can be seen by applying the same techniques as in Theorem 2.3. Also, it is easy to see that f'_h is continuous on $(0, 1)$ for every $h \in H$.

Finally, notice first that given any non-zero algebraic combination f of functions f_h, we have that f is differentiable in $(0, 1)$. It remains to prove that f has an extreme value at every $x \in C$ and the derivative does not make a simple change of sign in any neighborhood of x in $[0, 1]$. Fix $x \in C$ and assume that $x \neq 1$. Let f be a non-zero algebraic combination of functions f_h. If $y > x$ with $y \notin C$, then we have, by following the proof of Theorem 2.2 restricted to the points $(x, y) \setminus C$, that f' does not make a simple change of sign in every neighborhood of x in $[0, 1]$. Moreover, the same arguments as in the proof of Theorem 2.2 show that f has a local maximum or a local minimum at x. The case $x = 1$ can be handled analogously by considering $y < 1$ with $y \notin C$. □

The following results are stronger versions of [26, sections 5 and 6], where it is proven that $\mathcal{UD}_0$ is $\aleph_0$-lineable.

Theorem 2.5 *Let $a \in \mathbb{R}$. Then the family $\mathcal{UD}_a$ is strongly $\mathfrak{c}$-algebrable.*

Proof Fix $a \in \mathbb{R}$. Let $H \subset (0,1)$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. For every $h \in H$, let

$$f_h(x) = \begin{cases} (x-a)^2 e^{h(x-a)} \sin\left(e^{\frac{h}{x-a}}\right) & \text{if } x \neq a, \\ 0 & \text{if } x = a. \end{cases}$$

It is enough to show that the family $\mathcal{F} = \{f_h : h \in H\}$ is a system of generators algebraically independent such that $\mathcal{A}(\mathcal{F})\setminus\{0\} \subset \mathcal{UD}_a$. It is easy to see that f_h is differentiable and $f_h'(a) = 0$ for any $h \in H$, which implies that any algebraic combination is also differentiable and the derivative at a is also 0. Now, let $f_{h_1}, \dots, f_{h_n} \in \mathcal{F}$ and P be a polynomial of positive degree in $n \in \mathbb{N}$ variables and without independent term. Then, $f = P(f_{h_1}, \dots, f_{h_n})$ is of the form $\sum_{i=1}^k \alpha_i \prod f_{h_{s_i}}^{m_{s_i}}$ with $\alpha_1, \dots, \alpha_k \in \mathbb{R}\setminus\{0\}$, $m_{s_i} \in \mathbb{N}$, and $\prod f_{h_{s_i}}^{m_{s_i}}(x) = e^{\beta_i(x-a)}(x-a)^{p_i} \prod \sin^{m_{s_i}}\left(e^{\frac{h_i}{x-a}}\right)$ if $x \neq a$ and 0 otherwise, where $\beta_i = \sum m_{s_i} h_{s_i}$ and $p_i = 2\sum m_{s_i}$. Notice that, since $h_{s_i} \in H$ where H is a Hamel basis and $m_{s_i} \in \mathbb{N}$, we have $\beta_i \neq \beta_j$ provided $i \neq j$. Hence, there exists $i_0 \in \{1, \dots, k\}$ such that $\beta_{i_0} > \beta_i$ for every $i \neq i_0$, which implies that $\lim_{x\to+\infty} |f(x)| = +\infty$. Thus, f cannot be identically zero. Furthemore, notice that we can extend f to a holomorphic function on $\mathbb{C} \setminus \{a\}$. Therefore, f' has at most a countable number of zeros in $\mathbb{R}$.

It suffices to prove that f' is unbounded on the set $\left[a - \frac{1}{n}, a + \frac{1}{n}\right]$ for every $n \in \mathbb{N}$. Note that if $x \neq a$, then $f'(x)$ can be written as $\sum_{i=1}^k e^{\frac{h_i}{x-a}} g_i(x)$, where g_i is given by sums of functions that are multiples of products of powers of $x-a$, exponentials of $x-a$ and functions of the type $\sin\left(e^{\frac{h}{x-a}}\right)$, for any $i \in \{1, \dots, k\}$. Since f' has at most countably many zeros, there exists $J \subset \{1, \dots, k\}$ such that g_j is not the zero function for every $j \in J$. Let $h_{j_0} = \max\{h_j : j \in J\}$. Now, if d is the biggest common power from the powers of $x-a$, we have $g_{j_0}(x) = (x-a)^d g_{j_0}(x)^*$, where $g_{j_0}(x)^*$ is an entire function such that $g_{j_0}(a)^* \neq 0$. By Theorem 2.1, there exists a sequence $\{x_r\}_{r\in\mathbb{N}}$ converging to a such that $g_{j_0}(x_r)^* \nrightarrow 0$ when $r \to \infty$. Since $(x_r - a)^d e^{\frac{h_{j_0}}{x_r - a}}$ diverges when $r \to \infty$, the function $e^{\frac{h_{j_0}}{x-a}} g_{j_0}(x)$ is unbounded on a neighborhood of a. Therefore, by the choice of h_{j_0}, the function f' is unbounded on $\left[a - \frac{1}{n}, a + \frac{1}{n}\right]$ for every $n \in \mathbb{N}$. □

