

# Nonfactorization of four-quark condensates at low energies within chiral perturbation theory

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Four-quark correlators and the factorization hypothesis are analyzed in the meson sector within chiral perturbation theory. We define the four-quark condensate as  $\lim_{x \rightarrow 0} \langle T(\bar{q}q)(x)(\bar{q}q)(0) \rangle$ , which is equivalent to other definitions commonly used in the literature. Factorization of the four-quark condensate holds to leading and next to leading order. However, at next to next to leading order, a term with a nontrivial space-time dependence in the four-quark correlator yields a divergent four-quark condensate, whereas the two-quark condensate and the scalar susceptibility are finite. Such a nonfactorization term vanishes only in the chiral limit. We also comment on how factorization still holds in the large  $N_c$  limit, provided such a limit is taken before renormalization.

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## I. INTRODUCTION

Scalar condensates play a relevant role in QCD, since they are directly related to vacuum properties. The quark condensate  $\langle \bar{q}q \rangle$  is a parameter deeply related to spontaneous chiral symmetry breaking and the description of low-energy QCD. In principle, quark condensates of arbitrary order  $\langle (\bar{q}q)^n \rangle$  are also built out of chiral noninvariant operators with vacuum quantum numbers and are also related to chiral symmetry restoration. In addition, quark condensates appear directly in QCD sum rules, through the operator product expansion (OPE) approach [1], where the following hypothesis of factorization or vacuum saturation is customarily made:

$$\langle (\bar{q}q)^2 \rangle = \left(1 - \frac{1}{N}\right) \langle \bar{q}q \rangle^2. \quad (1)$$

Note that we have particularized to the case where the four-quark operator has the quantum numbers of the scalar, isoscalar, and colorless condensates that we are interested in. In addition,  $N = 4N_c N_f$ , where  $N_c$  and  $N_f$  denote the number of colors and flavors, respectively, and  $q$  is a Dirac spinor, flavor, and color vector. We remark that in the large- $N_c$  limit factorization simply reduces to  $\langle (\bar{q}q)^2 \rangle = \langle \bar{q}q \rangle^2$ . The second term in Eq. (1) comes from the contraction of indices (including color) between the first and second  $\bar{q}q$  operators.

The use of the factorization hypothesis is a key point in order to estimate the size of higher order condensates in the OPE. However, its justification is still a matter of debate. It was shown in [2] that factorization implies that  $\langle (\bar{q}q)^2 \rangle$  becomes dependent on the QCD renormalization scale. This means that for QCD sum rules including six-dimensional operators, like  $(\bar{q}q)^2$ , one cannot write a renormalization-group (RG) invariant four-quark conden-

sate, preventing RG improvements of such sum rules. This is not a problem when considering six-dimensional pure-gluon operators or quark operators with dimensions lower than six, like the RG-invariant  $\bar{q}\mathcal{M}q$  with  $\mathcal{M}$  the mass matrix. We will come back to this point in Sec. IV. The validity of vacuum saturation has also been questioned within the framework of finite-energy sum rules [3] and has been formally shown not to hold when dressed QCD vertices are considered [4].

In this work we will present a study of the scalar four-quark condensate within the framework of chiral perturbation theory (ChPT). Since ChPT relies only on symmetries and not on vacuum saturation or dominance assumptions, as in some of the approaches commented above, it will allow us to obtain low-energy model-independent results concerning the factorization hypothesis.

An important point concerns the definition of the quark condensate in terms of Green functions. In the chiral Lagrangian framework, one has access not to individual quark operators at a given space-time point  $x$ , but to the low-energy representation of the quark-antiquark operator  $\bar{q}q(x)$ , given by a functional derivative with respect to an external scalar source (see details in Sec. II). Therefore, a natural way to define the four-quark condensate is through the limit of the two-point function (four-quark correlator):

$$\langle (\bar{q}q)^2 \rangle = \lim_{x \rightarrow 0} \langle T(\bar{q}q)(x)(\bar{q}q)(0) \rangle. \quad (2)$$

This is the definition that we will choose to work with here, where all the divergencies will be treated within the  $\overline{\text{MS}}$  scheme in dimensional regularization, as it is customary in ChPT. However, from the comments above, it is not clear that the four-quark condensate itself has to be a scale-independent and finite object, which means that the  $x \rightarrow 0$  limit is ill defined and other definitions in terms of Green functions could give different answers. Actually, Eq. (2) is not the usual  $\overline{\text{MS}}$  definition when working, for instance, with four-quark vacuum expectation values in the

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context of electroweak penguin contributions [5,6], where the following prescription is used instead:

$$\begin{aligned}\langle(\bar{q}q)^2\rangle &= \int d^D x \langle T(\bar{q}q)(x)(\bar{q}q)(0) \rangle \delta^{(D)}(x) \\ &= \int \frac{d^D Q}{(2\pi)^D} \Pi(Q^2),\end{aligned}\quad (3)$$

where the integrals are defined in *Euclidean* space-time dimension  $D$  and  $\Pi(Q^2)$  is the Fourier transform of the correlator  $\langle T(\bar{q}q)(x)(\bar{q}q)(0) \rangle$ . In the ChPT framework, we will show (details are given in Appendix B) that this definition gives the same result as that obtained when using the definition in Eq. (2), meaning that factorization is spoiled at next to next to leading order (NNLO), which questions seriously the validity of the factorization hypothesis, now from the point of view of the low-energy representation.

The four-quark two-point correlator, apart from defining the four-quark condensate, is also related to the chiral or scalar susceptibility, defined as  $\chi = -\partial\langle\bar{q}q\rangle/\partial m_q$ , which can be written also in terms of  $\langle T(\bar{q}q)(x)(\bar{q}q)(0) \rangle$ . The susceptibility is a crucial observable regarding chiral symmetry restoration, since it is associated with thermal fluctuations and tends to grow near the critical point [7]. For us, the susceptibility will serve as a crucial consistency check, since we can calculate it directly as a quark mass derivative or through the four-quark correlator, and both should coincide and be finite and scale independent.

Therefore, we will give the complete results in ChPT for the four-quark correlators and four-quark condensates in SU(2) and SU(3) up to NNLO, performing a consistency check by calculating the scalar susceptibility and showing the robustness of the result under different definitions of the vacuum four-quark expectation value. In addition, the discussion of factorization breaking necessarily implies the calculation and renormalization of the two-quark condensate also at NNLO, which we will perform explicitly here. We will also carry out the large- $N_c$  analysis of the factorization breaking, which can also be performed from the low-energy representation and is formally relevant. These are the main results of this work.

The plan of the paper is the following: In Sec. II we present our calculation of the relevant four-quark correlators for two and three flavors. The details of the calculation are given for  $N_f = 2$ , for simplicity. The scalar susceptibility derived from the four-quark condensate is obtained in Sec. III. The factorization hypothesis is then examined in Sec. IV, whereas in Sec. V we discuss the large- $N_c$  limit of our results, regarding factorization. In Sec. VI we present a brief summary and our conclusions. Finally, in Appendix A we provide the detailed mathematical expressions for the two-quark condensates to NNLO in ChPT and discuss in detail their renormalization, whereas in Appendix B we show the equivalence of our definition of

the four-quark condensate with the usual one in the literature.

## II. FOUR-QUARK CORRELATORS

Our main object of study will be the time-ordered four-quark correlator  $\langle T(\bar{q}q)(x)(\bar{q}q)(0) \rangle$ . We will follow the external source method and write this four-quark correlator as a second functional derivative of the QCD generating functional  $Z_{\text{QCD}}[s]$  with respect to the scalar source  $s(x)$ , which, in general, will be a matrix-valued function in flavor space and couples to the QCD Lagrangian as

$$\begin{aligned}Z_{\text{QCD}}[s] &= \int \mathcal{D}\bar{q}\mathcal{D}q \dots \exp i \int d^4x \mathcal{L}_{\text{QCD}}[\bar{q}, q, s(x), \dots], \\ \mathcal{L}_{\text{QCD}}[s] &= \bar{q}(i\not{D} - s(x))q + \dots,\end{aligned}\quad (4)$$

where the rest of the Lagrangian terms and other fields, which are indicated by dots, are irrelevant for our purposes. A sum over  $N_f$  light flavors,  $N_c$  colors, and Dirac indices is assumed in  $\bar{q}q$ . The physical QCD Lagrangian and partition function correspond to setting  $s(x) = \mathcal{M}$ , the quark mass matrix, in the above equation.

We will consider the effective low-energy representation of  $Z_{\text{QCD}}[s]$  given by chiral perturbation theory [8], built from chiral symmetry invariance as an expansion in external momenta (derivatives) and meson masses:

$$\begin{aligned}Z_{\text{QCD}}[s] &\simeq Z_{\text{eff}}[s] = \int \mathcal{D}\phi^a \exp i \int d^4x \mathcal{L}_{\text{eff}}[\phi^a, s(x)], \\ \mathcal{L}_{\text{eff}} &= \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 \dots,\end{aligned}\quad (5)$$

where the subscript in the effective Lagrangian indicates the order in the derivative and mass expansion, formally  $\mathcal{L}_k = \mathcal{O}(p^k)$  [ $s = \mathcal{O}(p^2)$  in the standard ChPT power counting]. Note that  $\phi^a$  denote the Nambu-Goldstone boson (NGB) fields, usually collected in the  $SU(N_f)$  matrix  $U = \exp[i\lambda_a \phi^a/F]$ , where  $\lambda_a$  are the Gell-Mann or Pauli matrices for  $N_f = 3$  and  $N_f = 2$ , respectively, and  $F$  is the pion decay constant in the chiral limit. The Lagrangian  $\mathcal{L}_2$  is the nonlinear sigma model:

$$\mathcal{L}_2 = \frac{F^2}{4} \text{Tr}[\partial_\mu U^\dagger \partial^\mu U + \chi(U + U^\dagger)], \quad (6)$$

with  $\chi = 2B_0 s(x)$ . When  $s(x) = \mathcal{M}$ , the constants  $m_q$ ,  $F$ ,  $B_0$  appearing in  $\mathcal{L}_2$  are related to meson masses, decay constants, and the quark condensate. For simplicity, we will work in the isospin limit  $m_u = m_d \equiv m$ , so that, to lowest order in SU(2),  $M_{0\pi}^2 = 2mB_0(1 + \mathcal{O}(p^2))$ ,  $F_\pi = F(1 + \mathcal{O}(p^2))$ , and  $\langle\bar{q}q\rangle = B_0 F(1 + \mathcal{O}(p^2))$ . As usual,  $M_{0\pi,0K,0\eta}$  stand for the leading order meson masses, in terms of which we will express our results. Their relation to the physical masses is given in Eqs. (A9) and (A10) in Appendix A. In addition, and for our purposes here, Weinberg's chiral power counting [9], on which chiral perturbation theory relies, can be equivalently accounted

for by keeping trace of inverse powers of  $F$ , which will be used extensively in this work.

