

# On Twistor Solutions of the dKP Equation\*

Francisco Guil<sup>1</sup>, Manuel Mañas<sup>2</sup> and Luis Martínez Alonso<sup>3</sup>  
*Departamento de Física Teórica II, Facultad de Ciencias Físicas,  
 Universidad Complutense, 28040 Madrid, Spain*

<sup>1</sup>fguil@fis.ucm.es, <sup>2</sup>manuel@darboux.fis.ucm.es, <sup>3</sup>luism@fis.ucm.es

12th November 2013

## Abstract

The factorization problem for the group of canonical transformations close to the identity and the corresponding twistor equations for an ample family of canonical variables are considered. A method to deal with these reductions is developed for the construction classes of nontrivial solutions of the dKP equation.

*Key words:* Dispersionless integrable hierarchies, factorization problems, twistor methods.

*1991 MSC:* 58B20.

## 1 Introduction

This paper deals with the integration problem of the dispersionless Kadomtsev–Petviashvili (dKP) equation, the first member of the integrable hierarchy obtained from the ordinary KP hierarchy through the dispersionless limit. That limit coincides with the quasi-classical approximation for the underlying quantum space of differential operators of the KP equation which, for the dKP case, becomes the classical phase space endowed with a Poisson structure.

---

\*Partially supported by the DGESIC. Proyecto PB98-0821.

The theory and applications of the nonlinear models arising in the dispersionless (or quasiclassical) limits of the integrable systems of KdV-type have been active subjects of research for more than twenty years (see for example [1]-[14]). However, the theory of their solution methods seems far from being completed. Indeed, only for a few cases [15]-[17] the dispersionless limit of the inverse scattering method is available and *dispersionless versions* of ordinary direct methods like the  $\bar{\partial}$ -method are not yet fully developed [18].

In [3]-[4] Kodama and Gibbons provided a direct method for finding solutions of the dispersionless KP (dKP) equation and its associated dKP hierarchy of nonlinear systems. The main ingredient of their method is the use of reductions of the dKP hierarchy formulated in terms of hydrodynamic-type equations. An alternative direct method for solving the dKP hierarchy from its reductions was recently proposed in [19]. It is based on the characterization of reductions ((and hodograph solutions)) of the dKP hierarchy by means of certain systems of first-order partial differential equations. In [5] Takasaky and Takebe showed that the factorization problem for the group of canonical transformations in two-dimensional phase space provides a direct solution method for the dKP equation. Furthermore, that method has a twistor interpretation. The group of canonical transformations acts on the phase space, viewed as a part of the Lie algebra of that group, through the adjoint representation. Hence, the factorization problem in the group is represented in the phase space too. That representation is called the twistor formulation of the factorization problem and admits several geometric interpretations. This twistor formulation allows for the use of the canonical formalism of Classical Mechanics in the resolution of the factorization problem for solving the dKP equation. The canonical formalism enters in the form of generating functions for canonical transformations. When expressed in terms of appropriate variables, these generating functions provide us with a trivialization of the group structure in the sense that transform right-derivatives in the group of canonical transformations into ordinary derivatives. This seems to be a relevant fact in the whole integration scheme.

The Hamilton–Jacobi method of integration of the canonical equations appears in the present context of multi-time Hamiltonian formalism as a trivialization, again, of the group structure. The zero-curvature condition, which holds for the right-differential of a function in the group, proves to be equivalent to the requirement that the Poincaré–Cartan action 1-form associated to that right-differential be a closed differential form. The dependence of the momentum in coordinates and time, the main feature of the Hamilton–

Jacobi theory, compensates the non-commutative Poisson structure of the phase space and transforms the zero-curvature condition in an equivalent closed differential 1-form which is the differential of the mechanical action of the system.

In this paper a method for solving the dKP equation based in the resolution of the twistor equations is given. It is concerned with a class of canonical transformations which can be considered as defining the initial conditions for the factorization problem. One of the main ingredients in our construction lays in the observation that, regarding the twistor equations, negative powers in the momentum variable can be substituted by negative powers in the Lax function. This equivalence furnishes a method that allows for the reduction of the twistor equations, in the class of the chosen canonical variables, to a system of first-order ordinary differential equations for a finite number of the coefficients of a generating function; however, it reduces to an algebraic system in the simplest cases.

Our main result consists in the characterization and construction of a class of canonical transformations, the initial conditions for the factorization problem or twistor data in the terminology of [5], for which the twistor equations can be solved explicitly. Through such resolution we obtain the corresponding solutions of the dKP equation which are described by means of the generating function of the canonical transformation under consideration. The generating functions are characterized as series of fractionary powers in the momentum variable.

The organization and content of the paper are as follows. Next, in §2 we revisit the twistor equations from the factorization problem point of view. We first present the factorization problem in the group of canonical transformations close to the identity in a two-dimensional phase space. Then, the twistor equations are derived. We continue by emphasizing the role of generating functions in the factorization problem and the relevance of the expansions in negative powers of the Lax function. At the end of the section some of the symmetries of the dKP are rederived. In §3 we consider the general case characterized by canonical variables  $X$  and  $P$  of finite order at  $p = \infty$ . In particular, generating functions for these reductions are found and explicit solutions of the dKP equation are constructed.

## 2 The twistor equations

### 2.1 The factorization problem

Let  $\mathfrak{g}$  be the Lie algebra of functions  $F(p, x)$  on the two complex variables  $p$  and  $x$  given as the sum of two subalgebras  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$  which are the spaces of analytic functions on the variable  $p$  at  $p = 0$  and  $p = \infty$ , respectively, with a common domain of definition. Functions in  $\mathfrak{g}_-$  vanish at  $p = \infty$  and the commutator in  $\mathfrak{g}$  is defined by the Poisson bracket

$$\{F_1, F_2\} = \frac{\partial F_1}{\partial p} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial p}.$$

Let  $L$  be an element in  $\mathfrak{g}$  that depends on new variables  $t_2, t_3, \dots$  and has the prescribed form

$$L(p, x, t_2, t_3, \dots) = p + \sum_{j \geq 1} u_j(x, t_2, t_3, \dots) p^{-j}.$$

As usual, define the functions  $P_n$ ,  $n = 2, 3, \dots$  as the projections on  $\mathfrak{g}_+$  of the positive powers of  $L$ ,  $P_n = L^n|_+$  in terms of which the Lax–Sato hierarchy is given by the infinite set of equations

$$\frac{\partial L}{\partial t_n} = \{P_n, L\},$$

for  $n = 2, 3, \dots$  that in particular imply the dKP equation for the function  $u = 2u_1$

$$\left(u_t - \frac{3}{2}uu_x\right)_x = \frac{3}{4}u_{yy}$$

where  $u_1$  is the coefficient of  $p^{-1}$  in  $L$  and we have set  $t_2 = y, t_3 = t$ .

