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MULTIPLE BIDS IN A MULTIPLE-UNIT COMMON VALUE AUCTION: SIMULATIONS FOR THE SPANISH AUCTION*

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ABSTRACT

The Spanish Treasury is the only one in the world that uses a hybrid system of discriminatory and uniform price auctions to sell bonds. In the Spanish auction, winning bidders pay their bid price if it is lower than the weighted average price of winning bids, while all other winning bidders pay the weighted average price of winning bids. We adapt Gordy's (96) model of the discriminatory auction to the Spanish auction. The model is a discrete model of multiple bids in a multiple-unit common value auction. We use numerical simulations to find equilibria for the Spanish, the uniform and the discriminatory auction. Our results show that bidders in the Spanish and discriminatory auctions use bid spread to cover themselves against uncertainty, and that expected seller's revenue is larger on average in the former.

RESUMEN

El Tesoro español es el único en el mundo que usa un sistema híbrido de subastas discriminatoria y uniforme para subastar Letras del Tesoro. En la subasta Española, las pujas ganadoras pagan su precio si están por debajo del precio medio ponderado de las pujas ganadoras (WAP) y el WAP en otro caso. Adaptamos un modelo de Gordy(96) de subastas discriminatorias al caso español. El modelo es de múltiples pujas, discretas, con múltiples unidades y valoración común. Usamos simulaciones para encontrar equilibrios en la subasta Española, discriminatoria y uniforme. Nuestros resultados muestran que los pujadores en las subastas Española y discriminatoria usan *bid spread* para cubrise contra la incertidumbre, y que los ingresos esperados del vendedor son, en media, mayores en la primera.

JEL CLASSIFICATION: D44
Keywords: Treasury auctions, Spanish auction, Multi-unit auctions, Simulations.

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1. INTRODUCTION

One of the most important auction markets in the world is the market for government debt. Treasuries apply mainly two auction formats: discriminatory and uniform price auctions. In a discriminatory auction, used by the majority of the Treasuries around the world, winning bidders pay their bid price. A few Treasuries use uniform price auctions, where all winning bidders pay the same price for each unit, the minimum accepted price. But the Spanish Treasury is the only one that uses a hybrid system of discriminatory and uniform price auctions: winning bidders pay their bid price if it is lower than the weighted average price of winning bids, while all other winning bidders pay the weighted average of winning bids. With the Spanish format, the price that some bidders have to pay for certain units depends on the bids of all other winning bidders, including his own bids. This fact increases the players' strategic considerations with respect to discriminatory and uniform auctions, even in the more simple models.

The general director of the Spanish Treasury, Jaime Caruana, mentioned that "the adoption of the euro will establish a more efficient market, in which the Spanish debt will have to compete with other countries' debt on interest rates, credit quality and calendar" (El País, April 14, 1998). He did not mention that it has to compete with a different auction mechanism. But his statement calls attention to the fact that competition has increased after January 1999, and it is important to establish the characteristics of the Spanish auction mechanism, both from the point of view of the seller and the buyers.

This paper studies the Spanish Treasury auctions, and compares them with the discriminatory auctions, the format that is used by most Treasuries around the world. We consider two aspects of the Spanish auction. First, in Treasury auctions bidders are allowed to submit multiple bids, and they do. In Spain, the average number of bids per competitive bidder is 2.7, and both in the United States and Portugal, the median number of bids per bidder is three[1]. Gordy (96) conjectures that in discriminatory auctions, multiple bids can be used to hedge against winner's curse, as well as to express downward sloping demand due to risk aversion. We study the use of multiple bids in the Spanish auction, and whether Gordy's conjecture also holds for the Spanish auction. Second, a principal aim of auction theory is the ranking of different types of auctions with respect to the expected seller's revenue. We consider the ranking of the Spanish and the discriminatory auctions in terms of revenue.

We adapt Gordy's model for the discriminatory auction to the Spanish auction. The model is a discrete model, that allows explicitly for the use of multiple bids in a multiple-unit auction. It models Treasury auctions as common value auctions with asymmetric information. Using numerical simulations, we find all (if any) Bayesian Nash symmetric equilibria for the Spanish, the discriminatory and the uniform auction, for a wide range of parameter combinations.

Our main findings can be summarized as follows. First, Gordy's conjecture holds for the Spanish auction: multiple bidding in the Spanish auctions is used to hedge against the winner's curse, as well as to express downward sloping demand due to risk aversion. We find that this hedging against the winner's curse is stronger in the Spanish than in the discriminatory auction. There are two contributing factors. On the one hand, bidders can

[1] See Mazón and Nuñez (99) regarding Spain, and Gordy (96) regarding the U.S. and Portugal.

increase their bid on the first unit at a lower expected cost for the Spanish auction than for the discriminatory auction: if they have overvalued the good and they win, they only pay the weighted average price instead of their bid, as they do in the discriminatory case. On the other hand, they have an incentive to lower their bid for the second unit, since for the Spanish auction the low bid determines the price the bidder has to pay on the first unit if he wins two units: the lower his bid on the second unit is, the lower is the price that the bidder has to pay for the first unit. Second, it is not possible to offer a complete ranking of the Spanish and the discriminatory auctions with respect to expected seller's revenue. First of all, the ranking varies with the values of the parameters. And for some parameter values there are multiple equilibria, and the ranking depends upon which of the equilibria is examined. Nevertheless, on average across the multiple equilibria, the Spanish auction gives higher expected seller's revenue than the discriminatory auction. Note that, as we have argued, bidders bid more aggressively for the first unit in the Spanish than in the discriminatory auction, which tends to increase expected seller's revenue; but if they win with the highest bid, they only pay the weighted average price, lower than their bid that they pay on the discriminatory auction. Our results suggest that the first effect is higher than the second.

