SEMIPOSITIVE BUNDLES AND BRILL-NOETHER THEORY

VICENTE MUÑOZ AND FRANCISCO PRESAS

Abstract. We prove a Lefschetz hyperplane theorem for the determinantal loci of a morphism between two holomorphic vector bundles E and F over a complex manifold under the condition that $E^* \otimes F$ is Griffiths k-positive. We apply this result to find some homotopy groups of the Brill-Noether loci for a generic curve.

1. Introduction

The topological properties of the zero sets of sections of a very ample line bundle on a projective variety were studied by Lefschetz in the 20's, when he proved the renowned Lefschetz hyperplane theorem. Later on Bott, Andreotti and Fraenkel [Bo, AF] gave a new proof of the result using Morse theory, which has the byproduct of giving homotopy isomorphisms. The ampleness condition of the vectors bundles in these proofs can be weakened to a *semipositivity* condition. We recall the concept of Griffiths k-positivity [Si]

Definition 1.1. *A holomorphic vector bundle E over a complex manifold M is said to be (Griffiths) k-positive if there exists a hermitian metric h on E such that for every point $x \in M$ there exists a complex subspace $V_x \subset T_xM$ of dimension at least k where the curvature form Θ of the connection associated to h satisfies that $\Theta_{v,iv}$ is a definite positive quadratic form in the fiber E_x, for every non-zero $v \in V_x$.*

For $k = \dim_{\mathbb{C}} M$ we recover the definition of positivity of a vector bundle [Gr]. We have the following extension of the classical Lefschetz theorem

Theorem 1.2. *Let E be a rank r k-positive vector bundle over a complex compact manifold M and let s be a holomorphic section of E. Let $W = Z(s)$ be the zero set of s. Then $M - W$ has the homotopy type of a CW-complex of dimension $2n - (k - r) - 1$. When W is smooth the inclusion $W \to M$ induces isomorphisms on homology (resp. homotopy) groups H_p (resp. π_p) for $p < k - r$ and an epimorphism for $p = k - r$.*

Theorem 1.2 may be extended in the following way.

Date: July, 2001.

1991 *Mathematics Subject Classification.* 32Q55, 14M12, 14H51.

Key words and phrases. Ample bundle, Griffiths k-positive, Lefschetz hyperplane theorem, determinantal locus, Brill-Noether.

Partially supported by The European Contract Human Potential Programme, Research Training Network HPRN-CT-2000-00101.

Definition 1.3. *Let E be a holomorphic vector bundle over a complex manifold M and let s be a holomorphic section of E. Then the pair (E, s) is k-positive if there exists a hermitian metric h on E such that for every $x \in M$ which is a critical point for $|s|^2$ with $s(x) \neq 0$, there is a complex subspace $V_x \subset T_x M$ of dimension at least k satisfying $h(s, \Theta_{v,iv} s) > 0$ for every non-zero $v \in V_x$.*

Theorem 1.4. *Let E be a rank r vector bundle over a compact complex manifold M and s a holomorphic section such that (E, s) is a k-positive pair. Let $W = Z(s)$ be the zero set of s. Then $M - W$ has the homotopy type of a CW-complex of dimension $2n - (k - r) - 1$. When W is smooth the inclusion $W \to M$ induces isomorphisms on homology (resp. homotopy) groups H_p (resp. π_p) for $p < k - r$ and an epimorphism for $p = k - r$.*

Let us suppose now that we have two holomorphic vector bundles E and F over M of ranks e and f respectively, and let $\phi : E \to F$ be a vector bundle morphism. We can always assume that $e \leq f$ by changing to the transpose morphism if necessary. Fix a positive integer $r \leq e$. The r-determinantal subvariety of ϕ is defined as

$$D_r(\phi) = \{x \in M : \text{rank } \phi_x \leq r\}.$$

This can be constructed in the following way. Consider the grassmannian fibration $\pi : G = \text{Gr}(e - r, E) \to M$, and the composition

$$\phi_r : U \hookrightarrow \pi^* E \xrightarrow{\pi^* \phi} \pi^* F,$$

where the first map is the natural inclusion of the universal bundle U over G. The zero set of ϕ_r satisfies

$$\pi(Z(\phi_r)) = D_r(\phi).$$

In fact π is bijective over $D_r(\phi) - D_{r-1}(\phi)$. We have the following result which extends that of [FL, De]

Theorem 1.5. *Let ϕ be a morphism between the holomorphic vector bundles E and F over a compact complex manifold M, such that $E^* \otimes F$ is k-positive. Suppose that $Z(\phi_r)$ is smooth. Then the natural inclusion $Z(\phi_r) \hookrightarrow G = \text{Gr}(e - r, E)$ induces isomorphisms between the homology (resp. homotopy) groups $H_i(Z(\phi_r))$ and $H_i(G)$ (resp. $\pi_i(Z(\phi_r))$ and $\pi_i(G)$) for $i < k - (e - r)(f - r)$ and epimorphisms for $i = k - (e - r)(f - r)$.*

We apply the theory developed to recover Debarre's result [De] computing the homology of Brill-Noether loci over an algebraic curve. Our method gives us as well information about the homotopy groups. Let C be an algebraic curve. Consider the jacobian variety of degree d line bundles $\text{Jac}^d(C)$. The Brill-Noether moduli spaces, $W_d^k(C)$, are defined as

$$W_d^k(C) = \{L \in \text{Jac}^d(C) |\ h^0(L) \geq k + 1\}.$$

The varieties of linear systems are

$$G_d^k(C) = \{(L, V) | L \in W_d^k(C), V \subset H^0(L), \dim V = k + 1\}.$$

Clearly there is a map $G_d^k(C) \to W_d^k(C)$ which is one-to-one over $W_d^k(C) - W_d^{k+1}(C)$. For a generic curve C, the spaces $G_d^k(C)$ are smooth [ACGH] and of the expected dimension

$\rho = g - (k+1)(g-d+k)$. We prove the following results about the topology of $G^k_d(C)$ and $W^k_d(C)$.