The Lax–Sato equations provide, in their first and more usual interpretation, the compatibility conditions for the vanishing of the curvature of the connection

$$\omega_+ := \sum_{n \geq 2} P_n dt_n.$$

For that differential 1-form one finds

$$d\omega_+ = \frac{1}{2}\{\omega_+, \omega_+\} = \sum_{m < n} \{P_m, P_n\} dt_m \wedge dt_n$$

where the dKP equation is represented by  $\partial_3 P_2 - \partial_2 P_3 + \{P_2, P_3\} = 0$ . We can think also of the previous system as a collection of Hamiltonian flows in the phase space with canonical coordinates  $(p, x)$ . There is an infinite set of commuting flows defined by the Hamiltonians  $H_n = -P_n$ ,  $n = 2, 3, \dots$

$$\frac{\partial x}{\partial t_n} = \frac{\partial H_n}{\partial p}, \quad \frac{\partial p}{\partial t_n} = -\frac{\partial H_n}{\partial x},$$

whose compatibility conditions are precisely the equations of the dKP hierarchy for the potentials in the function  $L$ .

The structure underlying the Lax–Sato system can be conveniently understood by means of the group of canonical transformations connected with the identity. A transformation  $(p, x) \rightarrow (P, X)$  is canonical ( $dP \wedge dX = dp \wedge dx$ ) and connected with the identity if its of the form

$$\begin{aligned} P &= \exp(\text{ad}K).p = p - K_x - \frac{1}{2}\{K, K_x\} - \dots, \\ X &= \exp(\text{ad}K).x = x + K_p + \frac{1}{2}\{K, K_p\} + \dots, \end{aligned}$$

where  $K = K(p, x)$  is a function in the phase space and

$$K_x := \frac{\partial K}{\partial x}, \quad K_p := \frac{\partial K}{\partial p}.$$

These formulas admit also an interpretation in terms of the right-derivatives of a group element in the local Lie group  $G$  defined by the Lie algebra  $\mathfrak{g}$ . Let  $G = \exp \mathfrak{g}$  be the Lie group of canonical transformations connected with the identity. Then, for the action of an element  $k = \exp K$  on the canonical coordinates  $(p, x)$  we obtain the expressions

$$\begin{aligned} P &= \text{Ad}k.p = \exp(\text{ad}K).p = p - k_x k^{-1}, \\ X &= \text{Ad}k.x = \exp(\text{ad}K).x = x + k_p k^{-1}. \end{aligned}$$

Where we have used the right-differential

$$d k k^{-1} = \sum_{n \geq 0} \frac{1}{(n+1)!} (\text{ad}K)^n dK.$$

The relevance of this group in the theory of the dKP equation appears in connection with the formulation of the Lax–Sato system as a factorization

problem. Let  $G_{\pm} = \exp \mathfrak{g}_{\pm}$  be the local Lie groups with Lie algebras  $\mathfrak{g}_{\pm}$ , respectively, viewed as subgroups of the group  $G$ . Let  $t(p)$  be an element in  $\mathfrak{g}_+$  of the form

$$t(p) := t_2 p^2 + t_3 p^3 + \dots$$

and define the canonical transformation  $\psi = (\exp t(p))k$ . The factorization problem in the group  $G$  with respect to the subgroups  $G_{\pm}$  is then the equation  $\psi = \psi_-^{-1} \psi_+$  for the representation of a given element  $\psi$  as the product of a pair of group elements  $\psi_{\pm}$  in the subgroups  $G_{\pm}$  respectively. In particular, for the element  $\psi$  defined above we have the equation

$$e^{t(p)} k = \psi_-^{-1} \psi_+ \tag{2.1}$$

the solutions of which depend both on the time variables  $\{t_2, t_3, \dots\}$  and on the initial condition given by the constant element  $k$ . More precisely, the element  $k$  enters in the solutions  $\psi_{\pm}$  as the representative of a point in the double coset space  $\Gamma_- \backslash G / G_+$  where  $\Gamma_-$  is the centralizer of  $\exp t(p)$  in  $G_-$ .

In order to construct the Lax–Sato equations through the factorization problem we take the right-differential with respect to the time variables in (2.1) from which we get

$$d\psi_- \psi_-^{-1} + \text{Ad}\psi_- \cdot dt(p) = d\psi_+ \psi_+^{-1}. \tag{2.2}$$

If we identify the function  $L$  in  $\mathfrak{g}$  with a point in the orbit of  $p$  for the action of  $G_-$ ,  $L = \text{Ad}\psi_- \cdot p$ , the projection on  $\mathfrak{g}_+$  of

$$\text{Ad}\psi_- \cdot dt(p) = \sum_{n \geq 2} \text{Ad}\psi_- p^n dt_n = \sum_{n \geq 2} L^n dt_n \tag{2.3}$$

implies that the differential form  $\omega_+ = d\psi_+ \psi_+^{-1} = \sum_{n \geq 2} L^n|_+ dt_n$  is of zero curvature and hence the dKP equation obtains from the factorization problem. The meaning of these equations is that positive and negative projections of the form

$$\omega := \sum_{n \geq 2} L^n dt_n$$

in (2.3),  $\omega = \omega_+ - \omega_-$ , are of zero-curvature,  $\omega_{\pm} = d\psi_{\pm} \psi_{\pm}^{-1}$ , as follows from (2.2).

## 2.2 Twistor equations

The canonical formalism allows for a reformulation of the factorization problem (2.1) in the twistor language [5]. Let  $(P, X)$  be new canonical variables defined by the element  $k$  in (2.1),  $P = \text{Ad}k.p$ ,  $X = \text{Ad}k.x$ , and define in addition the canonical pair  $(L, M)$  through the action of the canonical transformation  $\psi_- \exp t(p)$ ,

$$L = \text{Ad}(\psi_- e^{t(p)}).p, \quad M = \text{Ad}(\psi_- e^{t(p)}).x. \quad (2.4)$$

For these variables, that are constants of motion for the Hamiltonian flows with Hamiltonians  $H_n = -P_n = L^n|_+$ ,  $n = 2, 3, \dots$ :

$$\frac{\partial L}{\partial t_n} + \{H_n, L\} = 0, \quad \frac{\partial M}{\partial t_n} + \{H_n, M\} = 0,$$

one finds the expressions

$$L = p - \frac{\partial \psi_-}{\partial x} \psi_-^{-1}, \quad M = \frac{\partial t(L)}{\partial L} + x + \frac{\partial \psi_-}{\partial p} \psi_-^{-1}. \quad (2.5)$$

As a consequence of formula (2.1) we obtain the relation

$$\psi_- e^{t(p)} k = \psi_+$$

whose action on the pair of canonical variables  $(p, x)$  results in an equivalent description of the factorization problem, namely

$$P(L, M) = \text{Ad}\psi_+.p, \quad X(L, M) = \text{Ad}\psi_+.x$$

which finally lead to the equations for the function  $L$

$$P(L, M)|_- = 0, \quad X(L, M)|_- = 0 \quad (2.6)$$

that represent the twistor form of the factorization problem (2.1).

