

ISOPERIMETRIC INEQUALITIES IN RIEMANN SURFACES AND GRAPHS

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ABSTRACT. A celebrated theorem of Kanai states that quasi-isometries preserve isoperimetric inequalities between uniform Riemannian manifolds (with positive injectivity radius) and graphs. Our main result states that we can study the (Cheeger) isoperimetric inequality in a Riemann surface by using a graph related to it, even if the surface has injectivity radius zero (this graph is inspired in Kanai's graph, but it is different from it). We also present an application relating Gromov boundary and isoperimetric inequality.

Keywords: Isoperimetric inequality; Cheeger isoperimetric constant; Riemann surface; Poincaré metric; Gromov hyperbolicity.

2010 AMS Subject Classification numbers: Primary 53C21, 53C23; Secondary 58C40.

1. INTRODUCTION

Isoperimetric inequalities are of interest in pure and applied mathematics (see, e.g., [14], [32]). There are close connections between isoperimetric inequality and some conformal invariants of Riemannian manifolds and graphs, namely Poincaré-Sobolev inequalities, the bottom of the spectrum of the Laplace-Beltrami operator, the exponent of convergence, and the Hausdorff dimensions of the sets of both bounded geodesics and escaping geodesics in a negatively curved surface (see [7], [10, p.228], [15], [17], [18], [19], [20], [21], [29], [31], [40, p.333]). The Cheeger isoperimetric inequality is closely related to the project of Ancona on the space of positive harmonic functions of Gromov-hyperbolic manifolds and graphs ([2], [3] and [4]). In fact, in the study of the Laplace operator on a hyperbolic manifold or graph X , Ancona obtained in these three last papers interesting results, under the additional assumption that the bottom of the spectrum of the Laplace spectrum $\lambda_1(X)$ is positive. The well-known Cheeger inequality $\lambda_1(X) \geq \frac{1}{4} h(X)^2$, where $h(X)$ is the isoperimetric constant of X , guarantees that $\lambda_1(X) > 0$ when $h(X) > 0$ (see [9] for a converse inequality). Hence, the results of this paper are useful in order to obtain these Ancona's results.

Given any Riemannian n -manifold M , the Cheeger isoperimetric constant of M is defined as

$$h(M) = \inf_{\Omega} \frac{\text{Vol}_{n-1,M}(\partial\Omega)}{\text{Vol}_{n,M}(\Omega)},$$

where Ω ranges over all non-empty bounded open subsets of M , and $\text{Vol}_{k,M}(B)$ denotes the k -dimensional Riemannian volume in M of the set B . We write $\text{Vol}_{2,M} = A_M$ and $\text{Vol}_{1,M} = L_M$.

Given any graph $\Gamma = (V, E) = (V(\Gamma), E(\Gamma))$, let us consider the natural length metric d_Γ where every edge has length 1. For any graph Γ , any vertex $v \in V$ and any $k \in \mathbb{N}$, let $B_k(v)$ and $\bar{B}_k(v)$ denote the open and closed balls, respectively.

The combinatorial Cheeger isoperimetric constant of Γ is defined to be

$$h(\Gamma) = \inf_A \frac{|\partial A|}{|A|},$$

where A ranges over all non-empty finite subsets of vertices in Γ , $\partial A = \{v \in \Gamma \mid d_\Gamma(v, A) = 1\}$ and $|A|$ denotes the cardinality of A .

Date: October 24, 2019.

⁽¹⁾ Supported in part by a grant from Ministerio de Ciencia, Innovación y Universidades (PGC2018-098321-B-I00), Spain.

⁽²⁾ Supported in part by two grants from Ministerio de Economía y Competitividad, Agencia Estatal de Investigación (AEI) and Fondo Europeo de Desarrollo Regional (FEDER) (MTM2016-78227-C2-1-P and MTM2017-90584-REDT), Spain.

We say that a Riemannian manifold or graph X satisfies the (Cheeger or linear) *isoperimetric inequality* (LII) if $h(X) > 0$, since in this case

$$(1.1) \quad \text{Vol}_{n,M}(A) \leq h(X)^{-1} \text{Vol}_{n-1,M}(\partial A),$$

for every bounded open set $A \subseteq X$ if X is a Riemannian n -manifold, and

$$|A| \leq h(X)^{-1} |\partial A|,$$

for every finite set $A \subseteq V(X)$ if X is a graph.

Along the paper, we just consider manifolds and graphs X which are connected. This is not a loss of generality, since if X has connected components $\{X_j\}$, then $h(X) = \inf_j h(X_j)$.

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \rightarrow Y$ is said to be an (α, β) -*quasi-isometric embedding*, with constants $\alpha \geq 1$, $\beta \geq 0$, if for every $x, y \in X$:

$$\alpha^{-1} d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta.$$

The function f is ε -*full* if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \leq \varepsilon$.

A map $f : X \rightarrow Y$ is said to be a *quasi-isometry*, if there exist constants $\alpha \geq 1$, $\beta, \varepsilon \geq 0$ such that f is an ε -full (α, β) -quasi-isometric embedding. Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry $f : X \rightarrow Y$. One can check that to be quasi-isometric is an equivalence relation.

A graph Γ is said to be μ -*uniform* if each vertex p of V has at most μ neighbors, i.e.,

$$\sup \{|N(p)| : p \in V(\Gamma)\} \leq \mu.$$

If a graph Γ is μ -uniform for some constant μ we say that Γ is *uniform*.

A complete Riemannian n -manifold M is *uniform* if it has positive injectivity radius and a lower bound on its Ricci curvature.

The *injectivity radius* $\text{inj}(x)$ of $x \in M$ is defined as the supremum of those $r > 0$ such that $B_r(x)$ is simply connected or, equivalently, as half the infimum of the lengths of the (homotopically non-trivial) loops based at x . The *injectivity radius* $\text{inj}(M)$ of M is the infimum over $x \in M$ of $\text{inj}(x)$.

A celebrated theorem of Kanai in [26] states that quasi-isometries preserve isoperimetric inequalities between uniform Riemannian manifolds and graphs. A main ingredient in Kanai's proof is the following interesting fact: we can study the LII in a Riemannian manifold M by studying the LII in a graph related to it, if M is uniform. Kanai's result also holds with different hypotheses in the context of non-exceptional Riemann surfaces (although it is not possible to relate the LII in the Riemann surface with the LII in the graph provided by Kanai) [12, Theorem 1.2], [23, Theorem 5.1].

A non-exceptional Riemann surface S is a Riemann surface whose universal covering space is the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, endowed with its Poincaré metric (also called the hyperbolic metric), i.e., the metric obtained by projecting the Poincaré metric of the unit disk

$$ds = \frac{2|dz|}{1 - |z|^2}.$$

With this metric, S is a complete Riemannian manifold with constant curvature -1 . The only Riemann surfaces which are left out (the exceptional Riemann surfaces) are the sphere, the plane, the punctured plane and the tori.

A natural context to apply Kanai's results are Riemann surfaces endowed with their Poincaré metrics, since they have constant negative curvature. However, these surfaces usually have isolated singularities which are cusps, and thus, injectivity radius equal to zero. That means that, unfortunately, it is not possible to apply Kanai's results in this case. Recall that the Poincaré metric plays a main role in geometric function theory, since if R, S are Riemann surfaces with their Poincaré metrics, then $d_S(f(w), f(z)) \leq d_R(w, z)$ for every holomorphic map $f : R \rightarrow S$ and every $z, w \in R$.

Our main result is Theorem 3.7, that states that we can study the LII in a Riemann surface by using a graph related to it, even if the surface has injectivity radius zero (this graph is inspired in Kanai's graph, but it is different from it). Thus, Theorem 3.7 shows that the main tool in Kanai's proof also works for all

non-exceptional Riemann surfaces, even if the injectivity radius is zero (although, unfortunately, the LII is not preserved by quasi-isometries between Riemann surfaces with injectivity radius zero, see [12, Example 2.3]). Although this result is interesting by itself, we present in the last section an application of Theorem 3.7 relating Gromov boundary and LII (see Theorem 4.12).

Finally, we want to remark that [24, Theorem 5.5] gives that every orientable and complete Riemannian surface with pinched negative curvature (with Gaussian curvature K satisfying $-k_2^2 \leq K \leq -k_1^2 < 0$) is bilipschitz equivalent to a non-exceptional Riemann surface (and therefore with constant negative curvature -1). [24, Theorem 5.5] shows that it suffices to work with surfaces of curvature -1 (instead of pinched negative curvature) in order to check LII, since this inequality is invariant by bilipschitz maps. This fact enlarges the scope of Theorems 3.7 and 4.12.

2. BACKGROUND AND TECHNICAL RESULTS

A *geodesic domain* in a non-exceptional Riemann surface S is a domain $G \subset S$ (which is not simply or doubly connected) such that ∂G consists of finitely many simple closed geodesics, and $A_S(G)$ is finite. G does not have to be relatively compact since it may “surround” finitely many cusps. We can think of a cusp as a boundary geodesic of zero length. Recall that if γ is a closed curve in S and $[\gamma]$ denotes its free homotopy class in S , then there is a unique simple closed geodesic of minimal length in the class, unless γ is homotopic to zero or surrounds only a cusp; in this case it is not possible to find such geodesic because there are curves in the class with arbitrarily small length.

In [21, Lemma 1.2] it was proved the following.

Lemma 2.1. *If a non-exceptional Riemann surface S satisfies*

$$A_S(G) \leq c L_S(\partial G)$$

for some constant c and every geodesic domain G in S , then S has LII.