Theorem 1.6. *Let C be a generic curve of genus $g \geq 2$ and let $d \geq 1$ and $k \geq 0$ be integers. If $\rho > 0$ then $G^k_d(C)$ and $W^r_d(C)$ are connected and non-empty. If $\rho > 1$ then $\pi_1(W^r_d(C)) = \pi_1(G^r_d(C))$ is free abelian of rank $2g$. In general we have that*

$$\pi_i(G^k_d(C)) = \pi_i(BU(k+1)),\ 2 \leq i \leq \rho - 1.$$

In particular, the corresponding rational homotopy groups of $G^k_d(C)$ are

$$\pi_i(G^k_d(C)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & i \leq 2k+2,\ i \equiv 0 \pmod 2 \\ 0 & \text{otherwise} \end{cases}$$

for $2 \leq i \leq \rho - 1$.

Corollary 1.7. *Let C be a generic curve of genus $g \geq 2$ and let $d \geq 1$, $k \geq 0$ and $l \geq 1$ be integers. Suppose that $G^{k+i}_d(C)$ are non-empty for $0 \leq i \leq l$. Then $\pi_1(W^k_d(C)) = \mathbb{Z}^{2g}$ and $\pi_i(W^k_d(C)) = 0$ for $2 \leq i \leq 2l$.*

Acknowledgements: We are specially indebted to Ignacio Sols for proposing us this problem. We are also grateful to Ignasi Mundet and Aniceto Murillo for very useful comments.

2. Proof of Theorems 1.2 and 1.4

The result of theorem 1.2 follows from that of theorem 1.4 since if E is k-positive then (E, s) is a k-positive pair for any holomorphic section s. Therefore we may suppose that we are under the assumptions of theorem 1.4. Then the result is a slight generalization of the classical proof using Morse theory. We consider the real function $f = \log h(s, s)$. Clearly f is well defined over $M - Z(s)$ where $Z(s)$ is the zero set of s. Our objective is to show that all the critical points of f have index at least $k - r + 1$, since then the result follows from standard Morse theory.

Now we compute

$$\partial f = \frac{1}{h(s,s)}(h(\bar{\partial} s, s) + h(s, \partial s)).$$

At a critical point this quantity vanishes and so we get

$$h(\bar{\partial} s, s) + h(s, \partial s) = 0.$$

Recalling that $\bar{\partial} s = 0$ we obtain

$$h(s, \partial s) = 0. \tag{1}$$

A second differentiation, omitting quantities that vanish at the critical point, gives us

$$\bar{\partial}\partial \log |s|^2 = \frac{1}{|s|^2}(h(\partial\bar{\partial} s, s) + h(\bar{\partial} s, \bar{\partial} s) + h(\partial s, \partial s) + h(s, \bar{\partial}\partial s)).$$

Now observe that the section is holomorphic and so $\bar{\partial} s = 0$. Also we have that $\bar{\partial}\partial + \partial\bar{\partial} = \Theta$, the curvature of the bundle E. Substituting into the equation we obtain

$$\bar{\partial}\partial \log |s|^2 = \frac{1}{|s|^2}(h(s, \Theta s) + h(\partial s, \partial s)).$$

Now we must control the second term in the sum. For this we define the subspace

$$W = \{v \in V_x | \ \partial s(v) = 0\}.$$

Using that for any $v \in T_xM$ we have that $h(\partial s(v), s) = 0$, we have that the complex dimension of W is bounded below by $k-r+1$. So $h(\partial s, \partial s) = 0$ in W. By the condition on the curvature, it follows that, restricting to $v \in W$, $\bar{\partial}\partial f(v, \mathbf{i}v) = -\mathbf{i}\alpha$, for some real number $\alpha > 0$. Let H_f be the Hessian of f. Then

$$H_f(v) + H_f(\mathbf{i}v) = -2\mathbf{i}\bar{\partial}\partial f(v, \mathbf{i}v).$$

This quantity is clearly negative for any non-zero $v \in W$. Suppose that the index of the critical point were strictly less than $k-r+1$. Then there would be a subspace $P \subset T_xM$ of dimension greater than $2n-k+r-1$, where H_f is definite positive. Here $n = \dim_{\mathbb{C}} M$. So $P \bigcap \mathbf{i}P$ would have real dimension at least $2n - 2k + 2r$. This subspace would intersect non-trivially to W providing a contradiction. So the index of the critical point is at least $k-r+1$.

Now a standard argument in Morse theory gives that $M - W$ has the homotopy type of a CW-complex of dimension $2n - (k-r+1)$.

