A Corrected Algorithm for Computing the Theoretical Autocovariance Matrices of a Vector ARMA Model

José Alberto Mauricio

Instituto Complutense de Análisis Económico

Universidad Complutense de Madrid

Campus de Somosaguas

28223 Madrid - SPAIN

Abstract: The algorithm of Kohn and Ansley (1982) is reconsidered here, in order to correct several implementation errors concerning the construction of the linear equations that must be solved for computing the theoretical autocovariance matrices of a vector ARMA model. This note presents a concise description of the corrected algorithm.

Resumen: En esta nota se corrigen algunos errores del algoritmo de Kohn y Ansley (1982), que tienen que ver con la construcción de un sistema de ecuaciones lineales para calcular las matrices de autocovarianzas teóricas de un modelo ARMA multivariante.

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1. INTRODUCTION

Consider the stationary vector ARMA(p,q) model:

$$\mathbf{\Phi}(B)\tilde{\mathbf{w}}_t = \mathbf{\Theta}(B)\mathbf{a}_t \ , \tag{1}$$

where

$$\mathbf{\Phi}(B) = \mathbf{I}_m - \mathbf{\Phi}_1 B - \mathbf{\Phi}_2 B^2 - \cdots - \mathbf{\Phi}_p B^p \ ,$$
$$\mathbf{\Theta}(B) = \mathbf{I}_m - \mathbf{\Theta}_1 B - \mathbf{\Theta}_2 B^2 - \cdots - \mathbf{\Theta}_q B^q \ ,$$

$\mathbf{\Phi}_i$ ($i = 1, ..., p$) and $\mathbf{\Theta}_i$ ($i = 1,..., q$) are $m\times m$ parameter matrices, $\mathbf{I}_m$ is the identity matrix of order m, $\boldsymbol{\mu}$ is the $m\times 1$ mean vector, $\tilde{\mathbf{w}}_t = \mathbf{w}_t - \boldsymbol{\mu}$ is the $m\times 1$ vector of mean-corrected observations at time t, B is the backshift operator ($B^k\mathbf{z}_t = \mathbf{z}_{t-k}$), and the $\mathbf{a}_t$'s are $m\times 1$ random vectors identically and independently distributed, with $E[\mathbf{a}_t] = \mathbf{0}$, $E[\mathbf{a}_t\mathbf{a}_t^T] = \mathbf{\Sigma}$ (symmetric and positive definite) and $E[\mathbf{a}_t\mathbf{a}_{t+k}^T] = \mathbf{0}$ ($k \neq 0$).

The theoretical autocovariance matrices of the stationary process $\mathbf{w}_t$ generated by (1) are defined by

$$\mathbf{\Gamma}_k = E[\tilde{\mathbf{w}}_t\tilde{\mathbf{w}}_{t+k}^T] = E[\tilde{\mathbf{w}}_{t-k}\tilde{\mathbf{w}}_t^T] = \mathbf{\Gamma}_{-k}^T \ ,$$

and can be shown to be given by the following expression (see, for example, Jenkins and Alavi 1981):

$$\mathbf{\Gamma}_k = \mathbf{\Gamma}_{k-1}\mathbf{\Phi}_1^T + \cdots + \mathbf{\Gamma}_{k-p}\mathbf{\Phi}_p^T + \mathbf{\Lambda}_k - \mathbf{\Lambda}_{k-1}\mathbf{\Theta}_1^T - \cdots - \mathbf{\Lambda}_{k-q}\mathbf{\Theta}_q^T \qquad (k \geq 0) \ , \tag{2}$$

where $\mathbf{\Lambda}_i = E[\tilde{\mathbf{w}}_t\mathbf{a}_{t+i}^T] = E[\tilde{\mathbf{w}}_{t-i}\mathbf{a}_t^T]$ is the theoretical cross-covariance matrix of order i between $\mathbf{w}_t$ and $\mathbf{a}_t$. The computational problem arises in the calculation of the $\mathbf{\Gamma}_k$'s, since, although the $\mathbf{\Lambda}_i$'s need to be obtained too, their calculation is straightforward (see Section 2).

