RING STRUCTURE OF THE FLOER COHOMOLOGY OF $\Sigma \times \mathbb{S}^1$

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Abstract. We give a presentation for the Floer cohomology ring $HF^*(\Sigma \times \mathbb{S}^1)$, where Σ is a Riemann surface of genus $g \geq 1$, which coincides with the conjectural presentation for the quantum cohomology ring of the moduli space of flat $SO(3)$-connections of odd degree over Σ. We study the spectrum of the action of $H_*(\Sigma)$ on $HF^*(\Sigma \times \mathbb{S}^1)$ and prove a physical assumption made in [1].

1. Introduction

Let $\Sigma = \Sigma_g$ be a Riemann surface of genus $g \geq 1$ and let $\mathcal{N}_g$ denote the moduli space of flat $SO(3)$-connections with nontrivial second Stiefel-Whitney class w_2. This is a smooth symplectic manifold of dimension $6g-6$. Alternatively, we can consider Σ as a smooth complex curve of genus g and $\mathcal{N}_g$ as the moduli space of odd degree rank two stable vector bundles on Σ with fixed determinant, which is a smooth complex variety of complex dimension $3g-3$. The symplectic deformation class of $\mathcal{N}_g$ only depends on the genus g and not on the particular complex structure on Σ.

We consider the following rings associated to the Riemann surface (we will always use $\mathbb{C}$-coefficients):

- $QH^*(\mathcal{N}_g)$ is the quantum cohomology of $\mathcal{N}_g$ (see [12]). This is well-defined since $\mathcal{N}_g$ is a positive symplectic manifold. As vector spaces, $QH^*(\mathcal{N}_g) = H^*(\mathcal{N}_g)$, but the multiplicative structure is different. The minimal Chern number of $\mathcal{N}_g$ is 2, so $QH^*(\mathcal{N}_g)$ is $\mathbb{Z}/4\mathbb{Z}$-graded (the grading comes from reducing mod 4 the $\mathbb{Z}$-grading of $H^*(\mathcal{N}_g)$). The ring structure of $QH^*(\mathcal{N}_g)$, called quantum multiplication, is a deformation of the usual cup product for $H^*(\mathcal{N}_g)$. It is associative and graded commutative. We remark that we do not introduce Novikov rings to define $QH^*(\mathcal{N}_g)$ as in [12, section 8] (otherwise

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said, as $H^2(\mathcal{N}_g) = \mathbb{Z}$, we should introduce an extra variable q of degree 4, then we equate $q = 1$).

- $HF_*^{\text{symp}}(\mathcal{N}_g)$ is the symplectic Floer homology of $\mathcal{N}_g$ (with the symplectomorphism $\phi = \text{id}$). The symplectic manifold $\mathcal{N}_g$ is connected, simply connected and $\pi_2(\mathcal{N}_g) = \mathbb{Z}$ (see [4, introduction]), so the groups $HF_*^{\text{symp}}(\mathcal{N}_g)$ are well-defined [5]. They are $\mathbb{Z}/4\mathbb{Z}$-graded. $HF_*^{\text{symp}}(\mathcal{N}_g)$ is endowed with the pair of pants product [11], which is an associative and graded commutative ring structure. The symplectic Floer cohomology of $\mathcal{N}_g$, $HF^*_{\text{symp}}(\mathcal{N}_g)$, is defined as the dual of the symplectic Floer homology. There is a Poincaré duality [11, remark 2.4] and a pairing $<,>$.
- $HF_*(\Sigma \times \mathbb{S}^1)$ is the instanton Floer homology of the three manifold $Y = \Sigma \times \mathbb{S}^1$ for the $SO(3)$-bundle with second Stiefel-Whitney class $w_2 = \text{P.D.}[\mathbb{S}^1] \in H^2(Y; \mathbb{Z}/2\mathbb{Z})$. This is defined in [6] and is $\mathbb{Z}/4\mathbb{Z}$-graded. We introduce a multiplication on $HF_*(Y)$ using a suitable four-dimensional cobordism [13, section 5]. Let X be the four manifold given as a pair of pants times Σ, which is a cobordism between $Y \sqcup Y$ and $-Y$. This gives a map $HF_*(Y) \otimes HF_*(Y) \to HF_*(Y)$, which is an associative and graded commutative ring structure on $HF_*(Y)$. Again, the instanton Floer cohomology of Y, $HF^*(Y)$, is the dual of $HF_*(Y)$. There is a pairing $<,>: HF^*(Y) \otimes HF^*(-Y) \to \mathbb{C}$. As $Y = \Sigma \times \mathbb{S}^1$ admits a orientation reversing self-diffeomorphism, we can identify $HF^*(-Y) \cong HF^*(Y)$, and hence we have a pairing on $HF^*(Y)$ (see [2] [3]). We will denote $HF_g^* = HF^*(\Sigma \times \mathbb{S}^1)$.

Theorem 1. *There are natural isomorphisms of vector spaces*

$$QH^*(\mathcal{N}_g) \cong HF^*_{\text{symp}}(\mathcal{N}_g) \cong HF^*(\Sigma \times \mathbb{S}^1). \tag{1}$$

Moreover the first isomorphism respects the ring structures.

Proof. The second isomorphism is due to Dostoglou and Salamon [4, theorem 10.1]. It is the particular case where one considers $\phi = \text{id} : \Sigma \to \Sigma$, in which the mapping torus of ϕ is $\Sigma \times \mathbb{S}^1$ and the $SO(3)$-bundle has $w_2 = \text{P.D.}[\mathbb{S}^1]$. The second isomorphism is a standard result obtained by Floer [5]. In [11, theorem 5.1] it is proved that the first isomorphism intertwines the products. □

Conjecture 2. *The second isomorphism in* (1) *is a ring isomorphism.*

In [13] D. Salamon announced a proof of conjecture 2 but finally he could not complete it. Many implications of this conjectural result to four-dimensional topology were given by Donaldson [3]. The author followed this program in [8] for small genus. Later he obtained very nice results on the behaviour of Donaldson invariants under the operation of connected sum along a Riemann surface [9] [10], exploiting only the isomorphism (1) as vector spaces.

Using physical methods, Vafa et al. [1] find a set of generators and relations for $QH^*(\mathcal{N}_g)$. There are two main assumptions in their argument. The first one is conjecture 2. The second one is that the spectrum of the action of $H_*(\Sigma)$ on $HF^*(\Sigma \times \mathbb{S}^1)$ can be read off from the Donaldson invariants of $\Sigma \times \mathbb{T}^2$.