2. SURVEY OF THE LITERATURE

An abundant literature exits on uniform and discriminatory auctions, and general results are established for the auctioning of a single, indivisible item, results that can be extended to settings with multiple units, if each bidder has a taste for only one item. But as Asubel and Cramton (98) mention, "in environments with multiple units and bidders who each may desire multiple units, general results about even the most common auction forms remain elusive". The reason is that the problem is very complicated. First of all, bidders have a very large strategy space. Second, there is a strategic component in bidding: in a uniform auction, bids on later units might determine the price the bidder pays for earlier units. And third, Treasury auctions are assumed to be common-value auctions, where there is a true value of each unit, unknown to the bidders at the time of the auction, and bidders receive private signals concerning the value of the asset. If bidders receive different signals, that is, if the model allows asymmetric information, equilibrium bids must address not only the strategic component of bidding, but the inference problem due to asymmetric information.

Most authors that study multiple-unit auctions where bidders demand more than one unit, follow the "share auctions" approach, proposed by Wilson (79), where the good is assumed to be perfectly divisible and a bid is a smooth demand schedule. Wang and Zender (98) characterized the equilibria for both the discriminatory and the uniform auction, when bidders possess private information, and conclude that the equilibrium bidding strategies take explicit account of the winner's curse. If all the bidders have the same information, they obtain an analytical solution and fully characterized the set of equilibria under risk neutrality and risk aversion utility, assuming a specific functional form for noncompetitive demand. They obtain a continuum of equilibria for the uniform auction, and only one equilibrium for the discriminatory auction if a reserve price of zero is imposed. Given the multiplicity of equilibria for the uniform auction, they conclude that it is not possible to rank both auction formats in terms of expected sellers' revenue. Asubel and Cramton (98) also follow the "share auction" approach, and provide several examples to demonstrate that auctions results are inefficient, and that the ranking of uniform and discriminatory auctions is ambiguous: they

provide examples with reasonable specifications of demand where the uniform auction dominates the discriminatory auction on expected seller revenues, and equally reasonable specifications where the reverse is true.

A different approach is taken by Gordy (96), which offers a discrete counterpart of Wilson's (79) continuous "share auction" model. He uses numerical simulations to find equilibria for the discriminatory auction, when two units of an indivisible good are auctioned to N bidders. He finds evidence that supports the conjecture that multiple-bidding can be used to hedge winner's curse, when bidders are risk averse.

Compared to the overwhelming amount of work about uniform and discriminatory auctions, very little has been said about the Spanish auction format. To our knowledge, the properties of the Spanish auction mechanism have been studied only by Salinas (90), Mazón and Nuñez (99), and Álvarez, Cerdá and Mazón (99). Salinas (90) presents a model where demand is restricted to one unit per bidder, and values are private. He uses the results of Maskin and Riley (89) to argue that the Spanish mechanism generates the same expected revenue as uniform and discriminatory auctions. Mazón and Nuñez (99) presents a stylized game theoretical model that captures the two distinct features of the Spanish auction: the hybrid system of uniform and discriminatory auctions used; and the uncertainty about the amount to be issued. They show that, under the assumptions of the model, the auction format used in Spain is equivalent in terms of revenue to the seller to the discriminatory format, and that both formats maximize the seller's revenue. And they present an empirical analysis, using data of Spanish bond auctions between 1993 and 1997. They find evidence of the good functioning of the market, and the relatively low price differentials paid by accepted bids, which is consistent with the results of the model. But the model assumes that demand functions are common knowledge, and a bid is a price-quantity pair; therefore, the model allows multiple-unit demands, but at the same price. Álvarez et al. (99) follow the "share auction" approach and characterize the set of linear equilibria for the Spanish auction.

3. MULTIPLE BIDS: THE DISCRIMINATORY AND THE SPANISH CASE

3.1 The model

We follow Gordy (96) and adapt his model for the discriminatory auction to the uniform and to the Spanish auctions. *N* bidders compete for two indivisible and identical units of a good. Each bidder submits two sealed bids, specifying a price, but not a particular unit. The two units are awarded to the two highest bids, and if there is a tie, there is randomization among the tie bids[2]. Payments depend on the auction type. In the discriminatory auction, winning bids pay the bid price. In the uniform auction, all winning bids pay the same price for each unit, the minimum accepted bid. In the Spanish auction, winning bids pay the bid price if it is lower than the weighted average of winning bids (WAP), and pay the WAP if the bid price is higher than the WAP.

[2] Pro rata distribution is used in real auctions. But we follow Gordy, that mentions that "the loss in realism is more than offset by a loss in computational efficiency", and that "limited exploration of the model suggest very similar results" with pro rata distribution rather than randomization among tie bids.

As an example, consider two bidders, A and B, bidding (4,1) and (3,2) respectively. That is, bidder A bids 4 for one unit and 1 for the other. Winning bids are 4 and 3, and hence each bidder gets one unit in the three auction formats. In the discriminatory auction, A pays 4 and B pays 3; in the uniform auction, both bidders pay 3; and in the Spanish auction A pays 0.5(4+3)=3.5, the WAP, and B pays 3.

The true unit value of the good for sale, v, is unknown to the bidders at the time of the auction. The prior distribution of v is $F(v)$, and it is public information. We assume $F(v)$ to be beta $(\alpha\mu, \alpha(1-\mu))$; this distribution has mean μ and variance decreasing in α. Hence, the larger α is, the more accurate is public information. Furthermore, each bidder draws a signal from the finite set $X \equiv \{0,1,\ldots,K\}$, with $K>0$. The probability distribution of the signal conditional on v is assumed to be binomial (K,v)[3]. Signals are independent across bidders and are private information: each bidder only observes his own signal. Bidders combine public information (the prior on v) and private information (the signal received) using Bayes rule. The posterior distribution of v, $F(v/x)$, is beta $(x+\alpha\mu, K-x+\alpha(1-\mu))$, where x is the signal that the bidder has received ($x \in X$), see DeGroot (70). The posterior distribution has conditional expected value $E(v/x)=(x+\alpha\mu)/(K+\alpha)$, and its variance is decreasing in K and α.

One possible interpretation of $E(v/x)$ is the following. It can be rewritten as: $E(v/x)=\delta(x/K)+(1-\delta)\mu$, where $\delta=(1+\alpha/K)^{-1}$. Furthermore x/K is the maximum likelihood estimator for v based only on private information. Therefore, $E(v/x)$ is a strictly convex combination of μ (public information) and x/K (private information), where the former receives more weight as α (accuracy of public information) increases or K (accuracy of private information) decreases.

