The SEIQS stochastic epidemic model with external source of infection.

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Abstract This paper deals with a stochastic epidemic model for computer viruses with latent and quarantine periods, and two sources of infection: internal and external. All sojourn times are considered random variables which are assumed to be independent and exponentially distributed. For this model extinction and hazard times are analyzed, giving results for their Laplace transforms and moments. The transient behavior is considered by studying the number of times that computers are susceptible, exposed, infectious and quarantined during a period of time $(0, t]$ and results for their joint and marginal distributions, moments and cross moments are presented. In order to give light this analysis, some numerical examples are showed.

Keywords: Computer virus; Markov chain; Epidemic; Quarantine; Extinction time; Hazard time.

1. Introduction

The first work about epidemiology application to computer viruses is due to Murray in 1988 [15], although he does not describe any model. Kephart and White [11, 12] have been the pioneers of modeling the spread behavior of computer viruses throughout a Susceptible-Infective-Susceptible (SIS) model. Since then epidemic models have been widely used in order to model the spread behavior of computer viruses by introducing modifications to the simplest models, SIS and SIR (Susceptible-Infective-Recovered). A lot of models applied to computer environment are deterministic, based on ordinary differential equations (ODEs) (see [16, 17, 21, 22, 23, 26, 27], for example). Piqueira et al [21] (see also [22]) deal with a modification of the traditional SIR model with an antidotal population compartment (A). Mishra and Saini [16] take into account a latent period where computers remain in the exposed class (E) before becoming infective (SEIRS epidemic model). Yao et al [27] implement the quarantine class (Q) to the model (SIQ model), and Mishra and Jha [17] consider a model with exposed and quarantine classes (SEIQRS model); Wang et al [23] also consider exposed and quarantine compartments and they analyze a more sophisticated SEIQRS model that presents more transitions and rates than the aforementioned SEIQRS model. Recently L. Yang and X. Yang [26] have described a new epidemic model by distinguishing between internal or external computers depending on whether they are connected to the Internet or not, and they also consider latent periods for viruses.

Stochastic epidemic models take into account the random nature of population events and they are more appropriate than deterministic models for small populations (see [1, 2, 7, 10]). We find different types of stochastic epidemic models applied to computer environment: stochastic differential equation models (see [28]), continuous-time Markov chain models (see [3, 4, 5, 19, 25]) and we can also find works focus on inference from a Bayesian perspective (see [13])

Zhang et al [28] introduce a random noise in ODEs of a deterministic SEIR model and transform it into a corresponding stochastic differential equation model.

We are concerned with stochastic models that employ continuous time Markov chains (CTMC) for modeling the propagation of viruses. In this setting, we point out that Weiss and Dishon [24] consider a continuous birth-and-death process to describe the SIS model. Wierman and Marchete [25] extend the stochastic SIS model by taking account of reintroduction of computer virus; Okamura et al [19] propose the stochastic KS model, i.e. the stochastic SIS model with kill signals; the idea of incorporating the kill signals to a stochastic epidemic model is also found in Amador and Artalejo [4]. Amador [5] describes and analyzes the stochastic SIRA model, i.e. the extension of traditional stochastic SIR model by including antidotal computers.

In this paper the interest is focused on the stochastic SIS model which incorporates latent and quarantine periods, i.e. stochastic SEIQS model, and it considers two different sources of infection, by direct contact with an infectious computer or by an external source. The description of this model is given in Section 2 by using a continuous-time Markov chain. Exponential distributions and independence of the involved random periods are two fundamental assumptions of a continuous-time Markov chain that make the probabilistic model tractable, so they are commonly assumed [4, 5, 14, 19, 24, 25]. Moreover, there are some studies with real epidemic data which verify the validity of these assumptions [18, 20]. An alternative approach is the block-structured state-dependent event (BSDE), introduced by Artalejo and Gomez-Corral [6], which is helpful to deal with non-exponential correlated flows. This approach has also been used in [3, 9]. The problem is that the BSDE version of an epidemic model augments the dimensionality of the original model and hence it can be intractable. For this model, it is interesting to study characteristics related with the first time at which all computers are infected or the first time at which no-one is infected: the hazard time and the extinction time, respectively. These first passage times are analyzed in Section 3. It is also important to know the situation of computers during a fixed period of time and this is done in Section 4. Some numerical examples are presented in Section 5 in order to illustrate the results of previous sections. Finally, Section 6 contains some conclusions of this work.

2. Model description

The stochastic $SEIQS$ model is an extension of the classic stochastic SIS model for which latent and quarantine periods are considered. More concretely, we deal with a closed population of size N (e.g. N computers) which is partitioned into subclasses of computers, namely susceptible, exposed (infected but not yet infectious), infectious and quarantined (infectious computers which are isolated). In this model we assume two sources of infection, internal infections caused by transmission from any infectious computer in the population and external infections coming from outside the computer network. When a susceptible computer is infected, there is a period of time during which this computer does not transmit the infection (latent period), after this time it becomes infectious and it can be isolated in order to avoid contagion. After a time, an infectious computer is recovered but it does not acquire immunity and it becomes susceptible immediately. Let $S(t)$, $E(t)$, $I(t)$ and $Q(t)$ be, respectively, the number of susceptible, exposed, infectious and quarantined computers at time t, where one of them can be expressed in terms of the other three, e.g. $Q(t) = N - S(t) - E(t) - I(t)$.

Let us assume an initial condition $(S(0), E(0), I(0)) = (i_0, j_0, k_0)$, with $k_0 \geq 1$ and $i_0+j_0+k_0 \leq N$; i.e. at time $t = 0$ there are at least one infected computer. If the state is (i, j, k) at time t, the possible transitions are as follows: towards the state $(i-1, j+1, k)$ when a susceptible computer is infected, with rate $(k\beta/N + \varepsilon)i$ $(i > 0)$, being ε the individual external infection contact rate and β the internal infection contact rate; other possibility is to go to the state $(i, j-1, k+1)$ which occurs when an exposed computer becomes infectious, with rate σj $(j > 0)$, σ is called individual incubation rate; the transition to the state $(i, j, k-1)$ happens when an infectious computer is isolated, with rate αk $(k > 0)$, α is called individual quarantine rate; other transition is to the state $(i+1, j, k-1)$ when an infected computer is recovered and then susceptible, with rate γk $(k > 0)$, γ is called individual recovery rate from infection; and the last one is to the state $(i+1, j, k)$ when a quarantined computer is recovered becoming susceptible, with rate $\delta(N-i-j-k)$ $((N-i-j-k) > 0)$, being δ the individual recovery rate from quarantine. Fig. 1 illustrates these transitions.

By assuming exponential distributions and independence of the involved random periods, the process $\mathbf{X} = \{(S(t), E(t), I(t)); t \geq 0\}$ is a tridimensional $CTMC$ on the state space $\mathbf{S} = \{(i, j, k) : 0 \leq i \leq N,\ 0 \leq j \leq N,\ 0 \leq k \leq N,\ i + j + k \leq N\}$. The infinitesimal generator of this $CTMC$, $\mathbf{Q} = \big(q_{(i,j,k),(i',j',k')} : \{(i, j, k), (i', j', k')\} \in \mathbf{S}\big)$, is a square matrix of order $(N+1)(N+2)(N+3)/6$, and its non-null entries $q_{(i,j,k),(i',j',k')}$ are given by

$$q_{(i,j,k),(i',j',k')} = \begin{cases} i(k\hat{\beta} + \varepsilon), & \text{if } (i', j', k') = (i-1, j+1, k), \\ j\sigma, & \text{if } (i', j', k') = (i, j-1, k+1), \\ k\alpha, & \text{if } (i', j', k') = (i, j, k-1), \\ k\gamma, & \text{if } (i', j', k') = (i+1, j, k-1), \\ (N-i-j-k)\delta, & \text{if } (i', j', k') = (i+1, j, k), \\ -q_{ijk} & \text{if } (i', j', k') = (i, j, k), \end{cases}$$

with $\hat{\beta} = \beta/N$ and $q_{ijk} = i(k\hat{\beta} + \varepsilon) + j\sigma + k\alpha + k\gamma + (N-i-j-k)\delta$.

The state space $\mathbf{S}$ can be partitioned as $\mathbf{S} = \bigcup_{i=0}^{N} l(i)$, where $l(i)$ is the $i-$th level of the process $\mathbf{X}$ containing the states with $S(t) = i$, i.e. $l(i) = \{(i, j, k) : 0 \leq j \leq N-i, 0 \leq k \leq N-i, j+k \leq N-i\}$, being its cardinality $|l(i)| = (N+1-i)(N+2-i)/2$. Moreover, every level $l(i)$ can be partitioned as $l(i) = \bigcup_{j=0}^{N-i} l(i, j)$, being $l(i, j) = \{(i, j, k) : 0 \leq k \leq N-i-j\}$ the $(i, j)-$sublevel of the process with cardinality $|l(i, j)| = N-i-j+1$. By taking this partition into account, we can express the infinitesimal generator $\mathbf{Q}$ in a block tridiagonal structure

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{00} & \mathbf{Q}_{01} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}_{10} & \mathbf{Q}_{11} & \mathbf{Q}_{12} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{21} & \mathbf{Q}_{22} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Q}_{N-1,N-1} & \mathbf{Q}_{N-1,N} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Q}_{N-1,N} & \mathbf{Q}_{NN} \end{pmatrix}, \tag{1}$$

where $\mathbf{0}$ is a matrix of zeros and the blocks $\mathbf{Q}_{ii'}$, whose dimension is $|l(i)|\times|l(i')|$, can be expressed in the block forms below

$$\mathbf{Q}_{ii}= \begin{pmatrix} \mathbf{Q}_{(i,0)(i,0)} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}_{(i,1)(i,0)} & \mathbf{Q}_{(i,1)(i,1)} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Q}_{(i,N-i-1)(i,N-i-1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Q}_{(i,N-i)(i,N-i-1)} & \mathbf{Q}_{(i,N-i)(i,N-i)} \end{pmatrix}, \text{ for } 0 \le i \le N,$$

$$\mathbf{Q}_{i,i-1}= \begin{pmatrix} \mathbf{0} & \mathbf{Q}_{(i,0)(i-1,1)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_{(i,1)(i-1,2)} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Q}_{(i,N-i)(i-1,N-i+1)} \end{pmatrix}, \text{ for } 1 \le i \le N,$$

$$\mathbf{Q}_{i,i+1}= \begin{pmatrix} \mathbf{Q}_{(i,0)(i+1,0)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{(i,1)(i+1,1)} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Q}_{(i,N-i-1)(i+1,N-i-1)} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}, \text{ for } 0 \le i \le N-1.$$