## 2.3 Generating functions

The most effective method to deal with canonical transformations is furnished by the formulation in terms of their generating functions. If we want to describe the canonical variables  $(L, M)$  of (2.5) it proves to be convenient to define a generating function for this transformation  $\Phi(L, x)$  such

that the differential of it is given by  $d\Phi(L, x) = M dL + p dx$ . For such a transformation one finds the expression

$$\Phi(L, x) = xL + t(L) + \phi(L, x) \quad (2.7)$$

where  $t(L) := t_2L^2 + t_3L^3 + \dots$  and  $\phi(L, x)$  is a negative power series in  $L$ ,  $\phi(L, x) = \sum_{n \geq 1} \phi_n(x)L^{-n}$ . With this definition we deduce the relations

$$p = \frac{\partial \Phi}{\partial x} = L + \frac{\partial \phi}{\partial x} \quad (2.8)$$

and

$$M = \frac{\partial \Phi}{\partial L} = x + \frac{\partial t}{\partial L} + \frac{\partial \phi}{\partial L}. \quad (2.9)$$

These formulas are to be compared with the corresponding expressions for  $L$  and  $M$  as given by (2.5) that imply the relations

$$\frac{\partial \phi}{\partial x}(L, x) = \frac{\partial \psi_-}{\partial x} \psi_-^{-1}(p, x), \quad \frac{\partial \phi}{\partial L}(L, x) = \frac{\partial \psi_-}{\partial p} \psi_-^{-1}(p, x) \quad (2.10)$$

provided  $p$  and  $L$  satisfy  $p = L + \phi_x$  as in (2.8).

From these equations we learn that functions  $\phi$  and  $\psi_-$  are equivalent negative power series to describe the solutions of the factorization problem. The main advantage of  $\phi$  as compared with  $\psi_-$  is that it allows for the use of ordinary partial derivatives instead the right-invariant derivatives necessary when dealing with  $\psi_-$ . Such a simplification is achieved by choosing the mixed independent variables  $(L, x)$  for the generating function  $\Phi$  of the canonical transformation. Greater simplification also is gained in describing the coefficients of  $\phi(L, x)$ , for if we represent  $\psi_- = \exp \Psi$  in terms of a negative power series  $\Psi = \sum_{j \geq 1} \Psi_j p^{-j}$ , the right-differential  $\omega_- = d\psi_- \psi_-^{-1}$  is then

$$\omega_- = d\Psi + \frac{1}{2}\{\Psi, d\Psi\} + \dots$$

Integrating this equation along a conveniently chosen closed path gives

$$d\Psi_1 = \frac{1}{2\pi i} \int \omega_-(p) dp,$$

from which we deduce, taking into account (2.2) and the definition of  $\omega$ , the relation

$$d\Psi_1 = -\frac{1}{2\pi i} \int \omega(p) dp = \frac{1}{2\pi i} \int \omega(L) \sum_{k \geq 1} kL^{-k-1} \phi_{kx} dL$$

after we change the integration variable according to formula (2.8). Upon substitution of  $\omega(L) = \sum_{j \geq 2} L^j dt_j$  we finally get the relations

$$\frac{\partial \Psi_1}{\partial t_k} = k \phi_{kx}$$

for  $k = 2, 3, \dots$  besides  $\Psi_1 = \phi_1$  that follows from (2.10).

## 2.4 $L^{-1}$ expansions for the twistor equations

The twistor form for the factorization problem as given by eqs. (2.6) implies for the unknown function  $\phi(L, x)$  the equations

$$P(L, x + \frac{\partial t}{\partial L} + \frac{\partial \phi}{\partial L}) \Big|_- = 0 \quad (2.11)$$

and

$$X(L, x + \frac{\partial t}{\partial L} + \frac{\partial \phi}{\partial L}) \Big|_- = 0. \quad (2.12)$$

In order to obtain a solution  $\phi(L, x)$  for a fixed pair of canonical variables  $(P, X)$  we should begin computing the negative parts, as power series in the variable  $p$ , of both of the eqs. (2.11) and (2.12). In this context the following observation seems to be crucial in the whole procedure of resolution developed in the sequel. Due to the connection (2.8) between the variables  $p$  and  $L$  a negative power series in  $p$  can be written as a negative power series in  $L$ , since  $p^{-1} = L^{-1}(1 + L^{-1}\phi_x)^{-1}$ , and reciprocally. The vanishing of the negative parts of  $P$  and  $X$  as power series in  $p$  is therefore equivalent to the vanishing the negative parts of  $P$  and  $X$  as power series in  $L$ . To take advantage of the simplification gained in solving eqs. (2.11) and (2.12) viewed as power series in  $L$ , mainly because they are naturally expressed in the variable  $L$  rather than in  $p$ , we should be able to compute the negative part of any power of  $L$ .

We shall presently proceed to compute the negative part of a power series in the variable  $L$ ; i.e., its projection onto the subalgebra  $\mathfrak{g}_-$  expressed in the variable  $L$ . In that case we obtain for the projection of a negative power of  $L$  the same negative power  $L^{-k}|_- = L^{-k}$  for  $k = 1, 2, \dots$  while for a positive power of  $L$  we find the recurrent formula

$$L^k|_- = [L^k - (L + \phi_x)^k]_- \quad (2.13)$$

for  $k = 0, 1, 2, \dots$ . This relation follows from the identity  $L^k|_- = [L^k - p^k]_-$  after substitution of  $p = L + \phi_x$  according to (2.8). To see why this is a recurrent formula we develop the r.h.s. member according to the binomial formula from which we get the desired relation

$$L^k|_- = -kL^{k-1}\phi_x|_- - \binom{k}{2}L^{k-2}\phi_x^2|_- - \dots - \phi_x^k.$$

## 2.5 Generating functions and symmetries

As we said before, solutions of the factorization problem furnish solutions of the dKP equation. The factorization problem is solved once we know the function  $\psi_-$  or equivalently, as we have just seen, the function  $\phi$  from which we obtain the dKP solution  $u$  as

$$u(x, t_2, t_3, \dots) = -2\phi_{1x}(x, t_2, t_3, \dots). \quad (2.14)$$

This formula follows from the relation  $L^2|_+ = p^2 - 2\phi_{1x} = p^2 + u$  which is a direct consequence of (2.8) and the definition of  $L$ .

