DONALDSON INVARIANTS FOR CONNECTED SUMS ALONG SURFACES OF GENUS 2

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Abstract. We prove a gluing formula for the Donaldson invariants of the connected sum of two four-manifolds along a surface of genus 2. We also prove a finite type condition for manifolds containing a surface of genus 2, self-intersection zero and representing an odd homology class.

1. Introduction

This paper tries to answer the question of the behaviour of the Donaldson invariants under connected sums along surfaces of genus 2. This has been treated by the author in [10] making use of a suitable version of the Atiyah-Floer Conjecture. The purpose of this paper is to remove the use of any conjecture as well as to make the argument direct and very simple. This was prompted by Tom Mrowka in the conference on four-manifolds in Oberwolfach (Germany) on May 96. We also remark that similar cases have been treated by Morgan and Szabó [9] [14], but our results are more general.

Let X be a smooth, compact, oriented four-manifold with $b^+ > 1$ and $b^+ - b_1$ odd. For any $w \in H^2(X;\mathbb{Z})$, D^w_X will denote the corresponding Donaldson invariant [8], which is defined as a linear functional on $\mathbb{A}(X) = \mathrm{Sym}^*(H_0(X) \oplus H_2(X))$ ($H_*(X)$ will always denote homology with rational coefficients, and similarly for $H^*(X)$). Let $x \in H_0(X)$ be the class of a point. Then Kronheimer and Mrowka [8] define X to be of simple type (with respect to w) when $D^w_X((x^2-4)z) = 0$ for all $z \in \mathbb{A}(X)$, and in that case define

$$\mathbb{D}^w_X(z) = D^w_X((1+\frac{x}{2})z),$$

for all $z \in \mathrm{Sym}^* H_2(X)$. The series $\mathbb{D}^w_X(e^{t\alpha})$, $\alpha \in H_2(X)$, is even or odd depending on whether $d_0 = d_0(X,w) = -w^2 - \frac{3}{2}(1-b_1+b^+)$ is even or odd. When $b_1 = 0$ and $b^+ > 1$, X is of simple type with respect to some w if and only if it is so with respect to any w. In such case, X is just called of simple type.

Proposition 1. *Let X be a manifold of simple type with $b_1 = 0$ and $b^+ > 1$ and odd. Then we have*

$$\mathbb{D}^w_X(e^\alpha) = e^{Q(\alpha)/2} \sum (-1)^{\frac{K_i \cdot w + w^2}{2}} a_i \, e^{K_i \cdot \alpha}$$

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for finitely many $K_i \in H^2(X;\mathbb{Z})$ (called basic classes) and rational numbers a_i (the collection is empty when the invariants all vanish). These classes are lifts to integral cohomology of $w_2(X)$. Moreover, for any embedded surface $S \hookrightarrow X$ of genus g and with $S^2 \geq 0$, one has $2g-2 \geq S^2 + |K_i \cdot S|$.

Analogously, we define X to be of finite type with respect to w whenever $D^w_X((x^2-4)^n z) = 0$ for all $z \in \mathbb{A}(X)$, and some $n > 0$. The order is the minimum of such n.

When X has $b^+ = 1$, the invariants depend on the metric through a structure of walls and chambers [7] and therefore we have to specify the metric.

Definition 2. (w, Σ) is an **allowable** pair if $w, \Sigma \in H^2(X;\mathbb{Z})$, $w \cdot \Sigma \equiv 1 \pmod 2$ and $\Sigma^2 = 0$. Then we define

$$D^{(w,\Sigma)}_X = D^w_X + D^{w+\Sigma}_X.$$

When $b^+ = 1$ we consider the invariants referring to the chambers defined by Σ, i.e. for metrics whose period points are in the (unique) chamber containing Σ in its closure (which is so since $w \cdot \Sigma \equiv 1 \pmod 2$). In fact, we would need a result saying that the invariants only depend on the metric through the period point. This is true for simply-connected manifolds and for $\Sigma \times \mathbb{CP}^1$, with Σ a Riemann surface, which are all the cases we need for our arguments.

$D^{(w,\Sigma)}_X$ depends only on Σ and $w \pmod \Sigma$, since $D^{w+2\Sigma}_X = D^w_X$. As $(w+\Sigma)^2 \equiv w^2 + 2 \pmod 4$, we can recover D^w_X and $D^{w+\Sigma}_X$ from $D^{(w,\Sigma)}_X$. The series $D^{(w,\Sigma)}_X(e^{t\alpha})$, $\alpha \in H_2(X)$, is even or odd according to whether d_0 is even or odd.

Proposition 3. *Suppose X is a manifold of simple type with $b_1 = 0$ and $b^+ > 1$ and odd. Write the Donaldson series as $\mathbb{D}^w_X(e^\alpha) = e^{Q(\alpha)/2} \sum (-1)^{\frac{K_j \cdot w + w^2}{2}} a_j \, e^{K_j \cdot \alpha}$. Then setting $d_0 = d_0(X,w) = -w^2 - \frac{3}{2}(1+b^+)$ we have*

$$D^{(w,\Sigma)}_X(e^\alpha) = e^{Q(\alpha)/2} \sum_{K_j \cdot \Sigma \equiv 2 \pmod 4} (-1)^{\frac{K_j \cdot w + w^2}{2}} a_j e^{K_j \cdot \alpha} + e^{-Q(\alpha)/2} \sum_{K_j \cdot \Sigma \equiv 0 \pmod 4} i^{-d_0} (-1)^{\frac{K_j \cdot w + w^2}{2}} a_j e^{i K_j \cdot \alpha}$$

So giving $\mathbb{D}^w_X$ is equivalent to giving $D^{(w,\Sigma)}_X$.

Proof. Note that $K_j \cdot \Sigma \equiv 0 \pmod 2$, for all basic classes K_j. Since $((w+\Sigma)^2 + K_j \cdot (w+\Sigma)) = (w^2 + K_j \cdot w) + 2(w \cdot \Sigma + K_j \cdot \Sigma/2)$ we have

$$\mathbb{D}^{w+\Sigma}_X(e^\alpha) = e^{Q(\alpha)/2} \sum_{K_j \cdot \Sigma \equiv 2 \pmod 4} (-1)^{\frac{K_j \cdot w + w^2}{2}} a_j \, e^{K_j \cdot \alpha} - e^{Q(\alpha)/2} \sum_{K_j \cdot \Sigma \equiv 0 \pmod 4} (-1)^{\frac{K_j \cdot w + w^2}{2}} a_j \, e^{K_j \cdot \alpha}$$

Now since the only powers in $D^w_X(e^{t\alpha})$ are those t^d with $d \equiv d_0 \pmod 4$ one has

$$D^w_X(e^{t\alpha}) = \frac{1}{2}(\mathbb{D}^w_X(e^{t\alpha}) + i^{-d_0} \mathbb{D}^w_X(e^{it\alpha}))$$

and analogously

$$D_X^{w+\Sigma}(e^{t\alpha}) = \frac{1}{2}(\mathbb{D}_X^{w+\Sigma}(e^{t\alpha}) - i^{-d_0}\mathbb{D}_X^{w+\Sigma}(e^{it\alpha}))$$

since $d_0(X, w+\Sigma) = d_0(X,w) + 2$. So we finally get

$$D_X^{(w,\Sigma)}(e^{\alpha}) = e^{Q(\alpha)/2} \sum_{K_j\cdot\Sigma\equiv 2 \pmod 4} (-1)^{\frac{K_j\cdot w + w^2}{2}} a_j e^{K_j\cdot\alpha} + i^{-d_0} e^{-Q(\alpha)/2} \sum_{K_j\cdot\Sigma\equiv 0 \pmod 4} (-1)^{\frac{K_j\cdot w+w^2}{2}} a_j e^{iK_j\cdot\alpha}$$

