

The Ising model and planar $\mathcal{N} = 4$ Yang-Mills

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Abstract

The scattering-matrix for planar Yang-Mills with $\mathcal{N} = 4$ supersymmetry relies on the assumption that integrability holds to all orders in perturbation theory. In this note we define a map from the spectral variables x^\pm , parameterizing the long-range magnon momenta, to couplings in a two-dimensional Ising model. Under this map integrability of planar $\mathcal{N} = 4$ Yang-Mills becomes equivalent to the Yang-Baxter equation for the two-dimensional Ising model, and the long-range variables x^\pm translate into the entries of the Ising transfer matrices. We explore the Ising correlation length which equals the inverse magnon momentum in the small momentum limit. The critical regime is thus reached for vanishing magnon momentum. We also discuss the meaning of the Kramers-Wannier duality transformation on the gauge theory, together with that of the Ising model critical points.

1 Introduction

During the last years our understanding of the AdS/CFT correspondence has benefited greatly from its apparent integrability. The identification of the one-loop planar dilatation operator for Yang-Mills with $\mathcal{N} = 4$ supersymmetry with the hamiltonian of an integrable spin chain [1, 2], enabled the use of Bethe ansatz techniques to compute anomalous dimensions for large composite gauge-invariant operators. In the $su(2)$ sector the integrable system reduces to the $XX_{1/2}$ Heisenberg spin chain, and the dilatation operator can thus be diagonalized by the Bethe ansatz, giving the allowed set of magnon momenta $\{p_j\}$ as solutions to

$$e^{ip_j L} = \prod_{i \neq j}^M S(p_i, p_j) , \quad (1.1)$$

where the scattering-matrix takes the form

$$S(p_i, p_j) = \frac{u(p_i) - u(p_j) + i}{u(p_i) - u(p_j) - i} , \quad (1.2)$$

with u , the spin chain rapidity, given by

$$u(p) = \frac{1}{2} \cot \frac{p}{2} . \quad (1.3)$$

Assuming integrability holds to all orders in perturbation theory, a long-range Bethe ansatz for asymptotically long spin chains was later on conjectured in [3]. The S-matrix in the long-range Bethe ansatz takes the same form as in (1.2), but the rapidity (1.3) is replaced by

$$u(p) = \frac{1}{2} \cot \frac{p}{2} \sqrt{1 + g^2 \sin^2 \frac{p}{2}} , \quad (1.4)$$

where $g = \frac{\sqrt{\lambda}}{\pi}$, and where $\lambda \equiv g_{YM}^2 N$ is 't Hooft's coupling constant.¹ This conjecture was subsequently extended to other sectors [4, 5]. In doing so, introducing a set of spectral variables x^+ and x^- proved convenient. They are defined through the relations

$$e^{ip} = \frac{x^+}{x^-} , \quad (1.5)$$

and

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{4i}{g} . \quad (1.6)$$

After a rescaling $u \rightarrow \frac{4u}{g}$, the long-range spin chain rapidity (1.4) takes a quite simple form in terms of x^\pm ,

$$u = \frac{1}{2} \left(x^+ + \frac{1}{x^+} + x^- + \frac{1}{x^-} \right) . \quad (1.7)$$

¹Note that g is rescaled, as compared to earlier conventions, and coincides with the γ of [6].

In [7] and [8], the long-range S-matrix for planar $\mathcal{N} = 4$ Yang-Mills was then constructed algebraically² by demanding invariance of the S-matrix under a centrally-extended $su(2|2)$ algebra. This algebraic construction is performed as follows. First of all one should identify magnons with $(2|2)$ irreducible representations of the centrally extended algebra. These irreps are parameterized by the eigenvalues of the central elements. Secondly, the action of the algebra must be lifted to two-magnon states. This introduces a co-multiplication rule that by consistency should be an algebra homomorphism. For a classical algebra this co-multiplication, or composition rule, takes the standard form $\Delta\mathfrak{J} = \mathfrak{J} \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{J}$, with \mathfrak{J} any algebra generator. Finally the two-magnon S-matrix is determined by imposing

$$S \Delta_{12}(\mathfrak{J}) = \Delta_{21}(\mathfrak{J}) S , \quad (1.8)$$

where $\Delta_{12}(\mathfrak{J})$ means the action of \mathfrak{J} on two incoming magnons, labelled 1 and 2. In order to have a non-trivial S-matrix, different from just a permutation, we need an asymmetric co-multiplication rule. This is indeed the typical situation in quantum deformed algebras. The crucial step in Beisert's algebraic construction of the long-range S-matrix was to define a non-symmetric co-multiplication for the generators of the centrally-extended $su(2|2)$ algebra by introducing a new generator, the magnon momentum [7]. The asymmetric co-multiplication for the central elements is given by [16] (see also [17]-[19])

$$\Delta\mathfrak{P} = \mathfrak{P} \otimes e^{i\hat{p}} + \mathbb{1} \otimes \mathfrak{P} , \quad (1.9)$$

$$\Delta\mathfrak{K} = \mathfrak{K} \otimes e^{-i\hat{p}} + \mathbb{1} \otimes \mathfrak{K} , \quad (1.10)$$

while the co-products for the rest of the generators of the algebra are taken to be compatible with those of the central charges [19, 20]. In principle these co-multiplication rules will define a Hopf algebra structure, with generators those in the centrally extended $su(2|2)$ algebra, together with the magnon momentum operator. Taking now into account that central elements commute with the S-matrix, condition (1.8) leads to the constraint $\Delta_{12}(\mathfrak{L}) = \Delta_{21}(\mathfrak{L})$, with \mathfrak{L} any central element of the algebra. Using the asymmetric co-multiplications defined above, these relations allow us to relate the labels of the magnon

²Algebraic considerations fix the S-matrix up to a global dressing phase factor. The dressing phase is constrained by the integrable structure of semiclassical strings [9], or by the first quantum correction [10] (see also [11]). A solution to the algebraic condition that crossing symmetry imposes on the dressing factor [12] allowed an all-order strong-coupling expansion [13], that lead to agreement [14] with a perturbative computation in the weak-coupling regime. To date there is however no general symmetry prescription to fix or determine unambiguously [15] the structure of the dressing phase factor.

irreps, *i.e.* the eigenvalues of the central elements, to the magnon momentum. In this setup, once we introduce the x^\pm variables through $\frac{x^+}{x^-} = e^{ip}$, we get Beisert's parametrization of the irrep in terms of the generalized rapidities [7]. It is important to keep in mind that hidden in the parametrization of the magnon irreps in terms of the x^\pm variables there is a non-symmetric co-multiplication. This co-multiplication, together with the introduction of the extra momentum generator, are two ingredients that by no means are contained in the classical centrally extended $su(2|2)$ algebra, encoding the *classical* symmetries of the problem.

