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BANACH-STONE THEOREMS FOR BANACH MANIFOLDS[1]

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ABSTRACT

We obtain here that, under fairly general conditions, the smooth structure of a Banach manifold (respectively, the isomorphic structure of a Banach space) is determined by its algebra of smooth real functions. The analogous problem is also studied in the real-analytic case.

Let X be a topological space, maybe endowed with a richer structure (for example, a metric space, or a smooth manifold), and consider the set $C(X)$ of all continuous real functions on X. Loosely speaking, by *a Banach-Stone theorem* we mean a result characterizing the topological structure of X (respectively, its metric structure or its smooth structure) in terms of a given algebraic or topological-algebraic structure in $C(X)$ or in a subfamily of $C(X)$. The origin of this terminology refers to the classical Banach-Stone theorem asserting that, for a compact space X, the linear metric structure of $C(X)$ (endowed with the *sup*-norm) determines the topology of X. A further result along this line was obtained by Hewitt [10], who proved that for a realcompact space X, the algebra structure of $C(X)$ also determines the topology of X. In the case of a complete metric space X, is it proved in [7] that its metric structure is determined by the lattice structure of the set $U(X)$ of all real uniformly continuous functions on X. Our aim in this work is to give analogous results for algebras of smooth and real-analytic functions defined on Banach spaces or Banach manifolds.

We start with some notation. If X is any topological space and $\mathcal{A} \subset C(X)$ is a subalgebra, we say that $\varphi: \mathcal{A} \to \mathbb{R}$ is an *algebra homomorphism* if it is linear and multiplicative. We denote by $Hom(\mathcal{A})$ the set of all non-zero algebra homomorphisms. We say that $Hom(\mathcal{A}) = X$ when every $\varphi \in Hom(\mathcal{A})$ is given by evaluation at some point of X, that is, there exists some $x \in X$ such that $\varphi(f) = f(x)$, for every $f \in \mathcal{A}$.

Our first result concerns algebras of functions on a Banach space. For a real Banach space E, we denote by $C^\infty(E)$ the algebra of infinitely differentiable real functions on E, in the usual Fréchet sense.

Theorem 1. *For $i = 1, 2$, let E_i be a Banach space, and let $\mathcal{A}_i \subset C^\infty(E_i)$ be a unital subalgebra containing the dual space E_i^*, and such that $Hom(\mathcal{A}_i) = E_i$. If $\mathcal{A}_1$ is isomorphic to $\mathcal{A}_2$ as algebras, then E_1 is isomorphic to E_2.*

Proof. Suppose that $T: \mathcal{A}_2 \to \mathcal{A}_1$ is an algebra isomorphism. It is easy to see that $T(1) = 1$ (see e.g. [8], Theorem 1.9). We define the function $h: E_1 \to E_2$ in the following way: for every $x_1 \in E_1$ consider the evaluation $\delta_{x_1} \in Hom(\mathcal{A}_1)$ given by $\delta_{x_1}(f) = f(x_1)$. Hence $\delta_{x_1} \circ T \in Hom(\mathcal{A}_2)$, and from the hypotheses the homomorphism $\delta_{x_1} \circ T$ is given by evaluation at some $x_2 \in E_2$. Note that the point x_2 is unique since $E_2^* \subset \mathcal{A}_2$ separates the points of E_2. Then we define $h(x_1) = x_2$. Analogously, since T is an isomorphism, the function h^{-1} exists.