Theorem 2.6 *The family of differentiable functions on the open interval* $(0,1)$ *that have an unbounded derivative on every neighborhood of the points of a set with positive measure is strongly* $\mathfrak{c}$*-algebrable.*

Proof Let $H \subset (0,1)$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ and $C \subset [0,1]$ be a fat Cantor set of the form constructed above. For every $h \in H$, let us define $f_h : [0,1] \to \mathbb{R}$ in the following way:

For every $I = [a,b] \subset \mathbb{R}$ non-degenerate interval, we define first the function $f_{h,I} : I \to \mathbb{R}$ as

(i) $f_{h,I}(a) = f_{h,I}(b) = 0$;
(ii) $f_{h,I}(x) = (x-a)^2 e^{h(x-a)} \sin\left(e^{\frac{h}{x-a}}\right)$ on $(a, x_{I,1})$, where $x_{I,1}$ is the largest number less than or equal to $\frac{a+b}{2}$ for which $(x-a)^2 e^{h(x-a)} \sin\left(e^{\frac{h}{x-a}}\right)$ has a local maximum value;

(iii) $f_{h,I}(x) = (b-x)^2 e^{h(b-x)} \sin\left(e^{\frac{h}{b-x}}\right)$ on $(x_{I,2}, b)$, where $x_{I,2}$ is the smallest number greater than or equal to $\frac{a+b}{2}$ for which $(b-x)^2 e^{h(b-x)} \sin\left(e^{\frac{h}{b-x}}\right)$ has a local maximum value;

(iv)

$$\begin{aligned} f_{h,I}(x) &= (x_{I,1}-a)^2 e^{h(x_{I,1}-a)} \sin\left(e^{\frac{h}{x_{I,1}-a}}\right) \\ &= (b-x_{I,2})^2 e^{h(b-x_{I,2})} \sin\left(e^{\frac{h}{b-x_{I,2}}}\right) \end{aligned}$$

on $[x_{I,1}, x_{I,2}]$.

Consider

$$f_h(x) = \begin{cases} f_{h,I}(x) & \text{if } x \in \operatorname{int}(I), \\ 0 & \text{if } x \in C, \end{cases}$$

where I is some interval $[a, b]$ such that $(a, b) \subset [0, 1] \setminus C$ with $a, b \in C$.

By the proof of Theorem 2.5 and using the same techniques as the ones in the proofs of Theorems 2.3 and 2.4, the family $\mathcal{F} = \{f_h : h \in H\}$ is a system of generators algebraically independent such that $\mathcal{A}(\mathcal{F})$ is a family of differentiable functions that have an unbounded derivative on every neighborhood of every $x \in C \setminus \{0, 1\}$. □

Remark 2.2 In the proof of Theorem 2.6 any non-zero linear combination f of functions f_h has unbonded derivative for every point $x \in C \setminus \{0, 1\}$. As $\lambda(C \setminus \{0, 1\}) = \lambda(C) > 0$, the result is proven. But f is defined on $[0, 1]$, which implies that the derivative at 0 and 1 is not well defined. However, the lateral derivatives are, and in this case we have by construction

$$f'_+(0) = \lim_{h \to 0^+} \frac{f(h)}{h} = \pm\infty,$$

and

$$f'_-(1) = \lim_{h \to 1^-} \frac{f(1+h)}{h} = \pm\infty.$$

Next, to complete this section, we shall tackle the algebrability problem regarding functions on $(0, 1)$ that are not derivatives.

Theorem 2.7 *The family of functions from $(0, 1)$ to $\mathbb{R}$ that are not derivatives is strongly $\mathfrak{c}$-algebrable.*

Proof For every $c > 1$, we define the function f_c by

$$f_c(x) \begin{cases} \dfrac{1}{\left|\frac{1}{2} - x\right|^c} & \text{if } x \in (0, 1) \setminus \{\frac{1}{2}\}, \\ 0 & \text{if } x = \frac{1}{2}. \end{cases}$$

Let $H \subset (1, 2)$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. It is enough to show that the family $\mathcal{F} = \{f_h : h \in H\}$ is a system of generators of an algebra, which are in fact algebraically independent, and $\mathcal{A}(\mathcal{F}) \setminus \{0\}$ is a family of functions that are not derivatives.