In fact, if we define $h_g(S)$ as

$$h_g(S) = \inf_G \frac{L_S(\partial G)}{A_S(G)},$$

where G ranges over all geodesic domains in S , then [28, Theorem 7] gives

$$h(S)^{-1} \leq h_g(S)^{-1} + 1.$$

Definition 2.2. *Given a non-exceptional Riemann surface S , a subset $\Omega \subset S$ and a positive constant δ , let us denote by $\partial_\delta \Omega$ the set of points $z \in \partial \Omega$ such that the connected component of $\partial \Omega$ containing z has length at least δ . A non-exceptional Riemann surface X is δ -regular if there exists a positive constant c such that every geodesic domain G in X satisfies that $L(\partial_\delta G) \geq c CC(\partial G \setminus \partial_\delta G)$ (where $CC(A)$ denotes the cardinality of the set of connected components of A). We say that X is regular if it is δ -regular for some $\delta > 0$.*

Theorem 2.3. *If a non-exceptional Riemann surface is not regular, then it does not have LII.*

Proof. Since the non-exceptional Riemann surface S is not regular, for each $n \in \mathbb{N}$ there exists a geodesic domain G_n in S with

$$L_S(\partial_{1/n} G_n) < \frac{1}{n} CC(\partial G_n \setminus \partial_{1/n} G_n).$$

Hence, since $L_S(\partial G_n \setminus \partial_{1/n} G_n) < \frac{1}{n} CC(\partial G_n \setminus \partial_{1/n} G_n)$, we have

$$(2.2) \quad L_S(\partial G_n) = L_S(\partial_{1/n} G_n) + L_S(\partial G_n \setminus \partial_{1/n} G_n) < \frac{2}{n} CC(\partial G_n \setminus \partial_{1/n} G_n).$$

If m , p and g denote the cardinality of the simple closed geodesics in ∂G_n , the cusps surrounded by G_n and the genus of G_n , respectively, it is well-known that Gauss-Bonnet Theorem gives $A_S(G_n) = 2\pi(m+p-2+2g)$ and so, $m+p-2+2g \geq 1$. Note that $m = CC(\partial G_n)$. We are going to prove

$$(2.3) \quad m \leq 3(m+p-2+2g).$$

If $g \geq 1$, then $m + p - 2 + 2g \geq m + p \geq m$. Assume that $g = 0$. If $p \geq 2$, then $m + p - 2 + 2g = m + p - 2 \geq m$. If $p = 1$, then, since $m - 1 = m + p - 2 + 2g \geq 1$, we have $m \geq 2$ and $m \leq 2(m - 1) = 2(m + p - 2 + 2g)$. If $p = 0$, then, since $m - 2 = m + p - 2 + 2g \geq 1$, we have $m \geq 3$ and $m \leq 3(m - 2) = 3(m + p - 2 + 2g)$. Therefore, (2.3) holds.

Now, (2.2) and (2.3) give

$$\frac{n}{2} L_S(\partial G_n) < CC(\partial G_n \setminus \partial_{1/n} G_n) \leq m = \frac{m}{2\pi(m + p - 2 + 2g)} A_S(G_n) \leq \frac{3}{2\pi} A_S(G_n),$$

$$A_S(G_n) > \frac{n\pi}{3} L_S(\partial G_n).$$

Thus, S does not have LII. \square

Remark 2.4. *Theorem 2.3 shows that, in order to study LII in non-exceptional Riemann surfaces, it suffices to consider regular surfaces.*

Next, we need to consider also bordered Riemann surfaces whose boundary is a finite union of disjoint simple closed curves. They will always arise as closed subsets of some open non-exceptional Riemann surface, and we give them the metric induced from the Poincaré metric of the host open surface. Such induced metric has of course curvature $K = -1$. An example of such a bordered Riemann surface is the closure of any geodesic domain.

A *Y-piece* is a compact bordered Riemann surface which is topologically a sphere without three open disks and whose boundary curves are simple closed geodesics. Given three positive numbers a, b, c , there is a unique (up to conformal mapping) Y-piece such that their boundary curves have lengths a, b, c (see, e.g., [36, p.410]). They are a standard tool for constructing Riemann surfaces. A clear description of these Y-pieces and their use is given in [13, Chapter X.3] and [10, Chapter 1].

A *generalized Y-piece* is a bordered or non-bordered Riemann surface which is topologically a sphere without n open disks and m points, with integers $n, m \geq 0$ such that $n + m = 3$, so that the n boundary curves are simple closed geodesics and the m deleted points are cusps. Observe that a generalized Y-piece is topologically the union of a Y-piece and m cylinders, with $0 \leq m \leq 3$. It is clear that the interior of every generalized Y-piece is a geodesic domain. Furthermore, it is known that the closure of every geodesic domain is a finite union (with pairwise disjoint interiors) of generalized Y-pieces [1, Proposition 3.2].

The following example shows that regularity is not a sufficient condition in order to have LII.

Example 2.5. *Let X be a non-exceptional Riemann surface built as follows: Consider the generalized Y-pieces $\{Y_i\}_{i \in \mathbb{Z}}$ with one cusp and such that $\partial Y_i = \eta_1^i \cup \eta_2^i$ with $L(\eta_1^i) = L(\eta_2^i) = 1$. Let X be the surface obtained by identifying η_2^i with η_1^{i+1} for every $i \in \mathbb{Z}$. It is clear that if $G_n = \cup_{i=1}^n Y_i$, then $A(G_n) = 2\pi n$, $L(\partial G_n) = 2$ and X does not have LII. However, for any geodesic domain G and any $0 < \delta < 1$, $L(\partial_\delta G) = L(\partial G) \geq 0 = CC(\partial G \setminus \partial_\delta G)$, and X is regular.*

A *collar* in a non-exceptional Riemann surface S about a simple closed geodesic γ is a doubly connected domain in S “bounded” by two Jordan curves (called the boundary curves of the collar) orthogonal to the pencil of geodesics emanating from γ ; such collar is equal to $\{p \in S : d_S(p, \gamma) < d\}$, for some positive constant d . The constant d is called the *width* of the collar.

Let S be a non-exceptional Riemann surface with a cusp r (if $S \subset \mathbb{C}$, every isolated point in ∂S is a cusp). A *collar* in S about r is a doubly connected domain in S “bounded” both by r and a Jordan curve (called the boundary curve of the collar) orthogonal to the pencil of geodesics emanating from r . It is well-known that the length of the boundary curve is equal to the area of the collar (see, e.g., [6]). A collar of area λ about r is called a λ -collar. We denote by $C(r, \lambda)$ the λ -collar of the cusp r with area $0 < \lambda \leq 2$.

We will use several times the following result known as Collar Lemma (see [36]).

Lemma 2.6. *If γ is a simple closed geodesic in a non-exceptional Riemann surface S , then there exists a collar about γ of width w , where $\cosh w = \coth(L_S(\gamma)/2)$.*

Remark 2.7. Along this paper, γ will denote a simple closed geodesic in S and w the width of the collar of γ , where $\cosh w = \coth(L_S(\gamma)/2)$.

Denote by $C(\gamma, h)$ the collar of γ of width h and by $C(\gamma)$ the collar of γ of width w . It is well-known that if γ_1 and γ_2 are disjoint simple closed geodesics, then $C(\gamma_1) \cap C(\gamma_2) = \emptyset$.

For each cusp there exists a 2-collar and 2-collars of different cusps are disjoint. Besides, the collar $C(\gamma)$ of the simple closed geodesic γ does not intersect the 2-collar of a cusp (see [36], [39] and [10, Chapter 4]).

We will use the thick-thin decomposition of Riemann surfaces given by Margulis Lemma (see, e.g., [5, p.107]). Concretely, for any $\varepsilon < \text{Arcsinh } 1$ any Riemann surface, S , can be partitioned into a thick part, $S_\varepsilon := \{z \in S : \text{inj}(z) \geq \varepsilon\}$, and a thin part, $S \setminus S_\varepsilon$, whose connected components are either collars of cusps or collars of simple closed geodesics of length less than 2ε . In fact, [12, Lemma 4.9] gives the following.

Lemma 2.8. *Let S be a non-exceptional Riemann surface and $z \in S$. If $\text{inj}(z) < \text{Arcsinh } 1$, then the shortest geodesic loop η with base point z is contained either in the 2-collar of a cusp or in the collar $C(\gamma)$ of a simple closed geodesic γ .*

We collect below a well-known hyperbolic trigonometric formula (see, e.g., [10, p.454]) which will be useful.

Proposition 2.9. *The following formula holds for polygons on the unit disk (and then for simply connected polygons on any non-exceptional Riemann surface).*

Let us consider a geodesic quadrilateral with three right angles and let ϕ the other angle. If α, β are the lengths of the sides which meet with angle ϕ and a is the length of the opposite side to the side with length α , then $\sinh \alpha = \sinh a \cosh \beta$.

Lemma 2.10. *Let S be a non-exceptional Riemann surface, $0 < \varepsilon < \text{Arcsinh } 1$ and C a connected component of $\{z \in S \mid \text{inj}(z) < \varepsilon\}$. By Margulis Lemma, C is a collar.*

- (1) *If C is a collar of a simple closed geodesic γ , then $C = C(\gamma, h) \subset C(\gamma)$ and $h = \text{Arccosh } \frac{\sinh \varepsilon}{\sinh(L_S(\gamma)/2)}$.*
- (2) *If C is a collar of a cusp r , then $C = C(r, \lambda) \subset C(r, 2)$ and $\lambda = 2 \sinh \varepsilon$.*

Proof. Assume first that $C = C(\gamma, h)$ is a collar of a simple closed geodesic γ . Fix $z \in \partial C$. Since $\text{inj}(z) = \varepsilon < \text{Arcsinh } 1$, Lemma 2.8 gives that the shortest geodesic loop σ with base point z is contained in $C(\gamma)$. Let $l := L_S(\gamma)$. By Proposition 2.9, we have

$$\sinh \varepsilon = \sinh \text{inj}(z) = \sinh(l/2) \cosh h, \quad h = \text{Arccosh } \frac{\sinh \varepsilon}{\sinh(l/2)}.$$

Collar Lemma gives that there exists a collar about γ of width w , where $\cosh w = \coth(l/2)$. Thus,

$$\cosh h = \frac{\sinh \varepsilon}{\sinh(l/2)} < \frac{1}{\sinh(l/2)} < \frac{\cosh(l/2)}{\sinh(l/2)} = \cosh w,$$

$h < w$ and C is contained in $C(\gamma)$.

Assume now that $C = C(r, \lambda)$ is a collar of a cusp r . Fix $z \in \partial C$. Since $\text{inj}(z) = \varepsilon < \text{Arcsinh } 1$, Lemma 2.8 gives that the shortest geodesic loop σ with base point z is contained in $C(r, 2)$.