Later Siebert and Tian [14] claimed to have found a mathematical proof for the presentation of $QH^*(\mathcal{N}_g)$ given in [1] but they could not yet finish their program. In this paper we prove that the set of generators and relations given in [1] is a presentation for $HF^*(\Sigma \times \mathbb{S}^1)$ (theorem 16), following a method inspired in that of Siebert and Tian [14]. This together with completion of the work [14] will produce a proof of conjecture 2 (although in a rather indirect way). We also prove the physical assumption on the spectrum of $HF^*(\Sigma \times \mathbb{S}^1)$ in [1] (proposition 20).

We leave the implications of theorem 16 to Donaldson invariants of four-manifolds (mostly in the case $b^+ = 1$) for future work.

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2. Ring structure of $H^*(\mathcal{N}_g)$

Let us recall the known description of the homology of $\mathcal{N}_g$ [7] [16]. Let $\mathcal{U} \to \Sigma \times \mathcal{N}_g$ be the universal bundle and consider the Künneth decomposition as in [7]

$$c_2(\mathrm{End}_0\,\mathcal{U}) = 2a[\Sigma] + 4\psi - b$$

with $\psi = \sum c_i \gamma_i^{\#}$, where $\{\gamma_1, \ldots, \gamma_{2g}\}$ is a symplectic basis of $H_1(\Sigma; \mathbb{Z})$ with $\gamma_i \gamma_{i+g} = [\Sigma]$ for $1 \le i \le g$, and $\{\gamma_i^{\#}\}$ is the dual basis of $H^1(\Sigma)$. In terms of the map $\mu : H_*(\Sigma) \to H^{4-*}(\mathcal{N}_g)$, given by $\mu(a) = -\frac{1}{4} p_1(\mathfrak{g}_{\mathcal{U}})/4$ (here $\mathfrak{g}_{\mathcal{U}} \to \Sigma \times \mathcal{N}_g$ is the associated universal $SO(3)$-bundle, and $p_1(\mathfrak{g}_{\mathcal{U}}) \in H^4(\Sigma \times \mathcal{N}_g)$ its first Pontrjagin class), we have

$$\begin{cases} a = 2\,\mu(\Sigma) \in H^2 \\ c_i = \mu(\gamma_i) \in H^3, \qquad 0 \le i \le 2g \\ b = -4\,\mu(x) \in H^4 \end{cases}$$

where $x \in H_0(\Sigma)$ is the class of the point, and $H^i = H^i(\mathcal{N}_g)$. These elements generate $H^*(\mathcal{N}_g)$ as a ring [7] [17]. So there is a basis $\{f_s\}_{s \in \mathcal{S}}$ for $H^*(\mathcal{N}_g)$ with elements of the form

$$f_s = a^n b^m c_{i_1} \cdots c_{i_r},$$

for a finite set $\mathcal{S}$ of multi-indices of the form $s = (n, m; i_1, \ldots, i_r)$, $n, m \ge 0$, $r \ge 0$, $1 \le i_1 < \cdots < i_r \le 2g$. The mapping class group $\mathrm{Diff}(\Sigma)$ acts on $H^*(\mathcal{N}_g)$, with the action factoring through the action of $\mathrm{Sp}\,(2g,\mathbb{Z})$ on $\{c_i\}$. The invariant part, $H_I^*(\mathcal{N}_g)$, is generated by a, b and $c = -2\sum_{i=0}^g c_i c_{i+g}$. Then

$$H_I^*(\mathcal{N}_g) = \mathbb{C}[a,b,c]/I_g, \tag{2}$$

where I_g is the ideal of relations satisfied by a, b and c. Here $\deg(a)=2$, $\deg(b)=4$, $\deg(c)=6$. Actually, a basis for $H_I^*(\mathcal{N}_g)$ is given by the monomials $\alpha^a\beta^b\gamma^c$, $a+b+c<g$ [16]. For $0\le k\le g$, the primitive component of $\Lambda^k H^3$ is

$$\Lambda_0^k H^3 = \ker(c^{g-k+1}: \Lambda^k H^3 \to \Lambda^{2g-k+2}H^3).$$

Then the $\mathrm{Sp}\,(2g,\mathbb{Z})$-decomposition of $H^*(\mathcal{N}_g)$ is [7]

$$H^*(\mathcal{N}_g) = \bigoplus_{k=0}^g \Lambda_0^k H^3 \otimes \mathbb{C}[a,b,c]/I_{g-k}.$$

Proposition 3 ([7][16])**.** *For $g=1$, let $q_1^1=a$, $q_1^2=b$, $q_1^3=c$. Define recursively, for $g\ge 1$,*

$$\begin{cases} q_{g+1}^1 = a q_g^1 + g^2 q_g^2 \\ q_{g+1}^2 = b q_g^1 + \frac{2g}{g+1} q_g^3 \\ q_{g+1}^3 = c q_g^1 \end{cases}$$

Then $I_g = (q_g^1, q_g^2, q_g^3)$, for all $g\ge 1$.

Proof. Define $\zeta_0=1$ and $\zeta_{n+1} = a\zeta_n + n^2 b\zeta_{n-1} + 2n(n-1)c\zeta_{n-2}$, for all $n\ge 0$. Theorem 3.1 in [7] establishes that $I_g = (\zeta_g, \zeta_{g+1}, \zeta_{g+2})$. The proposition follows from the obvious equalities

$$\begin{cases} q_g^1 = \zeta_g \\ q_g^2 = \frac{1}{g^2}(\zeta_{g+1} - a\zeta_g) \\ q_g^3 = \frac{1}{2g(g+1)}\left(\zeta_{g+2} - a\zeta_{g+1} - (g+1)^2 b\zeta_g)\right) \end{cases}$$

□

3. Ring structure of HF_g^*

With the aid of the basis $\{f_s\}_{s\in\mathcal{S}}$ for $H^*(\mathcal{N}_g)$ we are going to construct a basis for $HF_g^* = HF^*(\Sigma\times\mathbb{S}^1)$ to understand its ring structure. We need to use the gluing properties of the Floer homology of a three manifold. Put $Y=\Sigma\times\mathbb{S}^1$ and let $w_2 = \mathrm{P.D.}[\mathbb{S}^1] \in H^2(Y;\mathbb{Z}/2\mathbb{Z})$. Let us state the result that we shall use.