The form of the co-multiplication already provides some hints on the underlying physics. The quantity that is playing the role of a measure for the deformation of the algebra is the magnon momentum. In fact, for zero magnon momentum the co-multiplication becomes classical, and we should expect the S-matrix to be simply a permutation. If the S-matrix is expanded in the incoming magnon momenta p_1 and p_2 , it takes the form

$$\mathcal{S} = \mathbb{1} + S_{1,1}p_1 + S_{1,2}p_2 + \dots \quad (1.11)$$

The point is therefore that the x^+ and x^- variables describe the departure of the S-matrix from triviality, while the classical algebra determines the precise form of the entries of the S-matrix in terms of the x^\pm . But there is yet another motivation to clarify the precise meaning of the x^\pm spectral variables. The long-range Bethe ansatz is asymptotic, and its validity is in fact limited by wrapping effects (see for instance [21]). If one wishes to extend the integrable spin chain to non-asymptotically long chains it is crucial to clarify the meaning of the x^\pm variables.

The purpose of this note is to show that there is a way to map the x^\pm variables into Ising model couplings K and L . In this way a natural interpretation will arise for the long-range spin chain rapidity u in terms of Ising model quantities. Under this correspondance, the Yang-Baxter equations for the Ising model are completely equivalent to the closure relation (1.6), which we will prove to be equivalent (not just implying) to the $su(2|2)$ spin chain Yang-Baxter equations. Furthermore, we will show that the Ising model correlation length seems to be related to the deformation parameter of the Hopf algebra of the theory. There is also a possibility, as we will motivate, that the Kramers-Wannier duality of the model could play a role in the full, supposedly integrable, planar $\mathcal{N} = 4$ Yang-Mills theory.

2 The map to the Ising model

In this section we will exhibit how the dynamics of planar $\mathcal{N} = 4$ Yang-Mills can be mapped to the two-dimensional Ising model. The Ising model on a square lattice is defined in terms of horizontal and vertical couplings J and J' , the temperature T and Boltzmann's constant k_B . Following Baxter [22], we will define new couplings, K and L , by $K = J/k_B T$ and $L = J'/k_B T$. The Ising model partition function is then given by

$$Z = \sum \prod_{(i,j)_H} \prod_{(k,l)_V} e^{K\sigma_i \cdot \sigma_j + L\sigma_k \cdot \sigma_l}, \quad (2.1)$$

where $\sigma_i = \pm 1$ is the spin sitting at site i , the sum is taken over all spin configurations, and $\{(i, j)_H\}$ stands for the set of sites, adjacent in the horizontal direction, while $\{(i, j)_V\}$ is defined analogously for the vertical direction.

In this note we will propose a map from the x^\pm variables, describing a magnon in the long-range spin chain of [8], to Ising model couplings through the relation ³

$$x^\pm = e^{-2L} e^{\pm 2K}. \quad (2.2)$$

The main theme of this work will be the study of this map and see what light it sheds on the long-range spin chain for $\mathcal{N} = 4$ Yang-Mills. To begin with, it is immediate to relate the Ising couplings to more familiar quantities appearing in the spin chain. Since $e^{ip} \equiv \frac{x^+}{x^-}$, the coupling K is simply $\frac{ip}{4}$. Furthermore, from [8], the eigenvalue C of the central charge \mathfrak{C} , usually interpreted as the magnon energy, is given by $C = \frac{1}{2} \frac{1+1/x^+ x^-}{1-1/x^+ x^-}$, which, using the map (2.2), just becomes $-\frac{1}{2} \coth 2L$. In conclusion,

$$e^{ip} = e^{4K}, \quad C = -\frac{1}{2} \coth 2L. \quad (2.3)$$

It should also be noted that the two possible solutions of (1.6), in the limit $g \rightarrow \infty$, normally given as $x^+ = x^-$ or $x^+ = 1/x^-$, now correspond to letting $K \rightarrow 0$ or $L \rightarrow 0$, respectively. However, a more important consequence of (2.2) is that spectral variables x^\pm can be given a direct interpretation. To do so, we will study the Ising model transfer matrices.

2.1 Ising transfer matrices

A standard way of calculating the partition function (2.1) is by introducing transfer matrices V and W . Rotating the lattice by 45° , these can be described graphically as in figures 1

³This parameterization of x^\pm is similar to the one employed in [23] in terms of p and β .

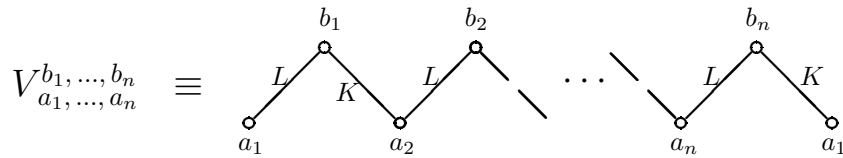


Figure 1: Graphical representation of the transfer matrix V .

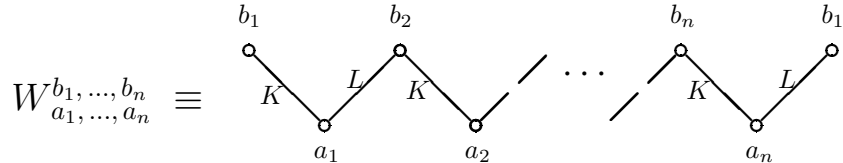


Figure 2: Graphical representation of the transfer matrix W .

and 2. From the spin configurations a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n , one obtains the matrix elements of the transfer matrices by multiplying the Boltzmann weights corresponding to the lines connecting each of the sites. The relevant Boltzmann weights are $e^{\pm L}$ for lines marked L , and $e^{\pm K}$ for lines marked K , with the plus sign for lines connecting spins of the same type, and the minus sign when the adjacent spins are of opposite type. When the total number of (diagonal) rows m is pair, the partition function is given by

$$Z = \text{Tr}[(VW)^{m/2}] . \quad (2.4)$$

Let us now consider a small, square lattice with only two rows of two sites each, and

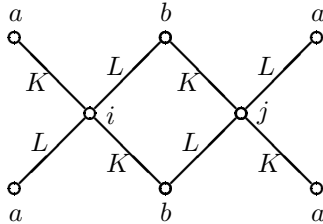


Figure 3: The lattice used in the interpretation of x^\pm .

periodic boundary conditions (see figure 3). We can then define V and W transfer matrices, as in the general case. The corresponding graphical representation is shown in figures 4 and 5.