Let $n \in \mathbb{N}$, $f_{h_1}, \ldots, f_{h_n} \in \mathcal{F}$ and P be a polynomial of positive degree in n variables and without independent term. Thus, for every $x \in (0,1) \setminus \left\{\frac{1}{2}\right\}$, we have $f(x) = P(f_{h_1}, \ldots, f_{h_n})(x) = \sum_{i=1}^{k} \alpha_i \prod_{j=1}^{k_i} f_{h_j}^{m_j}(x)$, where $\alpha_i \in \mathbb{R} \setminus \{0\}$ and $m_j, k_i \in \mathbb{N}$. Notice that $\prod_{j=1}^{k_i} f_{h_j}^{m_j} = f_{c_i}$, where $c_i = \sum_{j=1}^{k_i} m_j h_j$ for every $i \in \{1, \ldots, n\}$. Hence, $f = \sum_{i=1}^{k} \alpha_i f_{c_i}$, where $c_i > 1$ for every $i \in \{1, \ldots, k\}$ and $c_i \neq c_j$ provided $i \neq j$. Thus, $\lim_{x \to \frac{1}{2}} f(x) = \pm\infty$, where the sign depends on the sign of α_t with t being the subindex of c_i such that $c_t > c_j$ for every $j \neq t$. This proves that the functions in $\mathcal{F}$ are algebraically independent since, if $f \equiv 0$, the above limit would be 0 and we would have a contradiction.

It remains to prove that the function f is not the derivative of a function. As $\lim_{x \to \frac{1}{2}} |f(x)| = +\infty$, there exists a neighborhood $(a,b) \subset (0,1)$ of $\frac{1}{2}$ such that $|f(x)| > 1$ for every $x \in (a,b) \setminus \left\{\frac{1}{2}\right\}$. Now, by construction, $f\left(\frac{1}{2}\right) = 0$. Since for every $s \in (0,1)$, there does not exist $c \in (a,b)$ such that $f(c) = s$, by Darboux's Theorem, we have that f is not a derivative on $(0,1)$. □

3 On subdifferentials

We begin by introducing the definitions and notations of Fréchet, α-Hölder and limiting subdifferentials.

Definition 3.1 Let I be a non-degenerate interval and $f : I \to \mathbb{R}$. We say that f is lower semicontinuous if for every $x_0 \in I$ we have

$$f(x_0) \leq \liminf_{x \to x_0} f(x).$$

Definition 3.2 Let I be a non-degenerate interval, $f : I \to \mathbb{R}$ be a lower semicontinuous function, $x_0 \in I$ and $\alpha > 1$.

The Fréchet subdifferential of f at x_0, denoted by $\partial f(x_0)$, is the set

$$\partial f(x_0) := \left\{\xi \in \mathbb{R} : \liminf_{r \to 0} \frac{f(x_0 + r) - f(x_0) - \xi r}{|r|} \geq 0\right\}$$

The α-Hölder subdifferential of f, denoted by $\partial_{P,\alpha} f(x_0)$, is the set of numbers $\xi \in \mathbb{R}$ that satisfy the proximal α-Hölder inequality, that is, there exist $\delta, \sigma > 0$ such that

$$f(x_0 + r) - f(x_0) - \xi r \geq -\sigma |r|^{\alpha},$$

for every $r \in \mathbb{R}$ with $|r| < \delta$ and $x_0 + r \in I$. The particular case $\partial_{P,2} f(x_0)$ is known as proximal differential of f at x_0 and it is denoted by $\partial_P f(x_0)$.

The limiting subdifferential of f at x_0, denoted by $\partial_L f(x_0)$, is the set

$$\left\{\lim_{n \to +\infty} \xi_n : \xi_n \in \partial f(x_n) \text{ with } \lim_{n \to +\infty} x_n = x_0 \text{ and } \lim_{n \to +\infty} f(x_n) = f(x_0)\right\}.$$

Given $f : I \to \mathbb{R}$ a lower semicontinuous function, $x_0 \in I$ and $\alpha > 1$, it is known that $\partial_{P,\alpha} f(x_0) \subseteq \partial f(x_0)$ (see, for instance, [21]). The following result tackles the lineability of functions whose Fréchet subdifferential is non-empty at a point a but their α-Hölder sudifferential is empty at a.

