Milnor Number of Weighted-Lê–Yomdin Singularities

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At the beginning of the seventies, O. Zariski proposed several problems related with the (embedded) topology of a germ of a n-dimensional hypersurface singularity defined by the zero locus of a germ of a complex analytic function. The second one was roughly stated as "if two analytic hypersurface germs are topologically equivalent then their tangent cones must be homeomorphic and the homeomorphism must respect the topological equisingularity type at any point." In this paper, we give counter-examples for $n=3$ and 4 (even in a family). Our proof is mainly based on the study of the topology of weighted-Lê–Yomdin surface singularities which are a generalization of the well-known Lê–Yomdin singularities. We obtain a formula for the Milnor number of a weighted-Lê–Yomdin surface singularity and derive an equisingularity criterion for them.

In [19], Zariski proposed to study a series of problems (from A to H) related with the (embedded) topology of a germ of a hypersurface singularity $(V,0)\subset(\mathbb{C}^n,0)$ defined by the zero locus of a germ of a complex analytic function $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$. He defined two germs $(V_1, 0) \subset (\mathbb{C}^n, 0)$ and $(V_2, 0) \subset (\mathbb{C}^n, 0)$ to be *topologically equisingular* if there is a local homeomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\phi(V_1) = V_2$.

The A-problem (and the one which is still open) discussed by him was the so called Zariski's multiplicity question: does topological equisingularity of $(V_1, 0)$ and $(V_2, 0)$ imply that they have the same multiplicity? (For an updated survey paper, see for instance [5].) Given a hypersurface singularity $(V, 0)$ germ, we denote by CV its projectivized tangent cone and by BV be the blowup of V at p. The second problem by Zariski is to find out if the following assertion is true:

B-problem. Given two hypersurface singularities $(V_1, 0)$ and $(V_2, 0)$ which have the same embedded topological type, the following holds: there exists a (non-embedded) homeomorphism

$$h : CV_1 \to CV_2 \tag{B1}$$

of the projectivized tangent cones such that if $h(p_1) = p_2$, the following holds:

(1) The embedded topological types of (CV_1, p_1) and (CV_2, p_2) coincide.
(2) The embedded topological types of (BV_1, p_1) and (BV_2, p_2) coincide also. □

Zariski proved that this is true if $n = 2$. In 2005, Fernández de Bobadilla [6, Example 13] found a counterexample to this problem for a topologically equisingular family if $n \geq 5$.

In this paper, we give counterexamples if $n = 3$ and 4 (also in a family). In our examples and in those of [6, Example 13], there is no homeomorphism h from the projectivized tangent cones. Not having such a homeomorphism it does not even make sense to ask for the further properties (1) and (2).

Our proof is mainly based on the study of the topology of weighted-Lê–Yomdin singularities which are a generalization of the well-known Lê–Yomdin singularities, see [10, 12]. These singularities are deformations of weighted-homogeneous (non-isolated) singularities; some topological results about them can be found for instance in works by Dimca [4] and Massey and Siersma [14]. We obtain a formula for the Milnor number of a weighted-Lê–Yomdin surface singularity and derive an equisingularity criterion for them (see Theorem 3.2). This formula was proposed to us by Claus Hertling to whom we are very grateful.

1 Lê–Yomdin Singularities

We will study germs $(V, 0) \subset (\mathbb{C}^3, 0)$ of isolated hypersurface singularities defined by a convergent series $f \in \mathbb{C}\{x, y, z\}$, that is, $V = f^{-1}(0)$. Let $f := f_d + f_{d+k} + \ldots$ be the homogeneous decomposition of f and let $C_m \subset \mathbb{P}^2$ be the projective locus of zeroes of f_m. Thus, the tangent cone of V at 0 is C_d.

Definition 1.1. A hypersurface germ $(V, 0)$ is called a *Lê–Yomdin* singularity if

$$\mathrm{Sing}(C_d) \cap C_{d+k} = \emptyset$$

(note that its tangent cone C_d is reduced). □

Next result is due to Lê–Yomdin (see Luengo and Melle-Hernández [13] for arbitrary dimensions).

Proposition 1.2. If $(V, 0)$ is Lê–Yomdin, then its Milnor number μ satisfies:

$$\mu(V, 0) = (d-1)^3 + k \sum_{P \in \mathrm{Sing}(C_d)} \mu(C_d, P).$$

□

We are going to recall a topological proof of this fact since we will need it in the next section.

Proposition 1.3. [9] Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a (non-empty) singularity V. Let us denote by χ the Euler characteristic of its Milnor fiber. Let $\pi : T \to \mathbb{C}^n$ be a proper mapping which is an analytic isomorphism over $\mathbb{C}^n \setminus \{0\}$.

Let $E := \pi^{-1}(0)$ and let us suppose that there exists a stratification $\mathcal{S}$ of E satisfying the following property: $\forall S \in \mathcal{S}$ and $\forall p, q \in S$ the Euler characteristic of the Milnor fibers of the $f \circ \pi$ at p and q depends only on S and it is denoted by χ_S. Then,

$$\chi = \sum_{S \in \mathcal{S}} \chi(S)\chi_S.$$

□

For germs $(V, 0) \subset (\mathbb{C}^n, 0)$ of non-isolated hypersurface singularities Parusiński defined a *generalized Milnor number* [16], $\mu(V, 0) := (-1)^{n-1}(\chi(F_f) - 1)$ where the zero locus of $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ defines the singularity whose Milnor fiber is F_f.

