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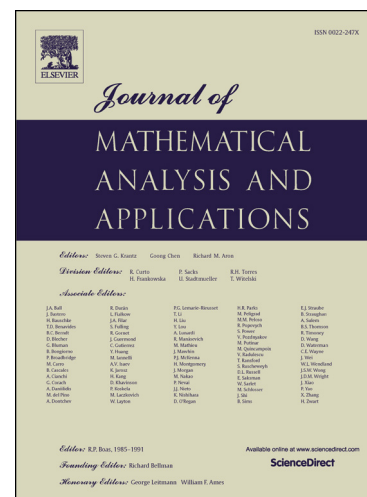
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A Bochner type characterization theorem for exceptional orthogonal polynomials

M^a Ángeles García-Ferrero

*Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), C/ Nicolas Cabrera 15,
28049 Madrid, Spain.*

David Gómez-Ullate

*Escuela Superior de Ingeniería, Universidad de Cádiz, Avda. Universidad de Cádiz,
Campus Universitario de Puerto Real, 11519, Spain.*

*Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), C/ Nicolas Cabrera 15,
28049 Madrid, Spain.*

*Departamento de Física Teórica, Universidad Complutense de Madrid, 28040 Madrid,
Spain.*

Robert Milson

*Department of Mathematics and Statistics, Dalhousie University, Halifax, NS, B3H 3J5,
Canada.*

Abstract

It was recently conjectured that every system of exceptional orthogonal polynomials is related to a classical orthogonal polynomial system by a sequence of Darboux transformations. In this paper we prove this conjecture, which paves the road to a complete classification of all exceptional orthogonal polynomials. In some sense, this paper can be regarded as the extension of Bochner's result for classical orthogonal polynomials to the exceptional class. As a supplementary result, we derive a canonical form for exceptional operators based on a bilinear formalism, and prove that every exceptional operator has trivial monodromy at all primary poles.

Keywords: Orthogonal polynomials, Darboux transformations, Sturm-Liouville problems, trivial monodromy.

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Email address: mag.ferrero@icmat.es, david.gomez-ullate@icmat.es, rmilson@dal.ca (Robert Milson)

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1. Introduction

Exceptional orthogonal polynomials are complete systems of orthogonal polynomials that satisfy a Sturm-Liouville problem. They differ from the classical families of Hermite, Laguerre and Jacobi in that there are a finite number of exceptional degrees for which no polynomial eigenfunction exists. The total number of gaps in the degree sequence is the *codimension* of the exceptional family. As opposed to their classical counterparts [47, 76], the differential equation contains rational instead of polynomial coefficients, yet the eigenvalue problem has an infinite number of polynomial eigenfunctions that form the basis of a weighted Hilbert space. Because of the missing degrees, exceptional polynomials circumvent the strong limitations of Bochner's classification theorem, which characterizes classical Sturm-Liouville orthogonal polynomial systems [3, 50].

The recent development of exceptional polynomial systems has received contributions both from the mathematics community working on orthogonal polynomials and special functions, and from mathematical physicists. Among the physical applications, exceptional polynomial systems appear mostly as solutions to exactly solvable quantum mechanical problems, describing both bound states [16, 29, 33, 34, 51, 61, 69, 71, 75] and scattering amplitudes [41, 80–82]. But there are also connections with super-integrability [55, 68] and higher order symmetry algebras [56–58], diffusion equations and random processes [6, 40, 44], quantum information entropy [17], exact solutions to Dirac equation [74] and finite-gap potentials [38].

Some examples of exceptional polynomials were investigated back in the early 90s, [9] but their systematic study started a few years ago, where a full classification was given for codimension one, [23, 25]. Soon after that, Quesne recognized the role of Darboux transformations in the construction process and wrote the first codimension two examples, [70], and Odake & Sasaki showed families for arbitrary codimension, [61, 62]. The role of Darboux transformations

was further clarified in a number of works, [24, 69, 73], and the next conceptual step involved the generation of exceptional families by multiple-step or higher order Darboux transformations, leading to exceptional families labelled by multi-indices, [27, 35, 63]. Other equivalent approaches to build exceptional polynomial systems have been developed in the physics literature, using the prepotential approach [39] or the symmetry group preserving the form of the Rayleigh-Schrödinger equation [36], leading to rational extensions of the well known solvable potentials.

In the mathematical literature, two main questions have centred the research activity in relation to exceptional polynomial systems: describing their mathematical properties and achieving a complete classification. Among the mathematical properties, the study of their zeros deserve particular attention. Zeros of exceptional polynomials are classified into two classes: *regular zeros* which lie in the interval of orthogonality and *exceptional zeros*, which lie outside this interval. Their interlacing, asymptotic behaviour, monotonicity as a function of parameters and electrostatic interpretation have been investigated in a number of works, [8, 28, 42, 45, 49], but there are still open problems in this direction.

A fundamental object in the theory of orthogonal polynomials is the recurrence relation. Classical orthogonal polynomials have a three term recurrence relation, but exceptional polynomial systems have recurrence relations whose order is higher than three. There is a set of recurrence relations of order $2N + 3$ where N is the number of Darboux steps [29, 64] with coefficients that are functions of x and n , and another set of recurrence relations whose coefficients are just functions of n (as in the classical case) and whose order is $2m + 3$ where m is the codimension, [14, 59, 66]. While the former relations are generally of lower order and thus more convenient for an efficient computation, the latter are more amenable to a theoretical interpretation in terms of the usual theory of Jacobi matrices and bispectrality. The spectral theoretic aspects of exceptional differential operators were first addressed in [18, 19] and developed more recently in a series of papers [52–54].

The quest for a complete classification of exceptional polynomials has been fundamental problem that is now close to being solved, and the results in the present paper are a key step towards this goal. The first attempts to classify exceptional polynomial systems proceeded by increasing codimension. Codimension one systems were classified in [23] and they included just one X_1 -Laguerre and one X_1 -Jacobi family. The classification for codimension two was performed in [26], based on an exhaustive case-by-case enumeration of invariant flags under a given symmetry group. Due to the combinatorial growth of complexity with increasing codimension, this original approach proved to be unfeasible for the purpose of achieving a complete classification. However, a fundamental idea towards the full classification was also launched in [26], namely that every exceptional polynomial system can be obtained from a classical system by applying a finite number of Darboux transformations. More precisely, the following conjecture was formulated:

Conjecture 1.1. [Gómez-Ullate, Kamran, Milson 2012] *Every exceptional orthogonal polynomial system of codimension m can be obtained by applying a sequence of at most m Darboux transformations to a classical orthogonal polynomial system.*

If the conjecture holds, then the program to classify exceptional polynomial systems becomes constructive: start from the three classical systems of Hermite, Laguerre and Jacobi and apply all possible Darboux transformations to describe the entire exceptional class. It should be stressed that only rational Darboux transformations need to be considered, i.e. those that map polynomial eigenfunctions into polynomial eigenfunctions, and this type of transformations are well understood and catalogued, and they are indexed by sequences of integers. This constructive approach has already been used to generate large classes of exceptional polynomial systems. The most general class obtained in this way can be labelled by two sets of indices or partitions (for the Laguerre and Jacobi classes) [4, 11, 21] or just one set (for the Hermite class) [20, 29] which can be conveniently represented in a Maya diagram [77], a representation that takes naturally into account a number of equivalent sets of indices that lead to the same exceptional system, [32, 65]. However, the question of whether this list contains all exceptional polynomials remained open.

In all examples known so far, the weight for the exceptional system $W(z)$ is a rational modification of a classical weight $W_0(z)$ having the following form:

$$W(z) = \frac{W_0(z)}{\eta(z)^2}, \quad (1)$$

where $\eta(z)$ is a polynomial in z (a Wronskian-like determinant) whose degree coincides with the codimension of the system. In this paper we prove that this is indeed the case for any possible exceptional polynomial system.

One important point remains, namely that of ensuring that the transformed weight gives rise to a well defined spectral problem, which we shall refer to as the *weight regularity* problem. This means studying the sequence of Darboux transformations and the range of parameters for which:

- i) the weight has the right asymptotic behaviour at the endpoints
- ii) $\eta(z)$ has no zeros inside the interval of orthogonality.

The regularity problem has been solved for the exceptional Hermite class [12, 29] based on results by Krein [48], and Adler [1], and also for the Laguerre class, [15], using a remarkable correspondence between exceptional polynomials and discrete Krall type polynomials, [11].

The main result of this paper is the following theorem, which is essentially a proof of Conjecture 1.1, albeit without a bound on the number of Darboux steps.

Theorem 1.2. *Every exceptional orthogonal polynomial system can be obtained by applying a finite sequence of Darboux transformations to a classical orthogonal polynomial system.*

The essential consequence of this result is that it places on safe ground the constructive approach to the full classification described above. The strategy of the proof involves several steps.

First, we establish a number of factorization results for second-order differential operators with rational coefficients. In particular, in Section 3 we show that every higher-order intertwiner can be factorized into a composition of first order operators, each of them corresponding to a one-step, rational Darboux transformation.

We introduce exceptional operators in Section 4, and prove a fundamental theorem that relates the codimension to the sum of certain integer indices at the poles of the operator. Next, in section 5 we prove that every exceptional operator admits a canonical formulation as a bilinear relation between two polynomials. The key technical tools are some results on the local behaviour of solutions around the singular points of the differential equations corresponding to exceptional operators. A further key step is the demonstration that an exceptional operator has trivial monodromy at almost every point $\zeta \in \mathbb{C}$. This result was already known for the exceptional Hermite class [29, 60], and we show that it can be extended to a general exceptional operator. The connection between trivial monodromy, bispectrality, Darboux transformations and the solvable character of Schrödinger operators has been discussed in a number of papers (see for instance [10, 22, 37, 60, 78, 79] and the references therein), and the results in this paper are one further piece of evidence of the close relationship among these concepts.

In Section 6 we build on the structural properties of exceptional operators to prove the existence of a higher order intertwiner between any exceptional operator and a classical operator, extending the proof given by Oblomkov [60] for the rational extensions of the harmonic oscillator.

Finally the proof of Theorem 1.2 is given in Section 7 making use of all the previous results. This section also contains Theorem 7.5 which states that the orthogonality weight for any exceptional polynomial system has the form (1).

2. Preliminaries

In this preliminary section we introduce some key definitions and notation, and prove some essential results about second-order differential operators with rational coefficients. Let $\mathcal{Q} = \mathbb{C}(z)$ denote the differential ring of univariate, complex-valued rational functions and $\mathcal{P} = \mathbb{C}[z]$ the subring of polynomials. Let $\mathcal{P}_n \subset \mathcal{P}$, $n \in \mathbb{N}$ denote the vector space of polynomials of degree $\leq n$, and $\mathcal{P}_n^* \subset \mathcal{P}_n$ the subset of polynomials whose degree is exactly equal to n . Similarly, let \mathcal{Q}_n denote the vector space of rational functions having degree $\leq n$, where the degree of a rational function is defined to be the difference of the degrees of the numerator and denominator.

Let $\text{Diff}(\mathcal{Q}) = \mathbb{C}(z)[D_z]$ denote the ring of linear differential operators with rational coefficients and $\text{Diff}(\mathcal{P}) = \mathbb{C}[z, D_z]$ the subring of operators with polynomial coefficients. Alternatively, $\text{Diff}(\mathcal{P})$ may be characterized as the subring

of $\text{Diff}(\mathcal{Q})$ that preserves \mathcal{P} . When needed, will use $\mathbb{R}\mathcal{Q}, \mathbb{R}\mathcal{P}, \mathbb{R}\mathcal{P}_n$ to denote the corresponding real-valued subrings and subspaces, and $\text{Diff}(\mathbb{R}\mathcal{Q}), \text{Diff}(\mathbb{R}\mathcal{P})$ the corresponding rings of real-valued differential operators.

For a sufficiently differentiable function y , we let $D_z^j y = y^{(j)}(z)$ denote the j^{th} derivative of $y(z)$ with respect to z . The notation $D_{zz} = D_z^2$ will also be employed. Let $\text{Diff}_\rho(\mathcal{Q})$ denote the set of ρ^{th} order differential operators; that is, operators of the form

$$L = \sum_{j=0}^{\rho} a_j(z) D_z^j, \quad a_j \in \mathcal{Q}, \quad a_\rho \neq 0, \quad (2)$$

with action

$$y \mapsto L[y] = \sum_{j=0}^{\rho} a_j(z) y^{(j)}(z), \quad y \in \mathcal{Q}. \quad (3)$$

Definition 2.1. We say that a function $\phi(z)$ is quasi-rational if its log-derivative

$$D_z \left[\log \phi(z) \right] = \frac{\phi'(z)}{\phi(z)}$$

is a rational function of z .

For $T \in \text{Diff}_2(\mathcal{Q})$, write

$$T = p(z)D_{zz} + q(z)D_z + r(z), \quad p, q, r \in \mathcal{Q} \quad (4)$$

and define the quasi-rational functions

$$P(z) = \exp \left(\int^z \frac{q(x)}{p(x)} dx \right), \quad (5a)$$

$$W(z) = \frac{P(z)}{p(z)}, \quad (5b)$$

$$R(z) = r(z)W(z). \quad (5c)$$

Multiplying the eigenvalue relation $T[y] = \lambda y$ by $W(z)$ gives an equivalent form as Sturm-Liouville type equation

$$(Py')' + Ry = \lambda Wy. \quad (6)$$

Proposition 2.2. *The operator T is formally symmetric with respect to W in the sense that*

$$\int^z T[f](x)g(x)W(x)dx - \int^z T[g](x)f(x)W(x)dx = P(z)(f'(z)g(z) - f(z)g'(z)), \quad (7)$$

where f, g are sufficiently differentiable functions.

Proof. This follows by (6) and integration by parts. \square

Definition 2.3. We say that two rational operators $T, \hat{T} \in \text{Diff}_2(\mathcal{Q})$ are gauge-equivalent if there exists a $\sigma \in \mathcal{Q}$ such that

$$\hat{T} = \sigma T \sigma^{-1}. \quad (8)$$

We will refer to σ as the *gauge-factor*.

Remark 2.4. Above we are using σ to denote both a rational function, and the multiplication operator $y \mapsto \sigma y$. The reason for the gauge-factor terminology is that the eigenvalue relation $T[y] = \lambda y$ is equivalent to the eigenvalue relation $\hat{T}[\hat{y}] = \lambda \hat{y}$, with $\hat{y} = \sigma y$.

Proposition 2.5. Suppose that $T, \hat{T} \in \text{Diff}_2(\mathcal{Q})$ satisfy (8). Letting $p, q, r, \hat{p}, \hat{q}, \hat{r}$ be the coefficients of T and \hat{T} as per (4), and W, \hat{W} the corresponding weights (5b) we have the following transformation laws

$$p = \hat{p} \quad (9a)$$

$$q = \hat{q} + \frac{2\sigma'}{\sigma} \hat{p} \quad (9b)$$

$$r = \hat{r} + \frac{\sigma'}{\sigma} \hat{q} + \frac{\sigma''}{\sigma} \hat{p} \quad (9c)$$

$$W = \sigma^2 \hat{W}. \quad (9d)$$

3. Rational Darboux transformations

The gauge-equivalence relation (8) is an intertwining relation of second-order operators by a zero-order multiplication operator. Consideration of higher-order intertwining relations leads naturally to the notion of a Darboux transformation.

Definition 3.1. For $T \in \text{Diff}_2(\mathcal{Q})$ a *rational factorization* is a relation of the form

$$T = BA + \lambda_0, \quad (10)$$

where $A, B \in \text{Diff}_1(\mathcal{Q})$ and $\lambda_0 \in \mathbb{C}$ is a constant. Given a rational factorization, we call the operator $\hat{T} \in \text{Diff}_2(\mathcal{Q})$ defined by

$$\hat{T} := AB + \lambda_0 \quad (11)$$

the *partner operator* and say that $T \mapsto \hat{T}$ is a *rational Darboux transformation*.

Proposition 3.2. Suppose that $T, \hat{T} \in \text{Diff}_2(\mathcal{Q})$ are related by a rational Darboux transformation. Then, the following intertwining relations hold

$$AT = \hat{T}A, \quad TB = B\hat{T}. \quad (12)$$

Proof. This is a direct consequence of (10) and (11). \square

Remark 3.3. The intertwining relation (12) implies that the eigenvalue relation $T[y] = \lambda y$ is formally equivalent to the eigenvalue relation $\hat{T}[\hat{y}] = \lambda \hat{y}$ where $\hat{y} = A[y]$.