Theorem 3.1 *Let $\alpha > 1$ and $a \in \mathbb{R}$. The family of differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that* $\operatorname{card}(\partial f(a)) = 1$ *and $\partial_{P,\alpha} f(a) = \varnothing$ is $\mathfrak{c}$-lineable.*

Proof Fix $\alpha > 1$ and $a \in \mathbb{R}$. Let $H \subset (1, \alpha)$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. For every $h \in H$, let f_h be defined as

$$f_h(x) = \begin{cases} |x-a|^h \sin\left(\frac{1}{x-a}\right) & \text{if } x \neq a, \\ 0 & \text{if } x = a. \end{cases}$$

Notice that f is differentiable at a and $f_h'(a) = 0$ for any $h \in H$. Hence, $\partial f_h(a) \supseteq \{0\}$. Furthermore, let $\xi \in \mathbb{R} \setminus \{0\}$. If $\xi > 0$, then $\lim_{r \to 0^+} \frac{f_h(a+r) - \xi r}{|r|} = -\xi < 0$; and if $\xi < 0$, then $\lim_{r \to 0^-} \frac{f_h(a+r) - \xi r}{|r|} = \xi < 0$. Thus, $\partial f_h(a) = \{0\}$ for any $h \in H$. As $\partial_{P,\alpha} f_h(a) \subseteq \partial f_h(a)$, if $\partial_{P,\alpha} f_h(a)$ were non-empty, we would have $\partial_{P,\alpha} f_h(a) = \{0\}$. Thus, there exist $\delta, \sigma > 0$ such that $f_h(a+r) \geq -\sigma |r|^\alpha$, for every $r \in \mathbb{R}$ with $0 < |r| < \delta$. Then $\sin\left(\frac{1}{r}\right) \geq -\sigma |r|^{\alpha - h}$, where $\alpha - h > 0$ and $r \in \mathbb{R}$ with $0 < |r| < \delta$. However, for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ with $\frac{2}{k\pi} < \delta$ such that $\sigma \left(\frac{2}{k\pi}\right)^{\alpha - h} < \varepsilon$ and $\sin\left(\frac{k\pi}{2}\right) = -1$, which is absurd. This implies that $\partial_{P,\alpha} f_h(a) = \varnothing$ for any $h \in H$.

It is enough to show that $\mathcal{F} = \{f_h \in h \in H\}$ is a family of linearly independent functions such that given any non-zero linear combination f of functions f_h we have that f is differentiable, $\operatorname{card}(\partial f(a)) = 1$ and $\partial_{P,\alpha} f(a) = \varnothing$. Let $f = \sum_{i=1}^n \beta_i f_{h_i}$, where $f_{h_1}, \ldots, f_{h_n} \in \mathcal{F}$ are distinct, $\beta_1, \ldots, \beta_n \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$. Assume, without loss of generality, that $h_1 < \cdots < h_n$. If f were identically zero, then $\sum_{i=1}^n \beta_i |x-a|^{h_i} = 0$ for every $x \neq a$. But the latter is not satisfied since $\lim_{x \to +\infty} \left|\sum_{i=1}^n \beta_i |x-a|^{h_i}\right| = +\infty$, and we have a contradiction. It is easy to see that f is differentiable everywhere. By using similar arguments as above, we have $\partial f(a) = \{0\}$. Thus, $\partial_{P,\alpha} f(a) \subseteq \{0\}$. Assume that $\partial_{P,\alpha} f(a) = \{0\}$. Then there exist $\delta, \sigma > 0$ such that $f(a+r) \geq -\sigma |r|^\alpha$, for every $r \in \mathbb{R}$ with $0 < |r| < \delta$. Hence $\sin\left(\frac{1}{r}\right) \sum_{i=1}^n \beta_i |r|^{h_i} \geq -\sigma |r|^\alpha$, for every $r \in \mathbb{R}$ with $0 < |r| < \delta$; which implies that $\sin\left(\frac{1}{r}\right) \left(\beta_1 + \sum_{i=2}^n \beta_i |r|^{h_i - h_1}\right) \geq -\sigma |r|^{\alpha - h_1}$, for every $r \in \mathbb{R}$ with $0 < |r| < \delta$. However, for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ with $\frac{2}{k\pi} < \delta$ such that $\left|\sum_{i=2}^n \beta_i \left(\frac{2}{k\pi}\right)^{h_i - h_1} + \sigma \left(\frac{2}{k\pi}\right)^{\alpha - h_1}\right| < \varepsilon$ and $\sin\left(\frac{k\pi}{2}\right) = -\operatorname{sign}(\beta_1)$, where $\operatorname{sign}(\cdot)$ denotes the sign of β_1. Thus, we have a contradiction which shows that $\partial_{P,\alpha} f(a) = \varnothing$. $\square$

Theorem 3.2 *Let $\alpha > 1$. The family of differentiable functions $f : [0,1] \to \mathbb{R}$ such that* $\operatorname{card}(\partial f(x)) = 1$ *and $\partial_{P,\alpha} f(x) = \varnothing$ at every point x of a set with positive measure is $\mathfrak{c}$-lineable.*