When W is smooth, we consider a small tubular neighbourhood $E(W)$ that retracts to W. Then M is obtained from $E(W)$ by attaching cells of index at least $k-r+1$. This means that the $(k-r)$-skeleta of M and W are the same (up to homotopy). Therefore we get that the natural map $\pi_p(W) \to \pi_p(M)$ is an isomorphism for $p < k-r$, and an epimorphism for $p = k-r$. The statement about homology groups holds by the same reason. □

3. Determinantal submanifolds

We are going to apply our results in the previous section to the study of determinantal subvarieties, aiming to prove theorem 1.5. Suppose we have a morphism ϕ between bundles E and F over a complex manifold M. We suppose that $\operatorname{rank} E = e \le f = \operatorname{rank} F$. The morphism ϕ can be interpreted as a holomorphic section of the bundle $E^* \otimes F$. Fix a positive integer $r \le e$, we can define the manifold $G = \mathrm{Gr}(e-r, E)$ which is a grassmannian fibration over M. The canonical projection of this fibration will be denoted by π. We consider the morphism $\phi_r : U \hookrightarrow \pi^*E \to \pi^*F$ on G, where U is the universal bundle. The r-determinantal subvariety of ϕ is

$$D_r(\phi) = \pi(Z(\phi_r)) = \{x \in M : \operatorname{rank} \phi_x \le r\}.$$

The subvarieties $\{D_r(\phi)\}_{i=0}^{e-1}$ have a natural structure of a stratified submanifold. In some situations, the strata are irreducible and reduced complex varieties of the expected dimension. In this case, $Z(\phi_r)$ are smooth subvarieties of G, so that they can be considered as a desingularization of $D_r(\phi)$. Note that $\dim G = \dim M + r(e-r)$ and $\operatorname{rank}(U^* \otimes \pi^*F) = f(e-r)$, so that the expected (complex) codimension of $Z(\phi_r)$ is $(e-r)(f-r)$. If this exceeds the dimension of M then $D_r(\phi)$ is empty.

We start by relating the curvature of the bundles over M and over G.

Proposition 3.1. *Let E be a rank e k-positive vector bundle over M. The vector bundle U on $G = \mathrm{Gr}(e-r, E)$ is $(k+r)$-positive.*

Proof. Let $x \in M$ and choose holomorphic coordinates $(z_1, \dots, z_n)$ at a neighborhood of the point. Fix a point $V \in G$ such that $\pi(V) = x$. Denote $p = e - r$. Choose a holomorphic frame $f = (u_1, \dots u_e)$ of the bundle E at a neighborhood of x. We can assume that $u_1, \dots u_p$ expand the subspace V and also that $f(0)$ is an orthonormal basis with respect to the hermitian metric h of E.

With these choices it is now easy to define holomorphic coordinates at a neighborhood of $V \in G$. The chart is defined as

$$\begin{array}{rcl} \mathbb{C}^n \times \mathbb{C}^{p(e-p)} & \to & G \\ (z_l, \lambda_{kj}) & \to & < u_1 + \sum_{k=p+1}^{e} \lambda_{k1} u_k, \dots, u_p + \sum_{k=p+1}^{e} \lambda_{kp} u_k > . \end{array}$$

A chart for the bundle U is given by

$$\begin{array}{rcl} \mathbb{C}^n \times \mathbb{C}^{p(e-p)} \times \mathbb{C}^p & \to & U \subset \pi^* E \\ (z_l, \lambda_{kj}, \beta_j) & \to & (u_1 + \sum_{k=p+1}^{e} \lambda_{k1} u_k, \dots, u_p + \sum_{k=p+1}^{e} \lambda_{kp} u_k) \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix} . \end{array} \tag{2}$$

With this chart the pull-back of the metric h to $\pi^* E$ is defined by the matrix $h(f) = (h(u_i, u_j))_{i,j}$. This matrix can be divided in the following blocks

$$h(f) = \begin{pmatrix} h^{11}(f) & h^{12}(f) \\ h^{21}(f) & h^{22}(f) \end{pmatrix},$$

according to the local decomposition $E = < u_1, \dots, u_p > \bigoplus < u_{p+1}, \dots, u_e >$.

For a point of G with coordinates (z_l, λ_{kj}) the restriction of the metric h to U with respect to the frame $f_U = (u_1 + \sum_{k=p+1}^{e} \lambda_{k1} u_k, \dots, u_p + \sum_{k=p+1}^{e} \lambda_{kp} u_k)$ is given by

$$h_U(f_U) = h^{11}(f) + h^{12}(f)\lambda + \lambda^* h^{21}(f) + \lambda^* h^{22}(f)\lambda,$$

where λ^* is the conjugate transpose matrix of λ. If we use a holomorphic frame f such that $h(z) = I + O(|z|^2)$ then we can compute the curvature using Lemma 2.3 in Chap. III of [We]. So the curvature in this case is $\Theta(0) = \bar\partial\partial h(0)$. We start by computing the derivative of h_U (we do not write the frame f to simplify the notation),

$$\partial h = \partial_z h^{11} + \partial_z h^{12}\lambda + h^{12} d\lambda + \lambda^* \partial_z h^{21} + \lambda^* \partial_z h^{22} \lambda + \lambda^* h^{22} d\lambda,$$

where by ∂_z we denote the derivatives in the (z_l) directions. Differentiating again

$$\begin{array}{rcl} \bar\partial\partial h & = & \bar\partial_z \partial_z h^{11} + \bar\partial_z \partial_z h^{12} \lambda + \bar\partial_z h^{12} \wedge d\lambda + d\lambda^* \wedge \partial_z h^{21} + \\ & & + \lambda^* \bar\partial_z \partial_z h^{21} + d\lambda^* \wedge \partial_z h^{22} \lambda + \lambda^* \bar\partial_z \partial_z h^{22} \lambda + \lambda^* \bar\partial_z h^{22} \wedge d\lambda + d\lambda^* \wedge h^{22} d\lambda. \end{array}$$