As an example, consider V in the case where $a = j = +$, $b = i = -$. Then, the line connecting a and i gives a factor of e^{-L} , the line connecting i and b gives e^K , the one

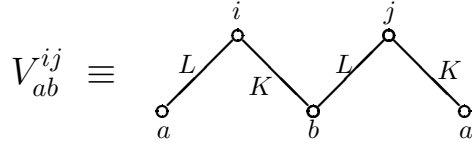


Figure 4: The V transfer matrix for rows of only two sites.

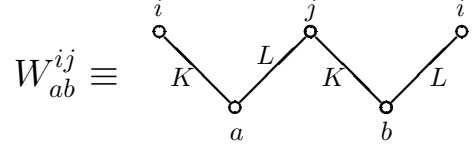


Figure 5: The W transfer matrix for rows of only two sites.

between b and j gives e^{-L} and the line between j and a gives e^K . Multiplying these four factors, we see that $V_{+-}^{-+} = e^{-2L}e^{2K} \equiv x^+$. In this way, all the elements of V and W are determined. The result is rather surprising: written in the basis $(++, +-, -+, --)$, we find

$$V = \begin{pmatrix} 1/x^- & 1 & 1 & x^- \\ 1 & 1/x^+ & x^+ & 1 \\ 1 & x^+ & 1/x^+ & 1 \\ x^- & 1 & 1 & 1/x^- \end{pmatrix}, \quad (2.5)$$

and

$$W = \begin{pmatrix} 1/x^- & 1 & 1 & x^- \\ 1 & x^+ & 1/x^+ & 1 \\ 1 & 1/x^+ & x^+ & 1 \\ x^- & 1 & 1 & 1/x^- \end{pmatrix}. \quad (2.6)$$

We thus see that the generalized rapidities x^\pm are simply the matrix elements of these transfer matrices! The reader might object that the case where the number n of sites per row is 2 is highly restrictive. However, as long as n is even the matrix elements of the corresponding transfer matrices can always be written as

$$(x^+)^a(x^-)^b, \quad \text{for integer } a \text{ and } b. \quad (2.7)$$

This is easily seen as follows:

1. The matrix elements $V_{+,+, \dots, +}^{+,+, \dots, +}$ and $W_{+,+, \dots, +}^{+,+, \dots, +}$ can obviously be written in this way, in the form $1/(x^-)^{n/2}$. All matrix elements can then be obtained from these two by flipping some of the spins on the upper and lower rows.

2. If a matrix element is of the form (2.7), then any element obtained by flipping a spin σ will also be. The site at which σ sits is connected to a spin ρ via an L -line, and to a spin τ via a K -line. If all three spins are equal, flipping σ will multiply the matrix element by $e^{-2L}e^{-2K} = x^-$. If $\sigma \neq \rho = \tau$, the flip multiplies the element by $e^{2L}e^{2K} = 1/x^-$. When $\sigma = \rho \neq \tau$, the multiple is $e^{-2L}e^{2K} = x^+$. And if $\sigma = \tau \neq \rho$, one obtains $e^{2L}e^{-2K} = 1/x^+$.

Thus in the general case the x^\pm are still natural variables for parameterizing the transfer matrices V and W .⁴

Crossing symmetry

Let us now address the issue of crossing symmetry at the level of the transfer matrices. There is evidence that the S-matrix in the AdS/CFT correspondence exhibits a crossing symmetry [12], under which the x^\pm -variables transform as

$$(x^\pm)^{\text{cr}} = \frac{1}{x^\pm}, \quad (2.8)$$

where the superscript ^{cr} denotes the crossing transformation. At this point, the reader might wonder why we have chosen to study the map (2.2), relating the x^\pm to the Ising model couplings K and L , when the map obtained after performing a crossing transformation,

$$x^\pm = e^{2L}e^{\mp 2K}, \quad (2.9)$$

should be on equal footing. The fact is that it really does not matter which map we chose, because the Ising model is invariant under this transformation. The easiest way to see this is by noting that crossing is equivalent to letting

$$K, L \mapsto -K, -L, \quad (2.10)$$

which leaves the partition function (2.1) invariant. However, if we study instead this invariance at the level of the transfer matrices V and W , the result turns out to be rather amusing. From the graphical representation of V and W in figures 1 and 2, we see that if we flip a spin b_i the contribution to the matrices from the attached lines changes from $e^{\pm K}$ and $e^{\pm L}$ to $e^{\mp K}$ and $e^{\mp L}$. Changing thus the sign of K and L globally is equivalent to

⁴The spectral variables x^\pm have appeared before in the algebraic Bethe ansatz solution of the Hubbard model [24] (see also [25] for a more direct relation with the Hubbard model in $\mathcal{N} = 4$ Yang-Mills [26]).

flipping all the spins on either the lower or the upper rows of V and W . Therefore, denoting the opposite of the spin b_i by \bar{b}_i and using collective indices such as $a \equiv (a_1, \dots, a_n)$, the transfer matrices transform as

$$(V^{\text{cr}})_a^b = V_a^{\bar{b}} = V_a^b, \quad (W^{\text{cr}})_a^b = W_a^{\bar{b}} = W_a^b. \quad (2.11)$$

Then

$$((VW)^{\text{cr}})_a^c = \sum_b (V^{\text{cr}})_b^c (W^{\text{cr}})_a^b = \sum_b V_b^c W_a^{\bar{b}} = \sum_{\bar{b}} V_b^c W_a^{\bar{b}} = (VW)_a^c. \quad (2.12)$$

From (2.4) it immediately follows that the partition function is invariant under crossing symmetry.

2.2 Yang-Baxter equation

We will now take the map from the long-range $\mathcal{N} = 4$ spin chain to the Ising model one step further. Let us associate a magnon, given by x_1^\pm , with the matrix V , and a second magnon, x_2^\pm , with W . Integrability of the Ising model, *i.e.* the existence of an infinite number of conserved charges, is encoded in the condition

$$V(x_1^\pm)W(x_2^\pm) = V(x_2^\pm)W(x_1^\pm). \quad (2.13)$$

This is true in general, but the connection with the spin chain is clearer for the two-site transfer matrices given in (2.5) and (2.6). Then,

$$[V(x_1^\pm)W(x_2^\pm)]_{ab}^{cd} = X(abcd; x_1^\pm, x_2^\pm) \cdot X(badc; x_1^\pm, x_2^\pm), \quad (2.14)$$

with

$$X(abcd; x_1^\pm, x_2^\pm) \equiv \sum_i \begin{array}{ccc} & c & d \\ & \circ & \circ \\ & K_2 & L_2 \\ & \diagdown & / \\ & \circ & i \\ & / & \diagdown \\ & L_1 & K_1 \\ \circ & & \circ \\ a & & b \end{array} \quad (2.15)$$

The commutation condition (2.13) is then equivalent to the existence of a new coupling M such that

$$X(abcd; x_2^\pm, x_1^\pm) e^{M(bd)} = e^{M(ac)} X(abcd; x_2^\pm, x_1^\pm), \quad (2.16)$$

where $M(ab) = \pm M$, depending on whether the spins a and b have the same or opposite orientation. Equation (2.16) is the standard Yang-Baxter equation for the Ising model [22].