Definition 3.4. For $T \in \text{Diff}_2(\mathcal{Q})$ and $\phi(z)$ quasi-rational, we will say that ϕ is a quasi-rational eigenfunction of T if

$$T[\phi] = \lambda_0 \phi, \quad \lambda_0 \in \mathbb{C}. \quad (13)$$

We observe that to every quasi-rational eigenfunction ϕ of T there corresponds a rational factorization, as shown by the following proposition.

Proposition 3.5. *Suppose that the eigenvalue relation (13) holds for a second-order $T \in \text{Diff}_2(\mathcal{Q})$ and a quasi-rational $\phi(z)$. Then, T admits a rational factorization relation (10) with $A, B \in \text{Diff}_1(\mathcal{Q})$ given by*

$$A = b(z)(D_z - w(z)), \quad B = \hat{b}(z)(D_z - \hat{w}(z)), \quad (14)$$

where

$$w = \frac{\phi'}{\phi}, \quad \hat{b} = \frac{p}{b}, \quad \hat{w} = -w - \frac{q}{p} + \frac{b'}{b}, \quad (15)$$

and where $b(z)$ is an arbitrary non-zero rational function. Moreover, (13) also implies the Riccati-type relation

$$p(w' + w^2) + qw + r = \lambda_0. \quad (16)$$

Conversely, suppose that a $T \in \text{Diff}_2(\mathcal{Q})$ admits a first-order rational factorization (10). Then, there exists a quasi-rational $\phi(z)$ such that (13) holds. Moreover, if w, \hat{w}, b, \hat{b} are defined as per (14), then (15) and (16) hold also.

Proof. Suppose that (13) holds. Hence,

$$\frac{p\phi''}{\phi} + \frac{q\phi'}{\phi} + r = \lambda_0.$$

The Riccati relation (16) follows immediately. Defining A, B as per (14) (15) and using (16), we have

$$\begin{aligned} (BA)[y] &= B[by' - bwy] \\ &= \hat{b}by'' + (\hat{b}b' - \hat{b}bw - b\hat{b}\hat{w})y' + (w\hat{w}\hat{b}\hat{b} - \hat{b}(bw)')y \\ &= py'' + \left(\frac{pb'}{b} + p\left(\frac{q}{p} - \frac{b'}{b}\right)\right)y' + \left(pw\left(-w - \frac{q}{p} + \frac{b'}{b}\right) - pw\frac{b'}{b} - pw'\right)y \\ &= py'' + qy' + (r - \lambda_0)y, \end{aligned}$$

which is a restatement of (10).

Conversely, suppose that a rational factorization (10) holds. Let b, \hat{b} be the coefficients of the first-order terms of A, B , and let w, \hat{w} be the ratios of the zeroth and first-order coefficients, as per (14). Define the quasi-rational function

$$\phi(z) = \exp\left(\int^z w(x)dx\right)$$

so that $w = \phi'/\phi$. By construction,

$$A[\phi] = 0,$$

which implies that

$$T[\phi] = (BA + \lambda_0)[\phi] = \lambda_0\phi;$$

i.e., (13) holds. Expanding BA , as above, shows that

$$p = \hat{b}b, \quad q = \hat{b}b' - \hat{b}b(w + \hat{w}), \quad r - \lambda_0 = w\hat{w}b\hat{b} - \hat{b}(bw)'.$$

From this (15) and (16) follow immediately. \square

The next proposition expresses the transformation law for the coefficients of a differential operator $T \in \text{Diff}_2(\mathcal{Q})$ under a rational Darboux transformation specified by the rational functions ϕ and b .

Proposition 3.6. *Suppose that $T, \hat{T} \in \text{Diff}_2(\mathcal{Q})$ are related by a rational Darboux transformation. Then, the coefficients of T and \hat{T} and the quasi-rational weights $W(z), \hat{W}(z)$, as defined by (5b), are related by*

$$\hat{p} = p \tag{17a}$$

$$\hat{q} = q + p' - \frac{2b'}{b}p, \tag{17b}$$

$$\hat{r} = r + q' + wp' - \frac{b'}{b}(q + p') + \left(2\left(\frac{b'}{b}\right)^2 - \frac{b''}{b} + 2w'\right)p \tag{17c}$$

$$\hat{W} = \frac{p}{b^2}W, \tag{17d}$$

where b and $w = (\log \phi)'$ are the rational functions defined in Proposition 3.5.

Proof. By (10)- (14), $p(z)$ is the second-order coefficient of both T, \hat{T} . Let $\hat{q}(z) \in \mathcal{Q}$ be the first-order coefficient of \hat{T} . Relation (17b) follows by (15) (14) and (11). Applying (5) and using (17b) gives (17d). Considering the hatted dual of (16) and applying (15) and (17b) gives

$$\begin{aligned} \hat{r} &= \lambda_0 - p(\hat{w}' + \hat{w}^2) - \hat{q}\hat{w} \\ &= \lambda_0 - p\left(\left(-w - \frac{q}{p} + \frac{b'}{b}\right)' + \left(-w - \frac{q}{p} + \frac{b'}{b}\right)^2\right) - \left(q + p' - \frac{2pb'}{b}\right)\left(-w - \frac{q}{p} + \frac{b'}{b}\right), \end{aligned}$$

which simplifies to the expression shown in (17c). \square

Next, we consider iterated rational Darboux transformations. In the context of Schrödinger operators, these are known as higher-order Darboux or Darboux-Crum transformations. [7].

Definition 3.7. Let $\hat{T}, T \in \text{Diff}_2(\mathcal{Q})$ be second-order operators with rational coefficients. We will say that \hat{T} is *Darboux connected* to T if there exists an operator $L \in \text{Diff}(\mathcal{Q})$ such that

$$\hat{T}L = LT. \tag{18}$$

Remark 3.8. Note that in the above definition the operator L could have any order, and that gauge-equivalent operators (8) are Darboux connected by definition, because they are related by a zero-th order intertwining relation.

Rational Darboux transformations can also be iterated, a concept that leads to the following definition.

Definition 3.9. We will say that $\hat{T}, T \in \text{Diff}_2(\mathcal{Q})$ are connected by a factorization chain if there exist second-order operators $T_i \in \text{Diff}_2(\mathcal{Q})$, $i = 0, 1, \dots, n$ with $T_0 = T$ and $T_n = \hat{T}$; first-order operators $A_i, B_i \in \text{Diff}_1(\mathcal{Q})$, $i = 0, 1, \dots, n-1$, and constants λ_i such that

$$T_i = B_i A_i + \lambda_i, \quad i = 0, 1, \dots, n-1 \quad (19)$$

$$T_{i+1} = A_i B_i + \lambda_i. \quad (20)$$

It is trivial to show that two operators connected by a factorization chain are also Darboux connected. The converse is also true [60][Theorem 1]. The just cited paper limits itself to the case of operators in Schrödinger form, but we state and prove the generalization for second-order operators with rational coefficients using essentially the same argument.

Theorem 3.10. *Two rational operators $T, \hat{T} \in \text{Diff}_2(\mathcal{Q})$ are Darboux connected if and only if they are either gauge-equivalent, or they are connected by a factorization chain.*

Proof. Suppose that T and \hat{T} are connected by a factorization chain. By assumption,

$$T_{i+1} A_i = A_i B_i A_i + \lambda_i A_i = A_i T_i, \quad i = 0, 1, \dots, n-1.$$

It follows by induction that

$$T_{i+1} A_i \cdots A_0 = A_i \cdots A_0 T_0.$$

Therefore, (18) is satisfied with

$$L = A_{n-1} \cdots A_1 \cdot A_0.$$

The proof of the converse is a modification of an argument given in [60]. If $\text{ord } L = 0$, then T and \hat{T} are gauge-equivalent. Thus, suppose that (18) holds and that $\text{ord } L \geq 1$.

Claim 1: the intertwining relation (18) implies that there exists a reduced operator \tilde{L} such that

$$\hat{T} \tilde{L} = \tilde{L} T \quad (21)$$

also and such that there does not exist a $\lambda \in \mathbb{C}$ such that \tilde{L} has a right factor of the form $T - \lambda$. To prove this, suppose that L has a right-factor $T - \lambda$ for some $\lambda \in \mathbb{C}$. Write

$$L = \tilde{L}(T - \lambda)^k,$$

with $k \in \mathbb{N}$ as large as possible. Since T commutes with $T - \lambda$, it follows that

$$\hat{T}\tilde{L}(T - \lambda)^k = \tilde{L}T(T - \lambda)^k.$$

The ring of differential expressions $\text{Diff } \mathcal{Q}$ has the right-cancellation property, meaning that $AX = BX$ implies that $A = B$. Hence, the right factor $(T - \lambda)^k$ may be removed from the above equation; this gives (21). Repeating this argument a finite number of times for all values of λ for which $T - \lambda$ is a right factor of L yields an intertwiner with the desired property. Therefore, no generality will be lost if we assume that L has been reduced so as to not have such right factors.

Claim 2: T leaves $\ker L$ invariant. By relation (18), if $y \in \ker L$, then

$$L[T[y]] = \hat{T}[L[y]] = 0,$$

so $T[y] \in \ker L$ also.

Claim 3: if $T[y] = \lambda y$, then

$$L[y] = F(z, \lambda)y + G(z, \lambda)y', \quad (22)$$

where F, G are polynomial in λ and rational in z . By assumption,

$$T[y] = p(z)y'' + q(z)y' + r(z)y = \lambda y$$

where $p(z), q(z), r(z)$ are rational in z . We have thus that

$$y'' = -\frac{q(z)}{p(z)}y' + \frac{\lambda - r(z)}{p(z)}y,$$

and hence a higher order derivative $y^{(k)}$, $k \geq 2$ can always be written as a linear combination of y and y' with coefficients that are polynomial in λ and rational in z .

Since $\ker L$ is finite-dimensional and invariant with respect to T , let us choose an eigenvector $\phi \in \ker L$ of T with eigenvalue λ_0 . It follows that

$$F(z, \lambda_0)\phi + G(z, \lambda_0)\phi' = 0,$$

with F, G defined above.

Claim 4: $G(z, \lambda_0)$ is not identically zero. If $\text{ord } L = 1$ the claim is trivial. For $\text{ord } L \geq 2$ we argue by contradiction and suppose that $G(z, \lambda_0) \equiv 0$. Then, $F(z, \lambda_0) \equiv 0$ also, which implies that

$$\ker(T - \lambda_0) \subset \ker L.$$

It can then easily be shown (see Theorem 1 in [60] and Section 5.4 of [46]) that

$$L = \tilde{L}(T - \lambda_0),$$

which violates the reducibility assumption established by Claim 1. Claim 4 is proved.

Thus, $G(z, \lambda_0)$ is *not* identically zero, and therefore,

$$\frac{\phi'(z)}{\phi(z)} = -\frac{F(z, \lambda_0)}{G(z, \lambda_0)}$$

is a rational function. Set $T_0 = T$ and $L_0 = L$. By Proposition 3.5, there exists a rational factorization

$$T = B_0 A_0 + \lambda_0$$

with $A_0[\phi] = 0$. Since $\phi \in \ker L$ we also have a rational factorization

$$L = L_1 A_0, \quad L_1 \in \text{Diff}(\mathcal{Q}).$$

Setting

$$T_1 = A_0 B_0 + \lambda_0$$

we have

$$(\hat{T}L_1 - L_1 T_1)A_0 = 0$$

which implies that

$$\hat{T}L_1 = L_1 T_1.$$

Claim 5: L_1 has no right factors of the form $T_1 - \lambda$. Suppose otherwise, so that

$$L_1 = \tilde{L}(T_1 - \lambda).$$

Then, setting $\tilde{\lambda} = \lambda - \lambda_0$ we have

$$L = L_1 A_0 = \tilde{L}(A_0 B_0 - \tilde{\lambda})A_0 = \tilde{L}A_0(B_0 A_0 - \tilde{\lambda}) = \tilde{L}A_0(T - \lambda),$$

which again violates the irreducibility assumption of Claim 1.

Continuing by induction, we have

$$\hat{T}L_i = L_i T_i, \quad i = 0, 1, \dots$$

with L_i reduced. Repeating the above argument, we construct rational factorizations

$$T_i = B_i A_i + \lambda_i, \quad T_{i+1} = A_i B_i + \lambda_i,$$

so that

$$L = L_{i+1} A_i \cdots A_0$$

and

$$\hat{T}L_{i+1} = L_{i+1} T_{i+1},$$

and so that L_{i+1} is reduced as per Claim 1. This process terminates when L_i is a first-order operator, because then we can take $L_i = A_i$, which gives $\hat{T} = T_{i+1}$, and completes the factorization chain that connects \hat{T} and T . \square

Corollary 3.11. *The property of being Darboux connected is an equivalence relation on $\text{Diff}_2(\mathcal{Q})$.*

Proof. Reflexivity of the relation is self-evident. We need to prove that the Darboux connected relation possesses both symmetry and transitivity. Suppose (18) holds. If $L = \mu$ is zero-order then,

$$T\mu^{-1} = \mu^{-1}\hat{T},$$

so that T is Darboux connected to \hat{T} . If $\text{ord } L \geq 1$ then \hat{T} and T are related by a factorization chain. By inspection of the definition, the property of being connected by a factorization chain is symmetric; one simply switches the A_i and the B_i and reverses the order of the factorization chain.

Next suppose that

$$T_1L_1 = L_1T_2, \quad T_2L_2 = L_2T_3,$$

where $T_1, T_2, T_3 \in \text{Diff}_2(\mathcal{Q})$ and $L_1, L_2 \in \text{Diff}(\mathcal{Q})$. Then, by associativity of operator composition,

$$T_1L_1L_2 = L_1T_2L_2 = L_1L_2T_3,$$

so that T_1 is Darboux connected to T_3 . \square

4. Exceptional operators and invariant polynomial subspaces

Definition 4.1. We will say that a second-order operator $T \in \text{Diff}_2(\mathcal{Q})$ is *exceptional* if T has a polynomial eigenfunction for all but finitely many degrees. The precise condition is that there exists a finite set of natural numbers $\{k_1, \dots, k_m\} \subset \mathbb{N}$ such that for all $k \notin \{k_1, \dots, k_m\}$, there exists a $y_k \in \mathcal{P}_k^*$ and a $\lambda_k \in \mathbb{C}$ such that

$$T[y_k] = \lambda_k y_k, \quad k \in \mathbb{N} - \{k_1, \dots, k_m\}$$

and such that no such polynomial exists if $k \in \{k_1, \dots, k_m\}$. We will refer to k_1, \dots, k_m as the *exceptional degrees*.

Remark 4.2. Note that in the above definition of an exceptional differential operator, m could be zero, i.e. exceptional operators include classical operators as a special case. In the recent literature on this subject, the adjective *exceptional* is usually reserved for the case $m > 0$ to differentiate them from the classical ones. However, for the purpose of this paper it is convenient to handle the general class. Thus, in order not to introduce further notation, we will stick to the term *exceptional* in this wider context, hoping that no confusion will arise.

As the following Proposition shows, no generality is lost by assuming that an exceptional operator has rational coefficients. Indeed, the existence of just 3 linearly independent polynomial eigenfunctions is enough to conclude that a second-order differential expression has rational coefficients.

Proposition 4.3. *Let $T = p(z)D_{zz} + q(z)D_z + r(z)$ be a second-order differential expression with the property that $T[f_k] \in \mathcal{P}$, $k = 1, 2, 3$ for 3 linearly independent polynomials $f_1, f_2, f_3 \in \mathcal{P}$. Then, necessarily the coefficients of T are rational functions.*

Proof. The three conditions read

$$g_k = T[f_k] = p(z)f_k'' + q(z)f_k' + r(z)f_k, \quad k = 1, 2, 3.$$

where g_k are polynomials. The coefficients p, q, r are the unique solutions of the following linear equation:

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} f_1'' & f_1' & f_1 \\ f_2'' & f_2' & f_2 \\ f_3'' & f_3' & f_3 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

It follows that p, q, r are rational functions. \square

Definition 4.4. For an exceptional operator $T \in \text{Diff}_2(\mathcal{Q})$, let $\mathcal{U} \subset \mathcal{P}$ denote the maximal invariant polynomial subspace, and ν the codimension of \mathcal{U} in \mathcal{P} .