Both the theoretical autocovariance and cross-covariance matrices are needed, for example, to compute the objective function during exact maximum likelihood estimation of (1) (see, among others, Hall and Nicholls 1980; Shea 1989; and Mauricio 1995). Other applications include the generation of independent realizations from a vector ARMA model (see, for example, Shea 1988), which is particularly useful in Monte Carlo studies. Thus, as Kohn and Ansley (1982) point out, it is important to have a fast method of obtaining those matrices, since they are recalculated many times during both estimation and simulation runs.

The method of Kohn and Ansley (1982) provides a fast means of computing the autocovariance matrices. In particular, it is faster than the methods of Hall and Nicholls (1980) and Ansley (1980). However, the implementation in the algorithmic section of Kohn and Ansley (1982, pp. 278-279) is incorrect. In particular, the guidelines on the construction of the linear equations, that must be solved in order to compute the first p autocovariance matrices, are wrong. A corrected version of that algorithm, based on the theoretical background of Section 2, is presented in Section 3. The construction of both the matrix and the right-hand-side vector of the system of linear equations for the first p autocovariance matrices, is described in detail. Some further comments in Section 4 complete this note.

2. THEORETICAL BACKGROUND

In order to motivate the algorithm of Section 3, a concise summary of the method proposed by Kohn and Ansley (1982), using the notation introduced in Section 1, is now given. First, postmultiplying (1) by $\mathbf{a}_{t+i}^T$, taking expectations, and noting that $E[\tilde{\mathbf{w}}_{t-i}\mathbf{a}_t^T] = E[\mathbf{a}_t\mathbf{a}_{t+i}^T] = \mathbf{0}$ for $i > 0$, it is straightforward to show that

$$\mathbf{\Lambda}_{-i} = -\mathbf{\Theta}_i\mathbf{\Sigma} + \sum_{j=1}^{i}\mathbf{\Phi}_j\mathbf{\Lambda}_{j-i} \quad (i \geq 1) \ , \tag{3}$$

where $\mathbf{\Phi}_j = \mathbf{0}$ if $j > p$, $\mathbf{\Lambda}_0 = \mathbf{\Sigma}$, and $\mathbf{\Lambda}_i = \mathbf{0}$ for $i > 0$. Then, writing out (2) for $k = 0$ and for $k \geq 1$:

$$\mathbf{\Gamma}_0 = \sum_{i=1}^{p} \mathbf{\Gamma}_{-i} \mathbf{\Phi}_i^T + \mathbf{\Lambda}_0 - \sum_{j=1}^{q} \mathbf{\Lambda}_{-j} \mathbf{\Theta}_j^T , \tag{4}$$

$$\mathbf{\Gamma}_k = \mathbf{W}_k + \sum_{j=1}^{p} \mathbf{\Gamma}_{k-j} \mathbf{\Phi}_j^T \quad (k \geq 1) , \tag{5}$$

where

$$\mathbf{W}_k = -\sum_{j=k}^{q} \mathbf{\Lambda}_{k-j} \mathbf{\Theta}_j^T \quad (k \geq 1) . \tag{6}$$

Noting that $\mathbf{\Gamma}_{-i} = \mathbf{\Gamma}_i^T$ and using (5), with $k = i$, in (4), it turns out that

$$\mathbf{\Gamma}_0 = \mathbf{W}_0 + \sum_{j=1}^{p} \sum_{i=1}^{p} \mathbf{\Phi}_i \mathbf{\Gamma}_{i-j} \mathbf{\Phi}_j^T , \tag{7}$$

where

$$\mathbf{W}_0 = \mathbf{\Sigma} - (\mathbf{B} + \mathbf{B}^T) + \sum_{j=1}^{q} \mathbf{\Theta}_j \mathbf{\Sigma} \mathbf{\Theta}_j^T , \tag{8}$$

$$\mathbf{B} = \sum_{i=1}^{p} \sum_{j=i}^{q} \mathbf{\Phi}_i \mathbf{\Lambda}_{i-j} \mathbf{\Theta}_j^T . \tag{9}$$