On the other hand, once we write out the matrix products in (2.13) using the transfer matrices (2.5) and (2.6), it is immediate to see that condition (2.13) is satisfied iff

$$x_1^+ + \frac{1}{x_1^+} - x_1^- - \frac{1}{x_1^-} = x_2^+ + \frac{1}{x_2^+} - x_2^- - \frac{1}{x_2^-} . \quad (2.17)$$

Denoting the common value on the LHS and the RHS of (2.17) by $\frac{4i}{g}$ (and allowing g to be an arbitrary complex variable), we see that the Yang-Baxter equations for the Ising model are equivalent to (1.6). In appendix A we will show that the Yang-Baxter equations of the $su(2|2)$ spin chain S-matrix are equivalent to a set of conditions of the form (2.17).⁵

For the Ising model, condition (2.17) is naturally expressed in terms of K and L . Using the map (2.2),

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = -4 \sinh 2K \sinh 2L . \quad (2.18)$$

Condition (2.17) thus implies that the expression $\sinh 2K_i \sinh 2L_i$ is the same for all i . This defines the elliptic modulus k of the Ising model,

$$\sinh 2K \sinh 2L = \frac{1}{k} . \quad (2.19)$$

Combining (1.6) and (2.19) we get

$$k = i g , \quad (2.20)$$

relating k and the 't Hooft coupling constant. Notice that this elliptic modulus is the same as the one used in [8] to parameterize (1.6) in terms of a rapidity z .

Let us now consider the partition function of the two-row Ising model in terms of x^\pm . From (2.4), taking $m = 2$, and using the representation given by (2.5) and (2.6), one finds that

$$Z = \text{Tr}(VW) = 2 \left[4 + \left(x^- + \frac{1}{x^-} \right)^2 \right] . \quad (2.21)$$

This also has a simple expression in terms of the long-range spin chain variable u . Using (1.7) and (1.6), u can be rewritten as

$$u = x^- + \frac{1}{x^-} + \frac{2i}{g} . \quad (2.22)$$

⁵In earlier works (such as [7] and [8]) it was shown that the closure (1.6) implied Yang-Baxter, but complete equivalence was, to our knowledge, never established. Notice also that (1.6) is equivalent to the Yang-Baxter equation for the $su(2|2)$ S-matrix. This is not the case for the $su(1|2)$ S-matrix [5], which satisfies Yang-Baxter automatically for any values of x^\pm .

The above identification of the Ising elliptic parameter $k = (\sinh 2K \sinh 2L)^{-1}$ with ig allows us to rewrite the partition function as

$$Z = 2 \left[4 + \left(u - \frac{2i}{g} \right)^2 \right]. \quad (2.23)$$

The spin chain variable u is thus the variable parameterizing the partition function.

3 The correlation length and Kramers-Wannier

In this section we will reinterpret the correlation length for the Ising model in terms of gauge theory variables. Using the Ising model formulae [22], and the expression

$$u = 2 \cosh 2K \cosh 2L, \quad (3.1)$$

obtained by inserting (2.2) into the long-range spin chain variable u , (1.7), the correlation length is given, for real and positive k , as ⁶

$$\xi^{-1} = \ln \left(\frac{u/2 + \frac{|1-k|}{k}}{u/2 - \frac{|1-k|}{k}} \right). \quad (3.2)$$

It is easy to check that, for real, positive k , this expression is invariant under Kramers-Wannier duality,

$$k \rightarrow 1/k, \quad u \rightarrow ku. \quad (3.3)$$

Notice also that at the self-dual point $k = 1$ the correlation length becomes infinity, indicating the existence of a critical point, whose meaning will be discussed in section 3.2. However, in our case k is taken as ig which, for real couplings, is obviously not a real, positive number, having as a consequence that (3.2) is no longer invariant under (3.3). We should however bear in mind that (2.2) does not define a mapping of the spin chain variables to the ordinary Ising model, but to an Ising model analytically continued in K and the elliptic modulus k . The expression (3.2) for the correlation length, although correct on the positive, real k -axis, should thus be analytically continued to the entire plane. The most natural way to define the analytic extension is to impose invariance under (3.3), because Kramers-Wannier is a symmetry of the partition function, and should therefore

⁶Strictly speaking, this formula is obtained in the thermodynamic limit. However, the qualitative conclusions we derive from it should be valid in general.

be present whether K , L and k are real or complex. Thus for general k , we will define ξ by

$$\xi^{-1} = \ln \left(\frac{u/2 + \frac{\sqrt{(1-k)^2}}{k}}{u/2 - \frac{\sqrt{(1-k)^2}}{k}} \right), \quad (3.4)$$

which, for a suitably chosen branch of the square roots, obviously coincides with the previous expression when k is real and positive, is invariant under Kramers-Wannier duality, and still exhibits a critical point at $k = 1$.

Our main reason for studying the correlation length is the limit which one obtains for small momenta and fixed coupling. Since $u = 2 \cosh 2K \cosh 2L$, using (2.19) and (3.1) we can write

$$u = 2 \cosh 2K \sqrt{1 + \frac{1}{k^2 \sinh^2 2K}}. \quad (3.5)$$

Then, when $|K| \ll |k|^{-1}$ we get

$$u \sim \pm \frac{2 \cosh 2K}{k \sinh 2K} = \mp \frac{2 \cos \frac{p}{2}}{g \sin \frac{p}{2}}, \quad \text{when } |p| \ll |g|^{-1}. \quad (3.6)$$

Inserted into (3.4), we then get

$$\xi^{-1} \sim \ln \left(\frac{\cos \frac{p}{2} \pm i(1 - ig) \sin \frac{p}{2}}{\cos \frac{p}{2} \mp i(1 - ig) \sin \frac{p}{2}} \right) \rightarrow \pm i(1 - ig)p, \quad \text{when } p \rightarrow 0. \quad (3.7)$$

From the previous expression we see that for generic and finite coupling the correlation length becomes infinity when the magnon momentum goes to zero. Moreover the real part of the correlation length in the limit of strong 't Hooft coupling becomes exactly the string momentum, $p_{\text{string}} = gp$ [6], while in the limit of small 't Hooft coupling the correlation length is completely determined by the magnon momentum. An amusing formal representation of the correlation length in the limit of small magnon momentum is thus $\xi^{-1} \sim (p_{\text{string}} + ip_{\text{chain}})$. After our previous discussion on the role of the magnon momentum as parameterizing the deformation of the Hopf algebra we observe that the algebra becomes classical precisely when the correlation length becomes infinity, *i.e.* at the critical points. As we will discuss in the next subsection the critical behaviour at $p = 0$, and for generic finite coupling, appears because in this limit the model becomes effectively one-dimensional.

