



Moduli spaces for principal bundles in arbitrary characteristic

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Abstract

In this article, we solve the problem of constructing moduli spaces of semistable principal bundles (and singular versions of them) over smooth projective varieties over algebraically closed ground fields of positive characteristic.

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1. Introduction

In this article, we introduce a formalism for dealing with principal bundles on projective manifolds defined over an algebraically closed ground field of arbitrary characteristic which enables us to construct and compactify the moduli space of Ramanan-stable principal bundles. As a major application, we obtain, under some restrictions on the characteristic of the base field, the solution of the long-standing problem of constructing the projective moduli space of semistable principal bundles (with semisimple structure group) on a smooth projective variety.

In general, we get different compactifications of the moduli space of Ramanathan-stable principal G -bundles for different representations $G \hookrightarrow \mathrm{GL}(V)$. In the curve case, all of them are equal, whereas in higher dimensions we use torsion free sheaves to compactify the moduli space of semistable principal G -bundles, so they naturally become different.

The theory of (semi)stable principal G -bundles starts for the structure group $G = \mathrm{GL}_r(\mathbb{C})$ as the theory of (semi)stable vector bundles. Based on his development of Geometric Invariant Theory, David Mumford proposed the notion of a (semi)stable vector bundle on a Riemann surface [38]. At about the same time, Narasimhan and Seshadri made the fundamental discovery that stable vector bundles on the Riemann surface X are precisely those arising from irreducible unitary representations of the fundamental group $\pi_1(X)$ [40]. (Recall that the relationship between vector bundles and representations of the fundamental groups was first investigated by A. Weil [58].) Finally, Seshadri gave the GIT construction of the moduli space of stable vector bundles on a Riemann surface together with its compactification by S-equivalence classes of semistable vector bundles [52]. This construction easily generalizes to ground fields of arbitrary characteristic.

Since its beginnings, the study of stable G -bundles has widely developed and interacted with other fields. The scope of the theory has been progressively enlarged by eliminating limitations on the “three parameters” of the theory, i.e., the structure group G , the base manifold X , and the ground field k . First, in the work of Gieseker [13] and Maruyama [36], the theory of stable vector bundles was enlarged to a theory of semistable torsion free sheaves on projective manifolds over fields of characteristic zero. Later, Simpson brought this theory into its final form [55]. In the work [31,32], the barriers of extending Simpson’s results to fields of positive characteristic were finally removed. The arguments given there improve the formalism even in characteristic zero.

At the time when the results of Gieseker and Maruyama were published, Ramanathan had also treated the theory of principal G -bundles on a compact Riemann surface X for an arbitrary connected reductive structure group G . In the paper [43], he introduced the notion of (semi)stability for a principal G -bundle \mathcal{P} on the Riemann surface X and generalized the results of the paper [40], i.e., linked the theory of semistable principal bundles on X to the study of representations of the fundamental group in a compact real form K of G . More important to us is the main result of his PhD thesis, finished at the Tata Institute in 1976. There, Ramanathan provides an ingenious GIT construction for the moduli space of semistable principal G -bundles on a compact Riemann surface X . Due to the untimely death of the author, this important result appeared in the posthumous publication [44]. At that time, the subject had become of general interest to mathematicians and physicists.

In the recent papers [16,17,45,47] two independent—although related—methods for generalizing Ramanathan’s theory to the case of higher dimensional base manifolds defined over the complex numbers were presented. More precisely, the moduli space of Ramanathan stable bundles was constructed and compactified with certain “generalized” principal bundles, satisfying a Gieseker type semistability condition.

It thus seemed natural to join the forces of the authors to cope with the problem of bringing these recent developments to base fields of arbitrary characteristic. In the present paper, we rewrite the theory of the paper [45] from scratch. We will see that the results of that paper are, in fact, true in positive characteristic. Furthermore, some of the fundamental discoveries of the papers [16,17,47] also remain valid over any algebraically closed field. In any case, we manage to construct our moduli spaces as open subschemes of the moduli spaces of “ δ -semistable pseudo G -bundles.” In a separate publication [15], we will explain how the approach via the ad-

joint representation of a connected reductive group G of [17,44], or more generally via faithful representations with kernel in the center of G , may be generalized to positive characteristic.

The main change of philosophy which made the progress possible is the following: Classically, as suggested by the work of Ramanan and Ramanathan [42], one studied semistability of principal bundles by relating it to the semistability of associated vector bundles. This works well in characteristic zero but makes the assumption of sufficiently high characteristic of the base field necessary while working over fields of positive characteristic. In the more recent work quoted above, we linked the semistability of a principal bundle to the semistability of an associated *decorated* vector bundle. Unfortunately, the theory of polynomial representations of the general linear group is more complicated in positive characteristic (see the book [18]), so that the set-up of [45,47] cannot be directly copied. Nevertheless, the basic idea of that work makes perfect sense over fields of positive characteristic. Thanks to the results of [31,32], one may adapt the fundamental arguments from characteristic zero.

Let us introduce a piece of notation, so that we may state our result in a precise form. In this paper, we will deal with moduli functors of the form

$$\underline{M}^{(s)s} : \underline{\text{Sch}}_k \rightarrow \underline{\text{Set}}$$

$$S \mapsto \{ \text{Isomorphism classes of families of (semi)stable objects} \}.$$

In each case, we define S-equivalence on the set of isomorphism classes of semistable objects (e.g., semistable sheaves or principal G -bundles with fixed numerical data) which, restricted to stable objects, reduces to isomorphism. Assuming we have the moduli functor and S-equivalence, we introduce the following convenient terminology: A *coarse moduli scheme* for the functors $\underline{M}^{(s)s}$ consists of a scheme M^{ss} , an open subscheme $M^s \subseteq M^{ss}$, and natural transformations of functors

$$\vartheta^{(s)s} : \underline{M}^{(s)s} \rightarrow h_{M^{(s)s}}$$

with the following properties:

1. The space $M^{(s)s}$ corepresents $\underline{M}^{(s)s}$ with respect to $\vartheta^{(s)s}$. It does so uniformly, if $\text{Char}(k) > 0$, and universally, if $\text{Char}(k) = 0$. (See [29, Definition 2.2.1]. Observe that “uniformly” refers to the base change property for *flat* morphisms φ in that definition.)
2. The map $\vartheta^s(k) : \underline{M}^s(k) \rightarrow M^s(k)$ is a bijection between the set of isomorphism classes of stable objects and the closed points of M^s .
3. The map $\vartheta^{ss}(k) : \underline{M}^{ss}(k) \rightarrow M^{ss}(k)$ induces a bijection between the set of S-equivalence classes of semistable objects and the closed points of M^{ss} .

The difference between positive and zero characteristic in the above definition comes from our use of Geometric Invariant Theory, as GIT quotients in positive characteristic are not necessarily universal categorical. For $G = \text{GL}(V)$, one can in fact show that, in positive characteristic, the moduli space of stable sheaves universally corepresents the moduli functor (see [31, Theorem 0.2]). This follows from the fact that stable sheaves are simple. However, even in characteristic zero, the sheaves corresponding to stable principal G -bundles on a curve are no longer simple (see [43, Remark 4.1]), so this proof fails in general. We now come to the more detailed presentation of the contents of our work.

1.1. Quasi-projective moduli spaces

Let G be a connected semisimple group. Fix a faithful representation $\varrho : G \rightarrow \text{GL}(V)$ and note that $\varrho(G) \subseteq \text{SL}(V)$. In characteristic zero, a theory for semistable singular principal G -bundles based on such a representation was developed in [45,47]. However, some characteristic zero gadgets such as the Reynolds operator, the instability flag, and normal forms for homogeneous polynomial representations were used. In this paper, we will rewrite the theory from the beginning, such that it becomes independent of the characteristic of the base field and works without decorated sheaves.

We will look at pairs (\mathcal{A}, τ) with a torsion free \mathcal{O}_X -module \mathcal{A} which has rank $\dim_k(V)$ and trivial determinant and a homomorphism $\tau : \mathcal{S}ym^*(\mathcal{A} \otimes V)^G \rightarrow \mathcal{O}_X$ of \mathcal{O}_X -algebras which is non-trivial in the sense that the induced section $\sigma : X \rightarrow \mathcal{I}pec(\mathcal{S}ym^*(\mathcal{A} \otimes V)^G)$ is not the zero section. Such a pair is called a *pseudo G -bundle*, and if, furthermore, $\sigma_U(U) \subset \mathcal{I}som(V \otimes \mathcal{O}_U, \mathcal{A}_{|U}^\vee)/G$, $U := U_{\mathcal{A}}$ being the maximal open subset where \mathcal{A} is locally free, we speak of a *singular principal G -bundle*.¹ In the case of a singular principal G -bundle (\mathcal{A}, τ) , we get a principal G -bundle $\mathcal{P}(\mathcal{A}, \tau)$ over U , defined by means of base change:

$$\begin{array}{ccc} \mathcal{P}(\mathcal{A}, \tau) & \longrightarrow & \mathcal{I}som(V \otimes \mathcal{O}_U, \mathcal{A}_{|U}^\vee) \\ \downarrow & & \downarrow \\ U & \xrightarrow{\sigma|_U} & \mathcal{I}som(V \otimes \mathcal{O}_U, \mathcal{A}_{|U}^\vee)/G. \end{array}$$

We now define the notion of semistability for a singular principal G -bundle (\mathcal{A}, τ) . For this, let $\lambda : \mathbb{G}_m(k) \rightarrow G$ be a one parameter subgroup of G . This yields a parabolic subgroup $Q_G(\lambda)$ (see (4) below) and a weighted flag $(V_\bullet(\lambda), \alpha_\bullet(\lambda))$ in the vector space V (see Section 2.1). A *reduction of (\mathcal{A}, τ) to λ* is a section $\beta : U' \rightarrow \mathcal{P}(\mathcal{A}, \tau)_{|U'}/Q_G(\lambda)$ over an open subset $U' \subseteq U$ with $\text{codim}_X(X \setminus U') \geq 2$. This defines a weighted filtration $(\mathcal{A}_\bullet(\beta), \alpha_\bullet(\beta))$ of \mathcal{A} . Here, $\alpha_\bullet(\beta) = (\alpha_t, \dots, \alpha_1)$, if $\alpha_\bullet(\lambda) = (\alpha_1, \dots, \alpha_t)$, and the filtration $\mathcal{A}_\bullet(\beta) : 0 \subsetneq \mathcal{A}_1 \subsetneq \dots \subsetneq \mathcal{A}_t \subsetneq \mathcal{A}$ is obtained as follows: The section

$$\beta' : U' \xrightarrow{\beta} \mathcal{P}(\mathcal{A}, \tau)_{|U'}/Q_G(\lambda) \hookrightarrow \mathcal{I}som(V \otimes \mathcal{O}_{U'}, \mathcal{A}_{|U'}^\vee)/Q_{\text{GL}(V)}(\lambda)$$

yields a filtration

$$0 \subsetneq \mathcal{A}'_1 \subsetneq \dots \subsetneq \mathcal{A}'_t \subsetneq \mathcal{A}_{|U'}^\vee$$

of $\mathcal{A}_{|U'}^\vee$ by subbundles with $\text{rk}(\mathcal{A}'_i) = \dim_k(V_i)$, $i = 1, \dots, t$. This is because $Q_{\text{GL}(V)}(\lambda)$ is the $\text{GL}(V)$ -stabilizer of the flag $V_\bullet(\lambda)$ and, thus, $\mathcal{I}som(V \otimes \mathcal{O}_{U'}, \mathcal{A}_{|U'}^\vee)/Q_{\text{GL}(V)}(\lambda) \rightarrow U'$ is the bundle of flags in the fibers of $\mathcal{A}_{|U'}^\vee$ having the same dimensions as the members of the flag $V_\bullet(\lambda)$. We define $\mathcal{A}''_i := \ker(\mathcal{A}_{|U'} \rightarrow \mathcal{A}_{|U'}^{\vee}_{t+1-i})$, $i = 1, \dots, t$, so that we obtain a filtration

$$0 \subsetneq \mathcal{A}''_1 \subsetneq \dots \subsetneq \mathcal{A}''_t \subsetneq \mathcal{A}_{|U'}$$

¹ Here, we deviate from the original terminology in [45,47].

of $\mathcal{A}|_{U'}$ by subbundles. Let $\iota: U' \rightarrow X$ be the inclusion and define \mathcal{A}_i as the saturation of $\mathcal{A} \cap \iota_* (\mathcal{A}_i'')$, $i = 1, \dots, t$. This is the filtration we denote by $\mathcal{A}_\bullet(\beta)$. It is worth noting that, if $\lambda' = g \cdot \lambda \cdot g^{-1}$ for some $g \in G$, then any reduction to λ may also be interpreted as a reduction to λ' . Now, we say that a singular principal G -bundle (\mathcal{A}, τ) is (semi)stable,² if for every non-trivial one parameter subgroup $\lambda: \mathbb{G}_m(k) \rightarrow G$ and every reduction β of (\mathcal{A}, τ) to λ , we have

$$M(\mathcal{A}, \tau; \beta) := M(\mathcal{A}_\bullet(\beta), \alpha_\bullet(\beta)) (\succ) 0,$$

where, for every weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of \mathcal{A} , we set

$$M(\mathcal{A}_\bullet, \alpha_\bullet) := \sum_{i=1}^t \alpha_i \cdot (\text{rk } \mathcal{A}_i \cdot P(\mathcal{A}) - \text{rk } \mathcal{A} \cdot P(\mathcal{A}_i)).$$

Finally, there is a notion of S-equivalence which will be explained in Section 5.2 and Remark 5.4.3. We have the implications

$$\begin{aligned} \mathcal{P}(\mathcal{A}, \tau) \text{ is Ramanathan-stable} &\Rightarrow (\mathcal{A}, \tau) \text{ is stable} \\ &\Rightarrow (\mathcal{A}, \tau) \text{ is semistable} \\ &\Rightarrow \mathcal{P}(\mathcal{A}, \tau) \text{ is Ramanathan-semistable.} \end{aligned}$$

More precisely, in our language, Ramanathan’s notion of (semi)stability becomes

$$L(\mathcal{A}, \tau; \beta) := L(\mathcal{A}_\bullet(\beta), \alpha_\bullet(\beta)) := \sum_{i=1}^t \alpha_i \cdot (\text{rk } \mathcal{A}_i \cdot \text{deg}(\mathcal{A}) - \text{rk } \mathcal{A} \cdot \text{deg}(\mathcal{A}_i)) (\geq) 0 \quad (1)$$

for every one parameter subgroup $\lambda: \mathbb{G}_m(k) \rightarrow G$ and every reduction β of (\mathcal{A}, τ) to λ . Here, deg stands for the degree with respect to the chosen polarization.

Remark. It is easy to check from the definition that the condition of semistability has to be verified only for the indivisible one parameter subgroups that define maximal parabolic subgroups.

For a fixed Hilbert polynomial P , we define the moduli functors

$$\begin{aligned} \underline{M}_P^{(s)s}(\varrho) &: \underline{\text{Sch}}_k \rightarrow \underline{\text{Set}} \\ S &\mapsto \left\{ \begin{array}{l} \text{Isomorphism classes of families of (semi)stable singular principal} \\ G\text{-bundles } (\mathcal{A}, \tau), \text{ such that } P(\mathcal{A}) = P \end{array} \right\}. \end{aligned}$$

Main theorem. *The coarse moduli space for the functors $\underline{M}_P^{(s)s}(\varrho)$ exists as a quasi-projective scheme $M_P^{SS}(\varrho)$.*

² If the word (semi)stable is used together with the symbol “ (\leq) ,” then there are two statements: One for “semistable” with “ \leq ” and one for “stable” with “ $<$.”

This moduli space contains the moduli space for Ramanathan-stable principal bundles (whose associated vector bundle has Hilbert polynomial P) as an open subscheme. In particular, we have constructed the moduli spaces for Ramanathan-stable principal bundles in any characteristic.

The moduli space $M_P^{ss}(\varrho)$ will be constructed inside a larger *projective* moduli space $M_P^{\tilde{\delta}\text{-}ss}(\varrho)$ of $\tilde{\delta}$ -semistable pseudo G -bundles, so that it always comes with a natural compactification.

Example. For $G = \mathrm{PGL}_r(k)$, principal G -bundles correspond to Azumaya algebras, so that our construction yields in particular a moduli space for Azumaya algebras and a compactification by “Azumaya algebra sheaves.” Such moduli spaces have become of interest recently (see [27], especially Proposition 4.1, and [59]). More examples can be found in [15].

Remark (*Non-emptiness of moduli spaces*). The above theorem is a mere existence statement. The next step is to investigate the geometry of the moduli spaces. If X is a curve of genus $g \geq 2$, then one can use the moduli stack of principal G -bundles. It is a smooth algebraic stack of dimension $(g - 1) \cdot \dim(G)$ (see [6, Corollary 8.1.9]). Its connected components are labeled by the elements of the fundamental group $\pi_1(G)$ (see [10, Proposition 5] and [28, Proposition 3.15]). Estimating the dimension of the locus of unstable principal G -bundles, one sees that there are stable principal G -bundles for any given topological type $\vartheta \in \pi_1(G)$ (see [28, Proposition 3.25]). (The reader may consult [43] for the topological argument over $k = \mathbb{C}$ and [24, Proposition 4.2.2], for an existence result on stable principal G -bundles with a quasi-parabolic structure.) Our main theorem shows that the moduli space $M_G^{ss}(\vartheta)$ of Ramanathan-semistable principal G -bundles of “topological type” $\vartheta \in \pi_1(G)$ exists. Using the moduli stack, one checks that it is an irreducible normal variety of dimension $(g - 1) \cdot \dim(G)$.

On higher dimensional base varieties, the geometry of the moduli spaces is completely unknown, even if the base field is \mathbb{C} . Note that $M_P^{ss}(\varrho)$ always contains the moduli space of slope stable principal G -bundles (of the respective numerical invariants) as an open subscheme. To prove non-emptiness of $M_P^{ss}(\varrho)$, it hence suffices to construct slope stable principal G -bundles. A natural approach is to use stable vector bundles and construct from them principal G -bundles by extension of the structure group. On a surface over the field $k = \mathbb{C}$, Balaji used this method to prove interesting existence results for slope stable (and thus stable) principal G -bundles [1, Theorem 7.10]. As he also announces [1, Remark 7.2], such existence results are likely to hold in large positive characteristic as well. The details will appear in [3].

1.2. Projectivity and the semistable reduction theorem

The projectivity of the moduli spaces is not built into our new approach. The remaining question is thus under which assumptions (on the representation or the characteristic of the base field), the moduli space $M_P^{ss}(\varrho)$ is projective. Since any connected semisimple group is over k isomorphic to one defined over the integers, one may assume that G is itself defined over the integers. Then, there is also a faithful representation $\varrho: G \rightarrow \mathrm{GL}(V)$ which is defined over the integers. Under this assumption, one may develop an elegant formalism which provides projective moduli spaces in any dimension, provided that the characteristic of the base field is either zero or greater than a constant which depends on ϱ . These results will appear in [15]. As remarked before, the moduli spaces will also be projective, if G is one of the classical groups and ϱ its standard representation.

Until very recently, the most general result in that direction was contained in the work of Balaji and Parameswaran [2] where the existence of moduli spaces of semistable principal G -bundles

on a smooth projective curve was established under the assumptions that G is semisimple and the characteristic of the base field is *sufficiently large*. After the first version of this paper containing an erroneous proof for semistable reduction appeared, Heinloth managed in [22] to adapt Langton’s algorithm [34] to the setting of principal G -bundles. His new approach is to work with the affine Grassmannian (see [12] for a discussion of this object in positive characteristic), so it depends heavily on the variety being a curve. In our approach, we show a semistable reduction theorem in all dimensions.

Theorem (Semistable reduction). *Assume that either $\varrho : G \rightarrow \mathrm{GL}(V)$ is of low separable index or G is an adjoint group, ϱ is the adjoint representation and it is of low height. Then, given a semistable singular principal G -bundle \mathcal{P}_K over $X \times \mathrm{Spec}(K)$, where K is the quotient field of the complete discrete valuation ring R , there exists a finite extension $R \subseteq R'$ such that the pullback $\mathcal{P}_{K'}$ of \mathcal{P}_K to $X \times \mathrm{Spec}(K')$, K' being the fraction field of R' , extends to a semistable singular principal G -bundle $\mathcal{P}_{R'}$ over $X \times \mathrm{Spec}(R')$.*

We also recover the following theorem of Heinloth from [22]:

Corollary (Heinloth). *The assertions of the above theorem hold if X is a curve and one of the following conditions is satisfied:*

- $\mathrm{Char}(k) = 0$.
- The simple factors of G ³ are of type A and k is arbitrary.
- The simple factors of G are of type A, B, C, D and $\mathrm{Char}(k) \neq 2$.
- The simple factors of G are of type A, B, C, D, G and $\mathrm{Char}(k) > 10$.
- The simple factors of G are of type A, B, C, D, G, F, E_6 and $\mathrm{Char}(k) > 22$.
- The simple factors of G are of type $A, B, C, D, G, F, E_6, E_7$ and $\mathrm{Char}(k) > 34$.
- The simple factors of G are of type A, B, C, D, G, F, E and $\mathrm{Char}(k) > 58$.

Then, given a semistable principal G -bundle \mathcal{G}_K over $X \times \mathrm{Spec}(K)$ where K is the spectrum of the complete discrete valuation ring R , there exists a finite extension $R \subseteq R'$, such that the pullback $\mathcal{G}_{K'}$ of \mathcal{G}_K to $X \times \mathrm{Spec}(K')$, K' being the fraction field of R' , extends to a semistable principal G -bundle $\mathcal{G}_{R'}$ over $X \times \mathrm{Spec}(R')$.

Proof. Note that semistability is preserved under extension of the structure group via a central isogeny. So to prove the corollary we can restrict to a simple group of adjoint type.

In the case of classical groups, the statement can be obtained by familiar methods. We will explain the idea when $G = \mathrm{PSO}_n(k)$. Then, we have a short exact sequence of groups

$$\{0\} \rightarrow \mathbb{G}_m \rightarrow \mathrm{GO}_n(k) \rightarrow \mathrm{PSO}_n(k) \rightarrow \{1\},$$

where $\mathrm{GO}_n(k)$ is the group of matrices, such that $A^t A = cI$, with $c \in \mathbb{G}_m$. Since $H^2(X_{\mathrm{fl}}, \mathbb{G}_m) = 0$ ($H^2(X_{\mathrm{fl}}, \mathbb{G}_m) = H^2(X, \mathcal{O}_X)$ by Hilbert’s theorem 90, [37, Chapter III, Proposition 4.9]), every principal $\mathrm{PSO}_n(k)$ -bundle reduces to $\mathrm{GO}_n(k)$. Giving a principal $\mathrm{GO}_n(k)$ -bundle is equivalent to fixing a line bundle L and giving a pair (E, φ) , where E is a vector bundle of rank n and

³ More precisely, we mean the simple factors of the adjoint form of G .

$\varphi: E \otimes E \rightarrow L$ is a symmetric non-degenerate bilinear form. By the theory of decorated vector bundles on curves, developed in [46] over \mathbb{C} and extended in this work to positive characteristic (see Remark 4.5.11 and [49]), there is a moduli space for such objects when dropping the condition on non-degeneracy. Here, the stability concept depends on a parameter δ . The fact that one gets the moduli space for principal $\mathrm{GO}_n(k)$ -bundles for large δ is exactly the same as for $\mathrm{SO}_n(k)$ -bundles. The latter explained is Example 5.4.5. In the case of other groups the theorem follows directly from the theorem and the remark below. \square

Note that the above proof works only, because in the curve case we need to check semistable reduction only for a single representation. The above statements imply projectivity of the moduli spaces.