As usual, consider a fundamental domain for S in the upper half-plane \mathbb{H} contained in $\{\zeta \in \mathbb{H} : 0 \leq \Re \zeta \leq 1\}$ and such that $\{\zeta \in \mathbb{H} : 0 \leq \Re \zeta \leq 1, \Im \zeta > 1/2\}$ corresponds to $C(r, 2)$. Thus, $\{\zeta \in \mathbb{H} : 0 \leq \Re \zeta \leq 1, \Im \zeta > 1/\lambda\}$ corresponds to $C(r, \lambda)$. Without loss of generality we can assume that i/λ corresponds to z . Thus, we can represent σ in the upper half-plane by means of a geodesic with endpoints i/λ and $1 + i/\lambda$. We have

$$\sinh \varepsilon = \sinh \text{inj}(z) = \sinh \frac{L_S(\sigma)}{2} = \sinh \frac{d_{\mathbb{H}}(i/\lambda, 1 + i/\lambda)}{2} = \frac{\lambda}{2},$$

$\lambda = 2 \sinh \varepsilon < 2$ and C is contained in $C(r, 2)$. □

Lemma 2.11. *Let S be a non-exceptional Riemann surface, $0 < \varepsilon < \text{Arcsinh } 1$ and C a connected component of $\{z \in S \mid \text{inj}(z) < \varepsilon\}$. If η is a connected component of ∂C , then $L_S(\eta) \leq 2 \sinh \varepsilon < 2$.*

Proof. By Margulis Lemma, C is a collar.

Assume first that C is a collar of a simple closed geodesic γ . Lemma 2.10 gives that $C = C(\gamma, h) \subset C(\gamma)$ and $h = \operatorname{Arccosh} \frac{\sinh \varepsilon}{\sinh(l/2)}$ with $l = L_S(\gamma)$. We have

$$L_S(\eta) = l \cosh h = l \frac{\sinh \varepsilon}{\sinh(l/2)}.$$

Since the function $f(t) = t - \tanh t$ satisfies $f'(t) = 1 - (\cosh t)^{-2} > 0$ for every $t > 0$, we conclude $f(t) > f(0) = 0$, $t > \tanh t$ and $t \cosh t > \sinh t$ for every $t > 0$. Since the function $g(t) = t/\sinh t$ satisfies

$$g'(t) = \frac{\sinh t - t \cosh t}{\sinh^2 t} < 0$$

for every $t > 0$, the function g is decreasing on $(0, \infty)$. Consequently, $g(t) < \lim_{x \rightarrow 0} g(x) = 1$ for every $t > 0$, and

$$L_S(\eta) = l \frac{\sinh \varepsilon}{\sinh(l/2)} < 2 \sinh \varepsilon < 2.$$

Assume now that C is a collar of a cusp r . Lemma 2.10 gives that $C = C(r, \lambda) \subset C(r, 2)$ and $\lambda = 2 \sinh \varepsilon$. Thus, $L_S(\eta) = L_S(\partial C) = \lambda = 2 \sinh \varepsilon < 2$. \square

Lemma 2.12. *Let S be a non-exceptional Riemann surface, $0 < \varepsilon < \operatorname{Arcsinh} 1$ and C a connected component of $\{z \in S \mid \operatorname{inj}(z) < \varepsilon\}$ that is a collar of a simple closed geodesic γ . Then*

$$A_S(C) = 2L_S(\gamma) \sqrt{\frac{\sinh^2 \varepsilon}{\sinh^2(L_S(\gamma)/2)} - 1} < 4 \sinh \varepsilon.$$

Furthermore, if $L_S(\gamma)/2 \leq \delta < \varepsilon$, then

$$A_S(C) \geq \frac{4\delta}{\sinh \delta} \sqrt{\sinh^2 \varepsilon - \sinh^2 \delta}.$$

Proof. Let $l := L_S(\gamma)$. Lemma 2.10 gives $C = C(\gamma, h) \subset C(\gamma)$ and $h = \operatorname{Arccosh} \frac{\sinh \varepsilon}{\sinh(l/2)}$. It is well-known that $A_S(C) = 2l \sinh h$, and so,

$$\begin{aligned} A_S(C) &= 2l \sinh h = 2l \sqrt{\cosh^2 h - 1} = 2l \sqrt{\frac{\sinh^2 \varepsilon}{\sinh^2(l/2)} - 1} \\ &= \frac{2l}{\sinh(l/2)} \sqrt{\sinh^2 \varepsilon - \sinh^2(l/2)}. \end{aligned}$$

Since the function $g(t) = t/\sinh t$ is decreasing on $(0, \infty)$, we have

$$A_S(C) < 4 \sqrt{\sinh^2 \varepsilon - \sinh^2(l/2)} < 4 \sinh \varepsilon.$$

If $l/2 \leq \delta < \varepsilon$, then

$$A_S(C) \geq \frac{4\delta}{\sinh \delta} \sqrt{\sinh^2 \varepsilon - \sinh^2 \delta}.$$

\square

Lemma 2.13. *Let S be a non-exceptional Riemann surface, $0 < \varepsilon < \operatorname{Arcsinh} 1$ and C a connected component of $\{z \in S \mid \operatorname{inj}(z) < \varepsilon\}$ that is a collar of a simple closed geodesic γ with*

$$\delta_0 \leq \log \frac{4}{3}, \quad L_S(\gamma) \leq 2 \operatorname{Arcsinh} \left(\frac{\sqrt{3}}{4} \sinh \varepsilon \right) =: 2\delta, \quad h = \operatorname{Arccosh} \frac{\sinh \varepsilon}{\sinh(L_S(\gamma)/2)}.$$

Then, $\delta_0 < h$ and

$$A_S(C(\gamma, h - \delta_0)) > \frac{1}{2} A_S(C(\gamma, h)) \geq \frac{2\delta}{\sinh \delta} \sqrt{\sinh^2 \varepsilon - \sinh^2 \delta}.$$

Proof. By Lemma 2.10 we have $C = C(\gamma, h)$. Define $h' = h - \delta_0$. We also have

$$\begin{aligned} L_S(\gamma) \leq 2 \operatorname{Arcsinh} \left(\frac{\sqrt{3}}{4} \sinh \varepsilon \right) &\Leftrightarrow \frac{4}{\sqrt{3}} \leq \frac{\sinh \varepsilon}{\sinh(L_S(\gamma)/2)} \\ \Leftrightarrow \log \frac{16}{3} \leq 2 \log \frac{\sinh \varepsilon}{\sinh(L_S(\gamma)/2)} &\Rightarrow \log \frac{16}{3} < 2 \operatorname{Arccosh} \frac{\sinh \varepsilon}{\sinh(L_S(\gamma)/2)} = 2h \\ \Rightarrow \delta_0 \leq \log \frac{4}{3} < 2h - \log 4 &\Rightarrow e^{\delta_0} < \frac{e^{2h}}{4} \\ \Leftrightarrow e^{-h'} = e^{\delta_0 - h} < \frac{e^h}{4} &\Rightarrow e^{h'} - e^{-h'} > e^{h'} - \frac{e^h}{4}. \end{aligned}$$

Besides, $\delta_0 \leq \frac{1}{2} \log \frac{16}{9} < \frac{1}{2} \log \frac{16}{3} < h$, and $h' = h - \delta_0$. We have

$$\begin{aligned} \delta_0 \leq \log \frac{4}{3} &\Leftrightarrow \frac{e^h}{e^{\delta_0}} \geq \frac{3}{4} e^h \\ \Leftrightarrow e^{h'} \geq \frac{3}{4} e^h &\Rightarrow 2 \left(e^{h'} - \frac{e^h}{4} \right) \geq e^h > e^h - e^{-h}. \end{aligned}$$

Denote by C' the collar $C' = C(\gamma, h')$. Since $A_S(C') = 2L_S(\gamma) \sinh h'$ and $A_S(C) = 2L_S(\gamma) \sinh h$, we conclude

$$\begin{aligned} A_S(C') &= 2L_S(\gamma) \sinh h' = L_S(\gamma) (e^{h'} - e^{-h'}) > L_S(\gamma) \left(e^{h'} - \frac{e^h}{4} \right) \\ &> \frac{1}{2} L_S(\gamma) (e^h - e^{-h}) = \frac{1}{2} 2L_S(\gamma) \sinh h = \frac{1}{2} A_S(C). \end{aligned}$$

Finally, by Lemma 2.12, since

$$L_S(\gamma)/2 \leq \delta = \operatorname{Arcsinh} \left(\frac{\sqrt{3}}{4} \sinh \varepsilon \right) < \varepsilon,$$

we have

$$A_S(C') > \frac{1}{2} A_S(C) \geq \frac{2\delta}{\sinh \delta} \sqrt{\sinh^2 \varepsilon - \sinh^2 \delta}.$$

□

Lemma 2.14. *Let S be a non-exceptional Riemann surface, $0 < \varepsilon < \operatorname{Arcsinh} 1$ and C_1, C_2 two different connected components of $\{z \in S \mid \operatorname{inj}(z) < \varepsilon\}$. Let K_1 be the collar given by the Collar Lemma corresponding to C_1 . Then*

$$d_S(C_1, \partial K_1) \geq \log \frac{1}{\sinh \varepsilon}, \quad d_S(C_1, C_2) \geq 2 \log \frac{1}{\sinh \varepsilon}.$$

Furthermore, if C_1 is a collar of a simple closed geodesic γ , then $d_S(C_1, \partial K_1)$ is an increasing function on $L_S(\gamma)$.

Proof. By Margulis Lemma, C_1 and C_2 are collars.