Note that the polynomial subspace \mathcal{U} includes the span of all polynomial eigenfunctions of T , but sometimes it can be larger (see Remark 7.2 and Example 7.3). It can also be characterized in the following manner:

Proposition 4.5. *An equivalent characterization of \mathcal{U} is*

$$\mathcal{U} = \{y \in \mathcal{P} : T^j[y] \in \mathcal{P} \text{ for all } j \in \mathbb{N}\}. \quad (23)$$

Proof. Let \mathcal{U}' denote the subspace defined by the right side of (23). For all $y \in \mathcal{U}'$ we have $T[y] \in \mathcal{U}'$ by definition. Hence, \mathcal{U}' is T -invariant, and hence $\mathcal{U}' \subset \mathcal{U}$. On the other hand, if $y \in \mathcal{U}$ then $T^j[y] \in \mathcal{U} \subset \mathcal{P}$ for all $j \in \mathbb{N}$. Therefore, $\mathcal{U} \subset \mathcal{U}'$, also. \square

We begin by collecting some basic results concerning polynomial subspaces $\mathcal{U} \subset \mathcal{P}$.

Definition 4.6. For a meromorphic function $f(z)$ we define $\text{ord}_\zeta f$, $\zeta \in \mathbb{C}$ to be the largest integer k such that $(z - \zeta)^{-k}f(z)$ is bounded as $z \rightarrow \zeta$; i.e., k is the degree of the leading term in the Laurent expansion of f . We will also employ the Landau O -notation to indicate local behaviour near $z = \zeta$. Thus,

$$f(z) \equiv g(z) + O((z - \zeta)^k), \quad z \rightarrow \zeta$$

means that $\text{ord}_\zeta(f - g) \geq k$, $k \in \mathbb{Z}$. When no ambiguity arises, we omit the $z \rightarrow \zeta$.

Definition 4.7. Let $\mathcal{U} \subset \mathcal{P}$ be a polynomial subspace. For a given $\zeta \in \mathbb{C}$, we define the *order sequence of T at ζ* as

$$I_\zeta = \{\text{ord}_\zeta y : y \in \mathcal{U}\}. \quad (24)$$

We define ν_ζ to be the cardinality of $\mathbb{N} \setminus I_\zeta$; that is, the number of gaps in the order sequence.

Proposition 4.8. *Let $\mathcal{U} \subset \mathcal{P}$ be a polynomial subspace. Then, for every $\zeta \in \mathbb{C}$, there exists a basis $\{y_k\}_{k \in I_\zeta}$ of \mathcal{U} such that $\text{ord}_\zeta y_k = k$, $k \in I_\zeta$.*

Proof. Fix $\zeta \in \mathbb{C}$. For every $k \in I_\zeta$ choose a polynomial $y_k \in \mathcal{U}$ such that $\text{ord}_\zeta y_k = k$ and such that $\deg y_k$ is as small as possible. We claim that $\{y_k\}_{k \in I_\zeta}$ is a basis of \mathcal{U} . Suppose not. Set $\mathcal{U}' = \text{span}\{y_k\}_{k \in I_\zeta}$ and let $f \in \mathcal{U} \setminus \mathcal{U}'$ be given. Since the order of a polynomial cannot exceed its degree, we can choose a $g \in \mathcal{U}'$ such that $\deg(f - g) - \text{ord}_\zeta(f - g)$ is as small as possible. Let $k = \text{ord}_\zeta(f - g)$. Since $f - g \in \mathcal{U}$ we must have $k \in I_\zeta$. Hence, there exists a $c \in \mathbb{C}$ such that $\text{ord}_\zeta(f - g - cy_k) > k$. By the way that y_k was chosen, $\deg y_k \leq \deg(f - g)$ and hence

$$\deg(f - g - cy_k) \leq \deg(f - g).$$

We have thus,

$$\deg(f - g - cy_k) - \text{ord}_\zeta(f - g - cy_k) < \deg(f - g) - \text{ord}_\zeta(f - g),$$

which contradicts the assumption regarding g . \square

Proposition 4.9. *Let $\mathcal{U} \subset \mathcal{P}$ be a polynomial subspace. Suppose that the codimension $\nu = \dim \mathcal{P}/\mathcal{U}$ is finite. Then, for every $\zeta \in \mathbb{C}$, we have $\nu_\zeta \leq \nu$.*

Proof. Let

$$y(z) = \sum_{n \notin I_\zeta} a_n (z - \zeta)^n, \quad a_n \in \mathbb{C}.$$

If $y \neq 0$, then $\text{ord}_\zeta y$ is the smallest element of $\mathbb{N} \setminus I_\zeta$, which means that $y \notin \mathcal{U}$. Hence, the ν_ζ polynomials $\{(z - \zeta)^n\}_{n \notin I_\zeta}$ are linearly independent modulo \mathcal{U} . By assumption, it is not possible to choose more than ν linearly independent polynomials in \mathcal{P}/\mathcal{U} and the claim is thus established. \square

Proposition 4.10. *Let $\mathcal{U} \subset \mathcal{P}$ be a finite-codimension polynomial subspace, $\zeta \in \mathbb{C}$ and $n = \max(\mathbb{N} \setminus I_\zeta)$, which is finite by the preceding Proposition. Then there exists a basis $\{\tilde{y}_j\}_{j \in I_\zeta}$ of \mathcal{U} such that*

$$\text{ord}_\zeta \tilde{y}_j = j, \quad \tilde{y}_j^{(j)}(\zeta) = 1, \quad j \in I_\zeta, \quad (25)$$

$$\tilde{y}_j^{(i)}(\zeta) = 0, \quad i, j \in I_\zeta, \quad j < i < n. \quad (26)$$

Proof. By Proposition 4.8, there exists a basis $\{y_j\}_{j \in I_\zeta}$ of \mathcal{U} such that $\text{ord}_\zeta y_j = j$. Let $m = \#\{j \in I_\zeta : j < n\}$ and let $j_1 < j_2 < \dots < j_m$ be the elements of $\{j \in I_\zeta : j < n\}$ arranged in ascending order. Consider the $m \times m$ matrix Y whose components are given by

$$Y_{ab} = y_{j_a}^{(j_b)}(\zeta)$$

The assumption that $\text{ord } y_j = j$, implies that Y is upper triangular, with non-zero diagonal entries. Hence, Y is invertible. Let $C = Y^{-1}$ and set

$$\begin{aligned}\tilde{y}_{j_a}(z) &= \sum_{b=1}^m C_{ab} y_{j_b}(z), \quad C_{ab} \in \mathbb{C}, \quad a = 1, \dots, m, \\ \tilde{y}_j(z) &= \frac{y_j(z)}{y_j^{(j)}(\zeta)}, \quad j > n, \quad j \in I_\zeta.\end{aligned}$$

Then, (25) and (26) hold by construction. \square

Definition 4.11. We define a differential functional with support at $\zeta \in \mathbb{C}$ to be a linear map $\alpha : \mathcal{P} \rightarrow \mathbb{C}$ of the form

$$\alpha[y] = \sum_{j=0}^k a_j y^{(j)}(\zeta), \quad a_j \in \mathbb{C}.$$

We define the order of α to be the largest j such that $a_j \neq 0$. For $\mathcal{U} \subset \mathcal{P}$ and $\zeta \in \mathbb{C}$ we define $\text{Ann}_\zeta \mathcal{U}$ to be the vector space of differential functionals with support at ζ that annihilate \mathcal{U} .

Proposition 4.12. *A differential functional supported at $\zeta \in \mathbb{C}$ cannot be given as a finite linear combination of differential functionals with support at other points.*

Proof. Let α_i , $i = 1, \dots, m$ be differential functionals of order k_i with support at $\zeta_i \in \mathbb{C}$. Set

$$g(z) = \prod_{i=1}^m (z - \zeta_i)^{k_i+1}.$$

Suppose that α is a differential functional of order k supported at $\zeta \notin \{\zeta_1, \dots, \zeta_m\}$. Let $L : \mathcal{P}_k \rightarrow \mathcal{P}_k$ be the linear transformation uniquely defined by the relation

$$\text{ord}_\zeta(L(f) - gf) \geq k + 1, \quad f \in \mathcal{P}_k.$$

Suppose that $L(f) = 0$, $f \in \mathcal{P}_k$. Then,

$$g(z)f(z) = (z - \zeta)^{k+1}h(z), \quad h \in \mathcal{P}.$$

This is only possible if $f = h = 0$. Hence, $\ker L$ is trivial and L is invertible. Since $\mathcal{P}_k \not\subset \ker \alpha$, it is possible to choose an $f \in \mathcal{P}_k$ such that $\alpha(L(f)) \neq 0$. Hence, $\alpha(fg) \neq 0$. By construction, $\alpha_i[fg] = 0$, $i = 1, \dots, m$. Therefore α cannot be given as a linear combination of $\alpha_1, \dots, \alpha_m$. \square

Proposition 4.13. *For every $\zeta \in \mathbb{C}$ we have $\dim \text{Ann}_\zeta \mathcal{U} = \nu_\zeta$.*

Proof. Suppose that $\nu_\zeta > 0$, $\zeta \in \mathbb{C}$. By Proposition 4.10 there exists a basis $\{\tilde{y}_j\}_{j \in I_\zeta}$ of \mathcal{U} such that (25) and (26) hold. Also, since $\text{ord}_\zeta \tilde{y}_j = j$, we must have

$$\tilde{y}_j^{(i)}(\zeta) = 0, \quad i < j.$$

For $k \notin I_\zeta$, set

$$\alpha_k[f] = f^{(k)}(\zeta) - \sum_{\substack{i < k \\ i \in I_\zeta}} \tilde{y}_i^{(k)}(\zeta) f^{(i)}(\zeta), \quad f \in \mathcal{P}. \quad (27)$$

We claim that $\alpha_k \in \text{Ann}_\zeta \mathcal{U}$. If $j > n$, then $\alpha_k[\tilde{y}_j] = 0$ because $\text{ord } \alpha_k < j$. If $j < n$, then

$$\alpha_k[\tilde{y}_j] = \tilde{y}_j^{(k)}(\zeta) - \sum_{\substack{i < k \\ i \in I_\zeta}} \tilde{y}_i^{(k)}(\zeta) \tilde{y}_j^{(i)}(\zeta) = \tilde{y}_j^{(k)}(\zeta) - \tilde{y}_j^{(k)}(\zeta) \tilde{y}_j^{(j)}(\zeta) = 0.$$

Next, we claim that the $\{\alpha_k\}_{k \notin I_\zeta}$ are a basis of $\text{Ann}_\zeta \mathcal{U}$. For $j \in I_\zeta$, $j < n$, let $p_j \in \mathcal{P}_n$, be the n th Taylor polynomial of $\tilde{y}_j(z)$ around $z = \zeta$; i.e.,

$$\tilde{y}_j(z) \equiv p_j(z) + O((z - \zeta)^{n+1}), \quad z \rightarrow \zeta.$$

Let $\mathcal{U}_{\zeta,n} \subset \mathcal{P}_n$ be the span of these p_j . Let $\tilde{\alpha}_j = \alpha_j|_{\mathcal{P}_n}$ denote the indicated restriction to \mathcal{P}_n . Observe that the α_k , $k \notin I_\zeta$ have distinct orders, and that all such orders are $\leq n$. Hence $\tilde{\alpha}_k$, $k \notin I_\zeta$ are also linearly independent. Let $\mathcal{U}_{\zeta,n}^\perp$ denote the vector space of linear forms on \mathcal{P}_n that annihilate $\mathcal{U}_{\zeta,n}$. Since $\mathcal{U}_{\zeta,n}$ has codimension ν_ζ in \mathcal{P}_n , we conclude that $\{\tilde{\alpha}_k\}_{k \notin I_\zeta}$ is a basis of $\mathcal{U}_{\zeta,n}^\perp$. Let $\alpha \in \text{Ann}_\zeta \mathcal{U}$ be given, and let $\tilde{\alpha} = \alpha|_{\mathcal{P}_n}$ be the indicated restriction. Observe that $\text{ord } \alpha \leq n$ because $\alpha[\tilde{y}_j] = 0$ for all $j > n$. Hence, α is fully determined by $\tilde{\alpha}$. Therefore, α belongs to the span of the α_k , $k \notin I_\zeta$. \square

Definition 4.14. Let $T \in \text{Diff}_2(\mathcal{Q})$ be an arbitrary exceptional operator. We say that $\zeta \in \mathbb{C}$ is a *pole of T* if it is a pole of any of its coefficients $p, q, r \in \mathcal{Q}$ as per (4).

Let $\zeta_1, \dots, \zeta_N \in \mathbb{C}$ be the poles of T , and let $\nu_i = \nu_{\zeta_i}$, $i = 1, \dots, N$ be the number of gaps in each order sequence, i.e. $\nu_i = \#\mathbb{N} \setminus I_{\zeta_i}$. We are now ready to state the first main result

Theorem 4.15. *Let T be an exceptional operator, \mathcal{U} its maximal polynomial invariant subspace and ν the codimension of \mathcal{U} in \mathcal{P} . We have $\nu_\zeta > 0$, $\zeta \in \mathbb{C}$ if and only if ζ is a pole of T . Moreover,*

$$\nu = \sum_{i=1}^N \nu_i. \quad (28)$$

Proof. For $j \in \mathbb{N}$, set

$$d_{ij} = \min\{\text{ord}_{\zeta_i} T^j[y] : y \in \mathcal{P}\}.$$

Consider the Laurent expansion of $T^j[y]$ at $z = \zeta_i$, and define differential functionals α_{kij} with support at ζ_i by means of the relation

$$T^j[y] \equiv \sum_{k=1}^{-d_{ij}} \alpha_{kij}[y](z - \zeta_i)^{-k} + O(1), \quad z \rightarrow \zeta_i. \quad (29)$$

By Proposition 4.5, \mathcal{U} is the joint kernel of the α_{kij} defined above. Since the codimension is finite, the joint kernel may be restricted to a finite number of triples (i, j, k) . Hence $\text{Ann}_\zeta \mathcal{U} \subset \text{span}\{\alpha_{kij}\}_{i,j,k}$ for all $\zeta \in \mathbb{C}$. By Proposition 4.13 if $\nu_\zeta > 0$, then $\text{Ann}_\zeta \mathcal{U}$ is non-trivial, and hence ζ must be a pole of T . Conversely, for every pole $\zeta_i \in \mathbb{C}$ is, there is at least one α_{kij} that annihilates \mathcal{U} . Hence $\nu_i > 0$ for all i . Therefore $\nu_\zeta > 0$ if and only if $\zeta \in \mathbb{C}$ is a pole of T . It also follows that \mathcal{U} is the joint kernel of $\bigoplus_{i=1}^N \text{Ann}_{\zeta_i} \mathcal{U}$. Relation (28) now follows by Proposition 4.12. \square

5. The structure theorem for exceptional operators

Let T be an exceptional operator with eigenpolynomials $y_k \in \mathcal{P}_k^*$. Observe that for every $\sigma \in \mathcal{P}_n^*$, $n \geq 1$, the gauge-equivalent operator $\hat{T} = \sigma T \sigma^{-1}$ is also exceptional with eigenpolynomials $\tilde{y}_{k+n} = \sigma y_k \in \mathcal{P}_{k+n}$. Thus every exceptional operator is gauge equivalent to infinitely many other exceptional operators. However, as we show below, every gauge-equivalent class of exceptional operators admit a distinguished gauge, as per the following.