The Lagrangians  $\mathcal{L}_4$  and  $\mathcal{L}_6$  are given in [8,10], respectively, where use has been made of different operator identities, partial integration, and the equations of motion to the relevant order. Those Lagrangians contain the so-called low-energy constants (LEC), multiplying each of the independent terms compatible with the symmetries. The  $\mathcal{L}_4$  LEC receive different names depending on whether they multiply terms containing  $U$  fields or not, respectively,  $L_i$  and  $H_i$  in the SU(3) case. The terms without  $U$  fields are contact terms containing just external sources and no fields, but they are needed to absorb some divergences coming from loop diagrams using  $\mathcal{L}_2$  vertices. The original SU(2) Lagrangians in [11] are written in terms of vector fields instead of matrix fields  $U$  as above, but they also use different names for the  $\mathcal{L}_4$  low-energy constants— $l_i$  and  $h_i$  in this case. However, it is possible to recast [12] these Lagrangians using matrix field notation, which we will use throughout this paper, and keep the same  $l_i$ ,  $h_i$  low-energy constants. The relation between the SU(3) and SU(2) low-energy constants is given in [8,13,14].

This name differentiation for the  $\mathcal{L}_6$  is not followed any longer [10]: All of them are called  $c_i$  in the SU(2) case and  $C_i$  in the SU(3) case. Note that the  $\mathcal{O}(p^6)$  LEC contained in  $\mathcal{L}_6$  absorb both two-loop divergences from  $\mathcal{L}_2$  and one-loop divergences in diagrams with  $\mathcal{L}_4$  vertices. All the details for renormalization of quark condensates up to the order we are considering here are given in Appendix A. We recall that the  $\mathcal{L}_4$  Lagrangian in SU(3) also contains the Wess-Zumino-Witten (WZW) [15] anomalous term, accounting for anomalous NGB processes, whose coefficient is fixed by topology arguments and is proportional to the number of colors  $N_c$ .

### A. Two flavors

For simplicity, we will discuss the full details of our approach in the simpler case  $N_f = 2$ . Thus we will denote by the subscript  $l$  the light quark correlator, and study  $\langle \bar{q}q \rangle_l \equiv \bar{u}u + \bar{d}d$ . Note that we have defined the scalar source  $s(x)$  as a matrix, but since for the physical partition function it corresponds to the mass matrix  $\mathcal{M}$ , which is diagonal, we are thus only interested in the diagonal elements of  $s(x)$  and we can set the rest of the source terms to zero. In particular, for the two flavor case  $\mathcal{M} = m\mathbb{1}_2$ , and we can write  $s(x) = s_0(x)\mathbb{1}_2$ , so that

$$\begin{aligned} \langle \bar{q}q \rangle_l &\equiv \frac{i}{Z_{\text{QCD}}[m]} \frac{\delta Z_{\text{QCD}}[s_0]}{\delta s_0(x)} \Big|_{s_0=m} \\ &\simeq \frac{i}{Z_{\text{eff}}[m]} \frac{\delta Z_{\text{eff}}[s_0]}{\delta s_0(x)} \Big|_{s_0=m} \equiv - \left\langle \frac{\delta \mathcal{L}_{\text{eff}}[s_0]}{\delta s_0(x)} \right\rangle_{s_0=m}. \end{aligned} \quad (7)$$

Following the same procedure, but now for the four light quark correlator, we get

$$\begin{aligned} \langle T(\bar{q}q)_l(x)(\bar{q}q)_l(0) \rangle &= - \frac{1}{Z_{\text{eff}}[m]} \frac{\delta}{\delta s_0(x)} \frac{\delta}{\delta s_0(0)} Z_{\text{eff}}[s_0] \Big|_{s_0=m} \\ &= -i \left\langle T \frac{\delta^2 \mathcal{L}_{\text{eff}}[s_0(x)]}{\delta s_0(x)^2} \right\rangle_{s_0=m} \delta^{(D)}(x) \\ &\quad + \left\langle T \frac{\delta \mathcal{L}_{\text{eff}}[s_0]}{\delta s_0(x)} \frac{\delta \mathcal{L}_{\text{eff}}[s_0]}{\delta s_0(0)} \right\rangle_{s_0=m}. \end{aligned} \quad (8)$$

We will regularize all our expressions in dimensional regularization with  $D = 4 - \epsilon$ , and for that purpose, we keep the  $D$  dependence in the  $\delta$ -function term above.

Now, from Eq. (8), and using the Lagrangians in [8,10], we obtain the following result:

$$\begin{aligned} \langle T(\bar{q}q)_l(x)(\bar{q}q)_l(0) \rangle_{\text{NLO}} &= 4B_0^2 F^4 \left\{ 1 + \frac{4M_{0\pi}^2}{F^2} (l_3^r + h_1^r) - 6\mu_\pi \right\}, \end{aligned} \quad (9)$$

$$\begin{aligned} \langle T(\bar{q}q)_l(x)(\bar{q}q)_l(0) \rangle_{\text{NNLO}} &= \langle T(\bar{q}q)_l(x)(\bar{q}q)_l(0) \rangle_{\text{NLO}} \\ &\quad + 4B_0^2 F^4 \left[ \frac{2M_{0\pi}^2}{F^2} (l_3^r + h_1^r) - 3\mu_\pi \right]^2 \\ &\quad + 8B_0^2 F^4 \left[ -\frac{3}{2} \mu_\pi^2 - \frac{3M_{0\pi}^2}{F^2} (\mu_\pi \nu_\pi + 4l_3^r \mu_\pi) \right. \\ &\quad \left. + \frac{3M_{0\pi}^4}{8F^4} (-16l_3^r \nu_\pi + \hat{c}_1^r) \right], \\ &\quad + B_0^2 [-8i(l_3 + h_1)\delta^{(D)}(x) + K^{(2)}(x)], \end{aligned} \quad (10)$$

where the NNLO constants  $\hat{c}_i$  are defined in Eq. (A3) and, as usual [8],

$$\begin{aligned} \mu_\pi &= \frac{M_{0\pi}^2}{32\pi^2 F^2} \log \frac{M_{0\pi}^2}{\mu^2}, \\ \nu_\pi &= F^2 \frac{\partial \mu_{0\pi}}{\partial M_{0\pi}^2} = \frac{1}{32\pi^2} \left( 1 + \log \frac{M_{0\pi}^2}{\mu^2} \right). \end{aligned} \quad (11)$$

Note that we have defined  $K^{(2)}(x)$  as the connected part of the four-pion correlator to leading order:

$$\begin{aligned} K^{(2)}(x) &= \langle T \phi^a(x) \phi_a(x) \phi^b(0) \phi_b(0) \rangle_{\text{LO}} \\ &\quad - \langle T \phi^a(0) \phi_a(0) \rangle_{\text{LO}}^2 \\ &= 6G_\pi^2(x), \end{aligned} \quad (12)$$

$G_\pi(x)$  being the pion propagator to leading order, and the factor of 6 =  $2(N_f^2 - 1)$  comes from the Wick contractions and is nothing but twice the number of NGB fields. The details of the renormalization and the dependence of the constants  $l_i^r$ ,  $h_i^r$ , and  $\hat{c}_i^r$  on the renormalization scale  $\mu$  are given in Appendix A.

To understand the structure of the different contributions to Eqs. (9) and (10) it is useful to recall the general form of the SU(2) low-energy Lagrangian terms depending on the external scalar source. For our NNLO calculation, we will need to keep terms up to  $\mathcal{O}(F^{-2})$ . Let us then separate the terms in the Lagrangian [8,10], according to their  $s$  dependence after expanding the  $U$  in NGB fields:

$$\begin{aligned} \mathcal{L}_{\text{eff}}[s_0] = & \left( \mathcal{L}_2^{0\phi} F^2 + \mathcal{L}_2^{2\phi} + \frac{1}{F^2} \mathcal{L}_2^{4\phi} + \frac{1}{F^2} \mathcal{L}_4^{2\partial\phi} \right) s_0 \\ & + \left( \mathcal{L}_4^{0\phi} + \frac{1}{F^2} \mathcal{L}_4^{2\phi} \right) s_0^2 + \frac{1}{F^2} \mathcal{L}_6^{0\phi} s_0^3 \\ & + \frac{1}{F^2} \tilde{\mathcal{L}}_6^{0\phi} \partial_\mu s_0 \partial^\mu s_0 + \mathcal{O}\left(\frac{1}{F^4}\right), \end{aligned} \quad (13)$$

where we have also made explicit the leading  $1/F^2$  dependence of each term. The superscripts “ $n\phi$ ” indicate the number of NGB fields or field derivatives on each Lagrangian contribution. Note that, since  $\mathcal{L}_k = \mathcal{O}(p^k)$  in derivatives or  $s$  powers [ $s = \mathcal{O}(p^2)$ ], it counts at least as  $\mathcal{O}(1/F^{k-4})$ , but the  $1/F^2$  order of each term grows when increasing the number of NGB fields,  $\phi$ . We have represented the vertices arising from the different pieces of the Lagrangian above in the left column of Fig. 1. Note that all  $\mathcal{L}_6$  terms in Eq. (13) have the  $0\phi$  superscript, because, to this order, they are simply constants. The constant  $\mathcal{L}_6^{0\phi}$  term enters in  $\langle \bar{q}q \rangle_{\text{NNLO}}^2$  and ensures that one can renormalize the full result so that the quark condensate is finite and scale independent. The term containing  $(\partial s_0)^2$  does not

contribute to this order. The details as well as the explicit expression of the condensates up to NNLO are given in Appendix A.