Lemma 1.4. Let V be the singularity in $(\mathbb{C}^3, 0)$ defined by $z^d(z^k + f(x, y))$, where $C : f(x, y) = 0$ is a germ of curve singularity with generalized Milnor number μ . Then, the Euler characteristic of the Milnor fiber of V is $(d+k)\mu$. $\square$

Proof. Let V_t be the Milnor fiber $z^d(z^k + f(x, y)) = t$. Let us consider the projection $(x, y, z) \mapsto (x, y)$. The discriminant of this projection restricted to V_t is shown in the equation:

$$\left(\frac{(-d)^{\frac{d}{e}} k^{\frac{k}{e}}}{(d+k)^{\frac{d+k}{e}}} f(x, y)^{\frac{d+k}{e}} - t^{\frac{k}{e}} \right)^e = 0,$$

where $e := \gcd(d, k)$. Then, the ramification locus of the projection is the disjoint union of $(d+k)/e$ Milnor fibers F_t of $f(x, y) = 0$, and on each point we loose exactly e points. By Riemann–Hurwitz, we have:

$$\chi(V_t) = (d+k) - e\frac{d+k}{e}\chi(F_t) = (d+k)(1 - 1 + \mu) = (d+k)\mu. \qquad \blacksquare$$

Proof of Proposition 1.2. We consider the blowing-up $\pi : T \to \mathbb{C}^3$ of the origin and let $\mathbb{P}^2 \cong E := \mathbb{P}^{-1}(0)$. We can consider a stratification $\mathcal{S}$ as in Proposition 1.3 as follows:

- There is a stratum S_0 of dimension 2 given by $\mathbb{P}^2 \setminus C_d$. The local equation of $\pi^*(V)$ at a point $p \in S_0$ is of type $z^d = 0$ and then $\chi_{S_0} = d$. We recall that

$$\chi(S_0) = 3 - d(3 - d) - \sum_{P \in \operatorname{Sing}(C_d)} \mu(C_d, P) = \frac{(d-1)^3 - 1}{d} - \sum_{P \in \operatorname{Sing}(C_d)} \mu(C_d, P).$$

- The space $\check{C}_d := C_d \setminus \operatorname{Sing}(C_d)$ is a union S of strata where the local equation of $\pi^*(V)$ at a point $p \in S$ is of type $xz^d = 0$ and then $\chi_{S_0} = 0$.
- The points $P \in \operatorname{Sing}(C_d)$ are strata. Since the local equation of $\pi^*(V)$ at P is as in Lemma 1.4, we know that $\chi_P = (d+k)\mu(C_d, P)$.

These data imply the statement. $\blacksquare$

We want to give a *weighted* version of this formula.

2 Weighted Projective Planes

Let $l \in \mathbb{N}$; we denote μ_l the group of lth roots of unity. We will consider actions of μ_l on $\mathbb{C}^m$, given by:

$$\zeta \cdot (x_1, \ldots, x_m) = (\zeta^{p_1} x_1, \ldots, \zeta^{p_m} x_m);$$

such an action is *primitive* if the quotient cannot be obtained from an action of a smaller cyclic group.

Definition 2.1. The class of the origin in $\mathbb{C}^m/\mu_l$ is a singular point of *index* l for a primitive action. $\square$

Definition 2.2. A *weight* is a triple $\omega := (p_x, p_y, p_z) \in \mathbb{N}^3$ such that $\gcd(p_x, p_y, p_z) = 1$. A polynomial f is ω-weighted-homogeneous of degree d if $f(t^{p_x} x, t^{p_y} y, t^{p_z} z) = t^d f(x, y, z)$. $\square$

Let us fix a weight ω. We can adapt the definition of the projective plane.

Definition 2.3. The *weighted projective plane* $\mathbb{P}^2_\omega$ is the normalization of the quotient of $\mathbb{C}^3 \setminus \{0\}$ by the action of $\mathbb{C}^*$ defined by:

$$t \cdot (x, y, z) := (t^{p_x} x, t^{p_y} y, t^{p_z} z).$$

The elements of $\mathbb{P}^2_\omega$ will be denoted by $[x : y : z]_\omega$. A weighted-homogeneous polynomial h defines in a natural way a curve C^ω in the weighted projective plane $\mathbb{P}^2_\omega$. $\square$

Notation 2.4. The point $P^{x,y}_\omega := [0 : 0 : 1]_\omega$ is a *vertex* of $\mathbb{P}^2_\omega$; we define accordingly the other vertices. The curve $X^\omega := \{[x : y : z]_\omega \in \mathbb{P}^2_\omega \mid x = 0\}$ is an *axis* of $\mathbb{P}^2_\omega$; we denote $\check{X}^\omega := X^\omega \setminus \{P^{x,y}_\omega, P^{x,z}_\omega\}$. We define the axes Y^ω, Z^ω in the same way. $\square$

Let us describe $\mathbb{P}^2_\omega$ using *multicharts*. The mapping

$$\begin{array}{rccl} \Phi^\omega_z : & \mathbb{C}^2 & \rightarrow & U_z := \{[x : y : z]_\omega \in \mathbb{P}^2_\omega \mid z \neq 0\} \\ & (x, y) & \mapsto & [x : y : 1]_\omega \end{array}$$

is holomorphic but not injective. If $\zeta \in \mu_{p_z}$, it is easily seen that

$$[x : y : 1]_\omega = [\zeta^{p_x} x : \zeta^{p_y} y : 1]_\omega.$$

In fact, if we consider the action of μ_{p_z} on $\mathbb{C}^2$ given by $\zeta \cdot (x, y) := (\zeta^{p_x} x, \zeta^{p_y} y)$, the mapping Φ_z^ω factorizes through $\mathbb{C}^2/\mu_{p_z}$ and we obtain an isomorphism

$$\Psi_z^\omega : \mathbb{C}^2/\mu_{p_z} \to U_z.$$

We define the same objects for the variables y, z.