At $\lambda = 0$ we have $h^{11} = I_p$, $h^{22} = I_{r-p}$, $h^{12} = 0$, $h^{21} = 0$ and $\partial h = \bar\partial h = 0$. Therefore

$$\Theta_U(f_U(0)) = \bar\partial\partial h^{11}(f)(0) + d\lambda^* \wedge d\lambda = \Theta(0)_{|U} + d\lambda^* \wedge d\lambda. \tag{3}$$

Now we study the grassmannian direction. We want to find a subspace in $< \frac{\partial}{\partial \lambda_{kj}} >$ where $d\lambda^* \wedge d\lambda$ is positive definite for any non trivial pair of directions v, iv in the base. So we need to impose that for any non-zero $\beta \in \mathbb{C}^p$,

$$\beta^* d\lambda^* \wedge d\lambda \, \beta > 0. \tag{4}$$

We may choose Δ to be the subspace of those $\lambda \in \mathbb{C}^{p(e-p)}$ which have all their columns $\lambda_j = (\lambda_{kj})_{k=p+1}^e$ equal. For any $\beta \neq 0$ and any non-zero $\lambda \in \Delta$ we have

$$\beta^* d\lambda^* \wedge d\lambda \, \beta = \sum_j |\beta_j|^2 d\lambda_j^* \wedge d\lambda_j > 0.$$

Take the subspace $H \subset T_x M$ where Θ is positive definite at x. Then the subspace $H \bigoplus \Delta$ is a subspace where Θ_U is positive definite. Recall that $r = e - p$. So we have proved that U is $(k+r)$-positive. It is easy to check that we have found the largest dimension for a subspace Δ satisfying the required properties. □

Using proposition 3.1 together with theorem 1.2 would not give a large range for Lefschetz isomorphisms for $Z(\phi_r) \subset G$. We get around this problem by using the notion of k-positive pair in definition 1.3 and the freedom of varying the hermitian metrics in E and F.

Proposition 3.2. *Let $\phi : E \to F$ be a morphism where $E^* \otimes F$ is k-positive and let $\phi_r : U \to \pi^* F$ be the induced map on $G = \mathrm{Gr}(e-r, E)$. Then $(U^* \otimes \pi^* F, \phi_r)$ is a $(k + r(e-r))$-positive pair.*

Proof. Let $V \in G$ be a critical point of $|\phi_r|^2$. As ϕ_r is holomorphic, this implies that $h(\phi_r, \nabla \phi_r) = 0$ at V, where h is the metric induced in $U^* \otimes F$ by the metrics in E and F. Let $x = \pi(V) \in M$ and choose holomorphic coordinates z_l in M with $z_1 = \cdots = z_n = 0$ at x. Fix holomorphic frames $f_E = (u_1, \ldots, u_e)$ and $f_F = (v_1, \ldots, v_f)$ for E and F respectively, orthonormal at x, such that $V = < u_1, \ldots, u_p >$, where $p = e - r$. The map $\phi : E \to F$ becomes, with respect to these frames, a map $s : \mathbb{C}^e \to \mathbb{C}^f$, which is decomposed as $s = \left[\begin{smallmatrix} s_1 \\ s_2 \end{smallmatrix}\right]$, with respect to $\mathbb{C}^e = \mathbb{C}^p \oplus \mathbb{C}^r$.

As in the proof of Proposition 3.1 we have coordinates (z_l, λ_{kj}) for G using the frame f_E. The bundle U has a natural trivialization given by (2) and $\phi_r : \mathbb{C}^p \to \mathbb{C}^f$ is written as the map $\phi_r = s_1 + s_2 \lambda$. Therefore $\nabla \phi_r = \nabla_z s_1 + s_2 d\lambda$, at the point $V \in G$ (since this point has $\lambda = 0$). The condition $h(\phi_r, \nabla \phi_r) = 0$ translates thus into

$$h(s_1, \nabla_z s_1) = 0 \qquad \text{and} \qquad s_1^* s_2 = 0.$$

We need to compute $h(\sigma, \Theta_{v,iv} \sigma)$, where Θ is the curvature form of $U^* \otimes F$ and the section is $\sigma = \phi_r$. By (3) we have

$$\Theta(0) = \Theta_{E^* \otimes F}(0)|_{U^* \otimes F} - d\lambda^* \wedge d\lambda.$$

Let $v = (u, W) \in T_V G = T_x M \oplus T_V \mathrm{Gr}(p, e)$ be a vector. We have

$$h(s_1, \Theta(0)_{v,iv} s_1) = h(s_1, \Theta_{E^* \otimes F}(0)_{u,iu} s_1) - \mathbf{i} \, \mathrm{Tr}(s_1^* W^* W s_1),$$

by the usual formula for the metric in the tensor product $E^* \otimes F$. If we can arrange that s_1 is injective then we would have $W s_1 \neq 0$ for any $W \neq 0$. Then we choose $H \subset T_x M$

where $\Theta_{E^*\otimes F}(0)_{u,iu}$ is definite positive and hence $\Theta_{v,iv}$ is positive on $H \oplus T_V \operatorname{Gr}(p,e)$, which has dimension $k + (e-p)p = k + r(e-r)$.