Once the structure of the vertices arising from the Lagrangian equation (13) are understood, we represent diagrammatically in Fig. 1 the different contributions to  $\langle T(\bar{q}q)(x)(\bar{q}q)(0) \rangle$ . On each diagram, the horizontal dotted line represents space-time, where each quark-antiquark bilinear stands at separate points 0 and  $x$ . To LO and NLO—respectively,  $\mathcal{O}(F^4)$  and  $\mathcal{O}(F^2)$ —all contributions are disconnected, as seen in diagrams (a), (b), and (c). The reason is that we can only use the  $\mathcal{L}_2^{2\phi}$  term once, and therefore, the NGB line has to close upon itself—a tadpole. This gives diagram (b) in Fig. 1. To NNLO [ $\mathcal{O}(F^0)$ ] we have all the possibilities shown in Fig. 1 in diagrams (d)–(j). If one of the vertices comes from  $\mathcal{L}_4$  or  $\mathcal{L}_6$ , once more there is at most one NGB line and the resulting diagram is disconnected. Note that among these is the  $\delta^{(D)}(x)$  term in Eq. (10) from diagram (h). With only  $\mathcal{L}_2$  vertices, one has a diagram with a double tadpole in one of the vertices, leading to a LO propagator squared at the same point [diagram (d)], two vertices with one tadpole

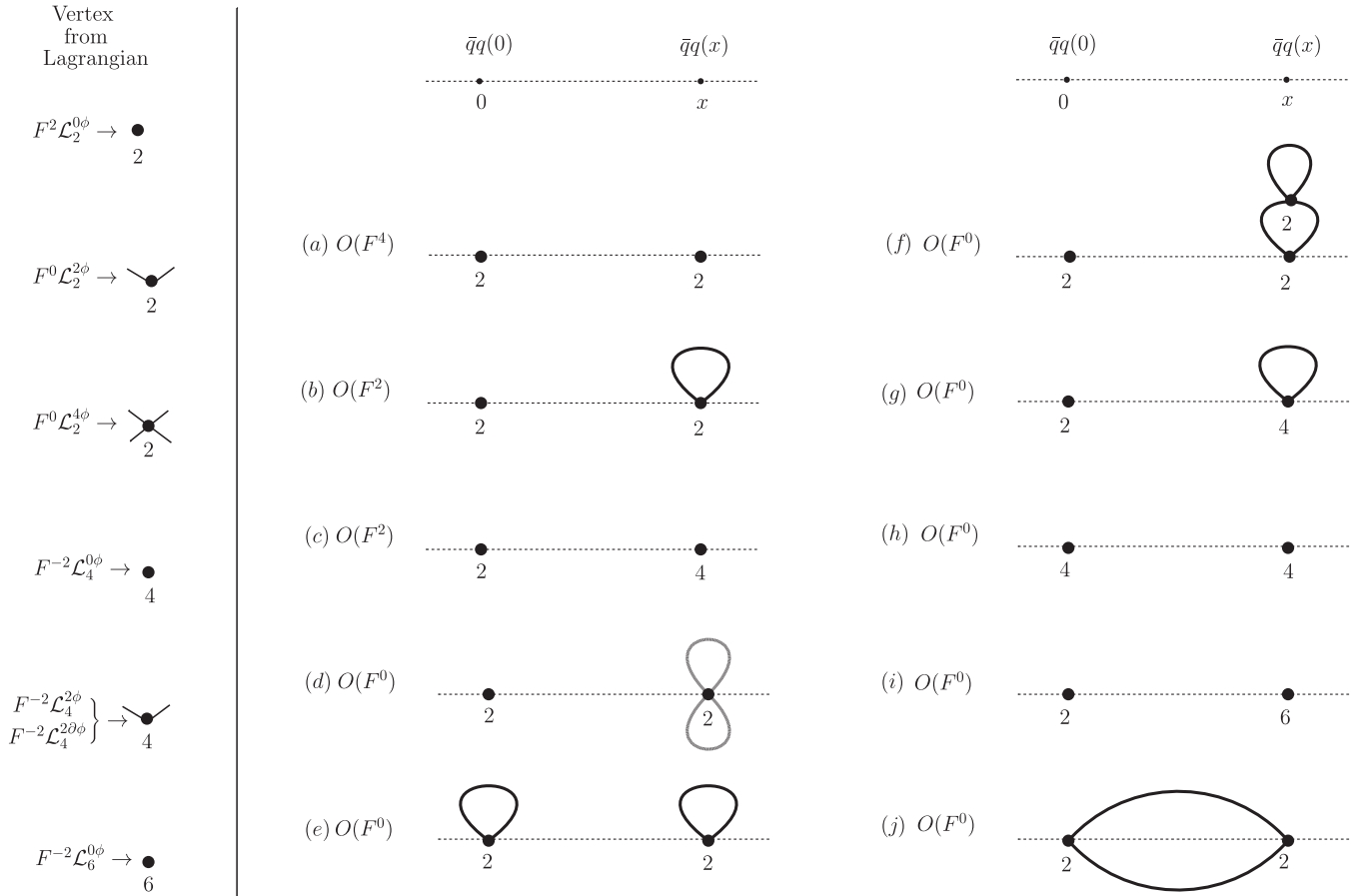


FIG. 1. In the left column we provide the diagrammatic representation of the vertices coming from the different terms of the Lagrangian in Eq. (13). The numbers attached to each vertex indicate the order of the Lagrangian. Diagrams (a) to (j) represent the different contributions to the four-quark correlator. The dotted horizontal line represents the space-time separation between 0 and  $x$ . Note that each NGB line decreases the order of the diagram by  $1/F^2$ . Diagram (j) is the first factorization-breaking term.



each [diagram (e)], a diagram like (b) but with the propagator renormalized to next to leading order (NLO) [diagram (f)], and another with two NGB lines on each vertex but joined to form a connected one-loop diagram, which is diagram (j). Actually, the latter is the only possible connected contribution to this order, and gives the  $G^2(x)$  term in Eq. (12). This whole discussion of vertices and diagrams will be valid also for the SU(3) case discussed below.

Let us now turn to the factorization hypothesis and the relation between the four-quark correlation function and the two-quark condensate. We have collected in Appendix A all the two-quark condensate ChPT expressions up to NLO [given also in [11] for SU(2) and in [8] for SU(3)] and up to NNLO, which have been given explicitly in [16] for SU(3). Numerical estimations including NNLO corrections are given in [16,17]. In view of Eqs. (7), (A11), and (A12), it is easy to check that

$$\begin{aligned}\langle T(\bar{q}q)_l(x)(\bar{q}q)_l(0) \rangle_{\text{NLO}} &= (\langle \bar{q}q \rangle_l^2)_{\text{NLO}}, \\ \langle T(\bar{q}q)_l(x)(\bar{q}q)_l(0) \rangle_{\text{NNLO}} &= (\langle \bar{q}q \rangle_l^2)_{\text{NNLO}} \\ &+ B_0^2[-8i(l_3 + h_1)\delta^{(D)}(x) + K^{(2)}(x)].\end{aligned}\quad (14)$$

We see that all contributions from disconnected diagrams in Fig. 1, other than the  $\delta^{(D)}$  term, can be absorbed in the two-quark condensate. Actually, up to NLO, we observe that  $\langle T(\bar{q}q)_l(x)(\bar{q}q)_l(0) \rangle$  in Eq. (9) is constant and equal to the NLO of the quark condensate squared, which leads to factorization in the  $N_c \rightarrow \infty$  limit (see Secs. IV and V). However, to NNLO the previous expression for  $x = 0$  contains the  $G^2(0)$  divergent contribution, even after the quark condensate has been renormalized and the  $\delta^{(D)}$  term regularized. We will show below that divergences cancel in physical quantities such as the scalar susceptibility, which is directly expressed in terms of observable quantities such as the free energy density. That is not the case for the four-quark condensate, which will remain divergent. Before analyzing these issues, let us extend the previous analysis to the SU(3) case.

