

On the principle of pseudo-linearized stability: application to some delayed nonlinear parabolic equations

A.C. Casal^a and J.I. Diaz^b

^aDept. de Matemática Aplicada, E.T.S.A., Univ. Politécnica de Madrid,
28040 Madrid, Spain

^bDept. de Matemática Aplicada, Fac. de Matemáticas, Univ. Complutense de Madrid,
28040 Madrid, Spain

1. Introduction

We study the stabilization, as $t \rightarrow \infty$, of the solutions of the nonlinear abstract functional differential equation

$$\begin{cases} \frac{du}{dt}(t) + Au(t) + Bu(t) \ni F(u_t(\cdot)) & \text{in } X, \\ u(s) = u_0(s) & s \in [-\tau, 0]. \end{cases} \quad (1)$$

on a Banach space X , where

$$u_t(\theta) = u(t + \theta), \theta \in [-\tau, 0],$$

to the associated equilibria: $w \in D(A) \subset D(B) \subset X$ such that

$$Aw + Bw \ni F(\widehat{w}(\cdot)),$$

where $\widehat{w} \in C := C([- \tau, 0] : X)$ is the function which takes constant values equal to w . Our main goal is to extend, to a broad class of nonlinear operators A , the usual linearized stability principle saying, roughly speaking, that for the special case of A linear (single valued) and B and F are differentiable, the asymptotic stability of the zero solution of the linearized equation,

$$\begin{cases} \frac{dv}{dt}(t) + Av(t) + DB(w)v(t) = DF(\widehat{w})v_t(\cdot) & \text{in } X, \\ v(s) = u_0(s) & s \in [-\tau, 0]. \end{cases}$$

implies that $u(t : u_0) \rightarrow w$ as $t \rightarrow \infty$, at least if $u_0(\cdot)$ is close enough to \widehat{w} . We point out that our results seem to be new even without the delayed and nonlocal term (i.e. for $F \equiv 0$).

Our main motivation comes from some previous works by the authors and collaborators ([16], [12]), [11]) dealing with the stabilization of the uniform oscillations for the *complex Ginzburg-Landau equation*. This stabilization takes place by means of some global delayed

feedback. If, for instance, we consider the case in which the domain is $\Omega = (0, L_1) \times (0, L_2)$ with periodic boundary conditions, and define the faces of the boundary

$$\Gamma_j = \partial\Omega \cap \{x_j = 0\}, \Gamma_{j+2} = \partial\Omega \cap \{x_j = L_j\}, j = 1, 2,$$

this problem can be stated as follows

$$(P_1) \begin{cases} \frac{\partial \mathbf{u}}{\partial t} - (1 + i\epsilon)\Delta \mathbf{u} = (1 - i\omega)\mathbf{u} - (1 + i\beta)|\mathbf{u}|^2 \mathbf{u} + \mu e^{i\chi_0} \mathbf{F}(\mathbf{u}, t, \tau) & \Omega \times (0, +\infty), \\ \mathbf{u}|_{\Gamma_j} = \mathbf{u}|_{\Gamma_{j+2}}, \left(-\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right)|_{\Gamma_j} = \frac{\partial \mathbf{u}}{\partial x_j}\bigg|_{\Gamma_j} = \frac{\partial \mathbf{u}}{\partial x_j}\bigg|_{\Gamma_{j+2}} \left(= \frac{\partial \mathbf{u}}{\partial \mathbf{n}}\bigg|_{\Gamma_{j+2}}\right) & \partial\Omega \times (0, +\infty), \\ \mathbf{u}(x, s) = \mathbf{u}_0(x, s) & \Omega \times [-\tau, 0], \end{cases}$$

where \mathbf{n} is the outpointing normal unit vector and

$$\mathbf{F}(\mathbf{u}, t, \tau) = [m_1 \mathbf{u}(t) + m_2 \bar{\mathbf{u}}(t) + m_3 \mathbf{u}(t - \tau, x) + m_4 \bar{\mathbf{u}}(t - \tau)] \text{ with } \bar{\mathbf{u}}(s) = \frac{1}{|\Omega|} \int_{\Omega} \mathbf{u}(s, x) dx.$$

Here the parameters $\epsilon, \beta, \omega, \mu, \chi_0, m_i$ and τ are real numbers, in contrast with the solution $\mathbf{u}(x, t) = u_1(x, t) + iu_2(x, t)$.