Remark 2.5. Let us denote $d_x := \gcd(p_y, p_z)$ and let us define in the same way d_y, d_z. We define $q_x := p_x/d_y d_z$ and q_y, q_z in the same way. The number of preimages for Φ_z^ω verifies:

- $\#(\Phi_z^\omega)^{-1}([x : y : 1]_\omega) = p_z$ if $xy \neq 0$.
- $\#(\Phi_z^\omega)^{-1}([x : 0 : 1]_\omega) = \frac{p_z}{d_y} = q_z d_x$ if $x \neq 0$.
- $\#(\Phi_z^\omega)^{-1}([0 : 0 : 1]_\omega) = 1$. □

Definition 2.6. The *reduced weight* associated to ω is $\eta := (q_x, q_y, q_z)$ (the components are pairwise coprime). □

Lemma 2.7. The mapping $\mathbb{P}^2_\omega \to \mathbb{P}^2_\eta$ defined by $[x : y : z]_\omega \mapsto [x^{d_x} : y^{d_y} : z^{d_z}]_\eta$ is an isomorphism. □

Remark 2.8. It is easily seen that $\mathbb{P}^2_\eta$ is smooth outside its vertices and $P_\eta^{x,y}$ is singular if and only if $q_z > 1$. In that case, q_z is the index of the singular point; we have similar statements for the other vertices. □

We can define *weighted blowups*. Let

$$T_\omega := \{((x, y, z), [u : v : w]_\omega) \in \mathbb{C}^3 \times \mathbb{P}^2_\omega \mid (x, y, z) \in [u : v : w]_\omega\},$$

where we identify the class $[u : v : w]_\omega$ with its closure in $\mathbb{C}^3$; let $\pi : T_\omega \to \mathbb{C}^3$ be the restriction of the first projection. As with usual blowup, π is an isomorphism outside the origin and $E := \pi^{-1}(0)$ is isomorphic to $\mathbb{P}^2_\omega \cong \mathbb{P}^2_\eta$. We can study T_ω using *multicharts*. Let us define:

$$\widetilde{\Phi}^\omega_z : \mathbb{C}^3 \to \widetilde{U}_z := \{((x, y, z), [u : v : w]_\omega) \in T_\omega \mid z \neq 0\},$$

where $\widetilde{\Phi}^\omega_z(x, y, z) := ((xz^{p_x}, yz^{p_y}, z^{p_z}), [x : y : 1]_\omega)$. If we consider the action of μ_{p_z} on $\mathbb{C}^3$ defined by

$$\zeta \cdot (x, y, z) := (x\zeta^{p_x}, y\zeta^{p_y}, z\bar{\zeta}),$$

then $\widetilde{\Phi}^\omega_z$ factorizes through an isomorphism $\widetilde{\Psi}^\omega_z : \mathbb{C}^3/\mu_{p_z} \to \widetilde{U}_z$.

Proposition 2.9. The variety T_ω satisfies:

1. If $p_x > 1$, then $(0, P^{y,z}_\omega)$ is singular of index p_x.
2. If $d_x > 1$, then, T_ω has singularities at $\{0\} \times \check{X}^\omega$ of index d_x.

Similar facts happen for the other axes and vertices. □

Following these facts, we will define weighted Milnor numbers.

Definition 2.10. Let f be a non-zero ω-weighted-homogeneous polynomial and let $C^\omega \subset \mathbb{P}^2_\omega$ be the associated curve. Let $P \in C^\omega$ and let us suppose that we can express it as $P = [x_0 : y_0 : 1]_\omega$. Let C_z be the zero locus in $\mathbb{C}^2$ of $f(x, y, 1)$ and let μ be the usual Milnor number of C_z at (x_0, y_0). We define:

- the *ω-Milnor number* as $\mu^\omega(C^\omega, P) := \mu/\nu_P$, where ν_P is the index of $(0, P)$ in T^ω, that is, it is the gcd of the weights of the non-zero coordinates of P;
- the *x-intersection multiplicity* $m^x(C^\omega, P)$ of C^ω at P as the intersection number of C_z and $x = 0$ at $(0, y_0)$, when $P = [0 : y_0 : 1]$.

We denote $m^{x,y}_x(C^\omega) := m^x(C^\omega, P^{x,y}_\omega)$. We naturally extend the definitions for other writings of P and we check that it does not depend on the choices. The *singular points* of C^ω are those where $\mu^\omega > 0$. □

Even if $\mathbb{P}^2_\omega$ and $\mathbb{P}^2_\eta$ are isomorphic, since we take into account their embeddings in T^ω and T^η, the concept of weighted Milnor number depends actually on the weight.