□

Definition 4. We say that (X,Σ) is **permissible** if X is a smooth compact oriented four-manifold and $\Sigma \hookrightarrow X$ is an embedded Riemann surface of genus 2 and self-intersection zero such that $[\Sigma] \in H_2(X;\mathbb{Z})$ is odd (its reduction modulo 2 is non-zero, or equivalently, it is an odd multiple of a primitive homology class). So we can consider $w \in H^2(X;\mathbb{Z})$ with $w\cdot\Sigma \equiv 1 \pmod 2$. Then (w,Σ) is an allowable pair. This implies that $b^+ > 0$. Let $N_\Sigma \cong A = \Sigma \times D^2$ be an open tubular neighbourhood of Σ and set $X^o = X - N_\Sigma$. Then $\partial X^o = Y \cong \Sigma \times \mathbb{S}^1$ (but the isomorphism is not canonical). We consider one such isomorphism fixed and (when necessary) we furnish X^o with a cylindrical end, i.e. we consider $X^o \cup (Y\times[0,\infty))$ (and keep on calling it X^o).

We call **identification** for $Y = \Sigma\times\mathbb{S}^1$ any (orientation preserving) bundle automorphism $\phi : Y \xrightarrow{\sim} Y$. Up to isotopy, ϕ depends only on the isotopy class of the induced diffeomorphism on Σ and on an element of $H^1(\Sigma;\mathbb{Z})$.

Definition 5. Let (X_1,Σ_1) and (X_2,Σ_2) be permissible. We pick orientations so that $\partial X_1^o = -\partial X_2^o = Y$ (minus means reversed orientation). Then $X = X(\phi) = X_1^o \cup_\phi X_2^o = X_1 \#_\Sigma X_2$ is a compact, naturally oriented, smooth four-manifold, called the **connected sum along** Σ of (X_1,Σ_1) and (X_2,Σ_2) (with identification ϕ). The induced homology classes $[\Sigma_1]$ and $[\Sigma_2]$ coincide and are induced by an embedded $\Sigma\hookrightarrow X$. Then (X,Σ) is permissible.

Choose $w_i \in H^2(X_i;\mathbb{Z})$, $i=1,2$, and $w\in H^2(X;\mathbb{Z})$ such that $w_i\cdot\Sigma_i \equiv 1 \pmod 2$, $w\cdot\Sigma \equiv 1 \pmod 2$, in a compatible way (i.e. the restricition of w to $X_i^o \subset X$ coincides with the restriction of w_i to $X_i^o \subset X_i$). We shall call w all of them, not making explicit to which manifold they refer. Also let $\mathcal{H} = \{D \in H_2(X)/D|_Y = k[\mathbb{S}^1] \in H_1(Y), \text{ for some } k\}$. Then for every $D\in\mathcal{H}$, we can choose $D_i \in H_2(X_i)$ agreeing with D (i.e. $D_i|_{X_i^o} = D|_{X_i^o}$, $i=1,2$) and with $D^2 = D_1^2 + D_2^2$. Furthermore, we can arrange $D \mapsto (D_1,D_2)$ to be linear. Once chosen one of these maps, any other is of the form $D\mapsto (D_1 - r\Sigma, D_2 + r\Sigma)$, for a rational number r.

A simple but important remark is that if $b_1(X_1) = b_1(X_2) = 0$ then $b_1(X) = 0$ and $b^+(X) > 1$. Now we are ready to state our main results.

Theorem 6. *Suppose (X_1,Σ_1) and (X_2,Σ_2) are permissible and X_1, X_2 have both $b_1=0$ and $b^+>1$ and are of simple type. Let $\mathbb{D}^w_{X_1}(e^\alpha)=e^{Q(\alpha)/2}\sum a_{i,w}e^{K_i\cdot\alpha}$ and $\mathbb{D}^w_{X_2}(e^\alpha)=e^{Q(\alpha)/2}\sum b_{j,w}e^{L_j\cdot\alpha}$. Let $X=X_1\#_\Sigma X_2$ (for some identification). Then X is of simple type and for every $D\in\mathcal{H}$, choose $D_i\in H_2(X_i)$ agreeing with D satisfying $D^2=D_1^2+D_2^2$, in such a way that $D\mapsto(D_1,D_2)$ is linear. Then*

$$\mathbb{D}^w_X(e^{tD})=$$
$$=e^{Q(tD)/2}\Big(\sum_{K_i\cdot\Sigma=L_j\cdot\Sigma=2}-32a_{i,w}b_{j,w}\,e^{(K_i\cdot D_1+L_j\cdot D_2+2\Sigma\cdot D)t}+\sum_{K_i\cdot\Sigma=L_j\cdot\Sigma=-2}32a_{i,w}b_{j,w}\,e^{(K_i\cdot D_1+L_j\cdot D_2-2\Sigma\cdot D)t}\Big),$$

(for appropriate homology orientations).

Remark 7. The reason for the different signs is easy to work out. First, w^2 for X is always congruent $\pmod 2$ with the sum of both of w^2 for X_i. Also $-\frac{3}{2}(1-b_1(X)+b^+(X))=-\frac{3}{2}(1-b_1(X_1)+b^+(X_1))-\frac{3}{2}(1-b_1(X_2)+b^+(X_2))-3(g-1)$. Therefore, as $g=2$, $d_0(X,w)\equiv d_0(X_1,w)+d_0(X_2,w)+1 \pmod 2$. Now the sign comes from the fact that the coefficient for the basic class $-\kappa$ is $(-1)^{d_0}c_\kappa$, being c_κ the coefficient for the basic class κ.

Corollary 8. *Suppose we are in the conditions of the former theorem. Write $\mathbb{D}_X(e^\alpha)=e^{Q(\alpha)/2}\sum c_\kappa e^{\kappa\cdot\alpha}$ for the Donaldson series for X and $\mathbb{D}_{X_1}(e^\alpha)=e^{Q(\alpha)/2}\sum a_i e^{K_i\cdot\alpha}$ and $\mathbb{D}_{X_2}(e^\alpha)=e^{Q(\alpha)/2}\sum b_j e^{L_j\cdot\alpha}$ for the Donaldson series for X_1 and X_2. Then given any pair $(K,L)\in H^2(X_1^o;\mathbb{Z})\oplus H^2(X_2^o;\mathbb{Z})$, we have*

$$\sum_{\{\kappa/\kappa|_{X_1^o}=K,\,\kappa|_{X_2^o}=L\}}c_\kappa=\pm32\,\Big(\sum_{K_i|_{X_1^o}=K}a_i\Big)\cdot\Big(\sum_{L_j|_{X_2^o}=L}b_j\Big)$$

whenever $K|_Y=L|_Y=\pm2P.D.[\mathbb{S}^1]$. Otherwise, the left hand side is zero.