Proof Fix $\alpha > 1$. Let $H \subset (1, \alpha)$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ and $C \subset [0,1]$ be a fat Cantor set of the form constructed in Section 2. For every $h \in H$, let $f_h : [0,1] \to \mathbb{R}$ be as follows:

For every $I = [a,b]$ non-degenerate interval, let $f_{h,I} : I \to \mathbb{R}$ as:

(i) $f_{h,I}(a) = f_{h,I}(b) = 0$;
(ii) $f_{h,I}(x) = (x-a)^h \sin\left(\frac{1}{x-a}\right)$ on $(a, x_{I,1})$, where $x_{I,1}$ is the largest number less than or equal to $\frac{a+b}{2}$ for which $(x-a)^h \sin\left(\frac{1}{x-a}\right)$ has a local maximum value;

(iii) $f_{h,I}(x) = (b-x)^h \sin\left(\frac{1}{b-x}\right)$ on $(x_{I,2}, b)$, where $x_{I,2}$ is the smallest number greater than or equal to $\frac{a+b}{2}$ for which $(b-x)^h \sin\left(\frac{1}{b-x}\right)$ has a local maximum value;

(iv) $f_{h,I}(x) = (x_{I,1}-a)^h \sin\left(\frac{1}{x_{I,1}-a}\right) = (b-x_{I,2})^h \sin\left(\frac{1}{b-x_{I,2}}\right)$ on $[x_{I,1}, x_{I,2}]$.

Finally set

$$f_h(x) = \begin{cases} f_{h,I}(x) & \text{if } x \in \operatorname{int}(I), \\ 0 & \text{if } x \in C, \end{cases}$$

where I is some interval $[a, b]$ such that $(a, b) \subset [0, 1] \setminus C$ with $a, b \in C$.

By the proofs of Theorems 2.4 and 3.1, we have that $\mathcal{F} = \{f_h : h \in H\}$ is family of differentiable functions that are linearly independent. Thus, $\operatorname{span}\{\mathcal{F}\}$ is a family of differentiable functions from $[0, 1]$ to $\mathbb{R}$. Hence, it is enough to prove that given any non-zero linear combination f of functions f_h we have $\operatorname{card}(\partial f(x)) = 1$ and $\partial_{P,\alpha} f(x) = \varnothing$ for any $x \in C \setminus \{0, 1\}$.

Let $f = \sum_{i=1}^n \beta_i f_{h_i}$, where $f_{h_1}, \ldots, f_{h_n} \in \mathcal{F}$ are distinct, $\beta_1, \ldots, \beta_n \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$. Assume, without loss of generality, that $h_1 < \cdots < h_n$. By the proof of Theorem 2.4, we have $f'(x) = 0$ for any $x \in C \setminus \{0, 1\}$. Thus, for every $x \in C$, we have $\partial f(x) \supseteq \{0\}$. Furthermore, since the points a, b are dense in C, we have that $\partial f(x) = \{0\}$ for every $x \in C \setminus \{0, 1\}$. Hence, $\partial_{P,\alpha} f(x) \subseteq \{0\}$. Fix $x \in C \setminus \{0, 1\}$ and assume that $\partial_{P,\alpha} f(x) = \{0\}$. Then, there exist $\delta, \sigma > 0$ such that $f(y) \geq -\sigma|x-y|^\alpha$, for every $y \in [0, 1] \setminus \{x\}$ with $|x - y| < \delta$. If $y \in C$, then $0 \geq -\sigma|x-y|^\alpha$, where $-\sigma|x-y|^\alpha < 0$. So consider only points $y \notin C$ with $y > x$ (the case $y < x$ can be done in a similar way). Then, there exists $I = (a, b) \subset [0, 1] \setminus C$ such that $y \in (a, b)$. Since $y - x < \delta$, we can assume that $y \in (a, x_{\overline{I},1})$. Hence, $\sin\left(\frac{1}{y-a}\right)\left(\beta_1 + \sum_{i=2}^n \beta_i (y-a)^{h_i - h_1}\right) \geq -\sigma \frac{(y-x)^\alpha}{(y-a)^{h_1}}$ for every $y \in (a, x_{\overline{I},1})$ with $y - x < \delta$, and where I is the interval contained in $[0, 1] \setminus C$ with left endpoint a and right in C. Now, as we can take $a > x$ as close to x as we want, for every $\varepsilon > 0$, there exist $a > x$ with $a \in C$ and $k \in \mathbb{N}$ with $\frac{2}{k\pi} + a \in (a, x_{\overline{I},1})$ and $0 < \frac{2}{k\pi} + a - x < \delta$ such that $\left|\sum_{i=2}^n \beta_i \left(\frac{2}{k\pi}\right)^{h_i - h_1} + \frac{\sigma}{2^{h_1}} (k\pi)^{h_1} \left(\frac{2}{k\pi} + a - x\right)^\alpha\right| < \varepsilon$ and $\sin\left(\frac{k\pi}{2}\right) = -\operatorname{sign}(\beta_1)$. We have reached a contradiction, thus $\partial_{P,\alpha} f(x) = \varnothing$ for every $x \in C \setminus \{0, 1\}$. □

The following result deals with functions whose Fréchet subdifferential has cardinality 1 but the limiting subdifferential has cardinality > 1 at a point $a \in \mathbb{R}$.