Matrices $\mathbf{Q}_{(i,j)(i',j')}$ are of dimension $|l(i,j)| \times |l(i',j')|$ and they are as follows:

$$\mathbf{Q}_{(i,j)(i,j)} = \begin{pmatrix} -q_{ij0} & 0 & \cdots & 0 & 0 \\ \alpha & -q_{ij1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (N-i-j)\alpha & -q_{ij,N-i-j} \end{pmatrix}, \text{ for } \ 0 \le i \le N, 0 \le j \le N-i,$$

$$\mathbf{Q}_{(i,j)(i,j-1)} = \begin{pmatrix} 0 & j\sigma & 0 & \cdots & 0 \\ 0 & 0 & j\sigma & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & j\sigma \end{pmatrix}, \text{ for } \ 0 \le i \le N, 1 \le j \le N-i,$$

$$\mathbf{Q}_{(i,j)(i-1,j+1)} = \begin{pmatrix} i\varepsilon & 0 & \cdots & 0 \\ 0 & i(\hat{\beta}+\varepsilon) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & i((N-i-j)\hat{\beta}+\varepsilon) \end{pmatrix}, \text{ for } \ 1 \leq i \leq N, 0 \leq j \leq N-i,$$

$$\mathbf{Q}_{(i,j)(i+1,j)} = \begin{pmatrix} (N-i-j)\delta & 0 & \cdots & 0 \\ \gamma & (N-i-j-1)\delta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta \\ 0 & 0 & \cdots & (N-i-j)\gamma \end{pmatrix},$$

$$\text{for } \ 0 \leq i \leq N-1, 0 \leq j \leq N-i-1.$$

3. Extinction time and hazard time

In this section the interest is focused on the distributions of two first passage times: the extinction time and the hazard time. The first one is defined as the first time at which viruses are eradicated, i.e. the first time at which all of N computers are susceptible: $T = \min\{t \geq 0 : (S(t), E(t), I(t)) = (N, 0, 0)\}$. The hazard time is defined as the first time at witch all computers are infected, i.e. the first time at which the number of susceptible computers is zero: $H = \min\{t \geq 0 : (S(t), E(t), I(t)) = (0, j, k), j+k \leq N\}$.

In order to obtain these distributions, we consider conditional first passage times T_{ijk} and $H_{ijk}, \forall (i,j,k) \in \mathbf{S}$, defined as the first time at which N or 0 computers, respectively, are susceptible, given that the current state is (i,j,k). Note that, in the case of the initial state (i_0, j_0, k_0), we have $T = T_{i_0 j_0 k_0}$ and $H = H_{i_0 j_0 k_0}$.

In what follows, we derive the Laplace transforms of the conditional extinction times. Then, we calculate their probability density functions by numerical inversion techniques [8]. Moments of these conditional extinction times are calculated by differentiating their corresponding Laplace transforms.

Let $\varphi^T_{ijk}(s)$ be the Laplace transform of T_{ijk} defined by $\varphi^T_{ijk}(s) = E[\exp\{-sT_{ijk}\}]$, for $\mathrm{Re}(s) \geq 0$ and $(i,j,k) \in \mathbf{S}$, with moments $m^l_{ijk}(T) = E[T^l_{ijk}]$, for $l \geq 1$. $m^0_{ijk}(T) = P\{T_{ijk} < \infty\} = 1$, for $(i,j,k) \in \mathbf{S}$, due to the state space is finite. In order to obtain Laplace transforms $\varphi^T_{ijk}(s)$ and since $\varphi^T_{N00}(s) = 1$, we consider the following notation associated to the set of states excluding $(N,0,0)$, so let us denote $\mathbf{S}_T = \mathbf{S} - l(N)$ and $\mathbf{Q}_T$ as matrix $\mathbf{Q}$ after deleting the last row and column.

As for the hazard time, $\varphi^H_{ijk}(s)$ and $m^l_{ijk}(H)$ denote the corresponding Laplace transform and moments of H_{ijk}. Moreover, $\mathbf{S}_H = \mathbf{S} - l(0)$ and

$$\mathbf{Q}_H = \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Q}_{N-1,N-1} & \mathbf{Q}_{N-1,N} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Q}_{N-1,N} & \mathbf{Q}_{NN} \end{pmatrix}.$$

The following theorem summarizes the main results corresponding to the extinction time and the hazard time.

Theorem 1. *i) Laplace transforms* $\{\varphi^T_{ijk}(s);\ (i,j,k) \in \mathbf{S}_T\}$ *satisfy the tridiagonal system of equations*

$$(\mathbf{Q}_T - s\mathbf{I}_g)\boldsymbol{\varphi}^T(s) = \mathbf{b}_T, \qquad (2)$$

where $\boldsymbol{\varphi}^T(s) = (\boldsymbol{\varphi}^T_0(s), ..., \boldsymbol{\varphi}^T_{N-1}(s))'$ *and* $\boldsymbol{\varphi}^T_i(s) = (\varphi^T_{i00}(s),...,\varphi^T_{i,0,N-i}(s),..., \varphi^T_{i,N-i,0}(s))'$, *for* $0 \leq i \leq N-1$, $\mathbf{I}_g$ *denotes the identity matrix of dimension* g *and* $\mathbf{b}_T = -(0, ..., 0, \delta, \gamma, 0)'$ *is a vector of dimension* g, *with* $g = (N+1)(N+2)(N+3)/6 - 1$.
ii) Moments $\{m^l_{ijk}(T);\ (i,j,k) \in \mathbf{S}_T\}$ *can be recursively computed from the system of equations*

$$\mathbf{m}^l(T) = -l\mathbf{Q}_T^{-1}\mathbf{m}^{l-1}(T),\ l \geq 1, \qquad (3)$$

where $\mathbf{m}^0(T) = \mathbf{1}_g$ *(*$\mathbf{1}_g$ *is a column vector of order* g *with elements equal to* 1), $\mathbf{m}^l(T) = (\mathbf{m}^l_0(T),...,\mathbf{m}^l_{N-1}(T))'$ *and* $\mathbf{m}^l_i(T) = (m^l_{i00}(T),...,m^l_{i,0,N-i}(T), ...,m^l_{i,N-i,0}(T))'$, *for* $l \geq 1$ *and* $0 \leq i \leq N-1$.
iii) Laplace transforms $\{\varphi^H_{ijk}(s);\ (i,j,k) \in \mathbf{S}_H\}$ *satisfy the tridiagonal system of equations*

$$(\mathbf{Q}_H - s\mathbf{I}_h)\boldsymbol{\varphi}^H(s) = \mathbf{b}_H,$$

where $\boldsymbol{\varphi}^H(s) = (\boldsymbol{\varphi}^H_1(s), ..., \boldsymbol{\varphi}^H_N(s))'$ *and* $\boldsymbol{\varphi}^H_i(s) = (\varphi^H_{i00}(s), ..., \varphi^H_{i,0,N-i}(s), ..., \varphi^H_{i,N-i,0}(s))'$, *for* $1 \leq i \leq N$, $\mathbf{b}_H$ *is a vector of dimension* h, *with* $h = N(N+1)(N+2)/6$, *defined as* $\mathbf{b}_H = (\mathbf{b}_1, 0, ..., 0)'$, *where* $\mathbf{b}_1 = -\mathbf{Q}_{10}\mathbf{1}_{|l(0)|}$.
iv) Moments $\{m^l_{ijk}(H);\ (i,j,k) \in \mathbf{S}_H\}$ *can be recursively computed from the system of equations*

$$\mathbf{m}^l(H) = -l\mathbf{Q}_H^{-1}\mathbf{m}^{l-1}(H),\ l \geq 1,$$

where $\mathbf{m}^0(H) = \mathbf{1}_h$, $\mathbf{m}^l(H) = \left(\mathbf{m}^l_1(H), ..., \mathbf{m}^l_N(H)\right)'$ *and* $\mathbf{m}^l_i(H) = (m^l_{i00}(H), ..., m^l_{i,0,N-i}(H), ..., m^l_{i,N-i,0}(H))'$, *for* $l \geq 1$ *and* $1 \leq i \leq N$.

Proof. By using first step analysis (i.e. by conditioning on the event which determines the exit from state (i,j,k)), we derive the equations

$$
\begin{aligned}
\varphi^{T}_{N00}(s) &= 1,\\
\varphi^{T}_{ijk}(s) &= \frac{i(k\widehat{\beta}+\varepsilon)}{s+q_{ijk}}\varphi^{T}_{i-1,j+1,k}(s) + \frac{j\sigma}{s+q_{ijk}}\varphi^{T}_{i,j-1,k+1}(s)\\
&+\frac{k\alpha}{s+q_{ijk}}\varphi^{T}_{i,j,k-1}(s) + \frac{k\gamma}{s+q_{ijk}}\varphi^{T}_{i+1,j,k-1}(s)\\
&+\frac{(N-i-j-k)\delta}{s+q_{ijk}}\varphi^{T}_{i+1,j,k}(s),\ (i,j,k)\in \mathbf{S}_T. \qquad (4)
\end{aligned}
$$

It can be readily verified that the system of equations (4), for $(i,j,k) \in \mathbf{S}_T$, can be expressed in matrix form as (2).