Finally, since $\mathbf{\Gamma}_{-k} = \mathbf{\Gamma}_k^T$, it is clear that, when (5) and (6) are written for $k = 1, ..., p-1$, only $\mathbf{\Gamma}_0$ through $\mathbf{\Gamma}_{p-1}$ (unknown) and $\mathbf{\Lambda}_0$ through $\mathbf{\Lambda}_{-q+1}$ (known) appear in equations (5) through (9). Thus, the following system of linear equations may be specified in order to calculate the first p unknown theoretical autocovariance matrices:

$$\mathbf{\Gamma}_0 - \sum_{i=1}^{p} \mathbf{\Phi}_i \mathbf{\Gamma}_0 \mathbf{\Phi}_i^T - \sum_{i=1}^{p-1} \sum_{j=1}^{p-i} [\, \mathbf{\Phi}_{i+j} \mathbf{\Gamma}_i \mathbf{\Phi}_j^T + \mathbf{\Phi}_j \mathbf{\Gamma}_i^T \mathbf{\Phi}_{i+j}^T \,] = \mathbf{W}_0 \tag{10}$$

$$\mathbf{\Gamma}_k - \sum_{i=1}^{k-1} \mathbf{\Gamma}_i \mathbf{\Phi}_{k-i}^T - \sum_{i=0}^{p-k} \mathbf{\Gamma}_i^T \mathbf{\Phi}_{k+i}^T = \mathbf{W}_k \quad (k = 1, ..., p-1) \tag{11}$$

Together, expressions (10) and (11) form a system of pm^2 linear equations with pm^2 unknowns (the m^2 elements of each of the p matrices $\mathbf{\Gamma}_k$, $k = 0, 1, ..., p-1$). But since $\mathbf{\Gamma}_0$ is symmetric, it contains only $m(m+1)/2$ different elements. Thus, letting $\tilde{\mathbf{\Gamma}}_0$ and $\tilde{\mathbf{W}}_0$ be the diagonals and upper triangles of $\mathbf{\Gamma}_0$ and $\mathbf{W}_0$, respectively, the $m(m+1)/2 + m^2(p-1)$ unknowns in (10) and (11) are the (unique) solution $\mathbf{x}$ of the linear

system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{x} = \text{vec}(\tilde{\mathbf{\Gamma}}_0, \mathbf{\Gamma}_1, ..., \mathbf{\Gamma}_{p-1})$, $\mathbf{b} = \text{vec}(\tilde{\mathbf{W}}_0, \mathbf{W}_1, ..., \mathbf{W}_{p-1})$ and $\mathbf{A}$ is the matrix of coefficients, which is constructed as described in the next section.

3. The Corrected Algorithm

In order to concentrate on the more relevant aspects of the algorithm (the construction of $\mathbf{A}$ and $\mathbf{b}$), it is assumed that $\mathbf{\Lambda}_0$ and $\mathbf{\Lambda}_{-1}$ through $\mathbf{\Lambda}_{-q+1}$, as well as $\tilde{\mathbf{W}}_0$, have already been computed using (3), (8) and (9). Now, it is shown how to construct $\mathbf{A}$ and $\mathbf{b}$ row by row. First, initialize $\mathbf{A}$ and $\mathbf{b}$ to zero; then, execute the following two steps:

Step 1. Compute the first $m(m+1)/2$ rows:

FOR $j = 1$ TO m

FOR $i = 1$ TO j

$row = j(\ j-1)/2 + i$

Step 1.1. Compute the first $m(m+1)/2$ columns within row:

FOR $l = 1$ TO m

FOR $k = 1$ TO l

$col = l(l-1)/2 + k$

IF $k = l$

$$\mathbf{A}(row,col) \ = \ -\sum_{r=1}^{p} \mathbf{\Phi}_r(i,k)\, \mathbf{\Phi}_r(\ j,l)$$

ELSE

$$\mathbf{A}(row,col) \ = \ -\sum_{r=1}^{p} [\ \mathbf{\Phi}_r(i,k)\, \mathbf{\Phi}_r(\ j,l) \ + \ \mathbf{\Phi}_r(i,l)\, \mathbf{\Phi}_r(\ j,k)\ ]$$