So it remains to prove that for suitable hermitian metrics on E and F we have that $h(s_1, \nabla s_1) = 0$ and $s_1^* s_2 = 0$ imply that s_1 is injective. For this we use a perturbation argument. Fix the metric on E and let $\mathcal{M}_F$ be the space of hermitian metrics for F, completed in a suitable Sobolev norm so that it is a Hilbert manifold and the elements $g \in \mathcal{M}_F$ have enough regularity. We define the map

$$\begin{array}{rcl} \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) : G \times \mathcal{M}_F & \rightarrow & T^*M \oplus \operatorname{Hom}(U, U^\perp) \\ (y, g) & \mapsto & \left(g(s_1, \nabla s_1)_{\pi(y)}, (s_2^{*g} s_1)_y\right), \end{array}$$

where the dual of s_2 is defined using g. We want to check that this map is a submersion. Fix (y, g_0) and let us compute the differential $D\mathcal{F}$ at this point. Fix a trivialization of G as in the proof of proposition 3.1 with coordinates (z, λ). Fix a hermitian trivialization of E (recall that we are only varying the metric of F) and a holomorphic trivialization of F such that $g_0(0) = I$. The points in $G \times \mathcal{M}_F$ are denoted by (z, λ, g), where g is a hermitian matrix valued function of z. The tangent vectors will be denoted as (v, A, b), where b is also a hermitian matrix valued function. The first component of $\mathcal{F}$ is $\mathcal{F}_1 = \partial |s_1|_g^2$, which in the chart gives $\mathcal{F}_1(0,0,g) = \operatorname{Tr}(s_1^* g d s_1 + s_1^* \partial g s_1)$. Thus

$$D\mathcal{F}_1(0,0,b) = \operatorname{Tr}(s_1^* b(0) d s_1 + s_1^* (\partial b)(0) s_1).$$

Also $\mathcal{F}_2(0, \lambda, h) = (s_2 - s_1 \lambda^*)^* g (s_1 + s_2 \lambda)$ and then

$$D\mathcal{F}_2(0, A, b) = \operatorname{Tr}(s_2^* b(0) s_1 - A s_1^* s_1 + s_2^* s_2 A).$$

If the range of $D\mathcal{F}$ is not the whole tangent space then there exist $w \in TM$ and $B \in \operatorname{Hom}(\mathbb{C}^p, \mathbb{C}^{n-p})$ with $< D\mathcal{F}(v, A, b), (w, B) >= 0$ for all (v, A, b). Choose $v = 0$, $A = 0$ and $b(0) = 0$ to get $\operatorname{Tr}(s_1^* \partial_w b(0) s_1) = 0$. Varying $\partial b(0)$ and using that $s_1 \neq 0$ we get that $w = 0$. Now choose $v = 0$, $A = 0$ to get $\operatorname{Tr}(B^* s_2^* b(0) s_1) = 0$. Varying $b(0)$ among all hermitian matrices, we get $s_2 B = 0$. Now choose $v = 0$ to get $\operatorname{Tr}(B^* A s_1^* s_1) = 0$ for all A. As $s_1 \neq 0$ we get $B = 0$, which completes the proof of the surjectivity of $D\mathcal{F}$.

Being $\mathcal{F}$ a submersion, then for generic metric $g \in \mathcal{M}_F$ the zero set of $\mathcal{F}$ is regular. Therefore the set of critical points of $|\sigma|_g^2$ is a finite collection of points of G. The metric g is not $\mathcal{C}^\infty$ in principle but we may approximate it by a smooth metric without losing the above property.

Also we want to check that the projection of the critical points to M is in generic position. This is achieved if we check that at any (y, g_0), for $v \in TM$ there exist $A \in \operatorname{Hom}(\mathbb{C}^p, \mathbb{C}^{n-p})$ and $b \in T\mathcal{M}_F$ such that $D\mathcal{F}(v, A, b) = 0$. For this choose first $(A, b(0))$ such that $D\mathcal{F}_2(v, A, b) = 0$ and then for such v and A we can choose $\partial b(0)$ such that $D\mathcal{F}_1(v, A, b) = 0$. Finally we take some $b \in T_{g_0}\mathcal{M}$ with the obtained $b(0)$ and $\partial b(0)$.

Now at a generic point of M we have that ϕ is injective. Therefore at the critical points of $|\sigma|_g^2$ the map ϕ_r is injective, which is what we wanted to show. □

Proof of Theorem 1.5. By Proposition 3.2, $(U^* \otimes F, \phi_r)$ is a $(k + r(e-r))$-positive pair. On the other hand the rank of the bundle $U^* \otimes F$ is $(e-r)f$. By Theorem 1.2 we get that $Z(\phi_r)$ satisfies the required isomorphisms with $i < k + r(e-r) - f(e-r) = k - (e-r)(f-r)$, and an epimorphism for $i = k - (e-r)(f-r)$. □

4. Application to Brill-Noether theory

We briefly recall all the tools that we are going to use following [ACGH]. We shall deal with the case of general rank since many of the constructions are valid in this case, and later we will particularize to the case of line bundles. Fix an algebraic curve C and denote by $\mathcal{M}_d^r(C)$ the moduli space of stable vector bundles of rank r and degree d. We choose d and r coprime so that $\mathcal{M}_d^r(C)$ is compact. Brill-Noether moduli spaces, $W_d^{r,k}(C)$, are defined as

$$W_d^{r,k}(C) = \{A \in \mathcal{M}_d^r(C) | \ h^0(A) \geq k+1\}.$$

It is also convenient to define the varieties

$$G_d^{r,k}(C) = \{(A,V) | A \in \mathcal{M}_d^r(C), V \subset H^0(A), \dim V = k+1\}.$$

The Brill-Noether loci for rank $r = 1$ are the ones defined in the introduction, with $\mathcal{M}_d^1(C) = \mathrm{Jac}^d(C)$.