The above result is valid for $|K| \ll |k|^{-1}$, but not if $g \rightarrow \infty$ faster than $p \rightarrow 0$. This is for instance the case in the near-flat limit considered in [27], where a non-trivial S-matrix is obtained. This phenomenon can also be understood from the Ising model point of view.

In fact if $g \rightarrow \infty$ faster than $p \rightarrow 0$, one obtains $u \sim 2 \cosh 2K = 2 \cos \frac{p}{2}$, and

$$\xi^{-1} \rightarrow \ln \left(\frac{\cos \frac{p}{2} + 1}{\cos \frac{p}{2} - 1} \right) \rightarrow \infty. \quad (3.8)$$

We thus see that in the limit $p \rightarrow 0$ the correlation length vanishes. In Ising model terms this case, where both K and L go to zero, is the high-temperature limit. Using (2.3), we see that the high-temperature regime in the Ising model corresponds in the spin chain to the $p \rightarrow 0$ and $C \rightarrow \infty$ regime. It is easy to see that in the high-temperature limit the S-matrix might not become trivial when $p \rightarrow 0$, although there are also ways of taking this limit which produce a trivial S-matrix. These issues will be discussed in the next subsection.

For completeness let us also note that a similar situation to the $K \rightarrow 0$ regime will arise when $K \rightarrow \infty$. This is due to the symmetry between K and L in the Ising model, since when $K \rightarrow \infty$ with k generic the L -coupling vanishes, $L \rightarrow 0$, and we will once again obtain an infinite correlation length (again with an exception, appearing now at weak-coupling, and corresponding to the low-temperature limit). This critical behaviour is however not observed on the spin chain side, since the latter is defined for real momentum p , and $K \rightarrow \infty$ would correspond to $p \rightarrow -i\infty$. It is interesting to observe how the analytical continuation in K has broken the original symmetry of the Ising model. This is fortunate for us since (2.2) implies that an interchange of K and L inverts x^+ , which is not a symmetry of the spin chain. It also implies that the high-temperature limit is observed from the spin chain, but not the low-temperature one.

3.1 The high-temperature limit and triviality of the S-matrix

In the previous section, we found that the Ising high-temperature limit $K, L \rightarrow 0$, had a correlation length ξ of zero, despite that one expects the S-matrix to become trivial for small momenta. In this section we will show that it is possible to obtain a non-trivial S-matrix in this limit and that it is the only limit which can be non-trivial for $K \rightarrow 0$. Let us start by showing this last statement. If $K \rightarrow 0$ and we are outside the high-temperature limit either L stays finite, or $L \rightarrow \infty$. In the first case, corresponding to the plane-wave limit, we can safely take K to zero and set $x^+ = x^-$, obtaining a trivial S-matrix. The second case, $L \rightarrow \infty$, arises when g stays finite, or goes to infinity slower than $p \rightarrow 0$. We thus see from (2.19) that $e^{-2L} \sim k \sinh 2K$ so that $x^\pm \sim -g \sin \frac{p}{2} e^{\pm ip/2}$. Inserting this into the S-matrix of [8] one finds that it becomes trivial when $p_1, p_2 \rightarrow 0$.

Consider now the high-temperature limit with $K_1 = tK'_1$, $K_2 = tK'_2$ and $k = \delta/t^2$, where K'_1 , K'_2 and δ are fixed and where we will let $t \rightarrow 0$. This is the near-flat limit, because $p\sqrt{g} \sim p\sqrt[4]{\lambda}$ remains constant [9, 27]. Using (2.19), the L -couplings can be expressed in terms of t , δ and the K -couplings. Inserting this into the S-matrix it is not difficult to check that, by adjusting K'_1 , K'_2 and δ , we can make a given matrix element take any value we like.

Moving on to a more general case, let $K_1 = tK'_1$ and $K_2 = tK'_2$, as above, but where, in the limit $t \rightarrow 0$, the dominating contribution to k is of the form $k = \delta/t^\alpha$, for some exponent α . In order to be in the high-temperature regime we must have $\alpha > 1$ (note that the near-flat limit corresponds to $\alpha = 2$). Using (2.19) we see that, when $t \rightarrow 0$,

$$2tK' \cdot 2L = \frac{t^\alpha}{\delta} \Rightarrow L = \frac{t^{\alpha-1}}{4\delta K'} , \quad (3.9)$$

so that

$$x^\pm = e^{-2L} e^{\pm 2K} \sim 1 \pm 2K't - \frac{t^{\alpha-1}}{2\delta K'} . \quad (3.10)$$

Inserting this expression into the S-matrix shows that it is also non-trivial for $\alpha > 2$. Some matrix elements can be chosen arbitrarily, by adjusting K'_1 , K'_2 and δ , while some take fixed, constant values. On the other hand, for $1 < \alpha < 2$, the S-matrix becomes trivial once again, despite being obtained in a high-temperature limit.

Summing up, we have the following results for the limit $K_1 = tK'_1$, $K_2 = tK'_2$ and $k = \delta/t^\alpha$:

$$\begin{aligned} \alpha \leq 1 &\Rightarrow \text{Trivial} \\ 1 < \alpha < 2 &\Rightarrow \text{Trivial, despite being high-temperature} \\ \alpha = 2 &\Rightarrow \text{Non-trivial, near-flat limit} \\ 2 < \alpha &\Rightarrow \text{Non-trivial} . \end{aligned}$$

Among these cases the near-flat limit stands out. Besides corresponding to the first non-trivial α , it is the only small-momentum limit where the value for the S-matrix element A (in the notation of [8]), which corresponds to the process $\phi\phi \rightarrow \phi\phi$, describing scattering of bosons of the same type, can be adjusted to an arbitrary value. In contrast, when $\alpha < 2$ we get $A = 1$, and when $\alpha > 2$, $A = -1$. Also, it is only for $\alpha = 2$ that the S-matrix is sensitive to the value of δ . For $\alpha > 2$ all terms containing δ fall out. However when

$\alpha > 2$ the limit has also a fascinating property: the matrix elements are such that there is virtually no difference between fermions and bosons.⁷

These results show that an infinite correlation length guarantees triviality in the small-momentum limit, but that there are also cases in which the S-matrix is trivial, despite ξ being finite. These are the plane-wave limit, where $\alpha = 1$ and L remains finite, so that $u \sim 2 \cosh 2L$, and

$$\xi^{-1} = \ln \left(\frac{\cosh 2L + 1}{\cosh 2L - 1} \right), \quad (3.11)$$

and the high-temperature limit when $1 < \alpha < 2$, where $\xi = 0$. When considering the correlation length these results seem slightly out of place. However, it might be worthwhile to study these limits carefully as their triviality might be a consequence of the simplifications used in constructing the theory, notably the assumption of asymptotically long spin chains.