Assume that C_1 is a collar $C(\gamma, h)$ for some simple closed geodesic γ . Collar Lemma gives that there exists a collar $C(\gamma) = K_1$ about γ of width w , where $\cosh w = \coth(L_S(\gamma)/2)$. Lemma 2.10 gives that C_1 is contained in K_1 and

$$d_S(C_1, \partial K_1) = d_S(\partial C_1, \partial C(\gamma)) = w - h = \operatorname{Arccosh} \frac{\cosh(L_S(\gamma)/2)}{\sinh(L_S(\gamma)/2)} - \operatorname{Arccosh} \frac{\sinh \varepsilon}{\sinh(L_S(\gamma)/2)}.$$

If $0 < t < \varepsilon$, then

$$\begin{aligned} \sinh^2 \varepsilon \cosh^2 t > \sinh^2 \varepsilon - \sinh^2 t, \quad & \frac{\sinh \varepsilon \cosh t}{\sqrt{\sinh^2 \varepsilon - \sinh^2 t}} > 1, \\ \frac{\sinh \varepsilon \cosh t}{\sinh t \sqrt{\sinh^2 \varepsilon - \sinh^2 t}} - \frac{1}{\sinh t} &> 0. \end{aligned}$$

If $0 < t < \varepsilon$, then we define the function

$$f(t) = \operatorname{Arccosh} \frac{\cosh t}{\sinh t} - \operatorname{Arccosh} \frac{\sinh \varepsilon}{\sinh t}.$$

Since

$$f'(t) = \frac{\sinh \varepsilon \cosh t}{\sinh t \sqrt{\sinh^2 \varepsilon - \sinh^2 t}} - \frac{1}{\sinh t} > 0,$$

$d_S(C_1, \partial K_1)$ is an increasing function in $L_S(\gamma)$. In particular, $f(t) > \lim_{x \rightarrow 0^+} f(x)$ for every $0 < t < \varepsilon$. Since

$$\operatorname{Arccosh} x - \operatorname{Arccosh} y = \operatorname{Arccosh} (xy - \sqrt{x^2 - 1} \sqrt{y^2 - 1}),$$

we consider

$$\begin{aligned} & \frac{\cosh t \sinh \varepsilon}{\sinh t \sinh t} - \sqrt{\frac{\cosh^2 t}{\sinh^2 t} - 1} \sqrt{\frac{\sinh^2 \varepsilon}{\sinh^2 t} - 1} \\ &= \frac{\cosh t \sinh \varepsilon - \sqrt{\sinh^2 \varepsilon - \sinh^2 t}}{\sinh^2 t} \frac{\cosh t \sinh \varepsilon + \sqrt{\sinh^2 \varepsilon - \sinh^2 t}}{\cosh t \sinh \varepsilon + \sqrt{\sinh^2 \varepsilon - \sinh^2 t}} \\ &= \frac{\sinh^2 \varepsilon + 1}{\cosh t \sinh \varepsilon + \sqrt{\sinh^2 \varepsilon - \sinh^2 t}}. \end{aligned}$$

Thus,

$$\begin{aligned} & \lim_{t \rightarrow 0^+} \left(\frac{\cosh t \sinh \varepsilon}{\sinh t \sinh t} - \sqrt{\frac{\cosh^2 t}{\sinh^2 t} - 1} \sqrt{\frac{\sinh^2 \varepsilon}{\sinh^2 t} - 1} \right) \\ &= \lim_{t \rightarrow 0^+} \frac{\sinh^2 \varepsilon + 1}{\cosh t \sinh \varepsilon + \sqrt{\sinh^2 \varepsilon - \sinh^2 t}} = \frac{\sinh^2 \varepsilon + 1}{2 \sinh \varepsilon}, \\ & \lim_{t \rightarrow 0^+} f(t) = \operatorname{Arccosh} \frac{\sinh^2 \varepsilon + 1}{2 \sinh \varepsilon} = \log \left(\frac{\sinh^2 \varepsilon + 1}{2 \sinh \varepsilon} + \sqrt{\frac{(\sinh^2 \varepsilon + 1)^2}{4 \sinh^2 \varepsilon} - 1} \right) \\ &= \log \left(\frac{\sinh^2 \varepsilon + 1}{2 \sinh \varepsilon} + \frac{1 - \sinh^2 \varepsilon}{2 \sinh \varepsilon} \right) = \log \frac{1}{\sinh \varepsilon}. \end{aligned}$$

Hence,

$$d_S(C_1, \partial K_1) = \operatorname{Arccosh} \frac{\cosh(L_S(\gamma)/2)}{\sinh(L_S(\gamma)/2)} - \operatorname{Arccosh} \frac{\sinh \varepsilon}{\sinh(L_S(\gamma)/2)} > \log \frac{1}{\sinh \varepsilon}.$$

Assume now that C_1 is a collar $C(r, \lambda)$ for some cusp r . Consider a fundamental domain for S in the upper half-plane \mathbb{H} contained in $\{\zeta \in \mathbb{H} : 0 \leq \Re \zeta \leq 1\}$ and such that $\{\zeta \in \mathbb{H} : 0 \leq \Re \zeta \leq 1, \Im \zeta > 1/2\}$ corresponds to $C(r, 2) = K_1$. Thus, $\{\zeta \in \mathbb{H} : 0 \leq \Re \zeta \leq 1, \Im \zeta > 1/\lambda\}$ corresponds to C_1 , and Lemma 2.10 gives that $\lambda = 2 \sinh \varepsilon < 2$ and C_1 is contained in $C(r, 2)$. Hence,

$$d_S(C_1, \partial K_1) = d_S(\partial C_1, \partial C(r, 2)) = d_{\mathbb{H}}(i/\lambda, i/2) = \log \frac{2}{\lambda} = \log \frac{1}{\sinh \varepsilon}.$$

Since collars of geodesics and cusps are pairwise disjoint by Remark 2.7, we conclude

$$d_S(C_1, C_2) \geq 2 \log \frac{1}{\sinh \varepsilon}.$$

□

Let X be a non-exceptional Riemann surface and $0 < \varepsilon < \operatorname{Arcsinh} 1$. Consider the thick-thin decomposition given by Margulis Lemma: $X_\varepsilon = \{z \in X \mid \operatorname{inj}(z) \geq \varepsilon\}$ and $X \setminus X_\varepsilon$.

Given the cusps $\{r_i\}_{i \in I}$ in X , let us consider for each r_i the collar in X about r_i with area $2 \sinh \varepsilon$, $C_i = C(r_i, 2 \sinh \varepsilon)$, and let $\sigma_i = \partial C(r_i, 2 \sinh \varepsilon)$.

Given the simple closed geodesics $\{\gamma_j\}_{j \in J}$ in X with $L_X(\gamma_j) < 2\varepsilon$, let us consider for each γ_j its collar K_j in X of width $h_j = \operatorname{Arccosh} \frac{\sinh \varepsilon}{\sinh(L_X(\gamma_j)/2)}$.

By Lemma 2.10, we have

$$(2.4) \quad X \setminus X_\varepsilon = \left(\cup_{i \in I} C_i \right) \cup \left(\cup_{j \in J} K_j \right).$$

The following result is well-known.

Lemma 2.15. *Let S be a non-exceptional Riemann surface, $z \in S$ and $r > 0$. Then*

$$A_S(B_r(z)) \leq 4\pi \sinh^2 \frac{r}{2}, \quad L_S(\partial B_r(z)) \leq 2\pi \sinh r.$$

Furthermore, if $0 < r \leq \text{inj}(z)$, then

$$A_S(B_r(z)) = 4\pi \sinh^2 \frac{r}{2}, \quad L_S(\partial B_r(z)) = 2\pi \sinh r.$$

3. SURFACES AND GRAPHS

We are going to construct a graph associated to any non-exceptional Riemann surface.

A subset A in a metric space (X, d) is called r -separated, $r > 0$, if $d(a, a') \geq r$ for any distinct $a, a' \in A$. Note that if A is maximal with this property, then the union $\cup_{a \in A} B_r(a)$ covers X . A maximal r -separated set A in a metric space X is called an r -approximation of X .

Fix $0 < \varepsilon < \text{Arcsinh } 1$. Given any ε -approximation A_ε of X , the graph $\Gamma_{A_\varepsilon} = (V, E)$ with $V = A_\varepsilon$ and $E := \{xy \mid x, y \in A_\varepsilon \text{ with } 0 < d(x, y) \leq 2\varepsilon\}$ is called an ε -net.

Given any $\delta > 0$ satisfying

$$(3.5) \quad \delta < \delta_1 := \min \left\{ \log \frac{1}{\sinh \varepsilon}, \text{Arcsinh} \left(\frac{\sqrt{3}}{4} \sinh \varepsilon \right) \right\},$$

let $J_\delta := \{j \in J \mid L(\gamma_j) < 2\delta\}$, where J denotes the index set of the collars about simple closed geodesics in the thick-thin decomposition from equation (2.4). Note that $\delta_1 < \varepsilon$ and so, $\delta < \varepsilon$. Let $[X_\varepsilon]_\delta = X_\varepsilon \cup (\cup_{j \in J \setminus J_\delta} K_j)$. Therefore, $[X_\varepsilon]_\delta$ is the surface X without the collars $C_i = C(r_i, 2 \sinh \varepsilon)$ about the cusps and the collars $K_j = C(\gamma_j, h_j)$ with $L_S(\gamma_j) < 2\delta$. Notice that $X_\varepsilon \subseteq [X_\varepsilon]_\delta \subseteq X_\delta$. Let $\{[X_\varepsilon]_\delta^r\}_r$ be the connected components of $[X_\varepsilon]_\delta$. For each r , let $[A]_\delta^r$ be a δ -approximation of $[X_\varepsilon]_\delta^r$ (with respect to the distance in X) and $\Gamma_{[A]_\delta^r}$ the corresponding δ -net. Let $[A]_\delta = \cup_r [A]_\delta^r$ and $\Gamma_{[A]_\delta} = \cup_r \Gamma_{[A]_\delta^r}$. To simplify the notation we make no distinction between the vertices of the graph and the corresponding points in the surface. For each cusp r_i let us define a vertex w_i and for each γ_j with $j \in J_\delta$ let us define a new vertex v_j . Let us define a graph $\Gamma_\delta(X)$ with vertices $[A]_\delta \cup \{w_i\}_{i \in I} \cup \{v_j\}_{j \in J_\delta}$, containing $\Gamma_{[A]_\delta}$ as a subgraph and such that w_i is adjacent to every vertex v in $[A]_\delta$ such that $d(v, C(r_i, 2 \sinh \varepsilon)) \leq \delta$ and v_j is adjacent to every vertex v in $[A]_\delta$ such that $d(v, K_j) \leq \delta$.