### B. Three flavors

In the SU(3) case,  $\bar{q}q \equiv \bar{u}u + \bar{d}d + \bar{s}s$ ,  $\mathcal{M} = \text{diag}(m, m, m_s)$ ,  $s(x) = \text{diag}[s_0(x), s_0(x), s_s(x)]$ , and

$$\langle \bar{q}q \rangle = -\left\langle \frac{\delta \mathcal{L}_{\text{eff}}[s_0]}{\delta s_0(x)} + \frac{\delta \mathcal{L}_{\text{eff}}[s_0, s_s]}{\delta s_s(x)} \right\rangle_{s=\mathcal{M}}, \quad (15)$$

$$\begin{aligned}\langle T(\bar{q}q)(x)(\bar{q}q)(0) \rangle &= -i \left\langle T \left( \frac{\delta}{\delta s_0(x)} + \frac{\delta}{\delta s_s(x)} \right)^2 \mathcal{L}_{\text{eff}}[s_0(x), s_s(x)] \right\rangle_{s=\mathcal{M}} \delta^{(D)}(x) \\ &+ \left\langle \left( \frac{\delta \mathcal{L}_{\text{eff}}[s_0, s_s]}{\delta s_0(x)} + \frac{\delta \mathcal{L}_{\text{eff}}[s_0, s_s]}{\delta s_s(x)} \right) \right. \\ &\times \left. \left( \frac{\delta \mathcal{L}_{\text{eff}}[s_0, s_s]}{\delta s_0(0)} + \frac{\delta \mathcal{L}_{\text{eff}}[s_0, s_s]}{\delta s_s(0)} \right) \right\rangle_{s=\mathcal{M}}.\end{aligned}\quad (16)$$

The  $s$ -dependent terms in the SU(3) effective Lagrangian are now the generalization of Eq. (13) to include  $s_s(x)$ , so that we have crossed terms like  $s_0 s_s$ ,  $s_0^2 s_s$ , and so on, but the general structure is the same. As in the SU(2) case, the derivative terms  $(\partial s)^2$  do not contribute to  $\langle T(\bar{q}q)(x) \times (\bar{q}q)(0) \rangle$ , and thus only four  $\mathcal{L}_6$  constant terms contribute to renormalization. As seen in Appendix A, they are proportional to the  $\hat{C}_i$  LEC given in Eq. (A3). Since we already presented the detailed discussion for the SU(2) case in the previous section, for the sake of brevity we cast our SU(3) results for  $\langle T(\bar{q}q)(x)(\bar{q}q)(0) \rangle$ , which are much longer than before, directly in terms of the two-quark condensates, namely,

$$\begin{aligned}\langle T(\bar{q}q)(x)(\bar{q}q)(0) \rangle_{\text{NLO}} &= (\langle \bar{q}q \rangle^2)_{\text{NLO}}, \\ \langle T(\bar{q}q)(x)(\bar{q}q)(0) \rangle_{\text{NNLO}} &= (\langle \bar{q}q \rangle^2)_{\text{NNLO}} \\ &+ B_0^2[-24i(12L_6 + 2L_8 + H_2)\delta^{(D)}(x) + K(x)],\end{aligned}\quad (17)$$

where  $K(x)$  is the extension of Eq. (12) to the SU(3) case:

$$\begin{aligned}K(x) &= \langle T\phi^a(x)\phi_a(x)\phi^b(0)\phi_b(0) \rangle_{\text{LO}} - \langle T\phi^a(0)\phi_a(0) \rangle_{\text{LO}}^2 \\ &= 2[3G_\pi^2(x) + 4G_K^2(x) + G_\eta^2(x)].\end{aligned}\quad (18)$$

The ChPT expressions for the four-quark condensates to NNLO given in Eqs. (14) and (17) (simplified in terms of the explicit expressions for  $\langle \bar{q}q \rangle_{\text{NNLO}}$ , which are given in Appendix A) are among the main results of the present work.

Note that, as it happened in the SU(2) case, the contribution Eq. (18) stems from  $2(N_f^2 - 1)$  NGB propagators, although this time they have different masses. Similarly, we can calculate separately the strange and nonstrange four-quark condensates, which also factorize up to NLO, whereas to NNLO we get

$$\begin{aligned}\langle T(\bar{q}q)_l(x)(\bar{q}q)_l(0) \rangle &= \langle \bar{q}q \rangle_{l,\text{SU}(3)}^2 + B_0^2 \left[ -16i(8L_6 + 2L_8 + H_2)\delta^{(D)}(x) \right. \\ &\left. + 6G_\pi^2(x) + 2G_K^2(x) + \frac{2}{9}G_\eta^2(x) \right] + \mathcal{O}\left(\frac{1}{F^2}\right),\end{aligned}\quad (19)$$

$$\begin{aligned}\langle T(\bar{s}s)(x)(\bar{s}s)(0) \rangle &= \langle \bar{s}s \rangle^2 + B_0^2 \left[ -8i(4L_6 + 2L_8 + H_2)\delta^{(D)}(x) \right. \\ &\left. + 2G_K^2(x) + \frac{8}{9}G_\eta^2(x) \right] + \mathcal{O}\left(\frac{1}{F^2}\right).\end{aligned}\quad (20)$$

Once again the explicit expressions for the renormalized  $\langle \bar{q}q \rangle_{\text{NNLO}}$  are given in Appendix A.

The  $\langle T(\bar{q}q)(x)(\bar{s}s)(0) \rangle$  correlator has been calculated up to NNLO in [18,19] in terms of the basis of the solutions to the Muskhelishvili-Omnès equations.

We remark that the four-quark correlators to NNLO given in Eqs. (14) and (17) are key ingredients to define

the four-quark condensate and study the factorization hypothesis, as explained in the Introduction.

### III. THE SCALAR SUSCEPTIBILITY

In this section we will provide a consistency check of our calculation by analyzing the chiral or scalar susceptibility to the first nontrivial order, which can be obtained either by differentiating the two-quark condensates or by integration of the four-quark ones. The susceptibility is defined in Euclidean space-time as

$$\chi_l \equiv -\frac{\partial}{\partial m} \langle \bar{q}q \rangle_l \quad (21)$$

and measures the condensate thermal fluctuations, growing dramatically near the chiral restoration, as confirmed by different lattice studies [7]. Therefore, let us consider the Euclidean (imaginary time  $t = -i\tau$ ) version of Eqs. (4) and (5), replacing  $i \int d^4x \rightarrow \int d\tau \int d^3\vec{x} \equiv \int_E d^4x$  and the  $(-, -, -, -)$  metric in the Lagrangian. Recall that the finite temperature  $T$  case, which we will analyze elsewhere [20], would correspond to  $\tau \in [0, \beta]$  with  $\beta = 1/T$ . In addition, in Eqs. (8) and (16) we have to replace  $-i\delta^D(x) \rightarrow \delta(\tau)\delta^{(D-1)}(\vec{x}) \equiv \delta_E^D(x)$ . With these replacements, we can now relate the susceptibility with the four-quark correlators in the nonstrange sector:

$$\begin{aligned} \chi_l &= \frac{1}{V_E} \frac{\partial^2}{\partial m^2} \log Z = \frac{1}{V_E} \left[ \frac{1}{Z} \frac{\partial^2 Z}{\partial m^2} - \left( \frac{1}{Z} \frac{\partial Z}{\partial m} \right)^2 \right] \\ &= \int_E d^D x [\langle T(\bar{q}q)_l(x)(\bar{q}q)_l(0) \rangle - \langle \bar{q}q \rangle_l^2], \end{aligned} \quad (22)$$

where  $V_E = \int_E d^D x$  is the  $D$ -dimensional Euclidean volume and  $Z = Z[s = \mathcal{M}] = e^{-zV_E}$  is the partition function, with  $z$  the free energy density.

The relation in Eq. (22) between  $\chi_l$  and the four-point function allows us to check our previous results. From Eqs. (14) and (19), taking into account that

$$\int_E d^D x [G_i(x)]^2 = -\frac{d}{dM_i^2} G_i(0), \quad (23)$$

and the expressions Eqs. (A1) and (A2), together with the renormalization of the LEC in Eqs. (A5) and (A6), we obtain, using the last integral in Eq. (22),

$$\chi_l^{\text{SU}(2)} = B_0^2 [8(l_3^r(\mu) + h_1^r(\mu)) - 12\nu_\pi] + \mathcal{O}\left(\frac{1}{F^2}\right), \quad (24)$$

$$\begin{aligned} \chi_l^{\text{SU}(3)} &= B_0^2 \left[ 16(8L_6^r(\mu) + 2L_8^r(\mu) + H_2^r(\mu)) - 12\nu_\pi \right. \\ &\quad \left. - 4\nu_K - \frac{4}{9}\nu_\eta \right] + \mathcal{O}\left(\frac{1}{F^2}\right), \end{aligned} \quad (25)$$

with  $\nu_i$  given in Eq. (A8).

This is the same result that we get by taking directly the mass derivative of the quark condensate to NLO in Eqs. (A11) and (A13) using the leading order relations

between meson and quark masses [8]. This represents a check of consistency of our calculation of the four-quark condensates to NNLO. In addition, we have explicitly checked [using Eq. (A5)] that the susceptibilities above are finite and independent of the scale  $\mu$ . Furthermore, with the conversion between the SU(2) and SU(3) LEC given in [8],

$$\begin{aligned} l_3^r(\mu) + h_1^r(\mu) &= 2 \left( 8L_6^r(\mu) + 2L_8^r(\mu) + H_2^r(\mu) \right. \\ &\quad \left. - \frac{1}{4}\nu_K - \frac{1}{36}\nu_\eta \right), \end{aligned} \quad (26)$$

we end up with

$$\chi_l^{\text{SU}(2)} = \chi_l^{\text{SU}(3)}$$

which is also consistent since the SU(3) susceptibility is given by constant plus logarithmic terms in the  $m_s \rightarrow \infty$  expansion, with no subleading terms in that expansion; therefore, the very same expression has to be exactly recovered by calculating directly in the SU(2) limit. Note also that the susceptibility to this order is independent of  $F$ .

Our result for the susceptibility is also consistent with a previous work [21], where only the leading infrared order in the chiral limit was calculated, namely, the  $\log M_{0\pi}^2$  term inside the  $\nu_\pi$  in Eq. (24). This is the expected behavior of the susceptibility from the  $O(4)$  model universality class near the chiral limit and below the critical temperature, namely,  $\chi \sim \log m$ , with  $m$  the mass of the nonstrange quark [7,21].