Lemma 2.11. Let C^ω be a curve in $\mathbb{P}^2_\omega$ and let C^η be the corresponding curve in $\mathbb{P}^2_\eta$. Let $[x_0 : y_0 : z_0]_\omega =: P^\omega \in C^\omega$ and let us denote $P^\eta = [x_0^{d_x} : y_0^{d_y} : z_0^{d_z}]_\eta$ the corresponding point in C^η. Then:

1. If P^η is not a vertex of $\mathbb{P}^2_\eta$, then $\mu^\eta(C^\eta, P^\eta)$ coincides with the usual Milnor number.
2. If P^ω is outside the axes, then $\mu^\omega(C^\omega, P^\omega) = \mu^\eta(C^\eta, P^\eta)$.
3. If $P^\omega \in \check{X}^\omega$, then

$$\mu^\omega(C^\omega, P^\omega) = \mu^\eta(C^\eta, P^\eta) + \frac{(d_x - 1)(m^x(C^\eta, P^\eta) - 1)}{d_x}$$

and $m^x(C^\omega, P^\omega) = m^x(C^\eta, P^\eta)$.

4. For the vertices,

$$\begin{aligned}\mu^\omega(C^\omega, P^{x,y}_\omega) = {} & \frac{d_x d_y}{d_z}\mu^\eta(C^\eta, P^\eta) + \frac{(d_x - 1)(d_y - 1)}{p_z} + \\ & + \frac{(d_x - 1)d_y(m_x^{x,y}(C^\eta) - 1) + (d_y - 1)d_x(m_y^{x,y}(C^\eta) - 1)}{p_z}\end{aligned}$$

and $m_x^{x,y}(C^\omega) = d_y m_x^{x,y}(C^\eta)$. □

The proof is straightforward from the previous considerations.

3 Weighted-Lê–Yomdin Singularities

Let us consider a germ $(W, 0) \subset (\mathbb{C}^3, 0)$ defined by a series $g \in \mathbb{C}\{x, y, z\}$; let $g := g_d + g_{d+k} + \ldots$ be the weighted-homogeneous decomposition of g with respect to a weight ω and let $C^\omega_m \subset \mathbb{P}^2_\omega$ be the weighted projective locus of zeroes of g_m.

Definition 3.1. We say that $(W, 0)$ is a *weighted-Lê–Yomdin singularity with respect to* ω if $\operatorname{Sing}(C^\omega_d) \cap C^\omega_{d+k} = \emptyset$. □

We are going to prove the formula proposed by C. Hertling.

Theorem 3.2. The Milnor number μ of a *weighted-Lê–Yomdin singularity* $(W, 0)$ *with respect to* ω satisfies the following equality:

$$\mu(W, 0) = \left(\frac{d}{p_x} - 1\right)\left(\frac{d}{p_y} - 1\right)\left(\frac{d}{p_z} - 1\right) + k \sum_{P \in \operatorname{Sing}(C^\omega_d)} \mu(C^\omega_d, P).$$

□

We will use the next result.

Lemma 3.3. Let $h : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function which is invariant for an action of μ_n on $\mathbb{C}^m$. Let p be the class of the origin in $Q := \mathbb{C}^m/\mu_n$. Note that h defines a germ $\widetilde{h} : (Q, p) \to (\mathbb{C}, 0)$. Let us suppose that $h^{-1}(0)$ contains the points of $\mathbb{C}^m$ where the isotropy group of the action is not trivial. Then if χ (resp. $\widetilde{\chi}$) is the Euler characteristic of the Milnor fiber of h (resp. $\widetilde{h}$) then $\chi = n\widetilde{\chi}$. □

Proof. The Milnor fiber of h is an unramified covering of n sheets of the Milnor fiber of $\widetilde{h}$. ■

In order to prove Theorem 3.2, we are going to set up notations. We will denote

$$\check{C}^{\omega}_{d} := C^{\omega}_{d} \setminus \left(X^{\omega} \cup Y^{\omega} \cup Z^{\omega}\right).$$

It is possible for the axes to be components of C^{ω}_{d}. We set $\varepsilon^{x} = 1$ (resp. 0) if $X^{\omega} \subset C^{\omega}_{d}$ (resp. $\not\subset$) and we define ε^{y} and ε^{z} in the same way. We denote by $\widetilde{C}^{\omega}_{d}$ the union of irreducible components of C^{ω}_{d} different from the axes. Let p be the degree of $\widetilde{C}^{\omega}_{d}$; note that:

$$p = d - (\varepsilon^{x} p^{x} + \varepsilon^{y} p^{y} + \varepsilon^{y} p^{y}).$$

Let $\operatorname{Sing}(C^{d}_{\omega}) \cap \check{C}^{\omega}_{d} = \{P_1, \dots, P_r\}$. For $i = 1, \dots, r$ we denote

$$\mu_i := \mu^{\omega}(C^{\omega}_{d}, P_i) = \mu^{\omega}(\widetilde{C}^{\omega}_{d}, P_i).$$

Let $\widetilde{C}^{\omega}_{d} \cap \check{X}^{\omega} = \{P^{x}_{1}, \dots, P^{x}_{n^{x}}\}$. For $i = 1, \dots, n^{x}$, we denote

$$\mu^{x}_{i} := \mu^{\omega}(C^{\omega}_{d}, P^{x}_{i}), \quad \widetilde{\mu}^{x}_{i} := \mu^{\omega}(\widetilde{C}^{\omega}_{d}, P^{x}_{i}), \text{ and } m^{x}_{i} := m^{x}(\widetilde{C}^{\omega}, P^{x}_{i}).$$

Note that

$$\mu^{x}_{i} = \widetilde{\mu}^{x}_{i} + \varepsilon^{x} \frac{2m^{x}_{i} - 1}{d_{x}}.$$

Replacing the superindex x by y and z, we refer to the other axes.

