Moments of the Wigner distribution of rotationally symmetric partially coherent light

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The Wigner distribution of rotationally symmetric partially coherent light is considered, and the constraints for its moments are derived. Although all odd-order moments vanish, these constraints lead to a drastic reduction in the number of parameters that we need to describe all even-order moments: whereas in general we have $(N+1)(N+2)(N+3)/6$ different moments of order N, this number reduces to $(1+N/2)^2$ in the case of rotational symmetry. A way to measure the moments as intensity moments in the output planes of (generally anamorphic) fractional Fourier-transform systems is presented. © 2003 Optical Society of America

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The Wigner distribution[1] of partially coherent light is defined in terms of the cross-spectral density[2,3] $\Gamma(x_1, y_1, x_2, y_2)$ by

$$W(x,y,u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \times \Gamma\left(x + \frac{1}{2}x', y + \frac{1}{2}y', x - \frac{1}{2}x', y - \frac{1}{2}y'\right) \times \exp[-j2\pi(ux' + vy')]\mathrm{d}x'\mathrm{d}y'. \quad (1)$$

The (real-valued) Wigner distribution $W(x,y,u,v)$ represents partially coherent light in a combined space or spatial-frequency domain, the so-called phase space, where u and v are the spatial-frequency variables associated with the space variables x and y, respectively. In previous papers the special but important case of rotational symmetry has been studied extensively; we mention studies of twisted Gaussian–Schell model light[4] and the characterization of rotationally symmetric light in terms of second-order moments.[5,6] In this Letter we present an extension to higher-order moments.

To formulate the rotational symmetry of the Wigner distribution $W(x,y,u,v)$ we express the space variables x and y in polar coordinates, $x = \rho\cos\phi$ and $y = \rho\sin\phi$, respectively, and with the angle ϕ as an offset we do the same with the spatial-frequency variables u and v, $u = \zeta\cos(\phi+\theta)$ and $v = \zeta\sin(\phi+\theta)$, respectively. We can then formulate an expression for $W(x,y,u,v)$ in terms of the four variables ρ, ϕ, ζ, and θ; for rotational symmetry we require that this expression does not depend on the angle ϕ:

$$W[\rho\cos\phi, \rho\sin\phi, \zeta\cos(\phi+\theta), \zeta\sin(\phi+\theta)] = W_\circ(\rho,\zeta,\theta). \quad (2)$$

The (normalized) moments μ_{pqrs} of the Wigner distribution are defined as

$$\mu_{pqrs}E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(x,y,u,v) \times x^p u^q y^r v^s \mathrm{d}x\mathrm{d}y\mathrm{d}u\mathrm{d}v, \quad (3)$$

where $\mu_{0000} = 1$ and normalization constant E represents the intensity of the light. In general there are $(N+1)(N+2)(N+3)/6$ moments[7] of order $N = p+q+r+s$. In the case of rotational symmetry, however, the number of parameters that we need to describe all even-order moments is reduced drastically to $(1+N/2)^2$. That this is so can easily be seen from Eq. (2), from which we conclude that the four-dimensional Wigner distribution $W(x,y,u,v)$ is completely determined by the three-dimensional function $W(x,0,u,v)$, where, moreover, $W(x,0,u,v)$ is an even function of x; this three-dimensional function has $(1+N/2)^2$ different nonvanishing moments of even order N.

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Using symmetry condition (2), we write

$$\mu_{pqrs}E = \int_0^\infty \int_0^\infty \int_0^{2\pi} W_\circ(\rho, \zeta, \theta)\rho^{p+r+1}\zeta^{q+s+1}\mathrm{d}\rho\mathrm{d}\zeta\mathrm{d}\theta \times \int_0^{2\pi} (\cos\phi)^p[\cos(\phi+\theta)]^q \times (\sin\phi)^r[\sin(\phi+\theta)]^s\mathrm{d}\phi\,. \tag{4}$$