To find the l-th moment of T_{ijk}, for $l \geq 1$, we use that $m^{l}_{ijk} = (-1)^{l}\varphi^{T(l}_{ijk}(s)\Big|_{s=0}$. By differentiating Eqs. (2) l times with respect to s, we get

$$(\mathbf{Q}_T - s\mathbf{I}_g)\boldsymbol{\varphi}^{T(l}(s) - l\boldsymbol{\varphi}^{T(l-1}(s) = \mathbf{0}_g, \qquad (5)$$

where $\mathbf{0}_g$ is a column vector or order g with elements equal to 0.

Multiplying Eqs. (5) by $(-1)^l$ and setting $s = 0$, we get (3).

The proof of *iii)* and *iv)* is similar, so it can be omitted. □

It should be noted that the computation of $\varphi^{T}_{ijk}(s)$ and $\varphi^{H}_{ijk}(s)$, at complex arguments s, is required for getting the probability density functions, $f_{T_{ijk}}(t)$ and $f_{H_{ijk}}$, of T_{ijk} and H_{ijk} by numerical inversion [8]. From the Tauberian result, the value of $f_{T_{ijk}}(0)$, for $(i,j,k) \in \mathbf{S}_T$, and the value of $f_{H_{ijk}}(0)$, for $(i,j,k) \in \mathbf{S}_H$, are as follows:

$$f_{T_{ijk}}(0) = \lim_{s\to\infty} s\varphi^{T}_{ijk}(s) = \begin{cases} \gamma, & \text{if } (i,j,k) = (N-1,0,1),\\ \delta, & \text{if } (i,j,k) = (N-1,0,0),\\ 0, & \text{otherwise.}\end{cases} \qquad (6)$$

$$f_{H_{ijk}}(0) = \lim_{s\to\infty} s\varphi^{H}_{ijk}(s) = \begin{cases} \widehat{\beta}k+\varepsilon & \text{if } i=1,\\ 0, & \text{otherwise.}\end{cases} \qquad (7)$$

4. Transient analysis

In the *SEIQS* stochastic model, with external source of infection, every computer can be susceptible, exposed, infectious and/or isolated more than once during a time period. In this section we have centered on analyzing the number of these events taking place in a fixed period of time $(0,t]$, i.e. the aim is to deal with their transient analysis. To this end, we complete the previous process $\mathbf{X} = \{(S(t), E(t), I(t)); t \geq 0\}$ by adding four counting components $(N_S(t), N_E(t), N_I(t), N_Q(t))$ defined as follows: $N_S(t)$ is the total number of times all computers have been susceptible until time t, excluding the initial number of susceptible computers, i.e. $N_S(0) = 0$. In a similar way, $N_E(t)$, $N_I(t)$ and $N_Q(t))$ are defined as $N_S(t)$ by substituting number of susceptible by numbers of exposed, infectious and quarantined, respectively. Obviously, for any $t > 0$, every component can be greater than N, then the sum of these four components can also be greater than N. The process $(\mathbf{X},\mathbf{N}) = \{(S(t), E(t), I(t), N_S(t), N_E(t), N_I(t), N_Q(t)); t \geq 0\}$ is a continuous time Markov chain. For each $t > 0$, let us denote the transient probabilities

$$p_{ijknmlh}(t) = P\{S(t) = i, E(t) = j, I(t) = k, N_S(t) = n, N_E(t) = m, N_I(t) = l, N_Q(t) = h\},$$

for $(i, j, k) \in \mathbf{S}$, $n \geq 0, m \geq 0, l \geq 0, h \geq 0$, and the initial probabilities

$$p_{ijknmlh}(0) = \begin{cases} 1, \text{ if } (i, j, k, n, m, l, h) = (i_0, j_0, k_0, 0, 0, 0, 0), \\ 0, \text{ otherwise.} \end{cases}$$

The objective is to obtain the distributions of these counting components and their moments. For example, the probability mass function of $N_S(t)$ is given by the probabilities $p^S_n(t) = P\{N_S(t) = n\}$, for $n \in \mathbb{N}$. Let us note that $p^S_n(t) = \sum_{(i,j,k)\in\mathbf{S}} p^S_{ijkn}(t) = \sum_{(i,j,k)\in\mathbf{S}} P\{S(t) = i, E(t) = j, I(t) = k, N_S(t) = n\}$. The moments of $N_S(t)$ are defined as $E[(N_S(t))^r] = \sum_{n=0}^{\infty} n^r p^S_n(t) = \sum_{(i,j,k)\in\mathbf{S}} \sum_{n=0}^{\infty} n^r p^S_{ijkn}(t)$. For the other counting components, $N_E(t)$, $N_I(t)$, $N_Q(t)$, the definitions are analogous.

Besides the marginal study of these random variables, we are interested in analyzing the correlation between each two of them. For $N_S(t)$ and $N_E(t)$, the cross moment is defined as $E[N_S(t)N_E(t)] = \sum\limits_{n=0}^{\infty}\sum\limits_{m=0}^{\infty} nmp^{S,E}_{nm}(t) = \sum_{(i,j,k)\in\mathbf{S}} \sum\limits_{n=0}^{\infty}\sum\limits_{m=0}^{\infty} nmp^{S,E}_{ijknm}(t)$, being $p^{S,E}_{nm}(t) = P\{N_S(t) = n, N_E(t) = m\}$ and $p^{S,E}_{ijknm}(t) = P\{S(t) = i, E(t) = j, I(t) = k, N_S(t) = n, N_E(t) = m\}$. The rest of the cross moments are defined in a similar way.

All of these probabilities and moments can be calculated by using Laplace transforms and inverting them numerically [8]. Notation for Laplace transforms of the previous quantities are as follows: For $\mathrm{Re}(s) \geq 0$,

$$\begin{aligned}
\tilde{p}^S_{ijkn}(s) &= \int_0^{\infty} e^{-st} p^S_{ijkn}(t)dt, \\
\tilde{p}^{S,E}_{ijknm}(s) &= \int_0^{\infty} e^{-st} p^{S,E}_{ijknm}(t)dt, \\
\tilde{S}^r_{ijk}(s) &= \int_0^{\infty} e^{-st} \sum_{n=0}^{\infty} n^r p^S_{ijkn}(t)dt, \\
\widetilde{SE}_{ijk}(s) &= \int_0^{\infty} e^{-st} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} nmp^{S,E}_{ijknm}(t)dt.
\end{aligned}$$

By grouping these Laplace transforms in vectors, we have the vectorial notation

$$\begin{aligned}
\tilde{\mathbf{p}}^S_n(s) &= (\tilde{\mathbf{p}}^S_{0n}(s), ..., \tilde{\mathbf{p}}^S_{Nn}(s))', \\
\tilde{\mathbf{p}}^S_{in}(s) &= (\tilde{p}^S_{i00n}(s), ..., \tilde{p}^S_{i0,N-i,n}(s), ..., \tilde{p}^S_{i,N-i,0n}(s))', \text{ for } i = 0, ..., N, \\
\tilde{\mathbf{S}}^r(s) &= (\tilde{\mathbf{S}}^r_0(s), ..., \tilde{\mathbf{S}}^r_N(s))', \\
\tilde{\mathbf{S}}^r_i(s) &= (\tilde{S}^r_{i00}(s), ..., \tilde{S}^r_{i0,N-i}(s), ..., \tilde{S}^r_{i,N-i,0}(s))', \text{ for } i = 0, ..., N, \\
\tilde{\mathbf{p}}^{S,E}_{nm}(s) &= (\tilde{\mathbf{p}}^{S,E}_{0nm}(s), ..., \tilde{\mathbf{p}}^{S,E}_{Nnm}(s))', \\
\tilde{\mathbf{p}}^{S,E}_{inm}(s) &= (\tilde{p}^{S,E}_{i00nm}(s), ..., \tilde{p}^{S,E}_{i,N-i,0nm}(s))', \text{ for } i = 0, ..., N, \\
\widetilde{\mathbf{SE}}(s) &= (\widetilde{\mathbf{SE}}_0(s), ..., \widetilde{\mathbf{SE}}_N(s))', \\
\widetilde{\mathbf{SE}}_i(s) &= (\widetilde{SE}_{i00}(s), ..., \widetilde{SE}_{i,N-i,0}(s))', \text{ for } i = 0, ..., N,
\end{aligned}$$

and analogous notation for the rest of variables.

The next theorems give recursive systems of equations to obtain these Laplace transforms. Theorem 2 shows a system of equations for Laplace transforms of probability mass functions for each descriptor, $N_S(t)$, $N_E(t)$, $N_I(t)$ and $N_Q(t)$. Theorem 3 gives the systems of equations satisfied by the Laplace transforms for the $r-$moments of the previous descriptors, for $r \geq 0$ and, eventually, Theorem 4 gives systems of equations for Laplace transforms of cross moments between two of them.

The notation δ_{nm} appears in the following theorems. It corresponds to the Kronecker's delta, defined as one if $n = m$, and zero otherwise.