Definition 5.1. We will say that an operator $T \in \text{Diff}_2(\mathcal{Q})$ with coefficients p, q, r as per (4) is a *natural* operator if $p \in \mathcal{P}_2$ and if there exist polynomials $s \in \mathcal{P}_1$ and $\eta \in \mathcal{P}$ such that $T[y] = 0$, when multiplied by η , is equivalent to the bilinear relation

$$p(\eta y'' - 2\eta' y' + \eta'' y) + \frac{1}{2} p'(\eta y' + \eta' y) + s(\eta y' - \eta' y) = 0. \quad (30)$$

Remark 5.2. An equivalent formulation of the above definition is that T is a natural operator if $p \in \mathcal{P}_2$ and the other two coefficients have the form

$$q = \frac{p'}{2} + s - \frac{2p\eta'}{\eta} \quad (31a)$$

$$r = \frac{p\eta''}{\eta} + \left(\frac{p'}{2} - s\right) \frac{\eta'}{\eta}. \quad (31b)$$

for some polynomials $s \in \mathcal{P}_1$ and $\eta \in \mathcal{P}$.

The main result in this Section is a structure theorem for the coefficients of an exceptional operator T .

Theorem 5.3. *Let $T = p(z)D_{zz} + q(z)D_z + r(z)$ be an exceptional operator. Then, $p \in \mathcal{P}_2$ while q has the form shown in (31a) for some $s \in \mathcal{P}_1$ and*

$$\eta(z) = \prod_{i=1}^N (z - \zeta_i)^{\nu_i} \quad (32)$$

where ζ_i , $i = 1 \dots, N$ are the poles of T , and $\nu_i = \nu_{\zeta_i}$ are the corresponding gap cardinalities. Moreover, T is gauge equivalent to a natural operator; i.e. modulo a gauge-transformation r has the form shown in (31b).

Remark 5.4. Note that while the above theorem states that an exceptional operator must have a very specific form, the final characterization of an exceptional T is even more restrictive. Indeed, the poles ζ_i of an exceptional operator cannot be chosen at will, but will need to satisfy a set of constraints that, in similar contexts, have been called the *locus equations* [2, 5]. Equivalently, every exceptional operator T is gauge equivalent to a natural operator, but not every natural operator is exceptional.

We devote the rest of this section to the proof of this theorem.

It turns out that every equivalence class of gauge-equivalent exceptional operators admits another distinguished gauge, as per the following.

Definition 5.5. Let $T \in \text{Diff}_2(\mathcal{Q})$ be an exceptional operator and $\mathcal{U} \subset \mathcal{P}$ the corresponding maximal invariant polynomial subspace. We will say that T is *reduced* if there *does not exist* a $\zeta \in \mathbb{C}$ such that $y(\zeta) = 0$ for all $y \in \mathcal{U}$.

Proposition 5.6. *Every exceptional operator is gauge-equivalent to a reduced exceptional operator.*

Proof. Suppose that $\tilde{T} \in \text{Diff}_2(\mathcal{Q})$ is exceptional with eigenpolynomials $\tilde{y}_k \in \mathcal{P}_k^*$. Let $\sigma \in \mathcal{P}$ be the polynomial GCD of the \tilde{y}_k . Then, the operator

$$T = \sigma^{-1}\tilde{T}\sigma,$$

admits polynomial eigenfunctions $\sigma^{-1}\tilde{y}_k \in \mathcal{P}_{k-\deg \sigma}^*$ which, by construction, do not possess a common root. \square

Example 5.7. Unreduced operators are, for all practical purposes, equivalent to their reduced counterparts. For example, consider the classical Hermite differential equation

$$y'' - 2zy' + 2ny = 0, \quad n = 0, 1, 2, \dots$$

whose polynomial solutions are the classical Hermite polynomials $y = H_n(z)$. One could instead consider the polynomials $\hat{H}_n(z) = (1+z^2)H_{n-2}(z)$, $n \geq 2$. By construction, $y = \hat{H}_n$ is a solution of the differential equation

$$y'' - 2\left(z + \frac{2z}{1+z^2}\right)y' + \left(4 + 2n + \frac{2}{1+z^2} - \frac{8}{(1+z^2)^2}\right)y = 0,$$

which is obtained by conjugating the classical Hermite operator by the multiplication operator $1+z^2$. The ordinary Hermite polynomials are orthogonal on $(-\infty, \infty)$ relative to the weight e^{-z^2} , and hence by construction the modified polynomials $\hat{H}_n(z)$ are orthogonal relative to the weight $e^{-z^2}/(1+z^2)^2$. Thus, \hat{H}_n , $n \geq 2$ constitute a family of exceptional orthogonal polynomials with 2 missing degrees. This type of construction is quite general, but does not produce genuinely new orthogonal polynomials.

Remark 5.8. The reduced gauge is not necessarily unique. As an example, consider the classical Laguerre operator

$$\mathcal{L}_\alpha = zD_{zz} + (1 + \alpha - z)D_z. \quad (33)$$

The Laguerre polynomials

$$L_n^{(\alpha)}(z) = \frac{z^{-\alpha} e^z}{n!} D_z^n [e^{-z} z^{n+\alpha}], \quad n \in \mathbb{N}$$

are polynomial eigenfunctions with

$$\mathcal{L}_\alpha [L_n^{(\alpha)}] = -nL_n^{(\alpha)}.$$

Since $L_0^{(\alpha)} = 1$ is a constant, \mathcal{L}_α is reduced for all α . Now suppose that $\alpha = m > 0$ is a positive integer. A direct calculation shows that

$$z^m \mathcal{L}_m z^{-m} = \mathcal{L}_{-m} + m.$$

Both \mathcal{L}_m and \mathcal{L}_{-m} are reduced, exceptional operators but they are related by a non-trivial gauge transformation. At the root of this non-uniqueness is the fact that \mathcal{L}_m possesses rational, non-polynomial eigenfunctions. Indeed by [76][Section 5.2],

$$L_n^{(-m)}(z) = (-z)^m \frac{(m-n)!}{n!} L_{n-m}^{(m)}(z), \quad n \geq m.$$

Therefore, $z^{-m} L_n^{(-m)}(z)$ is a rational eigenfunction of \mathcal{L}_m . In the absence of such rational, non-polynomial eigenfunctions it seems reasonable to conjecture that the reduced condition fixes a unique gauge, but we will not pursue this question here. For us the reduced gauge is an important, but technical condition that simplifies some of the arguments in the proof of Theorem 5.3.

Before moving on, we note that there may even be an infinite number of distinct, but gauge-equivalent reduced exceptional operators. Consider, for example, Euler operators, that is operators of the form

$$T = az^2 D_{zz} + bzD_z + c, \quad a, b, c \in \mathbb{C}.$$

For such operators, every monomial z^k , $k \in \mathbb{Z}$ is an eigenfunction. Thus, every Euler operator T is reduced, but so is $z^n T z^{-n}$ for every $n \in \mathbb{Z}$. Also note that all Euler operators are in natural form, so that uniqueness also fails in Theorem 5.3.

We see that every class of gauge-equivalent exceptional operators has at least two distinguished gauges: the natural gauge (Definition 5.1) and the reduced gauge (Definition 5.5). Usually, these two choices of gauge are the same, but this is not always the case, as illustrated by the example below. Lemma 5.26, proved below, shows that the natural and reduced gauges are not the same precisely when the denominator polynomial $\eta(z)$ of the exceptional weight has repeated roots [43, 72].

Example 5.9. The following example illustrates the difference between the natural and reduced gauge of an exceptional operator. The example is based on the following family of two-step exceptional Laguerre polynomials [27]. Let $L_n^{(\alpha)}(z)$ denote the classical Laguerre polynomial of degree n . For $n \geq 2$ set

$$\hat{L}_n^{(\alpha)}(z) := e^{-z} \text{Wr} [L_{n-2}^{(\alpha)}(z), L_1^{(\alpha)}(z), e^z L_2^{(\alpha)}(-z)]. \quad (34)$$

By construction, $\hat{L}_3^{(\alpha)}(z) = 0$, and so we obtain a codimension-3 family of polynomials with degrees $n = 2, 4, 5, 6, \dots$. These polynomials can also be given using the following form introduced by Durán [11]

$$\hat{L}_n^{(\alpha)}(z) = \begin{vmatrix} L_{n-2}^{(\alpha)}(z) & -L_{n-3}^{(\alpha+1)}(z) & L_{n-4}^{(\alpha+2)}(z) \\ L_1^{(\alpha)}(z) & -L_0^{(\alpha+1)}(z) & 0 \\ L_2^{(\alpha)}(-z) & L_2^{(\alpha+1)}(-z) & L_2^{(\alpha+2)}(-z) \end{vmatrix}, \quad n = 2, 4, 5, 6, \dots \quad (35)$$

where $L_j^{(\alpha)}(z)$ is understood to be zero for $j < 0$.

Let

$$\begin{aligned} \eta^{(\alpha)}(z) &= e^{-z} \text{Wr} [L_1^{(\alpha)}(z), e^z L_2^{(\alpha)}(-z)] \\ &= \begin{vmatrix} L_1^{(\alpha)}(z) & -1 \\ L_2^{(\alpha)}(-z) & L_2^{(\alpha+1)}(-z) \end{vmatrix} \\ &= -\frac{1}{2}(z^3 + (\alpha + 4)z^2 - (\alpha + 4)(\alpha + 1)z - (\alpha + 1)(\alpha + 2)(\alpha + 4)). \end{aligned}$$

The polynomial family $\hat{L}_n^{(\alpha)}(z)$, $n = 2, 4, 5, \dots$ is exceptional and in the natural gauge, because of the following bilinear relations:

$$\begin{aligned} z \left(\eta^{(\alpha)} \hat{L}_n^{(\alpha)''} - 2\eta^{(\alpha)'} \hat{L}_n^{(\alpha)'} + \eta^{(\alpha)''} \hat{L}_n^{(\alpha)} \right) + \frac{1}{2} \left(\eta^{(\alpha)} \hat{L}_n^{(\alpha)'} + \eta^{(\alpha)'} \hat{L}_n^{(\alpha)} \right) \quad (36) \\ + \left(-z + \alpha + \frac{5}{2} \right) \left(\eta^{(\alpha)} \hat{L}_n^{(\alpha)'} - \eta^{(\alpha)'} \hat{L}_n^{(\alpha)} \right) + (n - 3) \hat{L}_n^{(\alpha)} \eta^{(\alpha)} = 0 \end{aligned}$$

It is easy to check that $\eta^{(\alpha)}(z) \neq 0$ for $z \in [0, \infty)$ if and only if $\alpha \in (-\infty, -4) \cup (-2, -1)$. Hence, for $\alpha \in (-2, -1)$ the polynomials $\hat{L}_n^{(\alpha)}(z)$ are orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_0^\infty \frac{z^{\alpha+2} e^{-z}}{(\eta^{(\alpha)}(z))^2} f(z) g(z) dz.$$

The discriminant of $\eta^{(\alpha)}(z)$ is $\frac{1}{8}(\alpha + 1)(\alpha + 4)^2(4\alpha + 7)^2$. Hence, for $\alpha = -\frac{7}{4}$ the denominator polynomial has a multiple root. Indeed,

$$\eta^{(-\frac{7}{4})}(z) = -\frac{1}{2} \left(z + \frac{3}{4} \right)^3;$$

there is a single root with a triple multiplicity. Moreover,

$$\begin{aligned} L_2^{(-\frac{7}{4})}(-z) &= \frac{1}{2} \left(z + \frac{3}{4} \right) \left(z - \frac{1}{4} \right) \\ L_1^{(-\frac{7}{4})}(z) &= - \left(z + \frac{3}{4} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \hat{L}_n^{(-\frac{7}{4})}(z) &= -e^{-z} \left(z + \frac{3}{4} \right)^3 \operatorname{Wr} \left[\frac{L_{n-2}^{(-\frac{7}{4})}(z)}{z + \frac{3}{4}}, 1, \frac{1}{2} e^z \left(z - \frac{1}{4} \right) \right] \\ &= -\frac{1}{2} \left(z + \frac{3}{4} \right)^3 L_{n-4}^{(\frac{1}{4})}(z) - \frac{1}{2} \left(z + \frac{3}{4} \right)^2 \left(z + \frac{15}{4} \right) L_{n-3}^{(-\frac{3}{4})}(z) - \frac{1}{2} \left(z + \frac{3}{4} \right) \left(z + \frac{15}{4} \right) L_{n-2}^{(-\frac{7}{4})}(z) \end{aligned}$$

has a root at $z = -\frac{3}{4}$ for every n . Thus, for $\alpha = -\frac{7}{4}$ the natural gauge does not agree with reduced gauge.

Let us therefore introduce the reduced family of polynomials

$$\tilde{L}_n(z) = \left(z + \frac{3}{4} \right)^{-1} \hat{L}_{n+1}^{(-\frac{7}{4})}(z), \quad n = 1, 3, 4, \dots$$

This family of polynomials is exceptional and reduced. The reduced inner product is

$$\langle f, g \rangle = \int_0^\infty \frac{z^{\frac{1}{4}} e^{-z}}{\left(z + \frac{3}{4} \right)^4} f(z) g(z) dz.$$

To obtain the corresponding differential equation we conjugate (36) by $z + \frac{3}{4}$. Applying the gauge-transformation law (9), we obtain the differential equation

$$z \tilde{L}_n'' + \left(\frac{5}{4} - z \right) \tilde{L}_n' + (n-1) \tilde{L}_n - \frac{4z \tilde{L}_n' + \tilde{L}_n}{z + \frac{3}{4}} = 0.$$

In this way we recover the codimension 2 exceptional family first described in [27, Section 6.2.5]. This example also serves as an illustration of the principle that codimension very much depends on the choice of gauge. The generic family described above has codimension 3. However, for one particular value of the parameter, the “true” codimension, that is the codimension of the corresponding reduced family, is actually 2.

We begin with some Lemmas. Below p, q, r are the coefficients of $T \in \operatorname{Diff}_2(\mathcal{Q})$ as per (4).

Definition 5.10. We define the *Laurent decomposition* of T at a given $\zeta \in \mathbb{C}$ to be the sum

$$T = \sum_{j \geq d_\zeta} T_j, \quad (37)$$

where

$$T_j = p_{j+2}(z - \zeta)^{j+2}D_{zz} + q_{j+1}(z - \zeta)^{j+1}D_z + r_j(z - \zeta)^j, \quad (38)$$

with

$$\begin{aligned} p(z) &= \sum_{j \geq \text{ord}_\zeta p} p_j(z - \zeta)^j, \quad p_j \in \mathbb{C}, \\ q(z) &= \sum_{j \geq \text{ord}_\zeta q} q_j(z - \zeta)^j, \quad q_j \in \mathbb{C}, \\ r(z) &= \sum_{j \geq \text{ord}_\zeta r} r_j(z - \zeta)^j, \quad r_j \in \mathbb{C} \end{aligned}$$

the Laurent decompositions of p, q, r , respectively. The *leading order* of the expansion is the integer d_ζ given by

$$d_\zeta = \min\{\text{ord}_\zeta p - 2, \text{ord}_\zeta q - 1, \text{ord}_\zeta r\}. \quad (39)$$

Lemma 5.11. *If T is exceptional, then T_{d_ζ} , $\zeta \in \mathbb{C}$ preserves $\text{span}\{(z - \zeta)^k : k \in I_\zeta\}$.*

Proof. Let $\mathcal{U} \subset \mathcal{P}$ be the maximal invariant polynomial subspace as per (23). By Proposition 4.8, there exists a basis of \mathcal{U} of the form

$$y_k(z) \equiv (z - \zeta)^k + O((z - \zeta)^{k+1}), \quad z \rightarrow \zeta, \quad k \in I_\zeta.$$

Since \mathcal{U} is T invariant and T_{d_ζ} is the smallest order term of T , the desired conclusion follows. \square

Lemma 5.12. *If $T \in \text{Diff}_2(\mathcal{Q})$ is exceptional and $d_\zeta < 0$, $\zeta \in \mathbb{C}$, then for every natural number $j \notin I_\zeta$, there exists a natural number $n_j > 0$ such that*

- (i) $j, j - d_\zeta, \dots, j - (n_j - 1)d_\zeta \notin I_\zeta$;
- (ii) $j - d_\zeta n_j \in I_\zeta$;
- (iii) $T_{d_\zeta}[(z - \zeta)^{j - d_\zeta n_j}] = 0$.