Proposition 4 ([2] [3] [8])**.** *For any smooth oriented four-manifold X^o with boundary $\partial X^o = Y$, any $w\in H^2(X;\mathbb{Z})$ with $w|_Y = P.D.[\mathbb{S}^1]$ in $H^2(Y;\mathbb{Z}/2\mathbb{Z})$, and any $z\in\mathbb{A}(X^o)$, we have defined a relative invariant $\phi^w(X^o,z)\in HF_*(Y)$. These relative invariants enjoy the following gluing property, suppose $X = X_1^o\cup_Y X_2^o$ is a closed four-manifold split into two open four manifolds X_i^o with $\partial X_1^o = Y$, $\partial X_2^o = -Y$, and*

$w \in H^2(X;\mathbb{Z})$ satisfying $w|_Y = P.D.[\mathbb{S}^1]$ in $H^2(Y;\mathbb{Z}/2\mathbb{Z})$. Put $w_i = w|_{X_i}$. Then for $z_i \in \mathbb{A}(X_i^o)$, $i = 1,2$, we have

$$D_X^{(w,\Sigma)}(z_1 z_2) =< \phi^{w_1}(X_1^o, z_1), \phi^{w_2}(X_2^o, z_2) >, \tag{3}$$

where $D_X^{(w,\Sigma)} = D_X^w + D_X^{w+\Sigma}$ (D_X^w is the Donaldson invariant of X for w, see also [8] [9]). When $b^+ = 1$, the invariants are calculated for a long neck, i.e. we refer to the invariants defined by Σ.

Consider the manifold $A = \Sigma \times D^2$, with boundary $Y = \Sigma \times \mathbb{S}^1$, and let $\Delta = \mathrm{pt} \times D^2 \subset A$ be the horizontal slice with $\partial\Delta = \mathbb{S}^1$. Put $w = \mathrm{P.D.}[\Delta] \in H^2(A;\mathbb{Z})$. Clearly $\mathbb{A}(A) = \mathbb{A}(\Sigma) = \mathrm{Sym}^*(H_0(\Sigma) \oplus H_2(\Sigma)) \otimes \bigwedge^* H_1(\Sigma)$. For every $s \in \mathcal{S}$, $f_s = a^n b^m c_{i_1} \cdots c_{i_r}$, define

$$\begin{aligned} z_s &= \Sigma^n x^m \gamma_{i_1} \cdots \gamma_{i_r} \in \mathbb{A}(\Sigma), \\ e_s &= \phi^w(A, z_s) \in HF^*(Y) = HF_g^* \end{aligned}$$

(here we identify Floer homology and Floer cohomology through Poincaré duality). Then $\{e_s\}_{s\in\mathcal{S}}$ is a basis for HF_g^*. This is a consequence of [9, lemma 21]. The product $HF_g^* \otimes HF_g^* \to HF_g^*$ is given by $\phi^w(A, z_S)\phi^w(A, z_{S'}) = \phi^w(A, z_S z_{S'})$. Then $\phi^w(A, 1)$ defines the neutral element of the product. As a consequence, the following elements are generators of HF_g^*,

$$\begin{cases} \alpha = 2\,\phi^w(A,\Sigma) \in HF_g^2 & \\ \psi_i = \phi^w(A,\gamma_i) \in HF_g^3, & 0 \le i \le 2g \\ \beta = -4\,\phi^w(A,x) \in HF_g^4 & \end{cases} \tag{4}$$

Note that there is an obvious epimorphism of rings $\mathbb{A}(\Sigma) \twoheadrightarrow HF_g^*$.

Theorem 5. *Denote by $*$ the product induced in $H^*(\mathcal{N}_g)$ by the product in HF_g^* under the isomorphism $H^*(\mathcal{N}_g) \xrightarrow{\cong} HF_g^*$ given by $f_s \mapsto e_s$, $s \in \mathcal{S}$. Then $*$ is a deformation of the cup-product graded modulo 4, i.e. for $f_1 \in H^i(\mathcal{N}_g)$, $f_2 \in H^j(\mathcal{N}_g)$, it is $f_1 * f_2 = \sum_{r\ge 0} \Phi_r(f_1, f_2)$, where $\Phi_r \in H^{i+j-4r}(\mathcal{N}_g)$ and $\Phi_0 = f_1 \cup f_2$.*

Proof. First, for $s, s' \in \mathcal{S}$,

$$< e_s, e_{s'} >= D^{(w,\Sigma)}_{\Sigma\times\mathbb{CP}^1}(z_s z_{s'}) = 0,$$

unless $\deg(f_s) + \deg(f_{s'}) = 6g - 6 + 4r$, $r \ge 0$, as these are the only possible dimensions for the moduli spaces of anti-self-dual connections on $\Sigma \times \mathbb{CP}^1$. Moreover, when $\deg(f_s) + \deg(f_{s'}) = 6g - 6$, the moduli space is $\mathcal{N}_g$, so $< e_s, e_{s'} >= - < f_s f_{s'}, [\mathcal{N}_g] >= - < f_s, f_{s'} >$ (the minus sign is due to the different convention orientation for Donaldson invariants).

Now let f_s, $f_{s'}$ be basic elements of degrees i and j respectively. Put $f_s f_{s'} = \sum c_t f_t$ and $f_s * f_{s'} = \sum d_t f_t$. This means that $e_s e_{s'} = \sum d_t e_t$. Write $e_s e_{s'} = \sum_m g_m$, where

$g_m = \sum_{\deg(f_t)=m} d_t e_t$ are the homogeneous parts. Put $\hat{g}_m = \sum_{\deg(f_t)=m} d_t f_t$. Let M be the maximum m such that $g_m \neq 0$. Then there is $f_{s''}$ of degree $6g-6-M$ such that $< \hat{g}_M, f_{s''} > \neq 0$. Since

$$0 \neq - < \hat{g}_M, f_{s''} > = < g_M, e_{s''} > = < e_s e_{s'}, e_{s''} > = D^{(w,\Sigma)}_{\Sigma\times\mathbb{CP}^1}(z_s z_{s'} z_{s''}),$$

it is $\deg(f_s)+\deg(f_{s'})+\deg(f_{s''}) \geq 6g-6$, i.e. $M \leq i+j$. Now for $m=i+j$, any $f_{s''}$ of degree $6g-6-m$, it is $< \hat{g}_m, f_{s''} > = -D^{(w,\Sigma)}_{\Sigma\times\mathbb{CP}^1}(z_s z_{s'} z_{s''}) = < f_s f_{s'} f_{s''}, [\mathcal{N}_g] > = < f_s f_{s'}, f_{s''} >$. So $\hat{g}_{i+j} = f_s f_{s'}$.