Theorem 2. *i) For each $n \geq 0$, the Laplace transform vector $\widetilde{\mathbf{p}}^S_n(s)$ satisfies the recursive system of equations*

$$\widetilde{\mathbf{p}}^S_{in}(s)(\mathbf{Q}_{ii} - s\mathbf{I}_{|l(i)|}) + (1 - \delta_{iN})\widetilde{\mathbf{p}}^S_{i+1,n}(s)\mathbf{Q}_{i+1,i} = \widetilde{\mathbf{b}}^S_{in}, \quad i = 0, ...N, \tag{8}$$

where $\widetilde{\mathbf{b}}^S_{in} = -(1 - \delta_{i0})\widetilde{\mathbf{p}}^S_{i-1,n-1}(s)\mathbf{Q}_{i-1,i}$, for $n \geq 1$, and $\widetilde{\mathbf{b}}^S_{i0} = (\widetilde{b}^S_{i000}, ..., \widetilde{b}^S_{i0,N-i,0}, ..., \widetilde{b}^S_{i,N-i,00})$, whose components are 0 except $\widetilde{b}^S_{i_0j_0k_00} = -1$.

ii) For each $n \geq 0$, the Laplace transform vector $\widetilde{\mathbf{p}}^E_n(s)$ satisfies the recursive system of equations

$$(1 - \delta_{i0})\widetilde{\mathbf{p}}^E_{i-1,n}(s)\mathbf{Q}_{i-1,i} + \widetilde{\mathbf{p}}^E_{in}(s)(\mathbf{Q}_{ii} - s\mathbf{I}_{|l(i)|}) = \widetilde{\mathbf{b}}^E_{in}, \quad i = 0, ...N, \tag{9}$$

where $\widetilde{\mathbf{b}}^E_{in} = -(1 - \delta_{iN})\widetilde{\mathbf{p}}^E_{i+1,n-1}(s)\mathbf{Q}_{i+1,i}$, for $n \geq 1$, and $\widetilde{\mathbf{b}}^E_{i0} = (\widetilde{b}^E_{i000}, ..., \widetilde{b}^E_{i0,N-i,0}, ..., \widetilde{b}^E_{i,N-i,00})$, whose components are 0 except $\widetilde{b}^E_{i_0j_0k_00} = -1$.

iii) For each $n \geq 0$, the Laplace transform vector $\widetilde{\mathbf{p}}^I_n(s)$ satisfies the recursive system of equations

$$(1-\delta_{i0})\widetilde{\mathbf{p}}^I_{i-1,n}(s)\mathbf{Q}_{i-1,i}+\widetilde{\mathbf{p}}^I_{in}(s)\left(\mathbf{Q}^{*I}_{ii} - s\mathbf{I}_{|l(i)|}\right)+(1-\delta_{iN})\widetilde{\mathbf{p}}^I_{i+1,n}(s)\mathbf{Q}_{i+1,i} = \widetilde{\mathbf{b}}^I_{in}, \quad i = 0, ...N, \tag{10}$$

where

$$\mathbf{Q}_{ii}^{*I}(s)=\begin{pmatrix} \mathbf{Q}_{(i,0)(i,0)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{(i,1)(i,1)} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Q}_{(i,N-i)(i,N-i)} \end{pmatrix},\ i=0,...N,$$

and $\widetilde{\mathbf{b}}_{in}^{I}=-\widetilde{\mathbf{p}}_{i,n-1}^{I}(s)\left(\mathbf{Q}_{ii}-\mathbf{Q}_{ii}^{*I}\right)$, *for* $n\geq 1$, *and* $\widetilde{\mathbf{b}}_{i0}^{I}=(\widetilde{b}_{i000}^{I},...,\widetilde{b}_{i0,N-i,0}^{I},...,\widetilde{b}_{i,N-i,00}^{I})$, *whose components are* 0 *except* $\widetilde{b}_{i_0 j_0 k_0 0}^{I}=-1$.

iv) For each $n\geq 0$, *the Laplace transform vector* $\widetilde{\mathbf{p}}_{n}^{Q}(s)$ *satisfies the recursive system of equations*

$$(1-\delta_{i0})\widetilde{\mathbf{p}}_{i-1,n}^{Q}(s)\mathbf{Q}_{i-1,i}+\widetilde{\mathbf{p}}_{in}^{Q}(s)\left(\mathbf{Q}_{ii}^{*Q}-s\mathbf{I}_{|l(i)|}\right)+(1-\delta_{iN})\widetilde{\mathbf{p}}_{i+1,n}^{Q}(s)\mathbf{Q}_{i+1,i}=\widetilde{\mathbf{b}}_{in}^{Q},\quad i=0,...N, \tag{11}$$

where

$$\mathbf{Q}_{ii}^{*Q}(s)=\begin{pmatrix} \mathbf{Q}_{(i,0)(i,0)}^{*Q} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}_{(i,1)(i,0)} & \mathbf{Q}_{(i,1)(i,1)}^{*Q} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Q}_{(i,N-i)(i,N-i-1)} & \mathbf{Q}_{(i,N-i)(i,N-i)}^{*Q} \end{pmatrix},\ i=0,...N,$$

$$\mathbf{Q}_{(i,j)(i,j)}^{*Q}=\begin{pmatrix} -q_{ij0} & 0 & \cdots & 0 \\ 0 & -q_{ij1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -q_{ij,N-i-j} \end{pmatrix},\ j=0,...,N-i,$$

and $\widetilde{\mathbf{b}}_{in}^{Q}=-\widetilde{\mathbf{p}}_{i,n-1}^{Q}(s)\left(\mathbf{Q}_{ii}-\mathbf{Q}_{ii}^{*Q}\right)$, *for* $n\geq 1$, *and* $\widetilde{\mathbf{b}}_{i0}^{Q}=(\widetilde{b}_{i000}^{Q},...,\widetilde{b}_{i0,N-i,0}^{Q},...,\widetilde{b}_{i,N-i,00}^{Q})$, *whose components are* 0 *except* $\widetilde{b}_{i_0 j_0 k_0 0}^{Q}=-1$.

Proof. The forward Kolmogorov equations of the process $(\mathbf{X},\mathbf{N})$ are given by

$$\begin{gathered} \frac{d}{dt}p_{ijknmlh}(t)=-q_{ijk}p_{ijknmlh}(t) \\ +(1-\delta_{iN})(1-\delta_{j0})(1-\delta_{m0})(i+1)(k\hat{\beta}+\varepsilon)p_{i+1,j-1,kn,m-1,lh}(t) \\ +(1-\delta_{jN})(1-\delta_{k0})(1-\delta_{l0})(j+1)\sigma p_{i,j+1,k-1,nm,l-1,h}(t) \\ +(1-\delta_{kN})(1-\delta_{h0})(k+1)\alpha p_{ij,k+1,nml,h-1}(t) \\ +(1-\delta_{i0})(1-\delta_{kN})(1-\delta_{n0})(k+1)\gamma p_{i-1,j,k+1,n-1,mlh}(t) \\ +(1-\delta_{i0})(1-\delta_{n0})(N-i-j-k+1)\delta p_{i-1,jk,n-1,mlh}(t), \\ (i,j,k)\in\mathbf{S},n\geq 0,m\geq 0,l\geq 0,h\geq 0. \end{gathered} \tag{12}$$

In order to obtain the equations for the $N_S(t)$ component, we sum eq. (12) over $m\geq 0,l\geq 0,h\geq 0$ and we obtain the corresponding equations for the probabilities $p_{ijkn}^{S}(t),(i,j,k)\in\mathbf{S},n\geq 0$. If we take Laplace transforms on resulting equations and we use $\int_0^\infty e^{-st} \times \frac{d}{dt} p^S_{ijkn}(t)dt = s\widetilde{p}^S_{ijkn}(s) - p^S_{ijkn}(0)$, Laplace transforms satisfy the system of equations

$$
\begin{gathered}
s\widetilde{p}^S_{ijkn}(s) - p^S_{ijkn}(0) = -q_{ijk}\widetilde{p}^S_{ijkn}(s) \\
+(1-\delta_{iN})((1-\delta_{j0})(i+1)(k\hat{\beta}+\varepsilon)\widetilde{p}^S_{i+1,j-1,kn}(s) \\
+(1-\delta_{jN})(1-\delta_{k0})(j+1)\sigma\widetilde{p}^S_{i,j+1,k-1,n}(s) \\
+(1-\delta_{kN})(k+1)\alpha\widetilde{p}^S_{ij,k+1,n}(s) \\
+(1-\delta_{i0})(1-\delta_{kN})(1-\delta_{n0})(k+1)\gamma\widetilde{p}^S_{i-1,j+1,k,n-1}(s) \\
(1-\delta_{i0})(1-\delta_{n0})(N-i-j-k+1)\delta\widetilde{p}^S_{i-1,jk,n-1}(s), \\
(i,j,k) \in \mathbf{S}, n \geq 0, \qquad (13)
\end{gathered}
$$

which expressed in matrix form gives (8). The proof for (9), (10) and (11) are analogous and so omitted. □

Before enunciating Theorem 3 and for simplicity's sake, let us denote by $\widetilde{\mathbf{m}}^r(s)$ the row vector of Laplace transforms of the $r-$moments for every descriptor, i.e. $\widetilde{\mathbf{m}}^r(s)$ is $\widetilde{\mathbf{S}}^r(s)$, $\widetilde{\mathbf{E}}^r(s)$, $\widetilde{\mathbf{I}}^r(s)$ or $\widetilde{\mathbf{Q}}^r(s)$, depending on the referred descriptor.

Theorem 3. *For $r \geq 0$, the Laplace transform vector $\widetilde{\mathbf{m}}^r(s)$) satisfy the recursive system of equations*

$$
(1-\delta_{i0})\widetilde{\mathbf{m}}^r_{i-1}(s)\mathbf{Q}_{i-1,i}+\widetilde{\mathbf{m}}^r_i(s)(\mathbf{Q}_{ii}-s\mathbf{I}_{|l(i)|})+(1-\delta_{iN})\widetilde{\mathbf{m}}^r_{i+1}(s)\mathbf{Q}_{i+1,i} = \mathbf{c}^r_i,\ i = 0, ..., N, \qquad (14)
$$

where $\mathbf{c}^0_i = (c^0_{i00}, ..., c^0_{i0,N-i}, ..., c^0_{i,N-i,0})$ whose components are 0 except $c^0_{i_0 j_0 k_0} = -1$, and $\mathbf{c}^r_i$, for $r \geq 1$, is as follows:
i) $\mathbf{c}^r_i = -(1-\delta_{i0})\sum_{l=1}^r \binom{r}{l}\widetilde{\mathbf{m}}^{r-l}_{i-1}(s)\mathbf{Q}_{i-1,i}$, for $\widetilde{\mathbf{m}}^r(s) = \widetilde{\mathbf{S}}^r(s)$.
ii) $\mathbf{c}^r_i = -(1-\delta_{iN})\sum_{l=1}^r \binom{r}{l}\widetilde{\mathbf{m}}^{r-l}_{i+1}(s)\mathbf{Q}_{i+1,i}$, for $\widetilde{\mathbf{m}}^r(s) = \widetilde{\mathbf{E}}^r(s)$.
iii) $\mathbf{c}^r_i = -\sum_{l=1}^r \binom{r}{l}\widetilde{\mathbf{m}}^{r-l}_i(s)\left(\mathbf{Q}_{ii} - \mathbf{Q}^{*I}_{ii}\right)$, for $\widetilde{\mathbf{m}}^r(s) = \widetilde{\mathbf{I}}^r(s)$.
iv) $\mathbf{c}^r_i = -\sum_{l=1}^r \binom{r}{l}\widetilde{\mathbf{m}}^{r-l}_i(s)\left(\mathbf{Q}_{ii} - \mathbf{Q}^{*Q}_{ii}\right)$, for $\widetilde{\mathbf{m}}^r(s) = \widetilde{\mathbf{Q}}^r(s)$.