From the special form of the integral over ϕ, we conclude that all odd-order moments (i.e., $N = p + q + r + s$ is odd) are zero. Moreover, using the definition of the beta function $B(x,y) = 2\int_0^{\pi/2}(\cos\varphi)^{2x-1} \times (\sin\varphi)^{2y-1}\mathrm{d}\varphi = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ we can express this integral as

$$I_{pqrs}(\theta) = \frac{1}{\pi}\int_0^{2\pi}(\cos\phi)^p[\cos(\phi+\theta)]^q \times (\sin\phi)^r[\sin(\phi+\theta)]^s\mathrm{d}\phi = \frac{1}{2\pi}\sum_{k=0}^{q}\sum_{l=0}^{s}\binom{q}{k}\binom{s}{l}(-1)^k \times [1+(-1)^{p+q-k+l}][1+(-1)^{r+s+k-l}] \times B\left(\frac{p+q-k+l+1}{2}, \frac{r+s+k-l+1}{2}\right) \times (\cos\theta)^{q+s-k-l}(\sin\theta)^{k+l}. \tag{5}$$

Since nonvanishing values under the summation appear only if both $p + q - k + l$ and $r + s + k - l$ are even, we can use the property $\Gamma(n + 1/2)2^n/\sqrt{\pi} = (2n-1)!! = 1 \times 3 \times 5 \ldots (2n-1)$.

It is advantageous to write the moments as $\mu_{p,q,m-p,n-q}$, where $p + r = m$ and $q + s = n$, and to group those moments that have identical m and n together. For easy reference, $I_{pqrs}(\theta)$ is presented in Table 1 for second-order moments $[(m,n) = (2,0), (1,1)$, and $(0,2)]$ and fourth-order moments $[(m,n) = (4,0)$, $(3,1)$, $(2,2)$, $(1,3)$, and $(0,4)]$ such that equal m and n (with $m + n = 2$ and $m + n = 4$, respectively) are grouped together. Identical values of $I_{pqrs}(\theta)$ in the same block then lead to companion moments μ_{pqrs}. For different choices of p and q (but with constant $m = p + r$ and $n = q + s$) we can easily find relations between the different moments μ_{pqrs}. In particular, we find that in any (m,n) block, the number of nonvanishing parameters equals $1 + \min(m,n)$, leading to a total of $(1 + N/2)^2$ parameters to describe the moments of order $N = m + n$.

Let us consider the second-order moments, which can be represented elegantly in the usual form of a real, symmetric 4×4 matrix. As a consequence of the moment relations, this matrix now takes a special form[4] and is determined by four parameters instead of the ten parameters in the general case. In particular, we observe that three moments appear in pairs with a positive companion (μ_{2000}, μ_{1100}, and μ_{0200}), and one moment forms a pair with a negative companion (μ_{1001}); moreover, two moments vanish (μ_{1010} and μ_{0101}).

Let us now consider the fourth-order moments. From the moment relations we conclude that the moments are determined by 9 parameters, whereas in the general case we would need 35 parameters. In particular, we observe that two moments appear in pairs with a positive companion (μ_{2200} and μ_{2002}), two moments appear in triples with positive companions (μ_{4000} and μ_{0400}), five moments appear in

Table 1. I_{pqrs} for Second-Order Moments $[(m,n) = (2,0), (1,1), (0,2)]$ and Fourth-Order Moments $[(m,n) = (4,0), (3,1), (2,2), (1,3), (0,4)]$