Finally, $< e_s e_{s'}, e_{s''} > = D^{(w,\Sigma)}_{\Sigma\times\mathbb{CP}^1}(z_s z_{s'} z_{s''}) = 0$, whenever $\deg(f_s)+\deg(f_{s'})+\deg(f_{s''}) \not\equiv 6g-6 \pmod 4$, so $\hat{g}_m = 0$ unless $m \equiv i+j \pmod 4$. □

Remark 6. We do not claim that the isomorphism in theorem 5 is the one in (1). Actually this is not the case (see example 22).

There is again an action of $\mathrm{Diff}(\Sigma)$ on HF^*_g factoring through an action of $\mathrm{Sp}\,(2g,\mathbb{Z})$ on $\{\psi_i\}$. The invariant part $(HF^*_g)_I$ is generated by α, β and $\gamma = -2\sum_{i=0}^{g} \phi^w(A, \gamma_i\gamma_{i+g})$. The epimorphism $\mathbb{C}[\alpha,\beta,\gamma] \twoheadrightarrow (HF^*_g)_I$, $z \mapsto \phi^w(A,z)$, allows us to write

$$HF^*(\Sigma\times\mathbb{S}^1)_I = \mathbb{C}[\alpha,\beta,\gamma]/J_g, \tag{5}$$

where J_g is the ideal of relations of α, β and γ. Now $\deg(\alpha)=2$, $\deg(\beta)=4$, $\deg(\gamma)=6$, but J_g is not a homogeneous ideal.

Lemma 7. *Suppose $\gamma J_g \subset J_{g+1}$, for all $g \geq 1$. Then we have the $Sp\,(2g,\mathbb{Z})$-decomposition*

$$HF^*(\Sigma\times\mathbb{S}^1) = \bigoplus_{k=0}^{g} \Lambda^k_0 H^3 \otimes \mathbb{C}[\alpha,\beta,\gamma]/J_{g-k}.$$

Proof. The isomorphisms in theorem 1 respect the $\mathrm{Sp}\,(2g,\mathbb{Z})$-action and hence induce isomorphisms on the invariant parts. Then $\dim(HF^*_g)_I = \dim H^*_I(\mathcal{N}_g)$, for all $g \geq 1$. Now the lemma is a consequence of the argument in the proof of [7, proposition 2.2] and the discussion preceding it. □

4. A presentation for $(HF^*_g)_I$

Theorem 5 and the arguments in [15, section 2] imply that we can deform the relations of $H^*_I(\mathcal{N}_g)$ to get a presentation for $(HF^*_g)_I$. More explicitly,

Lemma 8. *It is $(HF^*_g)_I = \mathbb{C}[\alpha,\beta,\gamma]/(R^1_g, R^2_g, R^3_g)$, where $R^i_g = q^i_g+$ lower order terms of degrees $\deg q^i_g - 4r$, $r>0$, as polynomials in $\mathbb{C}[\alpha,\beta,\gamma]$ (q^i_g are defined in proposition 3).*

Proof. Suppose first that $g \geq 2$. Granted theorem 5, [15, theorem 2.2] implies that $J_g = (R_g^1, R_g^2, R_g^3)$, where R_g^i is q_g^i expressed in terms of α, β and γ and the multiplication of HF_g^*. Now we note that under the isomorphism $H^*(\mathcal{N}_g) \stackrel{\cong}{\to} HF_g^*$ of theorem 5, $a \mapsto \alpha$, $b \mapsto \beta$, $c \mapsto \gamma$ (it always can be arranged so that these elements are in the basis, as $g \geq 2$, see [16, proposition 4.2]). So R_g^i is equal to q_g^i plus lower order terms. The case $g = 1$ is computed directly in lemma 11. □

Lemma 9. *$J_{g+1} \subset J_g$, for all $g \geq 1$.*

Proof. Let Σ_g be a Riemann surface of genus g and consider

$$\Sigma_{g+1} \subset A_g = \Sigma_g \times D^2 \subset S = \Sigma_g \times \mathbb{CP}^1,$$

where Σ_{g+1} is given by Σ_g with a trivial handle added internally. Then the map $H_*(\Sigma_{g+1}) \to H_*(\Sigma_g)$ induces $\mathbb{A}(\Sigma_{g+1}) \to \mathbb{A}(\Sigma_g)$ which sends $(\alpha, \beta, \gamma) \mapsto (\alpha, \beta, \gamma)$. Put $A_{g+1} = \Sigma_{g+1} \times D^2 \subset A_g$. This gives a map $(HF_{g+1}^*)_I \to (HF_g^*)_I$, $\phi^w(A_{g+1}, z) \mapsto \phi^w(A_g, z)$. Put $S = S^o \cup_{\Sigma_{g+1} \times \mathbb{S}^1} A_{g+1}$. Let $z \in J_{g+1}$. Then $\phi^w(\Sigma_{g+1} \times D^2, z) = 0$. So for any $z_s \in \mathbb{A}(\Sigma_g)$, $s \in \mathcal{S}$,

$$D_S^{(w,\Sigma)}(z\, z_s) =< \phi^w(\Sigma_{g+1} \times D^2, z), \phi^w(S^o, z_s) >= 0.$$

This is equivalent to $z \in J_g$. □

Theorem 10. *There are numbers $c_{g+1}, d_{g+1} \in \mathbb{C}$ such that, for all $g \geq 1$,*

$$\left\{ \begin{array}{l} R_{g+1}^1 = \alpha R_g^1 + g^2 R_g^2 \\ R_{g+1}^2 = (\beta + c_{g+1}) R_g^1 + \frac{2g}{g+1} R_g^3 \\ R_{g+1}^3 = \gamma R_g^1 + d_{g+1} R_g^2 \end{array} \right.$$

Proof. We follow almost literally the argument of Siebert and Tian [14, proposition 3.2]. As $R_{g+1}^1 \in J_{g+1} \subset J_g$ is a relation on degree $2g+2$, it is a linear combination of αR_g^1 and R_g^2. Looking at the leading terms (proposition 3), we have $R_{g+1}^1 = \alpha R_g^1 + g^2 R_g^2$. Analogously, R_{g+1}^2 is a combination of $\alpha^2 R_g^1$, βR_g^1, αR_g^2 and R_g^1. Only the term R_g^1 has degree less than $2g+4$, so $R_{g+1}^2 = \beta R_g^1 + \frac{2g}{g+1} R_g^3 + c_{g+1} R_g^1$, for an unknown coefficient c_{g+1}. In the same fashion, R_{g+1}^3 is γq_g^1 plus a linear combination of R_g^2 and αR_g^1. Adding a suitable multiple of R_{g+1}^1 (which is always allowed without loss of generality), we have $R_{g+1}^3 = \gamma q_g^1 + d_{g+1} R_g^2$. □