Proof. We center on Laplace transforms of $r-$moments of the descriptor $N_S(t)$, i.e. the vector $\widetilde{\mathbf{S}}^r(s)$, for the other descriptors the proof is similar. If we multiply both sides of Eqs. (12) by n^r and we sum over n, m, l, and h, and we denote $E[S^r_{ijk}(t)] = \sum\limits_{n=0}^{\infty} n^r p^S_{ijkn}(t)$, we obtain the following equations

$$
\begin{gathered}
\frac{d}{dt}E[S^r_{ijk}(t)] = -q_{ijk}E[S^r_{ijk}(t)] + (1-\delta_{iN})(1-\delta_{j0})(i+1)(k\hat{\beta}+\varepsilon)E[S^r_{i+1,j-1,k}(t)] \\
+(1-\delta_{jN})(1-\delta_{k0})(j+1)\sigma E[S^r_{i,j+1,k-1}(t)] + (1-\delta_{kN})(k+1)\alpha E[S^r_{ij,k+1}(t)] \\
+(1-\delta_{i0})(1-\delta_{kN})(k+1)\gamma\sum_{l=0}^{r}\binom{r}{l}E[S^{r-l}_{i-1,j,k+1}(t)] \\
+(1-\delta_{i0})(N-i-j-k+1)\delta\sum_{l=0}^{r}\binom{r}{l}E[S^{r-l}_{i-1,jk}(t)],\ (i,j,k) \in \mathbf{S}. \qquad (15)
\end{gathered}
$$

Taking Laplace transforms on Eqs. (15) and expressing them in matrix form, the system of equations (14) is derived for the case *i)*. □

Theorem 4 gives equations to satisfy Laplace transforms of the cross moments between two of the descriptors $N_S(t)$, $N_E(t)$, $N_I(t)$ and $N_Q(t)$. Let us denote by $\widetilde{\mathbf{m}}(s)$ the row vector of Laplace transforms of cross moment for every couple of descriptors (i.e. $\widetilde{\mathbf{m}}(s)$ is $\widetilde{\mathbf{SE}}(s)$, $\widetilde{\mathbf{SI}}(s)$, $\widetilde{\mathbf{SQ}}(s)$, $\widetilde{\mathbf{EI}}(s)$, $\widetilde{\mathbf{EQ}}(s)$ or $\widetilde{\mathbf{IQ}}(s)$, depending on the case under study).

Theorem 4. *The Laplace transform vector of the cross moments,* $\widetilde{\mathbf{m}}(s) = (\widetilde{\mathbf{m}}_0(s), ..., \widetilde{\mathbf{m}}_N(s))$ *satisfy the recursive system of equations*

$$(1-\delta_{i0})\widetilde{\mathbf{m}}_{i-1}(s)\mathbf{Q}_{i-1,i}+\widetilde{\mathbf{m}}_i(s)(\mathbf{Q}_{ii}-s\mathbf{I}_{|l(i)|})+(1-\delta_{iN})\widetilde{\mathbf{m}}_{i+1}(s)\mathbf{Q}_{i+1,i} = \mathbf{d}_i,\ i=0,...,N, \tag{16}$$

where $\mathbf{d}_i = (\mathbf{d}_{i0}, ..., \mathbf{d}_{i,N-i})$, $\mathbf{d}_{ij} = (d_{ij0}, ..., d_{ij,N-i-j})$, *for* $i = 0, ..., N$ *and* $j = 0, ..., N-i$, *are as follows:*

i) $\mathbf{d}_i = -\left((1-\delta_{i0})\widetilde{\mathbf{E}}^1_{i-1}(s)\mathbf{Q}_{i-1,i} + (1-\delta_{iN})\widetilde{\mathbf{S}}^1_{i+1}(s)\mathbf{Q}_{i+1,i}\right)$, *for* $\widetilde{\mathbf{m}}(s) = \widetilde{\mathbf{SE}}(s)$.

ii) $\mathbf{d}_i = -\left((1-\delta_{i0})\widetilde{\mathbf{I}}^1_{i-1}(s)\mathbf{Q}_{i-1,i} + \widetilde{\mathbf{S}}^1_i(s)\left(\mathbf{Q}_{ii} - \mathbf{Q}^{*I}_{ii}\right)\right)$, *for* $\widetilde{\mathbf{m}}(s) = \widetilde{\mathbf{SI}}(s)$.

iii) $\mathbf{d}_i = -\left((1-\delta_{i0})\widetilde{\mathbf{Q}}^1_{i-1}(s)\mathbf{Q}_{i-1,i} + \widetilde{\mathbf{S}}^1_i(s)\left(\mathbf{Q}_{ii} - \mathbf{Q}^{*Q}_{ii}\right)\right)$, *for* $\widetilde{\mathbf{m}}(s) = \widetilde{\mathbf{SQ}}(s)$.

iv) $\mathbf{d}_i = -\left((1-\delta_{iN})\widetilde{\mathbf{I}}^1_{i+1}(s)\mathbf{Q}_{i+1,i} + \widetilde{\mathbf{E}}^1_i(s)\left(\mathbf{Q}_{ii} - \mathbf{Q}^{*I}_{ii}\right)\right)$, *for* $\widetilde{\mathbf{m}}(s) = \widetilde{\mathbf{EI}}(s)$.

v) $\mathbf{d}_i = -\left((1-\delta_{iN})\widetilde{\mathbf{Q}}^1_{i+1}(s)\mathbf{Q}_{i+1,i} + \widetilde{\mathbf{E}}^1_i(s)\left(\mathbf{Q}_{ii} - \mathbf{Q}^{*Q}_{ii}\right)\right)$, *for* $\widetilde{\mathbf{m}}(s)\widetilde{\mathbf{EQ}}(s)$.

vi) $\mathbf{d}_i = -\left((\widetilde{\mathbf{Q}}^1_i(s)\left(\mathbf{Q}_{ii} - \mathbf{Q}^{*I}_{ii}\right) + \widetilde{\mathbf{I}}^1_i(s)\left(\mathbf{Q}_{ii} - \mathbf{Q}^{*Q}_{ii}\right)\right)$, *for* $\widetilde{\mathbf{m}}(s) = \widetilde{\mathbf{IQ}}(s)$.

Proof. In order to prove the part i) we multiply both sides of Eqs (12) by nm and we sum them over n, m, l and h. After that, we take Laplace transforms on the resulting equations and we obtain the following system of equations for $\widetilde{SE}_{ijk}(s)$, $(i,j,k) \in \mathbf{S}$,

$$\begin{aligned}
&(-q_{ijk}-s)\widetilde{SE}_{ijk}(s) + (1-\delta_{iN})(1-\delta_{j0})(i+1)(k\hat{\beta}+\varepsilon)\widetilde{SE}_{i+1,j-1,k}(s) \\
&+(1-\delta_{jN})(1-\delta_{k0})(j+1)\sigma\widetilde{SE}_{i,j+1,k-1}(s) + (1-\delta_{kN})(k+1)\alpha\widetilde{SE}_{ij,k+1}(s) \\
&+(1-\delta_{i0})(1-\delta_{kN})(k+1)\gamma\widetilde{SE}_{i-1,j,k+1}(s) + (1-\delta_{i0})(N-i-j-k+1)\delta\widetilde{SE}_{i-1,jk}(s) \\
&= -(1-\delta_{iN})(1-\delta_{j0})(i+1)(k\hat{\beta}+\varepsilon)\widetilde{S}^1_{i+1,j-1,k}(s) \\
&-(1-\delta_{i0})(1-\delta_{kN})(k+1)\gamma\widetilde{E}^1_{i-1,j,k+1}(s),\ (i,j,k) \in \mathbf{S}.
\end{aligned} \tag{17}$$

If we express Eq. (17) in matrix form we obtain (16) for the case i). Parts ii)-vi) get proved in a similar way and they are omitted. □

Once Laplace transforms have been obtained, the marginal probabilities and the moments can be calculated, as in [8], by inverting numerically their corresponding Laplace transforms. For example, we obtain the probabilities $p^S_n(t) = P\{N_S(t) = n\}$, for $n \in \mathbb{N}$, by inverting numerically the sum $\sum_{(i,j,k)\in\mathbf{S}} \widetilde{p}^S_{ijkn}(s)$.

5. Numerical examples

In this section some numerical results are presented in order to provide insight on the behavior of the SEIQS model, with external source of infection. There are many ways to design the numerical analysis due to the large number of the parameters in the model. Depending on the set of parameters we take, the behavior of the analyzed characteristics will vary.

We structure this section in three subsections. The first one corresponds to the stationary distributions of the number of susceptible, exposed, infectious and quarantined computers. Secondly, some results about the extinction time and the hazard time are presented. The third subsection is dedicated to numerical results for the number of susceptible, exposed, infectious and quarantined computers during a fixed period of time, that is their transient behavior. Numerical results have been obtained by using Fortran programming language and the figures have been plotted by using Matlab.