Corollary. *Under the assumption of the above theorem or corollary, the moduli space $M_p^{\mathrm{ss}}(\varrho)$ is projective. In particular, in the curve case, the moduli space $M^{\mathrm{ss}}(G, t)$ of semistable principal G -bundles (as defined by Ramanathan) of “topological type” $t \in \pi_1(G)$ exists as a projective scheme over k .*

Remark. (i) The low height assumption for the adjoint representation amounts to the following restrictions on the characteristic of the base field:

- $\mathrm{Char}(k) > 2n$, if G contains a simple factor of type A_n .
- $\mathrm{Char}(k) > 4n - 2$, if G contains a simple factor of type B_n or C_n .
- $\mathrm{Char}(k) > 4n - 6$, if G contains a simple factor of type D_n .
- $\mathrm{Char}(k) > 10$, if G contains a simple factor of type G_2 .
- $\mathrm{Char}(k) > 22$, if G contains a simple factor of type F_4 or E_6 .
- $\mathrm{Char}(k) > 34$, if G contains a simple factor of type E_7 .
- $\mathrm{Char}(k) > 58$, if G contains a simple factor of type E_8 .

(ii) Heinloth significantly improved the bounds on the characteristic in his theorem in a recent paper [23].

(iii) In a joint project [24], Heinloth and the third author have applied the techniques of the current paper to construct moduli spaces for parabolic principal G -bundles which are projective under the same hypotheses as stated in the above corollary. In the set-up of parabolic bundles, Heinloth’s semistable reduction algorithm could be generalized only to structure groups with classical root systems. For exceptional groups, one had to recur to the approach to semistable reduction which we introduce in the current paper. The work [24] contains an application of moduli spaces of parabolic principal bundles to the cohomology of the moduli stack of principal bundles and may serve as a motivation to study the techniques of the present paper.

(iv) By the fundamental Theorem 5.5.1 of Seshadri’s (which we are going to discuss in Appendix A), the existence of $M_p^{\mathrm{ss}}(\varrho)$ as a *projective* scheme implies the semistable reduction theorem for semistable singular principal G -bundles. Over higher dimensional base varieties, one may also consider the problem of semistable reduction for *slope* semistable singular principal G -bundles. If $k = \mathbb{C}$, this variant of the semistable reduction theorem is established in [1, Theorem 1.1]. Generalizations of that theorem to positive characteristic are announced in [1, Remark 7.2]. Note however that, over base varieties of dimension at least two, slope semistability is *not equivalent* to semistability (which we are using), so that Balaji’s semistable reduction theorem has no implications on the projectivity of $M_p^{\mathrm{ss}}(\varrho)$.

1.3. Notation

We work over the algebraically closed field k of characteristic $p \geq 0$. A *scheme* will be a scheme of finite type over k . For a vector bundle \mathcal{E} over a scheme X , we set $\mathbb{P}(\mathcal{E}) := \text{Proj}(\mathcal{S}ym^*(\mathcal{E}))$, i.e., $\mathbb{P}(\mathcal{E})$ is the projective bundle of hyperplanes in the fibers of \mathcal{E} . An open subset $U \subseteq X$ is said to be *big*, if $\text{codim}_X(X \setminus U) \geq 2$. The degree $\text{deg}(\mathcal{E})$ and the Hilbert polynomial $P(\mathcal{E})$ of a torsion free coherent \mathcal{O}_X -module \mathcal{E} are taken with respect to the fixed polarization $\mathcal{O}_X(1)$. We set $[x]_+ := \max\{0, x\}$, $x \in \mathbb{R}$.

2. Preliminaries

In this section, we collect different results which will be needed throughout the construction of the moduli space for singular principal G -bundles for a semisimple group G via a faithful representation $G \rightarrow \text{GL}(V)$.

2.1. GIT

We recall some notation and results from Geometric Invariant Theory. Let G be a reductive group over the field k and $\kappa : G \rightarrow \text{GL}(W)$ a representation on the finite dimensional k -vector space W . This yields the action

$$\begin{aligned} \alpha : G \times W &\rightarrow W \\ (g, w) &\mapsto \kappa(g)(w). \end{aligned}$$

Recall that a one parameter subgroup is a homomorphism

$$\lambda : \mathbb{G}_m(k) \rightarrow G.$$

Such a one parameter subgroup defines a decomposition

$$W = \bigoplus_{\gamma \in \mathbb{Z}} W^\gamma$$

with

$$W^\gamma = \{w \in W \mid \kappa(\lambda(z))(w) = z^\gamma \cdot w, \forall z \in \mathbb{G}_m(k)\}, \quad \gamma \in \mathbb{Z}.$$

Let $\gamma_1 < \dots < \gamma_{t+1}$ be the integers with $W^\gamma \neq \{0\}$ and $\gamma_\bullet(\lambda) := (\gamma_1, \dots, \gamma_{t+1})$. We define the flag

$$W_\bullet(\lambda): \quad \{0\} \subsetneq W_1 := W^{\gamma_1} \subsetneq W_2 := W^{\gamma_1} \oplus W^{\gamma_2} \subsetneq \dots \subsetneq W_t := W^{\gamma_1} \oplus \dots \oplus W^{\gamma_t} \subsetneq W$$

and the tuple $\alpha_\bullet(\lambda) := (\alpha_1, \dots, \alpha_t)$ of positive rational numbers with

$$\alpha_i := \frac{\gamma_{i+1} - \gamma_i}{\dim_k(W)}, \quad i = 1, \dots, t.$$

If λ is a one parameter subgroup of the special linear group $SL(W)$, we will refer to $(W_\bullet(\lambda), \alpha_\bullet(\lambda))$ as the *weighted flag of λ* . For a point $w \in W \setminus \{0\}$, we define

$$\mu_\kappa(\lambda, w) := \max\{\gamma_i \mid w \text{ has a non-trivial component in } W^{\gamma_i}, i = 1, \dots, t + 1\}.$$

Note that, for $G = SL(V)$ and $\lambda_j : \mathbb{G}_m(k) \rightarrow G, j = 1, 2,$

$$\mu_\kappa(\lambda_1, w) = \mu_\kappa(\lambda_2, w) \quad \text{if } (V_\bullet(\lambda_1), \alpha_\bullet(\lambda_1)) = (V_\bullet(\lambda_2), \alpha_\bullet(\lambda_2)). \tag{2}$$

(See [39, Chapter 2, Proposition 2.7]. Note that we take the weighted flags in V and not in W .)

Suppose we are given a projective scheme X , a G -action $\bar{\sigma} : G \times X \rightarrow X$, and a linearization $\sigma : G \times \mathcal{L} \rightarrow \mathcal{L}$ of this action in the line bundle \mathcal{L} . For a point $x \in X$ and a one parameter subgroup λ , we get the point

$$x_\infty(\lambda) := \lim_{z \rightarrow \infty} \bar{\sigma}(\lambda(z), x).$$

This point is fixed under the action $\mathbb{G}_m(k) \times X \rightarrow X, (z, x) \mapsto \bar{\sigma}(\lambda(z), x)$. Therefore, $\mathbb{G}_m(k)$ acts on the fiber $\mathcal{L}\langle x_\infty(\lambda) \rangle$. This action is of the form $l \mapsto z^\gamma \cdot l, z \in \mathbb{G}_m(k), l \in \mathcal{L}\langle x_\infty(\lambda) \rangle$, and we set

$$\mu_\sigma(\lambda, x) := -\gamma.$$

For a representation κ of G as above, we obtain the action

$$\begin{aligned} \bar{\sigma} : G \times \mathbb{P}(W^\vee) &\rightarrow \mathbb{P}(W^\vee) \\ (g, [w]) &\mapsto [\kappa(g)(w)] \end{aligned}$$

together with an induced linearization σ in $\mathcal{O}_{\mathbb{P}(W^\vee)}(1)$. One checks that

$$\mu_\kappa(\lambda, w) = \mu_\sigma(\lambda, [w]), \quad \forall w \in W \setminus \{0\}, \lambda : \mathbb{G}_m(k) \rightarrow G. \tag{3}$$

Finally, we recall that a one parameter subgroup $\lambda : \mathbb{G}_m(k) \rightarrow G$ gives the parabolic subgroup

$$Q_G(\lambda) := \left\{ g \in G \mid \lim_{z \rightarrow \infty} \lambda(z) \cdot g \cdot \lambda(z)^{-1} \text{ exists in } G \right\}. \tag{4}$$

The unipotent radical of $Q_G(\lambda)$ is the subgroup

$$\mathcal{R}_u(Q_G(\lambda)) := \left\{ g \in G \mid \lim_{z \rightarrow \infty} \lambda(z) \cdot g \cdot \lambda(z)^{-1} = e \right\}.$$

Remark 2.1.1. In the book [56], one defines the parabolic subgroup

$$P_G(\lambda) := \left\{ g \in G \mid \lim_{z \rightarrow 0} \lambda(z) \cdot g \cdot \lambda(z)^{-1} \text{ exists in } G \right\},$$

i.e.,

$$P_G(\lambda) = Q_G(-\lambda).$$

Therefore, every parabolic subgroup of Q is of the shape $Q_G(\lambda)$ for an appropriate one parameter subgroup λ of G . We have chosen a different convention, because it is compatible with our GIT notation.

2.1.1. *Actions on homogeneous spaces*

Let H be a reductive algebraic group, G a closed reductive subgroup, and $X := H/G$ the associated affine homogeneous space. Then, the following holds true.

Proposition 2.1.2. *Suppose that $x \in X$ is a point and $\lambda : \mathbb{G}_m(k) \rightarrow H$ is a one parameter subgroup, such that $x_0 := \lim_{z \rightarrow \infty} \lambda(z) \cdot x$ exists in X . Then, $x \in \mathcal{R}_u(Q_H(\lambda)) \cdot x_0$.*

Proof. We may assume $x_0 = [e]$, so that λ is a one parameter subgroup of G . Define

$$Y := \left\{ y \in X \mid \lim_{z \rightarrow \infty} \lambda(z) \cdot y = x_0 \right\}.$$

This set is closed and invariant under the action of $\mathcal{R}_u(Q_H(\lambda))$. Note that viewing X as a variety with $\mathbb{G}_m(k)$ -action, x_0 is the unique point in Y with a closed $\mathbb{G}_m(k)$ -orbit, and by the first lemma in Section III of [35] (or Lemma 8.3 in [5], or 3.1 in [26]), there is a $\mathbb{G}_m(k)$ -equivariant morphism $f : X \rightarrow T_{x_0}(X)$ which maps x_0 to 0 and is étale in x_0 . Obviously, f maps Y to

$$\left\{ v \in T_{x_0}X \mid \lim_{z \rightarrow \infty} \lambda(z) \cdot v = 0 \right\} = \mathfrak{u}_H(\lambda)/\mathfrak{u}_G(\lambda) \subset \mathfrak{h}/\mathfrak{g}. \tag{5}$$

Here, $\mathfrak{u}_H(\lambda)$ and $\mathfrak{u}_G(\lambda)$ are the Lie algebras of $\mathcal{R}_u(Q_H(\lambda))$ and $\mathcal{R}_u(Q_G(\lambda))$, respectively, and \mathfrak{h} and \mathfrak{g} are the Lie algebras of H and G , respectively. Note that \mathfrak{h} and \mathfrak{g} receive their G -module structures through the adjoint representation of G , and, moreover, by definition,

$$\mathfrak{u}_H(\lambda) = \left\{ v \in \mathfrak{h} \mid \lim_{z \rightarrow \infty} \lambda(z) \cdot v = 0 \right\}.$$

This yields the asserted equality in (5).

On the other hand, the dimension of $\mathfrak{u}_H(\lambda)/\mathfrak{u}_G(\lambda)$ equals the one of the $\mathcal{R}_u(Q_H(\lambda))$ -orbit of x_0 at X . By [26, Theorem 3.4], f maps Y isomorphically onto $\mathfrak{u}_H(\lambda)/\mathfrak{u}_G(\lambda)$. Therefore, since $\mathcal{R}_u(Q_H(\lambda)) \cdot x_0 \subseteq Y$, the subset Y must agree with the closed orbit $\mathcal{R}_u(Q_H(\lambda)) \cdot x_0$, and we are done. \square

The proof of the above result was communicated to us by Kraft and Kuttler (cf. [47]). Its purpose is to characterize one parameter subgroups of G among the one parameter subgroups of $SL(V)$, given a faithful representation $\varrho : G \rightarrow GL(V)$.

2.1.2. *Some specific quotient problems*

A key of understanding classification problems for vector bundles together with a section in an associated vector bundle is to study the representation defining the associated vector bundle. In our case, we have to study a certain GIT problem which we will now describe.

As before, we fix a representation $\varrho : G \rightarrow GL(V)$ on the finite dimensional k -vector space V . We look at the representation

$$R : \text{GL}_r(k) \times G \rightarrow \text{GL}(k^r \otimes V)$$

$$(g, g') \mapsto (w \otimes v \in k^r \otimes V \mapsto (g \cdot w) \otimes \varrho(g')(v)).$$

The representation R provides an action of $G \times \text{GL}_r(k)$ on

$$(V \otimes k^r)^\vee = \text{Hom}(k^r, V^\vee) \quad \text{and} \quad \mathbb{P}(\text{Hom}(k^r, V^\vee)^\vee)$$

and induces a $\text{GL}_r(k)$ -action on the categorical quotients

$$\mathbb{H} := \text{Hom}(k^r, V^\vee) // G \quad \text{and} \quad \overline{\mathbb{H}} := \mathbb{P}(\text{Hom}(k^r, V^\vee)^\vee) // G = (\mathbb{H} \setminus \{0\}) // \mathbb{G}_m(k).$$

The coordinate algebra of \mathbb{H} is $\text{Sym}^*(k^r \otimes_k V)^G$. For $s > 0$, we set

$$\mathbb{W}_s := \bigoplus_{i=1}^s \mathbb{U}_i, \quad \mathbb{U}_i := (\text{Sym}^i(k^r \otimes_k V)^G)^\vee, \quad i \geq 0.$$

If s is so large that $\bigoplus_{i=0}^s \text{Sym}^i(k^r \otimes_k V)^G$ contains a set of generators for the algebra $\text{Sym}^*(k^r \otimes_k V)^G$, then we have a $\text{GL}_r(k)$ -equivariant surjection of algebras

$$\text{Sym}^*(\mathbb{W}_s^\vee) \rightarrow \text{Sym}^*(k^r \otimes_k V)^G,$$

and, thus, a $\text{GL}_r(k)$ -equivariant embedding

$$\iota_s : \mathbb{H} \hookrightarrow \mathbb{W}_s.$$

Set $\mathbb{I} := \text{Isom}(k^r, V^\vee) / G \cong \text{GL}_r(k) / G$. This is a dense open subset of \mathbb{H} . The semistability of points $\iota_s(h)$, $h \in \mathbb{H}$, with respect to the action of the *special* linear group $\text{SL}_r(k)$ is described by the following result.

Lemma 2.1.3.

- (i) Every point $\iota_s(i)$, $i \in \mathbb{I}$, is $\text{SL}_r(k)$ -polystable.
- (ii) A point $\iota_s(h)$, $h \in \mathbb{H} \setminus \mathbb{I}$, is not $\text{SL}_r(k)$ -semistable.

Proof. (Compare Lemma 4.1.1 in [47].) Ad (i). We choose a basis for V^\vee . This provides us with the $(\text{SL}_r(k) \times G)$ -invariant function $\vartheta : \text{Hom}(k^r, V^\vee) \rightarrow k$, $f \mapsto \det(f)$, which descends to a (non-constant) function on \mathbb{H} , called again ϑ . For any $i \in \mathbb{I}$, we clearly have $\vartheta(\iota_s(i)) \neq 0$, so that $\iota_s(i)$ is $\text{SL}_r(k)$ -semistable. Furthermore, for any $f \in \text{Isom}(k^r, V^\vee)$, the $(\text{SL}_r(k) \times G)$ -orbit of f is just a level set $\vartheta^{-1}(z)$ for an appropriate $z \in \mathbb{G}_m(k)$. In particular, it is closed. The image of this orbit is the $\text{SL}_r(k)$ -orbit of $i := [f]$ in \mathbb{H} which is, therefore, closed. Since ι_s is a closed, $\text{SL}_r(k)$ -equivariant embedding, the orbit of $\iota_s(i)$ is closed, too.

Ad (ii). It is obvious from the construction that the ring of $\text{SL}_r(k)$ -invariant functions on \mathbb{H} is generated by ϑ . This makes the asserted property evident. \square

A key result is now the following:

Proposition 2.1.4. Fix a basis for V in order to obtain a faithful representation $\varrho : G \rightarrow \mathrm{GL}_r(k)$ and a $\mathrm{GL}_r(k)$ -equivariant isomorphism

$$\varphi : \mathrm{GL}_r(k)/G \rightarrow \mathrm{Isom}(k^r, V^\vee)/G.$$

Suppose that $x = \iota_s(i)$ for some $i = \varphi(g) \in \mathbb{I}$. Then, for a one parameter subgroup $\lambda : \mathbb{G}_m(k) \rightarrow \mathrm{SL}_r(k)$, the following conditions are equivalent:

- (i) $\mu_{\kappa_s}(\lambda, x) = 0$, κ_s being the representation of $\mathrm{SL}_r(k)$ on \mathbb{W}_s .
- (ii) There is a one parameter subgroup $\lambda' : \mathbb{G}_m(k) \rightarrow g \cdot G \cdot g^{-1}$ with

$$(V_\bullet(\lambda), \alpha_\bullet(\lambda)) = (V_\bullet(\lambda'), \alpha_\bullet(\lambda')).$$

Proof. We may clearly assume $g = \mathbb{E}_r$. We first show “(ii) \Rightarrow (i).” Since G is the $\mathrm{GL}_r(k)$ -stabilizer of x , we have $\mu(\lambda', x) = 0$ for any one parameter subgroup $\lambda' : \mathbb{G}_m(k) \rightarrow G$. Now, formula (2) implies the claim.

We turn to the implication “(i) \Rightarrow (ii).” By Lemma 2.1.3(i), there exists an element $g' \in \mathrm{SL}_r(k)$, such that

$$x' := \lim_{z \rightarrow \infty} \lambda(z) \cdot x = \varphi(g').$$

By Proposition 2.1.2, we may choose $g' \in \mathcal{R}_u(Q_{\mathrm{SL}_r(k)}(\lambda))$. In particular, the element g' fixes the flag $V_\bullet(\lambda)$. Since λ fixes x' , it lies in $g' \cdot G \cdot g'^{-1}$. Setting $\lambda' := g'^{-1} \cdot \lambda \cdot g'$, we obviously have $(V_\bullet(\lambda), \alpha_\bullet(\lambda)) = (V_\bullet(\lambda'), \alpha_\bullet(\lambda'))$, and λ' is a one parameter subgroup of G . \square

Next, we look at the categorical quotient

$$\overline{\mathbb{H}} = \mathrm{Proj}(\mathrm{Sym}^*(k^r \otimes_k V)^G).$$

For any positive integer d , we define

$$\mathrm{Sym}^{(d)}(k^r \otimes_k V)^G := \bigoplus_{i=0}^{\infty} \mathrm{Sym}^{id}(k^r \otimes_k V)^G.$$

Then, by the Veronese embedding,

$$\mathrm{Proj}(\mathrm{Sym}^*(k^r \otimes_k V)^G) \cong \mathrm{Proj}(\mathrm{Sym}^{(d)}(k^r \otimes_k V)^G).$$

We can choose s , such that

- (a) $\mathrm{Sym}^*(k^r \otimes_k V)^G$ is generated by elements in degree $\leq s$.
- (b) $\mathrm{Sym}^{(s^1)}(k^r \otimes_k V)^G$ is generated by elements in degree 1, i.e., by the elements in the vector space $\mathrm{Sym}^{s^1}(k^r \otimes_k V)^G$.

Set

$$\mathbb{V}_s := \mathbb{V}_s(k^r) := \bigoplus_{\substack{(d_1, \dots, d_s): \\ d_i \geq 0, \sum d_i = s!}} (\text{Sym}^{d_1}((k^r \otimes_k V)^G) \otimes \dots \otimes \text{Sym}^{d_s}(\text{Sym}^s(k^r \otimes_k V)^G)). \quad (6)$$

Obviously, there is a natural surjection $\mathbb{V}_s \rightarrow \text{Sym}^{s!}(k^r \otimes_k V)^G$ and, thus, a surjection

$$\text{Sym}^*(\mathbb{V}_s) \rightarrow \text{Sym}^{(s!)}(k^r \otimes_k V)^G.$$

This defines a closed and $\text{GL}_r(k)$ -equivariant embedding

$$i_s : \overline{\mathbb{H}} \hookrightarrow \mathbb{P}(\mathbb{V}_s).$$

We also define

$$\mathcal{O}_{\overline{\mathbb{H}}}(s!) := i_s^*(\mathcal{O}_{\mathbb{P}(\mathbb{V}_s)}(1)).$$

Note that

$$\mathcal{O}_{\overline{\mathbb{H}}}((s + 1)!) = \mathcal{O}_{\overline{\mathbb{H}}}(s!)^{\otimes(s+1)}. \quad (7)$$

Lemma 2.1.5. *Let s be a positive integer, such that (a) and (b) as above are satisfied, and $f \in \text{Hom}(k^r, V^\vee)$ a G -semistable point. Set $h := \iota_s([f])$ and $\bar{h} := i_s([f])$. Then, for any one parameter subgroup $\lambda : \mathbb{G}_m(k) \rightarrow \text{SL}_r(k)$, we have*

$$\mu_{\kappa_s}(\lambda, h) > (= / <) 0 \iff \mu_{\sigma_s}(\lambda, \bar{h}) > (= / <) 0.$$

Here, κ_s is the representation of $\text{SL}_r(k)$ on \mathbb{W}_s , and σ_s is the linearization of the $\text{SL}_r(k)$ -action on $\overline{\mathbb{H}}$ in $\mathcal{O}_{\overline{\mathbb{H}}}(s!)$. In particular, \bar{h} is $\text{SL}_r(k)$ -semistable if and only if $f \in \text{Isom}(k^r, V^\vee)$.

Proof. Note that we have the following commutative diagram:

$$\begin{array}{ccc} \text{Hom}(k^r, V^\vee) // G & \xrightarrow{\iota_s} & \mathbb{W}_s \setminus \{0\} \\ \mathbb{G}_m(k) \downarrow \text{quotient} & & \downarrow \alpha \\ \mathbb{P}(\text{Hom}(k^r, V^\vee)^\vee) // G & \xrightarrow{i_s} & \mathbb{P}(\mathbb{V}_s). \end{array}$$

The morphism α factorizes naturally over the quotient with respect to the $\mathbb{G}_m(k)$ -action on \mathbb{W}_s which is given on \mathbb{U}_i by scalar multiplication with z^{-i} , $i = 1, \dots, s$, $z \in \mathbb{G}_m(k)$. The explicit description of α is as follows: An element $(l_1, \dots, l_s) \in \mathbb{W}_s$ with

$$l_i : \text{Sym}^i(k^r \otimes_k V)^G \rightarrow k, \quad i = 1, \dots, s,$$

is mapped to the class

$$\left[\bigoplus_{\substack{d=(d_1, \dots, d_s): \\ d_i \geq 0, \sum d_i = s!}} l_d \right] : \mathbb{V}_s \rightarrow k$$

$$l_d : (u_1^1 \cdots u_1^{d_1}) \cdots (u_s^1 \cdots u_s^{d_s}) \mapsto (l_1(u_1^1) \cdots l_1(u_1^{d_1})) \cdots (l_s(u_s^1) \cdots l_s(u_s^{d_s}))$$

on $\text{Sym}^{d_1}((W \otimes \mathbb{C}^{r \vee})^G) \otimes \cdots \otimes \text{Sym}^{d_s}(\text{Sym}^s(W \otimes \mathbb{C}^{r \vee})^G)$. With this description, one easily sees

$$\mu_{\kappa_s}(\lambda, h) > (= / <) 0 \iff \mu_{\sigma_s}(\lambda, \alpha(h)) > (= / <) 0$$

for all $\lambda : \mathbb{G}_m(k) \rightarrow \text{SL}_r(k)$ and all $h \in \mathbb{W}_s \setminus \{0\}$. Together with the above diagram, this implies the claim. \square

Let us conclude this section with a formula for the μ -function. Note that \mathbb{V}_s is a $\text{GL}_r(k)$ -submodule of

$$\mathbb{S}_s := \bigoplus_{\substack{(d_1, \dots, d_s): \\ d_i \geq 0, \sum d_i = s!}} (\text{Sym}^{d_1}(k^r \otimes_k V) \otimes \cdots \otimes \text{Sym}^{d_s}(\text{Sym}^s(k^r \otimes_k V))).$$

Since $\mathbb{G}_m(k)$ is a linearly reductive group, the weight spaces inside \mathbb{V}_s with respect to any one parameter subgroup $\lambda : \mathbb{G}_m(k) \rightarrow \text{SL}_r(k)$ are the intersection of the weight spaces for λ inside \mathbb{S}_s with the subspace \mathbb{V}_s .