Prices are restricted to a finite set $\Lambda \equiv \{0, 1/\lambda, 2/\lambda, \ldots 1\}$, λ being some positive integer. The prices are restricted to the interval (0,1) because the support of $F(v/x)$ is the interval (0,1). We allow only for a finite number of prices for two reasons. First, in practice, Treasury auctions in Spain (and in most countries) have restrictions on the set of bids permitted. Second, if the set of permitted bids is dense, pure strategy equilibria does not exist, as in Gordy (96).

Bidders are assumed to be risk averse, and to have a constant absolute risk aversion (CARA) utility function, $U(z)=-\exp(-\rho z)$, where ρ, strictly positive, is the coefficient of absolute risk aversion, common to all bidders.

A strategy for bidder i is a function $S^i: X \to \Lambda \times \Lambda$. That is, a strategy is a function that defines a pair of bids, $(s_1(x), s_2(x))$, for every possible signal x. Let $\Sigma \equiv \{S^1,\ldots,S^N\}$, and $\Sigma_{-i} \equiv \Sigma - \{S^i\}$. Without loss of generality, we assume $s_1(x) \geq s_2(x)$ and we refer to $s_1(x)$ ($s_2(x)$) as the high (low) bid for signal x.

In the Spanish auction, given a signal x, and a strategy profile for all bidders but bidder i, Σ_{-i}, bidder i chooses the pair of bids (s_1,s_2) to maximize his expected utility. There are four possible outcomes. He can win both units, and given the auction rules, he will pay the average price, $(s_1+s_2)/2$, for the first unit, and his lower bid, s_2, for the second unit; since the v is the true value of the good, his profit if he wins two units is $2v-(s_1+s_2)/2-s_2$. He can win one unit

[3] With a binomial distribution, the probability of a high signal increases with v, and the probability of a low signal decreases with v.

with a bid lower or equal to the other winning bid, so that he pays his high bid, s_1, and his profit is $v-s_1$. He can win one unit with a bid higher than the other winning bid, so that he pays the average price of his high bid and the other winning bid, denoted by $\bar{s}$, and his profit is $v-(s_1+\bar{s})/2$. And he can win zero units. Define $h_2(.)$ as the probability of winning exactly two units, $h_1(.)$ as the probability of winning one unit and that the other unit is not awarded at a lower price, $h_{1'}(.)$ as the probability of winning one unit and that the other unit is awarded at a lower price $\bar{s}$, and $h_0(.)$ as the probability of winning zero units. These probabilities depend on the i-th bidder bids, (s_1,s_2), rivals' strategies, Σ_{-i}, and the true value of the good[4], v. The expected utility of bidding (s_1,s_2) is given by the following expression:

$$
\begin{aligned}
E(U^s(s_1,s_2)) = \int_0^1 \Big(& U\big(2v-\tfrac{s_1+s_2}{2}-s_2\big)\, h_2(s_1,s_2,\Sigma_{-i},v) + \\
& + U(v-s_1)\, h_1(s_1,s_2,\Sigma_{-i},v) + \sum_{\bar{s}\in\Lambda} U\big(v-\tfrac{s_1+\bar{s}}{2}\big)\, h_{1'}(s_1,s_2,\bar{s},\Sigma_{-i},v) + \\
& + U(0)\, h_0(s_1,s_2,\Sigma_{-i},v)\Big)\, dF(v/x)
\end{aligned} \tag{1}
$$

Analogously, in the discriminatory auction, it is:

$$
\begin{aligned}
E(U^d(s_1,s_2)) = \int_0^1 \Big(& U(2v-s_1-s_2)\, h_2(s_1,s_2,\Sigma_{-i},v) + \\
& + U(v-s_1)\, h_{1''}(s_1,s_2,\Sigma_{-i},v) + U(0)\, h_0(s_1,s_2,\Sigma_{-i},v)\Big)\, dF(v/x)
\end{aligned} \tag{2}
$$

where $h_2(.)$ and $h_0(.)$ are defined as above, and $h_{1''}(.)$ is the probability of winning one unit. Contrarily to the Spanish case, the payment for one unit in the discriminatory case does not depend on whether the other unit is awarded at a lower price or not, hence $h_{1''}(.)=h_{1'}(.)+h_1(.)$.

In the uniform auction, it is:

$$
\begin{aligned}
E(U^u(s_1,s_2)) = \int_0^1 \Big(& U(2v-2s_2)\, h_2(s_1,s_2,\Sigma_{-i},v) + \\
& + U(v-s_1)\, h_1(s_1,s_2,\Sigma_{-i},v) + \sum_{\bar{s}\in\Lambda} U(v-\bar{s})\, h_{1'}(s_1,s_2,\bar{s},\Sigma_{-i},v) + \\
& + U(0)\, h_0(s_1,s_2,\Sigma_{-i},v)\Big)\, dF(v/x)
\end{aligned} \tag{3}
$$

where $h_2(.)$, $h_{1'}(.)$ and $h_{1''}(.)$ and $h_0(.)$ are defined as above.

The model we just presented is a simultaneous game of incomplete information, and the equilibrium concept we use is Bayesian Nash equilibrium. We only consider pure strategy equilibrium. An equilibrium is a set of N strategies, $\Sigma \equiv \{S^1,\ldots,S^N\}$, such that for all bidders and each signal x, bidder i maximizes his expected utility with bids $(s_1(x), s_2(x))$, given that all other bidders' strategies are $\Sigma_{-i} \equiv \Sigma - \{S^i\}$. We restrict our analysis to symmetric equilibria, that is, equilibria of the form $\Sigma \equiv \{S,\ldots,S\}$. Note that this does not imply that all bidders bid the same prices in equilibrium, but that all bidders *that receive the same signal* bid the same prices.

The model is intractable analytically, even in the simplest cases. Hence we solve the model numerically for different parameter values. The details on the numerical implementation are left to the Appendix.