For the sake of simplicity of the exposition, in all of the examples the values of the individual external infection contact rate, of the individual incubation rate and of the individual recovery rate from quarantine are fixed, more concretely, $\varepsilon = 0.05$, $\sigma = 1.0$, $\delta = 1.0$. Moreover, we consider the initial state $(i_0, j_0, k_0) = (N-1, 0, 1)$, that is the epidemic begins with one infectious computer and the rest of computers are susceptible.

5.1. Numerical results for stationary distributions

In this section results about the limiting probabilities mass functions of $S(t)$, $E(t)$, $I(t)$ and $Q(t)$, when t tends to infinite, are presented and also their corresponding expectations and standard deviations. A population of 40 computers is considered ($N = 40$). Firstly, Fig. 2 illustrates these limiting probability mass functions when the infection spreads very fast ($\beta = 10.0$) and the recovery is impossible unless quarantine occurs previously ($\gamma = 0.0$), distinguishing two cases: infectious computers are quickly isolated ($\alpha = 10.0$) or more slowly ($\alpha = 0.5$). We observe the following: when quarantine occurs slowly the distribution of the number of susceptible computers concentrates around the small values (i.e. 0 and 1) and the number of infectious does it about 20, however the situation turns around when quarantine occurs quickly, the distribution of the number of infectious computers concentrates close to 0, being 1 the most likely value, followed by 0. We can also see in Fig. 2 the values of the expectations and standard deviations of these limiting distributions, which are respectively denoted by $E[S(\cdot)]$ and $\sigma(S(\cdot))$, for $S(t)$, and the corresponding notation for the other descriptors.

The study of expectations and standard deviations of stationary distributions is extended to more scenarios. Table 1 shows both characteristics for combinations of different values for internal infection contact rate, β, quarantine rate, α, and recovery rate from infection, γ; more specifically we choose $\beta \in \{0.5, 10.0\}$, $\alpha \in \{0.5, 10.0\}$, and $\gamma \in \{0.0, 1.0\}$. First of all, we observe that, for any fixed (α, γ), the faster the infection spreads, the greater the mean number of infectious computers and lesser the mean number of susceptible computers are, such as we expected. We can also observe that, for any fixed (β, γ), the expected number of susceptible computers increases with α and the expected number of infectious decreases with it. These increase and decrease are great for $\gamma = 0.0$, i.e. the effectiveness of increasing the quarantine rate is better when infected computers only can be recovered after isolating them. In this sense, for $\gamma = 0.0$ we observe in Table 1 that if the quarantine rate is small, $\alpha = 0.5$, the largest expected value correspond to $E[I(\cdot)]$ ($E[I(\cdot)] = 14.6059$, for $\beta = 0.5$, $E[I(\cdot)] = 19.0539$, for $\beta = 10.0$); however, if we increase the quarantine rate to $\alpha = 10.0$, the largest value corresponds to the mean number of susceptible computers and the smallest one to the mean number of infectious computers.

5.2. Numerical results for extinction and hazard times

This section present numerical examples for the first passage times analyzed in Section 3, showing their density probability functions and their expectations and standard deviations for different set of parameters.

The following numerical results correspond to the extinction time which are presented by Tables 2 and 3 and by Fig. 3. Tables 2 and 3 show the expectations and standard deviations for the parameter set $\{\alpha \in \{0.5, 1.0\}, \beta \in \{1.0, 5.0\}, \gamma = 0, N \in \{5, 10, 15, 20, 25, 30\}\}$ (Table 2) and for the parameter set $\{\alpha \in \{0.5, 0.7, 0.9, 1.1, 1.3, 1.5\}, \beta \in \{1.0, 2.0, 5.0\}, \gamma = 0.05, N = 20\}$ (Table 3). Fig. 3 illustrates density functions of the extinction time for the parameter set $\{\alpha \in \{0.5, 1.0\}, \beta \in \{1.0, 5.0\}, \gamma = 0.05, N \in \{5, 10\}\}$.

In general, by seeing Tables 2 and 3, we can say that the expected extinction time and its standard deviation increase with the population size, also with the infection contact rate β but they decrease with the quarantine rate α. Results about these tables are get out in more detail below.

In order to see how the population size, N, has an effect on the extinction time, we let N go from 5 to 30 by 5, for different values of α ($\alpha = 0.5, 1.0$) and β ($\beta = 1.0, 5.0$), and the numerical results appear in Table 2. We observe that as the population size increases, both expectation and standard deviation increase; these increases are, in relative terms, bigger when α is small (fixed β) and they are also bigger when β is big (fixed α). Table 2 shows how the expected extinction time increases from 4.8847 to 345.2424, for $(\alpha, \beta) = (1.0, 1.0)$, that is the smallest relative increase, and how it increases from 56.1584 to 3.4249×10^{10}, for $(\alpha, \beta) = (0.5, 5.0)$, that is the biggest increase.

As for Table 3, expectations and standard deviations are showed for a population of $N = 20$ computers and a recovery rate from infection equals 0.05, the quarantine rate α goes from 0.5 to 1.5 in increments of 0.2 and the infection contact rate takes values $\beta = 1.0$, 2.0, 5.0. In this table we can observe the effect of isolating infectious computers depending on the spread of infection. Again, Table 3 reveals that the expectation of extinction time (and its standard deviation) decreases as α increases and it increases as β does it. In relative terms, the greater β is, the more the expected extinction time decreases from $\alpha = 0.5$ to $\alpha = 1.5$, more concretely, the decrease is of 97.30% for $\beta = 1.0$, of 99.75% for $\beta = 2.0$ and of 99.97% for $\beta = 5.0$. As a conclusion, if the infection spreads quickly, the increment of quarantine rate is more effective.

In order to get a more detailed vision of the extinction time distribution, not only their moments, Fig. 3 illustrates four density functions corresponding to two different values of quarantine rate ($\alpha = 0.5, 1.0$) and two different values of infection contact rate ($\beta = 1.0, 5.0$). These four density functions are plotted for $N = 5$ and for $N = 10$. In all cases we take $\gamma = 0.05$. First of all, we see that these distributions are asymmetric and they have heavier right tail for $\beta = 5.0$ than for $\beta = 1.0$, fixed α, we also observe that the right tail is heavier for $\alpha = 0.5$ than for $\alpha = 1.0$, fixed β. Moreover, the density functions for $N = 5$ and $\beta = 5.0$ present two modes, one of them is close to $t = 0$ and the another one is around $t = 5$, for $\alpha = 1.0$, and around $t = 7$ for $\alpha = 0.5$.

The next numerical results correspond to the hazard time, i.e. the first time at which the number of susceptible computers is 0. The results are displayed in Tables 4 and 5 and in Figs. 4 and 5. Both tables present expected values and standard deviations of the hazard time for different set of parameters, Fig. 4 plots expected hazard time as function of population size and Fig. 5 shows density functions of this first passage time. In order to know the influence of the population size on these characteristics, we vary N from 5 to 30 in increments of 5, fix $\gamma = 0.0$ and $(\alpha, \beta) \in \{(0.5, 1.0), (0.5, 5.0), (1.0, 1.0), (1.0, 5.0)\}$, results about the mean hazard time and its standard deviation for these parameters are shown in Table 4. We observe that expectations and standard deviations of hazard time and extinction time have opposite relative behavior as functions of N: the greater the population size is, the greater expectation and standard deviation of hazard time are; the relative increase is less for small values of quarantine rate, fixed the infection contact rate, and it is also less for big values of infection contact rate, fixed the quarantine rate. We can see in Table 4 that the values of these characteristics grow suddenly for $(\alpha, \beta) = (1.0, 1.0)$; however, the increase of these values is quite small for $(\alpha, \beta) = (0.5, 5.0)$. We can also observe this feature graphically in Fig. 4

In Table 5 we can easily observe the effect of the quarantine rate on the expectation and standard deviation of hazard time depending on the infection contact rate value, for $N = 20$ and $\gamma = 0.05$. It shows how the expectation and standard deviation increase with the quarantine rate whatever fixed infection contact rate is. These characteristics rise very fast for $\beta = 1.0$ and much slower for $\beta = 5.0$. We can also observe they decrease as functions of infection contact rate, for fixed quarantine rate, being the relative decrease greater for bigger values of quarantine rate.

The last numerical results about the hazard time are presented by Fig. 5. Four density functions of the hazard time are plotted corresponding to the parameters ($\beta = 5.0, \alpha = 0.5$), ($\beta = 5.0, \alpha = 1.0$), ($\beta = 10.0, \alpha = 0.5$) and ($\beta = 10.0, \alpha = 1.0$), for $N = 5$ and $N = 10$. All of them are asymmetric and unimodal, the mode is closer to $t = 0$ for $\beta = 10.0$ than for $\beta = 5.0$ and it takes the maximum value for ($\beta = 10.0, \alpha = 0.5$). The heaviest right tail is for ($\beta = 5.0, \alpha = 1.0$). By comparing the density functions for $N = 5$ and for $N = 10$, we observe they move to right side and the modes take smaller values for $N = 10$.

5.3. Numerical examples for the transient behavior of descriptors.

In this section we display some examples about the number of times that computers are in one of the four classes (susceptible, exposed, infectious, quarantine) for different periods of time. The parameters that have been used are $N = 15$, $\alpha = 1.0$, $\gamma = 0.0$ and $\beta \in \{1.0, 5.0\}$. First of all, numerical results for expected values and standard deviations of these four descriptors are examined (Table 6). Moreover, correlations between two of them are also inspected (Table 7), where ρ denotes the correlation coefficient of two variables, i.e. the covariance of the two variables divided by the product of their individual standard deviations. Having a look at Table 6, we see that the longer the period is, the greater the expected number of cases in each class and their standard deviations are, so the expectation of total number of events increases with the period of time. For a fixed period of time, the biggest expected value corresponds to the class of exposed computers and the smallest to the class of susceptible ones, unless for $\beta = 1.0$ and $t = 20$ where the biggest is obtained for the class of quarantined computers and the smallest for the class of infectious ones (in this last case the four expected values are very similar, having small relative differences between two of them). We also observe that by increasing the infection contact rate from $\beta = 1.0$ to $\beta = 5.0$ the expected number of events grows in the four classes and the growth is more pronounced in exposed and infectious classes. When $\beta = 5.0$, the biggest expected value is obtained for the class of exposed computers and the smallest for the class of susceptible computers, whatever time period is.