This type of equations (called as of Stuart-Landau in absence of the diffusion term) arise in the study of the stability of reaction diffusion equations such as $\frac{\partial \mathbf{X}}{\partial t} - \mathbf{D}\Delta \mathbf{X} = \mathbf{f}(\mathbf{X}; \eta)$ where $\mathbf{X} : \Omega \times (0, +\infty) \rightarrow \mathbb{R}^n$ and η is a real scalar parameter when the deviation \mathbf{v} from the uniform state solution \mathbf{X}_{∞} is developed asymptotically in terms of some multiple scales (see Kuramoto [19]). Coefficient ϵ measures the degree to which the diffusion matrix \mathbf{D} deviates from a scalar. With the basis of a sound experimental work, many recent studies of a more descriptive nature, but of a great originality and interest, have been written. In those studies the delay term $\mathbf{F}(\mathbf{u}, t, \tau)$ has been taken corresponding to $m_4 = 1, m_i = 0$ for $i = 1, 2, 3$ and introduced as a control mechanism (see Battogtokh and Mikhailov [6], Mertens *et al.* [21]).

If we focus our attention on the so called *slowly varying complex amplitudes* defined by $\mathbf{u}(x, t) = \mathbf{v}(x, t)e^{-i\omega t}$, thus, \mathbf{v} satisfy

$$(P_2) \begin{cases} \frac{\partial \mathbf{v}}{\partial t} - (1 + i\epsilon)\Delta \mathbf{v} = \mathbf{v} - (1 + i\beta)|\mathbf{v}|^2 \mathbf{v} + \\ + \mu e^{i\chi_0} [m_1 \mathbf{v} + m_2 \bar{\mathbf{v}} + e^{i\omega\tau} (m_3 \mathbf{v}(t - \tau, x) + m_4 \bar{\mathbf{v}}(t - \tau))] & \text{in } \Omega \times (0, +\infty), \\ \mathbf{v}|_{\Gamma_j} = \mathbf{v}|_{\Gamma_{j+2}}, \left(-\frac{\partial \mathbf{v}}{\partial \mathbf{n}}\right)|_{\Gamma_j} = \frac{\partial \mathbf{v}}{\partial x_j}\bigg|_{\Gamma_j} = \frac{\partial \mathbf{v}}{\partial x_j}\bigg|_{\Gamma_{j+2}} \left(= \frac{\partial \mathbf{v}}{\partial \mathbf{n}}\bigg|_{\Gamma_{j+2}}\right) & \text{on } \partial\Omega \times (0, +\infty), \\ \mathbf{v}(x, s) = \mathbf{u}_0(x, s)e^{i\omega s} & \text{on } \Omega \times [-\tau, 0]. \end{cases}$$

The existence and uniqueness of a solution of (P_1) can be proven once we assume, for instance, that $\mathbf{u}_0 \in \mathbf{C}([-\tau, 0] : \mathbf{L}^2(\Omega))$ (see Díaz, Padial, J.I. Tello, L. Tello, [16]). In the mentioned references we were interested in the stability analysis of the time-periodical function $\mathbf{v}_{uosc}(x, t) = \rho_0 e^{-i\theta t}$. We can reduce the study to the stability of stationary solutions of some auxiliary problem by introducing the change of unknown $\mathbf{z}(x, t) = \mathbf{v}(x, t)e^{i\theta t}$ where $\mathbf{v}(x, t)$ is a solution of (P_2) . Thus $\mathbf{z}(x, t)$ satisfies

$$(P_3) \begin{cases} \frac{\partial \mathbf{z}}{\partial t} - (1 + i\epsilon)\Delta \mathbf{z} = (1 + i\theta)\mathbf{z} - (1 + i\beta)|\mathbf{z}|^2 \mathbf{z} + \\ + \mu e^{i\chi_0} [m_1 \mathbf{z} + m_2 \bar{\mathbf{z}} + e^{i(\omega+\theta)\tau} (m_3 \mathbf{z}(t - \tau, x) + m_4 \bar{\mathbf{z}}(t - \tau))] & \text{in } \Omega \times (0, +\infty), \\ \mathbf{z}|_{\Gamma_j} = \mathbf{z}|_{\Gamma_{j+2}}, \left(-\frac{\partial \mathbf{z}}{\partial \mathbf{n}}\right)|_{\Gamma_j} = \frac{\partial \mathbf{z}}{\partial x_j}\bigg|_{\Gamma_j} = \frac{\partial \mathbf{z}}{\partial x_j}\bigg|_{\Gamma_{j+2}} \left(= \frac{\partial \mathbf{z}}{\partial \mathbf{n}}\bigg|_{\Gamma_{j+2}}\right) & \text{on } \partial\Omega \times (0, +\infty), \\ \mathbf{z}(x, s) = \mathbf{u}_0(x, s)e^{i(\omega-\theta)s} & \text{on } \Omega \times [-\tau, 0]. \end{cases}$$

Notice that now, $\mathbf{v}_{uosc}(x, t) = \rho_0 e^{-i\theta t}$ is an uniform oscillation if and only if $\mathbf{z}(x, t) = \mathbf{v}_{uosc}(x, t)e^{i\theta t} = \mathbf{y} = \rho_0$ is an stationary solution of (P_3) : i.e. $\mathbf{0} = (1 + i\theta)\mathbf{y} - (1 + i\beta)|\mathbf{y}|^2\mathbf{y} + \mu e^{i\chi_0} [m_1 + m_2 + e^{i(\omega+\theta)\tau}(m_3 + m_4)]\mathbf{y}$.