Lemma 11. *The starting relations (for $g = 1$) are $R_1^1 = \alpha$, $R_1^2 = \beta - 8$ and $R_1^3 = \gamma$.*

Proof. HF_1^* is of dimension 1, i.e. $HF_1^* = \mathbb{C}$ (see [3] [8]). Let S be the $K3$ surface and fix an elliptic fibration for S, whose fibre is be $\Sigma = \mathbb{T}^2$. The Donaldson invariants are, for $w \in H^2(S; \mathbb{Z})$ with $w \cdot \Sigma \equiv 1 \pmod 2$ (see [8]),

$$D_S^{(w,\Sigma)}(e^{tD}) = -e^{-Q(tD)/2}.$$

Then $D_S^{(w,\Sigma)}(1) = -1$ and $D_S^{(w,\Sigma)}(\Sigma^d) = 0$, for $d > 0$. Also from [10, remark 4], $D_S^{(w,\Sigma)}(x) = 2$. Pet S^o be the complement of an open tubular neighbourhood of Σ in S. Then $\phi^w(S^o, 1)$ generates HF_1^* and $\phi^w(S^o, \Sigma) = 0$, $\phi^w(S^o, x) = -2\,\phi^w(S^o, 1)$ and $\phi^w(S^o, \gamma_1\gamma_2) = 0$, i.e. $\alpha = 0$, $\beta - 8 = 0$ and $\gamma = 0$ in HF_1^* (recall (4)). $\square$

Proposition 12. *For $g \geq 2$, there exists a non-zero vector $v \in HF_g^*$ such that*

$$\begin{aligned} \alpha v &= \begin{cases} 4(g-1)\, v & g \text{ even} \\ 4(g-1)\sqrt{-1}\, v & g \text{ odd} \end{cases} \\ \beta v &= (-1)^{g-1} 8\, v \\ \gamma v &= 0 \end{aligned}$$

Proof. We shall construct such a vector as the relative invariants of an open four-manifold X^o with boundary $\partial X^o = Y = \Sigma \times \mathbb{S}^1$, where the closed four-manifold $X = X^o \cup_Y A$ is of simple type with $b^+ > 1$ and $b_1 = 0$. For concreteness, let X be the manifold C_g from [10, definition 25]. We recall its construction. Let S_g denote the elliptic surface of geometric genus $p_g = g-1$ and with no multiple fibres. It contains a section σ which is a rational curve of self-intersection $-g$. Let F be the elliptic fibre. Then $\sigma + gF$ can be represented by an embedded Riemann surface $\tilde{\Sigma}$ of genus g and self-intersection g. Blow-up S_g at g points in $\tilde{\Sigma}$ to get B_g with an embedded Riemann surface Σ_g of genus g and self-intersection zero. Then put $X = C_g = B_g \#_{\Sigma_g} B_g$ (the double of B_g along Σ_g). By [10, proposition 27], X is of simple type and $\mathbb{D}_X^w(e^\alpha) = D_X^w((1+\frac{x}{2})e^\alpha) = -2^{3g-5} e^{Q(\alpha)/2} e^{K\cdot\alpha} + (-1)^g 2^{3g-5} e^{Q(\alpha)/2} e^{-K\cdot\alpha}$, where $K \in H^2(X;\mathbb{Z})$ satisfies $K \cdot \Sigma_g = 2g-2$ ($w \in H^2(C_g;\mathbb{Z})$ is a particular element, which we do not need to specify here). Let us suppose from now on that g is even, the other case being similar. By [10, proposition 3],

$$D_X^{(w,\Sigma)}(e^\alpha) = -2^{3g-5} e^{Q(\alpha)/2} e^{K\cdot\alpha} + (-1)^g 2^{3g-5} e^{Q(\alpha)/2} e^{-K\cdot\alpha}.$$

We set $v = \phi^w(X^o, \Sigma + 2g - 2) \in HF^*(\Sigma \times \mathbb{S}^1) = HF_g^*$. Let us prove that this is the required element. For any $z_s = \Sigma^n x^m \gamma_{i_1} \cdots \gamma_{i_r}$, it is [10, remark 4],

$$< v, e_s >= D_X^{(w,\Sigma)}((\Sigma + 2g - 2) z_s) = \begin{cases} 0, & r > 0 \\ -2^{3g-4}(2g-2)^{n+1} 2^m, & r = 0 \end{cases}$$

Then $< \alpha v, e_s >=< \phi^w(X^o, 2\,\Sigma(\Sigma + 2g - 2)), \phi^w(A, z_s) >= D_X^{(w,\Sigma)}((\Sigma + 2g - 2) 2\,\Sigma\, z_s) = (4g-4) < v, e_s >$, for all $s \in \mathcal{S}$. Then $\alpha v = (4g-4)v$. Analogously, $\gamma v = 0$ and $\beta v = -8v$. $\square$

Notation 13. We set $R_0^1 = 1$, $R_0^2 = 0$ and $R_0^3 = 0$.

Theorem 14. *For all $g \geq 1$, $c_g = (-1)^g 8$ and $d_g = 0$.*

Proof. The result is true for $g = 1$ by lemma 11 and notation 13. Suppose it is true for $1 \leq r \leq g$, and let us prove it for $g+1$. By proposition 12, there exists $v \in HF^*_{g+1}$ with $\beta v = (-1)^g 8\, v$, $\gamma v = 0$ and $\alpha v = 4g\, v$ if g is odd and $\alpha v = 4g\sqrt{-1}\, v$ if g is even.