Let us consider now the vertices. Let us denote $\varepsilon^{x,y} := 1$ (resp. 0) if $P_\omega^{x,y} \in \widetilde{C}_d^\omega$ (resp. otherwise). We denote

$$\mu_\omega^{x,y} := \mu^\omega(C_d^\omega, P_\omega^{x,y}), \quad \widetilde{\mu}_\omega^{x,y} := \mu^\omega(\widetilde{C}_d^\omega, P_\omega^{x,y}), \quad m_x^{x,y} := m_x^{x,y}(\widetilde{C}_d^\omega), \quad m_y^{x,y} := m_y^{x,y}(\widetilde{C}_d^\omega).$$

If the vertex is $P_\omega^{x,y}$, the corresponding Milnor number will be supposed to be zero. Let us also denote $\eta^{x,y} := 1$ (resp. 0) if $P_\omega^{x,y} \in C_d^\omega$ (resp. $\notin$). It is easily seen that

$$\eta^{x,y} = 1 - (1 - \varepsilon^{x,y})(1 - \varepsilon^x)(1 - \varepsilon^y).$$

Note that:

$$\mu_\omega^{x,y} = \varepsilon^{x,y}\left(\widetilde{\mu}_\omega^{x,y} - \frac{1}{p_z} + 2\frac{\varepsilon^x m_x^{x,y} + \varepsilon^y m_y^{x,y}}{p_z}\right) + \frac{2\varepsilon^x\varepsilon^y - \varepsilon^x - \varepsilon^y + \eta^{x,y}}{p_z}.$$

For the other vertices, we act in the same way.

We can relate weighted and standard projective planes via the covering $\rho : \mathbb{P}^2 \to \mathbb{P}^2_\omega$, given by $\rho([x : y : z]) = [x^{p^x} : y^{p^y} : z^{p^z}]_\omega$.

Proposition 3.4. The mapping ρ is of degree $p^x p^y p^z$ and unramified outside the axes. Each point in $\check{X}^\omega$ has $p_y p_z / d^x$ preimages and each vertex has only one preimage.

The plane curve $C_d := \rho^{-1}(\widetilde{C}_d^\omega)$ is of degree p with the following (possible) singular points:

(1) For each $i = 1, \ldots, r$ there are $p_x p_y p_z$ singular points over P_i, all of them with Milnor number equal to μ_i.

(2) For each $i = 1, \ldots, n^x$ there are $p_y p_z / d_x$ singular points over P_i^x, all of them with Milnor number equal to $d_x p_x \widetilde{\mu}_i^x + (p_x - 1)(m_i^x - 1)$. The intersection number of each point over P_i^x with the axis $x = 0$ equals m_i^x. Similar statement works for the other axes.

(3) If $\varepsilon^{x,y} = 1$, there is a singular point over $P_\omega^{x,y}$ with Milnor number

$$p_x p_y p_z \widetilde{\mu}_\omega^{x,y} + (p_x - 1)(p_y - 1) + p_y(p_x - 1)(m_x^{x,y} - 1) + p_x(p_y - 1)(m_y^{x,y} - 1).$$

The intersection number of C_d at this vertex with the axis $x = 0$ (resp. $y = 0$) equals $p_y m_x^{x,y}$ (resp. $p_x m_y^{x,y}$). Similar statements work for the other vertices. □

Using intersection numbers, we obtain:

Corollary 3.5. The degree of $\widetilde{C}_d^{\omega}$ satisfies:

$$p = \frac{p_y p_z}{d_x} \sum_{i=1}^{n^x} m_i^x + \varepsilon^{x,y} p_y m_x^{x,y} + \varepsilon^{x,z} p_z m_x^{x,z}.$$

A similar statement works for any permutation of the variables. □

Lemma 3.6. The Euler characteristic of $\check{C}_d^{\omega}$ equals

$$\frac{p(p_x + p_y + p_z - p)}{p_x p_y p_z} + \sum_{i=1}^{r} \mu_i + \sum_{i=1}^{n^x} \left(\widetilde{\mu}_i^x - \frac{1}{d_x} \right) + \sum_{i=1}^{n^y} \left(\widetilde{\mu}_i^y - \frac{1}{d_y} \right) + \sum_{i=1}^{n^z} \left(\widetilde{\mu}_i^z - \frac{1}{d_z} \right)$$
$$+ \varepsilon^{x,y} \left(\widetilde{\mu}_\omega^{x,y} - \frac{1}{p_z} \right) + \varepsilon^{x,z} \left(\widetilde{\mu}_\omega^{x,z} - \frac{1}{p_y} \right) + \varepsilon^{y,z} \left(\widetilde{\mu}_\omega^{y,z} - \frac{1}{p_x} \right). \qquad \square$$

Proof. Because of Proposition 3.4, we know that $\chi(\rho^{-1}(\check{C}_d^{\omega})) = p^x p^y p^z \chi(\check{C}_d^{\omega})$ and it is clear that $\chi(\rho^{-1}(\check{C}_d^{\omega}))$ is equal to $\chi(C_d)$ minus the number of points in the union of the axes. Recall that the Euler characteristic of a curve of degree p is equal to $p(3-p)$ plus the sum of the Milnor numbers of its points, that is, of its singular points. Using again Proposition 3.4, we obtain:

