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STRICT CONTINUOUS HOMOMORPHISMS ON SPACES OF BOUNDED HOLOMORPHIC FUNCTIONS

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Let H_E^∞ be the space of all bounded holomorphic functions on the unit ball of the Banach space E. In this note we study the algebra homomorphisms on H_E^∞ which are strict continuous.

On the algebra H_E^∞ of all bounded, holomorphic functions on the unit ball, $\mathbf{B}_E$, of the complex Banach space E, we can consider the strict topology β_E, introduced by Buck in the one dimensional case (see [4]). We define this topology for arbitrary Banach spaces:

Consider the family $\mathbf{K}_E = \{ k:\mathbf{B}_E \longrightarrow [0,\infty)$ such that k is bounded, continuous, and for every $\varepsilon>0$ there exists $r\in(0,1)$ with $k(z)<\varepsilon$ if $\|z\|>r\}$. Each $k\in\mathbf{K}_E$ defines a seminorm p_k on H_E^∞, where $p_k(f) = \sup\{k(z)\cdot|f(z)|:z\in\mathbf{B}_E\}$. The strict topology β_E is the topology induced by the seminorms $\{p_k:k\in\mathbf{K}_E\}$.

It is known that when endowed with the strict topology, H_E^∞ becomes a topological, conmutative, unital algebra (see |8|, 1.39). Moreover, the space of continuous polynomials on E, P(E), is a dense subspace of H_E^∞ for the strict topology (see |8|, 2.32).

Following |3| and |6|, given an n-homogeneous polynomial on E, P, we consider the canonical extension to E^{**}, $\hat{P}$, which verifies that $\|P\| = \|\hat{P}\|$. Thus, given $\hat{f}\in H_E^\infty$ there exists a canonical extension $\hat{f}:\mathbf{B}_E{}^{**} \longrightarrow \mathbb{C}$ given by $\hat{f}(z) = \sum_{n=0}^{\infty} \hat{P}_n(z)$ for all $z\in\mathbf{B}_E{}^{**}$ (where the Taylor series expansion of f is $f(x) = \sum_{n=0}^{\infty} P_n(x)$. It is shown in |6| that a net $(x_\alpha)\subset\mathbf{B}_E(0,r)$ $(r\in(0,1))$ such that for some $z\in\mathbf{B}_E{}^{**}$ $(P(x_\alpha))$ tends to $\hat{P}(z)$ for every P in P(E), verifies that the net $(f(x_\alpha))$ tends to $\hat{f}(z)$ for every f in H_E^∞. In the following, $\hat{f}$ will denote this extension.

Trough this note, E and F will denote complex Banach spaces; the space of continuous n-homogeneous polynomials on E is denoted by

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$P(^nE)$. Two important subspaces are the polynomials of finite type, $P_f(^nE)$, (which are generated by the dual space E^*) and its closure in $P(^nE)$, the compact polynomials, denoted by $P_c(^nE)$. Homomorphism means here a nonzero algebra homomorphism. For details on the theory of holomorphic functions we refer to |5|.

Our first results study the complex valued homomorphisms.

1.<u>Proposition</u>. Let Φ be a complex valued homomorphism on H_E^∞ . Then, there exists a unique z in $\bar{\mathbf{B}}_E{}^{**}$ such that $\Phi(P) = \hat{P}(z)$ for every compact polynomial on E. Moreover, if Φ is strict continuous, then z belongs to $\mathbf{B}_E{}^{**}$. □

2.<u>Proposition</u>. Let z be in $\mathbf{B}_E{}^{**}$. The homomorphism $\Phi_z:H_E^\infty \longrightarrow \mathbb{C}$ given by $\Phi_z(f) = \hat{f}(z)$ is strict continuous. □

Thanks to propositions 1 and 2 and the density of P(E) in H_E^∞ we obtain the following theorem:

3.<u>Theorem</u>. Let E be a Banach space such that $P(^nE) = P_c(^nE)$ for every $n\in\mathbb{N}$. Then each strict continuous homomorphism $\Phi:H_E^\infty \longrightarrow \mathbb{C}$ is given by evaluation at some point in $\mathbf{B}_E{}^{**}$. □

<u>Remark</u>. Some important Banach spaces verify Theorem 3. Finite dimensional spaces are our first example. Moreover, we present two infinite dimensional cases: the original Tsirelson space T^* (see |2| and |1|) and the space of all complex sequences which tend to zero, c_0.

Another interesting problem is the description of homomorphisms between algebras of type H_E^∞.

4.<u>Proposition</u>. Let $\phi:\mathbf{B}_E \longrightarrow \mathbf{B}_F$ be a holomorphic mapping. Consider the homomorphism $\Phi:H_F^\infty \longrightarrow H_E^\infty$ defined by $\Phi(f) = \hat{f}\circ\phi$. Then Φ is continuous for the strict topologies if and only if for every $r\in(0,1)$ there exists $s\in(0,1)$ such that $\phi(\mathbf{B}_E(0,r)) \subset \mathbf{B}_F{}^{**}(0,s)$. □

From the Schwarz lemma and the above proposition, it is clear that Φ , defined by $\Phi(f) = \hat{f}\circ\phi$ for all $f\in H_F^\infty$, is continuous for every holomorphic mapping such that $\phi(0) = 0$.

The next theorem provides us an explicit description of homomorphisms between algebras H^∞, for certain Banach spaces.

5.<u>Theorem</u>. Let E, F be Banach spaces such that every polynomial on F is compact. If $\Phi:H_F^\infty \longrightarrow H_E^\infty$ is a continuous homomorphism for the strict topologies, then there exists a holomorphic mapping $\phi:\mathbf{B}_E \longrightarrow \mathbf{B}_F{}^{**}$ such that $\Phi(f) = \hat{f}\circ\phi$ for all $f\in H^{\infty}_{F}$. □

Now, fixed a sequence (a_n) in B_E , we consider the associated homomorphism $R:H^{\infty}_{E} \longrightarrow l^{\infty}$ given by $R(f) = (f(a_n))$. We will characterize the continuity of R and its range. Analogous problems for weakly differentiable functions have been solved in |7|.

6.<u>Proposition</u>. Fixed a sequence (a_n) in B_E , the associated homomorphism R is $\beta_E-\|\cdot\|_{\infty}$ continuous if and only if there exists $r\in(0,1)$ such that $(a_n)\subset B_E(0,r)$. □

7.<u>Proposition</u>. Let (a_n) in B_E; consider the associated homomorphism R .

i) If the range of R lies in the space of all converging complex sequences, **c** , then (a_n) is a weakly Cauchy sequence.

ii) If $(a_n)\subset B_E(0,r)$ is weakly Cauchy and every polynomial on E is compact, the range of R is contained in **c** . □

8.<u>Proposition</u>. Let E be a complex Banach space.

i) If E contains a copy of l^1 , then there exists (a_n) in B_E such that associated homomorphism is strict continuous and onto.

ii) If E does not contain a copy of l^1 and every polynomial on E is compact, then for any $r\in(0,1)$ and any (a_n) in $B_E(0,r)$ the associated restriction homomorphism is not onto l^{∞}. □

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