Theorem 3.3 *Let $a \in \mathbb{R}$. The family of differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{card}(\partial f(a)) = 1$ and $\operatorname{card}(\partial_L f(a)) > 1$ is $\mathfrak{c}$-lineable.*

Proof Fix $a \in \mathbb{R}$. Let $H \subset (0, 1)$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. For every $h \in H$, the functions

$$f_h(x) = \begin{cases} (x-a)^2 \sin\left(\frac{h}{x-a}\right) & \text{if } x \neq a, \\ 0 & \text{if } x = a, \end{cases}$$

defined in the proof Theorem 2.1 form a Hamel basis of differentiable functions such that any non-zero linear combination f of f_h satisfies $\operatorname{card}(\partial f(a)) = 1$ and $\operatorname{card}(\partial_L f(a)) > 1$. □

Theorem 3.4 *The family of differentiable functions $f : [0,1] \to \mathbb{R}$ such that* $\operatorname{card}(\partial f(x)) = 1$ *and* $\operatorname{card}(\partial_L f(a)) > 1$ *at every point x of a set with positive measure is $\mathfrak{c}$-lineable.*

Proof Let $H \subset (0,1)$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. It is sufficient to the consider the family $\mathcal{F} = \{f_h : h \in H\}$ defined in the proof of Theorem 2.3. □

4 On mappings that do not satisfy Rolle's Theorem

For the rest of this section, X will be a real infinite dimensional Banach space, ∂U denotes the boundary of a set $U \subset X$, and $\mathsf{B}(0,1)$ and $\mathsf{S}(0,1)$ will denote, respectively, the unit ball and unit sphere centered at 0. In [34], S.A. Shkarin first proved that if X is superreflexive with Fréchet differentiable norm or non-reflexive with Fréchet differentiable norm, then X does not satisfy Rolle's Theorem. This result has been extended to a large class of real infinite dimensional Banach spaces. For instance, if X has Fréchet differentiable norm (see [4, section 2]), then Rolle's Theorem fails. To be more precise, there exists $f : X \to \mathbb{R}$ continuously Fréchet differentiable (C^1) such that

(i) $f(x) = 0$ for every $x \in X \setminus \mathsf{B}(0,1)$, and
(ii) $f'(x) \neq 0$ for every $x \in \mathsf{B}(0,1)$.

Furthemore, the construction given in [4, section 2] guarantees that we can also assume that $f(x) > 0$ holds for every $x \in \mathsf{B}(0,1)$. For completeness we will show the construction from [4, section 2], let $\varphi : X \to X \setminus \{0\}$ be a C^1 diffeomorphism such that $\varphi(x) = x$ for every $x \in X$ with $\|x\| \leq \frac{1}{2}$. The function $f : \overline{\mathsf{B}}(0,1) \to [0,1]$ defined by

$$f(x) = 1 - \|\varphi(x)\|$$

is well-defined and $f \restriction \mathsf{B}(0,1)$ is C^1 with $f(x) = 0$ for every $x \in \mathsf{S}(0,1)$ and $f'(x) \neq 0$ for every $x \in \mathsf{B}(0,1)$. Moreover, $f(x) > 0$ for every $x \in \mathsf{B}(0,1)$. Indeed, assume that there exists $\tilde{x} \in \mathsf{B}(0,1)$ such that $f(\tilde{x}) = 0$, then f has a local minimum at $\tilde{x}$, which implies that $f'(\tilde{x}) = 0$ and we have a contradiction. To finish the proof, consider the zero extension of f to X.

The real infinite dimensional Banach spaces with Fréchet differentiable norm are not the only Banach spaces that do not satisfy Rolle's Theorem. In [29] there are Banach spaces that do not satisfy Rolle's Theorem whose norm is not Fréchet differentiable. However, in [5, theorem 4.1], the authors were able to characterize the real infinite dimensional Banach spaces that do not satisfy Rolle's Theorem:

Theorem 4.1 ([5]) *Let X be a real infinite dimensional Banach space. The following statements are equivalent:*

(i) X has a C^1 bump function (a C^1 function with bounded, non-empty support).
(ii) There exist a bounded connected open subset U of X and a continuous bounded function $f : \overline{U} \to \mathbb{R}$ such that f is $C^1(U)$, $f \equiv 0$ on ∂U and $f'(x) = 0$ for every $x \in U$.
(iii) There exist a C^1 bounded function $f : X \to \mathbb{R}$ and an open connected bounded subset U in X such that $f \equiv 0$ on $X \setminus U$ and $f'(x) \neq 0$ for every $x \in U$.