The freedom allowed by the dKP equation for the definition of the solution and coordinates can be conveniently described in terms of the action of the associated mechanical system. In Hamilton–Jacobi theory, the zero-curvature equation in phase space for the 1-form  $\omega_+ = \sum_{n \geq 2} P_n dt_n$  defined by the Hamiltonians  $H_n = -P_n$ , transforms in the condition that the Poincaré–Cartan 1-form  $p dx + \omega_+$  be a closed form in configuration space. The action  $S(x, t)$  is thereby locally defined according to the relation

$$dS = p dx + \omega_+$$

and the Poisson structure disappears. In particular, the dKP equation follows from

$$dS = p dx + (p^2 + u) dy + (p^3 + \frac{3}{2}up + v) dt. \quad (2.15)$$

Incidentally, the function  $S(x, y, t)$  satisfies a modified dKP equation

$$(S_t + \frac{1}{2}S_x^3)_x = \frac{3}{2}S_y S_{xx} + \frac{3}{4}S_{yy},$$

which transforms into the dKP equation through the Miura map,

$$u = S_y - S_x^2$$

as a consequence of (2.15). Going back to the definition of coordinates, it is readily observed that (2.15) is kept as a closed form under the transformations:

$$x = \tilde{x} + \alpha(t), \quad y = \tilde{y}, \quad t = \tilde{t} \quad (2.16)$$

$$\tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) = u(x, y, t) + \frac{2}{3}\dot{\alpha} \quad (2.17)$$

$$x = \tilde{x} + \frac{2}{3}\dot{\beta}\tilde{y} \quad y = \tilde{y} + \beta, \quad t = \tilde{t} \quad (2.18)$$

$$p = \tilde{p} + \frac{1}{3}\dot{\beta}$$

$$\tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) = u(x, y, t) - \frac{2}{9}\dot{\beta}^2 + \frac{4}{9}\ddot{\beta}\tilde{y} \quad (2.19)$$

and

$$x = \dot{\gamma}^{1/3}\tilde{x} + \frac{2}{9}\dot{\gamma}^{-2/3}\ddot{\gamma}\tilde{y}^2 \quad y = \dot{\gamma}^{2/3}\tilde{y}, \quad t = \gamma(\tilde{t}) \quad (2.20)$$

$$p = \dot{\gamma}^{-1/3}(\tilde{p} - \frac{2}{9}\dot{\gamma}^{-1}\ddot{\gamma}\tilde{y})$$

$$\tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) = \dot{\gamma}^{2/3}u(x, y, t) + \frac{2}{9}\dot{\gamma}^{-1}\ddot{\gamma}\tilde{x} + \frac{4}{27}(\dot{\gamma}^{-1}\gamma^{(3)} - \frac{4}{3}\dot{\gamma}^{-2}\ddot{\gamma}^2)\tilde{y}^2. \quad (2.21)$$

This was stated in [14] where these symmetries were obtained for the space-time metric used to derive the dKP equation.

## 3 Reductions and the resolution of the the twistor equations

### 3.1 Reductions

To continue the analysis of the twistor system (2.11), (2.12) we should make an explicit choice of the canonical variables  $P, X$ , which can be understood as reductions of the general situation. In what follows we shall consider canonical variables  $P, X$  of the form

$$P(p, x) = \sum_{k \geq 0} a_{m-k}(\rho)p^{m-k} \quad (3.1)$$

and

$$X(p, x) = \sum_{k \geq 0} a_{n-k}(\rho) p^{n-k} \quad (3.2)$$

where the power series depend on  $x$  through the new variable

$$\rho := \frac{x}{h'(p)}, \quad h(p) := p^{r+1} e^{R_-(p)} \quad (3.3)$$

defined in terms of the arbitrary negative power series in  $p$

$$R_-(p) := \sum_{j \geq 1} c_j p^{-j}.$$

Some comments are in order about the definitions made above. The structure of the transformation is fixed by a triple of positive integers  $\{m, n, r\}$ . The first two of them  $m, n$  determine the positive degree in  $p$  of the new variables  $P = a_m p^m + \dots$  and  $X = b_n p^n + \dots$  where the dots denote terms that contains of lower powers in  $p$ . The third integer  $r$  determines both the definition of the variable  $\rho$  as well as the degree of the function  $t(p)$  that we now take as a polynomial of degree  $r + 1$  in  $p$ ,

$$t(p) = t_{r+1} p^{r+1} + t_r p^r + \dots + t_2 p^2. \quad (3.4)$$

Regularity conditions on the coefficients  $a_j, b_j$  will be assumed in each concrete case. At this point we shall postpone the explicit construction of the canonical transformation  $(p, x) \rightarrow (P, X)$  to the next section and concentrate in the method for solving the twistor equations for the cases under consideration.

The first step in that direction consists in the substitution in (3.1) and (3.2) of  $(p, x)$  by  $(L, M)$  to get new series in  $L$  that we now denote by

$$F(L, x) := P(L, M) = \sum_{k \geq 0} a_{m-k}(\rho) L^{m-k} \quad (3.5)$$

and

$$G(L, x) := X(L, M) = \sum_{k \geq 0} b_{n-k}(\rho) L^{n-k} \quad (3.6)$$

where the variable  $\rho$  is, of course,  $\rho = M/h'(L)$  for which, taking into account the expression (2.9) for  $M$ , we find the formula

$$\rho = \frac{1}{h'(L)} \left( \frac{\partial t}{\partial L} + x + \phi_L \right)$$

so that  $\rho$  is analytic in  $L$  at  $L = \infty$ . We assume generically that the coefficients  $a_j(\rho)$  and  $b_j(\rho)$  are analytic functions of  $L$  at  $L = \infty$ . With that hypothesis both  $F$  and  $G$  will continue having expressions as power series in  $L$  with degrees  $m$  and  $n$  respectively. The assumptions made in defining the canonical variables  $(P, X)$  are enough to guarantee that the number of positive powers of  $L$  appearing in  $F(L, x)$  and  $G(L, x)$  is in both cases finite. This is an important requisite to carry out the present construction.