It is possible to choose a high degree divisor Γ on C, let us say of degree m, such that for each element $A \in \mathcal{M}_d^r$ the long exact sequence

$$H^0(A) \to H^0(A(\Gamma)) \stackrel{\phi}{\to} H^0(A(\Gamma)/A) \to H^1(A) \to H^1(A(\Gamma)) \to \cdots$$

satisfies that $H^1(A(\Gamma)) = 0$, since the bundles in $\mathcal{M}_d^r(C)$ form a bounded family. A bundle will be in $W_d^{r,k}(C)$ whenever $H^0(A)$ is big enough. This is equivalent to make rank ϕ small enough. Denote $n = rm + d$. Take a universal bundle $\mathcal{U}$ over $\mathcal{M}_d^r(C) \times C$ and consider the sequence

$$\mathcal{U} \to \mathcal{U}(\Gamma) \to \mathcal{U}(\Gamma)/\mathcal{U}. \tag{5}$$

If π is the projection of $\mathcal{M}_d^r \times C$ to $\mathcal{M}_d^r$, we obtain by pushing forward through π

$$\pi_*(\mathcal{U}(\Gamma)) \stackrel{\psi}{\longrightarrow} \pi_*(\mathcal{U}(\Gamma)/\mathcal{U}).$$

The determinantal varieties of ψ define the sequence of Brill-Noether loci $W_d^{r,k}(C)$. Moreover the spaces $G_d^{r,k}(C)$ are the spaces $Z(\psi_s)$ in the notation of Section 3. So we need to study the curvature of the vector bundles

$$E = \pi_*(\mathcal{U}(\Gamma)), \qquad F = \pi_*(\mathcal{U}(\Gamma)/\mathcal{U})$$

to understand the topology of $W_d^{r,k}(C)$. More precisely, $e = \mathrm{rank}\, E = d + mr + r(1-g)$ and $f = \mathrm{rank}\, F = mr$. Suppose for instance that $\frac{d}{r} \leq g-1$. Then $e \leq f$ and the map ψ is generically injective. Put $s = e-(k+1)$. Then the locus $Z(\psi_s) = G_d^{r,k}(C) \subset \mathrm{Gr}(e-s, E)$. The (expected) codimension of $Z(\psi_s)$ is $(e-s)(f-s) = (k+1)(k+1-d+r(g-1))$. The case $\frac{d}{r} > g-1$ is similar.

Proposition 4.1. *There is a (natural) metric on* $E \to \mathcal{M}_d^r(C)$ *such that its curvature is given by*

$$\langle t, \Theta_{u,v} t\rangle = -\mathbf{i}\int_C \langle G_{\mathrm{End}\,A}(\Lambda[\eta_u, \bar{\eta}_v])t, t\rangle\omega + \int_C \langle G_{A(\Gamma)}(\eta_u t), \eta_v t\rangle\omega + \beta(u,v)\int_C |t|^2\omega,$$

at a point $A \in \mathcal{M}_d^r(C)$*, where* $t \in E_A = H^0(\mathcal{U}(\Gamma)|_{\{A\}\times C}) \cong H^0(A(\Gamma))$*,* $u, v \in T_A\mathcal{M}_d^r(C)$*. Here* ω *is the area form on* C*,* β *is a* $(1,1)$*-form on* $\mathcal{M}_d^r(C)$*,* η_u *and* η_v *are the Kodaira-Spencer representatives, i.e., the harmonic elements of* $\Omega^{0,1}(\mathrm{End}\,A)$ *corresponding to* u, v *under the natural isomorphism* $T_A\mathcal{M}_d^r(C) \cong H^1(\mathrm{End}\,A)$*, and* $G_{\mathrm{End}\,A}$ *and* $G_{A(\Gamma)}$ *are the Green operators for the Laplacian* $\Delta_{\bar\partial}$ *on* $\Omega^0(\mathrm{End}\,A)$ *and* $\Omega^{0,1}(A(\Gamma))$*, respectively.*

Proof. We write $M = \mathcal{M}_d^r(C)$ for simplicity. Let us start by computing the curvature of the universal bundle $\mathcal{U} \to X = M \times C$. Put a hermitian metric on $\mathcal{U}$ so that it becomes a family of hermitian-Einstein line bundles over C. Let F stand for the curvature of $\mathcal{U}$ and decompose it as $F = F_{CC} + F_{CM} + F_{MC} + F_{MM}$, according to the decomposition $X = M \times C$. For instance, $F_{CM} \in \Omega_C^{1,0} \otimes \Omega_M^{0,1}(\mathrm{End}\,A)$, $F_{MM} \in \Omega_C^0 \otimes \Omega_M^{1,1}(\mathrm{End}\,A)$, etc. In particular $F_{CC} = \lambda\omega I$, where λ is a constant.