Proof. Consider the set $G = \{k \in \mathbb{N} : j - kd_\zeta \notin I_\zeta\}$. By Proposition 4.9 and the fact that the codimension is finite, we infer that G is finite set. Set $n_j = \max G + 1$. Then (i) and (ii) hold by construction. By Lemma 5.11, $T_{d_\zeta}[(z - \zeta)^{j - n_j d_\zeta}]$ belongs to the span of $\{(z - \zeta)^k : k \in I_\zeta\}$. However, $T_{d_\zeta}[(z - \zeta)^{j - n_j d_\zeta}]$ is a multiple of $(z - \zeta)^{j - (n_j - 1)d_\zeta}$, but $j - (n_j - 1)d_\zeta \notin I_\zeta$. This can only be true if (iii) holds. \square

Lemma 5.13. *If T is exceptional, then $d_\zeta \geq -2$ for every $\zeta \in \mathbb{C}$.*

Proof. Suppose that $d_\zeta < -2$. For each $j \in \{0, 1, 2\}$, if $j \in I_\zeta$ then $T_{d_\zeta}[(z - \zeta)^j] = 0$. If $j \notin I_\zeta$, then by Lemma 5.12 there exists an integer $n_j > 0$ such that $T_{d_\zeta}[(z - \zeta)^{j - n_j d_\zeta}] = 0$. In all cases, we see that T_{d_ζ} would be required to annihilate $(z - \zeta)^k$ for three different integers k , and since it is a second order operator, this is impossible. \square

Lemma 5.14. *If T is exceptional, then $p(z)$ is a polynomial, and the poles of $q(z)$ are simple.*

Proof. If $\zeta \in \mathbb{C}$ is a pole of $p(z)$, then by (39) we would have $d_\zeta \leq -3$ which is forbidden by Lemma 5.13. To prove the second claim, note that if $\text{ord}_\zeta q < -1$, then $d_\zeta \leq -3$, which is again forbidden by Lemma 5.13. \square

We now prove a number of structural Lemmas about reduced exceptional operators. Proposition 5.6 allows us to extend these results to exceptional operators that are not necessarily reduced.

Lemma 5.15. *Let $T \in \text{Diff}_2 \mathcal{Q}$ be a reduced, exceptional operator, and $\zeta \in \mathbb{C}$ a pole of $q(z)$. Then,*

$$\begin{aligned} I_\zeta &= \{2j : j \in \mathbb{N}, j \leq \nu_\zeta\} \cup \{n \in \mathbb{N} : n \geq 2\nu_\zeta + 1\} \\ &= \{0, 2, 4, \dots, 2\nu_\zeta, 2\nu_\zeta + 1, 2\nu_\zeta + 2, \dots\} \end{aligned} \quad (40)$$

with ν_ζ, I_ζ as per Definition 4.7. Moreover, $d_\zeta = -2$ and the leading term of the Laurent expansion (37) has the form

$$T_{-2} = p(\zeta) \left(D_{zz} - \frac{2\nu_\zeta}{(z-\zeta)} D_z \right), \quad p(\zeta) \neq 0. \quad (41)$$

Proof. By lemma 5.13, we have $d_\zeta \geq -2$. As per (38), write

$$\begin{aligned} T_{-2} &= p_0 D_{zz} + q_{-1} (z-\zeta)^{-1} D_z + r_{-2} (z-\zeta)^{-2}, \\ T_{-1} &= p_1 (z-\zeta) D_{zz} + q_0 D_z + r_{-1} (z-\zeta)^{-1}, \end{aligned}$$

Since ζ is a pole of $q(z)$, we must have $q_{-1} \neq 0$ and hence $d_\zeta = -2$. Since T is reduced, $0 \in I_\zeta$. Hence, by Lemma 5.11, $T_{-2}[1] = 0$. Also, by Lemma 5.11, either $T_{-2}[(z-\zeta)^1] = 0$ or $1 \notin I_\zeta$. Suppose that the first case holds. Since T_{-2} is a second-order operator, it does not annihilate any other monomials. By Lemma 5.12 this would imply that $I_\zeta = \mathbb{N}$. However, by Theorem 4.15 we have $\nu_\zeta > 0$, which is a contradiction. Set $\nu_\zeta = \max G + 1$ where G is the set defined in the proof of Lemma 5.12. Hence, by the just-mentioned Lemma,

$$\{1, 3, \dots, 2\nu_\zeta - 1\} \notin I_\zeta,$$

and T_{-2} annihilates $(z-\zeta)^{2\nu_\zeta+1}$. Since T_{-2} cannot annihilate any other monomials, we must have

$$I_\zeta = \mathbb{N} \setminus \{1, 3, \dots, 2\nu_\zeta - 1\},$$

which is the form shown in (40). Since T_{-2} annihilates $(z-\zeta)^0$ and $(z-\zeta)^{2\nu_\zeta+1}$ it must be a multiple of

$$D_{zz} - \frac{2\nu_\zeta}{(z-\zeta)} D_z.$$

The form for T_{-2} shown in (41) is an immediate consequence. \square

Lemma 5.16. *Suppose that T is exceptional and reduced. Then,*

- (i) the poles of $q(z)$ are distinct from the zeros of $p(z)$;
- (ii) the poles of $r(z)$ are also the poles of $q(z)$;
- (iii) the poles of $r(z)$ are simple.

Proof. Claim (i) follows from (41). Since T is reduced, there exists a $y_0 \in \mathcal{U}$ satisfying

$$y_0(z) \equiv 1 + a(z - \zeta) + O((z - \zeta)^2), \quad z \rightarrow \zeta.$$

Suppose that $z = \zeta$ is a pole of $r(z)$. Employing the notation of the proof of Lemma 5.15, we must have $r_{-2} = 0$ and

$$T[y_0] \equiv (aq_{-1} + r_{-1})(z - \zeta)^{-1} + O(1), \quad z \rightarrow \zeta.$$

Hence, $r_{-1} = -aq_{-1}$, which implies that $q_{-1} \neq 0$. This proves (ii) and (iii). \square

Recall that $z = \zeta$ is an *ordinary* point of the differential equation

$$y''(z) + \frac{q(z)}{p(z)}y'(z) + \frac{r(z)}{p(z)}y(z) = 0,$$

if q/p and r/p are analytic at $z = \zeta$. If the above quotients are singular, but if

$$\text{ord}_\zeta \left(\frac{q}{p} \right) \geq -1, \quad \text{ord}_\zeta \left(\frac{r}{p} \right) \geq -2, \quad (42)$$

then $z = \zeta$ is called a *regular singular* point of T . Also recall that the above differential equation admits two linearly independent series solutions, in the sense of the method of Frobenius, if and only if $z = \zeta$ is either an ordinary point or a regular singular point. For more details, see [46][Section 15.3, Section 16.1-16.3]. Finally, observe that in light of (39), condition (42) can be restated more simply as

$$d_\zeta = \text{ord}_\zeta p - 2. \quad (43)$$

By Lemmas 5.14 and 5.16 every $\zeta \in \mathbb{C}$ is either an ordinary point or a regular singular point of a reduced operator T . By Proposition 5.6, the same is true for a general exceptional operator. We therefore introduce the following terminology.

Definition 5.17. We say that $z = \zeta$ is

- i) a *primary pole* if it is a pole of $q(z)$ or $r(z)$;
- ii) a *secondary pole* if it is not a primary pole, but it is a zero of $p(z)$;
- iii) an *ordinary point* otherwise.

Remark 5.18. By Lemma 5.16, if T is reduced, then primary poles are the same as the poles of $q(z)$. As the following example shows, this need not be the case for unreduced exceptional operators.

Example 5.19. Let $m > 0$ be a positive integer and consider the conjugation of the classical Laguerre operator (33),

$$T = z^m \circ \mathcal{L}_\alpha \circ z^{-m} + m = zD_{zz} + (\alpha + 1 - 2m - z)D_z + \frac{m - \alpha}{z}.$$

By construction, this is an exceptional, albeit unreduced, operator with gaps in degrees $n = 0, 1, \dots, m - 1$. The unique pole is at $z = 0$, which also happens to be a zero of $p(z) = z$. Also note that in this case, $z = 0$ is a pole of the operator, but not a pole of $q(z)$.

We now recall some key notions relating to logarithmic singularities from the point of view of Frobenius' method.

Definition 5.20. We say that $T \in \text{Diff}_2(\mathcal{Q})$ has an *apparent singularity* at $\zeta \in \mathbb{C}$ if $T[y] = 0$ admits two linearly independent Laurent series solutions, i.e. if the general solution of $T[y] = 0$ is meromorphic in a neighbourhood of ζ . Moreover, if $T - \lambda$ has an apparent singularity at $\zeta \in \mathbb{C}$ for all $\lambda \in \mathbb{C}$ we say that T has *trivial monodromy* at ζ .

If T is reduced, then at a primary pole, it can be seen from (41) that the two roots of the indicial equation are 0 and $2\nu_\zeta + 1$. Since they differ by an integer, there is the possibility that one of the solutions has a logarithmic singularity. We now show that the assumption that T is exceptional precludes that possibility; i.e., that T has an apparent singularity at a primary pole. Trivial monodromy then follows by an additional argument.

Proposition 5.21. *Let $T = p(z)D_{zz} + q(z)D_z + r(z)$ be an exceptional operator. If $p(\zeta) \neq 0$, $\zeta \in \mathbb{C}$, then T has trivial monodromy at $z = \zeta$.*

Proof. We begin by showing that T has an apparent singularity away from the zeros of $p(z)$. Without loss of generality $p(\zeta) = 1$. By Proposition 5.6, there is no loss of generality, if we suppose that T is reduced. If $z = \zeta$ is not a pole of $q(z)$, then by Lemma 5.16 it is an ordinary point, in which case $T[y] = 0$ admits two independent power series solutions around $z = \zeta$. We therefore assume that $z = \zeta$ is a pole of $q(z)$, and hence that $\nu_\zeta > 0$. By Lemma 5.13, and by the assumption on $p(\zeta)$ we have $d_\zeta = -2$. Indeed, by Lemma 5.15,

$$T_{-2} = D_{zz} - \frac{2\nu_\zeta}{z - \zeta} D_z. \quad (44)$$

Use Proposition 4.8 to choose a basis $\{y_j\}_{j \in I_\zeta}$ of \mathcal{U} such that $\text{ord}_\zeta y_j = j$. Without loss of generality,

$$y_j(z) = (z - \zeta)^j + O((z - \zeta)^{j+1}), \quad z \rightarrow \zeta.$$

Hence, a formal series

$$a(z) = \sum_{i \in I_\zeta} a_i y_i(z), \quad a_i \in \mathbb{C}$$

defines a power series around $z = \zeta$, with the coefficient of $(z - \zeta)^k$, $k \in \mathbb{N}$ being a finite linear combination of the a_i , $i \in I_\zeta$ such that $i \leq k$. Since \mathcal{U} is T -invariant and $d_\zeta = -2$, for a given $i \in I_\zeta$ we have

$$T[y_i] = \sum_{\substack{j \geq i-2 \\ j \in I_\zeta}} B_{ij} y_j, \quad B_{ij} \in \mathbb{C},$$

with $B_{ij} = 0$ for j sufficiently large. Thus, $T[a] = 0$ if and only if

$$\sum_{\substack{i \leq j+2 \\ i \in I_\zeta}} a_i B_{ij} = 0$$

for all $j \in I_\zeta$. By (44),

$$B_{i,i-2} = i(i-1-2\nu_\zeta), \quad i \in I_\zeta.$$

Thus, $T[a] = 0$ if and only if

$$(j+2)(j+1-2\nu_\zeta)a_{j+2} + \sum_{\substack{i \leq j+1 \\ i \in I_\zeta}} B_{ij} a_i = 0 \quad (45)$$

for all $j \in I_\zeta$. By Lemma 5.15,

$$I_\zeta = \{0, 2, 4, \dots, 2\nu_\zeta - 2, 2\nu_\zeta, 2\nu_\zeta + 1, 2\nu_\zeta + 2, 2\nu_\zeta + 3, \dots\}.$$

Hence,

$$(j+2)(j+1-2\nu_\zeta) \neq 0, \quad j \in I_\zeta,$$

and relations (45) recursively define a_j , $j \in I_\zeta$ for arbitrary values of $a_0, a_{2\nu_\zeta+1}$. Since there are two linearly independent power series solutions of $T[y] = 0$ at $z = \zeta$, the operator T has trivial monodromy there.

Finally, observe that if $T \in \text{Diff}_2(\mathcal{Q})$ is exceptional, then so is $T - \lambda$ for every $\lambda \in \mathbb{C}$. It follows that $T - \lambda$ has an apparent singularity at a primary pole for every $\lambda \in \mathbb{C}$. It follows that T has trivial monodromy there. \square

Remark 5.22. For general second-order operators, the property of having trivial monodromy is a stronger condition than the property of having an apparent singularity. As the above argument indicates, at the primary poles of an exceptional operator, the two properties are actually logically equivalent.

Lemma 5.15 established the form of the T_{-2} term of a reduced, exceptional operator. The conclusion is that the Laurent expansion of $q(z)$ at $z = \zeta_i$ has the form

$$q(z) \equiv \frac{-2\nu_i p_{i0}}{z - \zeta_i} + q_{i0} + O((z - \zeta_i)), \quad z \rightarrow \zeta_i, \quad q_{i0} \in \mathbb{C} \quad (46)$$

so that

$$T_{-2} = p_{i0} \left(D_{zz} - \frac{2\nu_i}{z - \zeta_i} D_z \right),$$

where

$$p_{i0} = p(\zeta_i) \neq 0.$$

Using the trivial monodromy results we can now describe the T_{-1} term.

Lemma 5.23. *If $T \in \text{Diff}_2(\mathcal{Q})$ is reduced and $z = \zeta_i$ one of the primary poles, then*

$$T_{-1} = p_{i1} \left((z - \zeta_i)D_{zz} - \frac{\frac{1}{2}\nu_i(3\nu_i - 1)}{z - \zeta_i} \right) + q_{i0} \left(D_z - \frac{\nu_i}{z - \zeta_i} \right), \quad (47)$$

where $p_{i1} = p'(\zeta_i)$.

The proof is based on the following result characterizing monodromy-free Schrödinger operators [10, Proposition 3.3].