Proof. This is an immediate consequence of theorem 6, noting that

$$(-1)^{\frac{\kappa\cdot w+w^2}{2}}=-(-1)^{\frac{K_i\cdot w+w^2}{2}}(-1)^{\frac{L_j\cdot w+w^2}{2}}$$

whenever $\kappa|_{X_1^o}=K_i|_{X_1^o}=K$, $\kappa|_{X_2^o}=L_j|_{X_2^o}=L$. □

Theorem 9. *Let (X,Σ) be permissible. Then X is of finite type (with respect to any $w\in H^2(X;\mathbb{Z})$ with $w\cdot\Sigma\equiv1 \pmod 2$, and for the invariants given by Σ in case $b^+=1$).*

Now we introduce a very important example. Let B be the $K3$-surface blown-up in two points. Let $S\subset K3$ be a tight surface of genus 2 (which existence is guaranteed by [8]) and let E_1, E_2 be the two exceptional divisors in B. Then $\Sigma=S-E_1-E_2$ is the proper transform of S. So (B,Σ) is permissible. For any (X,Σ) permissible, write $\tilde{X}=X\#_\Sigma B$ (fixing some identification). It has $b^+(X)>1$ and $b_1(\tilde{X})=0$ whenever $b_1(X)=0$. Now for any embedded surface $D^o\subset X^o$ with $\partial D^o=\partial X^o\cap D^o$, we can choose cappings $D=D^o+D^o_B$ in $\tilde{X}$ (in general it is enough to suppose that D^o is a cycle and that, giving X^o a cylindrical end, $D^o \cap (Y \times [0,\infty)) = \gamma \times [0,\infty)$, with $\gamma \subset Y$ an embedded curve). Fix an embedded surface representing $E_1 + E_2 + \Sigma$ and intersecting Σ transversely in two points, and let K^o be its restriction to B^o. Then we always impose $D^o_B \cdot K^o = 0$ (this pairing makes sense as long as ∂K^o and ∂D^o_B are disjoint).

Theorem 10. *Let (X_i, Σ_i) be permissible with $b_1(X_i) = 0$, $i = 1, 2$ (not necessarily of simple type). Consider $\tilde{X}_i = X_i \#_\Sigma B$. Then $\tilde{X}_i$ are of simple type. Put $\mathbb{D}^w_{\tilde{X}_1}(e^\alpha) = e^{Q(\alpha)/2} \sum \tilde{a}_{i,w} e^{\tilde{K}_i \cdot \alpha}$ and $\mathbb{D}^w_{\tilde{X}_2}(e^\alpha) = e^{Q(\alpha)/2} \sum \tilde{b}_{j,w} e^{\tilde{L}_j \cdot \alpha}$. Let $X = X_1 \#_\Sigma X_2$ (for some identification). Then X is of simple type. For every $D \in H_2(X)$, consider any cappings $D_i \in H_2(\tilde{X}_i)$ with the condition above in such a way that $D \mapsto (D_1, D_2)$ is linear. Then*

$$\mathbb{D}^w_X(e^{tD}) = e^{Q(tD)/2} \Big(\sum_{\tilde{K}_i \cdot \Sigma = \tilde{L}_j \cdot \Sigma = 2} -\frac{1}{2} \tilde{a}_{i,w} \tilde{b}_{j,w} e^{t(\tilde{K}_i \cdot D_1 + \tilde{L}_j \cdot D_2)} + \sum_{\tilde{K}_i \cdot \Sigma = \tilde{L}_j \cdot \Sigma = -2} \frac{1}{2} \tilde{a}_{i,w} \tilde{b}_{j,w} e^{t(\tilde{K}_i \cdot D_1 + \tilde{L}_j \cdot D_2)} \Big).$$

Corollary 11. *Under the conditions of theorem 10, X has no basic classes κ with $\kappa \cdot \Sigma = 0$.*

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2. Applications

Now we pass on to give some nice and simple applications of theorem 10. Probably, many results like the following can be obtained in the same fashion. We only want to give some examples to show its usefulness.

Corollary 12. *Let (X_1, Σ_1) and (X_2, Σ_2) be permissible with $b_1(X_i) = 0$. Let ϕ and ψ be two different identifications for $Y = \Sigma \times \mathbb{S}^1$ and consider the two different connected sums along Σ, $X(\phi)$ and $X(\psi)$. Suppose that $\phi_* = \psi_* : H_1(Y) \to H_1(Y)$. Then there is an (non-canonical) isomorphism of vector spaces $H^2(X(\phi)) \xrightarrow{\sim} H^2(X(\psi))$ sending the basic classes of $X(\phi)$ to those of $X(\psi)$ such that the rational numbers attached to them coincide.*

Proof. First we observe that we have a natural identification of the images I_ϕ of $H_2(X^o_1) \oplus H_2(X^o_2) \to H_2(X(\phi))$ and I_ψ of $H_2(X^o_1) \oplus H_2(X^o_2) \to H_2(X(\psi))$ since the kernels coincide. Now consider a splitting $H_2(X_2(\phi)) \cong \operatorname{Im}(I_\phi) \oplus V$ with $V \xrightarrow{\sim} H_1(Y)$. Choose an integral basis $\{\alpha\}$ for $H_1(Y; \mathbb{Z})$. For every α we have an element $D_\alpha \in H_2(X(\phi))$ which can be split as $D_\alpha = D^o_1 + D^o_2$, for $D^o_i \subset X^o_i$ with $\partial D^o_1 = \gamma$,

$-\partial D_2^o = \phi(\gamma)$ and $\alpha = [\gamma]$. Now we leave D_1^o (and $D_1 \in H_2(\tilde{X}_1)$) fixed and modify D_2^o to glue it to D_1^o in $H_2(X(\psi))$. Write $D_2 = D_2^o + D_3^o \in H_2(\tilde{X}_2)$. The loops $\phi(\gamma)$ and $\psi(\gamma)$ are homologous and hence there is homology $C = \mathbb{S}^1 \times [0,1] \hookrightarrow \Sigma \subset \Sigma \times \mathbb{S}^1$ between them. Consider

$$(D')_3^o = \left[D_3^o \cup_{\phi(\gamma)} C \cup_{\psi(\gamma)} (\psi(\gamma) \times [0,\infty))\right] + n\Sigma \subset B^o$$

$$(D')_2^o = \left[D_2^o \cup_{\phi(\gamma)} (-C) \cup_{\psi(\gamma)} (-\psi(\gamma) \times [0,\infty))\right] - n\Sigma \subset X_2^o$$

where n is chosen so that $(D')_3^o \cdot K^o = 0$. So $D_2' = (D')_2^o + (D')_3^o = D_2$. Consider $D_\alpha' = D_1^o + (D')_2^o \in H_2(X(\psi))$. The map $D_\alpha \mapsto D_\alpha'$ gives the required isomorphism $H^2(X(\phi)) \xrightarrow{\sim} H^2(X(\psi))$. □

This corollary says that although in principle $X(\phi)$ and $X(\psi)$ might not be diffeomorphic (and probably in many cases this happens), they can not be distinguished by the number and coefficients of their basic classes. Still the polynomial invariants can differentiate both manifolds (maybe the intersection matrix of the basic classes could help). It would be desirable to find examples when this happens. The identifications to try out could be Dehn twists along separating curves in Σ.

Corollary 13. *Let (X_1, Σ_1) and (X_2, Σ_2) be permissible with $b_1(X_i) = 0$. Let ϕ and ψ be two different identifications for $Y = \Sigma \times \mathbb{S}^1$ and consider the two different connected sums along Σ, $X(\phi)$ and $X(\psi)$. Suppose that $X(\phi)$ has only two basic classes $\pm\kappa$. Then the same is true for $X(\psi)$ and the coefficients coincide (up to sign). Also if the invariants of $X(\phi)$ vanish (no basic classes), so do the invariants of $X(\psi)$.*

Proof. We do the case of two basic classes. The other one is analogous. Suppose $\phi = \mathrm{Id}$, put $X = X(\phi)$ and let $\pm\kappa$ be the two basic classes, with $\kappa \cdot \Sigma = 2$. Let $c_{\kappa,w}$ be its coefficient. We now want to prove that this implies that there is only one basic class $\tilde{K}_i$ with $\tilde{K}_i \cdot \Sigma = 2$ and only one basic class $\tilde{L}_j$ with $\tilde{L}_j \cdot \Sigma = 2$. The result follows from this applying theorem 10.