Decompose the differential given by the natural connection on $\mathcal{U}$ as $d = d_C + d_M$. Clearly $\bar\partial_C F_{MC} = 0$. Using the Bianchi identity $dF = 0$ and looking at the decomposition

$$\Omega_X^{p,q} = \bigoplus_{i,j} \Omega_M^{i,j} \otimes \Omega_C^{p-i,q-j},$$

we have that $\partial_C F_{MC} = -\partial_M F_{CC} = 0$. Therefore $F_{MC}|_{\{A\}\times C}$ is harmonic. Actually F_{MC} gives a complex linear map $T_A M \to \Omega_C^{0,1}(\mathrm{End}\,A)$, which represents the Kodaira-Spencer map [TW]. Also $F_{CM} = \overline{F}_{MC}$. Using again the Bianchi identity and $\partial_C F_{MC} = 0$ we have that

$$\partial_C\bar\partial_C F_{MM} = -\partial_C\bar\partial_M F_{MC} = -\bar\partial_M\partial_C F_{MC} - [F_{CM}, F_{MC}] = -[F_{CM}, F_{MC}] = -[F_{MC}, F_{CM}],$$

where this is a combination of the wedge product of forms and the Lie bracket of the endomorphisms of A. By the Kähler identities [We] we have that $\Delta_{\bar\partial} F_{MM} = -\mathbf{i}\Lambda[F_{MC}, F_{CM}]$, so

$$F_{MM} = -\mathbf{i}G_{\mathrm{End}\,A}(\Lambda[F_{MC}, F_{CM}]) + \pi^*\beta,$$

for a purely imaginary $(1,1)$-form β on M.

Now we compute the curvature of the push-forward $\pi_*(\mathcal{U}(\Gamma))$. Since $\mathcal{U}(\Gamma) \to \mathcal{M}_n^r(C)\times C$ is also a universal bundle, where $n = rm + d$, we may suppose that Γ is the zero divisor for this computation. The natural hermitian L^2-metric h_E on $E = \pi_*\mathcal{U}$ is given by $h_E(t_1, t_2) = \int_C h(t_1, t_2)\omega$, for $t_1, t_2 \in E_A = H^0(A)$. Now the proof of [TW, theorem 1] states that for the L^2 metric on $\pi_*\mathcal{U}$, we have for $u, v \in T_A M$ and $t \in H^0(A)$,

$$\begin{aligned}\langle t, \Theta_{u,v} t\rangle &= \int_C \langle F_{MM}(u,v))t, t\rangle\omega + \int_C \langle G_A(\eta_u t), \eta_v t\rangle\omega = \\ &= -\mathbf{i}\int_C \langle G_{\mathrm{End}\,A}(\Lambda[\eta_u, \bar\eta_v])t, t\rangle\omega + \int_C \langle G_A(\eta_u t), \eta_v t\rangle\omega + \beta(u,v)\int_C |t|^2\omega,\end{aligned}$$

since $\eta_u = F_{CM}(u)$. There is a different sign in our formula to that of [TW, theorem 1] due to our convention on the hermitian metric to be complex linear in the second variable. □

In order to get more specific information on the bundles E and F we need to restrict from now on to the case of line bundles, i.e., $r = 1$. In this case $\mathcal{M}_d^1(C) = \mathrm{Jac}^d(C)$ and the Brill-Noether loci are those $W_d^k(C)$ described in the introduction. We have the following

Corollary 4.2. *Suppose that $r = 1$. Then the bundle $E^* \otimes F$ on $\mathrm{Jac}^d(C)$ is positive.*

Proof. In this case the universal bundle is the Poincaré bundle $\mathcal{L} \to \mathrm{Jac}^d(C) \times C$. The bundle

$$F = \pi_*(\mathcal{L}(\Gamma)/\mathcal{L}) = \bigoplus_{i=1}^m \mathcal{L}|_{x_i \times \mathrm{Jac}^d(C)},$$

where $\Gamma = x_1 + \cdots + x_m$. So to check that $E^* \otimes F$ is positive we only need to check that $E^* \otimes \mathcal{L}|_{x_i \times \mathrm{Jac}^d(C)}$ is so. This is equivalent to trivialize $\mathcal{L}$ so that $\mathcal{L}|_{x_i \times \mathrm{Jac}^d(C)} \cong \mathcal{O}$ and then to check that E^* is positive.

In the case of line bundles, the formula for the curvature of E in Proposition 4.1 reduces because the Lie bracket is zero. Putting $v = \mathbf{i}u$ we get for the curvature of E, for $t \in E_A$ with $||t|| = 1$,

$$\langle t, (\Theta_E)_{u,\mathbf{i}u} t\rangle = \mathbf{i} \int_C \langle G_{A(\Gamma)}(\eta_u t), \eta_u t\rangle \omega + \beta(u, \mathbf{i}u).$$

Moreover the curvature of $\mathcal{L}$ in the Jacobian direction is then $F_{MM} = \pi^*\beta$. Restricting to $x_i \times \mathrm{Jac}^d(C)$ we get β is an exact form. We may then change the hermitian metric in $E \to \mathrm{Jac}^d(C)$ in order to eliminate this term from the formula of the curvature. So we get $\langle t, (\Theta_{E^*})_{u,\mathbf{i}u} t\rangle = -\mathbf{i} \int_C \langle G_{A(\Gamma)}(\eta_u t), \eta_u t\rangle \omega$. Now $\Delta_{\bar\partial}$ is a positive operator on $\Omega^{0,1}(A(\Gamma))$ since it is self-adjoint semi-positive with kernel $H^1(A(\Gamma)) = 0$, by choosing the degree of Γ very large. Therefore $G_{A(\Gamma)}$ is also positive and the result follows. □

Proof of Theorem 1.6. By [ACGH, page 214], for a general curve C of genus g, $G_d^k(C)$ is reduced and of pure dimension $\rho = g - (k+1)(g-d+k)$. Moreover $G_d^k(C)$ is smooth.