The module \mathbb{S}_s is a quotient module of $(W^{\otimes s!})^{\oplus N}$, $W := k^r$. Therefore, the weight spaces inside \mathbb{S}_s with respect to a one parameter subgroup $\lambda : \mathbb{G}_m(k) \rightarrow \text{SL}_r(k)$ are the projections of the corresponding weight spaces in $(W^{\otimes s!})^{\oplus N}$. The latter may be easily described. Given a one parameter subgroup $\lambda : \mathbb{G}_m(k) \rightarrow \text{SL}_r(k)$, we obtain the decomposition

$$W = \bigoplus_{i=1}^{t+1} W^i$$

into eigenspaces and the corresponding weights $\gamma_1 < \cdots < \gamma_{t+1}$. Set $I := \{1, \dots, t + 1\}^{\times s!}$, and for $(i_1, \dots, i_{s!}) \in I$ define

$$W_{i_1, \dots, i_{s!}} := W^{i_1} \otimes \cdots \otimes W^{i_{s!}}.$$

Then, all the weight spaces inside $(W^{\otimes s!})^{\oplus N}$ may be written as direct sums of some subspaces of the form $(W_{i_1, \dots, i_{s!}})^{\oplus N}$.

If we let $W_{i_1, \dots, i_{s!}}^\bullet$ be the image of $(W_{i_1, \dots, i_{s!}})^{\oplus N}$ in \mathbb{S}_s and $W_{i_1, \dots, i_{s!}}^*$ the intersection of $W_{i_1, \dots, i_{s!}}^\bullet$ with \mathbb{V}_s , $(i_1, \dots, i_{s!}) \in I$, we understand that all the weight spaces inside \mathbb{V}_s are direct sums of subspaces of the form $W_{i_1, \dots, i_{s!}}^*$, $(i_1, \dots, i_{s!}) \in I$.

This enables us to compute the weights in \mathbb{V}_s in terms of the weighted flag $(W_\bullet(\lambda), \alpha_\bullet(\lambda))$. Indeed, we define

$$W_{i_1} \cdots W_{i_{s!}}$$

as the image of $(W_{i_1} \otimes \cdots \otimes W_{i_s})^{\oplus N}$ under the projection map $(W^{\otimes s})^{\oplus N} \rightarrow \mathbb{S}_s$ and

$$W_{i_1} \star \cdots \star W_{i_s} := (W_{i_1} \cdots \cdots W_{i_s}) \cap \mathbb{V}_s, \quad (i_1, \dots, i_s) \in I.$$

Altogether, we compute for $[I] \in \mathbb{P}(\mathbb{V}_s)$ and $\lambda: \mathbb{G}_m(k) \rightarrow \mathrm{SL}_r(k)$ with weighted flag $(W_\bullet(\lambda), \alpha_\bullet(\lambda))$ as before

$$\mu_{\sigma_s}(\lambda, [I]) = -\min\{\gamma_{i_1} + \cdots + \gamma_{i_s} \mid (i_1, \dots, i_s) \in I: l_{|W_{i_1} \star \cdots \star W_{i_s}} \neq 0\}. \tag{8}$$

2.1.3. Good quotients

Suppose the algebraic group G acts on the scheme X . In the framework of his GIT, Mumford defined the notion of a *good quotient* [39]. Moreover, a *universal (uniform) categorical quotient* is a categorical quotient (Y, φ) for X with respect to the action of G , such that, for every (every flat) base change $Y' \rightarrow Y$, Y' is the categorical quotient for $Y' \times_Y X$ with respect to the induced G -action. In particular, $Y \times Z$ is the categorical quotient for $X \times Z$ with respect to the given G -action on the first factor.

Example 2.1.6. (i) Mumford’s GIT produces good, uniform (universal, if $\mathrm{Char}(k) = 0$) categorical quotients (see [39, Theorem 1.10, p. 38]).

(ii) If G and H are algebraic groups and we are given an action of $G \times H$ on the scheme X , such that the good, universal, or uniform categorical quotients $X//G$ and $(X//G)//H$ exist, then

$$X//(G \times H) = (X//G)//H.$$

This follows from playing around with the universal property of a categorical quotient. For good quotients, one might also use the argument from [41].

The following lemma is well known (see [13, Lemma 4.6] and [44, Lemma 5.1] (both in characteristic zero), [54, Theorem 2(ii)]). We recall the proof for the convenience of the reader.

Lemma 2.1.7. *Let G be a reductive linear algebraic group acting on the schemes X_1 and X_2 , and let $\psi: X_1 \rightarrow X_2$ be an affine G -equivariant morphism. Suppose that there exists a good quotient $X_2 \rightarrow X_2//G$. Then, there also exists a good quotient $X_1 \rightarrow X_1//G$, and the induced morphism $\bar{\psi}: X_1//G \rightarrow X_2//G$ is affine. Moreover, the following holds:*

1. *If ψ is finite, then $\bar{\psi}$ is also finite.*
2. *If ψ is finite and $X_2//G$ is a geometric quotient, then $X_1//G$ is also a geometric quotient.*

Proof. If $X_2//G$ is affine, then X_1 and X_2 are also affine, and the existence of $X_1//G$ is well known (see [39, Theorem A.1.1]). In general, the existence of $X_1//G$ affine over $X_2//G$ is an easy exercise on gluing affine quotients (see [44, proof of Lemma 5.1]). The only non-trivial statement in the lemma is 1. It follows from the last part of [39, Theorem A.1.1]. The point is that, if ψ is finite, then X_1 is the spectrum of the sheaf $\psi_*(\mathcal{O}_{X_1})$ of \mathcal{O}_{X_2} -algebras which is coherent as an \mathcal{O}_{X_2} -module. Hence, by the theorem cited above, $(\psi_*\mathcal{O}_{X_1})^G$ is a coherent $\mathcal{O}_{X_2//G}$ -module, which is also an $\mathcal{O}_{X_2//G}$ -algebra whose spectrum is $X_1//G$. Hence, $\bar{\psi}$ is a finite morphism. \square

2.2. Destabilizing one parameter subgroups

We have seen in Lemma 2.1.3 that a point $t_s(h) \in \mathbb{W}_s$ is not semistable for the $SL_r(k)$ -action, if $h = [f] \in \text{Hom}(k^r, V^\vee) // G$ is the image of a homomorphism $f : k^r \rightarrow V^\vee$ which is not an isomorphism. This conclusion still holds, if we replace k by a non-algebraically closed ground field K . What is, unfortunately, not automatic in positive characteristic is the fact that there exists a one parameter subgroup $\lambda : \mathbb{G}_m(K) \rightarrow SL_r(K)$ with $\mu_{\kappa_s}(\lambda, h) < 0$ in this case. This property is, however, needed in our approach to the semistable reduction theorem for semistable singular principal G -bundles. Therefore, we will now explain under which assumptions on the characteristic of the ground field we will be able to verify the existence of a one parameter subgroup λ with $\mu_{\kappa_s}(\lambda, h) < 0$.

2.2.1. Preliminaries

Let us collect two basic results. The first one is a generalization of Kempf’s results on the instability flag.

Theorem 2.2.1 (Hesselink). *Let K be a not necessarily algebraically closed field. Suppose we are given a representation $\kappa : SL_r(K) \rightarrow GL(W)$ on the finite dimensional K -vector space W , a point $h \in W$, a separable extension \tilde{K}/K , and a one parameter subgroup $\tilde{\lambda} : \mathbb{G}_m(\tilde{K}) \rightarrow SL_r(\tilde{K})$ with*

$$\mu(\tilde{\lambda}, h) < 0.$$

Then, there also exists a one parameter subgroup $\lambda : \mathbb{G}_m(K) \rightarrow SL_r(K)$ with $\mu(\lambda, h) < 0$.

Proof. This follows from Theorem 5.5 in [25]. \square

Proposition 2.2.2. *Let G be a reductive linear algebraic group and $\varrho : G \rightarrow GL(W)$ a rational representation. If W' is a G -invariant subspace which possesses a direct complement as G -module, then the categorical quotient $W' // G$ embeds into $W // G$.*

Proof. We have to show that the surjection

$$t^\# : \text{Sym}^*(W^\vee) = k[W] \rightarrow k[W'] = \text{Sym}^*(W'^\vee)$$

of locally finite G -modules also induces a surjection on the algebras of invariant elements. Our assumption says that $W = W' \oplus W''$ splits as a G -module. This shows that $\text{Sym}^*(W'^\vee)$ embeds as a G -submodule into $\text{Sym}^*(W^\vee)$, such that the restriction of $t^\#$ onto it is simply the identity. This easily yields the claim. \square

2.2.2. Digression on low height representations

The general references for the following assertions are [2,51]. Let $\varrho : G \rightarrow GL(V)$ be a representation of the reductive linear algebraic group G . Then one attaches to ϱ its *height* $\text{ht}_G(\varrho)$ ([51, p. 20], [2, Definition 1]) and its *separable index* $\psi_G(\varrho)$ [2, Definition 6].

Remark 2.2.3. By [2, Remark 10], one has the estimate

$$\psi_G(\varrho) \leq \text{rank}(G)! \cdot \text{ht}_G(\varrho)^{\text{rank}(G)}.$$

We say that ρ is a representation of low height (low separable index), if $\text{ht}_G(\rho) < p$ ($\psi_G(\rho) < p$). (Of course, p is the characteristic of the base field k .) Here is a list of properties that representations of low height and low separable index do enjoy.

Facts 2.2.4. Assume that ρ is a representation of low height.

- (i) The representation ρ is semisimple.
- (ii) The stabilizer of any point $v \in V$ is a saturated subgroup of G . (See [51, p. 22], for the definition of a saturated subgroup.)
- (iii) Suppose that ρ is also non-degenerate, i.e., the connected component of the kernel is a torus. Let $v \in V$ be a polystable point. Then, ρ is also semisimple as a G_v -module.
- (iv) If ρ is of low separable index, then the action of G on V that is induced by ρ is separable.

Proof. Ad (i). This is Theorem 6 in [51].

Ad (ii). The asserted property is evident from the definition of a saturated subgroup given in [51].

Ad (iii). This property results from (ii) and Theorems 8 and 9 in [51]. (Note that $\text{ht}_G(\rho) < p$ also implies that the Coxeter number h_G of G is at most p , by [51, p. 20].)

Ad (iv). This is Theorem 7 in [2]. \square

2.2.3. Digression on the slice theorem

The references for this section are the papers [5,9]. We assume that $\rho : G \rightarrow \text{GL}(V)$ is such that ρ^\vee is of low separable index and non-degenerate and look at the resulting action of G on $\text{Hom}(k^r, V^\vee)$, $r := \dim(V)$. (Note that the height and separable index of $\text{Hom}(k^r, V^\vee) \cong (V^\vee)^{\oplus r}$ agrees with the one of V^\vee .) The results collected in Facts 2.2.4 imply that the slice theorem of Bardsley and Richardson [5] may always be applied. Let us review the formalism. Suppose that $f \in \text{Hom}(k^r, V^\vee)$ is a polystable point. Then, its stabilizer G_f is a reduced, reductive, and saturated closed subgroup of G , by Facts 2.2.4. The tangent space $T_f(G \cdot f)$ to the orbit of f at f is a G_f -submodule of $\text{Hom}(k^r, V^\vee)$. By 2.2.4(iii), we may find a G_f -submodule N of $\text{Hom}(k^r, V^\vee)$, such that

$$\text{Hom}(k^r, V^\vee) = T_f(G \cdot f) \oplus N$$

as G_f -module. Then, Proposition 7.4 of [5] asserts that there is a function $h \in k[N]^{G_f}$, such that $S := N_h$ is an étale slice at f , i.e., we have the cartesian diagram

$$\begin{CD} G \times^{G_f} S @>\psi>> \text{Hom}(k^r, V^\vee) \\ @VVV @VVV \\ S // G_f @>\psi // G>> \text{Hom}(k^r, V^\vee) // G \end{CD}$$

in which ψ and $\psi // G$ are étale morphisms.

Next, we discuss the stratification given in [9, §2]. To this end, let \mathcal{T} be the set of conjugacy classes of stabilizers of closed points in $\text{Hom}(k^r, V^\vee)$. We say that a point $f \in \text{Hom}(k^r, V^\vee)$ is of type $\tau \in \mathcal{T}$, if the stabilizer $G_{f'}$ of a point $f' \in \overline{G \cdot f}$ with closed orbit belongs to τ . If

$\varphi \in \text{Hom}(k^r, V^\vee) // G$, then the *type* of φ is the type of a point f with closed orbit that maps to φ under the quotient morphism. For $\tau \in \mathcal{T}$, we set

$$\text{Hom}(k^r, V^\vee)_\tau := \{f \in \text{Hom}(k^r, V^\vee) \mid f \text{ is of type } \tau\}$$

and

$$(\text{Hom}(k^r, V^\vee) // G)_\tau := \{\varphi \in \text{Hom}(k^r, V^\vee) // G \mid \varphi \text{ is of type } \tau\}.$$

Similarly, we define S_τ and $(S // G_f)_\tau$, if S is an étale slice as above. Finally, for $\nu, \tau \in \mathcal{T}$, we write $\nu \preceq \tau$, if there are points f and l in $\text{Hom}(k^r, V^\vee)$ of type ν and τ , respectively, such that $G_f \supseteq G_l$. By [9, Propositions 2.4 and 2.5], we have the following result.

Proposition 2.2.5. *For any $\tau \in \mathcal{T}$, the set $(\text{Hom}(k^r, V^\vee) // G)_\tau$ is an irreducible locally closed subset of $\text{Hom}(k^r, V^\vee) // G$ with*

$$\overline{(\text{Hom}(k^r, V^\vee) // G)_\tau} = \bigcup_{\nu \preceq \tau} (\text{Hom}(k^r, V^\vee) // G)_\nu.$$

The last statement which we are going to need is the following:

Proposition 2.2.6. *Let $f \in \text{Hom}(k^r, V^\vee)$ be a point with closed orbit of type $\tau \in \mathcal{T}$.*

(i) *The morphism $\psi // G$ induces an étale morphism*

$$(S // G_f)_\tau \rightarrow (X // G)_\tau.$$

(ii) *The natural map*

$$\eta : S^{G_f} \rightarrow (S // G_f)_\tau$$

is an isomorphism.

Proof. Part (i) is Proposition 2.6(i) of [9]. Moreover, that proposition also shows that η is a bijection. Now, N^{G_f} is a G_f -submodule of N . Since $\text{Hom}(k^r, V^\vee)$ and hence N is a semisimple G -module, Proposition 2.2.2 proves that the natural map

$$N^{G_f} = N^{G_f} // G_f \rightarrow N // G_f$$

is a closed embedding. Finally, by construction of the étale slice, we have the cartesian diagram

$$\begin{array}{ccc} S = N_h & \xrightarrow{\quad} & N \\ \downarrow & & \downarrow \\ S // G_f = (N // G_f)_h & \xrightarrow{\quad} & N // G_f \end{array}$$

in which the horizontal maps are open embeddings. Using this diagram, one easily infers our claim. \square

2.2.4. Finding the destabilizing one parameter subgroup

Let Y be any (irreducible) quasi-projective variety, and E a vector bundle of rank r on Y . We define as usual

$$m : \mathcal{H} := \mathcal{H}om(E, V^\vee \otimes \mathcal{O}_Y) \rightarrow Y$$

and let

$$\bar{m} : \overline{\mathcal{H}} := \mathcal{H} // G \rightarrow Y$$

be its good whence categorical quotient. Set $\pi : \mathcal{H} \rightarrow \overline{\mathcal{H}}$ to be the quotient map.

Theorem 2.2.7. *Let*

$$\sigma : Y \rightarrow \mathcal{H}om(E, V^\vee \otimes \mathcal{O}_Y) // G$$

be a section. Then, there are a finite separable extension K of the function field $k(Y)$ of Y and a K -valued point $\tilde{\eta} \in \mathcal{H}$, such that

$$\pi(\tilde{\eta}) = \sigma(\eta), \quad \eta \text{ being the generic point of } Y.$$

Proof. By shrinking Y , we may assume that the vector bundle E is trivial, so that we define

$$\sigma' : Y \xrightarrow{\sigma} \overline{\mathcal{H}} \cong (\text{Hom}(k^r, V^\vee) // G) \times Y \rightarrow \text{Hom}(k^r, V^\vee) // G.$$

Let $\tau \in \mathcal{T}$ be minimal with respect to “ \preceq ,” such that

$$\sigma'(\eta) \in (\text{Hom}(k^r, V^\vee) // G)_\tau.$$

Then, we may choose a point $f \in \text{Hom}(k^r, V^\vee)$ with closed orbit, such that $\pi(f) \in \sigma'(Y) \cap \text{Hom}(k^r, V^\vee)_\tau$. Let S be an étale slice at f . Then, by Proposition 2.2.6(i), we have the étale map

$$\text{et} : (S // G_f)_\tau \rightarrow (\text{Hom}(k^r, V^\vee) // G)_\tau, \quad \tau \text{ the type of } f.$$

By construction, $\sigma'(Y)$ meets $\text{et}((S // G_f)_\tau)$, so that, in particular, $\sigma'(\eta)$ lies in the image of et . Hence, we find a finite separable extension K of $k(Y)$ and a K -valued point of $(S // G_f)_\tau$ that maps to $\sigma'(\eta)$ under et . We now conclude with Proposition 2.2.6(ii). \square

Corollary 2.2.8. *In the above situation, look at the embedding*

$$\iota_s : \text{Hom}(k(Y)^r, V^\vee \otimes_k k(Y)) // (G \times_{\text{Spec}(k)} \text{Spec}(k(Y))) \hookrightarrow \mathbb{W}_s \otimes_k k(Y).$$

If $\sigma(\eta)$ is not an element of $\text{Isom}(k(Y)^r, V^\vee \otimes_k k(Y)) // (G \times_{\text{Spec}(k)} \text{Spec}(k(Y)))$, then there is a one parameter subgroup which defined over $k(Y)$ and which destabilizes $(\iota_s \circ \sigma)(\eta)$.

Proof. The point $\pi(\tilde{\eta})$ gives a homomorphism

$$h : K^r \rightarrow V^\vee \otimes_k K$$

whose kernel B is non-trivial. There is a one parameter subgroup $\tilde{\lambda} : \mathbb{G}_m(K) \rightarrow \mathrm{SL}_r(K)$ with weighted flag $(0 \subsetneq B \subsetneq K^r, (1))$. It satisfies $\mu(\tilde{\lambda}, h) < \dim(B) - r < 0$. One easily sees that also

$$\mu_{\kappa_s}(\tilde{\lambda}, (t_s \circ \sigma)(\eta)) < 0.$$

The result therefore follows from Theorem 2.2.1. \square

2.2.5. An improvement for adjoint groups

Here, we assume that G is an adjoint simple group of exceptional type. Let p be such that the adjoint representation of G is of low height. This implies that p is also a good prime for G (i.e., $p \neq 2, 3$ for type E_6, E_7, F_4 , and G_2 , and $p \neq 2, 3, 5$ for type E_8).

By our previous discussion, it suffices to show that the action of G on $\mathrm{Hom}(k^r, \mathfrak{g}^\vee)$ is separable, \mathfrak{g} the Lie algebra of G . We recall

Theorem 2.2.9. *Under the assumption that the characteristic of k is a good prime for G , the Killing form on \mathfrak{g} is non-degenerate.*

Proof. This follows from the computation of the determinant of the Killing form in [50]. \square

Thus, we may split the G -module $\mathrm{End}(\mathfrak{g})$ as $\mathfrak{m} \oplus \mathfrak{g}$ and derive the Ad-equivariant map

$$\psi : G \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g}) \rightarrow \mathfrak{g}$$

with $(d\psi)_e = \mathrm{id}_{\mathfrak{g}}$. Note that ad is a left inverse to the last map.

Now, let $Y \in \mathfrak{g}$ be an element. Then, the Lie algebra of the (scheme theoretic) stabilizer G_Y of Y under the adjoint representation is

$$\mathfrak{g}_Y := \{X \in \mathfrak{g} \mid [X, Y] = 0\}.$$

On the other hand, we have the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Ad}} & \mathrm{GL}(\mathfrak{g}) \\ \psi \downarrow & & \downarrow \\ \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathrm{End}(\mathfrak{g}). \end{array} \tag{9}$$

We know

$$\mathrm{ad}(\mathrm{Ad}(g)(Y)) = \mathrm{Ad}(g) \cdot \mathrm{ad}(Y) \cdot \mathrm{Ad}(g)^{-1}, \quad g \in G, Y \in \mathfrak{g}.$$

Thus, $g \in G_Y$ if and only if

$$[\text{Ad}(g), \text{ad}(Y)] = 0.$$

By diagram (9), this is equivalent to

$$[\psi(g), Y] = 0.$$

Therefore, under the étale morphism $\psi : G \rightarrow \mathfrak{g}$, the stabilizer G_Y is the preimage of the Lie algebra \mathfrak{g}_Y . In particular, G_Y is a reduced group scheme. The same argument shows that the action of G on \mathfrak{g}^r is separable. By Theorem 2.2.9, we see

Corollary 2.2.10. *If the characteristic of the field k is a good prime for G , then the action of G on $\text{Hom}(k^r, \mathfrak{g}^\vee) \cong \text{Hom}(k^r, \mathfrak{g})$ is separable.*

Remark 2.2.11. (i) Under the hypothesis that the characteristic is good, the isogeny $G \rightarrow \text{Ad}(G)$ is separable for all simple exceptional groups (see [30, Chapter VI, Remark 1.7]). Thus, over curves we may use this result to deal with arbitrary simple groups.

(ii) For simple groups of type $A, B, C,$ and D and k of very good characteristic (i.e., $\neq 2$ for $B, C,$ and $D,$ and $n \not\equiv -1 \pmod p$ for A_n), there also exist invariant scalar products on $\text{End}(\mathfrak{g})$ which induce non-degenerate forms on \mathfrak{g} . These come from the trace form for the standard representation of the respective classical group. Since an adjoint group is the product of its simple factors, we get the result for all adjoint groups.