Remark 3.1. *From the definition of $\Gamma_\delta(X)$, it follows that no pair of vertices in $\{w_i\}_{i \in I} \cup \{v_j\}_{j \in J_\delta}$ can be adjacent in $\Gamma_\delta(X)$. By Lemma 2.14, $\{N(w_i)\}_{i \in I} \cup \{N(v_j)\}_{j \in J_\delta}$ are pairwise disjoint sets in $\Gamma_\delta(X)$.*

We have the following result in [26, Lemma 2.3].

Lemma 3.2. *Let X be a complete Riemannian n -manifold whose Ricci curvature is bounded from below by $-(n-1)K^2$ ($K > 0$), and let P be an ε -separated subset of X . Then we have $|\{p \in P : x \in B_r(p)\}| \leq \nu$ for all $r > 0$ and for all $x \in X$, where $\nu = \nu(n, K, \varepsilon, r) > 0$. Consequently, every ε -net in a complete Riemannian manifold whose Ricci curvature is bounded from below is uniform.*

Lemma 3.2 has the following consequence.

Lemma 3.3. *If X is a non-exceptional Riemann surface and $[A]_\delta$ is a δ -separated set in $[X_\varepsilon]_\delta$, then there is some constant $\mu(\delta)$ such that $|B_{3\delta}(p) \cap [A]_\delta| \leq \mu(\delta)$ for every $p \in [X_\varepsilon]_\delta$.*

Proposition 3.4. *If X is a non-exceptional Riemann surface, $0 < \varepsilon < \text{Arcsinh } 1$ and $0 < \delta < \delta_1$, then $\Gamma_\delta(X)$ is uniform.*

Proof. Let v be any vertex in $\Gamma_\delta(X)$. If $v \in [A]_\delta$, then its adjacent vertices w in $[A]_\delta$ satisfy $d(v, w) \leq 2\delta$ and, by Lemma 3.3, these are at most $\mu(\delta)$. Also, by Remark 3.1, v has at most one adjacent vertex v_j or w_i , corresponding to a collar K_j or C_i , respectively.

If $v = w_i$ is the vertex corresponding to a collar $C_i = C(r_i, 2 \sinh \varepsilon)$ then, by Lemma 2.11, $\partial C_i = \sigma_i$ satisfies $L(\sigma_i) \leq 2 \sinh \varepsilon < 2$. Let $\{x_1, \dots, x_n\}$ be a maximal δ -separated set in σ_i . Then, $n \leq \frac{2 \sinh \varepsilon}{\delta}$ and for any vertex w adjacent to v , we have $w \in [A]_\delta$, $d(w, \sigma_i) \leq \delta$ and so, w must be contained in some $B_{2\delta}(x_k)$. Thus, by Lemma 3.3, there are at most $\frac{2 \sinh \varepsilon}{\delta} \mu(\delta) < \frac{2}{\delta} \mu(\delta)$ vertices adjacent to v in $\Gamma_\delta(X)$.

If $v = v_j$ is the vertex corresponding to a collar $K_j = C(\gamma_j, h)$ then, by Lemma 2.11, $\partial K_j = \eta_1^j \cup \eta_2^j$ and $L(\eta_i^j) \leq 2 \sinh \varepsilon < 2$. Let $\{x_1, \dots, x_n\}$ be a maximal δ -separated set in $\eta_1^j \cup \eta_2^j$. Then, $n \leq \frac{4 \sinh \varepsilon}{\delta}$ and for any vertex w adjacent to v , we have $w \in [A]_\delta$, $d(w, \eta_1^j \cup \eta_2^j) \leq \delta$ and so, w must be contained in some $B_{2\delta}(x_k)$. Thus, by Lemma 3.3, there are at most $\frac{4 \sinh \varepsilon}{\delta} \mu(\delta) < \frac{4}{\delta} \mu(\delta)$ vertices adjacent to v in $\Gamma_\delta(X)$. \square

Let X be a non-exceptional Riemann surface and $\Gamma_\delta(X)$ the associated graph with vertex set $[A]_\delta \cup \{w_i\}_{i \in I} \cup \{v_j\}_{j \in J_\delta}$. Given a geodesic domain G in X , let $S_G = ([A]_\delta \cap G) \cup \{w_i : C(r_i, 2 \sinh \varepsilon) \subset G\} \cup \{v_j : C(\gamma_j, h_j) \cap G \neq \emptyset\}$.

Remark 3.5. *Let X be a non-exceptional Riemann surface and $\Gamma_\delta(X)$ the associated graph defined above. Given a geodesic domain G , then ∂S_G is the set of vertices $w \in [A]_\delta \setminus G$ satisfying one of the following conditions:*

- w is adjacent to a vertex $v \in [A]_\delta \cap G$ and either $\bar{B}_\delta(v) \cap \partial G \neq \emptyset$ or $\bar{B}_\delta(w) \cap \partial G \neq \emptyset$. In particular, we have either $\bar{B}_\delta(v) \cap \partial_{2\delta} G \neq \emptyset$ or $\bar{B}_\delta(w) \cap \partial_{2\delta} G \neq \emptyset$.
- w is adjacent to a vertex $v \in S_G$ with $v = v_j$ for some $j \in J_\delta$ and so, $C(\gamma_j, h_j) \cap \partial G \neq \emptyset$.

Notice that $w \neq v_j$ for every $j \in J_\delta$ with $C(\gamma_j, h_j) \cap G = \emptyset$, since, by Lemma 2.14, the distance from $C(\gamma_j, h_j)$ to G is at least $\log \frac{1}{\sinh \varepsilon} > \delta$.

Note also that, since the collars are pairwise disjoint, given a geodesic domain G and a collar C_i , we have either $C_i \subset G$ or $C_i \cap G = \emptyset$.

Let $\partial_{2\delta} S_G$ denote the set of vertices $w \in [A]_\delta \setminus G$ satisfying condition a) above.

Remark 3.6. *For every vertex $w \in \partial_{2\delta} S_G$, $d(w, \partial_{2\delta} G) \leq 2\delta$.*

Theorem 3.7. *Given a $(2\delta_2)$ -regular non-exceptional Riemann surface X , $0 < \varepsilon < \text{Arcsinh } 1$ and $0 < \delta < \min\{\delta_1, \delta_2\}$, X satisfies LII if and only if $\Gamma_\delta(X)$ satisfies LII.*

Proof. Suppose X satisfies LII. Consider any non-empty finite subset of vertices S in $\Gamma_\delta(X)$. Let us define

$$\Omega = \left(\bigcup_{v \in S \cap [X_\varepsilon]_\delta} B_\delta(v) \right) \cup \left(\bigcup_{\{i \in I : w_i \in S\}} C_i \right) \cup \left(\bigcup_{\{j \in J_\delta : v_j \in S\}} K_j \right).$$

Choose δ' satisfying

$$\delta' < \min \left\{ \delta, 2 \log \frac{4}{3} \right\} < \delta_1 < 2 \text{Arcsinh} \left(\frac{\sqrt{3}}{4} \sinh \varepsilon \right).$$

Then we have the following:

- for any pair of vertices $v_1, v_2 \in S \cap [X_\varepsilon]_\delta$, $B_{\delta'/2}(v_1) \cap B_{\delta'/2}(v_2) = \emptyset$,
- for any vertex $v \in S \cap [X_\varepsilon]_\delta$ and any $i \in I$, we have $B_{\delta'/2}(v) \cap C(r_i, \lambda') = \emptyset$, where $\lambda' = e^{-\delta'/2} 2 \sinh \varepsilon$,
- for any vertex $v \in S \cap [X_\varepsilon]_\delta$ and any $j \in J_\delta$, we have $B_{\delta'/2}(v) \cap C(\gamma_j, h_j - \delta'/2) = \emptyset$.

Let $A' = \frac{\delta'}{\sinh(\delta'/2)} \sqrt{\sinh^2 \varepsilon - \sinh^2(\delta'/2)}$ and $K = \min\{4\pi \sinh^2(\delta'/4), \lambda', A'\}$. Lemmas 2.13 (with $\delta_0 = \delta'/2$) and 2.15 give $\delta'/2 < h_j$ and $A(C(\gamma_j, h_j - \delta'/2)) > A'$ for every $j \in J_\delta$, and

$$K |S| \leq 4\pi \sinh^2 \frac{\delta'}{4} |S \cap [X_\varepsilon]_\delta| + \lambda' |S \cap \{w_i\}_{i \in I}| + A' |S \cap \{v_j\}_{j \in J_\delta}| \leq A(\Omega).$$

Now, let $\partial'S = \partial(\Gamma_\delta(X) \setminus S)$. By Lemmas 2.15 and 2.11,

$$\begin{aligned} L(\partial\Omega) &\leq \sum_{v \in \partial'S \cap [X_\varepsilon]_\delta} L(\partial B_\delta(v)) + \sum_{\{i \in I : w_i \in \partial'S\}} L(\partial C_i) + \sum_{\{j \in J_\delta : v_j \in \partial'S\}} L(\partial K_j) \\ &\leq 2\pi \sinh \delta |\partial'S \cap [X_\varepsilon]_\delta| + 2 \sinh \varepsilon |\partial'S \cap \{w_i\}_{i \in I}| + 4 \sinh \varepsilon |\partial'S \cap \{v_j\}_{j \in J_\delta}| \\ &\leq \max\{2\pi \sinh \delta, 4 \sinh \varepsilon\} |\partial'S|. \end{aligned}$$

Since, by Proposition 3.4, $\Gamma_\delta(X)$ is μ -uniform for some constant $\mu > 0$, we have $|\partial'S| \leq \mu |\partial S|$. Therefore,

$$L(\partial\Omega) \leq \max\{2\pi \sinh \delta, 4 \sinh \varepsilon\} \mu |\partial S|.$$

Thus, since X satisfies LII, there is some constant h such that $A(\Omega) \leq h L(\partial\Omega)$ and, hence,

$$|S| \leq \frac{1}{K} A(\Omega) \leq \frac{h}{K} L(\partial\Omega) \leq \frac{h\mu}{K} \max\{2\pi \sinh \delta, 4 \sinh \varepsilon\} |\partial S|,$$

and $\Gamma_\delta(X)$ satisfies LII.