3.2 Critical points and the one-dimensional Ising model

We will now discuss the meaning of the critical points found in section 3. The ordinary Ising model critical point $k = 1$, has a natural interpretation as a branch point of the magnon dispersion relation⁸

$$E = \sqrt{1 + g^2 \sin^2 \frac{p}{2}}. \quad (3.12)$$

We see that when the sine takes its maximum value, the argument inside the square-root becomes zero precisely when $g = \pm i$ or, equivalently, $k = \pm 1$. This critical behaviour can thus be found at the boundary of the domain of convergence of the planar gauge theory. In this sense the Ising phase transition can be reflecting the regime where the number of planar diagrams becomes dense (see [28] for a recent discussion).

The other critical points, obtained for small momentum, are not standard two-dimensional Ising model critical points, but they can be understood in terms of the one-dimensional Ising model. As we have seen, the critical behaviour arises when $K \rightarrow \infty$, or $L \rightarrow \infty$. This implies that, in general, along one of the directions the spins will always have the same

⁷The S-matrix of [8] has the additional parameters γ_i and α_B which must be, in order for the representations to be unitary, equal to, respectively, $\sqrt{x_i^+ - x_i^-}$ and 1, up to some phase factors. If we choose to include no additional phases the only difference between fermions and bosons is a minus sign in a single matrix element.

⁸We thank J. Minahan for pointing this out to us.

orientation, as it would cost an infinite amount of energy to let two adjacent spins have opposite orientations. This means that we effectively obtain a one-dimensional model, and the partition function for the two-dimensional Ising model becomes the one-dimensional partition function, up to an infinite constant. It is well known that the one-dimensional Ising model has a critical point at zero temperature, and it is precisely this point that is obtained.

For a one-dimensional Ising model, the correlation length is given, in terms of the coupling $K^{(1D)}$, as

$$\xi_{(1D)}^{-1} = \ln \left(\frac{\cosh K^{(1D)}}{\sinh K^{(1D)}} \right) = \ln \left(\frac{e^{2K^{(1D)}} + 1}{e^{2K^{(1D)}} - 1} \right). \quad (3.13)$$

In the case $K \rightarrow 0$, studied above, the correlation length was given by (3.7), which written in terms of K and k is

$$\xi^{-1} = \ln \left(\frac{\cosh 2K \pm (1 - k) \sinh 2K}{\cosh 2K \mp (1 - k) \sinh 2K} \right). \quad (3.14)$$

Combining (3.13) and (3.14) allows us to, for a given k , relate K and $K^{(1D)}$. In general, however, we can see that $K \rightarrow 0$ corresponds to $K^{(1D)} \rightarrow \infty$, which is the low-temperature limit of the one-dimensional model, where its only critical point can be found.

The relationship is especially simple in the limit of zero coupling, which for K fixed also gives $L \rightarrow \infty$. In this regime it is easy to check that ξ is given by (3.14), with $k = 0$, independently of the value of K . Taking the plus sign, we then have the identification

$$\frac{\cosh 2K}{\sinh 2K} = e^{2K_{k=0}^{(1D)}}. \quad (3.15)$$

At infinite coupling, for fixed K , we do not have $L \rightarrow \infty$, but rather $L \rightarrow 0$. This can also be interpreted as a one-dimensional model, because now the horizontal rows decouple from each other, and we get indeed a set of one-dimensional models. The correlation length is now as in (3.8), and

$$\cosh 2K = e^{2K_{g=\infty}^{(1D)}}, \quad (3.16)$$

and the critical point now corresponds to $K \rightarrow \infty$. In this limit, $e^{2K} \rightarrow e^{2K_{g=\infty}^{(1D)}}$.

3.3 Kramers-Wannier duality and the long-range Bethe ansatz

Before we conclude this section, let us study the Kramers-Wannier duality transformation a little more closely. We will first focus on the transformation of the long-range variable u .

From (3.3), we see that $u \rightarrow k u$ under the duality. The inverse is also true: if we impose the one-loop Heisenberg-model result, which rescaled in order to match our conventions takes the form

$$u_{\text{one-loop}}(p) = \frac{2}{g} \cot \frac{p}{2}, \quad (3.17)$$

then, as is shown in appendix B, imposing Kramers-Wannier will give the all-loop result

$$u(p) = \frac{2}{g} \cot \frac{p}{2} \sqrt{1 + g^2 \sin^2 \frac{p}{2}}, \quad (3.18)$$

at least as the minimal solution.

Let us now discuss in some detail the Kramers-Wannier duality as a modular transformation. The map into the Ising model that we have suggested provides indeed a natural way to analitically extend $\mathcal{N} = 4$ Yang-Mills to arbitrary complex coupling. When we consider the k -plane, with $k = ig$, the physical region with positive coupling is just the positive imaginary axis. The negative imaginary axis corresponds thus to the analytic extension $\sqrt{\lambda} \rightarrow -\sqrt{\lambda}$, that could be holographically interpreted as some sort of continuation into de Sitter space [28]. The Kramers-Wannier duality transformation $k \rightarrow 1/k$ maps one region into the other.⁹

We can lift transformations on the elliptic modulus k into $Sl(2, \mathbb{Z})$ modular transformations of the underlying elliptic curve, with complex moduli $\tau(k) = iK'(k)/K(k)$. The transformation $k \rightarrow ik/k'$, with $k^2 + k'^2 = 1$, corresponds to $\tau \rightarrow \tau + 1$, and $k \rightarrow k'$ to $\tau \rightarrow -1/\tau$. The Kramers-Wannier duality transformation can be represented as the composition of $k \rightarrow -k$, and $g \rightarrow 1/g$. The first one can be recovered by performing twice the transformation $k \rightarrow ik/k'$, and therefore can be interpreted as the modular transformation $\tau \rightarrow \tau + 2$. The second is a standard strong/weak-coupling transformation, for the 't Hooft coupling. Keeping in mind the last fact, that the AdS/CFT correspondence is a duality on the 't Hooft coupling, it is interesting to wonder how much of this strong/weak-coupling duality is captured by the Kramers-Wannier transformation. In fact the $g \rightarrow 1/g$ transformation can be approached from a different point of view [8]. From the modular transformations of τ one gets

$$\tau(-ik/k') = -\frac{1}{\tau(1/k')}, \quad (3.19)$$

⁹Another possibility is to consider BPS bound states [29]. The integrability constant is then replaced by $4in/g$, and the Kramers-Wannier transformation is interpreted as a strong/weak-coupling duality transforming the 't Hooft coupling into n^2/g , and a global change of sign.