2.3. Some G -linearized sheaves

Assume that $\rho : G \rightarrow \text{GL}(V)$ is any representation. Let B be a scheme and \mathcal{A} a coherent \mathcal{O}_B -module. Equip B with the trivial G -action. We obtain the G -linearized sheaf $\mathcal{A} \otimes V$. It follows easily from the universal property of the symmetric algebra [21, Section (9.4.1)] that $\mathcal{S}ym^*(\mathcal{A} \otimes V)$ inherits a G -linearization. Note that the algebra $\mathcal{S}ym^*(\mathcal{A} \otimes V)$ is naturally graded and that the G -linearization preserves this grading. Let $\mathcal{S}ym^*(\mathcal{A} \otimes V)^G$ be the subalgebra of G -invariant elements in $\mathcal{S}ym^*(\mathcal{A} \otimes V)$. The G -linearization provides a π -invariant action of G on

$$\mathcal{H}om(\mathcal{A}, V^\vee \otimes \mathcal{O}_B) := \mathcal{S}pec(\mathcal{S}ym^*(\mathcal{A} \otimes V)),$$

$\pi : \mathcal{S}ym^*(\mathcal{A} \otimes V) \rightarrow B$ being the natural projection. Then, the categorical quotient of the scheme $\mathcal{S}pec(\mathcal{S}ym^*(\mathcal{A} \otimes V))$ by the G -action is given through

$$\mathcal{S}pec(\mathcal{S}ym^*(\mathcal{A} \otimes V)) // G = \mathcal{S}pec(\mathcal{S}ym^*(\mathcal{A} \otimes V)^G) \xrightarrow{\bar{\pi}} B.$$

In characteristic zero, the construction commutes with base change. In positive characteristic, we have to be more careful. Let $f : A \rightarrow B$ be a morphism of schemes. The natural isomorphism

$$f^*(\mathcal{S}ym^*(\mathcal{A} \otimes V)) \rightarrow \mathcal{S}ym^*(f^*(\mathcal{A}) \otimes V)$$

of G -linearized sheaves gives rise to the homomorphism

$$\text{bc}(f) : f^*(\mathcal{S}ym^*(\mathcal{A} \otimes V)^G) \rightarrow \mathcal{S}ym^*(f^*(\mathcal{A}) \otimes V)^G.$$

Lemma 2.3.1. *If \mathcal{A} is locally free, then $\text{bc}(f)$ is an isomorphism.*

Proof. If \mathcal{A} is locally free of rank r , then $\mathcal{S}\text{ym}^*(\mathcal{A} \otimes V)^G$ is the algebra that is associated to \mathcal{A} and the $\text{GL}_r(k)$ -module $\text{Sym}^*(k^r \otimes V)^G$, and the assertion is clear. \square

We also note the following property.

Lemma 2.3.2. *Let $\psi : \mathcal{A}' \rightarrow \mathcal{A}$ be a surjective map of \mathcal{O}_B -modules. Assume that \mathcal{A}' and \mathcal{A} are locally free. Then, the induced homomorphism*

$$\mathcal{S}\text{ym}^*(\mathcal{A}' \otimes V)^G \rightarrow \mathcal{S}\text{ym}^*(\mathcal{A} \otimes V)^G$$

of \mathcal{O}_B -algebras is surjective as well.

Proof. This follows like Lemma 2.3.1, taking into account Proposition 2.2.2. \square

2.4. Polynomial representations

A representation $\kappa : \text{GL}_r(k) \rightarrow \text{GL}(U)$ is called a *polynomial representation*, if it extends to a (multiplicative) map $\bar{\kappa} : M_r(k) \rightarrow \text{End}(U)$. We say that κ is *homogeneous of degree $u \in \mathbb{Z}$* , if

$$\kappa(z \cdot \mathbb{E}_r) = z^u \cdot \text{id}_U, \quad \forall z \in \mathbb{G}_m(k).$$

Let $P(r, u)$ be the abelian category of homogeneous polynomial representations of $\text{GL}_r(k)$ of degree u . It comes with the duality functor

$$\begin{aligned} * : P(r, u) &\rightarrow P(r, u) \\ \kappa &\mapsto (\kappa \circ \text{id}_{\text{GL}_r(k)}^\vee)^\vee. \end{aligned}$$

Here, \vee stands for the corresponding dual representation. An example for a representation in $P(r, u)$ is the *uth divided power* $(\text{Sym}^u(\text{id}_{\text{GL}_r(k)}))^*$, i.e., the representation of $\text{GL}_r(k)$ on

$$D^u(W) := (\text{Sym}^u(W^\vee))^\vee, \quad W := k^r.$$

More generally, we look, for $u, v > 0$, at the $\text{GL}_r(k)$ -module

$$\mathbb{D}^{u,v}(W) := \bigoplus_{\substack{(u_1, \dots, u_v): \\ u_i \geq 0, \sum_{i=1}^v u_i = u}} (D^{u_1}(W) \otimes \dots \otimes D^{u_v}(W)). \tag{10}$$

Lemma 2.4.1. *Let $\kappa : \text{GL}_r(k) \rightarrow \text{GL}(U)$ be a homogeneous polynomial representation of degree u . Then, there exists an integer $v > 0$, such that U is a quotient of the $\text{GL}(U)$ -module $\mathbb{D}^{u,v}(W)$.*

Proof. For the proof, we refer to [24]. \square

2.5. *Extension of the structure group*

We will need Theorem 8.4 of [33]:

Theorem 2.5.1. *Let G be a connected reductive group and $\varrho: G \rightarrow \mathrm{GL}(V)$ a representation which maps the radical of G to the center of $\mathrm{GL}(V)$. Then, there are the following cases:*

- (i) *Assume either $\mathrm{Char}(k) = 0$ or $\mu_{\max}(\Omega_X) \leq 0$. If (U, \mathcal{P}) is a Ramanathan semistable rational principal G -bundle on X and (U, \mathcal{E}) is the rational vector bundle with fiber V associated to (U, \mathcal{P}) , then (U, \mathcal{E}) is (strongly) slope semistable.*
- (ii) *If $\mathrm{Char}(k) = p > 0$ and $\mu_{\max}(\Omega_X) > 0$, there is a constant $C(\varrho)$, depending only on ϱ , such that for any Ramanathan semistable rational principal G -bundle (U, \mathcal{P}) on X with associated rational vector bundle (U, \mathcal{E}) , one finds*

$$0 \leq \mu_{\max}(\mathcal{E}) - \mu_{\min}(\mathcal{E}) \leq L_{\max}(\mathcal{E}) - L_{\min}(\mathcal{E}) \leq C(\varrho) \cdot \frac{[L_{\max}(\Omega_X)]_+}{p}.$$

Corollary 2.5.2. *Let G be a connected reductive group and $\varrho: G \rightarrow \mathrm{GL}(V)$ a representation which maps the radical of G to the center of $\mathrm{GL}(V)$. There is a constant $D(\varrho)$ which depends only on ϱ , such that*

$$\mu(\mathcal{E}) - D(\varrho) \leq \mu_{\min}(\mathcal{E})$$

for any semistable rational principal G -bundle (U, \mathcal{P}) on X with associated rational vector bundle (U, \mathcal{E}) .

Proof. One has

$$\mu(\mathcal{E}) \leq \mu_{\max}(\mathcal{E}) = (\mu_{\max}(\mathcal{E}) - \mu_{\min}(\mathcal{E})) + \mu_{\min}(\mathcal{E}).$$

The assertion follows directly from this and the previous theorem. \square

2.6. *An extension property*

Proposition 2.6.1. *Let S be a scheme and \mathcal{F}_S a vector bundle on $S \times X$. Let $Z \subset S \times X$ be a closed subset, such that $\mathrm{codim}_X(Z \cap (\{s\} \times X)) \geq 2$ for every point $s \in S$. Denote by $\iota: U := (S \times X) \setminus Z \subseteq S \times X$ the inclusion. Then, the natural map*

$$\mathcal{F}_S \rightarrow \iota_*(\mathcal{F}_{S|U})$$

is an isomorphism.

Proof. (After Maruyama [36, p. 112].) Since this is a local question, we may clearly assume $\mathcal{F}_S = \mathcal{O}_{S \times X}$. Note that Z is “stable under specialization” in the sense of [19, (5.9.1), p. 109]. By [19, Theorem (5.10.5), p. 115], one has to show that $\inf_{x \in Z} \mathrm{depth}(\mathcal{O}_{S \times X, x}) \geq 2$. Since X is smooth, the morphism $\pi_S: S \times X \rightarrow X$ is smooth. Thus, by [20, Proposition (17.5.8), p. 70],

$$\dim(\mathcal{O}_{S \times X, x}) - \text{depth}(\mathcal{O}_{S \times X, x}) = \dim(\mathcal{O}_{S, s}) - \text{depth}(\mathcal{O}_{S, s})$$

for every point $x \in S \times X$ and $s := \pi_S(x)$. This implies

$$\text{depth}(\mathcal{O}_{S \times X, x}) \geq \dim(\mathcal{O}_{S \times X, x}) - \dim(\mathcal{O}_{S, s}) = \dim(\mathcal{O}_{\pi_S^{-1}(s), x}). \tag{11}$$

Since for any point $x \in \pi_S^{-1}(s)$, one has $\dim(\mathcal{O}_{\pi_S^{-1}(s), x}) = \text{codim}_{\pi_S^{-1}(s)}(\overline{\{x\}})$, we derive the desired estimate $\text{depth}(\mathcal{O}_{S \times X, x}) \geq 2$ for every point $x \in Z$ from the fact that $\text{codim}_{\pi_S^{-1}(s)}(Z \cap \pi_S^{-1}(s)) \geq 2$ and (11). \square

Corollary 2.6.2. *Suppose S is a scheme, \mathcal{E}_S is a coherent $\mathcal{O}_{S \times X}$ -module, and \mathcal{F}_S is a locally free sheaf on $S \times X$. Let $U \subseteq S \times X$ be an open subset whose complement Z satisfies $\text{codim}_X(Z \cap (\{s\} \times X)) \geq 2$ for every point $s \in S$. Then, for any homomorphism $\tilde{\varphi}_S : \mathcal{E}_S|_U \rightarrow \mathcal{F}_S|_U$, there is a unique extension*

$$\varphi_S : \mathcal{E}_S \rightarrow \mathcal{F}_S$$

to $S \times X$. In particular, for a base change morphism $f : T \rightarrow S$, we have

$$\varphi_T = (f \times \text{id}_X)^*(\varphi_S).$$

Here, φ_T is the extension of $(f \times \text{id}_X)_{|(f \times \text{id}_X)^{-1}(U)}^*(\tilde{\varphi}_S)$.

Proof. An extension is given by

$$\varphi_S : \mathcal{E}_S \rightarrow \iota_*(\mathcal{E}_S|_U) \xrightarrow{\iota_*(\tilde{\varphi}_S)} \iota_*(\mathcal{F}_S) \stackrel{\text{Proposition 2.6.1}}{=} \mathcal{F}_S.$$

Since \mathcal{E}_S can be written as the quotient of a locally free sheaf, the uniqueness also follows from Proposition 2.6.1. The final statement is clearly a consequence of the uniqueness property. \square

3. Fundamental results on semistable singular principal bundles

After reviewing several elementary properties, we show that a singular principal G -bundle (\mathcal{A}, τ) is slope semistable in the sense which has been defined in the introduction if and only if the associated rational principal G -bundle $(U, \mathcal{P}(\mathcal{A}, \tau))$ is semistable in the sense of Ramanan.

3.1. The basic formalism of singular principal bundles

Since G is a semisimple group, the basic formalism of pseudo G -bundles in positive characteristic is exactly the same as in characteristic zero. Therefore, we may refer the reader to [45, Section 3.1], for more details (be aware that in this reference the term ‘‘singular principal G -bundle’’ is used for our ‘‘pseudo G -bundle’’). We fix a faithful representation $\varrho : G \rightarrow \text{GL}(V)$. Then, a pseudo G -bundle (\mathcal{A}, τ) consists of a torsion free coherent \mathcal{O}_X -module \mathcal{A} of rank $\dim_k(V)$ with trivial determinant and a homomorphism $\tau : \text{Sym}^*(\mathcal{A} \otimes V)^G \rightarrow \mathcal{O}_X$ which is non-trivial in the sense that it is not just the projection onto the degree zero component. Let

$U \subseteq X$ be the maximal open subset where \mathcal{A} is locally free. Since $\varrho(G) \subseteq \text{SL}(V)$, we have the open immersion

$$\mathcal{I} \text{som}(\mathcal{A}|_U, V^\vee \otimes \mathcal{O}_U) / G \subset \mathcal{H} \text{om}(\mathcal{A}, V^\vee \otimes \mathcal{O}_X) // G.$$

Recall the following alternatives.

Lemma 3.1.1. *Let (\mathcal{A}, τ) be a pseudo G -bundle and*

$$\sigma : X \rightarrow \mathcal{H} \text{om}(\mathcal{A}, V^\vee \otimes \mathcal{O}_X) // G$$

the section defined by τ . Then, either

$$\sigma(U) \subset \mathcal{I} \text{som}(\mathcal{A}|_U, V^\vee \otimes \mathcal{O}_U) / G$$

or

$$\sigma(U) \subset (\mathcal{H} \text{om}(\mathcal{A}|_U, V^\vee \otimes \mathcal{O}_U) // G) \setminus (\mathcal{I} \text{som}(\mathcal{A}|_U, V^\vee \otimes \mathcal{O}_U) / G).$$

Proof. See [45, Corollary 3.4]. \square

In the former case, we call (\mathcal{A}, τ) a *singular principal G -bundle*. We may form the base change diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{A}, \tau) & \longrightarrow & \mathcal{H} \text{om}(\mathcal{A}|_U, V^\vee \otimes \mathcal{O}_U) \\ \downarrow & & \downarrow \\ U & \xrightarrow{\sigma|_U} & \mathcal{H} \text{om}(\mathcal{A}|_U, V^\vee \otimes \mathcal{O}_U) // G. \end{array}$$

If (\mathcal{A}, τ) is a singular principal G -bundle, then $\mathcal{P}(\mathcal{A}, \tau)$ is a principal G -bundle over U in the usual sense, i.e., a rational principal G -bundle on X in the sense of Ramanathan (see Section 3.2).

A *family of pseudo G -bundles parameterized by the scheme S* is a pair (\mathcal{A}_S, τ_S) which consists of an S -flat family \mathcal{A}_S of torsion free sheaves on $S \times X$ and a homomorphism $\tau_S : \mathcal{S} \text{ym}^*(\mathcal{A}_S \otimes V)^G \rightarrow \mathcal{O}_{S \times X}$. We say that the family $(\mathcal{A}_S^1, \tau_S^1)$ is *isomorphic* to the family $(\mathcal{A}_S^2, \tau_S^2)$, if there is an isomorphism $\psi_S : \mathcal{A}_S^1 \rightarrow \mathcal{A}_S^2$, such that the induced isomorphism $\mathcal{S} \text{ym}^*(\mathcal{A}_S^1 \otimes V)^G \rightarrow \mathcal{S} \text{ym}^*(\mathcal{A}_S^2 \otimes V)^G$ carries τ_S^2 into τ_S^1 .

We also need a more general looking concept. A *pre-family of pseudo G -bundles parameterized by the scheme S* is a pair (\mathcal{A}_S, τ_S) which is composed of an S -flat family \mathcal{A}_S of torsion free sheaves on $S \times X$ and a homomorphism $\tau_S : \mathcal{S} \text{ym}^*(\mathcal{A}_{S|U} \otimes V)^G \rightarrow \mathcal{O}_U$. Here, $U \subseteq S \times X$ is the maximal open subset where \mathcal{A}_S is locally free. The pre-family $(\mathcal{A}_S^1, \tau_S^1)$ is *isomorphic* to the pre-family $(\mathcal{A}_S^2, \tau_S^2)$, if there is an isomorphism $\psi_S : \mathcal{A}_S^1 \rightarrow \mathcal{A}_S^2$, such that the induced isomorphism $\mathcal{S} \text{ym}^*(\mathcal{A}_{S|U}^1 \otimes V)^G \rightarrow \mathcal{S} \text{ym}^*(\mathcal{A}_{S|U}^2 \otimes V)^G$ transforms τ_S^2 into τ_S^1 .

Lemma 3.1.2. *Let S be a scheme of finite type over k . Then, the assignment*

$$\left\{ \begin{array}{l} \text{Isomorphism classes of families} \\ \text{of pseudo } G\text{-bundles parameterized by } S \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{Isomorphism classes of pre-families} \\ \text{of pseudo } G\text{-bundles parameterized by } S \end{array} \right\}$$

$$(\mathcal{A}_S, \tau_S) \mapsto (\mathcal{A}_S, \tau_{S|U})$$

is a bijection.

Proof. If (\mathcal{A}_S, τ'_S) is a pre-family, denote by U the maximal open subset where \mathcal{A}_S is locally free and by $\iota: U \rightarrow S \times X$ the inclusion. Set

$$\tau_S: \mathcal{S}ym^*(\mathcal{A}_S \otimes V)^G \rightarrow \iota_* (\mathcal{S}ym^*(\mathcal{A}_{S|U} \otimes V)^G) \xrightarrow{\iota_*(\tau'_S)} \iota_*(\mathcal{O}_U) \stackrel{2.6.1}{=} \mathcal{O}_{S \times X}.$$

Then, (\mathcal{A}_S, τ_S) is a real family of pseudo G -bundles which maps to (\mathcal{A}_S, τ'_S) under the above map.

It remains to verify injectivity. Let \mathcal{A}_S be a flat family of torsion free coherent \mathcal{O}_X -modules on $S \times X$. Over every affine open subset $W \subset S \times X$, the algebra $\mathcal{S}ym^*(\mathcal{A}_{S|W} \otimes V)^G$ is finitely generated. Since $S \times X$ is according to our assumption quasi-compact, we may find an $s > 0$, such that $\mathcal{S}ym^*(\mathcal{A}_S \otimes V)^G$ is generated by the coherent $\mathcal{O}_{S \times X}$ -module

$$\mathbb{W}_s(\mathcal{A}) := \bigoplus_{i=1}^s \mathcal{S}ym^i(\mathcal{A}_S \otimes V)^G.$$

Since τ_S is determined by $\tau'_S := \tau_{S|\mathbb{W}_s(\mathcal{A})}$, it remains to show that τ'_S is determined by its restriction to U . But this is an immediate consequence of Corollary 2.6.2. \square

By Lemma 2.3.1, we have a pullback operation on pre-families of pseudo G -bundles with respect to base change morphisms $T \rightarrow S$. Using the above arguments, we also obtain a pullback operation for families of pseudo G -bundles.

3.2. Semistable rational principal G -bundles

We now review the formalism introduced by Ramanathan and compare it with our setup. A *rational principal G -bundle on X* is a pair (U, \mathcal{P}) which consists of a big open subset $U \subseteq X$ and a principal G -bundle \mathcal{P} on U . Such a rational principal G -bundle is said to be *(semi)stable*, if for every open subset $U' \subseteq U$ which is big in X , every parabolic subgroup P of G , every reduction $\beta: U' \rightarrow \mathcal{P}|_{U'}/P$ of the structure group of \mathcal{P} to P over U' , and every antidominant character (see below) χ on P , we have

$$\deg(\mathcal{L}(\beta, \chi)) (\geq) 0.$$

Note that the antidominant character χ and the principal P -bundle $\mathcal{P}|_{U'} \rightarrow \mathcal{P}|_{U'}/P$ define a line bundle on $\mathcal{P}|_{U'}/P$. Its pullback to U' via β is the line bundle $\mathcal{L}(\beta, \chi)$. Since U' is big in X , it makes sense to speak about the degree of $\mathcal{L}(\beta, \chi)$.

We fix a pair (B, T) which consists of a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. If P and P' are conjugate in G , a reduction β of a principal G -bundle to P may equally be interpreted as a reduction to P' . Thus, it suffices to consider parabolic subgroups of the type $P_G(\lambda)$, $\lambda : \mathbb{G}_m(k) \rightarrow T$ a one parameter subgroup, which contain B . Here, we use the convention (compare Remark 2.1.1)

$$P_G(\lambda) := Q_G(-\lambda) = \left\{ g \in G \mid \lim_{z \rightarrow 0} \lambda(z) \cdot g \cdot \lambda(z)^{-1} \text{ exists in } G \right\}.$$

Let $X_\star(T)$ and $X^\star(T)$ be the free \mathbb{Z} -modules of one parameter subgroups and characters of T , respectively. We have the canonical pairing $\langle \cdot, \cdot \rangle : X_\star(T) \times X^\star(T) \rightarrow \mathbb{Z}$. Set $X_{\star, \mathbb{K}}(T) := X_\star(T) \otimes_{\mathbb{Z}} \mathbb{K}$ and $X_{\mathbb{K}}^\star(T) := X^\star(T) \otimes_{\mathbb{Z}} \mathbb{K}$, and let $\langle \cdot, \cdot \rangle_{\mathbb{K}} : X_{\star, \mathbb{K}}(T) \times X_{\mathbb{K}}^\star(T) \rightarrow \mathbb{K}$ be the \mathbb{K} -bilinear extension of $\langle \cdot, \cdot \rangle$, $\mathbb{K} = \mathbb{Q}, \mathbb{R}$. Finally, suppose $(\cdot, \cdot)^\star : X_{\mathbb{R}}^\star(T) \times X_{\mathbb{R}}^\star(T) \rightarrow \mathbb{R}$ is a scalar product which is invariant under the Weyl group $W(T) = \mathcal{N}(T)/T$. This also yields the product $(\cdot, \cdot)_\star : X_{\star, \mathbb{R}}(T) \times X_{\star, \mathbb{R}}(T) \rightarrow \mathbb{R}$. We assume that $(\cdot, \cdot)^\star$ is defined over \mathbb{Q} .

The datum (B, T) defines the set of positive roots R and the set R^\vee of coroots (see [56]). Let $\mathcal{C} \subset X_{\star, \mathbb{R}}(T)$ be the cone spanned by the elements of R^\vee and $\mathcal{D} \subset X_{\mathbb{R}}^\star(T)$ the dual cone of \mathcal{C} with respect to $\langle \cdot, \cdot \rangle_{\mathbb{R}}$. Equivalently, the cone \mathcal{D} may be characterized as being the dual cone of the cone spanned by the elements in R with respect to $(\cdot, \cdot)^\star$. Indeed, one has $\langle \cdot, \alpha^\vee \rangle_{\mathbb{R}} = 2(\cdot, \alpha)^\star / (\alpha, \alpha)^\star$, $\alpha \in R$. Now, a character $\chi \in X^\star(T)$ is called *dominant*, if it lies in \mathcal{D} , and *antidominant*, if $-\chi$ lies in \mathcal{D} . A character χ of a parabolic subgroup containing B is called *(anti)dominant*, if its restriction to T is (anti)dominant.