The motivation to keep A nonlinear after the process of linearization (reason why we used the term of *pseudo-linearization principle*) comes from the fact that if we use the representation for the unknown of the delayed nonlinear equation (P_3) as $\mathbf{z}(x, t) = \rho(x, t)e^{i\phi(x, t)}$ then we arrive to a coupled nonlinear system of delayed equations for ρ and ϕ which can be described in terms of the representation operator given by $\mathbf{P} : \mathbb{R}^2 \rightarrow \mathbb{C}$, $\mathbf{P}(\rho, \phi) = \rho e^{i\phi}$. Indeed, notice that \mathbf{P} is nonlinear and that if $\mathbf{q} = (\rho, \phi)$ then $\mathbf{z}(x, t) = \mathbf{P}(\mathbf{q}(x, t))$ and the (P_3) can be formulated as $\frac{d\mathbf{P}(\mathbf{q}(\cdot, t))}{dt} + A\mathbf{P}(\mathbf{q}(\cdot, t)) + B\mathbf{P}(\mathbf{q}(\cdot, t)) = F(\mathbf{P}(\mathbf{q}(\cdot, t)))$. By using that the matrix $\mathbf{C}(\mathbf{q}(\cdot, t)) = \text{grad}\mathbf{P}(\mathbf{q}(\cdot, t))$ is not singular, we can arrive to the simpler formulation

$$\frac{d\mathbf{q}}{dt}(\cdot, t) + \mathbf{C}(\mathbf{q}(\cdot, t))^{-1}[A\mathbf{P}(\mathbf{q}(\cdot, t)) + B\mathbf{P}(\mathbf{q}(\cdot, t))] = \mathbf{C}(\mathbf{q}(\cdot, t))^{-1}F(\mathbf{P}(\mathbf{q}(\cdot, t))). \quad (2)$$

Notice that, although this delayed system can be also (formally) linearized (this is the procedure followed in Battogtokh and Mikhailov [6] and Mertens *et al.* [21]) the above diffusion operator $\mathbf{C}(\mathbf{q}(\cdot, t))^{-1}A\mathbf{P}(\mathbf{q}(\cdot, t))$ becomes now quasilinear in \mathbf{q} and thus the mathematical justification is much more delicate.

Other examples, given in Section 3, justify also the philosophy of keeping A non-linear after linearizing the rest of the terms of the equation. For instance, this is the case when A is multivalued, or nondifferentiable or a degenerate quasilinear operator. We point out that some relevant examples of nonlinear functional equations arise in the most different contexts (see, for instance, Díaz and Hetzer [15] for one example in Climatology, Chukwu [13] for a family of examples dealing with the wealth of nations and the general exposition made in Hale [17]).

Coming back to the abstract formulation, the structural assumptions we shall assume in this paper are the following

(H1): $A \in \mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$, with

$$\mathcal{A}(\omega : X) = \{A : D_X(A) \subset X \rightarrow \mathcal{P}(X) \text{ such that } A + \omega I \text{ is a m-accretive operator}\},$$

(see Brezis [14] for the case of $X = H$ a Hilbert space and the works by Benilan, Crandall, Pazy and others for the case of a general Banach space: see the monographs [8] and [24]),

(H2): the operators semigroup $T(t) : \overline{D_X(A)}^X \rightarrow X$, $t \geq 0$, generated by A , is compact

(see Vrabie [24]),

(H3): $B \in \mathcal{A}(0 : X)$, B is single valued, Fréchet differentiable, and B is dominated by A ; i.e.

$$\begin{aligned} & D_X(A) \subset D_X(B) \text{ and } |Bu| \leq k|A^0 u| + \sigma(|u|) \\ & \text{for any } u \in D_X(A) \text{ and for some } k < 1 \text{ and some continuous function } \sigma : \mathbb{R} \rightarrow \mathbb{R}, \end{aligned} \quad (3)$$

where, here and in what follows, $|\cdot|$ denotes the norm in the space X (in contrast with the norm in space C which will be denoted by $\|\cdot\|$ if there is no ambiguity, when handling two spaces X and Y the corresponding norms will be indicated), $|A^0 u| := \inf\{|\xi| : \xi \in Au\}$ for $u \in D_X(A)$,