$$\chi(C_d) = p(3-p) + p_x p_y p_z \sum_{i=1}^{r} \mu_i + \frac{p_y p_z}{d_x} \sum_{i=1}^{n^x} \left(d_x p_x \widetilde{\mu}_i^x + (p_x - 1)(m_i^x - 1) \right)$$
$$+ \frac{p_x p_z}{d_y} \sum_{i=1}^{n^y} \left(d_y p_y \widetilde{\mu}_i^y + (p_y - 1)(m_i^y - 1) \right) + \frac{p_x p_y}{d_z} \sum_{i=1}^{n^z} \left(d_z p_z \widetilde{\mu}_i^z + (p_z - 1)(m_i^z - 1) \right)$$
$$+ \varepsilon^{x,y} \left(p_x p_y p_z \widetilde{\mu}_\omega^{x,y} + (p_x - 1)(p_y - 1) + p_y(p_x - 1)(m_x^{x,y} - 1) + p_x(p_y - 1)(m_y^{x,y} - 1) \right)$$
$$+ \varepsilon^{x,z} \left(p_x p_y p_z \widetilde{\mu}_\omega^{x,z} + (p_x - 1)(p_z - 1) + p_z(p_x - 1)(m_x^{x,z} - 1) + p_x(p_z - 1)(m_z^{x,z} - 1) \right)$$
$$+ \varepsilon^{y,z} \left(p_x p_y p_z \widetilde{\mu}_\omega^{y,z} + (p_y - 1)(p_z - 1) + p_z(p_y - 1)(m_y^{y,z} - 1) + p_y(p_z - 1)(m_z^{y,z} - 1) \right).$$

Then:

$$\begin{aligned}\frac{\chi(\rho^{-1}(\check{C}_d^{\omega}))}{p_x p_y p_z} &= \frac{p(3-p)}{p_x p_y p_z} + \sum_{i=1}^{r} \mu_i + \sum_{i=1}^{n^x}\left(\widetilde{\mu}_i^x - \frac{1}{d_x} + \frac{(p_x-1)m_i^x}{d_x p_x}\right)\\
&\quad + \sum_{i=1}^{n^y}\left(\widetilde{\mu}_i^y - \frac{1}{d_y} + \frac{(p_y-1)m_i^y}{d_y p_y}\right) + \sum_{i=1}^{n^z}\left(\widetilde{\mu}_i^z - \frac{1}{d_z} + \frac{(p_z-1)m_i^z}{d_z p_z}\right)\\
&\quad + \varepsilon^{x,y}\left(\widetilde{\mu}_\omega^{x,y} - \frac{1}{p_z} + \frac{(p_x-1)m_x^{x,y}}{p_x p_z} + \frac{(p_y-1)m_y^{x,y}}{p_y p_z}\right)\\
&\quad + \varepsilon^{x,z}\left(\widetilde{\mu}_\omega^{x,z} - \frac{1}{p_y} + \frac{(p_x-1)m_x^{x,z}}{p_x p_y} + \frac{(p_z-1)m_z^{x,z}}{p_y p_z}\right)\\
&\quad + \varepsilon^{y,z}\left(\widetilde{\mu}_\omega^{y,z} - \frac{1}{p_x} + \frac{(p_y-1)m_y^{y,z}}{p_x p_y} + \frac{(p_z-1)m_z^{y,z}}{p_x p_z}\right).\end{aligned}$$

Using Corollary 3.5, we can compute the terms involving $p_x - 1$ and we obtain $(p_x-1)p/p_x p_y p_z$. The same argument applies to y, z and we obtain the formula. ■

Proof of Theorem 3.2. We are going to apply Proposition 1.3. The strata to be considered are:

- ($S1$) Let $S_2 := \mathbb{P}^2_\omega(\widetilde{C}_d^\omega \setminus (X^\omega \cup Y^\omega \cup Z^\omega)$. It is clear that $\chi(S_2) = -\chi(\check{C}_d^\omega)$ and since the local equation is z^d, we have $\chi_{S_2} = d$.
- ($S2$) The local equation for the union of strata $S_1 := \check{C}_d^\omega \setminus \mathrm{Sing}(\widetilde{C}_d^\omega)$ is $z^d x$ and then, $\chi_{S_1} = 0$.
- ($S3$) Let $S_x := \check{X}^\omega \setminus \widetilde{C}_d^\omega$; we have $\chi(S_x) = -n^x$. Since the local equation is $z^d x^{\varepsilon^x}$, and this stratum is of index d_x, then

$$\chi_{S_x} = \frac{(1-\varepsilon^x)d}{d_x}.$$

  A similar formula applies for the other axes.
- ($S4$) Applying Lemma 1.4 to P_i, $i = 1, \ldots, r$ we have:

$$\chi_{P_i} = (d+k)\mu_i.$$

- ($S5$) We apply Lemmas 1.4 and 3.3 to P_i^x, $i = 1, \ldots, n^x$:

$$\chi_{P_i^x} = (d+k)\left(\widetilde{\mu}_i^x + \varepsilon^x \frac{2m_i^x - 1}{d_x}\right).$$

A similar formula applies for the other axes.

($S6$) For each point $P_\omega^{x,y}$, we apply Lemmas 1.4 and 3.3:

$$(d+k)\left(\varepsilon^{x,y}\left(\widetilde{\mu}_\omega^{x,y}-\frac{1}{p_z}+2\frac{\varepsilon^x m_x^{x,y}+\varepsilon^y m_y^{x,y}}{p_z}\right)+\frac{2\varepsilon^x\varepsilon^y-\varepsilon^x-\varepsilon^y+\eta^{x,y}}{p_z}\right)+\frac{(1-\eta^{x,y})d}{p_z}.$$

A similar formula applies for the other vertices.

Let us study the contribution of the different parts.