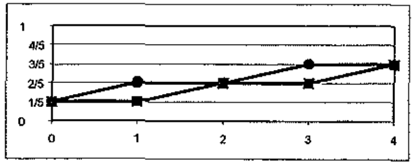
[4] To see the dependence on v, note that rivals' bids depend on Σ_{-i} and on rivals' signals, which in turn depend on v.

3.3 Equilibria of the model

The vector of parameters of the model is $(N,\mu,\alpha,\rho,\lambda,K)$. We present results for a selection of parameter combinations that we consider reasonable, given the computational limitations we have. With respect to the number of bidders, N, since there are only two indivisible units for sale, to consider $N>4$ means that, in any symmetric equilibrium, the probability that a single bidder gets at least one unit is very small. For this reason, we present results with N in $\{2,3,4\}$. We have set the *a priori* expected value of v, μ, equal to 0.75. Note that μ must lie in (0,1), and that if μ were too small, the *a posteriori* expected value of v would be too small for bidders to bid strictly positive bids for every possible signal. For the parameter of accuracy of prior information, α, we have selected values in $\{2,4,6,...,20\}$. For $\alpha=2$, that is, the maximum uncertainty case within the previous set, the standard deviation of the prior distribution on v is 0.25 (recall that this distribution is defined on (0,1) and that its average, μ, is 0.75). For $\alpha=20$, the standard deviation is approximately 0.1. For the parameter of risk aversion, ρ, we have selected values in $\{1,5,10\}$. We consider $\rho=1$ close to the risk neutral case, and consider higher values of ρ to study the effect of increasing risk aversion. Finally, the required computation time makes it unfeasible to explore a number of possible prices, $\lambda+1$, higher than 12, or a number of private signals, $K+1$, higher than 5. Thus, we have selected the combinations (λ,K) in $\{(5,2),(5,4),(9,2)\}$. Hence, for every auction format, we have explored 270 combinations of parameters[5].

Figure 1 shows an example of an equilibrium bidding strategy for the Spanish auction, when there are 2 players ($N=2$), the accuracy of private information is low ($\alpha=4$), risk aversion is low ($\rho=1$), there are 6 possible prices ($\lambda=5$), and there are five possible signals ($K=4$).

Figure 1: An equilibrium for the Spanish auction



Given $\lambda=5$, the possible prices are $\Lambda=\{0,1/5,2/5,3/5,4/5,1\}$, which are represented in the vertical axis; and given $K=4$, the possible signals are $X=\{0,1,2,3,4\}$, which are represented in the horizontal axis. We plot the bids, $s_1(x)$ and $s_2(x)$, circles and squares respectively, for every signal x in X.

[5] Gordy (96) uses similar parameter values for the model except for K, for which he uses values up to 7. The algorithm he uses requires a shorter computation time than ours, which in turn allows him to take larger values for K. However, given a vector of parameter values, he only finds an arbitrary subset of the equilibria while we find all. For a comparison of both algorithms, see the Appendix.

We characterize equilibria using two summary statistics. Let the bid spread, denoted by $\Delta(S)$, be the expected difference between a bidder's high and low bid, i.e.: $\Delta(S)=\sum_{x=0}^{K}(s_1(x)-s_2(x))\Pr(x)$. Since we have taken $s_1(x)\geq s_2(x)$, $\Delta(S)$ is positive. In addition, since $s_1(x)$ and $s_2(x)$ lie in (0,1), its difference also does. Hence, $\Delta(S)$ is a strictly convex combination of values in (0,1), and therefore $\Delta(S)$ is also in (0,1) for every S. For the example given in Figure 1, $\Delta(S)=0.07$.

Let $R(S)$ be the expected seller's revenue if all bidders play S, i.e., $R(S)=\sum_y r(y,S)\Pr(y)$ where $r(y,S)$ is the seller's revenue if bidders play S and the N-dimensional vector of private signals is y. The summation on the latter expression is over all possible vectors of signals. Since every bid lies in (0,1) and two units are awarded, $R(S)$ lies in (0,2) for every S, independently of the auction format[6]. For the example given in Figure 1, $R(S)=1.08$.

Next, we summarize the main findings.

Existence of equilibria. We check if every possible pure strategy is part of a symmetric Bayesian Nash equilibrium, and find that for the three auction formats, there is not a symmetric equilibrium for some sets of parameter values. The existence of equilibrium varies with the auction format: the model has at least an equilibrium in 51% of the cases for the Spanish auction, in 96% for the discriminatory auction, and only in 10% for the uniform auction. The non existence of equilibria is probably due to the discrete nature of the model, and we concentrate in comparing equilibria for sets of parameters for which we have at least an equilibrium for the Spanish and the discriminatory auction. In what follows, we do not report results on the uniform auction, given that it only exists for a few parameter combinations.

Why these differences on existence among auction types? We think that it is due to strategic considerations. In both the uniform and the Spanish auction, given that all other bidders are playing the same strategy, S, if bidder i, instead of responding with S, increases his high bid for a given signal, he increases the probability of winning at a low cost both for the uniform (the cut-off price is either his bid or does not change) and the Spanish auction (he pays his bid or the WAP, that may change slightly), and therefore increases his expected utility. This is not the case for the discriminatory auction. The profitability of such a deviation increases with the number of players, N, since the probability of winning at least one unit by playing the same strategy as all other players, in general, decreases with N. Thus, for all auction types, the number of parameter combinations for which there exist equilibria decreases with N. In particular, in the uniform format, there exist equilibria only for $N=2$.

Uniqueness of equilibria. When equilibrium exist, we often find more than one for both the Spanish and the discriminatory auctions. We think that multiplicity of equilibria is also due to the discrete nature of the model. The equilibrium is unique in 68% of the cases for the Spanish auction, and in 41% for the discriminatory auction. Multiple equilibria for the same parameter combination differ from one another in terms of bid spread and seller's expected

[6] The extreme case is a strategy S in which $s_1(x)=s_2(x)=1$ for all x, this leads to $R(S)=2$ in any auction. In general, as mentioned, given a strategy S, $R(S)$ depends on the auction format since $r(y,S)$ does.

revenue.