(H4): $F : C \rightarrow X$ satisfies a local Lipschitz condition, i.e.,

$$\begin{cases} \text{for any } R > 0 \text{ there exists } L(R) > 0 \text{ such that} \\ |F(\phi) - F(\psi)| \leq L(R) \|\phi - \psi\| \text{ for any } \phi, \psi \in C \text{ and } \|\phi\|, \|\psi\| \leq R. \end{cases} \quad (4)$$

(H5): there exists $\delta^F > 0$ such that $F : B_{\delta^F}^X(\widehat{w}) \rightarrow X$ is Fréchet differentiable with the Fréchet derivative $DF(\widehat{w})$ given by $D(F(\widehat{w}))\phi = \int_{-\tau}^0 d\eta(\theta)\phi(\theta)$, $\phi \in C$, for $\eta : [-\tau, 0] \rightarrow B(X, X)$ of bounded variation and the Fréchet derivative is locally Lipschitz continuous, where $B_{\delta^F}^X(\widehat{w}) = \{\phi \in C; \|\phi - \widehat{x}\| < \delta^F\}$,

We further assume the main condition of our arguments:

(H6): the operator $y \rightarrow Ay + By - DF(\widehat{w})(e^{\omega \cdot} y)$ belongs to $\mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$ with $\operatorname{Re} \omega = \gamma < 0$ where $e^{\omega \cdot} v \in C$ is defined by

$$(e^{\omega \cdot} v)(s) = e^{\omega s} \widehat{v}(s), \text{ with } \widehat{v}(s) = v, \text{ for any } s \in [-\tau, 0] \text{ for } v \in X. \quad (5)$$

In order to treat the case in which B is differentiable we introduce the conditions

(H7): there exists a Banach space Y and there exists $\delta^B > 0$ such that B is Fréchet differentiable as function from $B_{\delta^B}(w) = \{z \in D(B); |w - z| < \delta^B\}$ into Y , with the Fréchet derivative $DB(w)$ locally Lipschitz continuous,

and

(H8) the operator $y \rightarrow Ay + DB(w)y - DF(\widehat{w})(e^{\omega^* \cdot} y)$ belongs to $\mathcal{A}(\omega^* : Y)$, for some $\omega^* \in \mathbb{C}$ with $\operatorname{Re} \omega^* = \gamma^* < 0$.

2. The abstract results

Theorem 1 Assume (H1)-(H6). Then there exists $\alpha > 0$, $\epsilon > 0$ and $M \geq 1$ such that if $u_0 \in B_\epsilon^X(\widehat{w})$, $u_0(s) \in D_X(B)$ for any $s \in [-\tau, 0]$ then the solution $u(\cdot : u_0)$ of (1) exists on $[-\tau, +\infty)$ and

$$|u(t : u_0) - w| \leq M e^{-\alpha t} \|u_0 - \widehat{w}\|, \text{ for any } t > 0. \quad (6)$$

Moreover, if we also assume (H7), that (H1)-(H5) holds on the space Y and (H8) then there exists $\alpha^* > 0$, $\epsilon^* \in (0, \epsilon]$ and $M^* \geq 1$ such that if $u_0 \in B_{\epsilon^*}^{X \cap Y}(\widehat{w})$, $u_0(s) \in D_X(B) \cap D_Y(B)$ for any $s \in [-\tau, 0]$ then

$$|u(t : u_0) - w|_X + |u(t : u_0) - w|_Y \leq M^* e^{-\alpha^* t} (\|u_0 - \widehat{w}\|_X + \|u_0 - \widehat{w}\|_Y), \text{ for any } t > 0. \quad (7)$$

Proof. From assumptions (H4) and (H5)

$$F(\phi) = F(\hat{w}) + DF(\hat{w})(\phi - \hat{w}) + G^F(\hat{w}, \phi), \text{ for any } \phi \in B_{\delta^F}^X(\hat{w}).$$

Moreover since $DF(\hat{w})$ is locally Lipschitz continuous, there exists a continuous increasing functions b_X^F such that

$$|G^F(\hat{w}, \phi)| \leq b_X^F(\|\phi - \hat{w}\|) \|\phi - \hat{w}\|, \text{ for any } \phi \in B_{\delta^F}^X(\hat{w}). \quad (8)$$

Then

$$\frac{du}{dt}(t) - \frac{dw}{dt} + Au(t) - Aw + Bu(t) - Bw - DF(\hat{w})(u_t - \hat{w}) \ni -G^F(\hat{w}, u_t). \quad (9)$$