that for $k = ig$ is equivalent to the transformation $g \rightarrow 1/g$. Notice also that the self-dual point for this transformation, $g = 1$, does not correspond to a special point from the Ising model point of view. However at this point Kramers-Wannier corresponds to just the change of sign $k \rightarrow -k$. From the Ising model point of view the special point corresponds a complex value for the coupling, $g = i$, which is the place where the Ising phase transition takes place, and where as we have discussed above we reach the radius of converge of the planar series.

4 Conclusions

The present understanding of the map from $\mathcal{N} = 4$ dynamics to integrable spin chains is based on two key ingredients. One is the classical algebra of symmetries. The other is the set of kinematical relations defining the x^\pm variables. Both ingredients become entangled in the underlying Hopf algebra structure. What we see within the description of the long-range spin chain in terms of the Ising model is that the kinematical ingredients encoded in the central Hopf subalgebra are just defining a two-dimensional Ising model, and that the integrability of the spin chain is completely captured by the underlying Ising model Yang-Baxter equation. The results of this note can be summarized in the following table of correspondences between the spin-chain magnon dynamics and the two-dimensional Ising model. Probably the most interesting part of the correspondence is the

Magnon variables x^\pm	\Leftrightarrow	Transfer matrix elements
$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \text{invariant}$	\Leftrightarrow	Yang-Baxter condition
't Hooft coupling	\Leftrightarrow	Elliptic parameter
Departure from triviality	\Leftrightarrow	Inverse correlation length
Planar convergence radius	\Leftrightarrow	Ising critical point

Table 1: Equivalence of $\mathcal{N} = 4$ Yang-Mills variables to Ising model parameters.

one to one relation between the Yang-Baxter equation for the $su(2|2)$ spin chain magnon S-matrix and the Yang-Baxter equation for the two-dimensional Ising model. An obvious consequence of this correspondence is that both the $\mathcal{N} = 4$ Yang-Mills spin chain and the Ising model share the same elliptic curve. Another interesting consequence of this equivalence is the relation between the magnon momentum and the Ising model correlation

length. The magnon momentum operator is the additional piece that must be added to the symmetry algebra $su(2|2)$ in order to get a Hopf algebra determining a non-trivial S-matrix. On the other hand, the correlation length measures the departure from critical behaviour. We have described how critical behaviours, corresponding to infinite correlation lengths, are in correspondence with those zero-momentum limits of the spin chain where the S-matrix becomes trivial. There is however a piece in the scattering matrix that we have not considered at all in the present work. This is the global dressing phase factor, responsible for the interpolation from the strong to the weak-coupling regime. Hopefully the equivalence to the Ising model will also provide some light on the general structure of this dressing factor.

Acknowledgments

We are grateful to N. Dorey, F. Ferrari, J. Minahan and R. Roiban for comments and discussions. C. G. is also grateful to the Isaac Newton Institute for Mathematical Sciences for hospitality while this work was being completed. The work of J. G. is supported by a Spanish FPU grant and by a European fellowship through MRTN-CT-2004-005104. This work is also partially supported by the Spanish DGI under contracts FPA2003-02877 and FPA2003-04597 and by the CAM project HEPHACOS P-ESP-00346.

A Yang-Baxter and the closure condition

In this appendix we will demonstrate that the Yang-Baxter equations obtained from the S-matrix of [8], derived by imposing invariance under maximally centrally extended- $su(2|2)$, are equivalent to conditions of the form (2.13), or equivalently, to the closure (1.6). From the algebraic construction itself [8], or by direct computation using computer software, it follows that the closure condition (1.6) implies the Yang-Baxter equations of the S-matrix. The question that then arises is if the closure is necessary for Yang-Baxter to hold. The answer is not as obvious as it might seem. In [5], the S-matrix for the $su(1|2)$ -sector of the theory was constructed, also using the spectral variables x^+ and x^- (albeit scaled differently with respect to the convention used here), and it was found that the Yang-Baxter equation was satisfied without having to impose any relation between x^+ and x^- .

We will now settle the issue of the equivalence of Yang-Baxter and the closure. Fortunately, one of the Yang-Baxter equations takes an exceptionally simple form, solving

the problem for us. Using the matrix elements and the notation of [8], the equation corresponding to the process $|\phi^1\psi^1\phi^2\rangle \rightarrow |\psi^2\psi^1\psi^1\rangle$ is ¹⁰

$$\frac{1}{2}H_{12}L_{13}C_{23} + \frac{1}{4}G_{12}C_{13}(D_{23} - E_{23})\frac{x_2^-}{x_2^+} = \frac{1}{2}L_{23}C_{13}D_{12} , \quad (\text{A.1})$$

which, after plugging in the expressions for the matrix elements and simplifying, becomes

$$x_2^+ + \frac{1}{x_2^+} - x_2^- - \frac{1}{x_2^-} = x_3^+ + \frac{1}{x_3^+} - x_3^- - \frac{1}{x_3^-} . \quad (\text{A.2})$$

Furthermore, the equation for the process $|\phi^1\phi^2\psi^1\rangle \rightarrow |\psi^1\psi^1\psi^2\rangle$ is

$$\frac{1}{2}C_{12}(D_{13} - E_{13})D_{23} = \frac{1}{4}H_{23}C_{13}(D_{12} - E_{12}) + \frac{1}{2}G_{23}G_{13}C_{12}\frac{x_3^-}{x_3^+} , \quad (\text{A.3})$$

which becomes

$$\begin{aligned} & \left(x_2^- + \frac{1}{x_2^-} - x_1^- - \frac{1}{x_1^-}\right)\left(x_3^+ + \frac{1}{x_3^+} - x_1^+ - \frac{1}{x_1^+}\right) = \\ & = \left(x_2^+ + \frac{1}{x_2^+} - x_1^+ - \frac{1}{x_1^+}\right)\left(x_3^- + \frac{1}{x_3^-} - x_1^- - \frac{1}{x_1^-}\right) . \end{aligned} \quad (\text{A.4})$$

Using (A.2), this can be rewritten as

$$x_1^+ + \frac{1}{x_1^+} - x_1^- - \frac{1}{x_1^-} = x_3^+ + \frac{1}{x_3^+} - x_3^- - \frac{1}{x_3^-} . \quad (\text{A.5})$$

We have thus shown that the Yang-Baxter equation implies that the quantity $x_i^+ + \frac{1}{x_i^+} - x_i^- - \frac{1}{x_i^-}$, for $i = 1, 2, 3$, is equal to a common value, ¹¹ which we may call $4i/g$. Therefore the Yang-Baxter equation implies that

$$x_j^+ + \frac{1}{x_j^+} - x_j^- - \frac{1}{x_j^-} = \frac{4i}{g} , \quad j = 1, 2, 3 . \quad (\text{A.6})$$

This g does not have to be constant, though, and the Yang-Baxter equations are satisfied no matter how complicated g may be. If we want to interpret g as a coupling constant, we are forced to draw the conclusion that the physically admissible solutions of the Yang-Baxter equations are only a small part of the entire set of solutions.