Assume now that $\Gamma_\delta(X)$ satisfies LII. By Lemma 2.1, it suffices to check that X satisfies LII on every geodesic domain. Let G be any geodesic domain and consider the set $S_G \subset V(\Gamma_\delta(X))$. Then, since

$$G \subset \left(\cup_{v \in S_G \cap [X_\varepsilon]_\delta} B_{2\delta}(v) \right) \cup \left(\cup_{\{i \in I : w_i \in S_G\}} C_i \right) \cup \left(\cup_{\{j \in J_\delta : v_j \in S_G\}} K_j \right),$$

we have, by Lemmas 2.15, 2.10 and 2.12,

$$\begin{aligned} A(G) &\leq 4\pi \sinh^2 \delta |S_G \cap [X_\varepsilon]_\delta| + 2 \sinh \varepsilon |S_G \cap \{w_i : i \in I\}| + 4 \sinh \varepsilon |S_G \cap \{v_j : j \in J_\delta\}| \\ &\leq \max\{4\pi \sinh^2 \delta, 4 \sinh \varepsilon\} |S_G|. \end{aligned}$$

For any connected component γ of $\partial_{2\delta}G$ consider a maximal δ -separated set $\{x_1, \dots, x_n\}$. Notice that $n \leq \frac{L(\gamma)}{\delta}$. By Remark 3.6, for every point $v \in \partial_{2\delta}S_G$, $d(v, \partial_{2\delta}G) \leq 2\delta$ and since $\{x_1, \dots, x_n\}$ is a maximal δ -separated set, there is some x_i such that $d(v, x_i) < 3\delta$.

Therefore, by Lemma 3.3, there exists a constant $\mu(\delta)$ such that

$$(3.6) \quad |\partial_{2\delta}S_G| \leq \sum_{\gamma \subset \partial_{2\delta}G} \mu(\delta) \frac{L(\gamma)}{\delta} = \frac{\mu(\delta)}{\delta} L(\partial_{2\delta}G).$$

Also, by Remark 3.5, if $w \in \partial S_G \setminus \partial_{2\delta}S_G$, then w is adjacent to some v_j with $j \in J_\delta$ and, by Proposition 3.4, it follows that for every such v_j there exist at most Δ vertices adjacent to it. Hence,

$$(3.7) \quad |\partial S_G \setminus \partial_{2\delta}S_G| \leq \Delta CC(\partial G \setminus \partial_{2\delta}G).$$

Since X is $(2\delta_2)$ -regular and $0 < \delta < \delta_2$, X is also (2δ) -regular and there exists $c > 0$ such that $L(\partial_{2\delta}G) \geq c CC(\partial G \setminus \partial_{2\delta}G)$. Therefore, by inequalities (3.6) and (3.7), it follows that

$$\begin{aligned} |\partial S_G| &= |\partial_{2\delta}S_G| + |\partial S_G \setminus \partial_{2\delta}S_G| \leq \frac{\mu(\delta)}{\delta} L(\partial_{2\delta}G) + \Delta CC(\partial G \setminus \partial_{2\delta}G) \\ &\leq \frac{\mu(\delta)}{\delta} L(\partial_{2\delta}G) + \frac{\Delta}{c} L(\partial_{2\delta}G) \leq \left(\frac{\mu(\delta)}{\delta} + \frac{\Delta}{c} \right) L(\partial G). \end{aligned}$$

Thus, since $\Gamma_\delta(X)$ satisfies LII, there is some constant h such that $|S_G| \leq h |\partial S_G|$, and so,

$$\begin{aligned} A(G) &\leq \max\{4\pi \sinh^2 \delta, 4 \sinh \varepsilon\} |S_G| \leq \max\{4\pi \sinh^2 \delta, 4 \sinh \varepsilon\} h |\partial S_G| \\ &\leq \max\{4\pi \sinh^2 \delta, 4 \sinh \varepsilon\} \left(\frac{h\mu(\delta)}{\delta} + \frac{h\Delta}{c} \right) L(\partial G). \end{aligned}$$

Hence, X satisfies LII on every geodesic domain, finishing the proof. \square

Remark 3.8. Note that the hypothesis of regularity in Theorem 3.7 is not restrictive at all, since Theorem 2.3 gives that non-regular surfaces do not have LII.

4. GROMOV BOUNDARY AND LII

We present in this section an application of Theorem 3.7 relating Gromov boundary and LII (see Theorem 4.12). First of all, we need some background on Gromov hyperbolicity.

Let X be a metric space. Fix a base point $o \in X$ and for $x, x' \in X$ let

$$(x|x')_o = \frac{1}{2}(d(x, o) + d(x', o) - d(x, x')).$$

The number $(x|x')_o$ is non-negative and it is called the *Gromov product* of x, x' with respect to o .

Definition 4.1. *A metric space X is (Gromov) hyperbolic if it satisfies the δ -inequality*

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta$$

for some $\delta \geq 0$, for every base point $o \in X$ and all $x, y, z \in X$.

If X is a metric space and $I \subset \mathbb{R}$ is an interval, we say that the curve $\gamma : I \rightarrow X$ is a *geodesic* if we have $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$ (then γ is equipped with an arc-length parametrization). A *geodesic ray* is a geodesic defined on the interval $[0, \infty)$.

A *geodesic metric space* is a metric space such that for every couple of points there exists a geodesic joining them.

Let us recall the following from [27].

Definition 4.2. *A geodesic metric space X has a pole in a point v if there exists $M > 0$ such that each point of X lies in an M -neighborhood of some geodesic ray emanating from v .*

Let X be a hyperbolic space and $o \in X$ a base point.

The *relative geodesic boundary* of X with respect to the base point o is the set of equivalence classes

$$\partial^g X := \{[\gamma] \mid \gamma : [0, \infty) \rightarrow X \text{ is a geodesic ray with } \gamma(0) = o\},$$

where two geodesic rays γ_1, γ_2 are equivalent if there exists some $K > 0$ such that $d(\gamma_1(t), \gamma_2(t)) < K$, for every $t \geq 0$.

In fact, the definition above is independent from the base point. Therefore, the set of classes of geodesic rays is called *geodesic boundary* of X , $\partial^g X$. Herein, we do not distinguish between the geodesic ray and its image.

A sequence of points $\{x_i\} \subset X$ *converges to infinity* if

$$\lim_{i,j \rightarrow \infty} (x_i|x_j)_o = \infty.$$

This property is independent of the choice of o since

$$|(x|x')_o - (x|x')_{o'}| \leq d(o, o')$$

for any $x, x', o, o' \in X$.

Two sequences $\{x_i\}, \{x'_i\}$ that converge to infinity are *equivalent* if

$$\lim_{i \rightarrow \infty} (x_i|x'_i)_o = \infty.$$

Using the δ -inequality, we easily see that this defines an equivalence relation for sequences in X converging to infinity. The *sequential boundary at infinity* $\partial_\infty X$ of X is defined to be the set of equivalence classes of sequences converging to infinity.

Note that given a geodesic ray γ , the sequence $\{\gamma(n)\}$ converges to infinity and two equivalent rays induce equivalent sequences. Thus, in general, $\partial^g X \subseteq \partial_\infty X$.

We say that a metric space is *proper* if every closed ball is compact. Every uniform graph and every complete Riemannian manifold are proper geodesic metric spaces.

Proposition 4.3. [8, Chapter III.H, Proposition 3.1] *If X is a proper hyperbolic geodesic metric space, then the natural map from $\partial^g X$ to $\partial_\infty X$ is a bijection.*

For every $\xi, \xi' \in \partial_\infty X$, its Gromov product with respect to the base point $o \in X$ is defined as

$$(\xi|\xi')_o = \inf \liminf_{i \rightarrow \infty} (x_i|x'_i)_o,$$

where the infimum is taken over all sequences $\{x_i\} \in \xi$, $\{x'_i\} \in \xi'$.

A metric d on the sequential boundary at infinity $\partial_\infty X$ of X is said to be *visual*, if there are $o \in X$, $a > 1$ and positive constants c_1, c_2 , such that

$$c_1 a^{-(\xi|\xi')_o} \leq d(\xi, \xi') \leq c_2 a^{-(\xi|\xi')_o}$$

for all $\xi, \xi' \in \partial_\infty X$. In this case, we say that d is a visual metric with respect to the base point o and the parameter a .

Theorem 4.4. [11, Theorem 2.2.7] *Let X be a hyperbolic space. Then for any $o \in X$, there is $a_0 > 1$ such that for every $a \in (1, a_0]$ there exists a metric d on $\partial_\infty X$, which is visual with respect to o and a .*

Remark 4.5. *Notice that for any visual metric, $\partial_\infty X$ is bounded and complete.*

Definition 4.6. *Given a metric space (X, d) and a constant $S > 1$, (X, d) is S -uniformly perfect if there exists some $\varepsilon_0 > 0$ such that for every $x \in X$ and every $0 < \varepsilon \leq \varepsilon_0$ there exists a point $y \in X$ such that $\frac{\varepsilon}{S} < d(x, y) \leq \varepsilon$. (X, d) is uniformly perfect if there exists some S such that (X, d) is S -uniformly perfect.*

Theorem 4.7. [27, Theorem 4.15] *Given a hyperbolic uniform graph Γ with a pole, then Γ has LII if and only if $\partial_\infty \Gamma$ is uniformly perfect for some visual metric.*

Let X be a regular non-exceptional Riemann surface and consider some constants $0 < \delta < \varepsilon$ such that Theorem 3.7 holds. Then, for every collar $C(\gamma_j, h_j)$ of every simple closed geodesic γ_j with $L(\gamma_j) < 2\delta$, consider $\partial C(\gamma_j, h_j) = \eta_1^j \cup \eta_2^j$ (where η_i^j are Jordan curves) and let us define the (non-smooth) surface $[X]/\sim_\delta$ where $C(\gamma_j, h_j)$ is removed, η_1^j is glued along η_2^j by identifying the intersection in both curves with the same pencil geodesic, and the collars about the cusps $\{C(r_i, 2 \sinh \varepsilon)\}_{i \in I}$ are removed. Consider on $[X]/\sim_\delta$ the *inner distance* inherited from X :

$$d_{[X]/\sim_\delta}(x, y) = \inf \{L(\gamma) : \gamma \text{ is a curve in } [X]/\sim_\delta \text{ joining } x \text{ and } y\},$$

where L denotes the restriction to $[X]/\sim_\delta$ of the Poincaré length in X . One can check that the infimum in this definition is, in fact, a minimum, and that $[X]/\sim_\delta$ is a proper geodesic metric space.