Lemma 5.24 (Duistermaat-Grünbaum). *Let $U(x)$ be meromorphic in a neighbourhood of $x = 0$ with Laurent expansion*

$$U(x) = \sum_{j \geq -2} c_j x^j.$$

Then all eigenfunctions of the Schrödinger operator $H = -D_{xx} + U(x)$ are single-valued around $x = 0$ if and only if

$$c_{-2} = \nu(\nu + 1)$$

for some integer $\nu \geq 1$, and

$$c_{2j-1} = 0, \quad 0 \leq j \leq \nu.$$

Proof of Lemma 5.23. Since $p(\zeta_i) \neq 0$ we can find an analytic change of variables $z = \zeta(x)$ that satisfies

$$\zeta'(x)^2 = p(\zeta(x)), \quad \zeta(0) = \zeta_i. \quad (48)$$

Explicitly,

$$x = \int^{z=\zeta(x)} \frac{dz}{\sqrt{p(z)}}.$$

In this way

$$D_{xx} = p(z)D_{zz} + \frac{1}{2}p'(z)D_z.$$

Set

$$\mu(z) = \exp \left(\frac{1}{2} \int \frac{q(z) - \frac{1}{2}p'(z)}{p(z)} dz \right).$$

Observe that $\mu(z)$ is analytic at $z = \zeta_i$. A direct calculation shows that

$$\mu T \mu^{-1} = p(z)D_{zz} + \frac{1}{2}p'(z)D_z + V(z),$$

where

$$V(z) = \frac{p''(z)}{4} - \frac{q'(z)}{2} - \frac{(q(z) - \frac{1}{2}p'(z))(q(z) - \frac{3}{2}p'(z))}{4p(z)} + r(z).$$

Set

$$H = -D_{xx} - V(\zeta(x)),$$

so that $T[y] = \lambda y$ if and only if $H[\psi] = -\lambda\psi$, where

$$\psi(x) = \mu(\zeta(x))y(\zeta(x)).$$

Hence, T has trivial monodromy at $z = \zeta_i$ if and only if H has trivial monodromy at $x = 0$. Using (46) and a direct calculation, gives

$$V(z) \equiv -\frac{\nu_i(\nu_i + 1)p_{i0}}{(z - \zeta_i)^2} + \frac{\nu_i q_{i0} + r_{i,-1} + p_{i1}\nu_i(\nu_i - 1)}{(z - \zeta_i)} + O(1), \quad z \rightarrow \zeta_i,$$

where $r_{i,-1}$ is the residue of $r(z)$ at $z = \zeta_i$. Relation (48) implies

$$\begin{aligned} (\zeta(x) - \zeta_i)^{-1} &\equiv \frac{1}{\zeta'(0)}x^{-1} + O(1), \\ (\zeta(x) - \zeta_i)^{-2} &\equiv \frac{1}{\zeta'(0)^2}x^{-2} - \frac{\zeta''(0)}{\zeta'(0)^3}x^{-1} + O(1), \\ &\equiv \frac{1}{p_{i0}}x^{-2} - \frac{p_{i1}}{2p_{i0}\zeta'(0)}x^{-1} + O(1), \end{aligned}$$

with all relations holding as $x \rightarrow 0$. Hence,

$$U(x) \equiv \nu_i(\nu_i + 1)x^{-2} - \frac{1}{\zeta'(0)}(\nu_i q_{i0} + r_{i,-1} + \frac{1}{2}p_{i1}\nu_i(3\nu_i - 1))x^{-1} + O(1), \quad x \rightarrow 0.$$

By Lemma 5.24 the coefficient of x^{-1} must vanish. Hence,

$$r_{i,-1} = -\nu_i q_{i0} - \frac{1}{2}p_{i1}\nu_i(3\nu_i - 1).$$

Using the definitions (37) and (38) we can now write

$$T_{-1} = p_{i1}(z - \zeta_i)D_{zz} + q_{i0}D_z - \left(\nu_i q_{i0} + \frac{1}{2}p_{i1}\nu_i(3\nu_i - 1) \right) (z - \zeta_i)^{-1},$$

which coincides with (47). \square

Lemma 5.25. *If T is exceptional, then $\deg p \leq 2$, $\deg q \leq 1$, $\deg r \leq 0$.*

Proof. Use polynomial division to obtain the following decompositions

$$q(z) = q_p(z) + q_s(z), \quad r(z) = r_p(z) + r_s(z),$$

where $q_p, r_p \in \mathcal{P}$ and $q_s, r_s \in \mathcal{Q}$ with

$$\deg q_s, \deg r_s < 0, \quad \deg q_p = \deg q, \quad \deg r_p = \deg r.$$

Next consider the decomposition $T = T_p + T_s$, where

$$T_p = p(z)D_{zz} + q_p(z)D_z + r_p(z), \quad T_s = q_s(z)D_z + r_s(z).$$

By construction,

$$\deg T_s[y] < \deg y, \quad y \in \mathcal{P}.$$

Hence, if $y_k \in \mathcal{P}$ is an eigenpolynomial of degree k we must have

$$\deg T_p[y_k] \leq k.$$

The desired conclusion follows because this is true for infinitely many k . \square

Lemma 5.26. *Suppose that T is reduced and exceptional, and let*

$$\eta(z) = \prod_{i=1}^N (z - \zeta_i)^{\nu_i}, \quad (49)$$

$$\mu(z) = \prod_{i=1}^N (z - \zeta_i)^{\nu_i(\nu_i-1)/2}, \quad (50)$$

where $\zeta_i, i = 1, \dots, N$ are the poles of T , and $\nu_i = \nu_{\zeta_i}$ the corresponding gap cardinalities as per Definition 4.7. Then, for some $s \in \mathcal{P}_1$ and $c \in \mathbb{C}$ we have

$$q = \frac{1}{2}p' + s - \frac{2p\eta'}{\eta} \quad (51a)$$

$$r = \frac{p\eta''}{\eta} + \left(\frac{p'}{2} - s\right) \frac{\eta'}{\eta} + 2p \left(\frac{\mu''}{\mu} - \left(\frac{\mu'}{\mu}\right)^2 \right) + \frac{p'\mu'}{\mu} + c. \quad (51b)$$

Note that a reduced operator T would also be natural if $\mu = 1$, i.e. if $\nu_i = 1, i = 1, \dots, N$. If some $\nu_i > 1$ then a gauge transformation is needed to map the reduced T into natural form, as we see below. In practice, exceptional operators with poles $\nu_i > 1$ exist in the Laguerre and Jacobi cases, but only for a set of null measure in the parameters.

Proof. By Lemma 5.14 and by (41) of Lemma 5.15 we have

$$q(z) \equiv -\frac{2p_{i0}\nu_i}{z - \zeta_i} + O(1), \quad z \rightarrow \zeta_i \quad (52)$$

where $p_{i0} = p(\zeta_i)$. Set $s := q - p'/2 + 2p\eta'/\eta$, so that relation (51a) holds. By (49),

$$\frac{\eta'(z)}{\eta(z)} \equiv \frac{\nu_i}{z - \zeta_i} + O(1), \quad z \rightarrow \zeta_i.$$

Hence, $s(z)$ has vanishing residues at all primary poles $z = \zeta_i$. By Lemma 5.25, $\deg p \leq 2$, $\deg q \leq 1$, which implies that $s \in \mathcal{P}_1$.

Let $\tilde{r}(z)$ denote the right side of (51b). Since $\deg p \leq 2$, by inspection, $\deg \tilde{r} \leq 0$. By Lemma 5.25, $\deg r \leq 0$. By Lemma 5.16, $r(z)$ has simple poles at $z = \zeta_i$, $i = 1, \dots, N$. Hence, relation (51b) will follow once we show that $r(z)$ and $\tilde{r}(z)$ have the same residues at all $z = \zeta_i$. Set

$$\tau_i = \sum_{j \neq i} \frac{\nu_j}{\zeta_i - \zeta_j}, \quad i = 1, \dots, N,$$

so that

$$\begin{aligned} \frac{\eta'(z)}{\eta(z)} &\equiv \frac{\nu_i}{z - \zeta_i} + \tau_i + O((z - \zeta_i)), \\ p(z) \frac{\eta'(z)}{\eta(z)} &\equiv (p_{i0} + p_{i1}(z - \zeta_i)) \left(\frac{\nu_i}{z - \zeta_i} + \tau_i \right) + O((z - \zeta_i)), \\ &\equiv \frac{p_{i0}\nu_i}{z - \zeta_i} + p_{i0}\tau + p_{i1}\nu_i + O((z - \zeta_i)) \quad z \rightarrow \zeta_i. \end{aligned}$$

From (51a), which we have already established, it follows that

$$q_{i0} = p_{i1} \left(\frac{1}{2} - 2\nu_i \right) + s_{i0} - 2p_{i0}\tau_i, \quad s_{i0} = s(\zeta_i).$$

and by (47) of Lemma 5.23 we have

$$r(z) \equiv \frac{\frac{1}{2}p_{i1}\nu_i(1 - 3\nu_i) - (p_{i1}(\frac{1}{2} - 2\nu_i) + s_{i0} - 2p_{i0}\tau_i)\nu_i}{z - \zeta_i} + O(1), \quad z \rightarrow \zeta_i \quad (53)$$

Hence by (50) and a direct calculation we obtain

$$\begin{aligned} \frac{\mu'(z)}{\mu(z)} &\equiv \frac{\frac{1}{2}\nu_i(\nu_i - 1)}{z - \zeta_i} + O(1) \\ \frac{\mu''(z)}{\mu(z)} - \left(\frac{\mu'(z)}{\mu(z)} \right)^2 &\equiv -\frac{\frac{1}{2}\nu_i(\nu_i - 1)}{(z - \zeta_i)^2} + O(1), \\ \frac{\eta'(z)}{\eta(z)} + \frac{2\mu'(z)}{\mu(z)} &\equiv \frac{\nu_i^2}{z - \zeta_i} + O(1), \\ \frac{\eta''(z)}{\eta(z)} &\equiv \frac{\nu_i(\nu_i - 1)}{(z - \zeta_i)^2} + \frac{2\tau_i\nu_i}{z - \zeta_i} + O(1), \\ \tilde{r}(z) &= p \left(\frac{\eta''}{\eta} + 2 \left(\frac{\mu''}{\mu} - \left(\frac{\mu'}{\mu} \right)^2 \right) \right) + p' \left(\frac{\eta'}{2\eta} + \frac{\mu'}{\mu} \right) - \frac{s\eta'}{\eta} \\ &\equiv \frac{2p_{i0}\tau_i\nu_i + \frac{1}{2}p_{i1}\nu_i^2 - s_{i0}\nu_i}{z - \zeta_i} + O(1), \quad z \rightarrow \zeta_i, \end{aligned}$$

which agrees with (53). \square

We now show that the operator form shown in (51) is gauge-invariant.

Lemma 5.27. *Let $T = p(z)D_{zz} + q(z)D_z + r(z)$ where $p \in \mathcal{P}_2$ and*

$$q = \frac{p'}{2} + s - \frac{2p\eta'}{\eta} \quad (54a)$$

$$r = \frac{p\eta''}{\eta} + \left(\frac{p'}{2} - s\right) \frac{\eta'}{\eta} + 2p \left(\frac{\mu''}{\mu} - \left(\frac{\mu'}{\mu}\right)^2\right) + \frac{p'\mu'}{\mu}. \quad (54b)$$

for some $s \in \mathcal{P}_1$ and $\eta, \mu \in \mathcal{Q}$. Let $\sigma \in \mathcal{Q}$, and let $\tilde{T} = \sigma T \sigma^{-1}$ be the indicated, gauge-equivalent operator. Then the coefficients $\tilde{q}(z), \tilde{r}(z)$ of \tilde{T} have the form shown in (54), with

$$\tilde{\eta} = \sigma\eta, \quad \tilde{\mu} = \sigma^{-1}\mu, \quad (55)$$

in place of η, μ .

Proof. Set

$$H = \frac{\eta'}{\eta}, \quad M = \frac{\mu'}{\mu}, \quad S = \frac{\sigma'}{\sigma},$$

$$\tilde{H} = \frac{\tilde{\eta}'}{\tilde{\eta}} = H + S, \quad \tilde{M} = \frac{\tilde{\mu}'}{\tilde{\mu}} = M - S.$$

Applying (9), we have

$$\begin{aligned} \tilde{q} &= q - 2pS \\ &= \frac{p'}{2} + s - 2pH - 2pS \\ &= \frac{p'}{2} + s - 2\tilde{H}, \\ \tilde{r} &= r - qS + p(-S' + S^2), \\ &= p(H' + H^2) + \left(\frac{p'}{2} - s\right)H + 2pM' + p'M - \left(\frac{p'}{2} + s - 2pH\right)S + p(-S' + S^2), \\ &= p(H' + S' + (H + S)^2 + 2M' - 2S') + p' \left(\frac{H}{2} + \frac{S}{2} + M - S\right) - s(H + S), \\ &= p(\tilde{H}' + \tilde{H}^2) + \left(\frac{p'}{2} - s\right)\tilde{H} + 2p\tilde{M}' + p'\tilde{M}, \end{aligned}$$

which is the form shown in (54) but with η, μ replaced by $\tilde{\eta}, \tilde{\mu}$. \square

Proof of Theorem 5.3. Let $\tilde{T} = p(z)D_{zz} + \tilde{q}(z)D_z + \tilde{r}(z)$ be an exceptional operator with maximal invariant polynomial subspace $\tilde{\mathcal{U}}$. By Proposition 5.6, let $\sigma \in \mathcal{P}$ be a GCD of all polynomials in $\tilde{\mathcal{U}}$ so that $T = \sigma^{-1}\tilde{T}\sigma$ is reduced. Lemma 5.26 gives the form of T . By Lemma 5.27, \tilde{T} has the same form. Let ζ_1, \dots, ζ_N be the poles of T and ν_1, \dots, ν_N the gap cardinalities as per Definition 4.7. Write

$$\sigma(z) = \prod_{i=1}^N (z - \zeta_i)^{\alpha_i} \prod_{i=1}^M (z - \xi_i)^{\beta_i}$$

where ξ_1, \dots, ξ_M are the zeros of $\sigma(z)$ distinct from the ζ_i , and $\alpha_i \geq 0, \beta_i \geq 0$ the corresponding multiplicities. Let \mathcal{U} be the maximal invariant polynomial subspace of T .

We claim that $\tilde{\mathcal{U}} = \sigma\mathcal{U}$. The inclusion $\sigma\mathcal{U} \subset \tilde{\mathcal{U}}$ is obvious. We now prove that $\tilde{\mathcal{U}} \subset \sigma\mathcal{U}$. As was shown in the proof of Proposition 5.6, every element of $\tilde{\mathcal{U}}$ is divisible by σ . Let $\tilde{y} \in \tilde{\mathcal{U}}$ be given and set $y = \sigma^{-1}\tilde{y}$. Observe that

$$T^k[y] = (\sigma^{-1}\tilde{T}^k\sigma)[y] = \sigma^{-1}\tilde{T}^k[\tilde{y}], \quad k \in \mathbb{N}.$$

By definition, $\tilde{T}^k[\tilde{y}] \in \tilde{\mathcal{U}}$ for all $k \in \mathbb{N}$. Hence, $T^k[y] \in \mathcal{P}$ for all k . Therefore, $y \in \mathcal{U}$ by Proposition 4.5.

Having established the claim, we infer that the poles and the gap cardinalities of \tilde{T} are

$$\tilde{\zeta}_i = \begin{cases} \zeta_i, & i = 1, \dots, N \\ \xi_{i-N}, & i = N+1, \dots, N+M \end{cases}, \quad \tilde{\nu}_i = \begin{cases} \nu_i + \alpha_i, & i = 1, \dots, N \\ \beta_{i-N}, & i = N+1, \dots, N+M \end{cases}$$

By Lemmas 5.26 and 5.27,

$$\begin{aligned} \tilde{q}(z) &\equiv -2 \sum_{i=1}^N \frac{p(\zeta_i)\nu_i}{z - \zeta_i} - 2 \sum_{i=1}^N \frac{p(\zeta_i)\alpha_i}{z - \zeta_i} - 2 \sum_{i=1}^M \frac{p(\xi_i)\beta_i}{z - \xi_i} \pmod{\mathcal{P}_1} \\ &\equiv -2 \sum_{i=1}^N \frac{p(\zeta_i)(\nu_i + \alpha_i)}{z - \zeta_i} - 2 \sum_{i=1}^M \frac{p(\xi_i)\beta_i}{z - \xi_i} \pmod{\mathcal{P}_1} \\ &\equiv -2 \sum_{i=1}^{N+M} \frac{p(\tilde{\zeta}_i)\tilde{\nu}_i}{z - \tilde{\zeta}_i} \pmod{\mathcal{P}_1}, \end{aligned}$$

which proves the first assertion of the Theorem.

Next, set $\hat{T} = \mu T \mu^{-1}$, with μ as per (50). Let \hat{q}, \hat{r} be the corresponding first- and zero-order coefficients. By Lemma 5.27, \hat{r} has the form shown in (54b), but with $\hat{\eta} = \mu\eta$ and $\hat{\mu} = 1$ in place of η, μ . Hence,

$$\hat{r} = \frac{p\hat{\eta}''}{\hat{\eta}} + \left(\frac{p'}{2} - s \right) \frac{\hat{\eta}'}{\hat{\eta}},$$

which proves the second assertion of the Theorem. □

Before moving on to the next section, we make a remark and state two corollaries of Theorem 5.3 that generalize results for exceptional Hermite polynomials previously established in [31]. These results are not used elsewhere in the paper, but they may have some significance for future research, in particular for the derivation of recurrence relations for exceptional polynomials.

Remark 5.28. Since the roots of the indicial equation at a primary pole and at an ordinary point are non-negative, the general solution of $T[y] = \lambda y$ is not only meromorphic but holomorphic at such points. The only points at which

the general solution of $T[y] = \lambda y$ might not be meromorphic are the secondary poles of T , i.e. the roots of $p(z)$. In the case $p(z) = 1$ which corresponds to exceptional Hermite operators, the general solution is thus an entire function, as proved in [30].

Corollary 5.29. *Let $T \in \text{Diff}_2(\mathcal{Q})$ be a natural exceptional operator with poles ζ_1, \dots, ζ_N and corresponding gap multiplicities ν_1, \dots, ν_N . Let $\mathcal{U} \subset \mathcal{P}$ be the maximal polynomial invariant subspace of T , and $\eta \in \mathcal{P}$ be given by $\eta(z) = \prod_{i=1}^N (z - \zeta_i)^{\nu_i}$. Then $y \in \mathcal{U}$ if and only if*

$$2p\eta'y' - \left(p\eta'' + \frac{1}{2}p'\eta' - s\eta' \right) y \quad (56)$$

is divisible by η .