As N and λ increase, we observe other kinds of multiplicity of equilibria. For example, for $N=4$, $\lambda=9$, $K=2$, $\alpha=16$ and $\rho=10$, there are 45 equilibria both for the Spanish and the discriminatory auction, but there are only three different values for expected seller's revenue across the 45 equilibria. Equilibria with identical expected seller's revenue are characterized by a high bid that is identical across equilibria, and they differ only on the low bids, which are such that for any bidder, the probability of winning two units is zero (any low bid across all signals is lower than any high bid); as a consequence all combinations of low bids that are lower than any high bid are part of an equilibrium. Note that the probability of winning two bids, and hence the *relevance* of the low bid, decreases with N, and thus we observe more multiple equilibria of this kind for large N. Also, the number of possible combinations of *irrelevant* low bids increases with λ, and therefore multiple equilibria also increase with λ.

Bid Spread. Diagrams 1 to 3 present bid spread for the Spanish and discriminatory auction for the 270 combinations of parameters considered. Diagram 1 shows bid spread for $\lambda=5$ and $K=2$; each of the 9 figures on Diagram 1 represents a combination of ρ and N, and shows bid spread as a function of α. Diagrams 2 and 3 show the same information for $\lambda=5$ and $K=4$, and for $\lambda=9$ and $K=2$, respectively. Bid spread for the Spanish auction is represented by a square, and for the discriminatory auction by a circle. When there are multiple equilibria, we present the average value. Missing dots represent non-existence of equilibria for the corresponding auction format and combination of parameters.

Gordy (96) conjectures that multiple bids can be used to hedge against winner's curse, as well as to express downward sloping demand due to risk aversion, and finds that results for the discriminatory auction are, in general, consistent with the conjecture. He measures the use of multiple bids by bid spread. Our results show that the conjecture is also valid for the Spanish auction: bidders use bid spread to cover themselves against uncertainty. By spreading bids over a range of prices, bidders hedge against the risk of winning due to a misestimation of the value of the security. When they have overestimated the value of the good auctioned, they win at a lower average price than expected; when they have underestimated the value, they win at a higher average price than expected. To support this idea, note that on the diagrams, both for the discriminatory and the Spanish auction, there are many equilibria where bid spread is equal to 0, i.e., bidders submit the same bid for both units, when the parameter of risk aversion is low ($\rho = 1$), and the number of equilibria with zero bid spread decreases as ρ increases. Also zero bid spread occurs more often for larger values of α, that is, as public information becomes more accurate. Therefore, both in the discriminatory and the Spanish auction, as bidders are more risk averse and there is less public information, bidders spread their bids to cover themselves against uncertainty.

The diagrams also show that, on average, bid spread is higher or equal for the Spanish auction than for the discriminatory auction. There are two contributing factors. First, for the Spanish auction, bidders can increase the bid on the first unit at a lower cost than for the discriminatory auction: if they have overvalued the good and they win, they only pay the WAP instead of their bid as in the discriminatory case. Second, for the Spanish auction the low bid determines the price the bidder has to pay on the first unit if he wins two units[7]: the

[7] As mentioned, the probability of this event decreases with N and λ.

lower his bid on the second unit is, the lower is the WAP that the bidder has to pay on the first unit. Note that this strategic effect is not present in the discriminatory case, where the bid on the second unit has no effect on the price the bidder pays on the first unit. Since the high bid tends to be higher, and the lower bid lower in the Spanish than in the discriminatory case, bid spread is higher in the Spanish than in the discriminatory auction.

In 71% of the cases when there is equilibria for both the discriminatory and the Spanish auction, at least one of the equilibria is identical (same bids for each signal) for both auctions. In general, it occurs for higher values of α, that is, as public information becomes more accurate. We interpret this result as an indication that the Spanish auction works differently to the discriminatory auction when there is more uncertainty (small α), and therefore, more potential for the winner's curse.

Seller's Revenue. Table 1 illustrates how average expected seller's revenue changes with the different parameters. For each of the values considered for K, λ, ρ, N and α, the table shows average (across equilibria) expected seller's revenue both for the Spanish (S) and the discriminatory (D) auction formats. For example, for $K=2$ the average expected seller's revenue is 1.159 for the Spanish auction and 1.119 for the discriminatory auction. The averages are computed for the parameter combinations for which at least one equilibrium for both the Spanish and the discriminatory auctions exists. There are 136 such combinations.

Table 1: Average expected seller's revenue

	K		λ		ρ			N		
Par. Value	2	4	5	9	1	5	10	2	3	4
S	1.159	1.136	1.128	1.213	1.229	1.175	1.100	1.071	1.207	1.253
D	1.119	1.121	1.090	1.187	1.158	1.157	1.069	1.034	1.177	1.217
	α									
Par. Value	2	4	6	8	10	12	14	16	18	20
S	0.956	1.005	1.090	1.120	1.168	1.180	1.199	1.211	1.217	1.230
D	0.854	0.974	1.029	1.101	1.124	1.144	1.167	1.183	1.206	1.207

The expected seller's revenue increases for both auction formats as the number of possible prices, λ, the number of players, N, and the accuracy of public information, α, increases; and as the parameter of risk aversion, ρ, decreases. The results on N and α conform with the results for the Milgrom and Weber (82) single-unit model; also, as risk aversion decreases, bids are more aggressive and expected seller's revenue increases. The effect of increasing the number of possible signals, K, is ambiguous. Note that increasing the number of possible signals, on the one hand decreases i-th bidder's uncertainty on the true value of the good, but on the other hand increases his uncertainty on the other bidders' signals.