Now, for every function $g \in \mathcal{A}_2$, we have that $g \circ h = T(g)$. Indeed, for every $x_1 \in E_1$,

$$T(g)(x_1) = \delta_{x_1}\big(T(g)\big) = (\delta_{x_1} \circ T)(g) = \delta_{h(x_1)}(g) = g\big(h(x_1)\big).$$

In particular, for every $x_2^* \in E_2^*$, we have that $x_2^* \circ h \in C^\infty(E_1)$, and therefore, by [9], Corollary 3.8, h is C^∞-smooth. The same is true for h^{-1}, so E_1 and E_2 are C^∞-diffeomorphic. This implies that the differential of h at every point is an isomorphism between E_1 and E_2. □

From Theorem 1 we obtain that, in the class of Banach spaces E for which $Hom\big(C^\infty(E)\big) = E$, the algebra structure of $C^\infty(E)$ characterizes the isomorphic type of E. For instance, by Biström and Lindström [1], Corollary 3, we have:

Corollary 2. *Let E, F be Banach spaces which inject linear and continuously into $c_0(\Gamma)$, for some index set Γ*

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with nonmeasurable cardinal. The following are equivalent:

(a) *The algebras $C^\infty(E)$ and $C^\infty(F)$ are isomorphic.*

(b) *The spaces E and F are isomorphic.*

We note that Corollary 2 can be applied to a wide class of spaces. Recall that a Banach space E is said to be Weakly Compactly Generated (WCG) if there is a weakly compact subset whose linear span is dense in E. In particular, every separable and every reflexive space are WCG. Now, if either E or E^* is WCG (with nonmeasurable cardinal), then E injects linear and continuously into $c_0(\Gamma)$, for some Γ (with nonmeasurable cardinal) (see e.g. [5], p. 247). For further spaces satisfying the hypothesis of Corollary 2, we refer to Kriegl and Michor [12], 53.20 and 53.21.

We now consider smaller algebras of C^∞-smooth functions over a Banach space E. As usual, $P(E)$ denotes the algebra of all continuous real polynomials on E, and $R(E)$ the algebra of all rational functions of E, that is, the functions of the form P/Q where $P, Q \in P(E)$ and $Q(x) \neq 0$ for all $x \in E$. Recall that a real function f defined on an open set $U \subset E$ is said to be real-analytic on U if, for every $x \in U$ there exist a neighborhood W of 0 and a sequence (P_j), with each $P_j \in P(E)$ being j-homogeneous, such that $f(x + h) = \sum_{j=0}^{\infty} P_j(x + h)$, for every $h \in W$. Note that this series can be assumed to be uniformly convergent on W (see Bochnak and Siciak [2], Propositions 5.1 and 5.2, or Kriegl and Michor [12], Lemma 7.14). We denote by $C^\omega(U)$ the algebra of all real-analytic functions on U. It is clear that $R(E) \subset C^\omega(E) \subset C^\infty(E)$. Combining Theorem 1 with [6], Theorem 2.4, we obtain:

Corollary 3. *Let E, F be Banach spaces which inject linear and continuously into $\ell_p(\Gamma)$, for some $1 < p < \infty$ and some index set Γ with nonmeasurable cardinal. The following are equivalent:*

(a) *The algebras $C^\infty(E)$ and $C^\infty(F)$ are isomorphic.*

(b) *The algebras $C^\omega(E)$ and $C^\omega(F)$ are isomorphic.*

(c) *The algebras $R(E)$ and $R(F)$ are isomorphic.*

(d) *The spaces E and F are isomorphic.*

We point out that the class of spaces satisfying the hypothesis in Corollary 3 includes separable spaces and superreflexive spaces with nonmeasurable cardinal (see e.g. [6]).

On the other hand, the equivalence between (c) and (d) above should be compared with the results by Cabello, Castillo and García in [4] (see also Lassalle and Zalduendo [14]). They provide examples of separable nonisomorphic spaces E and F for which the algebras $P(E)$ and $P(F)$ do verify to be isomorphic.

Next we focus on smooth and real-analytic Banach manifolds. We refer to Lang [13] or Kriegl and Michor [12] for an account on this topic. Recall that a Banach space E is said to be C^∞-smooth if there exists a C^∞ real function on E with nonempty bounded support. If M is a Banach manifold modeled on C^∞-smooth Banach spaces, it is easy to see that $C^\infty(M)$ separates points and closed sets of M, that is, M carries the initial topology for the family $C^\infty(M)$ (see e.g. Bonic and Frampton [3]).