The structure of the system of equations (2.11), (2.12) in the present situation follows from the analysis of the series coefficients as we shall next show. Since it is assumed the existence of power series expansions for the coefficients  $a_k(\rho), b_k(\rho)$ , we have the series

$$a_k(\rho) = \sum_{j \geq 0} a_{kj}(x, t) L^{-j}, \quad b_k(\rho) = \sum_{j \geq 0} b_{kj}(x, t) L^{-j}$$

where  $t = (t_2, t_3, \dots, t_{r+1})$  represents the time variables. It is easy to see that the coefficients in these series are functions of the form

$$a_{kj}(x, t) = a_{kj}[x, t, \phi_1, \dots, \phi_{j-r-1}], \quad b_{kj}(x, t) = b_{kj}[x, t, \phi_1, \dots, \phi_{j-r-1}]$$

from which we deduce for  $F(L, x)$  and  $G(L, x)$  in (3.5), (3.6) expansions of the form

$$F(L, x) = a_{m,0} L^m + (a_{m,1} + a_{m-1,0}) L^{m-1} + (a_{m,2} + a_{m-1,1} + a_{m-2,0}) L^{m-2} + \dots$$

and

$$G(L, x) = b_{n,0} L^n + (b_{n,1} + b_{n-1,0}) L^{n-1} + (b_{n,2} + b_{n-1,1} + b_{n-2,0}) L^{n-2} + \dots$$

that imply the expressions

$$F(L, x) = \sum_{k \geq 0} f_k L^{m-k}, \quad G(L, x) = \sum_{k \geq 0} g_k L^{n-k}. \quad (3.7)$$

We have thus obtained the formulas

$$f_k(x, t) = f_k[x, t, \phi_1, \dots, \phi_{k-r-1}], \quad g_k(x, t) = g_k[x, t, \phi_1, \dots, \phi_{k-r-1}]$$

displaying the number of functions  $\phi$  contained in each coefficient. As in (2.13) we continue with the notation  $F|_-$  and  $G|_-$  for the projections on  $g_-$ , the negative part, of the series  $F$  and  $G$  of (3.7).

**Definition 3.1.** For the negative parts of the series (3.7) we define the coefficients  $F_k$  and  $G_k$  by the series

$$F|_{-} = \sum_{k \geq 1} F_k L^{-k}, \quad G|_{-} = \sum_{k \geq 1} G_k L^{-k}. \quad (3.8)$$

For instance, for  $F_1$  we find

$$F_1 = f_0 L^m|_{-1} + f_1 L^{m-1}|_{-1} + \cdots + f_{m-1} L|_{-1} + f_{m+1} \quad (3.9)$$

where  $L^k|_{-1}$  is the coefficient of  $L^{-1}$  in  $L^k|_{-}$  as given by (2.13). We are now in a position to formulate the main result of the present paper.

**Theorem 3.1.** Assume the factorization problem (2.1) has a solution described by the twistor equations corresponding to (3.7)

$$F(L, x)|_{-} = 0, \quad G(L, x)|_{-} = 0 \quad (3.10)$$

for the canonical transformation defined by (3.1), (3.2). Then, the solution  $u(x, t) = -2\phi_{1x}$ , as given by (2.14), for the dKP hierarchy can be found solving for  $\phi_1$  the nonlinear system of  $m+n-2$  ordinary differential equations for the  $m+n-2$  unknowns  $\phi_1, \phi_2, \dots, \phi_{m+n-2}$ ,

$$F_1 = F_2 = \cdots = F_{n-1} = 0, \quad (3.11)$$

$$G_1 = G_2 = \cdots = G_{m-1} = 0, \quad (3.12)$$

determined by the coefficients in (3.8).

*Proof.* Since the factorization problem is solvable there is a negative power series in  $L$ ,  $\phi(L, x)$ , defining the canonical transformation (2.7), and being a solution of (3.10). As follows from the expression for  $L^m|_{-1}$  in (2.13) this is a polynomial on the first  $m$  coefficients of  $\phi(L, x)$ ,  $\phi_{1x}, \phi_{2x}, \dots, \phi_{mx}$ , hence the coefficient  $F_1$  of  $F$  in  $L^{-1}$  (3.9) determines an equation of the form

$$F_1(x, t, \phi_1, \dots, \phi_{m-r}, \phi_{1x}, \dots, \phi_{mx}) = 0$$

where we have taken into account the structure of the coefficients  $f_j$  previously considered. By arguments of the same type one finds that  $F_k$  is of the form

$$F_k(x, t, \phi_1, \dots, \phi_{m+k-r-1}, \phi_{1x}, \dots, \phi_{m+k-1,x}) = 0$$

while for  $G$  we can write

$$G_k(x, t, \phi_1, \dots, \phi_{n+k-r-1}, \phi_{1x}, \dots, \phi_{n+k-1,x}) = 0.$$

Therefore, the first  $n-1$  equations for  $F$ , (3.11), and the first  $m-1$  equations for  $G$ , (3.12), define a nonlinear system of first-order ordinary differential equations containing as many equations as unknowns and hence our assertion follows.  $\square$

In practice, for the examples that one can reasonably compute, things become even simpler. The sought function  $\phi_{1x}$  can be found *algebraically* from system (3.11), (3.12). This is obviously always the case for  $r \geq m+n-2$  but it needs not to be so if  $r < m+n-2$ . An alternative interpretation of the theorem is as a *factorization criterion* for the problem (2.1). For a given canonical transformation  $(p, x) \rightarrow (P, X)$  one does not know in general whether there is a solution for the system (3.10). But if a solution  $\phi_{1x}$  determined by (3.11), (3.12) gives a solution  $u = -2\phi_{1x}$  of the dKP hierarchy, what one can check at least for the dKP equation, then the whole  $\phi(L, x)$  can be recovered. The question of whether or not the canonical transformation is factorizable, in order to the system (3.11), (3.12) admits a solution, does not seem a serious obstruction. For if we take transformations depending on arbitrary functions and parameters generically enough, there will be special values for the arbitrary data for which the solution ceases to exist. In that case the solution  $u$  of the dKP equation becomes singular for these special values of the free data.

### 3.2 The canonical transformation and the reduction

From what we have seen, one of the main ingredients in the whole procedure allowing for the construction of solutions in the twistor context is the canonical transformation  $(p, x) \rightarrow (P, X)$  in Section 2. Thus we need a description of the chosen canonical variables (3.1), (3.2) which are determined by the differential equation  $dP \wedge dX = dp \wedge dx$  that in terms of the variables  $(p, \rho)$ , where  $\rho$  is defined by (3.3), becomes:

$$dP \wedge dX = h'(p) dp \wedge d\rho.$$

Direct substitution of the series (3.1), (3.2) for  $P$  and  $X$  in this equation leads to a recurrent system of ordinary differential equations for the coefficients  $b_k(\rho)$  in terms of the arbitrarily given coefficients  $a_k(\rho)$  and viceversa.