Suppose that we can find $S_i \in H_2(\tilde{X}_i)$ with $\alpha = S_1 \cap [Y] = -S_2 \cap [Y] \in H_1(Y;\mathbb{Z})$ such that all the values $\tilde{K}_i \cdot S_1$ are different among them, and all the values $\tilde{L}_j \cdot S_2$ are also different among them (where $\tilde{K}_i$ and $\tilde{L}_j$ run through all the basic classes in $\tilde{X}_1$ and $\tilde{X}_2$ evaluating 2 on Σ). Then reorder the subindices in such a way that

$$\tilde{K}_1 \cdot S_1 < \tilde{K}_2 \cdot S_1 < \cdots < \tilde{K}_{n_1} \cdot S_1$$

$$\tilde{L}_1 \cdot S_2 < \tilde{L}_2 \cdot S_2 < \cdots < \tilde{L}_{n_2} \cdot S_2$$

We can easily arrange $D_i^o \subset X_i^o$ with $\partial D_1^o = -\partial D_2^o = \gamma$ with $[\gamma] = \alpha$ such that the corresponding D_i is S_i. Set $D = D_1^o + D_2^o \in H_2(X)$ and apply theorem 10. We have

$$c_{\kappa,w} e^{t\kappa \cdot D} = \sum_{\tilde{K}_i \cdot \Sigma = \tilde{L}_j \cdot \Sigma = 2} -\frac{1}{2} \tilde{a}_{i,w} \tilde{b}_{j,w} e^{t(\tilde{K}_i \cdot S_1 + \tilde{L}_j \cdot S_2)}$$

Considering the exponentials with the smallest and with the largest exponents, we see that we must have $\tilde{K}_1 \cdot S_1 + \tilde{L}_1 \cdot S_2 = \tilde{K}_{n_1} \cdot S_1 + \tilde{L}_{n_2} \cdot S_2$, from where the result.

To find the required collection of S_i, we consider all the differences $\alpha_{ij} = \tilde{K}_i - \tilde{K}_j$, $\beta_{ij} = \tilde{L}_i - \tilde{L}_j$, $i \neq j$. Consider $\alpha \in H_1(Y;\mathbb{Z})$ such that $\alpha \cdot \alpha_{ij} \neq 0$ for any α_{ij} which happens to be in the image of the homomorphism $H^1(Y) \cong H_2(Y) \hookrightarrow H_2(\tilde{X}_1) \cong H^2(\tilde{X}_1)$, and $\alpha \cdot \beta_{ij} \neq 0$ when β_{ij} is in the same condition with $\tilde{X}_2$ replacing $\tilde{X}_1$. Now we can choose $S_1 \in H_2(\tilde{X}_1)$ with $S_1 \cap [Y] = \alpha$ such that $\alpha_{ij} \cdot S_1 \neq 0$ (indeed the bad set is a finite union of hyperplanes). Analogously we choose S_2. □

3. Gluing theory

Let $X = X_1^o \cup_Y X_2^o$, $D \in H_2(X)$. Substitute D by a rational multiple if necessary so that $D|_Y \in H_1(Y;\mathbb{Z})$ and it is primitive. Represent D by a cycle so $D = D_1^o + D_2^o$, $D_i^o \subset X_i^o$, $\partial D_1^o = -\partial D_2^o = \gamma$, with $\gamma \subset Y$ an embedded curve in Y (when X_1^o has a cylindrical end, we suppose $D_1^o \cap (Y \times [0,\infty)) = \gamma \times [0,\infty)$, and analogously for X_2^o).

Proposition 14 ([2][10]). *Suppose $w|_Y$ odd. Then there are Fukaya-Floer homology groups $HFF_*(Y,\gamma)$ graded mod 4 such that (X_i^o, D_i^o) define relative invariants $\phi^{w_1}(X_1^o, e^{tD_1^o}) \in HFF_*(Y,\gamma)$, $\phi^{w_2}(X_2^o, e^{tD_2^o}) \in HFF_*(-Y,-\gamma)$. There is a natural pairing such that*

$$D_X^{(w,\Sigma)}(e^{tD}) = < \phi^{w_1}(X_1^o, e^{tD_1^o}), \phi^{w_2}(X_2^o, e^{tD_2^o}) > . \tag{1}$$

When $b^+ = 1$, the invariants are calculated for a long neck, i.e. we refer to the invariants defined by Σ.

In our case $Y \cong (-Y)$. Also, as explained in [2], $HFF_*(Y,\gamma)$ is the limit of a spectral sequence whose E_3-term is $HF_*(Y) \otimes \hat{H}_*(\mathbb{CP}^\infty)$ (the hat means the natural completion of $H_*(\mathbb{CP}^\infty)$), and d_3 is multiplication by $\mu(\gamma)$.

First, $HF_*(Y) = HF_*(\Sigma \times \mathbb{S}^1) \cong HF_*^{\text{symp}}(M_\Sigma^{\text{odd}}) \cong H_*(M_\Sigma^{\text{odd}})$ as vector spaces (we are using rational coefficients), where M_Σ^{odd} is the moduli space of odd degree rank two stable vector bundles on Σ (with the grading considered mod 4) (for the first isomorphism see [5], for the second see [12]). For $g = 2$, these groups were computed by Donaldson [4], finding that $H_*(M_\Sigma^{\text{odd}})$ has an even part of dimension 4 and an odd part of dimension 4 (in the even part the intersection product is symmetric, in the odd part it is antisymmetric).

There is a conjecture asserting that multiplication by $\mu(\gamma)$ is intertwined with quantum multiplication by $\mu(\gamma)$ (see [4] [10]). In [10, chapter 5], the author has studied the implications of such a conjecture. Here we want to avoid it altogether.

Essentially we have two cases to deal with, $\gamma = \mathbb{S}^1 \subset \Sigma \times \mathbb{S}^1 = Y$ and $\gamma \subset \Sigma \subset \Sigma \times \mathbb{S}^1 = Y$.

- $\gamma=\mathbb{S}^1\subset\Sigma\times\mathbb{S}^1$. Now all the differentials in the E_3 term of the spectral sequence are of the form $H_{\text{odd}}(M_\Sigma^{\text{odd}})\to H_{\text{even}}(M_\Sigma^{\text{odd}})$ and $H_{\text{even}}(M_\Sigma^{\text{odd}})\to H_{\text{odd}}(M_\Sigma^{\text{odd}})$. When the boundary cycle is $\gamma=\mathbb{S}^1$ and thus invariant under the action of the group $\text{Diff}(\Sigma)$ on $Y=\Sigma\times\mathbb{S}^1$, the differentials commute with the action of $\text{Diff}(\Sigma)$. As there are elements $\rho\in\text{Diff}(\Sigma)$ acting as -1 on $H^1(\Sigma)$, we have that ρ acts as -1 on $H_{\text{odd}}(M_\Sigma^{\text{odd}})$ and as 1 on $H_{\text{even}}(M_\Sigma^{\text{odd}})$. Therefore the differentials are zero and the spectral sequence degenerates in the third term. This implies that $HFF_{\text{even}}(Y,\gamma)=V_4[[t]]$, where $V_4=HF_{\text{even}}(Y)$ has dimension 4. The relative invariants will be $\phi^{w_1}(X_1^o,e^{tD_1^o})\in V_4[[t]]$. We do not consider the odd part since the pairing is antisymmetric in the odd part, but the expression (1) is symmetric.
- $\gamma\subset\Sigma\subset\Sigma\times\mathbb{S}^1$. The E_3 term of the usual spectral sequence is $HF_*(Y)\otimes\hat{H}_*(\mathbb{CP}^\infty)$, with differencital d_3 given by
$$\mu(\gamma):H_i(M_\Sigma^{\text{odd}})\otimes H_j(\mathbb{CP}^\infty)\to H_{i-3}(M_\Sigma^{\text{odd}})\otimes H_{j+2}(\mathbb{CP}^\infty).$$