We now use assumption (H6). We claim that we can find a constant constant $K \geq 1$ and such that

$$\|u_t - \hat{w}\| \leq Ke^{\gamma t} \|u_0 - \hat{w}\| + \int_0^t Ke^{\gamma(t-s)} |G^F(\hat{w}, u_s)| ds. \quad (10)$$

Indeed, as $u(t)$ and w are “integral solutions” in the sense of Benilan (see. e.g. [8]), then, by (H6), if we multiply (9) by $u(t) - w$ (by using the usual semi inner-braket $[\cdot, \cdot]$: see, for instance Benilan, Crandall and Pazy [8] or Vrabie [24] (Section 1.4)) we get that

$$|u(t) - w| \leq Ke^{\gamma(t-t_0)} |u(t_0) - w| + \int_{t_0}^t Ke^{\gamma(t-s)} |G^F(\hat{w}, u_s)| ds \quad (11)$$

for any $t \geq t_0 \geq 0$ (see, for instance, Benilan, Crandall and Pazy [8] or Vrabie [24] Theorem 1.7.5). Then,

$$|u(t) - w| \leq Ke^{\gamma t} \|u_0 - \hat{w}\| + \int_0^t Ke^{\gamma(t-s)} |G^F(\hat{w}, u_s)| ds \quad (12)$$

for any $t \geq 0$. Finally, since (12) holds trivially for $t \in [-\tau, 0]$ we get (10) by taking the maximum, in (11), on intervals of the form $[t - \tau, t]$ for any $t \geq 0$.

Now, let $R \in (0, \delta^F)$ be chosen so that

$$b_X^F(R) < (-\gamma)/(4K). \quad (13)$$

Define $\epsilon = \min \{R/(2K), \delta_X^F\}$. Let us show that if $u_0 \in B_\epsilon^X(\hat{w})$ then the associated solution u of (1) exists and $\|u_t - \hat{w}\| < R$ for all $t \geq 0$. Thanks to assumption (H2) we can apply some maximal continuation results (see, for instance, Chapter 3 of Vrabie [24], or Chapter 2 of Wu [25] when A is linear), it suffices to show that there exists no $t_1 > 0$ so that $\|u_{t_1}\| = R$ and $\|u_t\| < R$ for $t \in [0, t_1)$. By contradiction, if there exists such a t_1 , then on $[0, t_1]$ we have

$$\begin{aligned} \|u_t - \hat{w}\| &\leq Ke^{\gamma t} \|u_0 - \hat{w}\| + \int_0^t Ke^{\gamma(t-s)} |G^F(\hat{w}, u_s)| ds \\ &\leq Ke^{\gamma t} \|u_0 - \hat{w}\| + 2Kb_X^F(R) \int_0^t e^{\gamma(t-s)} \|u_s - \hat{w}\| ds. \end{aligned}$$

In particular, at $t = t_1$ we have

$$\|u_{t_1} - \hat{w}\| \leq K\epsilon + \frac{2Kb_X^F(R)}{(-\gamma)}R \leq R,$$

a contradiction to the choice of t_1 .

Finally, to end the proof, let $u_0 \in B_\epsilon^X(\hat{w})$, $u_0(s) \in D_X(B)$ for any $s \in [-\tau, 0]$ and let u the associated solution of (1). Since we have shown that $\|u_t - \hat{w}\| \leq R$ for all $t \geq 0$ we get that

$$\|u_t - \hat{w}\| \leq Ke^{\gamma t} \|u_0 - \hat{w}\| + Kb_X^F(R) \int_0^t e^{\gamma(t-s)} \|u_s - \hat{w}\| ds \quad (14)$$

holds for all $t \geq 0$. Thus, by using the Gronwall's inequality, we get

$$\begin{aligned} \|u_t - \hat{w}\| &\leq Ke^{[\gamma - Kb(R)]t} \|u_0 - \hat{w}\| \\ &\leq Ke^{(\gamma/2)t} \|u_0 - \hat{w}\|, u_0 \in B_\epsilon^X(\hat{w}) \end{aligned}$$

which shows (6).

In order to show the decay estimate (7), we repeat the same arguments as before but now on the space Y . Then, from assumptions (H3) on Y and (H7), there exist δ_Y^F and δ_X^B such that

$$\begin{aligned} B(z) &= B(w) + DB(w)(z - w) + G^B(w, z), \text{ for any } z \in B_{\delta_X^B}(w), \\ F(\phi) &= F(\hat{w}) + DF(\hat{w})(\phi - \hat{w}) + G^F(\hat{w}, \phi), \text{ for any } \phi \in B_{\delta_Y^F}^Y(\hat{w}). \end{aligned}$$

where now

$$B_{\delta_X^B}(w) = \{z \in D_X(B) \cap D_Y(B); |w - z| < \delta_X^B\},$$

$$B_{\delta_Y^F}^Y(\hat{w}) = \{\phi \in C; \|\phi - \hat{w}\|_Y < \delta_Y^F\}$$