Moreover, the proof of the previous result, [5, theorem 4.1], and a similar argument as above guarantees that the function f in Theorem 4.1 (iii) can also be assumed to satisfy $f(x) > 0$ for every $x \in U$. In this section, we will show that $\mathcal{NR}(X)$, the family of functions that do not satisfy Rolle's Theorem in X is strongly $\mathfrak{c}$-coneable. In particular, we will give an explicit construction in the case of ℓ_2.

Theorem 4.2 *If X is a real infinite dimensional Banach space with a C^1 bump function, then $\mathcal{NR}(X)$ is strongly $\mathfrak{c}$-coneable*[1].

Proof Let X be a real infinite dimensional Banach space that has a C^1 bump function and $H \subset (0,1)$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. By Theorem 4.1, there exist a C^1 bounded function $f : X \to \mathbb{R}$ and an open connected bounded subset U in X such that $f \equiv 0$ on $X \setminus U$ and $f'(x) \neq 0$ for every $x \in U$. Furthermore, assume that $f(x) > 0$ for every $x \in U$. For every $h \in H$, we define $f_h : X \to \mathbb{R}$ as:

$$f_h(x) = f(x)e^{hf(x)}.$$

We will show that the family $\mathcal{F} = \{f_h : h \in H\}$ is as needed.

Let $f_{h_1}, \ldots, f_{h_n} \in \mathcal{F}$ be distinct and P be a polynomial of positive degree in n variables without independent term. The function $g(x) = P(f_{h_1}, \ldots, f_{h_n})(x)$, which is of the form $\sum_{i=1}^{k} \alpha_i f(x)^{n_i} e^{\beta_i f(x)}$, where $\alpha_i \in \mathbb{R} \setminus \{0\}$, $n_i \in \mathbb{N}$ and $\beta_i = \sum k_j f_{h_j}$ with $k_j \in \mathbb{N}$, for every $i \in \{1, \ldots, n\}$, is not identically zero. Indeed, let us define the function $g^* : \mathbb{C} \to \mathbb{C}$ by $g^*(z) = \sum_{i=1}^{k} \alpha_i z^{n_i} e^{\beta_i z}$. Notice that g^* is holomorphic. Furthemore, g^* is not identically zero since the exponents β_i are different. Now, as f is continuous and non-constant, we have that $f[U]$ is an interval. Thus, g is not identically zero since otherwise, by the Identity Theorem, g^* would be identically zero.

Now, let $f_{h_1}, \ldots, f_{h_n} \in \mathcal{F}$ and P be a polynomial of positive degree in n variables without independent term and with positive coefficients. The function $P(f_{h_1}, \ldots, f_{h_n})$ is of the form $g(x) = \sum_{i=1}^{k} \alpha_i f(x)^{n_i} e^{\beta_i f(x)}$, where $\alpha_i > 0$, $n_i \in \mathbb{N}$ and $\beta_i = \sum k_j f_{h_j}$ with $k_j \in \mathbb{N}$, for every $i \in \{1, \ldots, n\}$. Notice that g is C^1 and $g(x) > 0$ for any $x \in U$. As $f(x) = 0$ for every $x \in X \setminus U$, we have $g(x) = 0$ for every $x \in X \setminus U$. It suffices to show that $g'(x) \neq 0$ for every $x \in U$. For each $x \in U$, we have

$$\begin{aligned} g'(x) &= f'(x) \sum_{i=1}^{k} \alpha_i \left(n_i f(x)^{n_i - 1} e^{\beta_i f(x)} + \beta_i f(x)^{n_i} e^{\beta_i f(x)} \right) \\ &= f'(x) \sum_{i=1}^{k} \alpha_i f(x)^{n_i - 1} e^{\beta_i f(x)} \left(n_i + \beta_i f(x) \right). \end{aligned}$$

Since $f'(x) \neq 0$ and $\sum_{i=1}^{k} \alpha_i f(x)^{n_i - 1} e^{\beta_i f(x)} \left(n_i + \beta_i f(x) \right) > 0$, we have the desired result. ☐

[1] Recall that a cone is a set endowed with two operations, addition and multiplication by positive scalars, which fulfill the usual properties (commutativity, associativity, existence of neutral element, etc.). If this set contains a linearly independent subset of infinite cardinality, the cone is said to be infinite. The dimension of the cone is the maximal posible cardinality of such a linearly independent set. In this paper, we shall say that a set M is strongly $\mathfrak{c}$-coneable whenever it contains (except for 0) a cone P of dimension $\mathfrak{c}$ and this cone contains $\mathfrak{c}$ algebraically independent elements as well (and, also, every non-trivial algebraic combination of elements of P, of positive coefficients, must belong to M).