Table 7 shows the correlation coefficients between every two descriptors. When the time period is long, all the correlation coefficients are big, however, for short periods of time these values are smaller, especially $\rho(N_S(t), N_E(t))$, $\rho(N_S(t), N_I(t))$ and $\rho(N_E(t), N_Q(t))$, being more marked these differences when the infection contact rate is smaller.

6. Discussion

In this paper, the $SEIQS$ stochastic epidemic model has been studied by using continuous time Markov chains. Firstly, the model has been formulated and some descriptors have been analyzed. We have obtained results for the distributions and moments of two first passage times, the extinction time and the hazard time, defined as the first time at which all computers are susceptible or none of them are, respectively. We have also presented some results about distributions and moments of random number of times that computers are susceptible, exposed, infectious or quarantined during a fixed period of time. Eventually, some numerical examples concerning the previous descriptors are showed and from them some conclusions can be drawn. Focus on the quarantine effectiveness, when quarantine occurs quickly the stationary distribution of the number of infectious computers concentrates close to 0, being 1 the most likely value, followed by 0; moreover, the expected number of susceptible computers increases with the quarantine rate and the expected number of infectious computers decreases with it. Numerical examples for the extinction time and for the hazard time reveal the effect of isolating infectious computers depending on the spread of infections. The expectation and standard deviation of extinction time decrease with the quarantine rate, being these decreases more pronounced, in relative terms, when the infection contact rate is greater. However, for the hazard time the expectation and standard deviation increase with the quarantine rate and decrease with the infection contact rate, being the relative decrease greater for bigger values of quarantine rate. As a conclusion, if the infection spreads quickly, the increment of quarantine rate is more effective. Numerical examples for the number of times that computers are susceptible, exposed, infectious or isolated, for different periods of time, show that their expectations and standard deviations increase with the period length and the correlations between two of them also increase with it.

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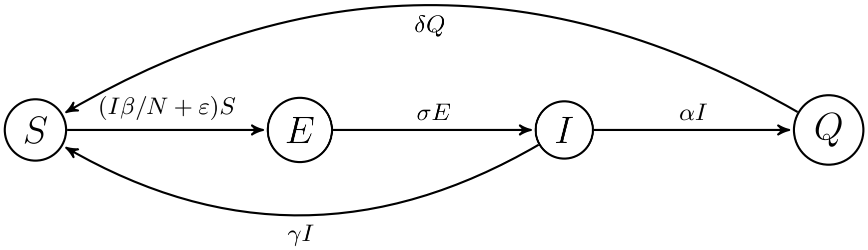

Figure 1: Compartmental flow chart.

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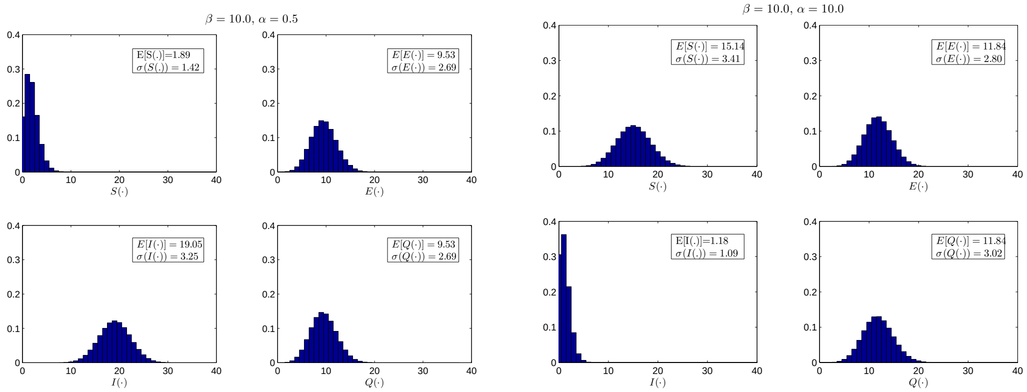

Figure 2: Stationary probability mass functions.

		$\gamma = 0.0$		$\gamma = 1.0$	
		$\alpha = 0.5$	$\alpha = 10.0$	$\alpha = 0.5$	$\alpha = 10.0$
$\beta = 0.5$	$\mathbf{E}[\mathbf{S}(\cdot)]$	**10.7882**	**19.2750**	**18.3919**	**19.7787**
	$\sigma(S(\cdot))$	2.9590	3.1738	3.2305	3.1730
	$\mathbf{E}[\mathbf{E}(\cdot)]$	**7.3029**	**9.8691**	**10.8040**	**10.1107**
	$\sigma(E(\cdot))$	2.3913	2.7173	2.7474	2.7399
	$\mathbf{E}[\mathbf{I}(\cdot)]$	**14.6059**	**0.9869**	**7.2027**	**0.9192**
	$\sigma(I(\cdot))$	3.1334	0.9819	2.4694	0.9483
	$\mathbf{E}[\mathbf{Q}(\cdot)]$	**7.3029**	**9.8691**	**3.6013**	**0.9192**
	$\sigma(Q(\cdot))$	2.4541	2.7338	1.8151	2.6666
$\beta = 10.0$	$\mathbf{E}[\mathbf{S}(\cdot)]$	**1.8922**	**15.1446**	**5.3839**	**15.8515**
	$\sigma(S(\cdot))$	1.4240	3.4062	2.3655	3.3850
	$\mathbf{E}[\mathbf{E}(\cdot)]$	**9.5270**	**11.8359**	**17.3080**	**12.0742**
	$\sigma(E(\cdot))$	2.6940	2.8050	2.8251	2.8222
	$\mathbf{E}[\mathbf{I}(\cdot)]$	**19.0539**	**1.1836**	**11.5387**	**1.0977**
	$\sigma(I(\cdot))$	3.2451	1.0863	3.0061	1.0455
	$\mathbf{E}[\mathbf{Q}(\cdot)]$	**9.5270**	**11.8359**	**5.7693**	**10.9766**
	$\sigma(Q(\cdot))$	2.6940	3.0176	2.2264	2.9320

Table 1: Stationary distribution characteristics of $S(t)$, $E(t)$, $I(t)$, $Q(t)$.

		$\beta = 1.0$		$\beta = 5.0$	
		$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 0.5$	$\alpha = 1.0$
$N = 5$	$E[T]$	11.8402	4.8847	56.1584	15.6697
	$\sigma(T)$	12.9649	5.0358	58.2943	15.8792
$N = 10$	$E[T]$	50.6058	10.6279	2374.1737	147.5390
	$\sigma(T)$	61.1338	2.5227	2541.9990	165.5275
$N = 15$	$E[T]$	247.3819	23.7753	129805.6922	1737.0183
	$\sigma(T)$	294.0406	29.0941	137051.2567	1931.0098
$N = 20$	$E[T]$	1342.2203	55.8316	7905477.8977	23141.2487
	$\sigma(T)$	1536.4801	67.9480	8262036.6821	25239.6949
$N = 25$	$E[T]$	7770.4414	136.6459	5.1068×10^8	329395.5822
	$\sigma(T)$	8621.5038	162.5324	5.2986×10^8	354004.7320
$N = 30$	$E[T]$	46879.1528	345.2425	3.4249×10^{10}	4882277.8917
	$\sigma(T)$	50871.4760	399.8585	3.5344×10^{10}	5189834.9549

Table 2: Characteristics of extinction time by varying N, α, β. $\gamma = 0.0$.

		$\beta = 1.0$	$\beta = 2.0$	$\beta = 5.0$
$\alpha = 0.5$	$E[T]$	748.4941	25288.7010	5236510.3118
	$\sigma(T)$	877.5104	28099.0354	5518793.0166
$\alpha = 0.7$	$E[T]$	149.9117	2185.7107	273827.4143
	$\sigma(T)$	181.4199	2512.1107	293575.8539
$\alpha = 0.9$	$E[T]$	63.1309	499.7540	40129.2789
	$\sigma(T)$	77.5151	591.3061	43757.5812
$\alpha = 1.1$	$E[T]$	37.2150	191.3863	10200.4409
	$\sigma(T)$	46.0034	231.4622	11308.8256
$\alpha = 1.3$	$E[T]$	26.0946	98.9793	3619.2993
	$\sigma(T)$	32.3920	121.5348	4077.6024
$\alpha = 1.5$	$E[T]$	20.2135	61.6232	1600.9976
	$\sigma(T)$	25.1812	76.4431	1831.6006

Table 3: Characteristics of extinction time by varying α, β. $\gamma = 0.05$, $N = 20$.