and, as before, $\|\cdot\|_Y$ denotes the norm on the space $C_Y := C([- \tau, 0] : Y)$. Moreover, there exists two continuous increasing functions b_X^B and b_Y^F such that

$$|G^B(w, z)|_Y \leq b_X^B(|w - z|) |w - z|, \text{ for any } z \in B_{\delta_X^B}(w), \quad (15)$$

$$|G^F(\hat{w}, \phi)|_Y \leq b_Y^F(\|\phi - \hat{w}\|_Y) \|\phi - \hat{w}\|_Y, \text{ for any } \phi \in B_{\delta_Y^F}^Y(\hat{w}). \quad (16)$$

Now

$$\frac{du}{dt}(t) - \frac{dw}{dt} + Au(t) - Aw + DB(w)(u(t) - w) - DF(\hat{w})(u_t - \hat{w}) \ni G^B(w, u(t)) - G^F(\hat{w}, u_t). \quad (17)$$

Thus, by using (H8) and arguing as in the first part we get that there exists a constant $K^* \geq 1$ such that

$$\|u_t - \hat{w}\|_Y \leq K^* e^{\gamma^* t} \|u_0 - \hat{w}\|_Y + \int_0^t K^* e^{\gamma^*(t-s)} (|G^B(w, u(s))|_Y + |G^F(\hat{w}, u_s)|_Y) ds \quad (18)$$

and then, by taking $\delta = \min(\delta_X^B, \delta_Y^F)$ and $R^* \in (0, \delta)$ such that

$$\max(b_X^B(R^*), b_Y^F(R^*)) < (-\gamma)/(4K), \quad (19)$$

we obtain that

$$\|u_t - \widehat{w}\|_Y \leq K^* e^{\gamma^* t} \|u_0 - \widehat{w}\|_Y + K^* \int_0^t e^{\gamma^*(t-s)} (b_X^B(R^*) \|u_s - \widehat{w}\|_X + b_Y^F(R^*) \|u_s - \widehat{w}\|_Y) ds. \quad (20)$$

We define

$$\widetilde{R} = \min(R, R^*), \widetilde{K} = \max(K, K^*), \widetilde{\gamma} = \max(\gamma, \gamma^*) < 0 \text{ and } \epsilon^* = \min \left\{ \widetilde{R}/(2\widetilde{K}), \delta \right\}.$$

Then, if $u_0 \in B_{\epsilon^*}^{X \cap Y}(\widehat{w})$, $u_0(s) \in D_X(B) \cap D_Y(B)$ for any $s \in [-\tau, 0]$ and we assume, for instance, that $\widetilde{\gamma} = \gamma$, by adding (14) and (20) we deduce that

$$\begin{aligned} \|u_t - \widehat{w}\|_X + \|u_t - \widehat{w}\|_Y &\leq \widetilde{K} e^{\widetilde{\gamma} t} (\|u_0 - \widehat{w}\|_X + e^{(\gamma^* - \gamma)t} \|u_0 - \widehat{w}\|_Y) + \\ &\widetilde{K} \int_0^t e^{\widetilde{\gamma}(t-s)} [(b_X^F(\widetilde{R}) + b_X^B(\widetilde{R}) e^{(\gamma^* - \gamma)t}) \|u_s - \widehat{w}\|_X + b_Y^F(R^*) e^{(\gamma^* - \gamma)t} \|u_s - \widehat{w}\|_Y] ds. \end{aligned}$$

and the estimate (7) follows, again, by Gronwall's inequality. ■

Remark 2 *It is not difficult to show that the assumption (H8) is implied (when A is linear) by the condition: “if $\lambda \in \mathbb{C}$ is given so that there exists $y \in D(B) \setminus \{0\}$ such that $Ay + DB(w)y - \lambda y \ni DF(\widehat{w})(e^\lambda y)$ then $\operatorname{Re} \lambda > 0$ ”. This allow to see Theorem 4.1 of Wu [25] (see also Parrot [22] and its references) as an special case of our abstract result with $B = 0$. In that case the “variation of the constants formula” can be used to get a different proof of the theorem since A is linear. Notice that if $B \neq 0$ and $D(B) \subsetneq X$ then the arguments of the proof of Wu [25] do not work (in spite of the claimed in the Example 4.8 given there).*

Remark 3 *When A is linear, as in the case without delay, assumption (H7) implies that the zero solution of the linearized problem $\frac{dU}{dt}(t) + AU(t) + DB(w)U(t) - DF(\widehat{w})U_t(\cdot) = 0$ in X , is locally asymptotically stable (Wu [25]).*