Step 1.2. Compute the remaining $m^2(p-1)$ columns within row:

FOR $s = 1$ TO $p-1$

FOR $l = 1$ TO m

FOR $k = 1$ TO m

$col = m(m+1)/2 + m^2(s-1) + m(l-1) + k$

$$\mathbf{A}(row,col) = -\sum_{r=1}^{p-s} [\ \mathbf{\Phi}_{r+s}(i,k)\,\mathbf{\Phi}_r(j,l) + \mathbf{\Phi}_{r+s}(j,k)\,\mathbf{\Phi}_r(i,l)\ ]$$

Step 1.3. Set up diagonal of $\mathbf{A}$ and right-hand-side $\mathbf{b}$:

$\mathbf{A}(row,row) = 1 + \mathbf{A}(row,row)$

$\mathbf{b}(row) = \tilde{\mathbf{W}}_0(i,j)$

Step 2. Compute the remaining $m^2(p-1)$ rows:

FOR $s = 1$ TO $p-1$

FOR $i = 1$ TO m

FOR $j = 1$ TO m

$row = m(m+1)/2 + m^2(s-1) + m(i-1) + j$

Step 2.1. Compute the first $m(m+1)/2$ columns within row:

FOR $l = 1$ TO m

IF $l \le j$

$col = j(j-1)/2 + l$

ELSE

$col = l(l-1)/2 + j$

$\mathbf{A}(row,col) = -\mathbf{\Phi}_s(i,l)$

Step 2.2. Compute the remaining $m^2(p-1)$ columns within *row*:

FOR $r = 1$ TO $p-1$

FOR $l = 1$ TO m

$col = m(m+1)/2 + m^2(r-1) + m(j-1) + l$

IF $r+s \leq p$

$\mathbf{A}(row,col) = -\mathbf{\Phi}_{r+s}(i,l)$

IF $s > r$

$col = m(m+1)/2 + m^2(r-1) + m(l-1) + j$

$\mathbf{A}(row,col) = \mathbf{A}(row,col) - \mathbf{\Phi}_{s-r}(i,l)$

Step 2.3. Set up diagonal of $\mathbf{A}$ and right-hand-side $\mathbf{b}$:

$\mathbf{A}(row,row) = 1 + \mathbf{A}(row,row)$

FOR $h = s$ TO q

$$\mathbf{b}(row) = \mathbf{b}(row) - \sum_{k=1}^{m} \mathbf{\Lambda}_{s-h}(j,k)\,\mathbf{\Theta}_h(i,k)$$

END.

Once $\mathbf{A}$ and $\mathbf{b}$ are constructed, the solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ may be obtained using standard numerical linear algebra routines (see, for example, Moler 1972; and Press et al. 1992). Then, the elements of the theoretical autocovariance matrices can be extracted from $\mathbf{x}$ as follows:

$$\mathbf{\Gamma}_0(i,j) = \mathbf{x}_{j(j-1)/2+i} \quad (i = 1, ..., m;\ j = i, ..., m)\ ;$$

$$\mathbf{\Gamma}_k(i,j) = \mathbf{x}_{m(m+1)/2+m^2(k-1)+m(j-1)+i} \quad (i = 1, ..., m;\ j = 1, ..., m\ )\ ,\ k = 1, ..., p-1\ .$$

For $k \geq p$, the autocovariances can be computed recursively using (5), (6) and (3).

4. Concluding Remarks

Note that, except for the first $m(m+1)/2$ coefficients in each of the first $m(m+1)/2$ rows of **A** (step 1.1 above), the implementation presented here differs from that of Kohn and Ansley (1982), which is incorrect. To note this, computer programs were written by the author, following the implementation in Kohn and Ansley (1982) and the one described in Section 3. The theoretical autocovariance matrices, calculated through the implementation in Kohn and Ansley (1982) for a wide range of vector ARMA models, always showed substantial differences with the results obtained through the implementation described here. Furthermore, the latter results always agreed with those obtained through standard procedures, such as that of Hall and Nicholls (1980) as implemented in Shea (1989).

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