Theorem 4.8. *Given a non-exceptional Riemannian surface X , $0 < \varepsilon < \text{Arcsinh } 1$ and $0 < \delta < \delta_1$, then $[X]/\sim_\delta$ and $\Gamma_\delta(X)$ are quasi-isometric.*

Proof. Let $\Gamma_\delta(X)/\sim_\delta$ be the graph defined by $\Gamma_\delta(X)$ where vertices $\{w_i\}_{i \in I}$ and $\{v_j\}_{j \in J_\delta}$ are removed and a new edge is defined between any pair of vertices adjacent to the same vertex v_j . Notice that the vertex set of $\Gamma_\delta(X)/\sim_\delta$ is $[A]_\delta$. Let us denote by $d_\Gamma(v, w)$ the path distance in $\Gamma_\delta(X)/\sim_\delta$ and by $d(v, w)$ the inner distance in $[X]/\sim_\delta$.

Since $\{N(w_i)\}_{i \in I} \cup \{N(v_j)\}_{j \in J_\delta}$ are pairwise disjoint sets by Remark 3.1, in order to prove that $\Gamma_\delta(X)/\sim_\delta$ is quasi-isometric to $\Gamma_\delta(X)$, it suffices to check that there is a constant $C > 0$ such that given any two vertices, u_1, u_2 , adjacent to some $v \in \{w_i\}_{i \in I}$ in $\Gamma_\delta(X)$, then $d_\Gamma(u_1, u_2) < C$.

Claim: Given any pair of points $v, w \in [A]_\delta$, $d_\Gamma(v, w) < \mu(\delta) \left(\frac{d(v, w)}{2\delta} + 2 \right)$, where $\mu(\delta)$ is the constant in Lemma 3.3. Consider any geodesic path γ in $[X]/\sim_\delta$ joining v to w and let $P_\gamma = \{u \in \Gamma_\delta(X)/\sim_\delta : B_\delta(v) \cap \gamma \neq \emptyset\}$. It is clear that $\{B_\delta(u) : u \in P_\gamma\}$ covers γ and $d_\Gamma(v, w) \leq |P_\gamma|$. Denote by $[t]$ the upper integer part of t , i.e., the smallest integer greater than or equal to t . Let $k = \lceil \frac{d(v, w)}{2\delta} \rceil$ and $v = x_0, \dots, x_k = w$ in γ such that $d(x_{i-1}, x_i) = \frac{d(v, w)}{k} \leq 2\delta$. Notice that for every vertex $u \in P_\gamma$, $u \in B_{2\delta}(x_i)$ for some $0 \leq i \leq k$. Hence, $d_\Gamma(v, w) \leq |P_\gamma| \leq \mu(\delta)(k+1) < \mu(\delta) \left(\frac{d(v, w)}{2\delta} + 2 \right)$.

If u_1, u_2 are adjacent to some $v \in \{w_i\}_{i \in I}$, it follows by Lemma 2.11 that $d(u_1, u_2) \leq 2\delta + \sinh \varepsilon$. Then, by the claim above $d_\Gamma(u_1, u_2) < \mu(\delta) \left(\frac{d(u_1, u_2)}{2\delta} + 2 \right) \leq \mu(\delta) \left(\frac{\sinh \varepsilon}{2\delta} + 3 \right)$ and $\Gamma_\delta(X)/\sim_\delta$ is quasi-isometric to $\Gamma_\delta(X)$.

Let us see now that $\Gamma_\delta(X)/\sim_\delta$ is quasi-isometric to $[X]/\sim_\delta$. Let $h: [A]_\delta \rightarrow [X]/\sim_\delta$ be the inclusion map. Now, consider any two vertices $v, w \in [A]_\delta$ and let $v = u_0, \dots, u_k = w$ be the shortest path in $\Gamma_\delta(X)/\sim_\delta$ joining them. For $1 \leq i \leq k$, if $u_{i-1}u_i$ is also an edge in $\Gamma_\delta(X)$, then $d(u_{i-1}, u_i) \leq 2\delta$. If $u_{i-1}u_i \notin E(\Gamma_\delta(X))$, then u_{i-1} and u_i are adjacent to the same vertex v_j in $\Gamma_\delta(X)$; therefore, since the length of each connected component of ∂K_j is at most $2 \sinh \varepsilon$ by Lemma 2.11, $d(u_{i-1}, u_i) \leq 2\delta + \sinh \varepsilon$. Thus, $d(u_{i-1}, u_i) \leq 2\delta + \sinh \varepsilon$ for every $1 \leq i \leq k$ and $d(v, w) \leq (2\delta + \sinh \varepsilon)k = (2\delta + \sinh \varepsilon)d_\Gamma(v, w)$.

By the claim above, $d_\Gamma(v, w) < \mu(\delta)\left(\frac{d(v, w)}{2\delta} + 2\right)$. Hence,

$$\frac{2\delta}{\mu(\delta)} d_\Gamma(v, w) - 4\delta \leq d(v, w) \leq (2\delta + \sinh \varepsilon) d_\Gamma(v, w)$$

and h is a $(\max\{\frac{\mu(\delta)}{2\delta}, 2\delta + \sinh \varepsilon\}, 4\delta)$ -quasi-isometric embedding. It is trivial to check that h is δ -full. \square

Theorem 4.9. [22, p.88] *If $f: X \rightarrow Y$ is a quasi-isometry between geodesic metric spaces, then X is hyperbolic if and only if Y is hyperbolic. Furthermore, if f is a c -full (a, b) -quasi-isometry and X (respectively, Y) is δ -hyperbolic, then Y (respectively, X) is δ' -hyperbolic, with δ' a universal constant which just depends on δ, a, b and c .*

Proposition 4.10. [27, Proposition 5.6] *Suppose X, Y are proper hyperbolic geodesic metric spaces and $f: X \rightarrow Y$ is a quasi-isometry. If X has a pole in v , then Y has a pole in $f(v)$.*

The following result follows from [11, Theorem 5.2.15] and [27, Proposition 5.10].

Theorem 4.11. *Let $f: X \rightarrow Y$ be a quasi-isometric map of proper hyperbolic geodesic metric spaces. Then $\partial_\infty X$ is uniformly perfect if and only if $\partial_\infty Y$ is uniformly perfect (with respect to any visual metrics).*

Theorem 4.12. *Let X be a $(2\delta_2)$ -regular non-exceptional Riemann surface, $0 < \varepsilon < \text{Arcsinh } 1$ and $0 < \delta < \min\{\delta_1, \delta_2\}$. Assume that $[X]/\sim_\delta$ is hyperbolic and has a pole. Then X has LII if and only if $\partial_\infty([X]/\sim_\delta)$ is uniformly perfect.*

Proof. Since $[X]/\sim_\delta$ is hyperbolic and has a pole, Theorems 4.8 and 4.9, and Proposition 4.10 give that $\Gamma_\delta(X)$ is hyperbolic and has a pole.

Theorems 4.8 and 4.11 give that $\partial_\infty([X]/\sim_\delta)$ is uniformly perfect if and only if $\partial_\infty \Gamma_\delta(X)$ is uniformly perfect. By Theorem 4.7, this holds if and only if $\Gamma_\delta(X)$ has LII. Finally, by Theorem 3.7, $\Gamma_\delta(X)$ has LII if and only if X has LII. \square

To determine if a geodesic metric space has a pole is not a difficult task. There are many results that allow to determine if a non-exceptional Riemann surface is hyperbolic (see, e.g., [33], [34], [35], [37], [38], [41]). Thus, Theorem 4.17 below is useful in order to apply Theorem 4.12, since it characterizes the hyperbolicity of $[X]/\sim_\delta$ in terms of the hyperbolicity of X .

In order to prove Theorem 4.17 we need some technical results. The following three propositions appear in [37, Theorems 2.1 and 2.4] and [34, Lemma 3.1].

Proposition 4.13. *Let X be a geodesic metric space, $\{K_n\}_n$ compact subsets of X and X' the quotient space obtained from X by identifying the points of each K_n in a single point k_n . Assume that there are positive constants c_1, c_2 , such that $\text{diam}_X K_n \leq c_1$ and $d_X(K_n, K_m) \geq c_2$ if $n \neq m$, and that X' is a geodesic metric space. Then X and X' are quasi-isometric, and X is hyperbolic if and only if X' is hyperbolic. Furthermore, if X (respectively, X') is δ -hyperbolic, then X' (respectively, X) is δ' -hyperbolic, with δ' a universal constant which just depends on δ, c_1 and c_2 .*

Let us consider a geodesic metric space X , a family of geodesic metric subspaces $\{X_n\}_n \subset X$ such that $\cup_n X_n = X$, $\eta_{nm} := \eta_{mn} := X_n \cap X_m$ are compact sets, and positive constants c_1, c_2 . We say that $\{X_n\}_n$ is a (c_1, c_2) -decomposition of X if $X \setminus \eta_{nm}$ is not connected for each non-empty set η_{nm} , $\text{diam}_{X_n}(\eta_{nm}) \leq c_1$ for every n, m , and $d_{X_n}(\eta_{nm}, \eta_{nk}) \geq c_2$ for every n and $m \neq k$ with non-empty sets η_{nm} and η_{nk} .