Lemma 15. *The image of $d_3:HF_3\to HF_0$ is one-dimensional and the kernel of $d_3:HF_2\to HF_3$ is one-dimensional.*

Proof. Let us see first that
$$\mu(\gamma):HF_*(Y)\to HF_*(Y)$$
is non-zero. We decompose $\Sigma\times\mathbb{CP}^1=A\cup_Y A$. From the definition of $\mu(\gamma)$ (see [3] [1] [10]), we have that for $X=X_1^o\cup_Y X_2^o$, $z_i\in\mathbb{A}(X_i^o)$, $\phi^w(X_i^o,z_i)\in HF_*(Y)$ and $\beta\in H_*(Y)$, it is $\phi^w(X_1^o,\beta z_1)=\mu(\beta)(\phi^w(X_1^o,z_1))$. Also $D_X^{(w,\Sigma)}(z_1z_2)$ $=<\phi^w(X_1^o,z_1),\phi^w(X_2^o,z_2)>$. We have thus
$$D^{(w,\Sigma)}_{\Sigma\times\mathbb{CP}^1}(\gamma_1\gamma_2)=<\phi^w(A,1),\mu(\gamma_1)\mu(\gamma_2)(\phi^w(A,1))>.$$
The invariant of the left hand side corresponds to the six-dimensional moduli space. This is in fact M_Σ^{odd}. From [4] [15] we know that this number is non-zero (actually $\epsilon_S(w)\,\gamma_1\cdot\gamma_2$, with $\epsilon_S(w)=(-1)^{\frac{K_S\cdot w+w^2}{2}}$). Therefore $\mu(\gamma)\neq 0$.

Under the intersection pairing, $HF_2\cong(HF_0)^*$ and $HF_3\cong(HF_3)^*$. also, $d_3:HF_3\to HF_0$ and $d_3:HF_2\to HF_3$ are dual maps, so the dimensions of $\ker(d_3:HF_2\to HF_3)$ and $\text{im}(d_3:HF_3\to HF_0)$ coincide. Since d_3 is non-zero, these dimensions are at least one. They cannot be two because that would imply that $HFF_{\text{even}}(Y,\gamma)=HF_0\oplus 0\oplus 0\oplus\cdots$ and hence
$$D_X^{(w,\Sigma)}(e^{tD})=<\phi^w(X_1^o,e^{tD_1^o}),\phi^w(X_2^o,e^{tD_2^o})>=0$$
for any case in which $D=D_1^o+D_2^o$, $D_i^o\subset X_i^o$, with $\partial D_1^o=-\partial D_2^o=\gamma$. In particular, the invariants $D_X^{(w,\Sigma)}$ would vanish whenever $X=X_1\#_\Sigma X_2$ with $b_1(X_i)=0$, $i=1,2$. But this is impossible, as we will see examples in the proof of theorem 10 when the invariants do not vanish (these examples are independent of the computation of $HFF_*(Y,\gamma)$ for $\gamma\subset\Sigma\subset Y$). $\square$

From this we write the even part of the E_5 term of the spectral sequence. Set $HF_2^{\text{red}} = \ker(d_3 : HF_2 \to HF_3)$, $HF_0^{\text{red}} = HF_0/\text{im}(d_3 : HF_3 \to HF_0)$. The even part of the E_5 term is

$$\begin{array}{ccccccccccc} (E_5)_0 & = & HF_0 & \oplus & 0 & \oplus & HF_2^{\text{red}} & \oplus & 0 & \oplus & HF_0^{\text{red}} & \oplus & \cdots \\ (E_5)_2 & = & HF_2^{\text{red}} & \oplus & 0 & \oplus & HF_0^{\text{red}} & \oplus & 0 & \oplus & HF_2^{\text{red}} & \oplus & \cdots \end{array}$$

The differential d_5 has to be zero (at least on the even part of E_5), since otherwise we would have again that the invariants $D_X^{(w,\Sigma)}$ vanish whenever $X = X_1 \#_\Sigma X_2$ with $b_1(X_i) = 0$, $i = 1, 2$.

Hence $HFF_{\text{even}}(Y, \gamma)$ is equal to this $(E_5)_{\text{even}}$. Write $HF_0 = \mathbb{R} \oplus HF_0^{\text{red}}$, where $\mathbb{R}$ is the orthogonal complement to HF_2^{red}. Then $HFF_{\text{even}}(Y, \gamma) = \mathbb{R} \oplus V_2[[t]]$, with $V_2 = HF_0^{\text{red}} \oplus HF_2^{\text{red}}$ of dimension 2, the pairing vanishing on the $\mathbb{R}$-summand. The relative invariants will be $\phi^w(X_1^o, e^{tD_1^o}) \in V_2[[t]]$. Again we do not consider the odd part, and we also ignore the extra $\mathbb{R}$-summand.

Proposition 16. 1. *There is a vector space V_4 of dimension 4 endowed with a symmetric bilinear form such that for every permissible (X, Σ) and $D^o \subset X^o$ with $\partial D^o = \mathbb{S}^1$, we have $\phi^w(X^o, e^{tD^o}) \in V_4[[t]]$. For $X = X_1^o \cup_Y X_2^o$, $D = D_1^o + D_2^o$, $\partial D_1^o = -\partial D_2^o = \mathbb{S}^1$, we have*

$$D_X^{(w,\Sigma)}(e^{tD}) = < \phi^{w_1}(X_1^o, e^{tD_1^o}), \phi^{w_2}(X_2^o, e^{tD_2^o}) > .$$

2. *There is a vector space V_2 of dimension 2 endowed with a symmetric bilinear form such that for every permissible (X, Σ) and $D^o \subset X^o$ with $\partial D^o = \gamma$ not representing in homology a multiple of $[\mathbb{S}^1]$, we have $\phi^w(X^o, e^{tD^o}) \in V_2[[t]]$. For $X = X_1^o \cup_Y X_2^o$, $D = D_1^o + D_2^o$, $\partial D_1^o = -\partial D_2^o = \gamma$, we have*

$$D_X^{(w,\Sigma)}(e^{tD}) = < \phi^{w_1}(X_1^o, e^{tD_1^o}), \phi^{w_2}(X_2^o, e^{tD_2^o}) > .$$

4. Proof of Theorems

Proof of Theorem 6.

The fact that X is of simple type will be proved in the proof of theorem 10. Let us analyse the following list of examples (we use proposition 3 for finding the invariants $D_X^{(w,\Sigma)}$).

- X a $K3$ surface blown-up twice with E_1 and E_2 the two exceptional divisors, $\Sigma = S - E_1 - E_2$ for S a tight surface of genus 2 in $K3$, $w = E_1$, D a cohomology class coming from the $K3$ such that $D \cdot S = 1$, $D^2 = 0$. We get $D_X^{(w,\Sigma)}(e^{s\Sigma + tD}) = -e^{ts} \frac{e^{2s} - e^{-2s}}{4}$.
- X, Σ, D as before, but now $w \in H^2(K3)$, with $w \cdot S = 1$. We will get $D_X^{(w,\Sigma)}(e^{s\Sigma + tD}) = (-1)^{\frac{w^2}{2}} e^{ts} \frac{e^{2s} + e^{-2s}}{4} - \frac{1}{2} e^{-ts}$.