Theorem 4. *Let M, N be paracompact Banach manifolds, modeled on C^∞-smooth Banach spaces. The following are equivalent:*

(a) *The algebras $C^\infty(M)$ and $C^\infty(N)$ are isomorphic.*

(b) *M and N are C^∞-diffeomorphic.*

Proof. (a) $\Rightarrow$ (b) Endow the set $Hom\big(C^\infty(M)\big)$ with the product topology as a subspace of $\mathbb{R}^{C^\infty(M)}$. As we mentioned before, the family $C^\infty(M)$ separates points and closed sets of M, and therefore M can be embedded as a topological subspace of $Hom\big(C^\infty(M)\big)$ by identifying each $x \in M$ with the point evaluation δ_x. Since $C^\infty(M)$ is closed under inversion, we have by [11], Theorems 1.1 and 1.4, that M is dense in $Hom\big(C^\infty(M)\big)$. Moreover, each $f \in C^\infty(M)$ admits a continuous extension $\hat{f}$ to $Hom\big(C^\infty(M)\big)$, given by $\hat{f}(\varphi) = \varphi(f)$, for each $\varphi \in Hom\big(C^\infty(M)\big)$. The same can be said for N.

Now suppose that $T\colon C^\infty(N) \to C^\infty(M)$ is an algebra isomorphism. Then the map $h\colon Hom\big(C^\infty(M)\big) \to Hom\big(C^\infty(N)\big)$ defined by $h(\varphi) = T \circ \varphi$ is a bijection, and in fact h is a homeomorphism since $pr_g \circ h = pr_{T(g)}$, for each $g \in C^\infty(N)$ (where pr denotes the corresponding projection on the product space). Next we are going to see that $h(M) = N$. This will be a consequence of the following:

Claim. *The point $\varphi \in Hom\big(C^\infty(M)\big)$ has a countable neighborhood basis in $Hom\big(C^\infty(M)\big)$ if, and only if, $\varphi \in M$.*

In order to prove the Claim, first note that every paracompact Banach manifold is completely metrizable (see [16], Theorem 3), so there exists a complete metric d on M which gives the topology.

Suppose first that $\varphi \in Hom\big(C^\infty(M)\big) \setminus M$ has a countable neighborhood basis. Since M is dense in $Hom\big(C^\infty(M)\big)$, we can choose a sequence (x_n) in M converging to φ. The completeness of the metric d implies that (x_n) has no d-Cauchy subsequence, and therefore there exist $\varepsilon > 0$ and a subsequence (x_{n_k}) such that $d(x_{n_k}, x_{n_j}) \geq \varepsilon$ for $k \neq j$. For each k, consider the d-ball $B_k = B(x_{n_k}, \frac{\varepsilon}{4})$, and a function $f_k \in C^\infty(M)$ such that the support of f_k is contained in B_k and $f_k(x_{n_k}) = 1$. Note that for every $x \in M$ the d-ball $B(x, \frac{\varepsilon}{4})$ meets at most one B_k. Thus the sum $f = \sum_{k=0}^{\infty} f_{2k}$ is locally finite and $f \in C^\infty(M)$. In addition, $f(x_{n_{2k}}) = 1$ and $f(x_{n_{2k+1}}) = 0$ for every k. But this

is a contradiction since f extends continuously to $Hom(C^\infty(M))$ and φ is in the closure of the sets $A = \{x_{n_{2k}}\}$ and $B = \{x_{n_{2k+1}}\}$.

Conversely, given $x \in M$, consider for each n the open d-ball $B_n = B(x, \frac{1}{n})$, and let W_n be the closure of B_n in $Hom(C^\infty(M))$. It is easy to check that the family $\{W_n\}$ is a countable neighborhood basis of x in $Hom(C^\infty(M))$. This completes the Claim.