In the definition of semistability, we may clearly assume that $(\chi, \alpha)^\star > 0$ for every $\alpha \in R$ with $\langle \lambda, \alpha \rangle > 0$, if $P = P_G(\lambda)$. Otherwise, we may choose λ' , such that $(\chi, \alpha)^\star > 0$ if and only if $\langle \lambda', \alpha \rangle > 0$. Then, $P_G(\lambda')$ is a parabolic subgroup which contains $P_G(\lambda)$, χ is induced by a character χ' on $P_G(\lambda')$, the reduction β defines the reduction $\beta' : \mathcal{P}|_{U'} / P_G(\lambda) \rightarrow \mathcal{P}|_{U'} / P_G(\lambda')$, and $\mathcal{L}(\beta, \chi) = \mathcal{L}(\beta', \chi')$. Every one parameter subgroup λ of T defines a character χ_λ of T , such that $\langle \lambda, \chi \rangle = (\chi_\lambda, \chi)^\star$ for all $\chi \in X^\star(T)$. Finally, observe that the cone \mathcal{C}' of one parameter subgroups λ of T , such that $B \subseteq P_G(\lambda)$, is dual to the cone spanned by the roots. Thus,

$$\forall \lambda \in X_\star(T): \quad B \subseteq P_G(\lambda) \iff \chi_\lambda \in \mathcal{D}.$$

If one of those conditions is verified, then $(P_G(\lambda), \chi_\lambda)$ consists of a parabolic subgroup containing B and a dominant character on it. Similarly, if $Q_G(\lambda)$ contains B , then χ_λ is an antidominant character on $Q_G(\lambda)$. Our discussion shows:

Lemma 3.2.1. *A rational principal G -bundle (U, \mathcal{P}) is (semi)stable if and only if for every open subset $U' \subseteq U$ which is big in X , every one parameter subgroup $\lambda \in \mathcal{C}'$, and every reduction $\beta : U' \rightarrow \mathcal{P}|_{U'} / Q_G(\lambda)$ of the structure group of \mathcal{P} to $Q_G(\lambda)$ over U' , we have*

$$\text{deg}(\mathcal{L}(\beta, \chi_\lambda)) (\geq) 0.$$

For any one parameter subgroup λ of G , we may find a pair (B', T') consisting of a Borel subgroup B' of G and a maximal torus $T' \subset B'$, such that $\lambda \in X_\star(T')$ and $B' \subseteq Q_G(\lambda)$. Then, there exists an element $g \in G$, such that $(g \cdot B \cdot g^{-1}, g \cdot T \cdot g^{-1}) = (B', T')$, and we obtain

$$\begin{aligned}
 (\dots)^{\star'} : X_{\mathbb{R}}^{\star}(T') \times X_{\mathbb{R}}^{\star}(T') &\rightarrow \mathbb{R} \\
 (\chi, \chi') &\mapsto (\chi(g^{-1} \dots g), \chi'(g^{-1} \dots g))^{\star}.
 \end{aligned}$$

Since $(\dots)^{\star}$ is invariant under $W(T)$, the product $(\dots)^{\star'}$ does not depend on g . In particular, we obtain a character χ_{λ} on $Q_G(\lambda)$. This character does not depend on (B', T') (see [47, (2.31)]). We conclude

Lemma 3.2.2. *A rational principal G -bundle (U, \mathcal{P}) is (semi)stable if and only if for every open subset $U' \subseteq U$ which is big in X , every one parameter subgroup $\lambda : \mathbb{G}_m(k) \rightarrow G$, and every reduction $\beta : U' \rightarrow \mathcal{P}|_{U'}/Q_G(\lambda)$ of the structure group of \mathcal{P} to $Q_G(\lambda)$ over U' , we have*

$$\text{deg}(\mathcal{L}(\beta, \chi_{\lambda})) (\geq) 0.$$

Suppose that $\varrho : G \rightarrow \text{GL}(V)$ is a faithful representation. We may assume that T maps to the maximal torus $\tilde{T} \subset \text{GL}(V)$, consisting of the diagonal matrices. The character group $X^{\star}(\tilde{T})$ is freely generated by the characters $e_i : \text{diag}(\lambda_1, \dots, \lambda_n) \mapsto \lambda_i, i = 1, \dots, n$. We define

$$\begin{aligned}
 (\dots)_{\tilde{T}}^{\star} : X_{\mathbb{R}}^{\star}(\tilde{T}) \times X_{\mathbb{R}}^{\star}(\tilde{T}) &\rightarrow \mathbb{R} \\
 \left(\sum_{i=1}^n x_i \cdot e_i, \sum_{i=1}^n y_i \cdot e_i \right) &\mapsto \sum_{i=1}^n x_i \cdot y_i.
 \end{aligned}$$

The scalar product $(\dots)_{\tilde{T}}^{\star}$ is clearly defined over \mathbb{Q} and invariant under the Weyl group $W(\tilde{T})$. The product $(\dots)_{\tilde{T}}^{\star}$ therefore restricts to a scalar product $(\dots)^{\star}$ on $X_{\mathbb{R}}^{\star}(T)$ with the properties we have asked for. We find a nice formula for $\text{deg}(\mathcal{L}(\beta, \chi_{\lambda}))$. Indeed, if (U, \mathcal{P}) is a rational principal G -bundle, and if \mathcal{E} is the vector bundle on U associated to \mathcal{P} by means of ϱ , then we have, for every one parameter subgroup $\lambda : \mathbb{G}_m(k) \rightarrow G$, the embedding

$$\iota : \mathcal{P}/Q_G(\lambda) \hookrightarrow \mathcal{I} \text{ som}(V \otimes \mathcal{O}_X, \mathcal{E})/Q_{\text{GL}(V)}(\lambda).$$

As usual, we obtain a weighted filtration $(V_{\bullet}(\lambda), \alpha_{\bullet}(\lambda))$ of V , and, for every reduction $\beta : U' \rightarrow \mathcal{P}|_{U'}/Q_G(\lambda)$ over a big open subset $U' \subseteq U$, the reduction $\iota \circ \beta$ corresponds to a filtration

$$\mathcal{E}_{\bullet}(\beta) : 0 \subsetneq \mathcal{E}_1 \subsetneq \dots \subsetneq \mathcal{E}_t \subsetneq \mathcal{E}|_{U'}$$

by subbundles with $\text{rk}(\mathcal{E}_i) = \dim_k(V_i), i = 1, \dots, t$. With the weighted filtration $(\mathcal{E}_{\bullet}(\beta), \alpha_{\bullet}(\lambda))$ of $\mathcal{E}|_{U'}$, we find

$$\text{deg}(\mathcal{L}(\beta, \chi_{\lambda})) = L(\mathcal{E}_{\bullet}(\beta), \alpha_{\bullet}(\lambda)) = \sum_{i=1}^t \alpha_i \cdot (\text{rk}(\mathcal{E}_i) \cdot \text{deg}(\mathcal{E}) - \text{rk}(\mathcal{E}) \cdot \text{deg}(\mathcal{E}_i)).$$

To see this, observe that the character χ_{λ} is, by construction, the restriction of a character χ of \tilde{T} , so that $\mathcal{L}(\beta, \chi_{\lambda}) = \mathcal{L}(\iota \circ \beta, \chi)$. The degree of the latter line bundle is computed in Example 2.15 of [48] and gives the result stated above. Thus, we conclude

Lemma 3.2.3. *A rational principal G -bundle (U, \mathcal{P}) is (semi)stable if and only if, for every open subset $U' \subseteq U$ which is big in X , every one parameter subgroup $\lambda: \mathbb{G}_m(k) \rightarrow G$, and every reduction $\beta: U' \rightarrow \mathcal{P}|_{U'}/Q_G(\lambda)$ of the structure group of \mathcal{P} to $Q_G(\lambda)$ over U' , we have*

$$L(\mathcal{E}_\bullet(\beta), \alpha_\bullet(\lambda)) (\geq) 0.$$

Remark 3.2.4. If (U, \mathcal{P}) is given as a singular principal G -bundle (\mathcal{A}, τ) , then, in the notation of the introduction, we have

$$\mathcal{A}|_U = \mathcal{E}^\vee, \quad \mathcal{A}|_U = \ker(\mathcal{E}^\vee \rightarrow \mathcal{E}_{t+1-i}^\vee) \quad \text{and} \quad \alpha_{i;\beta} = \alpha_{t+1-i}, \quad i = 1, \dots, t.$$

Then, one readily verifies

$$L(\mathcal{A}, \tau; \beta) = L(\mathcal{E}_\bullet(\beta), \alpha_\bullet(\lambda)).$$

This proves that the definition of slope semistability (1) given in the introduction is the original definition of Ramanathan. We have arrived at our notion of semistability, by replacing degrees by Hilbert polynomials. Thus, our semistability concept is a ‘‘Gieseker version’’ of Ramanathan semistability.

4. Dispo sheaves

In the papers [45,47], the theory of decorated sheaves was used to construct projective moduli spaces for singular principal G -bundles in characteristic zero. Due to the more difficult representation theory of general linear groups in positive characteristic, this approach is not available in (low) positive characteristic. Nevertheless, one may still associate to any singular principal bundle a more specific object than a decorated sheaf, namely a so-called ‘‘dispo sheaf.’’ The moduli theory of these dispo sheaves may be developed along the lines of the theory of decorated sheaves in [16,46], making several non-trivial modifications.

4.1. The basic definitions

For this section, we fix the representation $\varrho: G \rightarrow \text{SL}(V) \subseteq \text{GL}(V)$ and a positive integer s as in Section 2.1.2.

Suppose that \mathcal{A} is a coherent \mathcal{O}_X -module of rank $r := \dim_k(V)$. Then, the *sheaf of invariants in the symmetric powers of \mathcal{A}* is defined as

$$\mathbb{V}_s(\mathcal{A}) := \bigoplus_{\substack{(d_1, \dots, d_s): \\ d_i \geq 0, \sum d_i = s!}} (\mathcal{S}ym^{d_1}((\mathcal{A} \otimes_k V)^G) \otimes \dots \otimes \mathcal{S}ym^{d_s}(\mathcal{S}ym^s(\mathcal{A} \otimes_k V)^G)).$$

A *dispo⁴ sheaf (of type (ϱ, s))* is a pair (\mathcal{A}, φ) which consists of a torsion free sheaf \mathcal{A} of rank r with $\det(\mathcal{A}) \cong \mathcal{O}_X$ on X and a non-trivial homomorphism

$$\varphi: \mathbb{V}_s(\mathcal{A}) \rightarrow \mathcal{O}_X.$$

⁴ Decorated with invariants in symmetric powers.

Two dispo sheaves $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_2)$ are said to be *isomorphic*, if there exists an isomorphism $\psi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$, such that, with the induced isomorphism $\mathbb{V}_s(\psi) : \mathbb{V}_s(\mathcal{A}_1) \rightarrow \mathbb{V}_s(\mathcal{A}_2)$, one obtains

$$\varphi_1 = \varphi_2 \circ \mathbb{V}_s(\psi).$$

A *weighted filtration* $(\mathcal{A}_\bullet, \alpha_\bullet)$ of the torsion free sheaf \mathcal{A} consists of a filtration

$$0 \subsetneq \mathcal{A}_1 \subsetneq \dots \subsetneq \mathcal{A}_t \subsetneq \mathcal{A}_{t+1} = \mathcal{A}$$

of \mathcal{A} by saturated subsheaves and a tuple $\alpha_\bullet = (\alpha_1, \dots, \alpha_t)$ of positive rational numbers. Given such a weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$, we introduce the quantities

$$M(\mathcal{A}_\bullet, \alpha_\bullet) := \sum_{j=1}^t \alpha_j \cdot (\text{rk}(\mathcal{A}_j) \cdot P(\mathcal{A}) - \text{rk}(\mathcal{A}) \cdot P(\mathcal{A}_j)),$$

$$L(\mathcal{A}_\bullet, \alpha_\bullet) := \sum_{j=1}^t \alpha_j \cdot (\text{rk}(\mathcal{A}_j) \cdot \text{deg}(\mathcal{A}) - \text{rk}(\mathcal{A}) \cdot \text{deg}(\mathcal{A}_j)).$$

Next, let (\mathcal{A}, φ) be a dispo sheaf and $(\mathcal{A}_\bullet, \alpha_\bullet)$ a weighted filtration of \mathcal{A} . Fix a flag

$$W_\bullet: \quad 0 \subsetneq W_1 \subsetneq \dots \subsetneq W_t \subsetneq W := k^r \quad \text{with } \dim_k(W_i) = \text{rk}(\mathcal{A}_i), \quad i = 1, \dots, t.$$

We may find a small open subset U , such that

- $\varphi|_U$ is a surjection onto \mathcal{O}_U ;
- there is a trivialization $\psi : \mathcal{A}|_U \rightarrow W \otimes \mathcal{O}_U$ with $\psi(\mathcal{A}_i|_U) = W_i \otimes \mathcal{O}_U, i = 1, \dots, t.$

In presence of the trivialization ψ , the homomorphism $\varphi|_U$ provides us with the morphism

$$\beta : U \rightarrow \mathbb{P}(\mathbb{V}_s(\mathcal{A})) \xrightarrow{\mathbb{V}_s(\psi)} \mathbb{P}(\mathbb{V}_s) \times U \rightarrow \mathbb{P}(\mathbb{V}_s).$$

(Consult Section 2.1 for the notation “ \mathbb{V}_s .”) Finally, let $\lambda : \mathbb{G}_m(k) \rightarrow \text{SL}(W)$ be a one parameter subgroup with $(W_\bullet, \alpha_\bullet)$ as its weighted flag. With these choices made, we set

$$\mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) := \max\{\mu_{\sigma_s}(\lambda, \beta(x)) \mid x \in U\}.$$

(The linearization σ_s has been introduced in Lemma 2.1.5.) As in [46, p. 176], one checks that the quantity $\mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi)$ depends only on the data $(\mathcal{A}_\bullet, \alpha_\bullet)$ and φ .

Remark 4.1.1. (i) Let us outline another, intrinsic definition of the number $\mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi)$. First, observe that $\mathbb{V}_s(\mathcal{A})$ is a submodule of

$$\mathbb{S}_s(\mathcal{A}) := \bigoplus_{\substack{(d_1, \dots, d_s): \\ d_i \geq 0, \sum d_i = s!}} (\mathcal{S}ym^{d_1}(\mathcal{A} \otimes_k V) \otimes \dots \otimes \mathcal{S}ym^{d_s}(\mathcal{S}ym^s(\mathcal{A} \otimes_k V)))$$

and that $\mathbb{S}_s(\mathcal{A})$ is a quotient of $(\mathcal{A}^{\otimes s!})^{\oplus N}$. Let $(\mathcal{A}_\bullet, \alpha_\bullet)$ be a weighted filtration of \mathcal{A} . Set $I := \{1, \dots, t + 1\}^{\times s!}$ and $\mathcal{A}_{t+1} := \mathcal{A}$. For $(i_1, \dots, i_{s!}) \in I$, define

$$\mathcal{A}_{i_1} \cdots \mathcal{A}_{i_{s!}}$$

as the image of the subsheaf $(\mathcal{A}_{i_1} \otimes \cdots \otimes \mathcal{A}_{i_{s!}})^{\oplus N}$ of $(\mathcal{A}^{\otimes s!})^{\oplus N}$ in \mathbb{S}_s and

$$\mathcal{A}_{i_1} \star \cdots \star \mathcal{A}_{i_{s!}} := (\mathcal{A}_{i_1} \cdots \mathcal{A}_{i_{s!}}) \cap \mathbb{V}_s(\mathcal{A}).$$

The standard weight vectors are

$$\gamma_r^{(i)} := (\underbrace{i - r, \dots, i - r}_{i \times}, \underbrace{i, \dots, i}_{(r-i) \times}), \quad i = 1, \dots, r - 1.$$

Given a weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of the torsion free sheaf \mathcal{A} , we obtain the associated weight vector

$$(\underbrace{\gamma_1, \dots, \gamma_1}_{(\text{rk } \mathcal{A}_1) \times}, \underbrace{\gamma_2, \dots, \gamma_2}_{(\text{rk } \mathcal{A}_2 - \text{rk } \mathcal{A}_1) \times}, \dots, \underbrace{\gamma_{t+1}, \dots, \gamma_{t+1}}_{(\text{rk } \mathcal{A} - \text{rk } \mathcal{A}_t) \times}) := \sum_{j=1}^t \alpha_j \cdot \gamma_r^{(\text{rk } \mathcal{A}_j)}.$$

(We recover $\alpha_j = (\gamma_{j+1} - \gamma_j)/r$, $j = 1, \dots, t$.) For a dispo sheaf (\mathcal{A}, φ) and a weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of \mathcal{A} , we finally find with (8)

$$\mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) = -\min\{\gamma_{i_1} + \cdots + \gamma_{i_{s!}} \mid (i_1, \dots, i_{s!}) \in I: \varphi_{|\mathcal{A}_{i_1} \star \cdots \star \mathcal{A}_{i_{s!}}} \neq 0\}. \tag{12}$$

(ii) We need a variant of the former definition. Let (\mathcal{A}, φ) be a dispo sheaf. We look at the representation κ of $\text{GL}_r(k)$ on $\mathbb{V}_s(k^r)$. Let U be the maximal open subset on which \mathcal{A} is locally free. Then, $\mathbb{V}_s(\mathcal{A})|_U = \mathbb{V}_s(\mathcal{A}|_U)$ is the vector bundle that is associated to the vector bundle $\mathcal{A}|_U$ via the representation κ . Since κ is clearly a polynomial representation, we can write $\mathbb{V}_s(k^r)$ as a quotient of $\mathbb{D}^{s!,v}(k^r)$ for an appropriate integer $v > 0$. We let $\mathbb{D}^{s!,v}(\mathcal{A}|_U)$ be the vector bundle with fiber $\mathbb{D}^{s!,v}(k^r)$ that is associated to $\mathcal{A}|_U$. By construction, we have a surjection $\mathbb{D}^{s!,v}(\mathcal{A}|_U) \rightarrow \mathbb{V}_s(\mathcal{A}|_U)$, so that $\varphi|_U$ induces a homomorphism

$$\tilde{\varphi}: \mathbb{D}^{s!,v}(\mathcal{A}|_U) \rightarrow \mathcal{O}_U.$$

Note that $\mathbb{D}^{s!,v}(k^r)$ is a subrepresentation of $(k^r \otimes s!)^{\oplus N}$ for a suitable integer $N > 0$. Hence, $\mathbb{D}^{s!,v}(\mathcal{A}|_U)$ is a subbundle of $(\mathcal{A}|_U \otimes s!)^{\oplus N}$.

Let $(\mathcal{A}_\bullet, \alpha_\bullet)$ be a weighted filtration of \mathcal{A} . As before, $I = \{1, \dots, t + 1\}^{\times s!}$ and $\mathcal{A}_{t+1} = \mathcal{A}$. This time, we set

$$\mathcal{A}_{i_1} \star \cdots \star \mathcal{A}_{i_{s!}} := (\mathcal{A}_{i_1|U} \otimes \cdots \otimes \mathcal{A}_{i_{s!}|U})^{\oplus N} \cap \mathbb{D}^{s!,v}(\mathcal{A}|_U), \quad (i_1, \dots, i_{s!}) \in I.$$

Then,

$$\mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) = -\min\{\gamma_{i_1} + \cdots + \gamma_{i_{s!}} \mid (i_1, \dots, i_{s!}) \in I: \varphi_{|\mathcal{A}_{i_1} \star \cdots \star \mathcal{A}_{i_{s!}}} \neq 0\}. \tag{13}$$

Fix a positive polynomial $\delta \in \mathbb{Q}[x]$ of degree at most $\dim(X) - 1$. Now, we say that a dispo sheaf (\mathcal{A}, φ) is δ -(semi)stable, if the inequality

$$M(\mathcal{A}_\bullet, \alpha_\bullet) + \delta \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) (\succ) 0$$

holds for every weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of \mathcal{A} .

Let $\bar{\delta}$ be a non-negative rational number. We call a dispo sheaf (\mathcal{A}, φ) $\bar{\delta}$ -slope (semi)stable, if the inequality

$$L(\mathcal{A}_\bullet, \alpha_\bullet) + \bar{\delta} \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) (\geq) 0$$

holds for every weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of \mathcal{A} . Note that, for $\delta = \bar{\delta}/(\dim(X) - 1)! \cdot x^{\dim(X)-1} + \dots$ (where $n = \dim X$), we have

$$(\mathcal{A}, \varphi) \text{ is } \delta\text{-semistable} \implies (\mathcal{A}, \varphi) \text{ is } \bar{\delta}\text{-slope semistable.} \tag{14}$$

4.2. Global boundedness

Theorem 4.2.1. Fix a Hilbert polynomial P , a representation ϱ , and an integer s as above. Then, the set of isomorphism classes of torsion free sheaves \mathcal{A} on X with Hilbert polynomial P for which there do exist a positive rational number $\bar{\delta}$ and a $\bar{\delta}$ -slope semistable dispo sheaf (\mathcal{A}, φ) of type (ϱ, s) is bounded.

Proof. This is a slight modification of the proof of the corresponding result in [15]. We will use the notation in Remark 4.1.1(ii).

Suppose (\mathcal{A}, φ) is a dispo sheaf which is $\bar{\delta}$ -slope semistable for some $\bar{\delta} > 0$. Assume \mathcal{A} is not slope semistable as a sheaf and consider its slope Harder–Narasimhan filtration

$$\mathcal{A}_\bullet: \quad 0 = \mathcal{A}_0 \subsetneq \mathcal{A}_1 \subsetneq \mathcal{A}_2 \subsetneq \dots \subsetneq \mathcal{A}_t \subsetneq \mathcal{A}_{t+1} = \mathcal{A}.$$

We use the notation $\mathcal{A}^i = \mathcal{A}_i/\mathcal{A}_{i-1}$, $r_i := \text{rk}(\mathcal{A}_i)$, $r^i := \text{rk}(\mathcal{A}^i)$, and $\mu^i := \mu(\mathcal{A}^i)$, $i = 1, \dots, t + 1$. Define

$$C(\mathcal{A}_\bullet) = \left\{ \gamma = (\gamma_1, \dots, \gamma_{t+1}) \in \mathbb{R}^{t+1} \mid \gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_{t+1}, \sum_{i=1}^{t+1} \gamma_i \cdot r^i = 0 \right\}.$$

We equip \mathbb{R}^{t+1} with the maximum norm $\|\cdot\|$. For all $\gamma \in C(\mathcal{A}_\bullet) \setminus \{0\}$, we have

$$\sum_{i=1}^t \frac{\gamma_{i+1} - \gamma_i}{r} \cdot (r \cdot \text{deg}(\mathcal{A}_i) - r_i \cdot \text{deg}(\mathcal{A})) < 0,$$

so that the δ -semistability of (\mathcal{A}, φ) implies

$$f(\gamma) := \mu(\mathcal{A}_\bullet, \alpha_\bullet(\gamma); \varphi) > 0, \quad \alpha_\bullet(\gamma) := \left(\frac{\gamma_2 - \gamma_1}{r}, \dots, \frac{\gamma_{t+1} - \gamma_t}{r} \right).$$

Consider the set

$$K := C(\mathcal{A}_\bullet) \cap \{\gamma \in \mathbb{R}^{t+1} \mid \|\gamma\| = 1\}.$$

Obviously K is a compact set and f is piecewise linear whence continuous, so that f attains its infimum on K . It is easy to see that there are only finitely many possibilities for the function f , so that we may bound this infimum from below by a constant $C_0 > 0$ which depends only on the input data.