- X a $K3$ surface, Σ a tight torus with an added trivial handle to make it of genus 2, $w \in H^2(X;\mathbb{Z})$ such that $w \cdot \Sigma = 1$ and D with $D \cdot \Sigma = 1$, $D^2 = 0$. Then $D_X^{(w,\Sigma)}(e^{s\Sigma+tD}) = -e^{-ts}$.
- $S = \mathbb{CP}^1 \times \Sigma$, $w = \text{P.D.}[\mathbb{CP}^1]$, $D = \mathbb{CP}^1$. Then $g(t,s) = D_S^{(w,\Sigma)}(e^{s\Sigma+tD})$ is a non-zero function with monomials of degree at least three (since the smallest moduli space has dimension six).

We conclude that there are at least four functions, say $f_1 = e^{ts+2s}$,$f_2 = e^{ts-2s}$, $f_3 = e^{-ts}$ and $f_4 = g(t,s)$, appearing in some $D_X^{(w,\Sigma)}(e^{s\Sigma+tD})$ (for different permissible pairs (X,Σ) and $D \cdot \Sigma = 1$), and linearly independent over $\mathcal{F}(t)$, the field of (formal) Laurent series on t. We have the map

$$< \cdot, \phi^w(A, e^{t\Delta+s\Sigma}) >: V_4[[t]] \to \mathbb{R}^4[[t]] \tag{2}$$

which assigns to $\phi(t) \in V_4[[t]]$ a four-vector whose i-th coordinate (actually we should tensor $V_4[[t]]$ and $\mathbb{R}^4[[t]]$ with $\mathcal{F}(t)$, but we will not be explicit about this point) is the coefficient (in $\mathcal{F}(t)$) of f_i in $< \phi(t), \phi^w(A, e^{t\Delta+s\Sigma}) >$ (where $\Delta = \text{pt} \times D^2 \subset A$). Therefore $\phi^w(X^o, e^{tD^o})$ is sent to $(c_{X,i}(t))$, the coefficients of f_i in $D_X^{(w,\Sigma)}(e^{s\Sigma+tD})$, where $D = D^o + \Delta$ (so $D_X^{(w,\Sigma)}(e^{s\Sigma+tD}) = \sum c_{X,i}(t) \cdot f_i(t,s)$).

From the examples, the map above is an isomorphism (over $\mathcal{F}(t)$), so we can push the product from $V_4[[t]]$ to $\mathbb{R}^4[[t]]$ and we shall have a universal symmetric matrix $M(t) = (M_{ij}(t))$ such that

$$D_X^{(w,\Sigma)}(e^{tD}) = \sum_{i,j} c_{X_1,i}(t) M_{ij}(t) c_{X_2,j}(t).$$

Remark 17. Since the map (2) is an isomorphism, $D_X^{(w,\Sigma)}((x^2-4)e^{tD+s\Sigma}) = 0$ if and only if $\phi^w(X^o, (x^2-4)e^{tD^o}) = 0$.

The image of all possible $\phi^w(X^o, e^{tD^o})$ with X of simple type, $b_1 = 0$ and $b^+ > 1$, is exactly the three-dimensional subspace given by equating the last coordinate to zero. So when X_i are both of simple type with $b_1 = 0$ and $b^+ > 1$, write $\mathbb{D}_{X_1}^w(e^\alpha) = e^{Q(\alpha)/2} \sum a_{i,w} e^{K_i \cdot \alpha}$ and $\mathbb{D}_{X_2}^w(e^\alpha) = e^{Q(\alpha)/2} \sum b_{j,w} e^{L_j \cdot \alpha}$. Then

$$\begin{cases} c_{X_1,1}(t) = e^{Q(tD_1)/2} \sum\limits_{K_j \cdot \Sigma = 2} a_{j,w} e^{tK_j \cdot D_1} \\ c_{X_1,2}(t) = e^{Q(tD_1)/2} \sum\limits_{K_j \cdot \Sigma = -2} a_{j,w} e^{tK_j \cdot D_1} \\ c_{X_1,3}(t) = e^{-Q(tD_1)/2} \sum\limits_{K_j \cdot \Sigma = 0} i^{-d_0} a_{j,w} e^{ti\, K_j \cdot D_1} \\ c_{X_1,4}(t) = 0 \end{cases} \tag{3}$$

and

$$D_X^{(w,\Sigma)}(e^{tD}) = \sum_{1 \le i,j \le 3} c_{X_1,i}(t) M_{ij}(t) c_{X_2,j}(t). \tag{4}$$

This expression is valid for any $D \in H_2(X)$ with $D|_Y = [\mathbb{S}^1]$. For $D \in \mathcal{H}$ we have

$$D_X^{(w,\Sigma)}(e^{tD}) = \sum_{1\leq i,j\leq 3} c_{X_1,i}(t) M_{ij}(t(D\cdot\Sigma)) c_{X_2,j}(t).$$

Considering D, $D_1 + r\Sigma$, $D_2 - r\Sigma$ in (4), we get that $M_{ij}(t) = 0$ for $i \neq j$, $1 \leq i, j \leq 3$. Now consider the case in which both X_i and Σ_i are as in the third example of the list. Then $X = X_1 \#_\Sigma X_2$ splits off a $\mathbb{S}^2 \times \mathbb{S}^2$, so its invariants are zero. Therefore $M_{33}(t) = 0$. So finally we have (using also $D^2 = D_1^2 + D_2^2$),

$$D_X^{(w,\Sigma)}(e^{tD}) = e^{Q(tD)/2}\Big(\sum_{K_i\cdot\Sigma = L_j\cdot\Sigma = 2} M_{11}(t(D\cdot\Sigma)) a_{i,w} b_{j,w}\, e^{(K_i\cdot D_1 + L_j\cdot D_2)t} +$$

$$\sum_{K_i\cdot\Sigma = L_j\cdot\Sigma = -2} M_{11}(t(D\cdot\Sigma)) a_{i,w} b_{j,w}\, e^{(K_i\cdot D_1 + L_j\cdot D_2)t}\Big).$$

Let us now compute $M_{11}(t)$ and $M_{22}(t)$. By the universality and since all the manifolds involved can be chosen of simple type, one has $M_{11}(t) = \sum c_n\, e^{nt}$ and $M_{22}(t) = \sum d_n\, e^{nt}$, finite sums of exponentials. Let $S = \mathbb{CP}^2 \# 10\overline{\mathbb{CP}}^2$ be the rational elliptic surface blown-up once. Denote by $E_1, \ldots, E_{10}$ the exceptional divisors and let $T_1 = C - E_1 - \cdots - E_9$, $T_2 = C - E_1 - \cdots - E_8 - E_{10}$, where C is the cubic curve in $\mathbb{CP}^2$. So T_1 and T_2 can be represented by smooth tori of self-intersection zero and with $T_1 \cdot T_2 = 1$. We can glue two copies of S along T_1. The result is a K3 surface $S \#_{T_1} S$ blown-up twice. The T_2 pieces glue together to give a genus 2 Riemann surface Σ_2 of self-intersection zero which intersects T_1 in one point. This is actually the pair (B, Σ) we introduced before the statement of theorem 10. Now set $X = (S\#_{T_1} S) \#_{\Sigma_2} (S \#_{T_1} S)$, which is of simple type (by [8], since it contains a torus of self-intersection 0 intersecting an embedded (-2)-sphere transversely in one point). Now call $\Sigma = \Sigma_2$ and get D piecing together both T_1's in $S \#_{T_1} S$. So (choose $w = T_1$ on $S \#_{T_1} S$)