- The terms related with each P_i appear in several items. In ($S1$), we obtain $d\mu_i$. In ($S4$), we find $(d+k)\mu_i$. The final result is: $k\mu_i$.
- Let us consider now the terms related with P_i^x. In ($S1$), we have $-d(\widetilde{\mu}_i^x-(1/d_x))$. Adding term in ($S3$), we obtain $-d(\widetilde{\mu}_i^x-(\varepsilon^x/d_x))$. Summing with the term in ($S5$), we obtain:

$$k\left(\widetilde{\mu}_i^x+\varepsilon^x\frac{2m_i^x-1}{d_x}\right)+2\varepsilon^x d\frac{m_i^x}{d_x}=k\mu_i^x+2\varepsilon^x d\frac{m_i^x}{d_x}. \tag{3.1}$$

- Let us study what happens in $P_\omega^{x,y}$. The term in ($S1$) gives $-\varepsilon^{x,y}d(\widetilde{\mu}_\omega^{x,y}-(1/p_z))$. Combining with ($S6$):

$$k\mu_\omega^{x,y}+2\varepsilon^{x,y}d\frac{\varepsilon^x m_x^{x,y}+\varepsilon^y m_y^{x,y}}{p_z}+d\left(\frac{2\varepsilon^x\varepsilon^y-\varepsilon^x-\varepsilon^y+1}{p_z}\right). \tag{3.2}$$

Let us note that with the second term in (3.1) and with two second terms in (3.2) (and permutations), using Corollary 3.5, we obtain

$$2\varepsilon^x d\left(\sum_{i=1}^{n^x}\frac{m_i^x}{d_x}+\varepsilon^{x,y}\frac{m_x^{x,y}}{p_z}+\varepsilon^{x,z}\frac{m_x^{x,z}}{p_y}\right)=2\varepsilon^x\frac{dp}{p_y p_z}. \tag{3.3}$$

- Let us consider the remaining terms multiplied by $p_x p_y p_z$. In ($S1$), we have

$$\begin{aligned} dp(p-p_x-p_y-p_z)= \; & d(d-(\varepsilon^x p_x+\varepsilon^y p_y+\varepsilon^z p_z)) \\ & \times(d-((\varepsilon^x+1)p_x+(\varepsilon^y+1)p_y+(\varepsilon^z+1)p_z)). \end{aligned}$$

The terms in (3.3) give:

$$2d(\varepsilon^x p_x+\varepsilon^y p_y+\varepsilon^z p_z)(d-(\varepsilon^x p_x+\varepsilon^y p_y+\varepsilon^z p_z)).$$

If we sum these two terms, we obtain (recall that $\varepsilon^x = 0, 1$):

$$d(d - (\varepsilon^x p_x + \varepsilon^y p_y + \varepsilon^z p_z))(d + ((\varepsilon^x - 1)p_x + (\varepsilon^y - 1)p_y + (\varepsilon^z - 1)p_z)).$$

Let us take into account the third terms in (3.2) (and permutations):

$$dp_x p_y(2\varepsilon^x\varepsilon^y - \varepsilon^x - \varepsilon^y + 1) + dp_x p_z(2\varepsilon^x\varepsilon^z - \varepsilon^x - \varepsilon^z + 1) \\ + dp_y p_z(2\varepsilon^y\varepsilon^z - \varepsilon^y - \varepsilon^z + 1).$$

Summing up all these terms, we find $(d - p_x)(d - p_y)(d - p_z) + p_x p_y p_z$ and we finish the proof. ■

3.1 Euler characteristic formula

Let $\omega := (p_x, p_y, p_z) \in \mathbb{N}^3$ be a *weight* with $\gcd(p_x, p_y, p_z) = 1$. Let us denote $d_x := \gcd(p_y, p_z)$ and let us define in the same way d_y, d_z.

Let $\mathbb{P}^2_\omega$ be weighted projective space. The point $P^{x,y}_\omega := [0:0:1]_\omega$ is a *vertex* of $\mathbb{P}^2_\omega$; we define accordingly the other vertices. The curve $X^\omega := \{[x:y:z]_\omega \in \mathbb{P}^2_\omega \mid x = 0\}$ is an *axis* of $\mathbb{P}^2_\omega$; we denote $\check{X}^\omega := X^\omega \setminus \{P^{x,y}_\omega, P^{x,z}_\omega\}$. We define the axes Y^ω, Z^ω in the same way.

Let f be a non-zero ω-weighted-homogeneous polynomial of degree d and let $C^\omega \subset \mathbb{P}^2_\omega$ be the associated curve. With Lemma 3.6 we can compute the genus of the curve C^ω if it is irreducible. The smooth case was computed by Orlik and Wagreich [15, Proposition 3.5.1]. If C^ω coincides with one of the axis of the weighted projective space, then its genus is 0.

Otherwise consider

$$\check{C}^\omega := C^\omega \setminus \left(X^\omega \cup Y^\omega \cup Z^\omega\right)$$

and set $\operatorname{Sing}(C_\omega) \cap \check{C}^\omega = \{P_1, \ldots, P_r\}$.

Let $\widetilde{C}^\omega_d \cap \check{X}^\omega = \{P^x_1, \ldots, P^x_{n^x}\}$, and replacing the superindex x by y and z we refer to the other axes.

Let us consider now the vertices. Let us denote $\varepsilon^{x,y} := 1$ (resp. 0) if $P^{x,y}_\omega \in \widetilde{C}^\omega$ (resp. otherwise). For the other vertices, we act in the same way.