In first place, $\gamma v = 0$ implies $R^3_r v = 0$, for $1 \leq r \leq g$. In second place, $\beta v = (-1)^g 8\, v$ implies

$$\begin{array}{rcl} R^2_g v & = & (\beta + (-1)^g 8) R^1_{g-1} v = (-1)^g 16 R^1_{g-1} v, \\ R^2_{g-1} v & = & (\beta + (-1)^{g-1} 8) R^1_{g-1} v = 0, \\ R^2_{g-2} v & = & (-1)^g 16 R^1_{g-3} v, \\ R^2_{g-4} v & = & 0, \\ & \vdots & \end{array}$$

In third place, $R^1_g v = \alpha R^1_{g-1} v + (g-1)^2 R^2_{g-1} v = \alpha R^1_{g-1} v$, $R^1_{g-2} v = \alpha R^1_{g-3} v$, $\ldots$ Also

$$R^1_{g-1} v = \alpha R^1_{g-2} v + (g-2)^2 R^2_{g-2} v = (\alpha^2 + (g-2)^2 (-1)^g 16) R^1_{g-3} v.$$

So finally,

$$R^1_{g-1} v = \begin{cases} (\alpha^2 + (-1)^g 16 (g-2)^2) \cdots (\alpha^2 + (-1)^g 16 \cdot 1^2) v & g \text{ odd} \\ (\alpha^2 + (-1)^g 16 (g-2)^2) \cdots (\alpha^2 + (-1)^g 16 \cdot 2^2) \alpha v & g \text{ even} \end{cases}$$

As a conclusion $R^1_{g-1} v = \lambda v$, with $\lambda \neq 0$, and

$$\begin{cases} R^1_g v = \alpha R^1_{g-1} v \\ R^2_g v = (-1)^g 16 R^1_{g-1} v \\ R^3_g v = 0 \end{cases}$$

As $v \in HF^*_{g+1}$, we have $R^1_{g+1} v = 0$, $R^2_{g+1} v = 0$ and $R^3_{g+1} v = 0$. Evaluate the equations from theorem 10 on v to get $c_{g+1} = (-1)^{g+1} 8$ and $d_{g+1} = 0$. $\square$

Corollary 15. *We have $\gamma J_g \subset J_{g+1} \subset J_g$, for all $g \geq 1$.*

Proof. The second inclusion is lemma 9. For the first inclusion, note that $\gamma R^1_g = R^3_{g+1} \in J_{g+1}$ by the third equation in theorem 10. Then multiplying the first two equations in theorem 10 we get that $\gamma R^2_g, \gamma R^3_g \in J_{g+1}$. $\square$

Using this corollary in lemma 7, we have finally proved that

Theorem 16. *The Floer cohomology of $\Sigma \times \mathbb{S}^1$, for $\Sigma = \Sigma_g$ a Riemann surface of genus g, has a presentation*

$$HF^*(\Sigma \times \mathbb{S}^1) = \bigoplus_{k=0}^{g} \Lambda^k_0 H^3 \otimes \mathbb{C}[\alpha, \beta, \gamma]/J_{g-k}.$$

where $J_r = (R_r^1, R_r^2, R_r^3)$ *and* R_r^i *are defined recursively by setting* $R_0^1 = 1$, $R_0^2 = 0$, $R_0^3 = 0$ *and putting for all* $r \geq 0$

$$\begin{cases} R_{r+1}^1 = \alpha R_r^1 + r^2 R_r^2 \\ R_{r+1}^2 = (\beta + (-1)^{r+1} 8) R_r^1 + \frac{2r}{r+1} R_r^3 \\ R_{r+1}^3 = \gamma R_r^1 \end{cases}$$

Remark 17. The presentation obtained for HF_g^* is the conjectural presentation for $QH^*(\mathcal{N}_g)$ (see [14]).

Corollary 18. $\ker(\gamma : (HF_g^*)_I \to (HF_g^*)_I) = J_{g-1}/J_g \subset \mathbb{C}[\alpha, \beta, \gamma]/J_g = (HF_g^*)_I$.

Proof. By the corollary 15, γ factors as

$$\mathbb{C}[\alpha, \beta, \gamma]/J_g \twoheadrightarrow \mathbb{C}[\alpha, \beta, \gamma]/J_{g-1} \overset{\gamma}{\hookrightarrow} \mathbb{C}[\alpha, \beta, \gamma]/J_g.$$

The second map is a monomorphism since $\alpha^a \beta^b \gamma^c$, $a + b + c < g - 1$, form a basis for $\mathbb{C}[\alpha, \beta, \gamma]/J_{g-1}$, and their image under γ are linearly independent in $\mathbb{C}[\alpha, \beta, \gamma]/J_g$. The corollary follows. $\square$

For any $F \in \mathbb{C}[\alpha, \beta, \gamma]$ define the expectation value by $< F >_g = < F_g, 1 >_{HF_g^*}$, where $1 \in HF_g^*$ is the unit element. Therefore $< F_1, F_2 >_{HF_g^*} = < F_1 F_2 >_g$.

Corollary 19. *For any* $F \in \mathbb{C}[\alpha, \beta, \gamma]$, $< \gamma F >_g = -2g < F >_{g-1}$.

Proof. By corollary 15, the formula above holds for any $F \in J_{g-1}$, as both sides are zero. So it is enough to check it for a set of elements generating HF_{g-1}^*, i.e. for $F_{abc} = \alpha^a \beta^b \gamma^c$, $a + b + c < g - 1$. If $(a, b, c) \neq (0, 0, 0)$, it is $< F_{abc} >_{g-1} = 0$ and $< \gamma F_{abc} >_g = 0$ by degree reasons. Now $< \gamma^{g-1} >_{g-1} = - < c^{g-1}, [\mathcal{N}_{g-1}] >$ hence the corollary follows from $< c^g, [\mathcal{N}_g] > = -2g < c^{g-1}, [\mathcal{N}_{g-1}] >$ (see [17]). $\square$

5. Local ring decomposition of $HF^*(\Sigma \times \mathbb{S}^1)$

In [1] it is asserted that the only eigenvalues of the action of $\mu(\Sigma)$, $\mu(x)$ and $\mu(\gamma_i)$ on $HF^*(\Sigma \times \mathbb{S}^1)$ are the ones given by looking at the Donaldson invariants of the manifold $X = \Sigma \times \mathbb{T}^2$ i.e. if we denote by $W \subset HF_g^*$ the image $\phi^w(X^o, \mathbb{A}(\Sigma))$, where $X = X^o \cup_Y A$, then α, β and γ act on W and their eigenvalues are all the eigenvalues of their action on HF_g^*. The following result is a proof of this physical assertion.

Proposition 20. *The eigenvalues of* (α, β, γ) *in* $(HF_g^*)_I$ *are* $(0, 8, 0)$, $(\pm 4, -8, 0)$, $(\pm 8\sqrt{-1}, 8, 0)$, $\ldots$, $(\pm 4(g-1)\sqrt{-1}^g, (-1)^{g-1} 8, 0)$.