$$D_X^{(D,\Sigma)}(e^{tD+s\Sigma}) = e^{Q(tD+s\Sigma)/2}\Big(\sum_{K_i\cdot\Sigma = L_j\cdot\Sigma = 2} c_n a_i b_j\, e^{2s+nt} + \sum_{K_i\cdot\Sigma = L_j\cdot\Sigma = -2} d_n a_i b_j\, e^{-2s+nt}\Big) =$$

$$= e^{ts}\Big(\sum \frac{c_n}{16} e^{2s+nt} + \sum \frac{d_n}{16} e^{-2s+nt}\Big),$$

since T_1 evaluates 0 on basic classes being a torus of self-intersection zero (the coefficient $\frac{1}{16}$ appears from the explicit computation of the basic classes of the K3 surface blown-up in two points, see below (6)). The trick is now to use the symmetry fact that $X = (S\#_{T_2} S) \#_{\Sigma_1} (S\#_{T_2} S)$, where Σ_1 comes from gluing together both T_1's. Under this diffeomorphism $D = \Sigma_1$ and Σ comes from piecing together both T_2's in $S \#_{T_2} S$. Hence

$$D_X^{(\Sigma,D)}(e^{tD+s\Sigma}) = e^{ts}\Big(\sum \frac{c_n}{16} e^{2t+ns} + \sum \frac{d_n}{16} e^{-2t+ns}\Big).$$

Both expressions are equal, and equal to $\mathbb{D}_X^{D+\Sigma}(e^{tD+s\Sigma})$. From here we deduce that $c_n=0$ unless $n=\pm 2$ and $d_n=0$ unless $n=\pm 2$. Also $c_{-2}=d_2$. Put $l=c_2+c_{-2}$, so $D_X^{(D,\Sigma)}(e^{s\Sigma})=\frac{l}{16}e^{2s}-\frac{l}{16}e^{-2s}$ (note that $d_0(X,D)=-15$ is odd). So $c_2-d_{-2}=2\,l$. But $c_2=\pm d_{-2}$, so it has to be $c_{-2}=d_2=0$ and $c_2=-d_{-2}=l$. Thus

$$D_X^{(D,\Sigma)}(e^{tD+s\Sigma})=e^{ts}(\frac{l}{16}e^{2s+2t}-\frac{l}{16}e^{-2s-2t}). \tag{5}$$

So $M_{11}(t)=l\,e^{2t}$ and $M_{22}(t)=-l\,e^{-2t}$. To get the theorem it only remains to prove

Lemma 18. $l=-32$.

Proof. Let $e_i=\phi^w(A,\Sigma^i)\in HF_*(Y)$, $i=0,1,2,3$. Then $\{e_i\}$ is a basis for $HF_{\text{even}}(Y)$, since the latter is a vector space of dimension 4 and the intersection matrix for $(e_i\cdot e_j)$ is invertible. Actually, it is

$$N=\begin{pmatrix} 0 & 0 & 0 & -1/2 \\ 0 & 0 & -1/2 & 0 \\ 0 & -1/2 & 0 & -2 \\ -1/2 & 0 & -2 & 0 \end{pmatrix}.$$

To check this we note that $e_i\cdot e_j=<\phi^w(A,\Sigma^i),\phi^w(A,\Sigma^j)>=D_S^{(w,\Sigma)}(\Sigma^{i+j})$, $S=\Sigma\times\mathbb{CP}^1=A\cup_Y A$, $w=\mathbb{CP}^1$, so we only need to find $D_S^{(w,\Sigma)}(\Sigma^3)$ and $D_S^{(w,\Sigma)}(\Sigma^5)$. For the first one, the moduli space is M_Σ^{odd}, which is six-dimensional. Then $\mu(\Sigma)^3=1/2$, with $\mu(\Sigma)\in H^2(M_\Sigma^{\text{odd}})$, from [15]. This invariant is computed using the complex orientation of the moduli space which differs from the one we use by a factor $\epsilon_S(w)=(-1)^{\frac{K_S\cdot w+w^2}{2}}=-1$. So $D_S^{(w,\Sigma)}(\Sigma^3)=-1/2$. For the second one, the moduli space is ten-dimensional, corresponding to $w=\mathbb{CP}^1+\Sigma$ and polarisation close to Σ. For a polarisation close to $\mathbb{CP}^1$, the moduli space is empty [13]. There is only one wall corresponding to $\zeta=-\Sigma+\mathbb{CP}^1$. Now we can apply the formulas in [11] for wall-crossing when the irregularity is not zero, noting that ζ is a good wall. This gives $D_S^{(w,\Sigma)}(\Sigma^5)=-2$. Also this can be computed directly with an explicit description of the algebraic moduli space and we propose this calculation as a good exercise.

Now consider the pair (B,Σ). Recall that B is the $K3$ surface blown-up in two points. Let E_1 and E_2 be the two exceptional divisors. We have $\mathbb{D}_B(e^\alpha)=e^{Q(\alpha)/2}\sinh(E_1\cdot\alpha)\sinh(E_2\cdot\alpha)$, so by proposition 3 with $w=T_1$,

$$D_B^{(w,\Sigma)}(e^\alpha)=e^{Q(\alpha)/2}\frac{1}{2}\cosh((E_1+E_2)\cdot\alpha)+e^{-Q(\alpha)/2}\frac{1}{2}\cos((E_1-E_2)\cdot\alpha). \tag{6}$$

Then $D_B^{(w,\Sigma)}(e^{s\Sigma}) = \frac{1}{2}\cosh(2s)$, so in the basis dual to $\{e_i\}$, $\phi^w(B^o,1) = (1/2,0,2,0)$ and $\phi^w(B^o,\Sigma) = (0,2,0,8)$. Therefore for $C = B\#_\Sigma B$,

$$D_C^{(w,\Sigma)}(\Sigma) =< \phi^w(B^o,1), \phi^w(B^o,\Sigma) >= (1/2,0,2,0)N^{-1}\, N\, N^{-1} \begin{pmatrix} 0 \\ 2 \\ 0 \\ 8 \end{pmatrix} = -8.$$

From formula (5), we get $D_C^{(w,\Sigma)}(\Sigma) = \frac{l}{4}$, so $l = -32$. □

Proof of Theorem 9.

We are going to check that $D_X^{(w,\Sigma)}((x^2-4)^2 e^{tD}) = 0$, for all $D \in H_2(X)$ with $D \cdot \Sigma = 1$. This is clearly enough to infer the result. Put $D = D^o + \Delta$. If X is of simple type with $b_1 = 0$ and $b^+ > 1$, we have

$$0 = D_X^{(w,\Sigma)}((x^2-4)e^{tD+s\Sigma}) =< \phi^w(X^o, e^{tD^o}), \phi^w(A,(x^2-4)e^{t\Delta+s\Sigma}) > .$$

The vectors $\phi^w(X^o, e^{tD^o})$ (with X being of simple type with $b_1 = 0$ and $b^+ > 1$) generate a the 3-dimensional subspace $V_3[[t]]$ in $V_4[[t]]$ given by equating the last coordinate to zero. Then $\phi^w(A,(x^2-4)e^{t\Delta+s\Sigma})$ lies in the subspace orthogonal to $V_3[[t]]$. As the pairing in $V_4[[t]]$ is non-degenerate and $V_3[[t]]$ contains an isotropic vector (from the computation of the $M_{ij}(t)$ in the proof of theorem 6, the intersection matrix restricted to $V_3[[t]]$ is degenerate), $\phi^w(A,(x^2-4)e^{t\Delta+s\Sigma})$ is isotropic and hence

$$< \phi^w(A,(x^2-4)e^{t\Delta+s\Sigma}), \phi^w(A,(x^2-4)e^{t\Delta}) >= D^{(w,\Sigma)}_{\Sigma\times\mathbb{CP}^1}((x^2-4)^2 e^{t\mathbb{CP}^1+s\Sigma}) = 0,$$

from where $\phi^w(A,(x^2-4)^2 e^{t\Delta}) = 0$ (remark 17) and hence the result.