How do both auction formats compare in terms of expected seller's revenue? For all the parameters considered in Table 1, average expected seller's revenue is higher for the Spanish than for the discriminatory auction. However, there are many parameter combinations for which the ranking of equilibria is ambiguous, because there are multiple equilibria for one or both auction formats. For example, we can find parameter combinations for which there are two equilibria in the discriminatory case, with expected seller's revenue R_1^d and R_2^d

respectively, and one in the Spanish case, with R^s, and such that $R_1^d < R^s < R_2^d$; note that it can also be the case that $R^s > 0.5(R_1^d + R_2^d)$. Considering the expected seller's revenue for each parameter combination instead of averages, the Spanish format *dominates* the discriminatory one in 39% of the cases, out of the 136 parameter combinations where at least one equilibrium exist for both auction formats[8]. Note also that as mentioned earlier, in 71% of the cases when there is equilibria for both auction formats, at least one equilibria is identical for both auctions; in this case, expected seller's revenue is higher for the discriminatory auction, since the winner with the highest bid pays his bid, while in the Spanish one he only pays the WAP; but the difference in expected seller's revenue is small. What can we conclude then from the evidence? First, that the ranking of both auction formats on terms of expected seller's revenue is impossible, because there are many cases of multiple equilibria such that some equilibria for one auction format has a higher expected seller's revenue than some equilibria for the other, but the reverse holds for other equilibria. Second, that on average the Spanish auction gives higher expected seller's revenue: there are equilibria for the discriminatory auction that give low expected seller's revenue. Note that there are two different facts that could explain the different expected seller's revenue in the Spanish and the discriminatory auction: on the one hand, bidders in the Spanish auction bid more aggressively on the first unit, since they only pay the WAP instead of their bid if they win with the highest bid, which tends to increase expected seller's revenue; on the other hand they only pay the WAP, which tends to decrease expected seller's revenue. The results are consistent with the first fact dominating the second, on average.

For what parameter combinations does the Spanish auction dominate in terms of expected seller's revenue the discriminatory auction? Table 2 presents the percentage of parameter combinations for which the Spanish auction dominates the discriminatory. For example, in 42% of the parameter combinations with K=2 where there are equilibria for the Spanish and discriminatory auctions, the former dominates[9].

Table 2: Dominance in terms of expected seller's revenue for the Spanish auction

	K		λ		ρ			N		
Par. Value	2	4	5	9	1	5	10	2	3	4
	0.42	0.29	0.41	0.34	0.82	0.33	0.24	0.31	0.47	0.44
	α									
Par. Value	2	4	6	8	10	12	14	16	18	20
	0.63	0.25	0.40	0.27	0.33	0.23	0.41	0.29	0.44	0.69

The table shows that, in general, the Spanish auction dominates the discriminatory auction when risk aversion is low (for ρ=1, the Spanish auction dominates the discriminatory one in 82% of the cases). The intuition is simple. Bidders bid more aggressively in the Spanish case for the first unit than in the discriminatory auction since they only pay the WAP if they happen to win with the higher bid. This effect is specially strong if bidders have low risk aversion (small ρ). Despite the fact that bidders only pay the WAP for the higher bid, this more aggressive bidding gives higher expected revenue to the seller. The change in dominance with the other parameters is not clear.

[8] *Dominates* means that the worst Spanish equilibrium (in terms of expected seller's revenue) is not worse than the best discriminatory one.

[9] Recall that it does not imply that the latter dominates in the remaining 58%.

4. CONCLUSION

This paper develops a model of multiple bids in a common value auction for the Spanish auction format, following Gordy (96), that develops the model for the discriminatory auction format. The Spanish auction is a hybrid system of discriminatory and uniform price auctions: winning bidders pay their bid price if it is lower than the weighted average price of winning bids, while all other winning bidders pay the weighted average of winning bids. There are two units for sale, and bidders bid for both units. Both signals and bids are restricted to a finite set, which makes the problem solvable by simulations. We find equilibria for the Spanish, the uniform and the discriminatory auction, and compare them. Of course, we solve a special case, assuming functional forms for utility and distributions. But given the little that is known about the Spanish auction, we think that our results offer interesting insights about how it works.

Our main findings are the following. First, both in the Spanish and in the discriminatory auctions bidders use bid spread to cover themselves against uncertainty. Second, bid spread is higher in the Spanish auction for two reasons: on the one hand, since the cost of overbidding is lower due to the fact that a winning bidder with the highest bid only pays the weighted average price, while he pays his bid in a discriminatory auction, bidders bid more aggressively on their first unit; on the other hand, since the bid on the second unit could change the price paid on the first unit in the Spanish auction, bidders have an incentive to lower their second bid that is not present in the discriminatory auction. Third, expected seller's revenue for the seller is on average higher for the Spanish auction than for the discriminatory auction. As we have argued above, bidders bid more aggressively for the first unit on the Spanish auction, and expected revenues are higher as a result even if bidders only pay the weighted average of winning bids for the first unit.

Should the Spanish Treasury maintain the auction format they use? The answer is not clear. Even if revenue is on average higher for the Spanish auction, strategic considerations are more complicated, and this could imply that less participation occurs, specially since foreign bidders usually bid mainly in discriminatory auctions. This could lower participation and, as a consequence, decrease expected seller's revenue. Given that the discriminatory and the Spanish formats are similar when uncertainty decreases, a change to a discriminatory format at the same time as a commitment to better public information about the value of the good, could increase the Spanish Treasury's revenue.

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Appendix: Computational notes for the model

In this Appendix, we first present an overview of the algorithm we have used in the simulations[10]. Second, we concentrate on some mathematical results which simplify the calculations for the Spanish auction. We treat the discriminatory case as in Gordy (96), and adapt the uniform and the Spanish auctions from it; so we present results without proofs.

A.1 General framework

The key question to solve with the simulation is how to explore the set of all possible strategies, denoted by Ω, to find an equilibrium. Gordy suggests a tâtonemment algorithm: (1) take arbitrarily an element in Ω, say S; (2) assume that N-1 bidders play S and find the best reply of the N-th bidder to S, say S^*; (3) if $S=S^*$ then S is an equilibrium, go to (1), else set $S:=S^*$ and go to (2). As Gordy points out, this algorithm might enter in a loop of the form: $S\rightarrow S^*\rightarrow\ldots\rightarrow S$. Furthermore, the fact that S^* is the best reply to S does not imply that S^* is more likely to be an equilibrium than any other strategy.