m	n	$I_{pqrs}(\theta)$	μ_{pqrs}	Companion
2	0	1	μ_{2000}	
2	0	0	μ_{1010}	—
2	0	1	μ_{0020}	μ_{2000}
1	1	$\cos\theta$	μ_{1100}	
1	1	$\sin\theta$	μ_{1001}	
1	1	$-\sin\theta$	μ_{0110}	$-\mu_{1001}$
1	1	$\cos\theta$	μ_{0011}	μ_{1100}
0	2	1	μ_{0200}	
0	2	0	μ_{0101}	—
0	2	1	μ_{0002}	μ_{0200}
4	0	3/4	μ_{4000}	
4	0	0	μ_{3010}	—
4	0	1/4	μ_{2020}	$\mu_{4000}/3$
4	0	0	μ_{1030}	—
4	0	3/4	μ_{0040}	μ_{4000}
3	1	$3\cos\theta/4$	μ_{3100}	
3	1	$3\sin\theta/4$	μ_{3001}	
3	1	$-\sin\theta/4$	μ_{2110}	$-\mu_{3001}/3$
3	1	$\cos\theta/4$	μ_{2011}	$\mu_{3100}/3$
3	1	$\cos\theta/4$	μ_{1120}	$\mu_{3100}/3$
3	1	$\sin\theta/4$	μ_{1021}	$\mu_{3001}/3$
3	1	$-3\sin\theta/4$	μ_{0130}	$-\mu_{3001}$
3	1	$3\cos\theta/4$	μ_{0031}	μ_{3100}
2	2	$(2+\cos 2\theta)/4$	μ_{2200}	
2	2	$\sin 2\theta/4$	μ_{2101}	
2	2	$(2-\cos 2\theta)/4$	μ_{2002}	
2	2	$-\sin 2\theta/4$	μ_{1210}	$-\mu_{2101}$
2	2	$\cos 2\theta/4$	μ_{1111}	$(\mu_{2200} - \mu_{2002})/2$
2	2	$\sin 2\theta/4$	μ_{1012}	μ_{2101}
2	2	$(2-\cos 2\theta)/4$	μ_{0220}	μ_{2002}
2	2	$-\sin 2\theta/4$	μ_{0121}	$-\mu_{2101}$
2	2	$(2+\cos 2\theta)/4$	μ_{0022}	μ_{2200}
1	3	$3\cos\theta/4$	μ_{1300}	
1	3	$\sin\theta/4$	μ_{1201}	
1	3	$\cos\theta/4$	μ_{1102}	$\mu_{1300}/3$
1	3	$3\sin\theta/4$	μ_{1003}	$3\mu_{1201}$
1	3	$-3\sin\theta/4$	μ_{0310}	$-3\mu_{1201}$
1	3	$\cos\theta/4$	μ_{0211}	$\mu_{1300}/3$
1	3	$-\sin\theta/4$	μ_{0112}	$-\mu_{1201}$
1	3	$3\cos\theta/4$	μ_{0013}	μ_{1300}
0	4	3/4	μ_{0400}	
0	4	0	μ_{0301}	—
0	4	1/4	μ_{0202}	$\mu_{0400}/3$
0	4	0	μ_{0103}	—
0	4	3/4	μ_{0004}	μ_{0400}

quadruples, two moments appear with positive companions (μ_{3100} and μ_{1300}), and three moments appear with one positive and two negative companions (μ_{3001}, μ_{2101}, and μ_{1201}). Moreover, four moments vanish (μ_{3010}, μ_{1030}, μ_{0301}, and μ_{0103}), and moment μ_{1111} follows from the relation $\mu_{1111} = (\mu_{2200} - \mu_{2002})/2$.

Following the procedure described in Ref. 7, we can determine the moments from measurement of the intensity distribution $\Gamma(x, y, x, y)$ in the output plane of some (possibly anamorphic) fractional Fourier-transform systems, with a fractional angle α in the x direction and a fractional angle β in the y direction, say, for appropriately chosen values of α and β. In the output plane we then measure the intensity moments $\mu_{p0r0}^{\text{out}}(\alpha, \beta)$ [see Eq. (3) with $q = s = 0$], which are completely determined by the output intensity distribution. The general relationship between the output intensity moments and the moments in the input plane reads as[7]

$$\mu_{p0r0}^{\text{out}}(\alpha, \beta) = \sum_{k=0}^{p} \sum_{m=0}^{r} \binom{p}{k} \binom{r}{m} \mu_{p-k,k,r-m,m} \cos^{p-k} \alpha \times \sin^{k} \alpha \cos^{r-m} \beta \sin^{m} \beta \,. \tag{6}$$