We can follow the same procedure to obtain the strange quark susceptibility in terms of our strange four-quark correlation function:

$$\begin{aligned} \chi_s &\equiv -\frac{\partial}{\partial m_s} \langle \bar{s}s \rangle = \frac{1}{V_E} \frac{\partial^2}{\partial m_s^2} \log Z \\ &= \frac{1}{V_E} \left[ \frac{1}{Z} \frac{\partial^2 Z}{\partial m_s^2} - \left( \frac{1}{Z} \frac{\partial Z}{\partial m_s} \right)^2 \right] \\ &= \int_E d^4 x [\langle T(\bar{s}s)(x)(\bar{s}s)(0) \rangle - \langle \bar{s}s \rangle^2], \end{aligned} \quad (27)$$

which, from Eq. (20), gives

$$\chi_s = B_0^2 \left[ 8(4L_6^r + 2L_8^r + H_2^r) - 4\nu_K - \frac{16}{9}\nu_\eta \right]. \quad (28)$$

We have explicitly double checked this result by taking the derivative with respect to  $m_s$  of the NLO strange quark condensate in Eq. (A15). We remark that the results in Eqs. (24), (25), and (28) for the ChPT scalar susceptibilities have not been given elsewhere.

### IV. NONFACTORIZATION

As explained in the Introduction, we define the four-quark condensate through Eq. (2), although in Appendix B we show that this is equivalent to the more usual definition

of Eq. (3). Therefore, by taking the  $x \rightarrow 0$  limit in Eqs. (14) and (17), and despite the fact that  $\delta^{(D)}(0)$  vanishes identically in dimensional regularization [22] (now we are not integrating over  $x$  as for the scalar susceptibility), there is still a term that clearly breaks factorization, as defined in Eq. (1). In particular, we get in SU(2), from Eq. (14),

$$\frac{\langle(\bar{q}q)^2\rangle}{\langle\bar{q}q\rangle^2} = 1 + \frac{6}{F^4} G_\pi^2(0) + \mathcal{O}\left(\frac{1}{F^6}\right), \quad (29)$$

whereas in SU(3) from Eq. (18), we find

$$\begin{aligned} \frac{\langle(\bar{q}q)^2\rangle}{\langle\bar{q}q\rangle^2} &= 1 + \frac{2}{F^4} [3G_\pi^2(0) + 4G_K^2(0) + G_\eta^2(0)] \\ &+ \mathcal{O}\left(\frac{1}{F^6}\right), \end{aligned} \quad (30)$$

where the propagators  $G_i(0)$  are given in dimensional regularization in Eq. (A1).

The nonfactorization terms above are divergent and independent of the LEC, once the two-quark condensate  $\langle\bar{q}q\rangle$  has been rendered finite with the renormalization of the  $\mathcal{O}(p^4)$  and  $\mathcal{O}(p^6)$  LEC (see Appendix A). The renormalizability of  $\langle\bar{q}q\rangle$  is of course consistent with the fact that  $\bar{q}\mathcal{M}q$  is a QCD RG invariant. Therefore, our nonfactorization ChPT results in Eqs. (29) and (30) imply that the four-condensate is divergent, and hence the vacuum expectation value of  $(\bar{q}q)^2$  does not admit a meaningful low-energy representation.

Our result is consistent with the one-loop QCD RG analysis in [2], where only one flavor is considered. In that paper it is shown that factorization is incompatible with the renormalization group. Their argument goes as follows: The operator  $(\bar{q}q)^2$  mixes under renormalization with other four-quark operators, which can be chosen in combinations such that their vacuum expectation values would vanish if factorization holds. Then, assuming factorization for those other operators leads to the conclusion that  $\langle(\bar{q}q)^2\rangle$  is divergent, which, in particular, means that it does not factorize in terms of  $\langle\bar{q}q\rangle^2$  and that one cannot write any RG invariant made of four-quark operators.

Another interesting comment is that the factorization-breaking terms in Eqs. (29) and (30) vanish exactly in the chiral limit, since then all dimensionally regularized propagators  $G_\pi(0) = G_K(0) = G_\eta(0) = 0$ . In that case, we would be forced to examine the neglected NNNLO contributions in order to check the validity of factorization and the finiteness of the four-quark condensate. Recall that the arguments in [2] regarding four-quark operators actually hold for  $m = 0$ .

## V. LARGE $N_c$

Let us now discuss the  $N_f$  and  $N_c$  dependence for the regularized expression, namely, before taking the  $D = 4$  limit. As we have checked for the SU(2) and SU(3) cases, the  $K(x)$  contributions to the connected four-field functions

in Eqs. (12) and (18) are  $\mathcal{O}(N_{\text{GB}}) = \mathcal{O}(N_f^2)$ , where  $N_{\text{GB}} = N_f^2 - 1$  is the number of Goldstone bosons. In addition, the  $N_c$  leading behavior of the different ChPT constants is well known [8] from the QCD  $1/N_c$  expansion. In particular,  $F^2 = \mathcal{O}(N_c)$ . Therefore, the first term that breaks factorization in Eqs. (29) and (30) is  $\mathcal{O}(N_f^2/N_c^2)$ , which is rather different from the  $1/(4N_f N_c)$  scaling suggested in Eq. (1). Unfortunately, we cannot say much more about the  $N_f$  behavior of higher order terms, which could change the global  $N_f$  behavior. Note that the  $N_f$  dependence of the quark correlators has been studied in detail in [23] with a different motivation.

In the following, we will easily deduce the  $1/N_c$  behavior and, in particular, we can study the large  $N_c$  limit before renormalization. We will see that, in such a formal case, factorization holds for  $N_c \rightarrow \infty$ . First of all, contrary to Eq. (1), in Eqs. (29) and (30) there are no  $\mathcal{O}(1/N_c)$  terms. These could have arisen from contributions of the type  $L_i G(0)/F^4$ , when  $L_i$  is  $\mathcal{O}(N_c)$ , that actually appear in the calculation. However, as we have said before, the whole  $L_i$  dependence of the four-quark condensate is exactly that of the two-quark condensate squared, and thus such terms do not break factorization. The same happens with the  $\mathcal{O}(p^6)$   $c_i$  LEC in Eq. (A3). Still, one could wonder if  $\mathcal{O}(1/N_c)$  or larger  $N_c$  powers could arise from higher chiral orders that we have not calculated explicitly here.

Of course, as seen in Eqs. (29) and (30), these higher chiral orders count at least as  $\mathcal{O}(1/F^6)$ . Since  $F^2 = \mathcal{O}(N_c)$ , this already introduces a  $1/N_c^3$  factor, but it is not the only one, since the LEC can carry their own  $N_c$  behavior. In particular, we recall that, according to the chiral power counting discussed in Sec. II, the  $\mathcal{O}(1/F^n)$  contribution to the ratios in Eqs. (29) and (30) comes from connected diagrams with  $n = 2(L + 1) + \sum_d N_d(d - 2)$ , with  $L$  the number of loops and  $N_d$  the number of vertices from  $\mathcal{L}_d = \mathcal{L}_2, \mathcal{L}_4, \dots$ . Note that a nonfactorizing term requires at least  $L = 1$ , the leading contribution being the connected one-loop diagram (j) in Fig. 1 with two  $\mathcal{L}_2$  vertices. This diagram yields the factorization-breaking terms in Eqs. (29) and (30).

Now, the highest  $N_c$  scaling of the LEC from  $\mathcal{L}_d$  is  $\mathcal{O}(N_c^{(d-2)/2})$ . The reason is that these LEC, when divided by  $F^{d-4}$ , should yield  $\mathcal{O}(N_c)$  contributions at most, as expected from the large- $N_c$  behavior of the low-energy generating functional [8]. This includes the WZW term, which is the anomalous part of  $\mathcal{L}_4$  and is multiplied explicitly by  $N_c$  [15]. Although the WZW term does not depend on the quark mass, it could enter in this calculation through loop contributions. It is possible, of course, that some LEC do scale with a smaller  $N_c$  power. For instance, the  $L_1$  to  $L_{10}$  appearing in  $\mathcal{L}_4$  are known to scale as  $\mathcal{O}(N_c)$ , except  $L_4$ ,  $L_6$ , and  $L_7$ , which scale as  $\mathcal{O}(1)$ . These are model-independent QCD predictions obtained in [8], with the exception of  $L_7$ , which was taken there as  $\mathcal{O}(N_c^2)$ . This  $L_7$  counting corresponds to integrating the  $\eta'$  as a heavy

particle but then considering  $m_{\eta'}^2 \sim \mathcal{O}(1/N_c)$  and therefore a light particle. The consistent way of integrating the  $\eta'$  yields  $L_7 \approx \mathcal{O}(1)$  [24]. In summary, the  $L_i$  in  $\mathcal{L}_4$  are  $\mathcal{O}(N_c)$  at most, the  $c_i$  in  $\mathcal{L}_6$  are  $\mathcal{O}(N_c^2)$  at most, and so on.

Hence, if a diagram has  $N_d$  vertices from  $\mathcal{L}_d$ , they contribute, at most, with  $N_d(d-2)/2$  powers of  $N_c$ . Summing over all the  $d$ , the scaling of the LEC that contribute to that diagram is given, at most, by  $\sum_d N_d(d-2)/2$  powers of  $N_c$ . Taking into account that the  $1/F^n$  factors behave as  $\mathcal{O}(N_c^{-n/2})$ , we conclude that the nonfactorization terms should be  $\mathcal{O}(N_c^{\sum_d N_d(d-2)/2 - (n/2)}) = \mathcal{O}(N_c^{-(L+1)})$  at most. But since we noted that nonfactorization terms require  $L \geq 1$ , then the largest factorization-breaking contribution is  $\mathcal{O}(N_c^{-2})$ , at most. Actually, this is the behavior of the nonfactorization correction we explicitly calculated in Eqs. (29) and (30). This  $\mathcal{O}(N_c^{-2})$  counting of the factorization breaking, which we have formally showed here in the low-energy representation, confirms what had been suggested previously in the literature [25].