Proof. Let $\mathcal{U}' \subset \mathcal{P}$ be the polynomial subspace consisting of those $y \in \mathcal{P}$ such that (56) is divisible by η . If $y \in \mathcal{U}$, then $T[y] \in \mathcal{P}$ by Proposition 4.5. Decompose the operator in (31) as $T = T_0 + T_s$ where

$$\begin{aligned} T_0 &= pD_{zz} + \left(\frac{p'}{2} + s \right) D_z \\ T_s &= -\frac{2p\eta'}{\eta} D_z + \frac{p\eta''}{\eta} + \left(\frac{p'}{2} - s \right) \frac{\eta'}{\eta}. \end{aligned}$$

Since T_0 has polynomial coefficients, $T_s[y] \in \mathcal{P}$. Hence, $y \in \mathcal{U}'$, and therefore $\mathcal{U} \subset \mathcal{U}'$.

To obtain equality, we use a codimension argument. For $i = 1, \dots, N$, $j = 0, \dots, \nu_i - 1$, define the differential functionals $\alpha_i^{(j)} : \mathcal{P} \rightarrow \mathbb{C}$ by

$$y \mapsto D_z^j \left(2p(z)\eta'(z)y'(z) - \left(p(z)\eta''(z) + \frac{1}{2}p'(z)\eta'(z) - s(z)\eta'(z) \right) y(z) \right) \Big|_{z=\zeta_i}.$$

Observe that $y \in \mathcal{P}$ is divisible by η if and only if

$$y^{(j)}(\zeta_i) = 0$$

for the range of i, j given above. Hence, \mathcal{U}' is the joint kernel of the $\alpha_i^{(j)}$. By Proposition 4.12, these functionals are linearly independent, and therefore \mathcal{U}' has codimension $\sum_{i=1}^N \nu_i$ in \mathcal{P} . By Theorem 4.15, this is also the codimension of \mathcal{U} in \mathcal{P} , so we must have $\mathcal{U} = \mathcal{U}'$. \square

Corollary 5.30. *Let $T \in \text{Diff}_2(\mathcal{Q})$ be a natural exceptional operator, \mathcal{U} its maximal invariant polynomial subspace and η be the polynomial defined in (32). Suppose that $f \in \mathcal{P}$ is such that f' is divisible by η . Then, multiplication by f preserves \mathcal{U} ; i.e., $fy \in \mathcal{U}$ for every $y \in \mathcal{U}$.*

Proof. Suppose that f' is divisible by η . Replacing y with fy in (56) yields

$$2p\eta'(fy)' - \left(p\eta'' + \frac{1}{2}p'\eta' - s\eta' \right) fy = f \left[2p\eta'y' - \left(p\eta'' + \frac{1}{2}p'\eta' - s\eta' \right) y \right] + 2p\eta'f'y.$$

By Corollary 5.29, if $y \in \mathcal{U}$, then the above is divisible by η . \square

The above Corollary allows to build recurrence relations for exceptional polynomials, where the traditional multiplication by x is substituted by multiplication by the polynomial f satisfying the above condition, [14, 31, 59, 67]. The smallest order recurrence relations are obtained by taking $f = \int \eta$, the antiderivative of η . Since $\deg \eta = \nu$, these will be recurrence relations of order $2\nu + 3$.

6. Proof of the Conjecture

In this section we prove the previously conjectured result that every exceptional operator is Darboux connected to a classical operator. We begin with some preliminaries.

Definition 6.1. For $L \in \text{Diff}_\rho(\mathcal{Q})$ we define the degree of L to be

$$\deg L = \max\{\deg a_j - j : j = 0, 1, \dots, \rho\}, \quad (57)$$

where the $a_j \in \mathcal{Q}$ is the j^{th} order coefficient as per (2).

The degree of an operator has an alternative, but equivalent characterization. Let $L \in \text{Diff}_\rho(\mathcal{Q})$ and $k = \deg L$, as defined above. Express the coefficients of L as

$$a_j(z) \equiv c_j z^{j+k} \pmod{\mathcal{Q}_{j+k-1}}, \quad c_j \in \mathbb{C}.$$

and define the polynomial

$$\sigma(n) = \sum_{j=0}^{\rho} c_j n(n-1) \cdots (n-j+1).$$

Proposition 6.2. *We have*

$$\deg L[y] \leq \deg L + \deg y, \quad y \in \mathcal{Q}.$$

The inequality is strict if and only if $\deg y$ is a zero of σ .

Proof. It suffices to show that

$$L[z^n] \equiv \sigma(n) z^{n+k} \pmod{\mathcal{Q}_{n+k-1}}, \quad n \in \mathbb{N}.$$

Write $L = L_0 + L_1$, where

$$L_0 = \sum_{j=0}^{\rho} c_j z^{k+j} D_z^j,$$

is a homogeneous degree k operator. Hence, $\deg L_1 < k$ by construction, and

$$\deg L_1[z^n] \leq n + k - 1, \quad n \in \mathbb{N}.$$

The desired conclusion follows once we observe that

$$L_0[z^n] = \sigma(n) z^{n+k}, \quad n \in \mathbb{N}.$$

□

Proposition 6.3. *The degree of an operator $L \in \text{Diff}(\mathcal{Q})$ is the smallest $k \in \mathbb{N}$ such that $\deg L[y] \leq k + \deg y$ for all $y \in \mathcal{P}$.*

Proof. For $y \in \mathcal{P}$, we have

$$\deg L[y] = \deg \sum_{j=0}^{\rho} a_j y^{(j)} \leq \max\{\deg a_j - j + \deg y : j = 0, 1, \dots, \rho\}$$

It follows that if $k \geq \max_j(\deg a_j - j)$, then

$$\deg L[y] \leq k + \deg y.$$

Choose an $n \in \mathbb{N}$ such that $\sigma(n) \neq 0$. By Proposition 6.2,

$$\deg L[z^n] = \deg L + n.$$

It follows that if $k < \max_j(\deg a_j - j)$, then

$$\deg L[z^n] > k + n,$$

which establishes our claim. \square

Definition 6.4. We say that $T \in \text{Diff}_2(\mathcal{P})$ is a Bochner operator (or classical operator) if $\deg T = 0$.

Before stating the main result of this section, we note the following.

Proposition 6.5. *Every Bochner operator is exceptional.*

Proof. Let T be a Bochner operator. By Proposition 6.2 6.3,

$$T[z^k] \equiv \sigma(k)z^k \pmod{\mathcal{P}_{k-1}}$$

where $\sigma(k)$ is a non-zero polynomial of degree ≤ 2 . Hence, $T - \sigma(k)$ maps \mathcal{P}_k into \mathcal{P}_{k-1} for every $k \in \mathbb{N}$. By the rank-nullity theorem, this linear map has a non-trivial kernel, which means that, for every $k \in \mathbb{N}$, there exists a $y_k \in \mathcal{P}_k$ such that

$$T[y_k] = \sigma(k)y_k.$$

However $\deg y_k$ may be strictly less than k , which means that y_k may coincide with an eigenpolynomial of lower degree. However, this can happen only if $\sigma(k) = \sigma(k')$ for some $k' \neq k$; i.e. if the eigenvalue is not simple. Since $\sigma(k)$ is at most a quadratic function, and k is a positive integer, this can happen at most finitely many times. Therefore, a co-finite number of eigenvalues $\sigma(k)$ are simple, which means that there are eigenpolynomials for a co-finite number of degrees k . Therefore, T is an exceptional operator according to Definition 4.1. \square

Remark 6.6. Note that Bochner operators need not have polynomial eigenfunctions for every degree $k \in \mathbb{N}$. See for example Remark 7.2 and a counter-example in Example 7.3.

The main result of this section is the following theorem.

Theorem 6.7. *Let $T \in \text{Diff}_2(\mathcal{Q})$ be an exceptional operator with primary poles ζ_1, \dots, ζ_N and corresponding gap cardinalities ν_1, \dots, ν_N . Then, T is Darboux connected to a Bochner operator $T_B \in \text{Diff}_2(\mathcal{P})$. Moreover, if $p \in \mathcal{P}_2$ is the second-order coefficient of T , and W, W_B the weights associated by (5b) to T, T_B , we have the relation*

$$W(z) = W_B(z) \frac{\chi(z)}{\eta(z)^2}, \quad \chi \in \mathcal{Q}, \eta \in \mathcal{P}, \quad (58)$$

where

$$\eta(z) = \prod_{i=1}^N (z - \zeta_i)^{\nu_i}, \quad \frac{\chi'(z)}{\chi(z)} = \frac{k}{p(z)}, \quad k \in \mathbb{C}. \quad (59)$$

The proof of Theorem 6.7 requires a number of preliminary results. Let $T \in \text{Diff}_2(\mathcal{Q})$ be an exceptional operator and consider the vector space

$$\mathcal{L} := \{L \in \text{Diff}(\mathcal{P}) : T^k L \in \text{Diff}(\mathcal{P}) \text{ for all } k \in \mathbb{N}\}.$$

The following is an equivalent characterization of \mathcal{L} .

Lemma 6.8. *For $L \in \text{Diff}(\mathcal{P})$, we have $L \in \mathcal{L}$ if and only if $L[\mathcal{P}] \subset \mathcal{U}$.*

Proof. One direction is trivial; we prove the converse. Suppose that $L \in \mathcal{L}$ so that we have $T^k[L[y]] \in \mathcal{P}$ for all $y \in \mathcal{P}$ and all $k \geq 1$. By Definition 4.4, this implies that $L[y] \in \mathcal{U}$, as was to be shown. \square

Next, define the subspace

$$\mathcal{L}^{(\rho)} := \{L \in \mathcal{L} : \text{ord } L \leq \rho, \text{deg } L \leq 0\}$$

where it is clear that $\mathcal{L}^{(\rho_1)} \subset \mathcal{L}^{(\rho_2)}$ for $\rho_1 < \rho_2$. We will first show that at least one $\mathcal{L}^{(\rho)}$ is non-trivial.

Lemma 6.9. *Let ζ_1, \dots, ζ_N be the primary poles of T , and ν_1, \dots, ν_N the corresponding gap cardinalities. Then, $\dim \mathcal{L}^{(n)} > 0$ where*

$$n = \sum_{i=1}^N 2\nu_i.$$

Proof. Set

$$f(z) = \prod_{i=1}^N (z - \zeta_i)^{2\nu_i}. \quad (60)$$

By construction, for every $y \in \mathcal{P}$

$$\alpha_{ki}[fy] = 0, \quad i = 1, \dots, N, k \notin I_{\zeta_i},$$

where $\{\alpha_{ki}\}_{k \notin I_{\zeta_i}}$ is the basis of $\text{Ann}_{\zeta_i} \mathcal{U}$ defined in (27). Hence, by the proof of Proposition 4.13, $fy \in \mathcal{U}$ for all $y \in \mathcal{P}$, and Lemma 6.8 implies that the differential operator

$$L = f(z)D_z^n,$$

belongs to \mathcal{L} . By Proposition 4.15 its degree is zero, so $L \in \mathcal{L}^{(n)}$ as was to be proved. \square

Now, let ρ_{\min} be the minimum positive integer such that $\dim \mathcal{L}^{(\rho)} > 0$, i.e. $\dim \mathcal{L}^{(\rho_{\min})} > 0$ but $\dim \mathcal{L}^{(\rho)} = 0$ for all $\rho < \rho_{\min}$.

Lemma 6.10. *For all non-zero $L \in \mathcal{L}^{(\rho_{\min})}$ we have $\text{ord } L = \rho_{\min}$ and $\deg L = 0$ exactly.*

Proof. The order equality holds by the minimality assumption on ρ_{\min} . Similarly, suppose that there exists a non-zero $L \in \mathcal{L}^{(\rho_{\min})}$ such that $\deg L = -d < 0$. Since L has polynomial coefficients, such an operator would necessarily be of the form $L = \tilde{L}D_z^d$, where $\tilde{L} \in \text{Diff}(\mathcal{P})$. This would imply that $\tilde{L} \in \mathcal{L}^{(\rho_{\min}-d)}$, which would again contradict the minimality assumption for ρ_{\min} . \square

Lemma 6.11. $\dim \mathcal{L}^{(\rho_{\min})} \leq \rho_{\min} + 1$.

Proof. Observe that $\rho_{\min} + 1$ is the dimension of the space of degree homogeneous differential operators of order ρ_{\min} . Hence if $\dim \mathcal{L}^{(\rho_{\min})}$ were to exceed this bound, we would be able to construct an operator $L \in \mathcal{L}^{(\rho_{\min})}$ having strictly negative degree, which is impossible by Lemma 6.10. \square

Lemma 6.12. *Let T be an exceptional operator. Then, there exist a decomposition*

$$T = T_0 + T_s,$$

where $T_0 \in \text{Diff}_2(\mathcal{P})$ is a Bochner operator, and $T_s \in \text{Diff}_1(\mathcal{Q})$ has negative degree.

Proof. Let p, q, r be the coefficients of T , as per (4). By Lemmas 5.14 and 5.25, we can write

$$q = q_1 + q_s, \quad r = r_0 + r_s,$$

with $q_1 \in \mathcal{P}_1$, $r_0 \in \mathbb{C}$, $q_s, r_s \in \mathcal{Q}$, with

$$\deg q_s, \deg r_s < 0.$$

Taking

$$T_0 = pD_{zz} + q_1D_z + r_0, \quad T_s = q_sD_z + r_s$$

gives the desired decomposition. \square

Lemma 6.13. *Let T be an exceptional operator and T_0, T_s its decomposition into Bochner and singular part according to Lemma 6.12. If $L \in \mathcal{L}^{(\rho_{\min})}$ is non-zero, then*

$$\deg(TL - LT_0) < 0. \tag{61}$$

Proof. By Lemma 6.10, $\deg L = 0$. Hence, for $y \in \mathcal{P}_n$ we have

$$T[y] \equiv \sigma_1(n) y, \quad \text{mod } \mathcal{Q}_{n-1} \quad (62)$$

$$T_0[y] \equiv \sigma_1(n) y, \quad \text{mod } \mathcal{Q}_{n-1} \quad (63)$$

$$L[y] \equiv \sigma_2(n) y \quad \text{mod } \mathcal{Q}_{n-1}. \quad (64)$$

where, $\sigma_1(n), \sigma_2(n)$ are polynomials defined by Proposition 6.2. Hence,

$$(TL)[y] \equiv T[\sigma_2(n) y] \equiv \sigma_1(n)\sigma_2(n)y, \quad \text{mod } \mathcal{Q}_{n-1} \quad (65)$$

$$(LT_0)[y] \equiv L[\sigma_1(n) y] \equiv \sigma_2(n)\sigma_1(n)y \quad \text{mod } \mathcal{Q}_{n-1}, \quad (66)$$

which establishes (61). \square

Lemma 6.14. *Let T be an exceptional operator and T_0, T_s its decomposition according to Lemma 6.12. Then, there exists a linear transformation $\mathcal{A} : \mathcal{L}^{(\rho_{\min})} \rightarrow \mathcal{L}^{(\rho_{\min})}$ such that*

$$\mathcal{A}(L)D_z = TL - LT_0, \quad L \in \mathcal{L}^{(\rho_{\min})}.$$

Proof. Since T is second-order,

$$\text{ord}(TL - LT) \leq \text{ord } L + 1, \quad L \in \text{Diff}(\mathcal{P}).$$

By construction, $T - T_0$ is a first-order operator, and hence

$$\text{ord}(TL - LT_0) \leq \text{ord } L + 1, \quad L \in \text{Diff}(\mathcal{P})$$

also. By Lemma 6.13, if $L \in \mathcal{L}^{(\rho_{\min})}$, then

$$TL - LT_0 = \tilde{L}D_z$$

for some unique operator $\tilde{L} \in \text{Diff}(\mathcal{P})$. By construction, $(\tilde{L}D_z)[\mathcal{P}] \subset \mathcal{U}$ which means that that $\tilde{L}[\mathcal{P}] \subset \mathcal{U}$ as well. Hence, Lemma 6.8 implies that $\tilde{L} \in \mathcal{L}$. If $L \in \mathcal{L}^{(\rho_{\min})}$ then by the above results we see that $\text{ord } \tilde{L} \leq \text{ord } L = \rho_{\min}$ and $\deg \tilde{L} \leq 0$. This implies that $\tilde{L} \in \mathcal{L}^{(\rho_{\min})}$. Our claim is established once we set $\mathcal{A}(L) := \tilde{L}$. \square

Proof of Theorem 6.7. Let T, T_0 be as in the preceding Lemma. By Lemma 6.11, $\mathcal{L}^{(\rho_{\min})}$ is finite dimensional. Hence, there exists an eigenvector $L \in \mathcal{L}^{(\rho_{\min})}$ with eigenvalue γ of the linear transformation \mathcal{A} defined in Lemma 6.14. This means that $L \in \text{Diff}(\mathcal{P})$ and $\mathcal{A}(L) = \gamma L$ so that

$$TL = L(T_0 + \gamma D_z).$$

Therefore $T_B = T_0 + \gamma D_z$ is the desired Bochner operator.