Proposition 4.14. *Let X be a geodesic metric space and $\{X_n\}_n \subset X$ a family of geodesic metric spaces which is a (c_1, c_2) -decomposition of X . Then X is hyperbolic if and only if there exists a constant c_3 such that X_n is c_3 -hyperbolic for every n . Furthermore, if there exists such a constant c_3 , then X is δ -hyperbolic with δ a universal constant which only depends on c_1, c_2 and c_3 ; if X is δ -hyperbolic, then c_3 is a universal constant which only depends on c_1, c_2 and δ .*

Let X be a non-exceptional Riemann surface. For each choice of doubly connected domains $\{V_j\}_j$ in X we define

$$D_X(\{V_j\}_j) := \sup_j \{d_{V_j}(\eta_1^j, \eta_2^j) : \eta_1^j, \eta_2^j \text{ are the connected components of } \partial V_j$$

$$\text{and } X \setminus \eta_1^j \text{ is connected}\}.$$

Proposition 4.15. *Let X be a non-exceptional Riemann surface, $\{V_j\}_j$ doubly connected domains in X , and r, s positive constants with $L(\partial V_j) \leq r$ for every j , and $d(V_j, V_k) \geq s$ for every $j \neq k$. If $D_X(\{V_j\}_j) = \infty$, then X is not hyperbolic.*

Lemma 4.16. *Let X be a non-exceptional Riemann surface and $0 < \delta < \varepsilon < \text{Arccsinh } 1$. Then there exists a universal constant Δ such that C_i and K_j are Δ -hyperbolic for every $i \in I$ and $j \in J_\delta$.*

Proof. Given $i \in I$, let us consider a geodesic ray g_i in C_i joining ∂C_i with the cusp r_i , and the inclusion $h : g_i \rightarrow C_i$. Given $j \in J_\delta$, let us consider a geodesic γ_j in K_j joining the two connected components of ∂K_j , and the inclusion $h : \gamma_j \rightarrow K_j$. If η is a connected component of ∂C_i or ∂K_j , then Lemma 2.11 gives that $L(\eta) < 2$. Thus, in both cases h is a 1-full $(1, 0)$ -quasi-isometry and, since g_i and γ_j are 0-hyperbolic, Theorem 4.9 gives the result. \square

For a non-exceptional Riemann surface X , let us define

$$J_\delta^c = \{j \in J_\delta : X \setminus \gamma_j \text{ is connected}\}, \quad \Lambda(X) = \inf \{L(\gamma_j) : j \in J_\delta^c\}.$$

Theorem 4.17. *Let X be a non-exceptional Riemann surface, $0 < \varepsilon < \text{Arccsinh } 1$ and $0 < \delta < \delta_1$. Then X is hyperbolic if and only if $[X]/\sim_\delta$ is hyperbolic and $\Lambda(X) > 0$.*

Proof. Lemma 2.11 gives that if η is a connected component of $(\cup_{i \in I} \partial C_i) \cup (\cup_{j \in J_\delta} \partial K_j)$, then $L(\eta) < 2$, and so, $\text{diam } \eta \leq \frac{1}{2}L(\eta) < 1$. For each collar $K_j = C(\gamma_j, h_j)$ of a simple closed geodesic γ_j with $j \in J_\delta$, consider $\partial K_j = \eta_1^j \cup \eta_2^j$ (where η_i^j are Jordan curves). We have, for $j \in J_\delta$,

$$d_{K_j}(\eta_1^j, \eta_2^j) = 2h_j = 2 \text{Arccosh} \frac{\sinh \varepsilon}{\sinh(L(\gamma_j)/2)} > 2 \text{Arccosh} \frac{\sinh \varepsilon}{\sinh \delta} > 0,$$

since $0 < \delta < \delta_1 < \varepsilon$.

Assume first that $\Lambda(X) = 0$. Since $\inf_{j \in J_\delta^c} L(\gamma_j) = 0$, we have

$$D_X(\{K_j\}_{j \in J_\delta^c}) = \sup_{j \in J_\delta^c} d_{K_j}(\eta_1^j, \eta_2^j) = \sup_{j \in J_\delta^c} 2h_j = \sup_{j \in J_\delta^c} 2 \text{Arccosh} \frac{\sinh \varepsilon}{\sinh(L(\gamma_j)/2)} = \infty,$$

and we conclude, by Proposition 4.15, that X is not hyperbolic.

Consequently, it suffices to prove that if $\Lambda(X) > 0$, then X is hyperbolic if and only if $[X]/\sim_\delta$ is hyperbolic. Denote by $\{X_r\}_r$ the connected components of $X \setminus ((\cup_{i \in I} \partial C_i) \cup (\cup_{j \in J_\delta \setminus J_\delta^c} \partial K_j))$.

Lemma 2.14 gives that if $\sigma_1, \sigma_2 \subset \partial X_r$ are connected components of $(\cup_{i \in I} \partial C_i) \cup (\cup_{j \in J_\delta \setminus J_\delta^c} \partial K_j)$ and they belong to the closure of different collars, then $d_{X_r}(\sigma_1, \sigma_2) \geq d_X(\sigma_1, \sigma_2) \geq 2 \log \frac{1}{\sinh \varepsilon}$.

Therefore, if we define $c_2 = \min\{2 \text{Arccosh} \frac{\sinh \varepsilon}{\sinh \delta}, 2 \log \frac{1}{\sinh \varepsilon}\}$, we have that $\{X_r\}_r, \{C_i\}_{i \in I}, \{K_j\}_{j \in J_\delta \setminus J_\delta^c}$ is a $(1, c_2)$ -decomposition of X . By Proposition 4.14 and Lemma 4.16, X is hyperbolic if and only if there exists a constant c_3 such that X_r is c_3 -hyperbolic for every r .

For each r , let X'_r be the geodesic metric space obtained from X_r by identifying the points in the collar K_j in a single point x_j for every $j \in J_\delta^c$ with $K_j \subset X_r$. Let us define the (non-smooth) surface Y_r obtained from X_r by removing the collar K_j and by gluing η_1^j and η_2^j for every $j \in J_\delta^c$ with $K_j \subset X_r$; consider on Y_r

the inner metric inherited from X_r . Let Y'_r be the geodesic metric space obtained from Y_r by identifying the points in ∂K_j in a single point x_j for every $j \in J_\delta^\varepsilon$ with $K_j \subset X_r$.

Since $\Lambda(X) > 0$, we have

$$D_X(\{K_j\}_{j \in J_\delta^\varepsilon}) = \sup_{j \in J_\delta^\varepsilon} 2h_j = \sup_{j \in J_\delta^\varepsilon} 2 \operatorname{Arccosh} \frac{\sinh \varepsilon}{\sinh(L(\gamma_j)/2)} = 2 \operatorname{Arccosh} \frac{\sinh \varepsilon}{\sinh(\Lambda(X)/2)} < \infty$$

and so,

$$\sup_{j \in J_\delta^\varepsilon} \operatorname{diam} K_j \leq \frac{1}{2} L(\eta_1^j) + \sup_{j \in J_\delta^\varepsilon} 2h_j + \frac{1}{2} L(\eta_2^j) \leq 2 \operatorname{Arccosh} \frac{\sinh \varepsilon}{\sinh(\Lambda(X)/2)} + 2 < \infty.$$

This fact and Proposition 4.13 give that X_r is hyperbolic if and only if X'_r is hyperbolic, and Y_r is hyperbolic if and only if Y'_r is hyperbolic, with uniform hyperbolicity constants. Since $X'_r = Y'_r$, we conclude that there exists a constant c_3 such that X_r is c_3 -hyperbolic for every r if and only if there exists a constant c'_3 such that Y_r is c'_3 -hyperbolic for every r .

Since $\{Y_r\}_r$ is a $(1, c_2)$ -decomposition of $[X]/\sim_\delta$, Proposition 4.14 gives that $[X]/\sim_\delta$ is hyperbolic if and only if there exists a constant c'_3 such that Y_r is c'_3 -hyperbolic for every r .

Hence, X is hyperbolic if and only if $[X]/\sim_\delta$, and the conclusion holds. \square

Our last result shows that the hypotheses on Theorem 4.12 do not depend on δ .

Proposition 4.18. *Let X be a non-exceptional Riemann surface, $0 < \varepsilon < \operatorname{Arcsinh} 1$ and $0 < \delta, \hat{\delta} < \delta_1$. The following statements hold:*

- (1) $[X]/\sim_{\hat{\delta}}$ is hyperbolic if and only if $[X]/\sim_\delta$ is hyperbolic.
- (2) $[X]/\sim_{\hat{\delta}}$ has a pole if and only if $[X]/\sim_\delta$ has a pole.

Proof. By Theorem 4.9 and Proposition 4.10, it suffices to prove that $[X]/\sim_{\hat{\delta}}$ and $[X]/\sim_\delta$ are quasi-isometric.

Without loss of generality we can assume that $\delta < \hat{\delta}$. Thus, $J_\delta \subseteq J_{\hat{\delta}}$.

Let $([X]/\sim_\delta)'$ be the geodesic metric space obtained from $[X]/\sim_\delta$ by identifying the points in the closure of the collar K_j in a single point x_j for every $j \in J_{\hat{\delta}} \setminus J_\delta$, and $([X]/\sim_{\hat{\delta}})'$ the geodesic metric space obtained from $[X]/\sim_{\hat{\delta}}$ by identifying the points in $\eta_1^j \cong \eta_2^j$ in a single point x_j for every $j \in J_{\hat{\delta}} \setminus J_\delta$. Thus, $([X]/\sim_{\hat{\delta}})' = ([X]/\sim_\delta)'$.

Lemma 2.14 gives $d_{[X]/\sim_\delta}(K_{j_1}, K_{j_2}) \geq 2 \log \frac{1}{\sinh \varepsilon}$ for every $j_1, j_2 \in J_{\hat{\delta}} \setminus J_\delta$. Also, Lemmas 2.10 and 2.11 give

$$\begin{aligned} \operatorname{diam}_{[X]/\sim_\delta} K_j &\leq \frac{1}{2} L(\eta_1^j) + 2h_j + \frac{1}{2} L(\eta_2^j) \leq 2 \operatorname{Arccosh} \frac{\sinh \varepsilon}{\sinh(L(\gamma_j)/2)} + 2, \\ &\leq 2 \operatorname{Arccosh} \frac{\sinh \varepsilon}{\sinh \delta} + 2, \end{aligned}$$

for every $j \in J_{\hat{\delta}} \setminus J_\delta$.

These inequalities and Proposition 4.13 give that $[X]/\sim_\delta$ and $([X]/\sim_\delta)'$ are quasi-isometric, and $[X]/\sim_{\hat{\delta}}$ and $([X]/\sim_{\hat{\delta}})'$ are quasi-isometric. Since $([X]/\sim_{\hat{\delta}})' = ([X]/\sim_\delta)'$, we conclude that $[X]/\sim_{\hat{\delta}}$ and $[X]/\sim_\delta$ are quasi-isometric. \square

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