¹⁰Here, we have set the marker variable $\xi_k = 1$. This is permitted since all the ξ_k cancel from the Yang-Baxter equations, and we thus get the same result as with the Hopf algebra compatible value $\xi_k = \sqrt{\frac{x_k^+}{x_k^-}}$.

¹¹It should be noted that there is a subtlety in this calculation. Some of the matrix elements, presented in [8], were simplified using (1.6). In fact, the S-matrix of [8] does not satisfy (1.8), if one does not impose (1.6). This means that the equations that we have just derived could just as well be an artifact of this simplification, and that a non-simplified S-matrix would yield trivially satisfied Yang-Baxter equations. Fortunately, this is not the case. We have re-derived the matrix elements, as determined by equation (1.8), but without using (1.6) to simplify them, and checked that the closure is indeed necessary for Yang-Baxter to be satisfied.

B Proof that Kramers-Wannier duality determines u

In this appendix we will show that imposing Kramers-Wannier duality (3.3), and the one-loop result

$$u_{\text{one-loop}}(p) = \frac{2}{g} \cot \frac{p}{2}, \quad (\text{B.1})$$

gives the all-loop result

$$u(p) = \frac{2}{g} \cot \frac{p}{2} \sqrt{1 + g^2 \sin^2 \frac{p}{2}}. \quad (\text{B.2})$$

To show this, we must use the transformation properties of K under Kramers-Wannier duality. In fact, there are two ways to define K^* , the dual of K , compatible with (3.3), $\sinh 2K^* = \frac{1}{\sinh 2K}$, and $\sinh 2K^* = k \cdot \sinh 2K$. Here we will show the statement for

$$\sinh 2K^* = k \cdot \sinh 2K \Leftrightarrow \sin \frac{p^*}{2} = ig \sin \frac{p}{2}, \quad (\text{B.3})$$

but the other option produces the same result. Let us now set $x \equiv \sin \frac{p}{2}$. From (B.3), Kramers-Wannier is then

$$\begin{aligned} x^* &= kx, \\ k^* &= k, \\ u^* &= h(k)u, \end{aligned} \quad (\text{B.4})$$

where we, for the moment, leave a function $h(k)$ free. The one-loop expression (B.1) implies that in general u takes the form

$$u = \frac{2i}{k} \frac{\sqrt{1-x^2}}{x} \cdot f(x, k), \quad (\text{B.5})$$

where $f(x, 0) = 0$ for all possible values of x . Now, applying (B.4) to (B.5) and using the transformation properties of u gives

$$\frac{2ih(k)}{k} \frac{\sqrt{1-x^2}}{x} \cdot f(x, k) = \frac{\sqrt{1-k^2x^2}}{x} \cdot f(kx, k^{-1}), \quad (\text{B.6})$$

which leads to the following equations for the function f ,

$$\begin{aligned} \frac{f(kx, k^{-1})}{f(x, k)} &= \frac{k\sqrt{1-x^2}}{h(k)\sqrt{1-k^2x^2}}, \\ f(x, 0) &= 1. \end{aligned} \quad (\text{B.7})$$

If we now define a new function m by

$$m(x, y) \equiv \frac{f(x, y)}{\sqrt{1 - (xy)^2}}, \quad (\text{B.8})$$

equation (B.7) becomes

$$\frac{m(kx, k^{-1})}{m(x, k)} = \frac{k}{h(k)},$$

$$m(x, 0) = 1. \quad (\text{B.9})$$

The dependence in x must cancel from the left-hand side of (B.9). There are two ways in which this can happen, the first being the direct cancellation of the entries in the first argument of m , and the second being if m has Kramers-Wannier invariant factors, such as $(1 + \sqrt{k}x)$. For the moment, let us suppose that m does not have any Kramers-Wannier invariant factor.

In order for the arguments in the first entry of m to cancel, it must then take the form

$$m(x, y) = x^\eta g(y), \quad (\text{B.10})$$

and (B.9) then implies that $\eta = 0$. Thus m is a momentum-independent function. Retracing our steps, we find

$$u = \frac{2i}{k} \frac{\sqrt{1-x^2}}{x} \sqrt{1-(xk)^2} m(k) = \frac{2}{g} \cot \frac{p}{2} \sqrt{1+g^2 \sin^2 \frac{p}{2}} m(ig). \quad (\text{B.11})$$

Inserting the Kramers-Wannier invariant factors that we, optionally, could have in (B.9), we arrive at the complete solution

$$u = \frac{2}{g} \cot \frac{p}{2} \sqrt{1+g^2 \sin^2 \frac{p}{2}} m(ig) q \left(\sin \frac{p}{2}, g \right), \quad (\text{B.12})$$

where q is Kramers-Wannier invariant and where $\lim_{g \rightarrow 0} m(ig) q \left(\sin \frac{p}{2}, g \right) = 1$.¹² There is no way that the function q can cancel the factors $\cot \frac{p}{2}$, or $\sqrt{1+g^2 \sin^2 \frac{p}{2}}$, because if it contains one of them, it must contain the other, and a correction by \sqrt{g} , in order to be invariant. Then, it can impossibly have the correct weak-coupling behaviour. In our case, we have $h(k) = k$,¹³ allowing us to take the minimal solution $m q = 1$, which produces (B.2).

¹²This condition is more constraining than it first might seem. For example, $\tilde{u} = \sqrt{g}u$ is a Kramers-Wannier invariant, which however cannot appear in q . The reason for this is that by controlling the behaviour of p when $g \rightarrow 0$, \tilde{u} can be made to take any value that we like, and q can thus impossibly satisfy the weak-coupling condition.

¹³In fact, the problem has solutions for all $h(k)$ satisfying $h(k^{-1}) = h(k)^{-1}$.

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