As usual, we let U be the maximal open subset where \mathcal{A} is locally free. We have the induced homomorphism $\tilde{\varphi} : \mathbb{D}^{s!,v}(\mathcal{A}_U) \rightarrow \mathcal{O}_U$. Take a tuple $(i_1, \dots, i_{s!})$ with $\tilde{\varphi}|_{\mathcal{A}_{i_1} \star \dots \star \mathcal{A}_{i_{s!}}} \neq 0$ which is minimal with respect to the lexicographic ordering of the index set I . Define

$$\overline{\mathcal{A}}_{i_1, \dots, i_{s!}}$$

as the quotient of $\mathcal{A}_{i_1} \star \dots \star \mathcal{A}_{i_{s!}}$ by the subbundle that is generated by the $\mathcal{A}'_{i_1} \star \dots \star \mathcal{A}'_{i_{s!}}$ for the index tuples $(i'_1, \dots, i'_{s!})$ which are strictly smaller than $(i_1, \dots, i_{s!})$ in the lexicographic ordering. By construction, $\tilde{\varphi}$ factorizes over a non-zero homomorphism

$$\bar{\varphi} : \overline{\mathcal{A}}_{i_1, \dots, i_{s!}} \rightarrow \mathcal{O}_U,$$

whence

$$\mu_{\min}(\overline{\mathcal{A}}_{i_1, \dots, i_{s!}}) \leq 0.$$

In order to compute $\mu_{\min}(\overline{\mathcal{A}}_{i_1, \dots, i_{s!}})$, we observe that $\overline{\mathcal{A}}_{i_1, \dots, i_{s!}}$ is a subbundle of

$$(\mathcal{A}_{|U}^{i_1} \otimes \dots \otimes \mathcal{A}_{|U}^{i_{s!}})^{\oplus N}.$$

In fact, $\overline{\mathcal{A}}_{i_1, \dots, i_{s!}}$ is the vector bundle that is associated to the vector bundle

$$\mathcal{A}_{|U}^{i_1} \oplus \dots \oplus \mathcal{A}_{|U}^{i_{s!}}$$

by means of a representation $\varrho_{i_1, \dots, i_{s!}}$ of $\mathrm{GL}_{r^1}(k) \times \dots \times \mathrm{GL}_{r^{s!}}(k)$ which is a subrepresentation of the representation on

$$(k^{r^{i_1}} \otimes \dots \otimes k^{r^{i_{s!}}})^{\oplus N}.$$

This already shows

$$\mu(\overline{\mathcal{A}}_{i_1, \dots, i_{s!}}) = \mu^{i_1} + \dots + \mu^{i_{s!}}.$$

Now, we can apply Corollary 2.5.2 to see that

$$\mu^{i_1} + \dots + \mu^{i_{s!}} \leq \mu(\overline{\mathcal{A}}_{i_1, \dots, i_{s!}}) + D(\varrho_{i_1, \dots, i_{s!}}).$$

There are only finitely many possibilities for the representation $\varrho_{i_1, \dots, i_{s!}}$, so that we may replace the constant $D(\varrho_{i_1, \dots, i_{s!}})$ in the above inequality by a constant C which depends only on ϱ and s . Altogether, we have demonstrated

$$\mu^{i_1} + \dots + \mu^{i_{s!}} \leq C. \tag{15}$$

Take the point

$$\gamma := (\mu(\mathcal{A}) - \mu^1, \dots, \mu(\mathcal{A}) - \mu^{t+1}) = (-\mu^1, \dots, -\mu^{t+1}) \in \mathbb{R}^{t+1}.$$

By construction, $\gamma \in C(\mathcal{A}_\bullet) \setminus \{0\}$ and

$$f(\gamma) = \mu(\mathcal{A}_\bullet, \alpha_\bullet(\gamma); \varphi) \leq C.$$

But f is linear on each ray, so

$$f(\gamma) = \|\gamma\| \cdot f\left(\frac{\gamma}{\|\gamma\|}\right) \geq C_0 \cdot \|\gamma\|.$$

Now, this shows that either $\mu^1 = \|\gamma\| \leq C' := C/C_0$ or $-\mu^{t+1} = \|\gamma\| \leq C'$, i.e.,

$$\text{either } \mu_{\max}(\mathcal{A}) \leq \mu(\mathcal{A}) + C' \text{ or } \mu_{\min}(\mathcal{A}) \geq \mu(\mathcal{A}) - C'.$$

The theorem finally follows from the boundedness theorem of Maruyama–Langer [31]. \square

Corollary 4.2.2. *Fix the background data as in the theorem. There is a polynomial δ_∞ , such that for every polynomial $\delta > \delta_\infty$ and every dispo sheaf (\mathcal{A}, φ) of type (ϱ, s) in which \mathcal{A} has Hilbert polynomial P , the following conditions are equivalent:*

- (i) (\mathcal{A}, φ) is δ -(semi)stable.
- (ii) For every weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of \mathcal{A} , one has

$$\mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) \geq 0$$

and

$$M(\mathcal{A}_\bullet, \alpha_\bullet) (\succ) 0,$$

for every weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ with $\mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) = 0$.

Proof. Let us call a dispo sheaf (\mathcal{A}, φ) which satisfies (i) *asymptotically (semi)stable*. Using [15], one can find a polynomial δ_0 , such that for every dispo sheaf (\mathcal{A}, φ) of type (ϱ, s) in which \mathcal{A} has Hilbert polynomial P , the following holds true:

- If (\mathcal{A}, φ) is asymptotically (semi)stable, then it is δ -(semi)stable for every polynomial $\delta > \delta_0$.
- Assume $\delta_0 < \delta_1 < \delta_2$. If (\mathcal{A}, φ) is δ_2 -(semi)stable, it is also δ_1 -(semi)stable.

Note also: If (\mathcal{A}, φ) is not asymptotically (semi)stable, then it will not be δ -(semi)stable for any polynomial $\delta \gg 0$.

What remains to show is that we can find δ_∞ , such that, for every dispo sheaf (\mathcal{A}, φ) of type (ϱ, s) in which \mathcal{A} has Hilbert polynomial P and for every two polynomials $\delta_\infty < \delta_1 < \delta_2$, the implication

$$(\mathcal{A}, \varphi) \text{ is } \delta_1\text{-semistable} \implies (\mathcal{A}, \varphi) \text{ } \delta_2\text{-semistable}$$

is also correct. In [15], we referred to the instability flag for this. This is only adequate, if the characteristic of the base field is very large. We cannot assume this here. Therefore, we will give a different argument which relies only on general properties of semistability.

As before, we will use the finite set \mathcal{T} which depends only on ϱ and s , such that the condition of semistability of a dispo sheaf (\mathcal{A}, φ) has to be tested only for weighted filtrations $(\mathcal{A}_\bullet, \alpha_\bullet)$ of \mathcal{A} with $((\text{rk } \mathcal{A}_1, \dots, \text{rk } \mathcal{A}_t), \alpha_\bullet) \in \mathcal{T}$. If $\delta_0 < \delta_1 < \delta_2$ and (\mathcal{A}, φ) is a dispo sheaf of type (ϱ, s) and Hilbert polynomial P which is δ_1 -semistable but not δ_2 -semistable, there are a weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of \mathcal{A} with $((\text{rk } \mathcal{A}_1, \dots, \text{rk } \mathcal{A}_t), \alpha_\bullet) \in \mathcal{T}$ and a polynomial $\delta_1 \preceq \delta_\star < \delta_2$, such that

- $M(\mathcal{A}_\bullet, \alpha_\bullet) + \delta_1 \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) \succ 0$, $M(\mathcal{A}_\bullet, \alpha_\bullet) + \delta_2 \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) < 0$, and $M(\mathcal{A}_\bullet, \alpha_\bullet) + \delta_\star \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) = 0$. (Note that this implies $M(\mathcal{A}_\bullet, \alpha_\bullet) > 0$ and $\mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) \leq 0$.)
- (\mathcal{A}, φ) is δ_\star -semistable.

There is the admissible deformation $\text{df}_{(\mathcal{A}_\bullet, \alpha_\bullet)}(\mathcal{A}, \varphi) = (\mathcal{A}_{\text{gr}}, \varphi_{\text{gr}})$. It is performed with respect to the stability parameter δ_\star , and $(\mathcal{A}_{\text{gr}}, \varphi_{\text{gr}})$ is still δ_\star -semistable. We have

$$\mathcal{A}_{\text{gr}} = \bigoplus_{i=1}^{t+1} \mathcal{A}^i \quad \text{with } \mathcal{A}^i = \mathcal{A}_i / \mathcal{A}_{i-1}, \quad i = 1, \dots, t + 1.$$

If we define $\mathcal{A}_{\text{gr}, \bullet}$ via $\mathcal{A}_{\text{gr}, i} := \bigoplus_{j=1}^i \mathcal{A}^j$, $i = 1, \dots, t$, it is clear that $M(\mathcal{A}_{\text{gr}, \bullet}, \alpha_\bullet) = M(\mathcal{A}_\bullet, \alpha_\bullet)$ and $\mu(\mathcal{A}_{\text{gr}, \bullet}, \alpha_\bullet; \varphi_{\text{gr}}) = \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi)$. Since $(\mathcal{A}_{\text{gr}}, \varphi_{\text{gr}})$ is δ_\star -semistable, \mathcal{A}_{gr} belongs to a bounded family of torsion free sheaves with Hilbert polynomial P , by Theorem 4.2.1. Moreover, \mathcal{T} is finite, so that there are only finitely many possibilities for the polynomial $M(\mathcal{A}_\bullet, \alpha_\bullet)$. There are evidently only finitely many choices for $\mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi_{\text{gr}})$. The equation

$$M(\mathcal{A}_\bullet, \alpha_\bullet) + \delta_\star \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi_{\text{gr}}) = 0$$

leaves therefore only finitely many options for δ_\star . If we choose δ_∞ larger than the maximal possible value for δ_\star , the assertion of the corollary will hold. \square

4.3. S-equivalence

An important issue is the correct definition of S-equivalence of properly semistable dispo sheaves. For this, suppose we are given a δ -semistable dispo sheaf (\mathcal{A}, φ) and a weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of \mathcal{A} with

$$M(\mathcal{A}_\bullet, \alpha_\bullet) + \delta \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) \equiv 0.$$

We want to define the *associated admissible deformation* $\text{df}_{(\mathcal{A}_\bullet, \alpha_\bullet)}(\mathcal{A}, \varphi) = (\mathcal{A}_{\text{df}}, \varphi_{\text{df}})$. Of course, we set $\mathcal{A}_{\text{df}} = \bigoplus_{i=0}^t \mathcal{A}_{i+1}/\mathcal{A}_i$. Let U be the maximal (big!) open subset where \mathcal{A}_{df} is locally free. We may choose a one parameter subgroup $\lambda: \mathbb{G}_m(k) \rightarrow \text{SL}_r(k)$ whose weighted flag $(W_\bullet(\lambda), \alpha_\bullet(\lambda))$ in k^r satisfies:

- $\dim_k(W_i) = \text{rk } \mathcal{A}_i, i = 1, \dots, t$, in $W_\bullet(\lambda): 0 \subsetneq W_1 \subsetneq \dots \subsetneq W_t \subsetneq k^r$;
- $\alpha_\bullet(\lambda) = \alpha_\bullet$.

Then, the given filtration \mathcal{A}_\bullet corresponds to a reduction of the structure group of $\mathcal{I} \text{som}(\mathcal{O}_U^{\oplus r}, \mathcal{A}|_U)$ to $Q(\lambda)$. On the other hand, λ defines a decomposition

$$\mathbb{V}_s = U^{\gamma_1} \oplus \dots \oplus U^{\gamma_{u+1}}, \quad \gamma_1 < \dots < \gamma_{u+1}.$$

Now, observe that $Q(\lambda)$ fixes the flag

$$0 \subsetneq U_1 := U^{\gamma^1} \subsetneq U_2 := (U^{\gamma^1} \oplus U^{\gamma^2}) \subsetneq \dots \subsetneq U_u := (U^{\gamma^1} \oplus \dots \oplus U^{\gamma_u}) \subsetneq \mathbb{V}_s. \quad (16)$$

Thus, we obtain a $Q(\lambda)$ -module structure on

$$\bigoplus_{i=0}^u U_{i+1}/U_i \cong \mathbb{V}_s. \quad (17)$$

Next, we write $Q(\lambda) = \mathcal{R}_u(Q(\lambda)) \rtimes L(\lambda)$ where $L(\lambda) \cong \text{GL}(W_1/W_0) \times \dots \times \text{GL}(k^r/W_t)$ is the centralizer of λ . Note that (17) is an isomorphism of $L(\lambda)$ -modules. The process of passing from \mathcal{A} to \mathcal{A}_{df} corresponds to first reducing the structure group to $Q(\lambda)$, then extending it to $L(\lambda)$ via $Q(\lambda) \rightarrow Q(\lambda)/\mathcal{R}_u(Q(\lambda)) \cong L(\lambda)$, and then extending it to $\text{GL}_r(k)$ via the inclusion $L(\lambda) \subset \text{GL}_r(k)$. By (16), $\mathbb{V}_s(\mathcal{A}|_U)$ has a filtration

$$0 \subsetneq \mathcal{U}_1 \subsetneq \mathcal{U}_2 \subsetneq \dots \subsetneq \mathcal{U}_u \subsetneq \mathbb{V}_s(\mathcal{A}|_U),$$

and, by (17), we have a canonical isomorphism

$$\mathbb{V}_s(\mathcal{A}|_U) \cong \bigoplus_{i=1}^{u+1} \mathcal{U}_i/\mathcal{U}_{i-1}.$$

Now, for i_0 with $\gamma_{i_0} = -\mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi)$, the restriction φ_{i_0} of $\varphi|_U$ to \mathcal{U}_{i_0} is non-trivial, and thus we may define $\tilde{\varphi}_{\text{df}}$ as the map induced by φ_{i_0} on $\mathcal{U}_{i_0}/\mathcal{U}_{i_0-1}$ and as zero on the other components. Then, we finally obtain

$$\varphi_{\text{df}}: \mathbb{V}_s(\mathcal{A}) \rightarrow \iota_\star(\mathbb{V}_s(\mathcal{A}|_U)) \xrightarrow{\iota_\star(\tilde{\varphi}_{\text{df}})} \iota_\star(\mathcal{O}_U) = \mathcal{O}_X,$$

$\iota: U \rightarrow X$ being the inclusion. A dispo sheaf (\mathcal{A}, φ) is said to be δ -*polystable*, if it is δ -semistable and isomorphic to every admissible deformation $\text{df}_{(\mathcal{A}_\bullet, \alpha_\bullet)}(\mathcal{A}, \varphi) = (\mathcal{A}_{\text{df}}, \varphi_{\text{df}})$ associated to a filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of \mathcal{A} with

$$M(\mathcal{A}_\bullet, \alpha_\bullet) + \delta \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) \equiv 0.$$

By the GIT construction of the moduli space which will be given in Section 4.5, one has the following:

Lemma 4.3.1. *Let (\mathcal{A}, φ) be a δ -semistable dispo sheaf. Then, there is a δ -polystable admissible deformation $\text{gr}(\mathcal{A}, \varphi)$ of (\mathcal{A}, φ) . The dispo sheaf $\text{gr}(\mathcal{A}, \varphi)$ is unique up to isomorphism.*

In general, not every admissible deformation will immediately lead to a polystable dispo sheaf, but any iteration of admissible deformations (leading to non-isomorphic dispo sheaves) will do so after finitely many steps. We call two δ -semistable dispo sheaves (\mathcal{A}, φ) and (\mathcal{A}', φ') *S-equivalent*, if $\text{gr}(\mathcal{A}, \varphi)$ and $\text{gr}(\mathcal{A}', \varphi')$ are isomorphic.

Remark 4.3.2. Another way of looking at S-equivalence is the following: With the notation as above, we may choose an open subset $U \subseteq X$ (no longer big), such that φ is surjective over U and we have an isomorphism $\psi : \mathcal{A}|_U \cong k^r \otimes \mathcal{O}_U$ with $\psi(\mathcal{A}_i) = W_i \otimes \mathcal{O}_U$ for $i = 1, \dots, t$. For such a trivialization, we obtain, from $\varphi|_U$, the morphism

$$\beta : U \rightarrow \mathbb{P}(\mathbb{V}_s(\mathcal{A}|_U)) \xrightarrow{\mathbb{V}_s(\psi)} \mathbb{P}(\mathbb{V}_s) \times U \rightarrow \mathbb{P}(\mathbb{V}_s).$$

For the morphism $\beta_{\text{df}} : U \rightarrow \mathbb{P}(\mathbb{V}_s)$ associated to $\varphi_{\text{df}|U}$, we discover the relationship

$$\beta_{\text{df}}(x) = \lim_{z \rightarrow \infty} \lambda(z) \cdot \beta(x), \quad x \in U. \tag{18}$$

4.4. The main theorem on dispo sheaves

With the definitions which we have encountered so far, we may introduce the moduli functors

$$\begin{aligned} \underline{M}_P^{\delta-(s)s}(\varrho, s) : \underline{\text{Sch}}_k &\rightarrow \underline{\text{Sets}} \\ S &\mapsto \left\{ \begin{array}{l} \text{Isomorphism classes of families of} \\ \delta\text{-}(\text{semi})\text{stable dispo sheaves } \mathcal{A}_S \text{ of} \\ \text{type } (\varrho, s) \text{ with Hilbert polynomial } P \\ \text{parameterized by the scheme } S \end{array} \right\} \end{aligned}$$

for δ - (semi) stable dispo sheaves (\mathcal{A}, φ) of type (ϱ, s) on X with Hilbert polynomial $P(\mathcal{A}) = P$.

Theorem 4.4.1. *Given the input data P, ϱ, s , and δ as above, then the moduli space $M_P^{\delta\text{-ss}}(\varrho, s)$ for δ -semistable dispo sheaves (\mathcal{A}, τ) of type (ϱ, s) with $P(\mathcal{A}) = P$ exists as a projective scheme.*

4.5. The proof of the main theorem on dispo sheaves

In this section, we will outline how a GIT construction may be used for proving the main auxiliary result Theorem 4.4.1. Once one has the correct set-up, the details become mere applications of the techniques of the papers [46] or [16] and [32].

4.5.1. Construction of the parameter space

As we have seen in Theorem 4.2.1, there is a constant C , such that $\mu_{\max}(\mathcal{A}) \leq C$ for every δ -semistable dispo sheaf (\mathcal{A}, φ) with $P(\mathcal{A}) = P$, i.e., \mathcal{A} lives in a bounded family. Thus, we may choose an $n_0 \gg 0$ with the following properties: For every sheaf \mathcal{A} with Hilbert polynomial P and $\mu_{\max}(\mathcal{A}) \leq C$ and every $n \geq n_0$, one has

- $H^i(\mathcal{A}(n)) = \{0\}$ for $i > 0$;
- $\mathcal{A}(n)$ is globally generated.

We also fix a k -vector space U of dimension $P(n)$. Let \mathfrak{Q} be the quasi-projective scheme which parameterizes quotients $q: U \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{A}$ where \mathcal{A} is a torsion free sheaf with Hilbert polynomial P and $H^0(q(n))$ an isomorphism. Let

$$q_{\mathfrak{Q}}: U \otimes \pi_X^*(\mathcal{O}_X(-n)) \rightarrow \mathcal{A}_{\mathfrak{Q}}$$

be the universal quotient. Setting

$$\mathbb{V}_s(U) := \bigoplus_{\substack{(d_1, \dots, d_s): \\ d_i \geq 0, \sum d_i = s!}} (\text{Sym}^{d_1}((U \otimes_k V)^G) \otimes \dots \otimes \text{Sym}^{d_s}(\text{Sym}^s(U \otimes_k V)^G)),$$

there is a homomorphism

$$\mathbb{V}_s(U) \otimes \pi_X^*(\mathcal{O}_{\mathfrak{Q} \times X}(-s! \cdot n)) \rightarrow \mathbb{V}_s(\mathcal{A}_{\mathfrak{Q}}),$$

which is surjective over the open subset where $\mathcal{A}_{\mathfrak{Q}}$ is locally free (see Lemma 2.3.2). For a point $[q: U \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{A}] \in \mathfrak{Q}$, any homomorphism $\varphi: \mathbb{V}_s(\mathcal{A}) \rightarrow \mathcal{O}_X$ is determined by the induced homomorphism

$$\mathbb{V}_s(U) \rightarrow H^0(\mathcal{O}_X(s! \cdot n))$$

of vector spaces. Hence, our parameter space should be a subscheme of

$$\mathfrak{D}^0 := \mathfrak{Q} \times \underbrace{\mathbb{P}(\text{Hom}(\mathbb{V}_s(U), H^0(\mathcal{O}_X(s! \cdot n)))^\vee)}_{=:\mathbb{P}}.$$

Note that, over $\mathfrak{D}^0 \times X$, there is the universal homomorphism

$$\varphi''' : \mathbb{V}_s(U) \otimes \mathcal{O}_{\mathfrak{D}^0 \times X} \rightarrow H^0(\mathcal{O}_X(s! \cdot n)) \otimes \pi_{\mathbb{P}}^*(\mathcal{O}_{\mathbb{P}}(1)).$$

Let $\varphi'' = \text{ev} \circ \varphi'''$ be the composition of φ''' with the evaluation map

$$\text{ev}: H^0(\mathcal{O}_X(s! \cdot n)) \otimes \mathcal{O}_{\mathfrak{D}^0 \times X} \rightarrow \pi_X^*(\mathcal{O}_X(s! \cdot n)).$$

We twist φ'' by $\text{id}_{\pi_X^*(\mathcal{O}_X(-s! \cdot n))}$ in order to obtain

$$\varphi' : \mathbb{V}_s(U) \otimes \pi_X^*(\mathcal{O}_X(-s! \cdot n)) \rightarrow \pi_{\mathbb{P}}^*(\mathcal{O}_{\mathbb{P}}(1)).$$

Set $\mathcal{A}_{\mathcal{D}^0} := \pi_{\Omega \times X}^*(\mathcal{A}_{\Omega})$. We have the homomorphism $S: \mathbb{V}_s(U) \otimes \pi_X^*(\mathcal{O}_X(-s! \cdot n)) \rightarrow \mathbb{V}_s(\mathcal{A}_{\mathcal{D}^0})$. Therefore, we may define a closed subscheme \mathcal{D} of \mathcal{D}^0 by the condition that φ' vanishes on $\ker(S)$. Declaring $\mathcal{A}_{\mathcal{D}} := (\mathcal{A}_{\mathcal{D}^0})|_{\mathcal{D} \times X}$, there is thus the homomorphism

$$\varphi_{\mathcal{D}}: \mathbb{V}_s(\mathcal{A}_{\mathcal{D}}) \rightarrow \pi_{\mathbb{P}}^*(\mathcal{O}_{\mathbb{P}}(1))$$

with $\varphi|_{\mathcal{D} \times X} = \varphi_{\mathcal{D}} \circ S$. (To be precise, we first get $\varphi_{\mathcal{D}}$ on the maximal open subset $V \subset \mathcal{D} \times X$ where $\mathcal{A}_{\mathcal{D}}$ is locally free and then extend it to $\mathcal{D} \times X$, using Corollary 2.6.2. By the same token, $\varphi|_{\mathcal{D} \times X} = \varphi_{\mathcal{D}} \circ S$ is true, because it holds over V .) The family $(\mathcal{A}_{\mathcal{D}}, \varphi_{\mathcal{D}})$ is the *universal family of dispo sheaves parameterized by \mathcal{D}* . By its construction, it has the features listed below.

Proposition 4.5.1 (*Local universal property*). *Let S be a scheme and $(\mathcal{A}_S, \varphi_S)$ a family of δ -semistable dispo sheaves with Hilbert polynomial P parameterized by S . Then, there exist a covering of S by open subschemes S_i , $i \in I$, and morphisms $\beta_i: S_i \rightarrow \mathcal{D}$, $i \in I$, such that the family $(\mathcal{A}_{S_i}, \varphi_{S_i})$ is isomorphic to the pullback of the universal family on $\mathcal{D} \times X$ by $\beta_i \times \text{id}_X$ for all $i \in I$.*

4.5.2. The group action

There is a natural action of $\text{GL}(U)$ on the quot scheme Ω and on \mathcal{D}^0 . This action leaves the closed subscheme \mathcal{D} invariant, and therefore yields an action

$$\Gamma: \text{GL}(U) \times \mathcal{D} \rightarrow \mathcal{D}.$$

Proposition 4.5.2 (*Gluing property*). *Let S be a scheme and $\beta_i: S \rightarrow \mathcal{D}$, $i = 1, 2$, two morphisms, such that the pullback of the universal family via $\beta_1 \times \text{id}_X$ is isomorphic to its pullback via $\beta_2 \times \text{id}_X$. Then, there is a morphism $\Xi: S \rightarrow \text{GL}(U)$, such that β_2 equals the morphism*

$$S \xrightarrow{\Xi \times \beta_1} \text{GL}(U) \times \mathcal{D} \xrightarrow{\Gamma} \mathcal{D}.$$

4.5.3. Good quotients of the parameter space

For a point $z \in \mathcal{D}$, we let $(\mathcal{A}_z, \varphi_z)$ be the dispo sheaf obtained from the universal family by restriction to $\{z\} \times X$. It will be our task to show that the set $\mathcal{D}^{\delta\text{-}(s)\text{s}}$ parameterizing those points $z \in \mathcal{D}$ for which $(\mathcal{A}_z, \varphi_z)$ is δ -(semi)stable are open subsets of \mathcal{D} which possess a good or geometric quotient. This can be most conveniently done by applying GIT. To this end, we first have to exhibit suitable linearizations of the group action. We will use here the approach by Gieseker in order to facilitate the computations. The experienced reader should have no problem in rewriting the proof in Simpson’s language.