The Euler characteristic of the (possible singular) irreducible curve C^{ω} is given by the formula:

$$\chi(C^{\omega}) = \frac{d(p_x+p_y+p_z-d)}{p_xp_yp_z} + \sum_{i=1}^{r}\mu_i + \sum_{i=1}^{n^x}\left(\mu_i^x - \frac{1}{d_x}+1\right) + \sum_{i=1}^{n^y}\left(\mu_i^y - \frac{1}{d_y}+1\right)$$
$$+\sum_{i=1}^{n^z}\left(\mu_i^z - \frac{1}{d_z}+1\right) + \varepsilon^{x,y}\left(\mu_{\omega}^{x,y} - \frac{1}{p_z}+1\right) + \varepsilon^{x,z}\left(\mu_{\omega}^{x,z} - \frac{1}{p_y}+1\right)$$
$$+\varepsilon^{y,z}\left(\mu_{\omega}^{y,z} - \frac{1}{p_x}+1\right).$$

where:

- μ_i^* is the ω-*Milnor number* defined as $\mu^{\omega}(C^{\omega}, P) := \mu/\nu_P$;
- ν_P is the index of $(0, P)$, that is, it is the gcd of the weights of the non-zero coordinates of P.

If $P \in C^{\omega}$ can be expressed as $P = [x_0 : y_0 : 1]_{\omega}$ and C_z is the zero locus in $\mathbb{C}^2$ of $f(x, y, 1)$ then μ is the usual Milnor number of C_z at (x_0, y_0).

4 Example

In [19], Zariski proposed to study some open problems related with the (embedded) topology of a germ of a hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$ defined by the zero locus of a germ of a complex analytic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. He defined two germs $(V_1, 0) \subset (\mathbb{C}^n, 0)$ and $(V_2, 0) \subset (\mathbb{C}^n, 0)$ to be *topologically equisingular* if there is a local homeomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\phi(V_1) = V_2$.

A family of functions germs $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ holomorphically depending on the parameter t is said to be topologically trivial if there exists a family of homeomorphism germs $\varphi_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ depending continuously on the parameter such that $f_t \circ \varphi_t = f_0$.

The B-problem proposed by Zariski in [19, page 484] is the following: *if two analytic hypersurface germs are topologically equivalent then their tangent cones must be homeomorphic and the homeomorphism must respect the topological equisingularity type at any point*. Zariski proved that this is true if $n = 1$. In 2005, the second author [6] found a counterexample to this problem for a topologically equisingular family if $n = 5$.

The following example gives a negative answer to Question B of Zariski for germs of surface singularities in $\mathbb{C}^3$. We consider the holomorphic uni-parametric family $\{V_t\}$

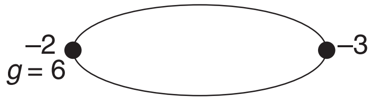

Fig. 1. The minimal resolution graph.

of hypersurface singularities defined as zero locus of $f_t = z^{12} + zy^3x + ty^2x^3 + x^6 + y^5$. The following properties hold:

(1) Every member of the family V_t defines a weighted-Lê–Yomdin singularity with respect to the weights $\omega = (\omega_x, \omega_y, \omega_z) = (2, 3, 1)$.

(2) Using Theorem 3.2, the Milnor number $\mu_t = 166$ for all t.

(3) The Minor number of the generic hyperplane section is changing: $\mu_0^2 = 18$ and $\mu_t^2 = 17$ for $t \neq 0$ (we have computed this with SINGULAR [8]). So the family is not Whitney-equisingular see [3, 18].

(4) The Newton polyhedra of V_0 and V_t are distinct from each other, but both are non-degenerate.

(5) The family f_t is topologically trivial by Parusiński criterion for families $f + tg$ (see [17]) or by Abderrahmane's main result in [1] which states: every μ-constant deformation of isolated singularities where the Newton polyhedra are non-degenerated is topologically trivial and equimultiple.

(6) One can compute the minimal embedded resolution graph; it is the same in both cases and it appears in Figure 1.
This implies that the abstract link is constant in the family. Therefore, the family is equisingular at the normalization in the sense of [7] and by the main result in the same paper the family is topologically trivial.

(7) The family of tangent cones is $C_t = \{y^2(zyx + tx^3 + y^3) = 0\}$. The cone C_0 is not homeomorphic to C_t because the reduction of C_t has only one singular point and the reduction of C_0 has two.

Remark 4.1. The tangent cones C_t are non-reduced curves. It would be interesting to find a counterexample with reduced tangent cones. □

The family $g_t := f_t + w^5$ answers negatively to Zariski's Question B if $n = 4$. This family has constant Milnor number by Thom–Sebastiani and therefore it is topologically trivial by Parusiński criterion. The family of tangent cones is $D_t := \{y^2(zyx + tx^3 + y^3) +$

$w^5 = 0\}$. The singular locus of D_t is independent of t and equals to $\Sigma := \{y = w = 0\}$. The generic transversal type at a generic point of Σ is (1) the curve singularity $y^2 + w^5$ for $t \neq 0$ and (2) the curve singularity $y^3 + w^5$ for $t = 0$. Any homeomorphism germ $\varphi : \mathbb{C}^5 \to \mathbb{C}^5$ which sends D_0 to D_t must leave Σ invariant because the Betti numbers of the local Milnor fibers are topological invariants of a germ by [11]. For the same reason, the homeomorphism cannot send a point of Σ where D_0 has general transversal type into a point of Σ where D_t has general transversal type.

Notice that in this case each of the tangent cones is irreducible and reduced.

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