Finally, if we compare with the original QCD factorization hypothesis Eq. (1), we conclude that factorization of the four-quark condensate as the square of the two-quark condensate holds formally in the  $N_c \rightarrow \infty$  limit. This is of course only a formal statement, since we have just seen that in the low-energy calculation the factorization-breaking terms diverge.

## VI. CONCLUSIONS

In this work we have addressed the issue of the four-quark condensate factorization into the two-quark condensate squared, within the low-energy representation of those condensates provided by chiral perturbation theory.

Our main result is the formal model-independent proof of the nonvalidity of the factorization or vacuum saturation hypothesis for the low-energy sector of QCD. A detailed calculation of the NNLO two-quark and four-quark condensates for both two and three flavors shows that, to that order, factorization is broken by terms which cannot be rendered finite with the usual renormalization procedure, ensuring that the two-quark condensate is finite and scale independent. This breaking of the factorization assumption at low energies is then a model-independent result, since it relies only on the effective Lagrangian formalism, and is consistent with previous observations regarding the incompatibility of the factorization hypothesis with the QCD renormalization-group evolution. In addition, the very same nonfactorization term is obtained by using more conventional definitions of the quark condensate within the  $\overline{\text{MS}}$  scheme in dimensional regularization. As a consistency check of our analysis, we have derived the light and strange susceptibilities from the calculated four-quark correlators, showing that they agree with a direct derivative with respect to the quark masses of the two-quark condensates. The explicit renormalized and scale-independent

expressions for the ChPT NNLO susceptibilities are not given elsewhere, to our knowledge. Factorization holds formally in the  $N_c \rightarrow \infty$  limit, as we have been able to show to any order in the chiral expansion, since the leading term that breaks factorization scales as  $\mathcal{O}(1/N_c^2)$ .

We believe that these results can be useful for workers in the field, in particular, concerning the OPE and sum-rule approach. A natural extension of this work is to consider finite temperature effects to see how they affect factorization and its connection with the chiral susceptibility, which in the thermal case plays a crucial role near chiral restoration [20].

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## APPENDIX A: QUARK CONDENSATES TO NNLO IN CHPT AND THEIR RENORMALIZATION

In this section we will give our NNLO results for the two-quark condensates. As explained in the text, the corresponding four-quark condensates cannot be obtained just by squaring these results, but one also has to add the nonfactorizing contributions described in Eqs. (14) and (17).

The free meson propagator in dimensional regularization is given by [8]

$$G_i(0) = 2M_{0i}^{D-2}\lambda, \quad (\text{A1})$$

with

$$\lambda = \frac{\Gamma[1 - \frac{D}{2}]}{2(4\pi)^{D/2}}, \quad (\text{A2})$$

and  $D = 4 - \epsilon$ .

The SU(3)  $\mathcal{L}_4$  ChPT Lagrangian is well known [8] and we do not reproduce it here. The relevant terms for the calculation of the condensates in the  $\mathcal{O}(p^6)$  Lagrangian [10] are only those dependent on the quark masses to leading order in the Goldstone boson fields. Here, we will follow, for simplicity, a different notation than in [10] to denote the  $\mathcal{L}_6$  low-energy constants involved in the mass terms:

$$\begin{aligned} \mathcal{L}_6^{m_q, \text{SU}(2)} &= \frac{B_0^3}{F^2} \hat{c}_1 m^3, \\ \mathcal{L}_6^{m_q, \text{SU}(3)} &= \frac{B_0^3}{F^2} (\hat{C}_1 m^3 + \hat{C}_2 m^2 m_s + \hat{C}_3 m m_s^2 + \hat{C}_4 m_s^3). \end{aligned} \quad (\text{A3})$$



Recall that our  $\hat{c}_i$  are linear combinations of the LEC considered in [10,26] whose precise form is not relevant here. Nevertheless, we still follow the convention in [26] for the renormalization of the  $\mathcal{O}(p^4)$  and  $\mathcal{O}(p^6)$  LEC in the  $\overline{\text{MS}}$  scheme:

$$\begin{aligned} l_i &= (c\mu)^{D-4} [l_i^r(\mu) + \gamma_i \Lambda], \\ h_i &= (c\mu)^{D-4} [h_i^r(\mu) + \delta_i \Lambda], \\ \hat{c}_i &= (c\mu)^{2(D-4)} [\hat{c}_i^r(\mu) - \hat{\gamma}_i^{(sq)} \Lambda^2 - (\hat{\gamma}_i^{(0)} + \hat{\gamma}_i^L(\mu)) \Lambda], \\ L_i &= (c\mu)^{D-4} [L_i^r(\mu) + \Gamma_i \Lambda], \\ H_i &= (c\mu)^{D-4} [H_i^r(\mu) + \Gamma_i^H \Lambda], \\ \hat{C}_i &= (c\mu)^{2(D-4)} [\hat{C}_i^r(\mu) - \hat{\Gamma}_i^{(sq)} \Lambda^2 - (\hat{\Gamma}_i^{(0)} + \hat{\Gamma}_i^L(\mu)) \Lambda], \end{aligned} \quad (\text{A4})$$

where  $\mu$  is the renormalization scale,  $\Lambda^{-1} = 16\pi^2(D-4)$ ,  $\log c = -[\log(4\pi) - \gamma + 1]/2$ ,  $\gamma = -\Gamma'[1]$ ,  $\gamma_i$ ,  $\delta_i$ ,  $\Gamma_i$ ,  $\Gamma_i^H$ ,  $\hat{\gamma}_i^{(sq)}$ ,  $\hat{\Gamma}_i^{(sq)}$ ,  $\hat{\gamma}_i^{(0)}$ , and  $\hat{\Gamma}_i^{(0)}$  are numerical coefficients, whereas  $\hat{\gamma}_i^L$ ,  $\hat{\Gamma}_i^L$  are linear combinations of the  $L_i^r(\mu)$ . The above expression for the  $\hat{c}_i$  shows that these constants have to absorb both two-loop divergences with  $\mathcal{L}_2$  vertices and one-loop ones with one  $\mathcal{L}_4$  and one  $\mathcal{L}_2$  vertex.

The renormalization of the  $L_i$  in Eq. (A4) coincides with that in [8] up to  $\mathcal{O}(1)$  in the  $\epsilon$  expansion:

$$L_i = L_i^r(\mu) + \Gamma_i \mu^{D-4} \lambda + \mathcal{O}(\epsilon), \quad (\text{A5})$$

and so on for the  $H_i$ , whereas the  $l_i$ ,  $h_i$  renormalizations coincide with [11] to that order. For the renormalization of the one-loop effective action, the  $\mathcal{O}(\epsilon)$  in Eqs. (A4) and (A5) can be neglected. However, when two-loop diagrams are considered, as in our case here for the quark condensates [e.g., diagram (d) in Fig. 1] products of the form  $L_i G(0)$  yield finite contributions that do not vanish in the  $\epsilon \rightarrow 0^+$  limit. The  $\mathcal{O}(\epsilon)$  has to also be kept in the expansion of  $\lambda$  in Eq. (A2) when expanding  $G_i(0)$  in Eq. (A1) in  $G_i(0)^2$  contributions.

As for the  $\mu$  scale dependence, the  $L_i$ ,  $l_i$  and the  $\hat{C}_i$ ,  $\hat{c}_i$  are scale independent so that the scale dependence of the  $L_i^r(\mu)$ ,  $l_i^r(\mu)$ ,  $\hat{C}_i^r(\mu)$ ,  $\hat{c}_i^r(\mu)$  is canceled with the explicit  $\mu$  dependence appearing in Eq. (A4). This allows us to express all the logarithms of the masses in terms of  $\log(M_i^2/\mu^2)$ , so that the final result for the observables should be finite and scale independent.

We also recall that to the order we are calculating, the propagators are renormalized to NLO (tadpole corrections) and one has to include the wave-function and mass renormalization to that order. The renormalized masses are given in [8], while the explicit wave-function renormalization can be found, for instance, in [27]. We recall that we should now include up to  $\mathcal{O}(\epsilon)$  in those tadpole corrections, for the reasons just explained.

With these renormalization conventions, we turn to the NNLO quark condensates. The  $\Gamma_i$  coefficients appearing in the calculation are [8]

$$\begin{aligned} \gamma_3 &= -1/2, & \delta_1 &= 2, & \Gamma_4 &= 1/8, \\ \Gamma_5 &= 3/8, & \Gamma_6 &= 11/144, & \Gamma_7 &= 0, \\ \Gamma_8 &= 5/48, & \Gamma_2^H &= 5/24. \end{aligned} \quad (\text{A6})$$

Recall that in SU(3),  $L_6$ ,  $L_8$ , and  $H_2$  come explicitly from the  $\mathcal{L}_4$  vertex contributions to the condensate and are therefore the only LEC appearing to NLO. The mass and wave-function renormalization introduce a dependence on  $L_4$ ,  $L_5$ , and  $L_7$  in the final result.  $L_7$  only appears in the  $\eta$  mass renormalization. In the pure SU(2) case, only  $l_3$  and  $h_1$  enter in the calculation.

Once the above LEC renormalization is performed, we have checked that one can choose the  $\hat{c}_i$  and  $\hat{C}_i$  in Eq. (A3), renormalized through Eq. (A4), so that the final result for the two-quark condensates is finite and scale independent. We obtain

$$\begin{aligned} \hat{\gamma}_1^{(sq)} &= 12, & \hat{\gamma}_1^L &= -48l_3^r, & \hat{\Gamma}_1^{(sq)} &= 896/81, & \hat{\Gamma}_2^{(sq)} &= 32/27, & \hat{\Gamma}_3^{(sq)} &= 64/9, & \hat{\Gamma}_4^{(sq)} &= 160/81, \\ \hat{\Gamma}_1^L &= \frac{32}{27}(444L_4^r + 191L_5^r - 6(148L_6^r + 4L_7^r + 65L_8^r)), & \hat{\Gamma}_2^L &= \frac{32}{9}(162L_4^r + 31L_5^r - 324L_6^r - 62L_8^r), \\ \hat{\Gamma}_3^L &= \frac{32}{9}(96L_4^r + 35L_5^r - 192L_6^r + 24L_7^r - 62L_8^r), & \hat{\Gamma}_4^L &= \frac{32}{27}(78L_4^r + 43L_5^r - 6(26L_6^r + 8L_7^r + 17L_8^r)), \end{aligned} \quad (\text{A7})$$

and all the linear terms  $\hat{\gamma}_i^{(0)} = \hat{\Gamma}_i^{(0)} = 0$  for the above LEC.