There is a projective subscheme $\mathfrak{A} \hookrightarrow \text{Pic}(X)$, such that the morphism $\det: \Omega \rightarrow \text{Pic}(X)$, $[q: U \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{A}] \mapsto [\det(\mathcal{A})]$ factorizes over \mathfrak{A} . We choose a Poincaré sheaf $\mathcal{P}_{\mathfrak{A}}$ on $\mathfrak{A} \times X$. Then, there is an integer n_1 , such that for every integer $n \geq n_1$ and every line bundle \mathcal{L} on X with $[\mathcal{L}] \in \mathfrak{A}$, the bundle $\mathcal{L}(rn)$ is globally generated and satisfies $h^i(\mathcal{L}(rn)) = 0$ for all $i > 0$. For such an n , the sheaf

$$\mathcal{G} := \pi_{\mathfrak{A}*}(\mathcal{P}_{\mathfrak{A}} \otimes \pi_X^*(\mathcal{O}_X(rn)))$$

is locally free. We then form the projective bundle

$$\mathbb{G}_1 := \mathbb{P} \left(\mathcal{H}om \left(\bigwedge^r U \otimes \mathcal{O}_{\mathfrak{A}}, \mathcal{G} \right)^\vee \right)$$

over the scheme \mathfrak{A} . For our purposes, we may always replace the Poincaré sheaf $\mathcal{P}_{\mathfrak{A}}$ by its tensor product with the pullback of the dual of a sufficiently ample line bundle on \mathfrak{A} , so that we can achieve that $\mathcal{O}_{\mathbb{G}_1}(1)$ is ample. The homomorphism

$$\bigwedge^r (q_{\Omega} \otimes \text{id}_{\pi_X^*(\mathcal{O}_X(n))}) : \bigwedge^r U \otimes \mathcal{O}_{\Omega \times X} \rightarrow \det(\mathcal{A}_{\Omega}) \otimes \pi_X^*(\mathcal{O}_X(rn))$$

defines a $\text{GL}(U)$ -equivariant and injective morphism

$$\Omega \rightarrow \mathbb{G}_1.$$

We declare

$$\mathbb{G}_2 := \mathbb{P}(\text{Hom}(\mathbb{V}_s(U), H^0(\mathcal{O}_X(s! \cdot n)))^\vee) \quad \text{and} \quad \mathbb{G} := \mathbb{G}_1 \times \mathbb{G}_2.$$

Then, we obtain the injective and $\text{SL}(U)$ -equivariant morphism

$$\text{Gies} : \mathfrak{D} \rightarrow \mathbb{G}.$$

The ample line bundles $\mathcal{O}_{\mathbb{G}}(v_1, v_2)$, $v_1, v_2 \in \mathbb{Z}_{>0}$, are naturally $\text{SL}(U)$ -linearized, and we choose v_1 and v_2 in such a way that

$$\frac{v_1}{v_2} = \frac{p - s! \cdot \delta(n)}{r \cdot \delta(n)}. \tag{19}$$

Theorem 4.5.3. *There exists $n_7 \in \mathbb{Z}_{>0}$, such that for all $n \geq n_7$ the following property is verified: For a point $z \in \mathfrak{D}$, the Gieseker point $\text{Gies}(z) \in \mathbb{G}$ is (semi)stable with respect to the above linearization if and only if $(\mathcal{A}_z, \varphi_z)$ is a δ -(semi)stable dispo sheaf of type (ϱ, s) .*

In the following, we will prove the theorem in several stages. As the first step, we establish the following result.

Proposition 4.5.4. *There is an $n_2 > 0$, such that the following holds true: The set \mathfrak{S} of isomorphism classes of torsion free sheaves \mathcal{A} with Hilbert polynomial P for which there exist an $n \geq n_2$ and a point $z = ([q : U \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{A}], \varphi) \in \mathfrak{D}$, such that $\text{Gies}(z)$ is semistable with respect to the above linearization, is bounded.*

Proof. We would like to find a lower bound for $\mu_{\min}(\mathcal{A})$ for a sheaf \mathcal{A} as in the proposition. Then, we may conclude with Theorem 4.2 of [31].

Let $\mathcal{Q} = \mathcal{A} / \mathcal{B}$ be a torsion free quotient sheaf of \mathcal{A} . We have the exact sequence

$$0 \rightarrow H^0(\mathcal{B}(n)) \rightarrow H^0(\mathcal{A}(n)) \rightarrow H^0(\mathcal{Q}(n)).$$

Let $\lambda : \mathbb{G}_m(k) \rightarrow \text{SL}(U)$ be a one parameter subgroup with weighted flag

$$(U_\bullet(\lambda): 0 \subsetneq U_1 := H^0(q(n))^{-1}(H^0(\mathcal{B}(n))) \subsetneq U, \alpha_\bullet(\lambda) = (1)).$$

Define $\mathcal{B}' := q(U_1 \otimes \mathcal{O}_X(-n))$. If $\text{Gies}(z) = ([M], [L])$, then

$$\mu(\lambda, [M]) = P(n) \cdot \text{rk}(\mathcal{B}') - h^0(\mathcal{B}(n)) \cdot r \leq P(n) \cdot \text{rk}(\mathcal{B}) - h^0(\mathcal{B}(n)) \cdot r.$$

Similarly to the proof of Theorem 4.2.1, one finds

$$\mu(\lambda, [L]) \leq s! \cdot (P(n) - h^0(\mathcal{B}(n))).$$

The assumption that $\text{Gies}(z)$ is semistable thus gives

$$\begin{aligned} 0 &\leq \frac{v_1}{v_2} \cdot \mu(\lambda, [M]) + \mu(\lambda, [L]) \\ &\leq \frac{P(n) - s! \cdot \delta(n)}{r \cdot \delta(n)} \cdot (P(n) \cdot \text{rk}(\mathcal{B}) - h^0(\mathcal{B}(n)) \cdot r) + s! \cdot (P(n) - h^0(\mathcal{B}(n))) \\ &= \frac{P(n)^2 \cdot \text{rk}(\mathcal{B})}{r \cdot \delta(n)} - \frac{P(n) \cdot h^0(\mathcal{B}(n))}{\delta(n)} - \frac{s! \cdot P(n) \cdot \text{rk}(\mathcal{B})}{r} + s! \cdot P(n). \end{aligned}$$

We multiply this by $r \cdot \delta(n)/P(n)$ and find

$$\begin{aligned} P(n) \text{rk}(\mathcal{B}) - r h^0(\mathcal{B}(n)) + \delta(n) s! (r - 1) &\geq P(n) \text{rk}(\mathcal{B}) - r h^0(\mathcal{B}(n)) + \delta(n) s! (r - \text{rk}(\mathcal{B})) \\ &\geq 0. \end{aligned}$$

The first exact sequence implies $h^0(\mathcal{B}(n)) \geq P(n) - h^0(\mathcal{Q}(n))$. This enables us to transform the above inequality into

$$\frac{h^0(\mathcal{Q}(n))}{r} \geq \frac{P(n)}{r} - \frac{\delta(n) \cdot s! \cdot (r - 1)}{\text{rk}(\mathcal{Q}) \cdot r} \geq \frac{P(n)}{r} - \frac{\delta(n) \cdot s! \cdot (r - 1)}{r}. \tag{20}$$

For a semistable sheaf \mathcal{E} with $\mu(\mathcal{E}) \geq 0$, [32] provides the estimate

$$\begin{aligned} \frac{h^0(\mathcal{E})}{\text{rk}(\mathcal{E})} &\leq \text{deg}(X) \cdot \left(\frac{\mu(\mathcal{E})}{\text{deg}(X)} + f(r) + \text{dim}(X) \right) \\ &\leq \frac{\text{deg}(X)}{\text{dim}(X)!} \cdot \left(\frac{\mu(\mathcal{E})}{\text{deg}(X)} + f(r) + \text{dim}(X) \right)^{\text{dim}(X)}. \end{aligned} \tag{21}$$

If $\mu(\mathcal{E}) < 0$, we have of course $h^0(\mathcal{E}) = 0$. The right-hand side $R(n)$ of (20) is a positive polynomial of degree $\text{dim}(X)$ with leading coefficient $\text{deg}(X)/\text{dim}(X)!$. We can bound it from below by a polynomial of the form

$$\frac{\text{deg}(X)}{\text{dim}(X)!} \cdot \left(\frac{C}{\text{deg}(X)} + f(r) + \text{dim}(X) + n \right)^{\text{dim}(X)}.$$

Assume that n_2 is so large that the value of this polynomial is positive and smaller than $R(n)$ for all $n \geq n_2$. Then, (20), applied to the minimal destabilizing quotient \mathcal{Q} of \mathcal{A} , together with (21) yields

$$\mu_{\min}(\mathcal{A}) \geq C,$$

and we are done. \square

Theorem 4.5.5. *There is an n_3 , such that for every $n \geq n_3$ and every point $z \in \mathcal{D}$ with (semi)stable Gieseker point $\text{Gies}(z) \in \mathbb{G}$, the dispo sheaf $(\mathcal{A}_z, \varphi_z)$ is δ -(semi)stable.*

Proof. As in [48, Proposition 2.14], one may show that there is a finite set

$$\mathcal{T} = \left\{ (r_{\bullet}^j, \alpha_{\bullet}^j) \mid r_{\bullet}^j = (r_1^j, \dots, r_{t_j}^j): 0 < r_1^j < \dots < r_{t_j}^j < r; \right. \\ \left. \alpha_{\bullet}^j = (\alpha_1^j, \dots, \alpha_{t_j}^j): \alpha_i^j \in \mathbb{Q}_{>0}, i = 1, \dots, t_j, j = 1, \dots, t \right\},$$

depending only on the $\text{GL}_r(k)$ -module \mathbb{V}_s , such that the condition of δ -(semi)stability of a dispo sheaf (\mathcal{A}, φ) of type (ϱ, s) with $P(\mathcal{A}) = P$ has to be verified only for weighted filtrations $(\mathcal{A}_{\bullet}, \alpha_{\bullet})$ with

$$((\text{rk}(\mathcal{A}_1), \dots, \text{rk}(\mathcal{A}_t)), \alpha_{\bullet}) \in \mathcal{T}.$$

We may prescribe a constant C' . Then, there exists a constant C'' , such that for every dispo sheaf (\mathcal{A}, φ) of type (ϱ, s) with $P(\mathcal{A}) = P$ and $[\mathcal{A}] \in \mathfrak{S}$ and every weighted filtration $(\mathcal{A}_{\bullet}, \alpha_{\bullet})$, such that $((\text{rk}(\mathcal{A}_1), \dots, \text{rk}(\mathcal{A}_t)), \alpha_{\bullet}) \in \mathcal{T}$ and

$$\mu(\mathcal{A}_i) \leq C'', \quad \text{for one index } i \in \{1, \dots, t\}, \tag{22}$$

one has

$$L(\mathcal{A}_{\bullet}, \alpha_{\bullet}) > C'.$$

It is easy to determine a constant C''' which depends only on \mathbb{V}_s with

$$\mu(\mathcal{A}_{\bullet}, \alpha_{\bullet}; \varphi) \geq -C'''$$

for every weighted filtration $(\mathcal{A}_{\bullet}, \alpha_{\bullet})$ of a sheaf \mathcal{A} as above with $((\text{rk}(\mathcal{A}_1), \dots, \text{rk}(\mathcal{A}_t)), \alpha_{\bullet}) \in \mathcal{T}$. We choose $C' \geq \bar{\delta} \cdot C'''$. Then, for a dispo sheaf (\mathcal{A}, φ) of type (ϱ, s) with $[\mathcal{A}] \in \mathfrak{S}$ and a weighted filtration $(\mathcal{A}_{\bullet}, \alpha_{\bullet})$, such that $((\text{rk}(\mathcal{A}_1), \dots, \text{rk}(\mathcal{A}_t)), \alpha_{\bullet}) \in \mathcal{T}$ and (22) holds, one has

$$L(\mathcal{A}_{\bullet}, \alpha_{\bullet}) + \bar{\delta} \cdot \mu(\mathcal{A}_{\bullet}, \alpha_{\bullet}; \varphi) > C' - \bar{\delta} \cdot C''' \geq 0,$$

so that also

$$M(\mathcal{A}_{\bullet}, \alpha_{\bullet}) + \delta \cdot \mu(\mathcal{A}_{\bullet}, \alpha_{\bullet}; \varphi) > 0.$$

Thus, the condition of δ -(semi)stability has to be verified only for weighted filtrations $(\mathcal{A}_\bullet, \alpha_\bullet)$ with $((\text{rk}(\mathcal{A}_1), \dots, \text{rk}(\mathcal{A}_t)), \alpha_\bullet) \in \mathcal{T}$ for which (22) fails. But these live in bounded families. We conclude

Corollary 4.5.6. *There is a positive integer $n_4 \geq n_3$, such that any $n \geq n_4$ has the following property: For every dispo sheaf (\mathcal{A}, φ) of type (ϱ, s) for which $[\mathcal{A}]$ belongs to the bounded family \mathfrak{S} , the conditions stated below are equivalent.*

1. (\mathcal{A}, φ) is δ -(semi)stable.
2. For every weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ with $((\text{rk}(\mathcal{A}_1), \dots, \text{rk}(\mathcal{A}_t)), \alpha_\bullet) \in \mathcal{T}$, such that $\mathcal{A}_j(n)$ is globally generated and $h^i(\mathcal{A}_j(n)) = 0$ for all $i > 0, j = 1, \dots, t$, one has

$$\sum_{j=1}^t \alpha_j \cdot (h^0(\mathcal{A}(n)) \cdot \text{rk}(\mathcal{A}_j) - h^0(\mathcal{A}_j(n)) \cdot \text{rk}(\mathcal{A})) + \delta(n) \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) (\geq) 0.$$

We assume that $n \geq n_4$. Now, let $z = ([q : U \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{A}], \varphi) \in \mathfrak{D}$ be a point with (semi)stable Gieseker point $\text{Gies}(z)$. Then, $[\mathcal{A}]$ belongs to the bounded family \mathfrak{S} . Therefore, it suffices to check Criterion 2. in Corollary 4.5.6 for establishing the δ -(semi)stability of (\mathcal{A}, φ) .

Let, more generally, $(\mathcal{A}_\bullet, \alpha_\bullet)$ be a weighted filtration of \mathcal{A} , such that

- $\mathcal{A}_j(n)$ is globally generated, $j = 1, \dots, t$;
- $h^i(\mathcal{A}_j(n)) = 0, i > 0, j = 1, \dots, t$.

Since $H^0(q(n))$ is an isomorphism, we define the subspaces

$$U_j := H^0(q(n))^{-1}(H^0(\mathcal{A}_j(n))) \subsetneq U, \quad j = 1, \dots, t.$$

Define the *standard weight vectors*

$$\gamma_p^{(i)} := (\underbrace{i - p, \dots, i - p}_{i \times}, \underbrace{i, \dots, i}_{(p-i) \times}), \quad i = 1, \dots, p - 1,$$

and choose a basis $\underline{u} = (u_1, \dots, u_p)$ of U , such that

$$\langle u_1, \dots, u_{l_j} \rangle = U_j, \quad l_j = \dim_k(U_j) = h^0(\mathcal{A}_j(n)), \quad j = 1, \dots, t.$$

These data yield the weight vector

$$\underline{\gamma} = (\gamma_1, \dots, \gamma_p) := \sum_{j=1}^t \alpha_j \cdot \gamma_p^{(l_j)}$$

and the one parameter subgroup $\lambda := \lambda(\underline{u}, \underline{\gamma}) : \mathbb{G}_m(k) \rightarrow \text{SL}(U)$ with

$$\lambda(z) \cdot \sum_{i=1}^p c_i \cdot u_i = \sum_{i=1}^p z^{\gamma_i} \cdot c_i \cdot u_i, \quad z \in \mathbb{G}_m(k).$$

Similarly, we define the one parameter subgroups $\lambda^j := \lambda(\underline{u}, \gamma_p^{(l_j)})$, $j = 1, \dots, t$. Let

$$L : \mathbb{V}_s(U) \rightarrow H^0(\mathcal{O}_X(s! \cdot n))$$

be a linear map that represents the second component of Gies(z). We wish to compute $\mu(\lambda, [L])$. First, we note that the choice of the basis \underline{u} provides an identification

$$\text{gr}(U) := \bigoplus_{j=1}^{t+1} U^j \cong U, \quad U^j := U_j/U_{j-1}, \quad j = 1, \dots, t + 1,$$

which we will use without further mentioning in the following. Define $I := \{1, \dots, t + 1\}^{\times s!}$. In analogy to the considerations at the very end of Section 2.1.2, we introduce the subspaces

$$U_{i_1, \dots, i_{s!}}^* \subset \mathbb{V}_s(U), \quad (i_1, \dots, i_{s!}) \in I.$$

As before, we check that all weight spaces with respect to the one parameter subgroup λ inside $\mathbb{V}_s(U)$ are direct sums of some of these subspaces. In addition, the subspaces $U_{i_1, \dots, i_{s!}}^*$ are eigenspaces for the one parameter subgroups $\lambda^1, \dots, \lambda^t$. More precisely, λ^j acts on $U_{i_1, \dots, i_{s!}}^*$ with the weight

$$s! \cdot l_j - v_j(i_1, \dots, i_{s!}) \cdot p, \quad (i_1, \dots, i_{s!}) \in I, \quad j = 1, \dots, t.$$

In that formula, we have used

$$v_j(i_1, \dots, i_{s!}) := \#\{i_k \leq j \mid k = 1, \dots, s!\}.$$

Thus, we find

$$\mu(\lambda, [L]) = -\min \left\{ \sum_{j=1}^t \alpha_j \cdot (s! \cdot l_j - v_j(i_1, \dots, i_{s!}) \cdot p) \mid U_{i_1, \dots, i_{s!}}^* \not\subseteq \ker(L), (i_1, \dots, i_{s!}) \in I \right\}. \tag{23}$$

Fix an index tuple $(i_1^0, \dots, i_{s!}^0) \in I$ for which the minimum is achieved.

Let

$$M : \bigwedge^r U \rightarrow H^0(\det(\mathcal{A})(rn))$$

represent the first component of Gies(z). It is well known that

$$\begin{aligned} \mu(\lambda, [M]) &= \sum_{j=1}^t \alpha_j \cdot (h^0(\mathcal{A}(n)) \cdot \text{rk}(\mathcal{A}_j) - h^0(\mathcal{A}_j(n)) \cdot \text{rk}(\mathcal{A})) \\ &= \sum_{j=1}^t \alpha_j \cdot (p \cdot \text{rk}(\mathcal{A}_j) - h^0(\mathcal{A}_j(n)) \cdot r). \end{aligned}$$

Since we assume $Gies(z)$ to be (semi)stable, we have

$$\begin{aligned} 0 & (\leq) \frac{v_1}{v_2} \cdot \mu(\lambda, [M]) + \mu(\lambda, [L]) \\ & = \frac{v_1}{v_2} \sum_{j=1}^t \alpha_j (p \operatorname{rk}(\mathcal{A}_j) - h^0(\mathcal{A}_j(n))r) - \sum_{j=1}^t \alpha_j (s!l_j - v_j(i_1^0, \dots, i_{s!}^0)p) \\ & = \frac{p - s!\delta(n)}{r\delta(n)} \sum_{j=1}^t \alpha_j (p \operatorname{rk}(\mathcal{A}_j) - h^0(\mathcal{A}_j(n))r) - \sum_{j=1}^t \alpha_j (s!l_j - v_j(i_1^0, \dots, i_{s!}^0)p) \\ & = \sum_{j=1}^t \alpha_j \left(\frac{p^2 \operatorname{rk}(\mathcal{A}_j)}{r\delta(n)} - \frac{ps! \operatorname{rk}(\mathcal{A}_j)}{r} - \frac{ph^0(\mathcal{A}_j(n))}{\delta(n)} \right) + \sum_{j=1}^t \alpha_j v_j (i_1^0, \dots, i_{s!}^0)p. \end{aligned}$$

For the last equation, we have used $l_j = h^0(\mathcal{A}_j(n))$. We multiply this inequality by $r \cdot \delta(n)/p$. This leads to the inequality

$$\begin{aligned} & \sum_{j=1}^t \alpha_j \cdot (p \cdot \operatorname{rk}(\mathcal{A}_j) - h^0(\mathcal{A}_j(n)) \cdot r) \\ & + \delta(n) \cdot \left(- \sum_{j=1}^t \alpha_j \cdot (s! \cdot \operatorname{rk}(\mathcal{A}_j) - v_j(i_1^0, \dots, i_{s!}^0) \cdot r) \right) (\geq) 0. \end{aligned}$$

To conclude, we have to verify

$$\mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) \geq - \sum_{j=1}^t \alpha_j \cdot (s! \cdot \operatorname{rk}(\mathcal{A}_j) - v_j(i_1^0, \dots, i_{s!}^0) \cdot r). \tag{24}$$

If $\underline{\gamma}$ is the weight vector with the distinct weights $\gamma_1 < \dots < \gamma_{t+1}$ which is associated to $(\mathcal{A}_\bullet, \alpha_\bullet)$ as in Remark 4.1.1, then one easily checks that

$$\gamma_{i_1^0} + \dots + \gamma_{i_{s!}^0} = \sum_{j=1}^t \alpha_j \cdot (s! \cdot \operatorname{rk}(\mathcal{A}_j) - v_j(i_1^0, \dots, i_{s!}^0) \cdot r).$$

In view of (12), it remains to show that

$$\varphi|_{\mathcal{A}_{i_1^0} \star \dots \star \mathcal{A}_{i_{s!}^0}} \neq 0. \tag{25}$$

To this end, note that, up to a scalar, L is given as

$$(\iota : \mathbb{V}_s(U) \rightarrow H^0(\mathbb{V}_s(\mathcal{A})(s! \cdot n))) \xrightarrow{H^0(\varphi \otimes \operatorname{id}_{\mathcal{O}_X(s! \cdot n)}} H^0(\mathcal{O}_X(s! \cdot n)).$$

The image of $U_{i_1^0, \dots, i_{s!}^0}^*$ lies in the subspace $H^0((\mathcal{A}_{i_1^0} \star \dots \star \mathcal{A}_{i_{s!}^0})(s! \cdot n))$ and that shows (25). \square

We now turn to the converse direction in the proof of Theorem 4.5.3. Again, we need a preparatory result.

Proposition 4.5.7. *There is a positive integer n_5 , such that any δ -(semi)stable dispo sheaf (\mathcal{A}, φ) of type (ϱ, s) with $P(\mathcal{A}) = P$ satisfies*

$$\sum_{j=1}^t \alpha_j \cdot (P(n) \cdot \text{rk}(\mathcal{A}_j) - h^0(\mathcal{A}_j(n)) \cdot r) + \delta(n) \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) (\geq) 0$$

for every weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of \mathcal{A} and every $n \geq n_5$.

Proof. By Theorem 4.2.1, we know that there is a bounded family \mathfrak{G}' of torsion free sheaves with Hilbert polynomial P , such that $[\mathcal{A}] \in \mathfrak{G}'$ for any δ -semistable dispo sheaf (\mathcal{A}, φ) of type (ϱ, s) with $P(\mathcal{A}) = P$. We choose a constant C , such that $\mu_{\max}(\mathcal{A}) \leq C$ for every torsion free sheaf \mathcal{A} on X with $[\mathcal{A}] \in \mathfrak{G}'$. Given an additional positive constant C' , we subdivide the class of torsion free sheaves \mathcal{B} which might occur as saturated subsheaves of \mathcal{O}_X -modules \mathcal{A} with $[\mathcal{A}] \in \mathfrak{G}$ into two classes:

- A. $\mu(\mathcal{B}) \geq -C'$.
- B. $\mu(\mathcal{B}) < -C'$.