In the case of second-order moments the set of relevant equations in which the intensity moments $\mu_{2000}^{\text{out}}(\alpha, \beta)$, $\mu_{1010}^{\text{out}}(\alpha, \beta)$, and $\mu_{0020}^{\text{out}}(\alpha, \beta)$ at the output of a fractional Fourier-transform system with fractional angles α and β are expressed in terms of the input moments reduces to

$$\mu_{2000}^{\text{out}}(\alpha, \beta) = \mu_{2000} \cos^2 \alpha + 2\mu_{1100} \cos \alpha \sin \alpha + \mu_{0200} \sin^2 \alpha \,, \tag{7}$$

$$\mu_{1010}^{\text{out}}(\alpha, \beta) = \mu_{1001} \sin(\beta - \alpha) \,. \tag{8}$$

To measure moment μ_{1001} from intensity moment $\mu_{1010}^{\text{out}}(\alpha, \beta)$ we clearly need an anamorphic system, $\alpha \neq \beta$. Together with two additional isotropic systems, $\alpha = \beta$, we can then construct four equations from measurements of the intensity distributions in the three output planes, and we conclude that the four second-order moments can be determined from knowledge of the intensity distributions in the output plane of three fractional Fourier-transform systems, where one of them has to be anamorphic; see Ref. 6. We would not need the anamorphic system if the rotationally symmetric light satisfied the additional condition that $W_{\circ}(\rho, \zeta, \theta)$ is an even function of θ.

In the case of fourth-order moments, the set of relevant equations for the output intensity moments[7] reduces to

$$\begin{aligned} \mu_{4000}^{\text{out}}(\alpha, \beta) = {} & \mu_{4000} \cos^4 \alpha + 4\mu_{3100} \cos^3 \alpha \sin \alpha \\ & + 6\mu_{2200} \cos^2 \alpha \sin^2 \alpha \\ & + 4\mu_{1300} \cos \alpha \sin^3 \alpha \\ & + \mu_{0400} \sin^4 \alpha \,, \end{aligned} \tag{9}$$

$$\begin{aligned} \mu_{3010}^{\text{out}}(\alpha, \beta) = {} & (\mu_{3001} \cos^2 \alpha + 3\mu_{2101} \cos \alpha \sin \alpha \\ & + 3\mu_{1201} \sin^2 \alpha) \sin(\beta - \alpha) \,, \end{aligned} \tag{10}$$

$$\begin{aligned} 3\mu_{2020}^{\text{out}}(\alpha, \beta) = {} & \mu_{4000} \cos^2 \alpha \cos^2 \beta \\ & + 2\mu_{3100} \cos \alpha \cos \beta \sin(\alpha + \beta) \\ & + 6\mu_{2200} \cos \alpha \sin \alpha \cos \beta \sin \beta \\ & + 3\mu_{2002} \sin^2(\beta - \alpha) \\ & + 2\mu_{1300} \sin \alpha \sin \beta \sin(\alpha + \beta) \\ & + \mu_{0400} \sin^2 \alpha \sin^2 \beta \,. \end{aligned} \tag{11}$$

To determine moments μ_{3001}, μ_{2101}, and μ_{1201} from intensity moment $\mu_{3010}^{\text{out}}(\alpha, \beta)$ and moment μ_{2002} from intensity moment $\mu_{2020}^{\text{out}}(\alpha, \beta)$ we obviously need three anamorphic systems. Together with two additional isotropic systems, we can then construct nine equations from measurements of the intensity distributions in the five output planes, with which the nine moments can be determined. We note that, even in the highly symmetric case in which $W_{\circ}(\rho, \zeta, \theta)$ is an even function of θ, we still need an anamorphic system. Such a system would not be necessary if $W_{\circ}(\rho, \zeta, \theta)$ did not depend on θ at all, in which case only the strictly even-order moments (i.e., p, q, r, and s are even) remain and all other moments vanish.

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