Applying theorem 1.5 using corollary 4.2 we get that the natural inclusion of $G_d^k(C) = Z(\psi_s)$ in $G = \mathrm{Gr}(e-s, E)$ induces isomorphisms in the homology groups of order less than or equal to $\rho - 1$ and epimorphisms for order ρ. The same holds for the homotopy groups.

Now G is the total space of a fibration $\mathrm{Gr}(e-s, e) \to G \to \mathrm{Jac}^d(C) = (\mathbb{S}^1)^{2g}$. Therefore $\pi_1(G) = \mathbb{Z}^{2g}$ and $\pi_i(G) = \pi_i(\mathrm{Gr}(e-s,e))$ for $i > 1$, since the grassmannian is simply connected. Now $e - s = k+1$ and e is very large (taking the degree of Γ very large), so $\mathrm{Gr}(e-s,e) = \mathrm{Gr}(k+1,e) \hookrightarrow \mathrm{Gr}(k+1,\infty) \cong BU(k+1)$ induces isomorphisms in homotopy and homology groups up to some arbitrarily large i_0. This proves the statements of the theorem regarding $G_d^k(C)$.

It only remains to prove that $\pi_1(G_d^k(C)) = \pi_1(W_d^k(C))$. Using $\pi_1(G) \cong \pi_1(G_d^k(C)) \to \pi_1(W_d^k(C)) \to \pi_1(\mathrm{Jac}^d(C))$, we get injectivity on the fundamental groups. We claim that the surjectivity follows from the fact that $W_d^{k+i}(C)$, $i \geq 0$, form a Whitney stratification of $W_d^k(C)$ and that the fibers of $\pi^{-1}(W_d^{k+i}(C)) \cap G_d^k(C) \to W_d^{k+i}(C)$ are connected. In fact, given a representative $\gamma : S^1 \to W_d^k(C)$ of a class in the fundamental group, we can always assure that it intersects the singular strata of $W_d^k(C)$ in isolated points $p_1, \dots, p_k$. Outside these points we can lift γ to $\hat{\gamma}$ in $G_d^k(C)$. As the preimage of p_i is connected, $\hat{\gamma}$ can be extended to a loop, also denoted $\hat{\gamma}$, mapping to γ. This completes the proof. □

It is also interesting to study the higher homotopy groups of the Brill-Noether varieties $W_d^k(C)$ in the case they are not smooth, i.e., when $W_d^{k+1}(C) \neq \emptyset$.

Proof of Corollary 1.7. The idea is that $W_d^{k+l} \neq \emptyset$ implies that there is a $\mathrm{Gr}(k+1, k+l+1) \hookrightarrow G_d^k$ mapping to a point in W_d^k. Since $\pi_i(\mathrm{Gr}(k+1, k+l+1)) \xrightarrow{\cong} \pi_i(G_d^k) \cong \pi_i(\mathrm{Gr}(k+1, e))$ for $i \leq 2l$, we have that $\pi_i(G_d^k) \to \pi_i(W_d^k)$ is the zero map. Instead of showing the surjectivity of such map, we will take a shortcut. By [De, theorem 2.2] the inclusion $W_d^k \hookrightarrow \mathrm{Jac}^d$ induces isomorphism on homology up to degree $2l$. By theorem 1.6 it also induces isomorphism on the fundamental group. This implies that it induces isomorphism on the homotopy groups up to degree $2l$. □

References

[AF] A. Andreotti and T. Fraenkel, The Lefschetz theorem on hyperplane sections, *Annals Math. (2)* **69** (1959) 713–717.

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of algebraic curves,* Springer-Verlag, New York (1985).

[Bo] R. Bott, On a theorem of Lefschetz, *Michigan Math. J.* **6** (1959) 211–216.

[De] O. Debarre, Lefschetz theorems for degeneracy loci, *Bull. Soc. Math. France* **128** (2000) 283–308.

[FL] W. Fulton and R. Lazarsfeld, On the connectedness of degeneracy loci and special divisors, *Acta Math.* **146** (1981) 271–283.

[Gr] P. A. Griffiths, *Hermitian differential geometry, Chern classes and positive vector bundles* in Global Analysis, Princeton University Press, Princeton, N.J. (1969).

[Si] Y-T. Siu, Complex-analyticity of harmonic maps, vanishing and Lefschetz theorems, *J. Diff. Geom.* **17** (1982) 55–138.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry,* Wiley-Interscience, New York (1978)

[TW] W-K. To and L. Weng, Curvature of the L^2-metric on the direct image of a family of Hermitian-Einstein vector bundles, *Amer. J. Math.* **120** (1998) 649–661.

[We] R. Wells, *Differential Analysis on Complex Manifolds,* Prentice-Hall, Englewood Cliffs, N.J., 2nd ed. Springer-Verlag (1973).

Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, 28049 Madrid, Spain

E-mail address: vicente.munoz@uam.es

Departamento de Álgebra, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain

E-mail address: fpm@eucmos.sim.ucm.es