Proof. Put $V = (HF_g^*)_I$. As $\gamma J_{g-1} \subset J_g$, one has $\gamma^g \in J_g$, i.e. $\gamma^g = 0$ in V, so the only eigenvalue of γ is zero. To compute the eigenvalues of α, β we can restrict to $V/\gamma V$ (if p is a polynomial with $p(\alpha) = 0$ in $V/\gamma V$, then $p(\alpha)$ is a multiple of γ in V and $p(\alpha)^g = 0$ in V). Now the ideal of relations of V can be written as $J_g = (\zeta_g, \zeta_{g+1}, \zeta_{g+2})$, where $\zeta_0 = 1$ and $\zeta_{r+1} = \alpha\zeta_r + r^2(\beta + (-1)^r 8)\zeta_{r-1} + 2r(r-1)\gamma\zeta_{r-2}$, for all $r \geq 0$ (see proposition 3). So

$$V/\gamma V = \mathbb{C}[\alpha, \beta]/(\bar{\zeta}_g, \bar{\zeta}_{g+1}),$$

where $\bar{\zeta}_0 = 1$, $\bar{\zeta}_{r+1} = \alpha\bar{\zeta}_r + r^2(\beta + (-1)^r 8)\bar{\zeta}_{r-1}$, for $r \geq 0$. From $r^2(\beta + (-1)^r 8)\bar{\zeta}_{r-1} = \bar{\zeta}_{r+1} - \alpha\bar{\zeta}_r$ we infer that $(\beta + (-1)^r 8)\bar{\zeta}_{r-1} \in (\bar{\zeta}_r, \bar{\zeta}_{r+1})$. Continuing in this way,

$$(\beta + (-1)^g 8)(\beta + (-1)^{g-1} 8) \cdots (\beta - 8) \in (\bar{\zeta}_g, \bar{\zeta}_{g+1}),$$

which implies that the only eigenvalues of β in $V/\gamma V$, and hence in V, are ± 8. Let us study the eigenvalues of α for $\beta = 8$, $\gamma = 0$. Again we only need to study $V/(\gamma, \beta - 8)V = \mathbb{C}[\alpha]/(\hat{\zeta}_g, \hat{\zeta}_{g+1})$, where now $\hat{\zeta}_0 = 1$, $\hat{\zeta}_{r+1} = \alpha\hat{\zeta}_r + r^2(8 + (-1)^r 8)\hat{\zeta}_{r-1}$. Then

$$\begin{cases} \hat{\zeta}_r = (\alpha^2 + (r-2)^2 16) \cdots (\alpha^2 + 2^2 16)\alpha^2 & r \text{ even} \\ \hat{\zeta}_r = (\alpha^2 + (r-1)^2 16) \cdots (\alpha^2 + 2^2 16)\alpha & r \text{ odd} \end{cases}$$

from where the eigenvalues of α will be $0, \pm 8\sqrt{-1}, \pm 16\sqrt{-1}, \ldots, \pm 8\left[\frac{g-1}{2}\right]\sqrt{-1}$. We leave the other case to the reader. $\square$

Remark 21. As mentioned in [1], by the very definition of $\gamma = -2\sum \phi^w(A, \gamma_i\gamma_{i+g})$, it is $\gamma^{g+1} = 0$ in HF_g^*, so the only eigenvalue of γ is zero.

Proposition 20 says that $(HF_g^*)_I$ can be decomposed as a sum of local artinians rings

$$(HF_g^*)_I = \bigoplus_{i=-(g-1)}^{g-1} R_{g,i}, \tag{6}$$

where $R_{g,i}$ is a local artinian ring with maximal ideal $\mathfrak{m} = (\alpha - 4i, \beta - (-1)^i 8, \gamma)$ if i is odd, $\mathfrak{m} = (\alpha - 4i\sqrt{-1}, \beta - (-1)^i 8, \gamma)$ if i is even. Also HF_g^* is decomposed as

$$HF_g^* = \bigoplus_{k=0}^{g} \bigoplus_{i=-(g-k-1)}^{g-k-1} \Lambda_0^k H^3 \otimes R_{g-k,i} = \bigoplus_{i=-(g-1)}^{g-1} \bigoplus_{k=0}^{g-|i|-1} \Lambda_0^k H^3 \otimes R_{g-k,i}. \tag{7}$$

We recall from lemma 11 that $HF_1^* = \mathbb{C}[\alpha, \beta, \gamma]/(\alpha, \beta - 8, \gamma)$. Let us see the next cases.

Example 22. For $g = 2$, $J_2 = (\alpha^2 + \beta - 8, \alpha(\beta + 8) + \gamma, \alpha\gamma)$. In $(HF_2^*)_I$, $\gamma = -\alpha(\beta + 8)$ and $\gamma\alpha = 0$ yield $\alpha^2(\beta + 8) = 0$. Now $\alpha^2 = -(\beta - 8)$ so $(\beta - 8)(\beta + 8) = 0$ and $\alpha^2(\alpha^2 - 16) = 0$. Also $\gamma J_1 \subset J_2$ implies $\gamma\alpha = \gamma^2 = \gamma(\beta - 8) = 0$. Finally

$(\gamma+16\alpha)(\alpha^2-16) = -16\gamma + 16\alpha(\alpha^2-16) = -16(\gamma+\alpha(\beta+8)) = 0$. All together proves

$$(HF_2^*)_I = \frac{\mathbb{C}[\alpha,\beta,\gamma]}{(\alpha-4,\beta+8,\gamma)} \oplus \frac{\mathbb{C}[\alpha,\beta,\gamma]}{(\alpha^2,\beta-8,\gamma+16\alpha)} \oplus \frac{\mathbb{C}[\alpha,\beta,\gamma]}{(\alpha+4,\beta+8,\gamma)}.$$

We want to remark that $HF_2^* \xrightarrow{\cong} QH^*(\mathcal{N}_2)$ (see [11, example 5.3] for a presentation of the latter ring). The isomorphism sends $\alpha \mapsto h_2$, $\beta \mapsto -4(h_4-1)$, $\gamma \mapsto 4(h_6-h_2)$, where h_2, h_4, h_6 are the generators of QH^2, QH^4, QH^6 respectively. This was conjectured in [8, conjecture 1.22].

Example 23. For $g=3$, $J_3 = \Big(\alpha(\alpha^2+\beta-8)+4(\alpha\beta+8\alpha+\gamma), (\beta-8)(\alpha^2+\beta-8)+\frac{4}{3}\alpha\gamma, \gamma(\alpha^2+\beta-8)\Big)$. Put $V=(HF_3^*)_I$. Then

$$V/\gamma V = \mathbb{C}[\alpha,\beta]/(\alpha(\alpha^2+\beta-8)+4(\alpha\beta+8\alpha), (\beta-8)(\alpha^2+\beta-8)).$$

In $V/\gamma V$, the first relation yields $-5\alpha(\beta-8) = \alpha^3+64\alpha$ and the second $\alpha(\beta-8)(\alpha^2+\beta-8)=0$. This implies $\alpha(\alpha^2-16)(\alpha^2+64)=0$. Also $(\beta-8)\alpha(\alpha^2-16)=0$. Using $(\beta-8)\alpha^2 = -(\beta-8)^2$, we get $\alpha(\beta-8)(\beta+8)=0$.