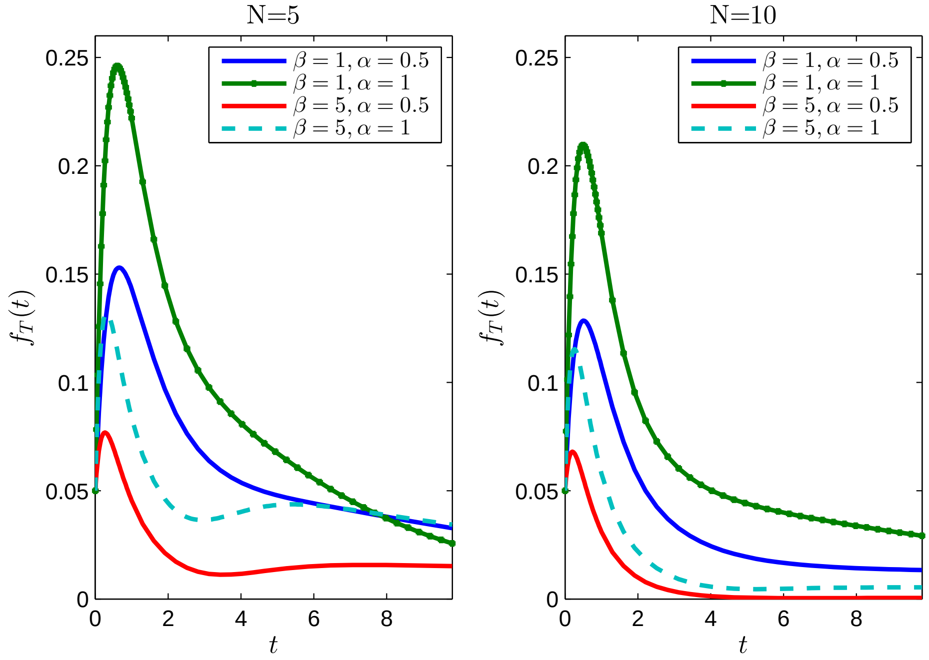

Figure 3: Density functions of extinction time

		$\beta = 1.0$		$\beta = 5.0$	
		$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 0.5$	$\alpha = 1.0$
$N = 5$	$E[H]$	23.4089	86.2161	2.5825	4.7508
	$\sigma(H)$	23.0248	86.6816	3.1064	5.7978
$N = 10$	$E[H]$	70.1726	1187.3086	3.1225	5.6651
	$\sigma(H)$	65.7012	1184.2290	1.9246	4.3663
$N = 15$	$E[H]$	291.7755	23912.6672	3.6720	7.8766
	$\sigma(H)$	285.7479	23908.1599	1.8325	5.9019
$N = 20$	$E[H]$	1493.2586	566988.5015	4.2490	12.0804
	$\sigma(H)$	1486.4430	566983.1737	2.0811	9.9594
$N = 25$	$E[H]$	8520.6362	1.4607×10^7	4.9346	20.2703
	$\sigma(H)$	8513.3305	1.4607×10^7	2.6083	18.2191
$N = 30$	$E[H]$	51673.1897	3.9561×10^8	5.8241	36.6350
	$\sigma(H)$	51665.5391	3.9561×10^8	3.4561	34.7598

Table 4: Characteristics of hazard time by varying N, α, β. $\gamma = 0.0$.

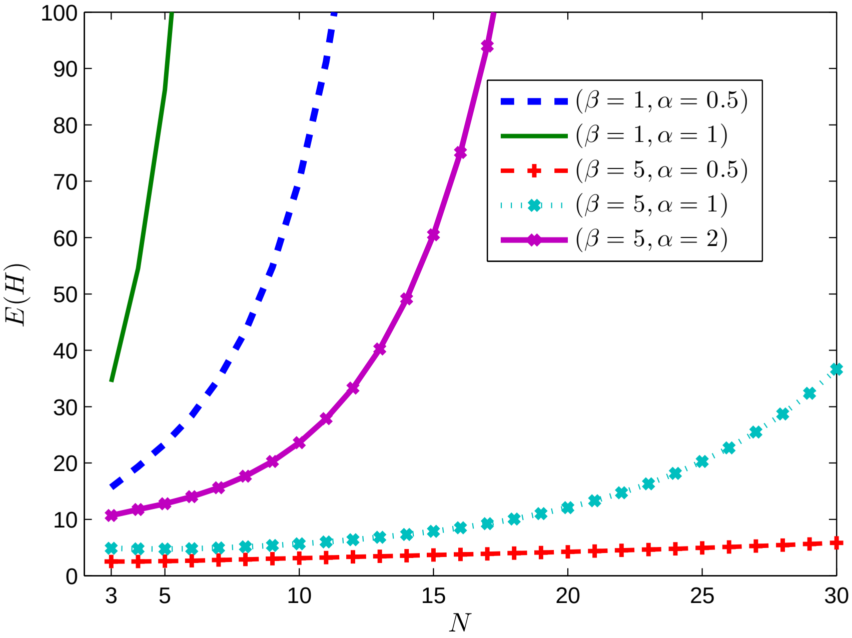

Figure 4: Expected hazard time versus population size.

		$\beta = 1.0$	$\beta = 2.0$	$\beta = 5.0$
$\alpha = 0.5$	$E[H]$	1493.2586	35.7430	4.2490
	$\sigma(H)$	1486.4430	31.5156	2.0811
$\alpha = 1.0$	$E[H]$	566988.5015	1143.4199	12.0804
	$\sigma(H)$	566983.1737	1139.0271	9.9594
$\alpha = 1.5$	$E[H]$	4.1071×10^7	29154.7565	46.5444
	$\sigma(H)$	4.1071×10^7	29150.6272	44.3157
$\alpha = 2.0$	$E[H]$	1.0025×10^9	464231.7113	190.6968
	$\sigma(H)$	1.0025×10^9	464227.9164	188.2568
$\alpha = 2.5$	$E[H]$	1.2269×10^{10}	4833186.9312	769.1839
	$\sigma(H)$	1.2269×10^{10}	4833183.3841	766.5680
$\alpha = 3.0$	$E[H]$	9.3972×10^{10}	3.5854×10^7	2956.2507
	$\sigma(H)$	9.3972×10^{10}	3.5854×10^7	2953.5288

Table 5: Characteristics of hazard time by varying α, β. $\gamma = 0.05$, $N = 20$.

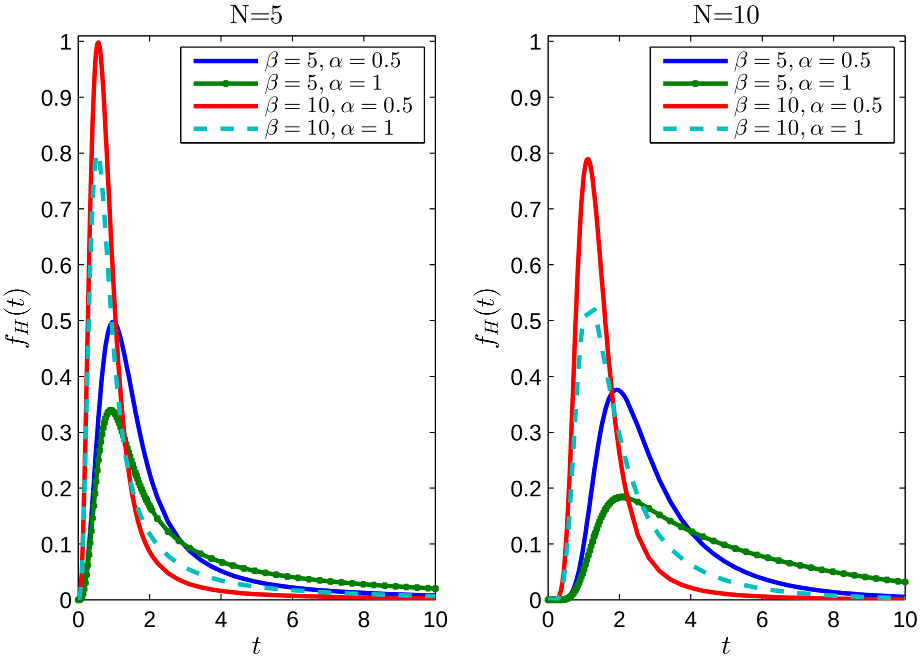

Figure 5: Density functions of hazard time.

	$\beta = 1.0$				$\beta = 5.0$			
	$t = 1$	$t = 2$	$t = 5$	$t = 20$	$t = 1$	$t = 2$	$t = 5$	$t = 20$
$E[N_S(t)]$	0.2984	0.8932	3.7845	7.7945	0.3730	1.5779	10.2201	66.8312
$\sigma(N_S(t))$	0.4758	0.7259	2.0148	3.2020	0.5430	1.1575	4.1703	7.61502
$E[N_E(t)]$	1.3503	2.5899	6.5555	7.9225	4.4690	8.8237	20.3083	77.2245
$\sigma(N_E(t))$	1.2533	1.9290	3.4650	3.4878	2.9796	4.5490	5.6809	7.8444
$E[N_I(t)]$	0.5148	1.4973	5.2253	7.5465	1.6646	5.0581	16.5339	73.4267
$\sigma(N_I(t))$	0.7504	1.3953	3.0338	3.40235	1.57656	3.2586	5.4497	7.8047
$E[N_Q(t)]$	0.7810	1.5934	4.9558	8.1705	1.1082	3.2923	13.7826	70.6289
$\sigma(N_Q(t))$	0.6078	0.9431	2.5153	3.30185	0.8349	1.9327	4.9104	7.6965

Table 6: Characteristics of transient behavior of descriptors: expectations and standard deviations.

	$\beta = 1.0$				$\beta = 5.0$			
	$t = 1$	$t = 2$	$t = 5$	$t = 20$	$t = 1$	$t = 2$	$t = 5$	$t = 20$
$\rho(N_S(t), N_E(t))$	-0.0569	0.1814	0.7285	0.9388	-0.0847	0.3961	0.8578	0.9523
$\rho(N_S(t), N_I(t))$	0.0353	0.2821	0.7949	0.9589	0.0442	0.4626	0.8713	0.95315
$\rho(N_S(t), N_Q(t))$	0.4314	0.5391	0.8853	0.9808	0.4054	0.6427	0.9233	0.9728
$\rho(N_E(t), N_I(t))$	0.6882	0.8445	0.9457	0.9853	0.8164	0.9104	0.9553	0.9790
$\rho(N_E(t), N_Q(t))$	0.0772	0.4982	0.8403	0.9614	0.18318	0.6732	0.8854	0.9518
$\rho(N_I(t), N_Q(t))$	0.2789	0.6565	0.9108	0.9809	0.4059	0.7809	0.9312	0.9699

Table 7: Characteristics of transient behavior of descriptors: correlation coefficients.