In this way we obtain that $h: M \to N$ is a homeomorphism and $T(g) = g \circ h \in C^\infty(M)$, for every $g \in C^\infty(N)$. This implies that h is C^∞-smooth. Indeed, let $x_0 \in M$ and consider $\phi: U \to E$ and $\psi: V \to F$ charts around x_0 and $h(x_0)$, respectively. Using that F is a C^∞-smooth space, we can find open neighborhoods $U_0 \subset U$ and $V_0 \subset V$, of x_0 and $h(x_0)$ respectively, and $\theta \in C^\infty(N)$ such that $h(U_0) \subset V_0$, $\theta = 1$ on V_0, and the support of θ is contained in V. In order to see that $\psi \circ h \circ \phi^{-1}$ is C^∞ on $\phi(U_0)$, it is enough to check that $v^* \circ \psi \circ h \circ \phi^{-1}$ is C^∞ for every functional $v^* \in F^*$ (see [9], Corollary 3.8). Now, given $v^* \in F^*$, define $g: V \to \mathbb{R}$ by

$$g(y) = \begin{cases} \theta(y) \cdot v^*(\psi(y)) & \text{if } y \in V \\ 0 & \text{otherwise.} \end{cases}$$

Then $g \in C^\infty(N)$, $g \circ h \in C^\infty(M)$ and $v^* \circ \psi \circ h \circ \phi^{-1} = g \circ h \circ \phi^{-1}$ is C^∞ on $\phi(U_0)$.

Working with h^{-1} we obtain that $h: M \to N$ is a C^∞-diffeomorphism. Finally, the converse (b) $\Rightarrow$ (a) is clear. □

The above result was obtained by Pursell [17], Theorem 5.2, in the case of finite-dimensional manifolds. For related results, see Li Pi Su [15] and references therein.

In the real-analytic setting, we restrict ourselves to submanifolds of certain Banach spaces.

Theorem 5. *Let E, F be Banach spaces which admit a continuous linear injection into $\ell_p(\Gamma)$, for some $1 < p < \infty$ and some index set Γ with nonmeasurable cardinal. Let M, N be real-analytic submanifolds of E and F, respectively. The following are equivalent:*

(a) *The algebras $C^\omega(M)$ and $C^\omega(N)$ are isomorphic.*

(b) *M and N are real-analytically isomorphic.*

Proof. (a) $\Rightarrow$ (b) First note that by [6], Theorem 2.4, we have $Hom(C^\omega(M)) = M$ and $Hom(C^\omega(N)) = N$. If $T: C^\omega(N) \to C^\omega(M)$ is an algebra isomorphism, we obtain as before a bijection $h: M \to N$ such that $T(g) = g \circ h \in C^\omega(M)$, for every $g \in C^\omega(N)$. We are going to see that $h: M \to N$ is real-analytic. Since N is a real-analytic submanifold of F, it is enough to prove that $h: M \to F$ is real-analytic. Now consider $x_0 \in M$ and a chart (U, ϕ) around x_0. Since $F^* \subset C^\omega(N)$, for every functional $v^* \in F^*$, we have that $v^* \circ h$ is real-analytic on M. Then $v^* \circ h \circ \phi^{-1}$ is real-analytic on $\phi(U)$, for every $v^* \in F^*$. Therefore by [2], Theorem 7.4, $h \circ \phi^{-1}: \phi(U) \to F$ is real-analytic on $\phi(U)$. In the same way we obtain that h^{-1} is also real-analytic.

The converse (b) $\Rightarrow$ (a) is clear. □

Our next result reveals that Theorem 5 does not hold for arbitrary Banach manifolds. This shows that there is no general real-analytic Banach-Stone theorem.