The most effective method of integration for these equations is furnished by a generating function for the canonical transformation, namely the function  $J(P, \rho)$  the differential of which is given by

$$X dP + h(p) d\rho = dJ(P, \rho).$$

The differential of this relation is the primitive equation for  $P$  and  $X$  that will be identically fulfilled provided we have a solution for the implicit equations

$$h(p) = \frac{\partial J}{\partial \rho}, \quad X = \frac{\partial J}{\partial P} \quad (3.13)$$

in terms of the arbitrary function  $J(P, \rho)$ . The degree of arbitrariness for the generating function  $J(P, \rho)$  is determined by the form of the new variables  $(P, X)$  fixed by (3.1), (3.2). One should distinguish here between two cases. For a given set of positive integers  $\{m, n, r\}$  assume first we have  $m+n \geq r+2$ . In that case, it is easy to see that we shall obtain the correct dependence of  $P$  and  $X$  on the variables  $(p, \rho)$  if we define the function  $J(P, \rho)$  according to the formula

$$J(P, \rho) = \sum_{k=r+2}^{m+n} \gamma_k P^{\frac{k}{m}} + \sum_{k \geq 0} J_{r+1-k}(\rho) P^{\frac{r+1-k}{m}}. \quad (3.14)$$

Here the fractionary powers are defined in terms of a fixed  $m$ th-root of  $P$ , the coefficients  $\gamma_k$  are arbitrary constants with the restriction  $\gamma_{n+m} \neq 0$  for the coefficient of the leading term. Analogously, for the arbitrary functions  $J_k(\rho)$  we impose the condition that the derivative of the first term be different from zero,  $J'_{r+1}(\rho) \neq 0$ . With these assumptions we shall prove the existence of the announced canonical variables (3.1), (3.2). Direct substitution of (3.14) in equations (3.13) gives the equation for  $P$

$$h(p) = \sum_{k \geq 0} J'_{r+1-k}(\rho) P^{\frac{r+1-k}{m}} \quad (3.15)$$

and also defines the variable  $X$  as

$$X = \frac{1}{m} \sum_{k=r+2}^{m+n} k \gamma_k P^{\frac{k-m}{m}} + \frac{1}{m} \sum_{k \geq 0} (r+1-k) J_{r+1-k}(\rho) P^{\frac{r+1-m-k}{m}}. \quad (3.16)$$

In order to write the explicit form of the sought solution  $P$  to (3.15) the variable  $P$  can be conveniently represented as follows,

$$P(p, \rho) = a_m(\rho) p^m e^{A-(p, \rho)},$$

where  $A_-(p, \rho)$  is a negative power series in  $p$ ,

$$A_-(p, \rho) = \sum_{k \geq 1} A_k(\rho) p^{-k}.$$

The coefficients  $a_k(\rho)$  for the series (3.1) defining  $P$  obtain from the series  $A_-(p, \rho)$  which in turn is computed through (3.15) according to the equation

$$e^{R_-} = J'_{r+1} a_m^{\frac{r+1}{m}} e^{\frac{r+1}{m} A_-} + J'_r a_m^{\frac{r}{m}} e^{\frac{r}{m} A_-} p^{-1} + \dots$$

The Taylor series in the variable  $\xi = p^{-1}$  at  $\xi = 0$  gives for the first coefficients the formulas

$$a_m(\rho) = \frac{1}{J'_{r+1}(\rho)^{m/r+1}}, \quad A_1(\rho) = \frac{m}{r+1} \left( c_1 - \frac{J'_r(\rho)}{J'_{r+1}(\rho)^{r/r+1}} \right),$$

and so on. By substitution of the known  $P$  into the definition (3.16) of  $X$  we get the final formula

$$X(p, \rho) = \frac{n+m}{m} \gamma_{n+m}(a_m(\rho)) \frac{n}{m} e^{\frac{n}{m} A_-(p, \rho)} p^n + \dots$$

As we said before, the structure of the canonical transformation varies depending on the relative values of the integers in the set  $\{m, n, r\}$ . If instead  $m+n \geq r+2$ , the case we have just considered, we would have  $m+n < r+2$  then we should take for the generating function  $J$  the simpler expression

$$J(P, \rho) = \sum_{k \geq 0} J_{r+1-k}(\rho) P^{\frac{r+1-k}{m}}. \quad (3.17)$$

Then, it readily follows that the method given for computing the variables  $(P, X)$  remains still valid for the new function  $J$ , but observe that  $n = r+1-m$ .

In any case, although  $J$  is given by the series (3.14), (3.17) it should be observed that only a finite number of terms are enough to determine the solution of the dKP equation according to eqs. (3.11), (3.12) in theorem 3.1. A simple counting argument on the number of terms needed to find the first  $m+n-1$  coefficients in the series (3.7) for  $F$  and  $G$  leads to the following result.

**Theorem 3.2.** *Let  $\{m, n, r\}$  a set of positive integers and  $(P, X)$  the corresponding canonical variables (3.1), (3.2) that define the twistor equations (3.10) for the solutions of the dKP equation. Then,  $(P, X)$  can be found through (3.13) in terms of the generating function*

$$J(P, \rho) = \sum_{k=r+2}^{m+n} \gamma_k P^{\frac{k}{m}} + \sum_{k=0}^{m+n-2} J_{r+1-k}(\rho) P^{\frac{r+1-k}{m}}$$

if  $m + n \geq r + 2$ , while for  $m + n < r + 2$  it is  $n = r + 1 - m$  and we have

$$J(P, \rho) = \sum_{k=0}^{m+n-2} J_{r+1-k}(\rho) P^{\frac{r+1-k}{m}}.$$

### 3.3 Explicit solutions

In this section we shall apply the previous results to the construction of explicit solutions of the dKP equation

$$(u_t - \frac{3}{2}uu_x)_x = \frac{3}{4}u_{yy}$$

in the simplest cases. To have a glimpse of the meaning of theorems (3.1) and (3.2) we shall analyze three examples of increasing complexity.