Let us next see an example that will help to illustrate what we proved earlier.

Remark 4.1 Consider the separable Hilbert space ℓ_2 of all square-summable real sequences with the Euclidean norm $\|\cdot\|_2$. In [22], the author constructed a C^1 function from ℓ_2 to $\mathbb{R}$ such that $f(x) = 0$ for every $x \in \mathsf{S}(0,1)$, but $f'(x) \neq 0$ for every $x \in \mathsf{B}(0,1)$. More precisely, the construction obtained in [22] is the following:

Let $R : \ell_2 \to \ell_2$ be the right shift operator, i.e.,

$$Rx = (0, x_1, x_2, \ldots),$$

for every $x \in \ell_2$. Now, let $T : \ell_2 \to \ell_2$ be the mapping defined for every $x \in \ell_2$ by

$$Tx = \left(\frac{1}{2} - \|x\|_2^2\right) e_1 + Rx,$$

where $e_1 = (1, 0, \ldots)$ is the vector with 1 in the first coordinate and the rest are 0. Finally, let us define f by

$$f(x) = \frac{1 - \|x\|_2^2}{\|x - Tx\|_2^2}.$$

The mapping f is C^1 and does not satisfy Rolle's Theorem since $f(x) = 0$ for every $x \in \mathsf{S}(0,1)$ and $f'(x) \neq 0$ for every $x \in \mathsf{B}(0,1)$. Furthermore, $f(x) > 0$ for every $x \in \mathsf{B}(0,1)$. Hence, letting $H \subset (0,1)$ be a Hamel basis, for every $h \in H$, we define

$$f_h(x) = \frac{1 - \|x\|_2^2}{\|x - Tx\|_2^2} \exp\left\{h \frac{1 - \|x\|_2^2}{\|x - Tx\|_2^2}\right\}.$$

Thus, by the proof of Theorem 4.2, the family $\{f_h : h \in H\}$ is algebraically independent and forms a cone contained in $\mathcal{NR}(\ell_2)$.

Remark 4.2 As a final remark, let us show a result that does not revolve around differentiability but rather continuity. Let us prove that the family of functions with finitely many points of continuity is strongly $\mathfrak{c}$-algebrable. Indeed, let $H \subset (0, \infty)$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. For every $h \in H$ and $m \in \mathbb{N}$, let us define $f_{h,m}$ by

$$f_{h,m}(x) = \begin{cases} |x|^m e^{h|x|} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

The family $\mathcal{F} = \{f_{h,m} : h \in H,\ m \in \mathbb{N}\}$ is a system of generators of an algebra that are algebraically independent such that $\mathcal{F} \setminus \{0\}$ are functions that have finitely many points of continuity.

To see this, let $f_{h_1}, \ldots, f_{h_r} \in \mathcal{F}$, where $r \in \mathbb{N}$, and P be a polynomial in r variables without independent term. Then, $f = P(f_{h_1}, \ldots, f_{h_r})$ can be written as $\sum_{i=1}^k \alpha_i |x|^{n_i} e^{\beta_i |x|}$, where $\alpha_i \in \mathbb{R} \setminus \{0\}$, $n_i \in \mathbb{N}$ and β_i is the sum of some h_j for every $i \in \{1, \ldots, k\}$. Notice that $\lim_{x \to +\infty,\ x \in \mathbb{R} \setminus \mathbb{Q}} f(x) = \pm\infty$, where the sign depends on the coefficient α_j such that β_j is the largest of the β_i. Hence, f cannot be identically zero.

It remains to prove that f has finitely many points of continuity. Let $g : \mathbb{C} \to \mathbb{C}$ be the holomorphic function $g(z) = \sum_{i=1}^k \alpha_i z^{n_i} e^{\beta_i z}$. Notice that $g(x) = f(x)$ for any $x \in \mathbb{R}^+ \setminus \mathbb{Q}$. Hence the set of zeros of g does not have an accumulation point since otherwise g would be identically 0, i.e., f would be the zero function, which would be absurd. Considering that $\lim_{x\to+\infty,\ x\in\mathbb{R}\setminus\mathbb{Q}} f(x) = +\infty$ (the same can be applied to $-\infty$) it follows that g can only have finitely many zeros in $\mathbb{R}^+$. Since for every point of continuity x_0 of f we obviously have, by a density argument, that $f(x_0) = g(x_0) = 0$, it follows that the set of all points of continuity of f in $\mathbb{R}^+$ is finite. Using the symmetry of f yields the desired result.

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