Proof of Theorem 10.

Recall the permissible pair (B,Σ), where B is the $K3$ surface blown-up in two points with E_1 and E_2 the exceptional divisors, and $\Sigma = S - E_1 - E_2$ is the proper transform of a tight embedded surface $S \subset K3$ of genus 2. Call $C = B\#_\Sigma B$ the double of B, i.e. the connected sum of B with itself with the identification which is given by the natural orientation reversing diffeomorphism of $Y = \partial B^o$ to itself. As in the proof of theorem 6, we choose $D \subset C$ to be the embedded surface obtained by piecing together two fibres of the natural elliptic fibration of B. Then D is a genus 2 Riemann surface of self-intersection zero. Also take $w = \text{P.D.}[D] \in H^2(X;\mathbb{Z})$. Then equation (5) gives

$$D_C^{(w,\Sigma)}(e^{tD+s\Sigma}) = -e^{ts}(2\, e^{2s+2t} - 2\, e^{-2s-2t}).$$

We can take a collection α_i, $1 \le i \le 4$, of loops in a fibre $\Sigma \subset \partial B^o$, which together with $\mathbb{S}^1$ form a basis for $H_1(Y)$, such that they can be capped off with embedded (-1)-discs D_i (writing $B = S\#_{T_1}S$, as in the proof of theorem 6, we consider the vanishing discs of the elliptic fibration of S with fibre T_2, see [6, page 167], since they do not intersect T_1). Now these discs can be glued together pairwise when forming

$C = B^o \cup_Y B^o$ to give a collection of (-2)-embedded spheres $S_i = D_i \cup_{\alpha_i} D_i$. Every one of these discs has a dual torus T_i, by considering another loop in $\Sigma \subset \partial B^o$, say β_i, with $\alpha_i \cdot \beta_i = 1$, and putting $T_i = \beta_i \times \mathbb{S}^1 \subset \Sigma \times \mathbb{S}^1$. Then the elements $S_i + T_i$ are represented by embedded tori of self-intersection zero. Hence the manifold C is of simple type [8], and the basic classes evaluate zero on T_i and on $S_i + T_i$. Our conclusion is

$$D_C^{(w,\Sigma)}(e^\alpha) = -4\, e^{Q(\alpha)/2} \sinh(K \cdot \alpha),$$

with $K \in H^2(C;\mathbb{Z})$ being the only cohomology class with

- $K \cdot \alpha = (E_1 + E_2) \cdot \alpha$ for $\alpha \in H_2(B^o)$.
- $K \cdot \Sigma = K \cdot D = 2$.
- $K \cdot S_i = K \cdot T_i = 0$, for all i.

We split K into two symmetric pieces $K^o \subset B^o$. The boundary of K^o is $\partial K^o = 2\mathbb{S}^1$ and $(K^o)^2 = 2$ since $K^2 = 4$.

Analogously, the manifold $C_2 = C \#_\Sigma B$ is of simple type and

$$D_{C_2}^{(w,\Sigma)}(e^\alpha) = 32\, e^{Q(\alpha)/2} \cosh(K_2 \cdot \alpha).$$

for a unique $K_2 \in H^2(C_2;\mathbb{Z})$. Let D_2 be obtained gluing the D coming from C with one fibre of the elliptic fibration of B. Then

$$D_{C_2}^{(w,\Sigma)}(e^{tD_2 + s\Sigma}) = e^{ts}(16\, e^{2s+2t} + 16\, e^{-2s-2t}).$$

So there are two functions, $\tilde{f}_1 = e^{ts+2s}$,$\tilde{f}_2 = e^{ts-2s}$, appearing in some $D_{\tilde{X}}^{(w,\Sigma)}(e^{s\Sigma+tD})$ (for different permissible (X,Σ) with $b_1 = 0$, $\tilde{X} = X^o \cup_Y B^o$, $D \in H_2(\tilde{X})$, $D = D^o + D_B^o$, $\partial D^o = \gamma$), and linearly independent over $\mathcal{F}(t)$. Now we mimic the reasoning of the proof of theorem 6. We have a map

$$< \cdot, \phi^w(B^o, e^{tD_B^o + s\Sigma}) >: V_2[[t]] \to \mathbb{R}^2[[t]]$$

which is an isomorphism (over $\mathcal{F}(t)$), such that $D_{\tilde{X}}^{(w,\Sigma)}(e^{s\Sigma+tD}) = \sum c_{\tilde{X},i}(t) \cdot \tilde{f}_i(t,s)$. We shall have a universal symmetric matrix $\tilde{M}(t) = (\tilde{M}_{ij}(t))$ such that

$$D_X^{(w,\Sigma)}(e^{tD}) = \sum_{i,j} c_{\tilde{X}_1,i}(t) \tilde{M}_{ij}(t) c_{\tilde{X}_2,j}(t).$$

This expression is valid for any $D \in H_2(X)$ with $D|_Y = [\gamma] \in H_1(Y)$, $\gamma \subset \Sigma \subset Y$ an embedded curve. Now $\phi^w(B^o, (x^2-4)e^{tD_B^o}) = 0$, since $D_C^{(w,\Sigma)}((x^2-4)e^{tD+s\Sigma}) = 0$, as C is of simple type. Therefore $\tilde{X}_i = X_i^o \cup_Y B^o$ are of simple type. Also this implies that $\phi^w(X_i^o, (x^2-4)e^{tD^o}) = 0$ and hence that $X = X_1^o \cup_Y X_2^o$ is of simple type. Now

$$\begin{cases} c_{\tilde{X}_1,1}(t) = e^{Q(tD_1)/2} \displaystyle\sum_{\tilde{K}_i \cdot \Sigma = 2} \tilde{a}_{i,w} e^{t\tilde{K}_i \cdot D_1} \\ c_{\tilde{X}_1,2}(t) = e^{Q(tD_1)/2} \displaystyle\sum_{\tilde{K}_i \cdot \Sigma = -2} \tilde{a}_{i,w} e^{t\tilde{K}_i \cdot D_1} \end{cases} \tag{7}$$

Again, as in the proof of theorem 6, we get that $\tilde{M}_{ij}(t) = 0$ for $i \neq j$. Consider $X_1 = B$, $X_2 = B$, $X = C$, $\tilde{X}_1 = C$, $\tilde{X}_2 = C$, let $D_C = D_B^o + D_B^o \in H_2(C)$ and put $D_1 = D_2 = D_C$, $D = D_C$. We get that $\tilde{M}_{11}(t) = -\frac{1}{2}e^{Q(tD_C)/2}$, $\tilde{M}_{22}(t) = \frac{1}{2}e^{Q(tD_C)/2}$. Now we have that for any $X = X_1^o \cup_Y X_2^o$ and $D = D_1^o + D_2^o$ with $\partial D_1^o = -\partial D_2^o = \gamma$, it is $Q(tD_C)/2 = Q(tD)/2 - Q(tD_1)/2 - Q(tD_2)/2$, where $D_i = D_i^o + D_B^o \in H_2(\tilde{X}_i)$. From this we get the sought expression in the statement of theorem 10, for any $D \in H_2(X)$ with $D|_Y \in H_1(Y)$ satisfying that $p_*(D|_Y) \in H_1(\Sigma)$ ($p : Y \to \Sigma$ the projection) is primitive and non-zero. Since we have chosen the map $D \mapsto (D_1, D_2)$ to be linear, this finishes the proof.

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