1. When one of the canonical variables  $(P, X)$ , say  $P$ , is of degree  $m = 1$  then the solution to the dKP hierarchy follows directly from  $F(L, x)|_- = 0$  in (3.11) independently of the concrete form of the generating function  $J(P, \rho)$ . In this case

$$F(L, x) = a_1 \left(\frac{\rho}{r+1}\right) L + a_0 \left(\frac{\rho}{r+1}\right) + a_{-1} \left(\frac{\rho}{r+1}\right) L^{-1} + \dots,$$

and

$$\rho = (r+1)t_{r+1} + rt_r L^{-1} + (r-1)t_{r-1} L^{-2} + \dots$$

The vanishing of the coefficient of  $L^{-1}$  leads to the equation,

$$\begin{aligned} \frac{1}{2} \left[ \frac{2(r-1)}{r+1} a_1'(t_{r+1}) t_{r-1} + \left(\frac{r}{r+1}\right)^2 a_1''(t_{r+1}) t_r^2 \right] \\ + \frac{r}{r+1} a_0'(t_{r+1} t_r) - a_1(t_{r+1}) \phi_{1x} = 0. \end{aligned}$$

For the solution  $u = -2\phi_{1x}$  we find the expression

$$u = I_2 t_{r-1} + \left( \frac{r^2}{2(r^2 - 1)} I_2' - \frac{r^2}{4(r - 1)^2} I_2^2 \right) t_r^2 + I_1 t_r + I_0,$$

where the arbitrary functions  $I_j(t_{r+1})$  are an equivalent parametrization to that of the functions  $a_j(t_{r+1})$  for the solution  $u$ .

Higher order coefficients  $\phi_{2x}, \phi_{3x}, \dots$  can be found from the equations corresponding to higher negative powers of  $L$ . In particular, for  $r = 2$  it results the solution of the dKP equation,

$$u(x, y, t) = I_0(t) + I_1(t)y + I_2(t)x + \left( \frac{2}{3} I_2'(t) - I_2(t)^2 \right) y^2.$$

Observe that this solution is of a very simple nature. In fact, it can be obtained from the zero solution  $u = 0$  by performing the two symmetries of the dKP equation described in the previous section.

Consideration of higher values of  $m$  and  $n$  leads to nice parametrizations for the solutions of nonlinear ordinary differential equations. The parallelism with the theory of algebraic curves for the KP equation, as a matter of fact, appears manifestly in the present construction.

2. It is observed that for values of the set of integers  $\{m, n, r\}$  in theorem (3.2) which are  $\{2, 3, 2\}$  or  $\{3, 2, 2\}$ ,  $\{3, 3, 2\}$ ,  $\{2, 4, 2\}$  or  $\{4, 2, 2\}$  the solutions of the dKP equation, as we shall see below, are of the form

$$u = I + \sqrt{K},$$

where  $I$  and  $K$  represent polynomials in  $x, y$  with coefficients depending on  $t$ ,

$$I(x, y, t) = I_0(t) + I_1(t)x + I_2(t)y + I_3(t)y^2,$$

and

$$K(x, y, t) = K_0(t) + K_1(t)x + K_2(t)y + K_3(t)y^2.$$

If one introduces this form of the solution in the dKP equation one gets the coefficients of  $I$  in terms of the coefficients of  $K$ , besides a Riccati equation for the coefficient  $K_3(t)$ , namely

$$K_3' = \frac{2}{K_1} K_3^2 + \frac{16}{15} \frac{K_1'}{K_1} K_3 + \frac{2}{15} K_1'' - \frac{8}{45} \frac{(K_1')^2}{K_1}.$$

A particular solution of this Riccati equation is

$$K_3 = \frac{2}{15}K_1';$$

hence, the general solution is easily found to be

$$K_3 = \frac{K_1^{8/5}}{C - 2 \int^t K_1(t)^{3/5} dt} + \frac{2}{15}K_1'.$$

Finally, the formula for the solution of the dKP equation we were looking for is:

$$u = \frac{1}{K_1^2} \left[ -\frac{1}{2}K_2^2 + \frac{2}{3}K_1K_0' - \frac{2}{9}K_0(3K_3 + 2K_1') \right] \quad (3.18)$$

$$+ \frac{2}{9}(K_1K_1' - 3K_1K_3)x \quad (3.19)$$

$$+ \frac{2}{9}(3K_1K_2' - 2K_2(6K_3 + K_1'))y \quad (3.20)$$

$$+ \frac{4}{3} \left( -K_3^2 + \frac{1}{5}K_3K_1' - \frac{8}{90}(K_1')^2 + \frac{2}{30}K_1K_1'' \right) y^2 \quad (3.21)$$

$$+ \sqrt{K_0 + K_1x + K_2y + K_3y^2} \quad (3.22)$$

This family of solutions lies in the orbit of a simpler solution under the action of the previously mentioned symmetries of the dKP equation. By these symmetries we can take  $K_0 = K_2 = K_3 = 0$ , so that

$$\frac{2}{15}K_1'' - \frac{8}{45} \frac{(K_1')^2}{K_1} = 0.$$

whose solution is

$$K_1(t) = \frac{1}{(at + b)^3},$$

with  $a$  and  $b$  arbitrary constants. The corresponding solution of the dKP equation is

$$u := -\frac{2bx}{3(a+bt)} + \sqrt{\frac{x}{(a+bt)^3}}.$$

Observe that this is a a solution of the of stationary type associated to the dKdV flow not depending on the  $y$  variable.

We now proceed to give the  $K$ 's in some examples

(a)  $\{2, 3, 2\}$  We take in this case the complete form for  $J$  and  $h$ :

$$J(P, \rho) = P^2\gamma_4 + P^{5/2}\gamma_5 + J_0(\rho) + P^{1/2}J_1(\rho) + PJ_2(\rho) + P^{3/2}J_3(\rho),$$

$$h = \exp(c_1p^{-1} + c_2p^{-2} + c_3p^{-3} + c_4p^{-4})p^3$$

and the corresponding  $K$ 's are

$$K_0 = -\frac{4j_3^{2/3}}{675\gamma_5^2}(-60c_1\gamma_5\dot{j}_2j_3^{1/3}$$

$$- 9(3J_3^2 - 10\gamma_5(J_1 - c_2tj_3^{1/3}))j_3^{2/3} + 5c_1^2\gamma_5t(3\dot{J}_2 + 4t\ddot{J}_3)),$$

$$K_1 = -\frac{8j_3^{5/3}}{15\gamma_5},$$

$$K_2 = -\frac{16j_3^{2/3}}{45\gamma_5}(\dot{j}_2j_3^{1/3} - c_1(\dot{J}_3 + \frac{2}{3}t\ddot{J}_3)),$$

$$K_3 = -\frac{16j_3^{2/3}\ddot{J}_3}{135\gamma_5}$$

Observe that  $c_3, c_4, \gamma_4, J_0$  do not appear in the solution.