For convenience and following the same notation as [8], we define

$$\begin{aligned} \mu_i &= \frac{M_{0i}^2}{32\pi^2 F^2} \log \frac{M_{0i}^2}{\mu^2}, \\ \nu_i &= F^2 \frac{\partial \mu_i}{\partial M_{0i}^2} = \frac{1}{32\pi^2} \left( 1 + \log \frac{M_{0i}^2}{\mu^2} \right). \end{aligned} \quad (\text{A8})$$

In SU(2) the leading order pion mass is related to the physical one by

$$M_\pi^2 = M_{0\pi}^2 \left( 1 + \mu_\pi + \frac{4M_{0\pi}^2}{F^2} l_3^r \right), \quad (\text{A9})$$

and in SU(3),

$$\begin{aligned}
M_\pi^2 &= M_{0\pi}^2 \left[ 1 + \mu_\pi - \frac{\mu_\eta}{3} + \frac{16M_{0K}^2}{F^2}(2L_6^r - L_4^r) + \frac{8M_{0\pi}^2}{F^2}(2L_6^r + 2L_8^r - L_4^r - L_5^r) \right], \\
M_K^2 &= M_{0K}^2 \left[ 1 + \frac{2\mu_\eta}{3} + \frac{8M_{0\pi}^2}{F^2}(2L_6^r - L_4^r) + \frac{8M_{0K}^2}{F^2}(4L_6^r + 2L_8^r - 2L_4^r - L_5^r) \right], \\
M_\eta^2 &= M_{0\eta}^2 \left[ 1 + 2\mu_K - \frac{4}{3}\mu_\eta + \frac{8M_{0\eta}^2}{F^2}(2L_8^r - L_5^r) + \frac{8}{F^2}(2M_{0K}^2 + M_{0\pi}^2)(2L_6^r - L_4^r) \right] \\
&\quad + M_{0\pi}^2 \left[ -\mu_\pi + \frac{2}{3}\mu_K + \frac{1}{3}\mu_\eta \right] + \frac{128}{9F^2}(M_{0K}^2 - M_{0\pi}^2)^2(3L_7^r + L_8^r).
\end{aligned} \tag{A10}$$

The relation between the leading order pion decay constant and the physical one up to two loops is given in [28] for SU(2) and in [29] for SU(3).

The final expressions for the two-quark condensates, finite and scale independent, up to NNLO, that have been calculated previously in [16] for SU(3), are given by

$$\langle \bar{q}q \rangle_{l,\text{NLO}}^{\text{SU}(2)} = -2B_0F^2 \left\{ 1 + \frac{2M_{0\pi}^2}{F^2}(h_1^r + l_3^r) - 3\mu_\pi \right\}, \tag{A11}$$

$$\langle \bar{q}q \rangle_{l,\text{NNLO}}^{\text{SU}(2)} = \langle \bar{q}q \rangle_{l,\text{NLO}}^{\text{SU}(2)} - 2B_0F^2 \left[ -\frac{3}{2}\mu_\pi^2 - \frac{3M_{0\pi}^2}{F^2}(\mu_\pi\nu_\pi + 4l_3^r\mu_\pi) + \frac{3M_{0\pi}^4}{8F^4}(-16l_3^r\nu_\pi + \hat{c}_1^r) \right], \tag{A12}$$

$$\langle \bar{q}q \rangle_{l,\text{NLO}}^{\text{SU}(3)} = -2B_0F^2 \left\{ 1 + \frac{4}{F^2}[(H_2^r + 4L_6^r + 2L_8^r)M_{0\pi}^2 + 8L_6^rM_{0K}^2] - 3\mu_\pi - 2\mu_K - \frac{1}{3}\mu_\eta \right\}, \tag{A13}$$

$$\begin{aligned}
\langle \bar{q}q \rangle_{l,\text{NNLO}}^{\text{SU}(3)} &= \langle \bar{q}q \rangle_{l,\text{NLO}}^{\text{SU}(3)} - 2B_0F^2 \left\{ -\frac{3}{2}\mu_\pi^2 + \frac{1}{18}\mu_\eta^2 + \mu_\pi\mu_\eta - \frac{4}{3}\mu_K\mu_\eta + \frac{1}{F^2} \left[ -3M_{0\pi}^2\mu_\pi\nu_\pi + \frac{1}{3}M_{0\pi}^2\mu_\pi\nu_\eta \right. \right. \\
&\quad \left. \left. - \frac{8}{9}M_{0K}^2\mu_K\nu_\eta + M_{0\pi}^2\mu_\eta\nu_\pi - \frac{4}{3}M_{0K}^2\mu_\eta\nu_K + \frac{1}{27}(16M_{0K}^2 - 7M_{0\pi}^2)\mu_\eta\nu_\eta \right] \right. \\
&\quad + \frac{24}{F^2}\mu_\pi[(3L_4^r + 2L_5^r - 6L_6^r - 4L_8^r)M_{0\pi}^2 + 2(L_4^r - 2L_6^r)M_{0K}^2] \\
&\quad + \frac{16}{F^2}\mu_K[(L_4^r - 2L_6^r)M_{0\pi}^2 + 2(3L_4^r + L_5^r - 6L_6^r - 2L_8^r)M_{0K}^2] \\
&\quad + \frac{8}{9F^2}\mu_\eta[(-3L_4^r - 2L_5^r + 6L_6^r - 48L_7^r - 12L_8^r)M_{0\pi}^2 + 2(15L_4^r + 4L_5^r - 30L_6^r + 24L_7^r)M_{0K}^2] \\
&\quad + \frac{24M_{0\pi}^2}{F^4}\nu_\pi[(L_4^r + L_5^r - 2L_6^r - 2L_8^r)M_{0\pi}^2 + 2(L_4^r - 2L_6^r)M_{0K}^2] \\
&\quad + \frac{16M_{0K}^2}{F^4}\nu_K[(L_4^r - 2L_6^r)M_{0\pi}^2 + (2L_4^r + L_5^r - 4L_6^r - 2L_8^r)M_{0K}^2] \\
&\quad + \frac{8}{27F^4}\nu_\eta[(-3L_4^r + L_5^r + 6L_6^r - 48L_7^r - 18L_8^r)M_{0\pi}^4 + 2(3L_4^r - 4L_5^r - 6L_6^r + 48L_7^r + 24L_8^r)M_{0\pi}^2M_{0K}^2 \\
&\quad + 8(3L_4^r + 2(L_5^r - 3(L_6^r + L_7^r + L_8^r)))M_{0K}^4] \\
&\quad \left. + \frac{1}{8F^4}[(3\hat{c}_1^r - 2\hat{c}_2^r + \hat{c}_3^r)M_{0\pi}^4 + 4(\hat{c}_2^r - \hat{c}_3^r)M_{0\pi}^2M_{0K}^2 + 4\hat{c}_3^rM_{0K}^4] \right\},
\end{aligned} \tag{A14}$$

$$\langle \bar{s}s \rangle_{\text{NLO}} = -B_0F^2 \left\{ 1 + \frac{4}{F^2}[-(H_2^r - 4L_6^r + 2L_8^r)M_{0\pi}^2 + 2(H_2^r + 4L_6^r + 2L_8^r)M_{0K}^2] - 4\mu_K - \frac{4}{3}\mu_\eta \right\}, \tag{A15}$$

$$\begin{aligned}
\langle \bar{s}s \rangle_{\text{NNLO}} = & \langle \bar{s}s \rangle_{\text{NLO}} - B_0 F^2 \left\{ \frac{8}{9} \mu_\eta^2 - \frac{8}{3} \mu_K \mu_\eta \right. \\
& + \frac{1}{F^2} \left[ \frac{4}{3} M_{0\pi}^2 \mu_\pi \nu_\eta - \frac{32}{9} M_{0K}^2 \mu_K \nu_\eta - \frac{8}{3} M_{0K}^2 \mu_\eta \nu_K + \frac{4}{27} (16 M_{0K}^2 - 7 M_{0\pi}^2) \mu_\eta \nu_\eta \right] \\
& + \frac{48}{F^2} \mu_\pi (L_4^r - 2 L_6^r) M_{0\pi}^2 + \frac{32}{F^2} \mu_K [(L_4^r - 2 L_6^r) M_{0\pi}^2 + 2(2 L_4^r + L_5^r - 4 L_6^r - 2 L_8^r) M_{0K}^2] \\
& + \frac{16}{9 F^2} \mu_\eta [(3 L_4^r - 4 L_5^r - 6 L_6^r + 48 L_7^r + 24 L_8^r) M_{0\pi}^2 + 8(3 L_4^r + 2(L_5^r - 3(L_6^r + L_7^r + L_8^r))) M_{0K}^2] \\
& + \frac{32 M_{0K}^2}{F^4} \nu_K [(L_4^r - 2 L_6^r) M_{0\pi}^2 + (2 L_4^r + L_5^r - 4 L_6^r - 2 L_8^r) M_{0K}^2] \\
& + \frac{32}{27 F^4} \nu_\eta [(-3 L_4^r + L_5^r + 6 L_6^r - 48 L_7^r - 18 L_8^r) M_{0\pi}^4 + 2(3 L_4^r - 4 L_5^r - 6 L_6^r + 48 L_7^r + 24 L_8^r) M_{0\pi}^2 M_{0K}^2 \\
& + 8(3 L_4^r + 2(L_5^r - 3(L_6^r + L_7^r + L_8^r))) M_{0K}^4] \\
& \left. + \frac{1}{4 F^4} [(\hat{C}_2^r - 2 \hat{C}_3^r + 3 \hat{C}_4^r) M_{0\pi}^4 + 4(\hat{C}_3^r - 3 \hat{C}_4^r) M_{0\pi}^2 M_{0K}^2 + 12 \hat{C}_4^r M_{0K}^4] \right\}, \quad (\text{A16})
\end{aligned}$$

where the Gell-Mann-Okubo relation  $3M_{0\eta}^2 = 4M_{0K}^2 - M_{0\pi}^2$  for the SU(3) leading order masses has been used, and the renormalized  $L_i^r$ ,  $l_i^r$  and  $\hat{C}_i^r$  constants depend on the scale  $\mu$  as explained above.