Proposition 6. *Consider the Banach space $c_0(\Gamma)$, where Γ is an uncountable set. Let $M = c_0(\Gamma)$ and $N = c_0(\Gamma) \setminus \{0\}$. Then:*

(i) *The algebras $C^\omega(M)$ and $C^\omega(N)$ are isomorphic.*

(ii) *M and N are not real-analytically isomorphic.*

Proof. (i) By [6], Example 2.6, each $f \in C^\omega(N)$ admits a continuous extension $\hat{f}$ to M given by $\hat{f}(0) = \lim_{x \to 0} f(x)$. In order to obtain an algebra isomorphism between $C^\omega(N)$ and $C^\omega(M)$, we have to prove that $\hat{f}$ is analytic on M. By [12], Theorem 10.1, it suffices to prove that $\hat{f}$ is C^∞-smooth on each C^∞-smooth curve, and real-analytic on each real-analytic curve.

So let $c: \mathbb{R} \to M$ be a C^∞-smooth curve (respectively real-analytic curve) and consider $Z = c^{-1}(0)$. If either Z is empty or $Z = \mathbb{R}$, the result is clear. Otherwise, let $\{t_m\}$ be a countable dense subset of $\mathbb{R} \setminus Z$, and for each m let $v_m = c(t_m)$. Denote by V the closed linear subspace of $c_0(\Gamma)$ spanned by the v_m's. For each $m \in \mathbb{N}$, since $v_m \neq 0$ there exist $\varepsilon_m > 0$ and a sequence of continuous polynomials $(P_n^m)_{n \in \mathbb{N}}$ such that $f(v_m + h) = f(v_m) + \sum_{n=1}^\infty P_n^m(h)$, for $\|h\| < \varepsilon_m$. As proved in [6], Example 2.6, using that any P_n^m can be obtained as uniform limit, on bounded sets, of a sequence of finite type polynomials, we can select $\gamma \in \Gamma$ and a unitary vector e_γ, so that $e_\gamma \notin V$ and $P_n^m(e_\gamma) = 0$, for all $m, n \in \mathbb{N}$. Now define the curve $d: \mathbb{R} \to M$ by $d(t) = c(t) + e_\gamma$. First note that $d(t) \neq 0$, for every $t \in \mathbb{R}$, since $c(\mathbb{R}) \subset V$. For each m, since $v_m + te_\gamma \neq 0$ for all $t \in \mathbb{R}$, the function given by $t \to f(v_m + te_\gamma)$ is real-analytic on $\mathbb{R}$ and coincides with $f(v_m)$, for $|t| < \varepsilon_m$. Therefore $f(v_m + te_\gamma) = f(v_m)$ for all $t \in \mathbb{R}$. In particular, $f(v_m + e_\gamma) = f(v_m)$, for all $m \in \mathbb{N}$.

Now we are going to see that $\hat{f}(c(t)) = f(d(t))$, for every $t \in \mathbb{R}$. This is clear if $t \in \mathbb{R} \setminus Z$, from the density of $\{t_m\}$. On the other hand, if $t \in Z$, it follows from the fact that $f(e_\gamma) = \hat{f}(0)$. Indeed, choosing $t_0 \in Z$ such that t_0 belongs to the closure of $\mathbb{R} \setminus Z$, there exists a subsequence (t_{m_j}) converging to t_0, and then $\hat{f}(0) = \hat{f}(c(t_0)) = \lim f(c(t_{m_j})) = \lim f(c(t_{m_j}) + e_\gamma) = f(e_\gamma)$.

(ii) By [6], Theorem 2.7, we have that $Hom(C^\omega(M)) = M$. On the other hand, it is easy to see that if M is real-analytically isomorphic to N and $Hom(C^\omega(M)) = M$, then $Hom(C^\omega(N)) = N$. But it is not possible since the

homomorphism $\varphi: C^{\omega}(N) \to \mathbb{R}$ defined by $\varphi(f) = \lim_{x \to 0} f(x)$ is not given by evaluation at any point of N. □

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