Since T is an exceptional operator, from Theorem 5.3 it follows that its first order coefficient $q(z)$ is given by (31a), with η determined by (32). The weight $W(z)$ is then determined by (5) to be,

$$W(z) = \exp\left(\int^z \frac{ax+b}{p(x)} dx\right) \eta(z)^{-2}, \quad a, b \in \mathbb{C}.$$

By definition of T_0 in Lemma 6.12 we see that the weight $W_B(z)$ associated to T_B must have the form

$$W_B(z) = \exp\left(\int^z \frac{ax+c}{p(x)} dx\right), \quad c \in \mathbb{C}.$$

By (17d) of Proposition 3.6, $W(z)/W_B(z)$ is a rational function, which implies that

$$\chi(z) = \eta(z)^2 \frac{W(z)}{W_B(z)} = \exp\left(\int^z \frac{b-c}{p(x)} dx\right)$$

is a rational function. Therefore, by inspection, (59) holds with $k = b - c$. \square

Remark 6.15. Observe that in Lemma 6.12 the decomposition $T = T_0 + T_s$ is not unique. Indeed, for every $\gamma_0 \in \mathbb{C}$ the operators

$$T'_0 = T_0 + \gamma_0 D_z, \quad T'_s = T_s - \gamma_0 D_z$$

give another valid decomposition of $T = T_0 + T_s$ into Bochner and degree-lowering summands. The eigenvalue γ utilized in the above proof then undergoes a corresponding shift to compensate for this: $\gamma' = \gamma - \gamma_0$.

7. Exceptional Orthogonal Polynomial Systems

In all of the previous sections the differential operator T was treated at a purely formal level, the emphasis being on the algebraic conditions leading to the existence of an infinite number of polynomial eigenfunctions. In this section, analytic conditions will be further imposed, in order to select those operators that have a self-adjoint action on a suitably defined Hilbert space.

Definition 7.1. Let T be an exceptional operator. We say that T is *polynomially semi-simple* if the action of T on every finite-dimensional, invariant polynomial subspace is diagonalizable. We will say that T is *polynomially regular* if there exists a positive-definite inner product on \mathcal{P} relative to which the action of T is symmetric.

Remark 7.2. By (23), \mathcal{U} contains all eigenpolynomials of T , which means that $\nu \leq m < \infty$, where m is the number of exceptional degrees as per Definition 4.1. If T is also polynomially semi-simple, then \mathcal{U} may be characterized as the span of the eigenpolynomials of T , in which case $\nu = m$. However, in general \mathcal{U} may contain polynomials that are not in the span of the eigenvectors of T , in which case $\nu < m$ strictly.

The polynomial semi-simplicity condition has not been considered previously in the literature. Rather in the context of orthogonal polynomial systems, the usual assumption is that T is a self-adjoint operator of Sturm-Liouville type, which implies that T is polynomially regular. By the finite-dimensional Spectral Theorem, if T is polynomially regular, as per Definition 7.1, then the T -action on invariant, finite-dimensional, polynomial subspaces is diagonalizable. In other words, regularity implies semi-simplicity.

Theorem 1.2 applies more generally to operators and polynomials with complex coefficients. This is necessary to establish the existence of an eigenvalue of a certain linear operator, as per the proof presented in the preceding section. However, if one were to restrict the context to actual orthogonal polynomials and self-adjoint operators the need for complex coefficients and eigenvalues would be obviated.

To illustrate the above remark, consider the following example.

Example 7.3. The operator

$$T[y] = (1 - z^2)y'' + 2(z - 2)y'$$

is the $\alpha = 0, \beta = -4$ instance of the classical Jacobi operator. This instance is degenerate, because the leading coefficient of the classical Jacobi polynomials is

$$P_n^{\alpha, \beta}(z) = \binom{\alpha + \beta + 2n}{n} 2^{-n} z^n + O(z^{n-1}), \quad z \rightarrow \infty.$$

Indeed, with the above choice of the α, β parameters, the third-degree Jacobi polynomial $P_3^{\alpha, \beta}$ degenerates to a constant. The constant $y = 1$ is an eigenfunction, but observe that

$$T[z^3 + 6z^2 + 21z] = -72.$$

Hence, the vector space spanned by $z^3 + 6z^2 + 21z$ and 1 is T -invariant, but the action is not diagonalizable. However, the Jacobi polynomials of all other degrees are eigenfunctions, so T does fit the definition of an exceptional operator. Regularity for Jacobi polynomials requires that $\alpha, \beta > -1$. Since our example violates this assumption, there is no well-defined inner product. This lack of an inner-product permits an operator with an action that is not semi-simple. Thus in this example, $\mathcal{U} = \mathcal{P}$ but there is no eigenvector of degree 3, so $m = 1$ but $\nu = 0$.

The above remarks motivate the following.

Definition 7.4. We say that a co-finite, real-valued polynomial sequence $y_k \in \mathbb{R}\mathcal{P}_k^*$, $k \notin \{k_1, \dots, k_m\}$ forms a Sturm-Liouville orthogonal polynomial system (SL-OPS) provided

- (i) the y_k are the eigenpolynomials of an operator $T \in \text{Diff}_2(\mathbb{R}\mathcal{Q})$,
- (ii) there is an open interval $I \subset \mathbb{R}$ such that
 - (ii-a) the associated weight function $W(z)$, as defined by (5b), is positive, single valued, and integrable on I ;
 - (ii-b) all moments are finite, i.e.

$$\int_I z^j W(z) dz < \infty, \quad j \in \mathbb{N};$$

(ii-c) $y(z)p(z)W(z) \rightarrow 0$ at the endpoints of I for every polynomial $y \in \mathcal{P}$

(iii) the vector space $\text{span}\{y_k : k \in \mathbb{N} \setminus \{k_1, \dots, k_m\}\}$ is dense in the weighted Hilbert space $L^2(W(z)dz, I)$.

Assumption (i) means that T is an exceptional operator. By Proposition 2.5 and Theorem 5.3, no generality is lost if we assume that T is in the natural gauge; i.e., that T has the form (30), where η is given by (32).

Proposition 2.2 and (ii-c) ensures that T is polynomially regular and that y_k are *orthogonal*

$$\int_I W(z)y_i(z)y_j(z)dz = c_i\delta_{ij}, \quad i, j \notin \{k_1, \dots, k_m\}, \quad c_i > 0.$$

As it was already mentioned in Remark 7.2, regularity implies semi-simplicity, which means that \mathcal{U} , the maximal invariant polynomial subspace, coincides with the span of the eigenpolynomials y_k , $k \notin \{k_1, \dots, k_m\}$, and $\nu = m$. Therefore, by assumption (iii), operator T is essentially self-adjoint on \mathcal{U} .

It has already been noted in all examples of exceptional orthogonal polynomials published in the literature, that the orthogonality weight for the exceptional OPS is a classical weight multiplied by a rational function. This can now be considered as a result.

Proposition 7.5. *The orthogonality weight $W(z)$ of a SL-OPS has the form*

$$W(z) = \frac{W_B(z)}{\eta(z)^2} \quad (67)$$

where

$$W_B(z) = \exp\left(\int^z \frac{s(x)}{p(x)}dx\right), \quad p \in \mathbb{R}\mathcal{P}_2, \quad s \in \mathbb{R}\mathcal{P}_1$$

is the weight of a classical OPS, and where $\eta \in \mathbb{R}\mathcal{P}_m^*$.

Proof. Expression (67) follows by (31a) and (5). By the SLOPS assumptions, both $W_B(z)$ and $\eta(z)$ must be real valued. Since T is polynomially regular, we have $\nu = m$. Therefore, $\deg \eta = m$ by (32) and Theorem 4.15. \square

Remark 7.6. The polynomial s above encodes the weight parameters for the Laguerre and Jacobi families. In the case of the Hermite family all parameters can be normalized away by means of a scaling and a translation. In the case of Laguerre families one of the parameters can be normalized by means of a scaling.

Remark 7.7. If an SL-OPS has polynomial eigenfunctions for all degrees, i.e. $m = 0$ in Definition 7.4, then it defines a classical orthogonal polynomial system, which up to an affine transformation must be Hermite, Laguerre or Jacobi [3, 50].

Since every SL-OPS has an associated exceptional operator T , the notion of Darboux connectedness for operators can be naturally extended to SL-OPS.

Definition 7.8. We say that two SL-OPS are Darboux connected if their associated exceptional operators, modulo a multiplicative constant and a spectral shift, are Darboux connected as per Definition 3.7.

The weights associated with a SL-OPS fall into the same three broad categories as do classical orthogonal polynomials.

Definition 7.9. We say that a SL-OPS is of, respectively, Hermite, Laguerre, and Jacobi type if the corresponding interval $I = (a, b)$ and weight $W(z)$, $z \in I$ have the form

$$I = (-\infty, \infty), \quad W_H(z) = \frac{e^{-z^2}}{\eta(z)^2}, \quad (68a)$$

$$I = (0, \infty) \quad W_L(z) = \frac{z^\alpha e^{-z}}{\eta(z)^2}, \quad \alpha > -1, \quad (68b)$$

$$I = (-1, 1) \quad W_J(z) = \frac{(1-z)^\alpha (1+z)^\beta}{\eta(z)^2}, \quad \alpha, \beta > -1, \quad (68c)$$

where $\eta \in \mathbb{R}\mathcal{P}$ is a real-valued polynomial which is non-vanishing on I .

Proposition 7.10. *Up to an affine transformation of the independent variable, every SL-OPS belongs to one of the three types shown above.*

Proof. Up to an affine change of variable, the second-order coefficient of an exceptional operator takes one of the following forms:

$$1, z, z^2, 1 + z^2, 1 - z^2.$$

Applying (5b) and (31), we see that cases 1,2, and 5 correspond to weights of Hermite, Laguerre, and Jacobi type, respectively. It therefore suffices to rule out the remaining possibilities. These correspond to, respectively, weights of the following form:

$$W(z) = \frac{z^a e^{\frac{b}{z}}}{\eta(z)^2},$$

$$W(z) = \frac{e^{a \arctan(z)} (1 + z^2)^b}{\eta(z)^2},$$

where $a, b \in \mathbb{R}$ are real constants. By inspection, there does not exist a choice of constants or an interval $I \subset \mathbb{R}$ such that of these forms can satisfy requirement (ii) in the definition of a SL-OPS. \square

The analysis of the regularity of the exceptional weight amounts to studying the range of parameters and the combination of Darboux transformations such that $\eta(z)$ has no zeros on I , and such that the classical portion of the weight is integrable on I . For the case of exceptional Hermite polynomials, this was done in [12, 30], for exceptional Laguerre polynomials in [11, 15], and for exceptional Jacobi polynomials in [13].

Applying (30) with $p(z) = 1, z, 1 - z^2$, respectively, we arrive at the following bilinear relations for the exceptional polynomials associated to the above 3 classes of SL-OPS:

$$(\eta \hat{H}_k'' - 2\eta' \hat{H}_k' + \eta'' \hat{H}_k) - 2z(\eta \hat{H}_k' - \eta' \hat{H}_k) + 2(k - m)\eta \hat{H}_k = 0 \quad (69)$$

$$z(\eta \hat{L}_k'' - 2\eta' \hat{L}_k' + \eta'' \hat{L}_k) + (1 + \alpha - z)\eta \hat{L}_k' + (z - \alpha)\eta' \hat{L}_k + (k - m)\eta \hat{L}_k = 0, \quad (70)$$

$$(1 - z^2)(\eta \hat{P}_k'' - 2\eta' \hat{P}_k' + \eta'' \hat{P}_k) + (-(2 + \alpha + \beta)z + \beta - \alpha)\eta \hat{P}_k' + ((\alpha + \beta)z - \beta + \alpha)\eta' \hat{P}_k + (k - m)(\alpha + \beta + 1 + k - m)\eta \hat{P}_k = 0, \quad (71)$$

Here, $\hat{H}_k(z), \hat{L}_k(z), \hat{P}_k(z)$ denote, respectively, exceptional Hermite, Laguerre, and Jacobi polynomials of degree k corresponding to a particular choice of $\eta(z) \in \mathcal{P}_m^*$, and valid for all $k \notin \{k_1, \dots, k_m\}$. Setting $m = 0$ in the above equations recovers the usual Hermite, Laguerre, and Jacobi differential equations. It therefore makes sense to regard (69) (70) and (71) as the exceptional generalizations of these 3 classical equations.

Theorem 6.7 states that every exceptional operator is Darboux connected to a Bochner operator, and holds for a general class of operators defined at a purely formal level. However, Theorem 1.2 is a statement about orthogonal polynomial systems, so it remains to show that the Darboux connection is guaranteed to be maintained between the more restricted class of essentially self adjoint exceptional operators that define an SL-OPS.

Proof of Theorem 1.2. Let $T \in \mathbb{R} \text{Diff}(\mathcal{Q})$ be the exceptional operator associated with a SL-OPS. By Theorem 6.7, T is Darboux connected to a Bochner operator T_B with the corresponding weights related by (58). Since $p \in \mathbb{R}\mathcal{P}_2$ is the same for both operators, the W and W_B belong to the same class of weights. In Proposition 7.5, we established that the polynomial $\eta(z)$ is real-valued. Therefore, the rational factor $\chi(z)$ in (58) must also be real-valued, by (59), and T_B has real coefficients.

It remains to show that the weight parameters in W_B satisfy the conditions in (68), so that the resulting measure has finite moments. We do not claim that T_B is necessarily regular, but we show next that T_B is always Darboux connected to a regular Bochner operator.

For the Hermite class, there is nothing to prove, because $p(z) = 1$, and hence $\chi(z)$ in (58) must be a constant.

Let us consider the Laguerre class next. Write

$$T_\alpha = zD_{zz} + (1 + \alpha - z)D_z = (zD_z + 1 + \alpha - z) \circ D_z.$$

The corresponding weight is $z^\alpha e^{-z}$. Performing a Darboux transformation gives

$$T_\alpha \mapsto D_z \circ (zD_z + 1 + \alpha - z) = T_{\alpha+1} - 1.$$

Therefore, T_α is Darboux connected to $T_{\alpha+1}$, and more generally to $T_{\alpha+n}$, where n is an arbitrary integer. Hence, even though the T_B produced by Theorem 6.7

may not be regular, it is Darboux connected to a regular Bochner operator, and hence so is T .

Finally, let us consider the Jacobi class. Write

$$T_{\alpha,\beta} = (1-z^2)D_{zz} + (-(2+\alpha+\beta)z + \beta - \alpha)D_z = ((1-z^2)D_z - (2+\alpha+\beta)z + \beta - \alpha) \circ D_z.$$

Performing a Darboux transformation gives

$$T_{\alpha,\beta} \mapsto D_z \circ ((1-z^2)D_z - (2+\alpha+\beta)z + \beta - \alpha) = T_{\alpha+1,\beta+1} - 2 - \alpha - \beta.$$

Therefore, $T_{\alpha,\beta}$ is Darboux connected to $T_{\alpha+n,\beta+n} - (2+\alpha+\beta)n$ for every integer n . By taking n sufficiently large, we can ensure that $T_{\alpha+n,\beta+n}$ is regular. \square

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