## APPENDIX B: FOUR-QUARK CONDENSATES IN THE USUAL $\overline{\text{MS}}$ DEFINITION

Here we consider the definition in Eq. (3) of the four-quark condensate in Euclidean space. Let us restrict to SU(2) since it will become clear that the argument can be straightforwardly extended to the SU(3) case. The four-quark correlator to NNLO is given in Eq. (14), so that its Euclidean Fourier transform to this order is (see our Euclidean space-time conventions in Sec. III)

$$\begin{aligned}
\Pi(Q^2) = & (2\pi)^D \langle \bar{q}q \rangle^2 \delta^{(D)}(Q) + 2B_0^2 [4(l_3 + h_1) \\
& + 3J_\pi(Q^2)] \quad (\text{B1})
\end{aligned}$$

with  $Q^2 = \sum_{i=1}^D Q_i^2$  and

$$J_\pi(Q^2) = \int \frac{d^D K}{(2\pi)^D} G_\pi(K) G_\pi(K - Q), \quad (\text{B2})$$

which is nothing but the one-loop integral appearing in pion-pion scattering, dimensionally regularized in [11]. Its divergent part is contained in  $J_\pi(0) = -2M_\pi^{D-4} \lambda - 1/(16\pi^2)$ , with  $\lambda$  defined in Eq. (A2), while  $\bar{J}(Q^2) = J_\pi(Q^2) - J_\pi(0)$  is finite. Note also that  $J_\pi(Q^2)$  defined in Euclidean space is real. The imaginary part in  $\bar{J}_\pi$  giving the usual unitarity cut in scattering amplitudes arises when the analytical continuation of  $Q^2$  to Minkowski space-time is performed, but here we should keep the Euclidean version, since we are following the prescription in Eq. (3) to perform the additional momentum integral.

Before proceeding to the calculation of the four-quark condensate, let us note that the divergent part of the  $J_\pi$  in

Eq. (B1) cancels exactly with the LEC contribution since  $l_3 + h_1 = l_3^r(\mu) + h_1^r(\mu) + (3/2)\mu^{D-4}\lambda$  [see Eqs. (A4) and (A6)]. Thus,  $\Pi(Q^2)$  is finite and scale independent before integration in  $Q$ . This is actually a welcomed check, since the scalar susceptibility given in Eq. (22) can be written also as  $\chi_l = \tilde{\Pi}(0)$  with  $\tilde{\Pi}(Q^2) = \Pi(Q^2) - (2\pi)^D \langle \bar{q}q \rangle^2 \delta^{(D)}(Q)$  and should be finite and scale independent.

However, we will immediately see that the additional integration in  $Q$  in Eq. (3) generates an extra divergence which cannot be removed, and in the end gives the same divergent factorization-breaking result as the definition in Eq. (2). For that purpose, let us follow the standard dimensional regularization procedure [22] and write

$$\begin{aligned}
J_\pi(Q^2) = & \frac{1}{(4\pi)^{D/2}} \int_0^1 dx \int_0^\infty d\lambda \lambda^{1-D/2} \\
& \times \exp\{-\lambda[M_\pi^2 + Q^2 x(1-x)]\} \quad (\text{B3})
\end{aligned}$$

which is valid within the domain  $\text{Re}[D] < 4$ . Now, before performing the  $x$  and  $\lambda$  integrals above, we integrate over  $Q$  so that

$$\begin{aligned}
\int \frac{d^D Q}{(2\pi)^D} J_\pi(Q^2) = & \frac{1}{(4\pi)^D} \left\{ \int_0^1 dx [x(1-x)]^{-D/2} \right\} \\
& \times \left\{ \int_0^\lambda d\lambda \lambda^{1-D} e^{-\lambda M_\pi^2} \right\} \\
= & \frac{(M_\pi^2)^{D-2}}{(4\pi)^D} \left[ \Gamma\left(1 - \frac{D}{2}\right) \right]^2 = G_\pi^2(0), \quad (\text{B4})
\end{aligned}$$

where the one-dimensional integrals are solved for  $\text{Re}[D] < 2$  and we have used standard properties of the Gamma function. Since the result is analytic in  $D$ , it can be extended to  $D = 4 - \epsilon$  with  $\epsilon \rightarrow 0^+$ . Therefore, integrating in Eq. (B1) over  $Q$  according to Eq. (3), and taking into

account that  $\int d^D Q / (2\pi)^D = \delta^{(D)}(0) = 0$ , gives exactly the same divergent factorization-breaking result for the four-quark condensate as the one using the prescription of Eq. (2).

Another way to arrive at the same conclusion is to perform the change of variables  $Q \rightarrow Q + K$  in the double  $D$ -integral  $\int d^D Q \int d^D K$  in the region of  $D$  where it

converges, which in this case is  $\text{Re}[D] < 2$ , which follows by direct power counting in  $Q$  and  $K$  of the propagators in Eq. (B2) in the large  $Q^2$  and  $K^2$  Euclidean region.

It is clear that the same equivalence between the two definitions holds in the SU(3) case simply by considering  $J_K$  and  $J_\eta$  apart from  $J_\pi$ , since the results of the correlators in Eqs. (19) and (20) do not mix different meson species.

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- [1] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Nucl. Phys.* **B147**, 385 (1979).
  - [2] S. Narison and R. Tarrach, *Phys. Lett.* **125B**, 217 (1983).
  - [3] R. A. Bertlmann, C. A. Dominguez, M. Loewe, M. Perrottet, and E. de Rafael, *Z. Phys. C* **39**, 231 (1988).
  - [4] H. s. Zong, D. k. He, F. y. Hou, and W. M. Sun, *Int. J. Mod. Phys. A* **23**, 1507 (2008).
  - [5] J. Bijnens, E. Gamiz, and J. Prades, *J. High Energy Phys.* **10** (2001) 009.
  - [6] V. Cirigliano, J. F. Donoghue, E. Golowich, and K. Maltman, *Phys. Lett. B* **555**, 71 (2003).
  - [7] F. Karsch (RBC-Bielefeld Collaboration), *Nucl. Phys.* **A820**, 99C (2009).
  - [8] J. Gasser and H. Leutwyler, *Nucl. Phys.* **B250**, 465 (1985).
  - [9] S. Weinberg, *Physica A (Amsterdam)* **96**, 327 (1979).
  - [10] J. Bijnens, G. Colangelo, and G. Ecker, *J. High Energy Phys.* **02** (1999) 020.
  - [11] J. Gasser and H. Leutwyler, *Ann. Phys. (N.Y.)* **158**, 142 (1984).
  - [12] S. Scherer, *Adv. Nucl. Phys.* **27**, 277 (2003).
  - [13] J. Gasser, C. Haefeli, M. A. Ivanov, and M. Schmid, *Phys. Lett. B* **652**, 21 (2007).
  - [14] J. Gasser, C. Haefeli, M. A. Ivanov, and M. Schmid, *Phys. Lett. B* **675**, 49 (2009).
  - [15] J. Wess and B. Zumino, *Phys. Lett.* **37B**, 95 (1971); E. Witten, *Nucl. Phys.* **B223**, 422 (1983).
  - [16] G. Amoros, J. Bijnens, and P. Talavera, *Nucl. Phys.* **B585**, 293 (2000); **B598**, 665(E) (2001).
  - [17] G. Amoros, J. Bijnens, and P. Talavera, *Nucl. Phys.* **B602**, 87 (2001).
  - [18] B. Moussallam, *J. High Energy Phys.* **08** (2000) 005.
  - [19] J. Bijnens, *Prog. Part. Nucl. Phys.* **58**, 521 (2007).
  - [20] A. Gómez Nicola, J. R. Peláez, and J. Ruiz de Elvira (work in progress).
  - [21] A. V. Smilga and J. J. M. Verbaarschot, *Phys. Rev. D* **54**, 1087 (1996).
  - [22] G. Leibbrandt, *Rev. Mod. Phys.* **47**, 849 (1975).
  - [23] S. Descotes-Genon, L. Girlanda, and J. Stern, *J. High Energy Phys.* **01** (2000) 041.
  - [24] S. Peris and E. de Rafael, *Phys. Lett. B* **348**, 539 (1995).
  - [25] B. L. Ioffe, *Prog. Part. Nucl. Phys.* **56**, 232 (2006).
  - [26] J. Bijnens, G. Colangelo, and G. Ecker, *Ann. Phys. (N.Y.)* **280**, 100 (2000).
  - [27] A. Gómez Nicola and J. R. Peláez, *Phys. Rev. D* **65**, 054009 (2002).
  - [28] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, and M. E. Sainio, *Nucl. Phys.* **B508**, 263 (1997); **B517**, 639(E) (1998).
  - [29] G. Amoros, J. Bijnens, and P. Talavera, *Nucl. Phys.* **B568**, 319 (2000).