Therefore, in V, $\alpha(\alpha^2-16)(\alpha^2+64)$ and $\alpha(\beta-8)(\beta+8)$ are multiples of γ. As $\gamma J_2 \subset J_3$, we have $\gamma\alpha^2(\alpha^2-16)=0$ and $\gamma\alpha^2(\beta-8)=0$ by example 22. So $\alpha^3(\alpha^2-16)^2(\alpha^2+64)=0$, $\alpha^3(\alpha^2-16)(\beta-8)(\beta+8)=0$ and $\alpha^3(\beta-8)^2(\beta+8)=0$. It can be checked now that

$$(HF_3^*)_I = \frac{\mathbb{C}[\alpha,\beta,\gamma]}{(\alpha-8\sqrt{-1},\beta-8,\gamma)} \oplus \frac{\mathbb{C}[\alpha,\beta,\gamma]}{((\alpha-4)^2,\beta+8,\gamma+8(\alpha-4))} \oplus$$
$$\oplus \frac{\mathbb{C}[\alpha,\beta,\gamma]}{(\alpha^3,\alpha(\beta-8),(\beta-8)^2-\frac{64}{3}\alpha^2,\gamma+16\alpha)} \oplus \frac{\mathbb{C}[\alpha,\beta,\gamma]}{((\alpha+4)^2,\beta+8,\gamma+8(\alpha+4))} \oplus$$
$$\oplus \frac{\mathbb{C}[\alpha,\beta,\gamma]}{(\alpha+8\sqrt{-1},\beta-8,\gamma)}.$$

6. Conjecture

We state the following conjecture, which first occurred to Paul Seidel and the author in mid'96.

Conjecture 24. *The decomposition in equation* (7) *is*

$$HF_g^* \cong H^*(s^0\Sigma) \oplus H^*(s^1\Sigma) \oplus \cdots \oplus H^*(s^{g-2}\Sigma) \oplus$$
$$\oplus H^*(s^{g-1}\Sigma) \oplus H^*(s^{g-2}\Sigma) \oplus \cdots \oplus H^*(s^0\Sigma),$$

where $s^i\Sigma$ is the i-th symmetric product of Σ. Here $H^*(s^i\Sigma)$ is isomorphic to the eigenspace of eigenvalues $(\pm 4(g-1-i)\sqrt{-1}^{g-i}, (-1)^{g-1-i}8, 0)$. The isomorphism respects only $\mathbb{Z}/2\mathbb{Z}$-grading and is Diff(Σ)-equivariant.

Simple computations establish that the dimensions of both vector spaces appearing in conjecture 24 are the same, i.e. $g2^g$. The Euler characteristic are both vanishing. Moreover the dimensions of the invariant parts coincide $\binom{g+2}{3}$. Examples 22 and 23 agree with the conjecture.

A deeper reason for the above conjecture is the fact that HF_g^* is the space for a gluing theory of Donaldson invariants associated to the three manifold $Y = \Sigma \times \mathbb{S}^1$. The gluing theory of Seiberg-Witten invariants should be based on the Seiberg-Witten-Floer homology groups of $\Sigma \times \mathbb{S}^1$, which are indexed by a line bundle L (the determinant line bundle of the spin^c-structure on Y). The only possibilities are $c_1(L) = \pm(g-1-i)[\mathbb{S}^1]$, $0 \leq i \leq g-1$ (see [8, section 6]). It is believed that the Seiberg-Witten-Floer groups for L are isomorphic to $H^*(s^i\Sigma)$.

References

1. M. Bershadsky, A. Johansen, V. Sadov and C. Vafa, Topological reduction of 4D SYM to 2D σ-models, Preprint, 1995.
2. S. K. Donaldson, On the work of Andreas Floer, *Jahresber. Deutsch. Math. Verein,* **95** 1993, 103-120.
3. S. K. Donaldson, Floer homology and algebraic geometry, *Vector bundles in algebraic geometry,* London Math. Soc. Lecture Notes Series, **208** Cambridge University Press, Cambridge, 1995, 119-138.
4. S. Dostoglou and D. Salamon, Self-dual instantons and holomorphic curves, *Annals of Mathematics,* **139** 1994, 581-640.
5. A. Floer, Symplectic fixed points and holomorphic spheres, *Comm. Math. Physics,* **120** 1989, 575-611.
6. A. Floer, Instanton homology and Dehn surgery, *The Floer memorial volume. Progress in mathematics,* **133** 1994, 77-97.
7. A. D. King and P. E. Newstead, On the cohomology ring of the moduli space of rank 2 vector bundles on a curve, Liverpool Preprint, 1994.
8. V. Muñoz, Oxford D. Phil. Thesis, Gauge Theory and Complex Manifolds, 1996.
9. V. Muñoz, Donaldson invariants for connected sums along Riemann surfaces of genus 2, dg-ga/9702004.
10. V. Muñoz, Gluing formulae for Donaldson invariants for connected sums along surfaces, dg-ga/9702002.
11. S. Piunikhin, D. Salamon and M. Schwarz, Symplectic Floer-Donaldson theory and quantum cohomology, Warwick Preprint, 1995.
12. Y. Ruan and G. Tian, A mathematical theory of quantum cohomology, *Jour. Diff. Geom.* **42** 1995, 259-367.
13. D. Salamon, Lagrangian intersections, 3-manifolds with boundary and the Atiyah-Floer conjecture, Proceedings of the International Congress of Mathematicians, **1** Birkhäuser-Verlag, 1994, 526-536.
14. B. Siebert, An update on (small) quantum cohomology, Preprint, 1997.
15. B. Siebert and G. Tian, On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator, submitted to Duke Math. Journal.

16. B. Siebert and G. Tian, Recursive relations for the cohomology ring of moduli spaces of stable bundles, *Proceedings of 3rd Gökova Geometry-Topology Conference 1994.*
17. M. Thaddeus, Conformal field theory and the cohomology of the moduli space of stable bundles, *Jour. Differential Geometry,* **35** 1992, 131-150.

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