(b)  $\{3, 3, 2\}$  While the general forms for  $J$  and  $h$  are:

$$J(P, \rho) = P^{4/3}\gamma_4 + P^{5/3}\gamma_5 + P^2\gamma_6$$

$$+ J_{-1}(\rho)P^{-1/3} + J_0(\rho) + J_1(\rho)P^{1/3} + J_2(\rho)P^{2/3} + J_3(\rho)P,$$

$$h = \exp(c_1p^{-1} + c_2p^{-2} + c_3p^{-3} + c_4p^{-4} + c_5p^{-5})p^3$$

we again concentrate in the case when  $c_4 = c_5 = \gamma_4 = 0$  and the corresponding  $K$ 's are

$$K_0 = \frac{1}{45\gamma_5}(2j_3^{2/3}(-12c_1t\dot{j}_2j_3^{1/3} + 18(J_1j_3^{2/3} - c_2t\dot{J}_3) + c_1^2t(3\dot{J}_3 + 4t\ddot{J}_3)))$$

$$K_1 = \frac{4j_3^{5/3}}{5\gamma_5},$$

$$K_2 = \frac{8j_3^{2/3}}{5\gamma_5}\left(\frac{1}{3}\dot{j}_2j_3^{1/3} - \frac{c_1}{3}(\dot{J}_3 + \frac{2}{3}t\ddot{J}_3)\right),$$

$$K_3 = \frac{8j_3^{2/3}\ddot{J}_3}{45\gamma_5}$$

Observe that now  $c_3, c_4, J_0$  do not appear in the solution.

3. For different values of  $m, n$  we get different type of solutions, for example if we take  $m = 5, n = 2$  and  $J(P, \rho) = \gamma P^{7/5} + \rho^2 P^{3/5}$ ,  $h = p^3$ , we get the solution

$$u = \frac{2}{2835\gamma^4 t^2} \left( \frac{A}{f} + B + Cf \right),$$

where

$$A := (21)^{2/3} \gamma^7 (-21870(2)^{1/3} t^{22/3} + 14\gamma(9tx - 8y^2)^2),$$

$$B := 7(9tx - 8y^2)\gamma^4,$$

$$C := (21)^{1/3},$$

$$f := (24800580(2)^{1/3} t^{22/3} (2tx + y^2)\gamma^{11} + 343\gamma^{12} (9tx - 8y^2)^3 + 5(7)^{1/2} \gamma^{21/2} g^{1/2})^{1/3},$$

$$g := 2510484768720 t^{22} + 2410616376(2)^{2/3} \gamma t^{44/3} (5751 t^2 x^2 + 5976 txy^2 + 1394 y^4) + 7715736(2)^{1/3} \gamma^2 t^{22/3} (9tx - 8y^2)^3 (27tx + 11y^2) + 343\gamma^3 (9tx - 8y^2)^6.$$

The same type is exhibited by the solution corresponding to  $m = 5, n = 2$  and  $J(P, \rho) = \gamma P^{7/5} + \rho P^{3/5}$ ,  $h = p^3 \exp(c/p)$

$$u = \frac{1}{315\gamma^4} \left( \frac{A}{f} + B + Cf \right),$$

where now

$$A := (21)^{2/3} \gamma^7 (-14c^4 \gamma + 540t),$$

$$B := 7c^2 \gamma^4,$$

$$C := -(21)^{1/3},$$

$$f := (-343\gamma^{11} c^6 + 5670\gamma^{11} (30x - 20cy + 11c^2 t) + 5(7)^{1/2} \gamma^{21/2} g^{1/2})^{1/3},$$

$$g := 18895680t^3 + 183708\gamma^7 (900x^2 - 1200cxy + 20c^2(33tx + 20y^2) - 440c^3 ty + 113c^4 t^2) - 15876c^6 \gamma^2 (42x - 28cy + 13c^2 t) + 343c^{12} \gamma^3,$$

Cubic type solutions similar to those just presented here were considered in [20].

## References

- [1] D. Lebedev and Yu. Manin, Phys. Lett. **74A**, 154 (1979)

- [2] V. E. Zakharov, Func. Anal. Priloz. **14**, 89 (1980); Physica **3D**, 193 (1981)
- [3] Y. Kodama, Phys.Lett. **129A**, 223 (1988); Prog. Theor. Phys. Suppl. **95**, 184 (1988).
- [4] Y. Kodama and J.Gibbons, Phys. Lett. **135A**, 167 (1989).
- [5] T. Takasaki and T. Takebe, Int. J. Mod. Phys. **A7** Suppl. **1B**, 889 (1992); Rev.in Math. Phys. **7**, 743 (1995)
- [6] B. A. Kupershmidt , J. Phys. A: Math. Gen. **23**, 871 (1990).
- [7] I. M. Krichever, Commun. Pure. Appl. Math. **47**, 437 (1992).
- [8] I. M. Krichever, Function. Anal. Appl. **22**, 200 (1989); Commun. Math. Phys. **143**, 415, (1992).
- [9] M. V. Saveliev, Theor. Math. Phys. **92**,457 (1992)
- [10] V. E. Zakharov, Dispersionless limit of integrable systems in 2 + 1 dimensions *Singular limits of dispersive waves* (eds. N. M. Ercolani et al), Nato Adv. Sci. Inst. Ser. B Phys. *320* , Plenum, New York (1994)
- [11] J.Gibbons and S.P.Tsarev, Phys. Lett. **211A**, 19, (1996); *ibid* **258A**, 263 (1999).
- [12] M.Mineev-Weinstein, P.B.Wiegmann and A.Zabrodin, Phys. Rev. Lett. **84**, 5106 (2000).
- [13] P. B. Wiegmann and A. Zabrodin, Commun. Math. Phys **213**, 523 (2000)
- [14] M. Dunajski, L. J. Mason and P. Tod, J. Geom. Phys. *37*, 63-93 (2001)
- [15] V. V. Geogdzhaev, Sov. Phys. Dokl. **30**, 10, 840 (1985)
- [16] Y. Kodama, Phys.Lett. **147A**, 477 (1990)
- [17] V. V. Geogdzhaev, Physica **87D**, 168 (1987)
- [18] B. Konopelchenko and L. Martinez Alonso, Phys. Lett. **286A**,161 (2001)

- [19] M. Mañas, L. Martinez Alonso and E. Medina, J. Phys. A : Math. Gen. **35**, 401 (2002)
- [20] M. Dunajski and P. Tod, [arXiv:nlin.SI/0204043](https://arxiv.org/abs/nlin.SI/0204043) (2002).