Remark 4 *It is possible to prove the existence of global solutions for a general class of initial data (not necessarily near \widehat{w}) by using that $A + B \in \mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$, some truncation of the nonlocal term $F(u_t)$ and passing to the limit by the compactness of the semigroup generated by A (see Vrabie [24] for some related results).*

An easy adaptation of the above proof leads to the following linearization result (now on a possibly smaller neighborhood of w) when A is differentiable

Theorem 5 *The conclusion of the above result remains true if we assume, additionally, that condition (H7) also holds for A and we replace condition (H8) by*

(H9): the operator $y \rightarrow DA(w)y + DB(w)y - DF(\hat{w})(e^{\omega \cdot} y)$ belongs to $\mathcal{A}(\omega)$, for some $\omega \in \mathbb{C}$ with $\operatorname{Re} \omega = \gamma < 0$ ■

Remark 6 We claim that our arguments keeping A nonlinear after linearizing the rest of the terms (and in particular the way in which we apply Gronwall inequality) allow to extend, to the case of quasilinear equations, the so called “method of quasilinearization” which, introduced by Bellman and Kalaba [7], we used to find solutions of a parabolic semilinear problem through the iteration of solutions of the linearized equation when starting in a super and a subsolution of the original semilinear problem (see, e.g., Lakshmikantham and Leela [20], Carl and Lakshmikantham [10] and their references). This will be the subject of a future work by the authors.

3. Some examples

3.1. Example 1. The complex Ginzburg-Landau equation with a global delayed mechanism

Motivated by the special form of the nonlinear term of the equation in (P_3) we shall take $X = \mathbf{L}^4(\Omega)$ and $Y = \mathbf{L}^{4/3}(\Omega)$ (notice that, in contrast with the case of scalar equations (see Parrot [22]) the space $\mathbf{L}^\infty(\Omega)$ is not suitable space to check assumption (H1): see [5]). A detailed analysis of the associated diffusion operator is consequence of some previous results in the literature: see, for instance, Amann ([3])). Notice that the operator $A\mathbf{u}$ can be formulated matricially as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \rightarrow \begin{pmatrix} \Delta & -\epsilon\Delta \\ \epsilon\Delta & \Delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

So, if $\epsilon \neq 0$ the diffusion matrix has a nonzero antisymmetric part. In particular, A is the generator of a semigroup of contractions $\{T(t)\}_{t \geq 0}$ on X and the compactness of the semigroup is consequence of the compactness of the inclusion $D(A) \subset X$ (notice that, since $N = 2$, $\mathbf{W}^{1,4}(\Omega) \subset \mathbf{W}^{1,4/3}(\Omega) \subset \mathbf{C}(\overline{\Omega})$ with compact imbedding) and some regularity results for nonsymmetric systems.

Concerning the rest of the terms of the equation in (P_3) , we define $B\mathbf{u} = (1 + i\beta) |\mathbf{u}|^2 \mathbf{u}$ with $D(B) = \mathbf{L}^{12}(\Omega)$. By using the characterization of the semi inner-bracket $[\cdot, \cdot]$ for the spaces $L^p(\Omega)$ (see, for instance Benilan, Crandall and Pazy [8]) it is easy to see that B verifies (H3). Moreover, by the results on the Frechet differentiability of Nemitsky operators (see Theorem 2.6 (with $p = 4$) of Ambrosetti and Prodi [4]) we get that (H7) holds, with $DB(\mathbf{y})\mathbf{v} = 3(1 + i\beta) |\mathbf{y}|^2 \mathbf{v}$, if we take $Y = \mathbf{L}^{4/3}(\Omega)$. It can be found in the above mentioned reference that assumption (H7) does not hold if we take $X = Y = \mathbf{L}^2(\Omega)$.

The nonlocal term is defined, by

$$F(\mathbf{u}_t) = (1 + i\theta)\mathbf{u}(t) + \mu e^{i\chi_0} [m_1 \mathbf{u}(t) + m_2 \overline{\mathbf{u}}(t) + e^{i(\omega+\theta)\tau} (m_3 \mathbf{u}(t-\tau) + m_4 \overline{\mathbf{u}}(t-\tau))],$$

is locally Lipschitz continuous and its Frechet derivative is given by

$$DF(\hat{\mathbf{y}})\mathbf{v}(t) = -(1 + i\theta)\mathbf{v}(t) - \mu e^{i\chi_0} [m_1 \mathbf{v}(t) + m_2 \overline{\mathbf{v}}(t) - e^{i(\omega+\theta)\tau} (m_3 \mathbf{v}(t-\tau) - m_4 \overline{\mathbf{v}}(t-\tau))] \quad (21)$$