We take a different approach. We list, in an arbitrary order, all the elements (strategies) in Ω, say $\Omega=\{S_0,S_1,S_2,\ldots,S_p\}$. Notice that, given the characteristics of the model, Ω is a finite set. We check whether each element is an equilibrium or not following the previous order. Hence we check first S_0, second S_1, and so forth up to S_p. In particular, after checking, say S_0, we check S_1 independently of whether S_0 is an equilibrium or not. In other words, the order in which we check the elements is fixed arbitrarily and independently of what the equilibria are.

If we list, without repetition, all the elements in Ω, all its elements are checked. Hence all equilibria (if any) are found, and each element is checked only once. The main problem with this approach is the computation time it requires[11]. We reduce it in two ways. First, when checking a strategy, we do not look for the best reply. Notice that, given a strategy S in which $(s_1(x),s_2(x))$ are the bids when the signal is x, finding *better* bids than $(s_1(x),s_2(x))$ is, in general, computationally faster than finding the *best* bids for signal x. Second, Gordy (96) only finds equilibria in which the high bid, $s_1(x)$, is non-decreasing in x (recall that $E(v/x)$ increases with x), and we only check for strategies with that property.

How to list all the elements in Ω which are non-decreasing in the high bid? We have programmed a function, ϕ, presented below, which maps Ω into itself. Denote as S_0 the strategy: $s_1(x)= s_2(x)=0$ for all x, then ϕ satisfies that all the elements in Ω satisfying the previous property can be listed, without repetition, as $\{S_0, \phi(S_0), \phi^2(S_0),\ldots, \phi^p(S_0)\}$, where ϕ^j denotes the composition of ϕ with itself j times, i.e.: $\phi^2(S_0)=\phi(\phi(S_0))$. The last element, $\phi^p(S_0)$, is the strategy: $s_1(x)=s_2(x)=1$ for all x. So, to generate these elements we use the algorithm: (i) set $S=S_0$; (ii) if $S=\phi^p(S_0)$, then stop; (iii) set $S=\phi(S)$ and go to step (ii).

To present the function ϕ, let us rewrite a strategy S as $\{s_1,s_2\}$, where $s_1=(s_1(0),\ldots,s_1(K))^T$ and $s_2=(s_2(0),\ldots,s_2(K))^T$, such that $s_1(x)$ and $s_2(x)$ are, as before, the high and the low bid, respectively, when the observed signal is x. Let $\{s_1,s_2\}$ be given, and set:

$$\phi(\{s_1,s_2\})=\begin{cases}\left\{\phi_1(s_1),(0,\ldots,0)^T\right\} & \text{if } \quad s_1=s_2\\ \left\{s_1,\phi_2(s_2)\right\} & \text{otherwise}\end{cases}$$

[10] The final version was written in Turbo Pascal. The exe file will be provided by the authors upon request.

[11] The computation time depends on: i) the number of elements in Ω to be checked, which is determined by λ and K, ii) the average time required to check if an element is an equilibrium, which is determined by λ, K and N. Combinations with $\lambda\geq 10$, $K\geq 5$ and $N\geq 5$ simultaneously are unfeasible.

where the functions ϕ_1 and ϕ_2 are defined below. In words, when generating $\phi(\{s_1,s_2\})$, if $s_1=s_2$, we update s_1 using ϕ_1 and we set the new vector of low bids to $(0,\ldots,0)^T$. If $s_1\neq s_2$ we update s_2 using ϕ_2. Denote the x-th component of $\phi_1(s_1)$ as $\phi_1(s_1)(x)$, and analogously for $\phi_2(s_2)$, we set:

$$\phi_1(s_1)(x)=\begin{cases} s_1(x)+\lambda^{-1} & \text{if } \; x\leq x^{*} \\ s_1(x) & \text{otherwise}\end{cases} \qquad \phi_2(s_2)(x)=\begin{cases} 0 & \text{if } \; x^{**}>0 \text{ and } x<x^{**} \\ s_2(x)+\lambda^{-1} & \text{if } \; x=x^{**} \\ s_2(x) & \text{otherwise}\end{cases}$$

where $x^{*}=\min\{x : s_1(x)<1\}$ and $x^{**}=\min\{x : s_2(x)<s_1(x)\}$.

A.2 An algorithm to evaluate bidder's utility in the Spanish auction

As we have mentioned above, this part follows trivially from Gordy(96), so we present the results without proofs.

Given a signal, x, for the N-th bidder, the utility of a bid depends on rivals' signals. Furthermore, since bidders are anonymous, the relevant fact is the vector of rivals' signals but not *who* receives each signal. For example, if N=4, the vectors of rivals' signals $(y_1,y_2,y_3)=(5,1,0)$ and $(y_1,y_2,y_3)=(1,5,0)$ are observationally equivalent for the other bidder. So, when computing the expected utility of a bid, it suffices to distinguish between vectors of rivals' signals that are different after ordering the components within each vector in decreasing order $(y_1\geq y_2\geq y_3)$. The probability function for vector y of rivals' signals (after ordering decreasingly), conditional on N-th bidder's signal x, $g_y(y/x)$, is:

$$g_y(y/x)=M(y)\prod_{i=1}^{N-1}\binom{K}{y_i}\frac{B(x+\alpha\mu,K-x+\alpha(1-\mu))}{B(\alpha\mu,\alpha(1-\mu))}$$

where $M()$ and $B()$ are the multinomial and the beta function respectively, see DeGroot (70).

Assume that all bidders except the i-th play S, that is, $\Sigma_{-i}=\{S,\ldots,S\}$. For bidder i, the expected utility of the pair $s=(s_1,s_2)$, in the Spanish case, given a signal x, is given by equation (1). We decouple the right hand term in (1) as:

$$E(U^s(s))=U_2(s,S,x)+U_1(s,S,x)+U_{1'}(s,S,x)+U_0(s,S,x) \tag{A1}$$

where, for instance, $U_0(s/S,x)$ is the expected utility of winning exactly zero units by the probability of that event, that is:

$$U_0(s,S,x)=\int_0^1 U(0)\; h_0(s_1,s_2,\{S,\ldots,S\},v)\, dF(v/x)$$

The other terms in the right side of (A1) are analogous for the events 1,1' and 2, defined in the model. Proceeding as Gordy (96), we have:

$$U_0(s,S,x)=-\sum_y p_0(s,S,y)g_y(y/x)$$

where the summation is over all possible decreasingly ordered vectors of N-1 rivals' signals, and $p_0(s,S,y)$ is the probability of winning exactly zero units given s, rivals' signals y and that rivals play $\Sigma_{-i}=\{S,\ldots,S\}$. Analogously:

$$U_1(s,S,x)=-\exp(\rho(1-s_1))\sum_y p_1(s,S,y)\,{}_1F_1(NK-\Sigma(y,x)+\alpha(1-\mu),NK+\alpha,\rho)g_y(y/x)$$

where ${}_1F_1()$ is the confluent hypergeometric function, see Abramowitz and Stegun (72), and $\Sigma(y,x)$ is the summation of the components of the vector y and the signal x. Also:

$$U_{1'}(s,S,x)=-\sum_{\bar{s}\in\Lambda}\exp(\rho(1-0.5(s_1+\bar{s}))\sum_y p_{1'}(s,S,y)\,{}_1F_1(NK-\Sigma(y,x)+\alpha(1-\mu),NK+\alpha,\rho)g_y(y/x)$$

$$U_2(s,S,x)=-\exp(\rho(2-1.5s_1-0.5s_2))\sum_y p_2(s,S,y)\,{}_1F_1(NK-\Sigma(y,x)+\alpha(1-\mu),NK+\alpha,2\rho)g_y(y/x)$$

Notice that the terms $p(.)$ are easily computable. More importantly, the terms ${}_1F_1(.)$ and $g_y(.)$ do not depend on the strategy, that is, they can be computed at the start of the program and then used repeatedly when checking every strategy.

Gordy (96) derives expressions for (2), the discriminatory auction. Expressions for (3), the uniform case, are straightforward from it.

Square: Spanish Circle: Discriminatory
Missing values denote non-existence of equilibrium. When multiple equilibria, we represent average value.
Horizontal axis represents α, from 2 to 20, with increments of 2.

DIAGRAM 1 λ=5, K=2, Bid Spread

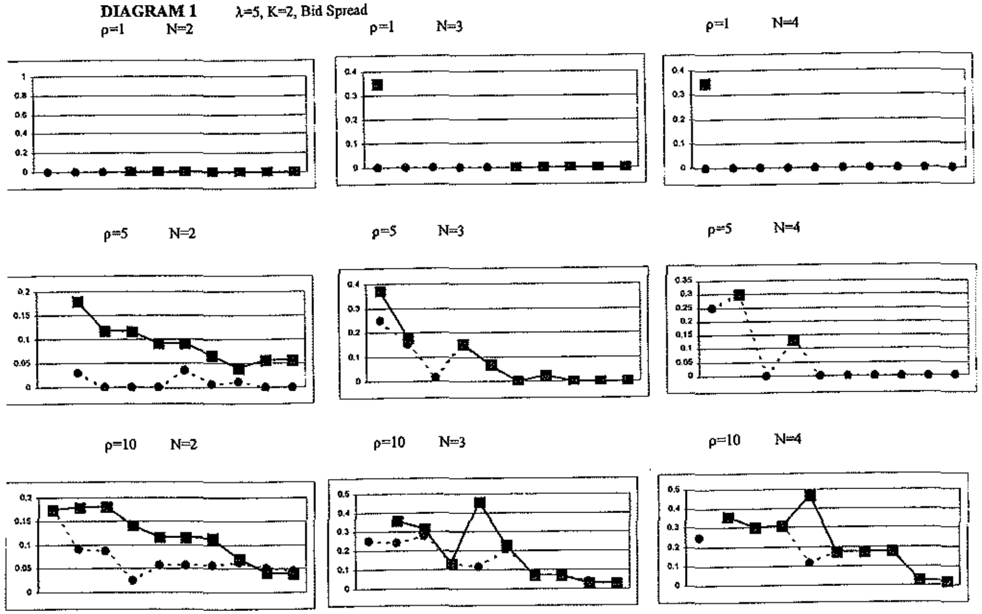

DIAGRAM 2 λ=5, K=4, Bid Spread

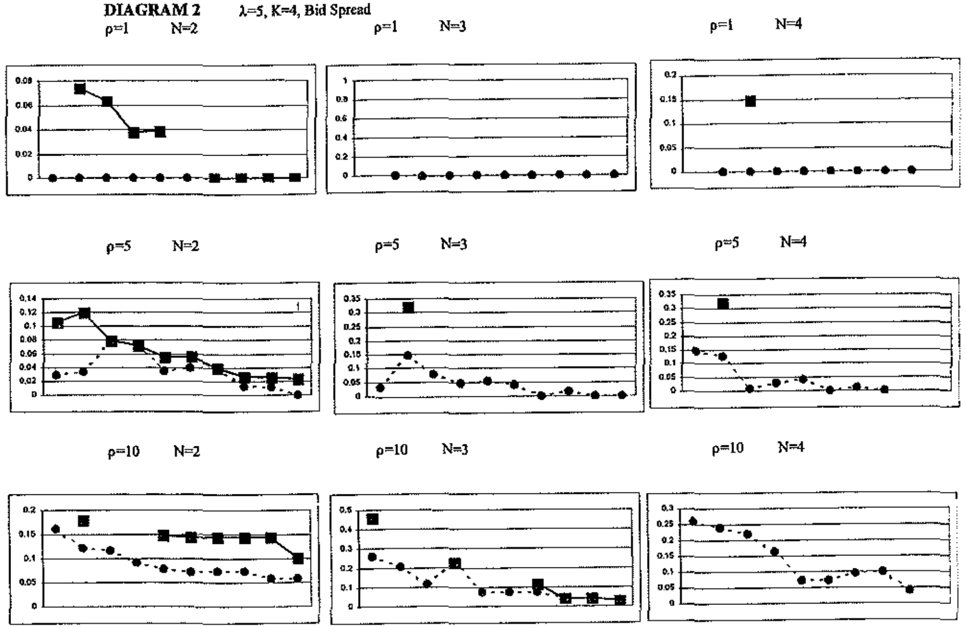

DIAGRAM 3 λ=9, k=2 Bid spread