since for any $\phi \in C$, the non-local operator $\phi \rightarrow \frac{1}{|\Omega|} \int_{\Omega} \phi(s) dx$ is linear and we can write $DF(\widehat{\mathbf{y}})\phi = \int_{-\tau}^0 d\eta(s)\phi(s)$, with

$$d\eta(s)\mathbf{v}(s) = \delta_0(s)(1+i\theta)\mathbf{v}(s) + \mu e^{i\chi_0} \left[\delta_0(s)(m_1\mathbf{v}(s) + m_2\overline{\mathbf{v}}(s)) + e^{i(\omega+\theta)\tau} \delta_{-\tau}(s)(m_3\mathbf{v}(s) + m_4\overline{\mathbf{v}}(s)) \right] \quad (22)$$

for any $\mathbf{v} \in C([-\tau, \infty); \mathbf{L}^4(\Omega))$ and any $s \in [-\tau, \infty)$, where $\delta_0(s), \delta_{-\tau}(s)$ denote the Dirac delta at the points $s = 0$ and $s = -\tau$ respectively. By well-known results, we have that $\eta : [-\tau, 0] \rightarrow B(X, X)$ has a bounded variation and so, conditions (H4) and (H5) hold (and analogously replacing X by Y).

Finally, assumption (H6) can be read as a condition on the stationary state \mathbf{y} (a study of the eigenvalue of operator A can be found, for instance, in Temam [23]).

Remark 7 By introducing the representation operator $\mathbf{P} : \mathbb{R}^2 \rightarrow \mathbb{C}$, $\mathbf{P}(\rho, \phi) = \rho e^{i\phi}$ it is clear that the quasilinear operator $A\mathbf{P}(\mathbf{q})$ obtained from the operator $A\mathbf{u} = -(1+i\epsilon)\Delta\mathbf{u}$ satisfies also condition $A \in \mathcal{A}(\omega)$ (since \mathbf{P} is merely a change of variables). We point out that,

$$A\mathbf{P}(\mathbf{q}) = -(1+i\epsilon)[\Delta\rho - \rho|\nabla\phi|^2 + i(2\nabla\rho \cdot \nabla\phi + \rho\Delta\phi)]e^{i\phi}.$$

Then, the “formal linearization” of the operator $\mathbf{E}(\mathbf{q}) := A\mathbf{P}(\mathbf{q})$ at $\mathbf{q}^*(x, y) := \mathbf{y} \equiv \rho_0$ becomes

$$D\mathbf{E}(\mathbf{q}^*)(\rho e^{i\phi}) = -(1+i\epsilon)[\Delta\rho + i\rho_0\Delta\phi]e^{i\phi}.$$

Notice that the linearization of $\mathbf{C}(\mathbf{q})^{-1}A\mathbf{P}(\mathbf{q})$ needs a slight modification of the above linear expression. ■

3.2. Example 2. Case in which A is nonlinear and nondifferentiable

It is not difficult to adapt the results of the first example to the case in which the vectorial operator is given by

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \rightarrow \begin{pmatrix} A_1 & -\epsilon\Delta \\ \epsilon\Delta & A_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (23)$$

with $A_i : D(A_i) \rightarrow \mathcal{P}(L^4(\Omega))$ two (possibly different) m-accretive operators in $L^4(\Omega)$, as for instance,

$$\left\{ \begin{array}{l} A_i u = -\operatorname{div}(|\nabla u|^{p_i-2} \nabla u) + \beta_i(u) \\ D(A_i) = \{u \in W^{1,1}(\Omega) \cap L^4(\Omega), u(x) \in D(\beta) \text{ a.e. } x \in \Omega, A_i u \in L^4(\Omega) \\ \text{and } -|\frac{\partial u}{\partial n}|^{p_i-2} \frac{\partial u}{\partial n} \in \gamma_i(u) \text{ on } \partial\Omega\} \end{array} \right.$$

where $p_i \in (1, +\infty)$ and β_i, γ_i are maximal monotone graphs of \mathbb{R}^2 (not necessarily associated to differentiable functions). We refer the reader to Vrabie [24] (and its references) for the study of the assumptions (H1) and (H2) for each of the nonlinear operators A_i . We point out that the structure of the nonlinear diffusion operator (23) allows to guarantee that the diffusion operator is m-accretive in $\mathbf{L}^4(\Omega)$. The same holds also on $\mathbf{L}^{4/3}(\Omega)$. ■

Remark 8 For some (partial) results on the linearization of the p -Laplacian operator see Aftalion and Pacella [1], [2]. See also Bermejo and Infante [9] for the application to numerical analysis of some associated stationary equations. Finally, we mention the linearization results by Hernández, Mancebo and Vega de Prada [18] for some semilinear equations with a singular zeroth order term.

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