By a lemma of Grothendieck’s [29, Lemma 1.7.9], the sheaves \mathcal{B} falling into class A live again in bounded families, so that we may always assume that our n is large enough, such that any such sheaf \mathcal{B} satisfies $h^i(\mathcal{B}(n)) = 0, i > 0$.

If \mathcal{E} is a torsion free sheaf on X with Harder–Narasimhan filtration $0 =: \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \dots \subsetneq \mathcal{E}_t \subsetneq \mathcal{E}_{t+1} := \mathcal{E}$, then

$$h^0(\mathcal{E}) \leq \sum_{i=1}^{t+1} h^0(\mathcal{E}_i/\mathcal{E}_{i-1}),$$

so that (21) gives, with $F(r) := \max\{f(i) \mid i = 1, \dots, r\}$ and $\text{rk}(\mathcal{E}) \leq r$,

$$\begin{aligned} h^0(\mathcal{E}) &\leq (\text{rk}(\mathcal{E}) - 1) \cdot \frac{\text{deg}(X)}{\text{dim}(X)!} \cdot \left(\frac{\mu_{\max}(\mathcal{E})}{\text{deg}(X)} + F(r) + \text{dim}(X) \right)^{\text{dim}(X)} \\ &\quad + \frac{\text{deg}(X)}{\text{dim}(X)!} \cdot \left(\frac{\mu(\mathcal{E})}{\text{deg}(X)} + F(r) + \text{dim}(X) \right)^{\text{dim}(X)}. \end{aligned}$$

For any sheaf \mathcal{A} with $[\mathcal{A}] \in \mathfrak{G}'$ and any saturated subsheaf \mathcal{B} of \mathcal{A} which belongs to class B, we thus find

$$\begin{aligned} h^0(\mathcal{B}(n)) &\leq (\text{rk}(\mathcal{B}) - 1) \cdot \frac{\text{deg}(X)}{\text{dim}(X)!} \cdot \left(\frac{C}{\text{deg}(X)} + F(r - 1) + \text{dim}(X) + n \right)^{\text{dim}(X)} \\ &\quad + \frac{\text{deg}(X)}{\text{dim}(X)!} \cdot \left(\frac{-C'}{\text{deg}(X)} + F(r - 1) + \text{dim}(X) + n \right)^{\text{dim}(X)} \\ &=: R(\text{rk}(\mathcal{B}), C')(n). \end{aligned} \tag{26}$$

We choose C' so large that

$$\begin{aligned}
 P \cdot \text{rk}(\mathcal{B}) - h^0(\mathcal{B}(n)) \cdot r &\geq P \cdot \text{rk}(\mathcal{B}) - R(\text{rk}(\mathcal{B}), C') \cdot r = K \cdot x^{\dim(X)-1} + \dots \\
 &> \delta \cdot s! \cdot (r - 1).
 \end{aligned}
 \tag{27}$$

For all $n \gg 0$, (27) remains true when evaluated at n .

Now, let $(\mathcal{A}_\bullet, \alpha_\bullet)$ be any weighted filtration of \mathcal{A} . Write $\{1, \dots, t\} = I_A \sqcup I_B$ with $i \in I_{A/B}$ if and only if \mathcal{A}_i belongs to class A/B. Let $1 \leq i_1^{A/B} < \dots < i_{t_{A/B}}^{A/B} \leq t$ be the indices in $I_{A/B}$ and define the weighted filtrations $(\mathcal{A}_\bullet^{A/B}, \alpha_\bullet^{A/B})$ with

$$\begin{aligned}
 \mathcal{A}_\bullet^{A/B}: \quad 0 \subsetneq \mathcal{A}_1^{A/B} &:= \mathcal{A}_{i_1^{A/B}} \subsetneq \dots \subsetneq \mathcal{A}_{i_{t_{A/B}}^{A/B}} := \mathcal{A}_{i_{t_{A/B}}^{A/B}} \subsetneq \mathcal{A}, \\
 \alpha_\bullet^{A/B} &= (\alpha_1^{A/B}, \dots, \alpha_{i_{t_{A/B}}^{A/B}}^{A/B}) := (\alpha_{i_1^{A/B}}, \dots, \alpha_{i_{t_{A/B}}^{A/B}}).
 \end{aligned}$$

It is easy to see that

$$\mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) \geq \mu(\mathcal{A}_\bullet^A, \alpha_\bullet^A; \varphi) - s! \cdot (r - 1) \cdot \sum_{j=1}^{t_B} \alpha_j^B.
 \tag{28}$$

Now, we compute

$$\begin{aligned}
 &\sum_{j=1}^t \alpha_j \cdot (P(n) \cdot \text{rk}(\mathcal{A}_j) - h^0(\mathcal{A}_j(n)) \cdot r) + \delta(n) \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) \\
 &\stackrel{(28)}{\geq} \sum_{j=1}^{t_A} \alpha_j^A \cdot (P(n) \cdot \text{rk}(\mathcal{A}_j^A) - h^0(\mathcal{A}_j^A(n)) \cdot r) + \delta(n) \cdot \mu(\mathcal{A}_\bullet^A, \alpha_\bullet^A; \varphi) \\
 &\quad + \sum_{j=1}^{t_B} \alpha_j^B \cdot (P(n) \cdot \text{rk}(\mathcal{A}_j^B) - h^0(\mathcal{A}_j^B(n)) \cdot r) - \delta(n) \cdot s! \cdot (r - 1) \cdot \sum_{j=1}^{t_B} \alpha_j^B \\
 &\stackrel{n \gg 0}{\cong} M(\mathcal{A}_\bullet^A, \alpha_\bullet^A)(n) + \delta(n) \cdot \mu(\mathcal{A}_\bullet^A, \alpha_\bullet^A; \varphi) \\
 &\quad + \sum_{j=1}^{t_B} \alpha_j^B \cdot ((P(n) \cdot \text{rk}(\mathcal{A}_j^B) - h^0(\mathcal{A}_j^B(n)) \cdot r) - \delta(n) \cdot s! \cdot (r - 1)) \\
 &\stackrel{(27) \& n \gg 0}{\geq} M(\mathcal{A}_\bullet^A, \alpha_\bullet^A)(n) + \delta(n) \cdot \mu(\mathcal{A}_\bullet^A, \alpha_\bullet^A; \varphi) \stackrel{n \gg 0}{(\geq)} 0.
 \end{aligned}$$

The last estimate results from the condition of δ -(semi)stability, applied to the weighted filtration $(\mathcal{A}_\bullet^A, \alpha_\bullet^A)$. We still have to justify that, in this last estimate, n can be uniformly chosen for all polynomials of the form $M(\mathcal{A}_\bullet, \alpha_\bullet) + \delta \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi)$ where all the members of the filtration \mathcal{A}_\bullet belong to class A. We use again the set \mathcal{T} which has been introduced in the proof of Theorem 4.5.5. Then, any polynomial of the above form can be written as a positive rational linear combination of polynomials $M(\mathcal{A}_\bullet^i, \alpha_\bullet^i) + \delta \cdot \mu(\mathcal{A}_\bullet^i, \alpha_\bullet^i; \varphi)$ where

$((\text{rk}(\mathcal{A}_1^i), \dots, \text{rk}(\mathcal{A}_t^i)), \alpha_\bullet^i) \in \mathcal{T}$ and all the members of the filtration \mathcal{A}_\bullet^i belong to class A, $i = 1, \dots, \mu$. By the boundedness of the set of isomorphy classes of sheaves in class A, it now follows that there are only finitely many polynomials of the form $M(\mathcal{A}_\bullet, \alpha_\bullet) + \delta \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi)$ where all the members of \mathcal{A}_\bullet belong to class A and $((\text{rk}(\mathcal{A}_1), \dots, \text{rk}(\mathcal{A}_t)), \alpha_\bullet) \in \mathcal{T}$. This proves our last claim and the proposition. \square

Theorem 4.5.8. *There exists a positive integer n_6 , enjoying the following property: If $n \geq n_6$ and (\mathcal{A}, φ) is a δ -(semi)stable dispo sheaf of type (ϱ, s) with $P(\mathcal{A}) = P$, then, for a point $z \in \mathcal{D}$ of the form $z = ([q : U \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{A}], \varphi)$, the associated Gieseker point $\text{Gies}(z)$ is (semi)stable for the given linearization.*

Proof. Let $\lambda : \mathbb{G}_m(k) \rightarrow \text{SL}(U)$ be a one parameter subgroup and suppose $\text{Gies}(z) = ([M], [L])$. Then, we have to verify that

$$\frac{\nu_1}{\nu_2} \cdot \mu(\lambda, [M]) + \mu(\lambda, [L]) (\geq) 0.$$

The one parameter subgroup λ provides the weighted flag $(U_\bullet(\lambda), \beta_\bullet(\lambda))$ with

$$U_\bullet(\lambda): \quad 0 =: U_0 \subsetneq U_1 \subsetneq \dots \subsetneq U_\tau \subsetneq U_{\tau+1} := U; \quad \beta_\bullet(\lambda) = (\beta_1, \dots, \beta_\tau).$$

For each $h \in \{1, \dots, \tau\}$, we let $\tilde{\mathcal{A}}_h$ be the saturated subsheaf that is generically generated by $q(U_h \otimes \mathcal{O}_X(-n))$. There may be improper inclusions among the $\tilde{\mathcal{A}}_h$'s. After clearing these, we obtain the filtration

$$\mathcal{A}_\bullet: \quad 0 =: \mathcal{A}_0 \subsetneq \mathcal{A}_1 \subsetneq \dots \subsetneq \mathcal{A}_t \subsetneq \mathcal{A}_{t+1} := \mathcal{A}.$$

For $j = 1, \dots, t$, we define

$$T(j) := \{h \in \{1, \dots, \tau\} \mid \tilde{\mathcal{A}}_h = \mathcal{A}_j\}$$

and

$$\alpha_j := \sum_{h \in T(j)} \beta_h.$$

This gives the weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of \mathcal{A} . By Proposition 4.5.7,

$$\sum_{j=1}^t \alpha_j \cdot (P(n) \cdot \text{rk}(\mathcal{A}_j) - h^0(\mathcal{A}_j(n)) \cdot r) + \delta(n) \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) (\geq) 0. \tag{29}$$

Recall from (12) that

$$\mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) = - \min \{ \gamma_{i_1} + \dots + \gamma_{i_s!} \mid (i_1, \dots, i_s!) \in I : \varphi|_{\mathcal{A}_{i_1} \star \dots \star \mathcal{A}_{i_s!}} \neq 0 \}. \tag{30}$$

Let $(i_1^0, \dots, i_{s!}^0) \in I = \{1, \dots, t+1\}^{\times s!}$ be an index tuple which computes the minimum. With

$$v_j(i_1, \dots, i_{s!}) := \#\{i_k \leq j \mid k = 1, \dots, s!\},$$

one calculates

$$\gamma_{i_1^0} + \dots + \gamma_{i_{s!}^0} = \sum_{j=1}^t \alpha_j \cdot (s! \cdot \text{rk}(\mathcal{A}_j) - v_j(i_1^0, \dots, i_{s!}^0) \cdot r).$$

Thus, (29) transforms into

$$\begin{aligned} & \sum_{j=1}^t \alpha_j \cdot (P(n) \cdot \text{rk}(\mathcal{A}_j) - h^0(\mathcal{A}_j(n)) \cdot r) \\ & + \delta(n) \cdot \left(- \sum_{j=1}^t \alpha_j \cdot (s! \cdot \text{rk}(\mathcal{A}_j) - v_j(i_1^0, \dots, i_{s!}^0) \cdot r) \right) (\geq) 0. \end{aligned}$$

A computation as in the proof of Theorem 4.5.5, but performed backwards, shows that this implies

$$\begin{aligned} & \frac{v_1}{v_2} \cdot \left(\sum_{j=1}^t \alpha_j \cdot (P(n) \cdot \text{rk}(\mathcal{A}_j) - h^0(\mathcal{A}_j(n)) \cdot r) \right) \\ & - \sum_{j=1}^t \alpha_j \cdot (s! \cdot h^0(\mathcal{A}_j(n)) - v_j(i_1^0, \dots, i_{s!}^0) \cdot p) (\geq) 0. \end{aligned} \tag{31}$$

First, we see that

$$\begin{aligned} \mu(\lambda, [M]) &= \sum_{h=1}^{\tau} \beta_h \cdot (P(n) \cdot \text{rk}(\tilde{\mathcal{A}}_h) - \dim_k(U_h) \cdot r) \\ &\geq \sum_{j=1}^t \alpha_j \cdot (P(n) \cdot \text{rk}(\mathcal{A}_j) - h^0(\mathcal{A}_j(n)) \cdot r). \end{aligned} \tag{32}$$

We need a little more notation. For $j = 0, \dots, t + 1$, we introduce

$$\begin{aligned} \underline{h}(j) &:= \min\{h = 1, \dots, \tau \mid \tilde{\mathcal{A}}_h = \mathcal{A}_j\}; & \underline{U}_j &:= U_{\underline{h}(j)}, \\ \bar{h}(j) &:= \max\{h = 1, \dots, \tau \mid \tilde{\mathcal{A}}_h = \mathcal{A}_j\}; & \bar{U}_j &:= U_{\bar{h}(j)}, \end{aligned}$$

as well as

$$\tilde{U}_j := \underline{U}_j / \bar{U}_{j-1}, \quad j = 1, \dots, t + 1.$$

For an index tuple $(i_1, \dots, i_{s!}) \in I$, we find the vector space

$$\tilde{U}_{i_1, \dots, i_{s!}} := (\tilde{U}_{i_1} \otimes \dots \otimes \tilde{U}_{i_{s!}})^{\oplus N}.$$

Using a basis \underline{u} of U which consists of eigenvectors for the one parameter subgroup λ , we identify these spaces with subspaces of $(U^{\otimes s!})^{\oplus N}$. Then, $\tilde{U}_{i_1, \dots, i_{s!}}^\bullet$ stands for the image of $\tilde{U}_{i_1, \dots, i_{s!}}$ in

$$\bigoplus_{\substack{(d_1, \dots, d_s): \\ d_i \geq 0, \sum id_i = s!}} (\text{Sym}^{d_1}(U \otimes_k V) \otimes \dots \otimes \text{Sym}^{d_s}(\text{Sym}^s(U \otimes_k V)))$$

and $\tilde{U}_{i_1, \dots, i_{s!}}^\star$ for the intersection of $\tilde{U}_{i_1, \dots, i_{s!}}^\bullet$ with $\mathbb{V}_s(U)$. A similar construction, generalizing the one at the end of Section 2.1, associates to a collection of subspaces $Y_1, \dots, Y_{s!}$ of U the subspace $Y_1 \star \dots \star Y_{s!}$ of $(U^{\otimes s!})^{\oplus N}$. Note that

$$\lambda = \lambda(\underline{u}, \underline{\gamma}) \quad \text{with } \underline{\gamma} = \sum_{h=1}^{\tau} \beta_h \cdot \gamma_p^{(\dim_k(U_h))}.$$

We define $\lambda^h := \lambda(\underline{u}, \gamma_p^{(\dim_k(U_h))})$, $h = 1, \dots, \tau$. The effect of our definition is that the spaces $\tilde{U}_{i_1, \dots, i_{s!}}^\star$, $(i_1, \dots, i_{s!}) \in I$, are weight spaces for λ as well as for $\lambda^1, \dots, \lambda^\tau$. We associate to an index $h \in \{1, \dots, \tau\}$ the index $j(h) \in \{1, \dots, t\}$ with $\tilde{\mathcal{A}}_h = \mathcal{A}_{j(h)}$. Then, $\underline{h}(j) \leq h$ holds if and only if $j \leq j(h)$, and one verifies that λ acts on $\tilde{U}_{i_1^0, \dots, i_{s!}^0}^\star$ with the weight

$$\begin{aligned} & - \sum_{h=1}^{\tau} \beta_h \cdot (s! \cdot \dim_k(U_h) - \nu_{j(h)}(i_1^0, \dots, i_{s!}^0) \cdot p) \\ & \geq - \sum_{j=1}^t \alpha_j \cdot (s! \cdot h^0(\mathcal{A}_j(n)) - \nu_j(i_1^0, \dots, i_{s!}^0) \cdot p). \end{aligned} \tag{33}$$

In view of the estimates (31), (32), and (33), it is now sufficient to ascertain that the restriction of L to $\tilde{U}_{i_1^0, \dots, i_{s!}^0}^\star$ is non-trivial. If it were trivial, then there would have to be an index tuple $(i'_1, \dots, i'_{s!})$ with $i'_l \leq i_l^0$, $l = 1, \dots, s!$, at least one inequality being strict, such that

$$L|_{\underline{U}_{i'_1} \star \dots \star \underline{U}_{i'_{s!}}} \neq 0. \tag{34}$$

This is because L restricts to a non-zero map on $\underline{U}_{i_1^0} \star \dots \star \underline{U}_{i_{s!}^0}$, as $\varphi|_{\mathcal{A}_{i_1^0} \star \dots \star \mathcal{A}_{i_{s!}^0}}$ is non-trivial. Now, if (34) holds true, then we must also have

$$\varphi|_{\mathcal{A}_{i'_1} \star \dots \star \mathcal{A}_{i'_{s!}}} \neq 0. \tag{35}$$

(Compare the arguments at the end of the proof of Theorem 4.5.5.) But, then the tuple $(i_1^0, \dots, i_{s!}^0)$ would not give the minimum in (30), a contradiction. \square

By Theorem 4.5.3, the subsets $\mathfrak{D}^{\delta-(s)s}$ of δ -(semi)stable dispo sheaves are the preimages of the sets of GIT-(semi)stable points in \mathbb{G} under the Gieseker morphism. Therefore, they are open subsets.

Proposition 4.5.9. *The restricted Gieseker morphism*

$$\text{Gies}_{|\mathcal{D}^{\delta\text{-ss}}} : \mathcal{D}^{\delta\text{-ss}} \rightarrow \mathbb{G}^{\text{ss}}$$

is proper. Since it is also injective, it is finite.

Proof. This is pretty standard, so we can be a bit sketchy. We apply the valuative criterion of properness. Let $\overline{\mathcal{Q}}$ be the closure of \mathcal{Q} in the quot scheme of $U \otimes \mathcal{O}_X(-n)$. Then, the parameter space \mathcal{D} may also be compactified to $\overline{\mathcal{D}} \rightarrow \overline{\mathcal{Q}}$. Given a discrete valuation ring R , a morphism $\eta : C := \text{Spec}(R) \rightarrow \mathbb{G}^{\text{ss}}$ which lifts over $C^* := \text{Spec}(\text{Quot}(R))$ to a morphism $\eta^* : C^* \rightarrow \mathcal{D}^{\delta\text{-ss}}$, we may first extend η^* to a morphism $\tilde{\eta} : C \rightarrow \overline{\mathcal{D}}$. This morphism is associated to a family

$$(q_C : U \otimes \pi_X^*(\mathcal{O}_X(-n)) \rightarrow \mathcal{A}_C, \varphi_C : \mathbb{V}_s(\mathcal{A}_C) \rightarrow \mathcal{O}_{C \times X})$$

on $C \times X$ where the restriction of \mathcal{A}_C to the special fiber $\{0\} \times X$ may have torsion. (As usual, one gets φ_C first on the open subset where \mathcal{A}_C is locally free and then extends it to X .) Let Z be the support of that torsion. Then, the family q_C may be altered to a family $\tilde{q}_C : U \otimes \pi_X^*(\mathcal{O}_X(-n)) \rightarrow \tilde{\mathcal{A}}_C$ where $\tilde{\mathcal{A}}_C$ is now a C -flat family of torsion free sheaves, \tilde{q}_C agrees with q_C on $(C \times X) \setminus Z$, but $\tilde{q}_C|_{\{0\} \times X}$ may fail to be surjective in points of Z . Let $\iota : (C \times X) \setminus Z \rightarrow C \times X$ be the inclusion. Define $\tilde{\varphi}_C$ as the composition

$$\mathbb{V}_s(\tilde{\mathcal{A}}_C) \rightarrow \iota_*(\mathbb{V}_s(\tilde{\mathcal{A}}_C|_{(C \times X) \setminus Z})) = \iota_*(\mathbb{V}_s(\mathcal{A}_C|_{(C \times X) \setminus Z})) \xrightarrow{\iota_*^{(\varphi_C|_{(C \times X) \setminus Z})}} \iota_*(\mathcal{O}_{(C \times X) \setminus Z}) = \mathcal{O}_{C \times X}.$$

The family $(\tilde{q}_C, \tilde{\varphi}_C)$ also defines a morphism to \mathbb{G} which coincides with η . Let $(\tilde{q} : U \otimes \mathcal{O}_X(-n) \rightarrow \tilde{\mathcal{A}}, \tilde{\varphi})$ be the restriction of the new family to $\{0\} \times X$. One checks the following results:

- $H^0(\tilde{q}(n))$ must be injective;
- Since $(\tilde{q}, \tilde{\varphi})$ defines a semistable point, $\tilde{\mathcal{A}}$ belongs to a bounded family (this is an easy adaptation of the proof of Proposition 4.5.4);
- The techniques of the proof of Theorem 4.5.5 may also be used to show that $(\tilde{\mathcal{A}}, \tilde{\varphi})$ must be δ -semistable. In particular, the higher cohomology groups of $\tilde{\mathcal{A}}(n)$ vanish, so that $H^0(\tilde{q}(n))$ is indeed an isomorphism.

The family $(\tilde{q}, \tilde{\varphi})$ is thus induced by a morphism $\tilde{\eta}$ which lifts η and extends η^* . This finishes the argument. \square

Since \mathbb{G}^{ss} possesses a projective quotient, Proposition 4.5.9 and Lemma 2.1.7 show that the good quotient

$$M_P^{\delta\text{-ss}}(\varrho) := \mathcal{D}^{\delta\text{-ss}} // \text{SL}(U)$$

exists as a projective scheme. Likewise, the geometric quotient

$$M_P^{\delta\text{-s}}(\varrho) := \mathcal{D}^{\delta\text{-s}} / \text{SL}(U)$$

exists as an open subscheme of $M_p^{\delta\text{-ss}}(\varrho)$. By Propositions 4.5.1 and 4.5.2, and the universal property of a categorical quotient, the space $M_p^{\delta\text{-ss}}(\varrho)$ is indeed a coarse moduli space. \square

Remark 4.5.10 (*S-equivalence*). Recall that two points in $\mathfrak{D}^{\delta\text{-ss}}$ are mapped to the same point in the quotient if and only if the closures of their orbits intersect in $\mathfrak{D}^{\delta\text{-ss}}$. Given a point $y \in \mathfrak{D}^{\delta\text{-ss}}$, let $y' \in \mathfrak{D}^{\delta\text{-ss}}$ be the point whose orbit is the unique closed orbit in $\overline{\text{SL}(U) \cdot y} (\subseteq \mathfrak{D}^{\delta\text{-ss}})$. Then, there is a one parameter subgroup $\lambda : \mathbb{G}_m(k) \rightarrow \text{SL}(U)$ with $\lim_{z \rightarrow \infty} \lambda(z) \cdot y \in \text{SL}(U) \cdot y'$. For this one parameter subgroup, one has of course $\mu(\lambda, y) = 0$. Thus, the equivalence relation that we have to consider on the closed points of $\mathfrak{D}^{\delta\text{-ss}}$ is generated by $y \sim \lim_{z \rightarrow \infty} \lambda(z) \cdot y$ for all one parameter subgroups $\lambda : \mathbb{G}_m(k) \rightarrow \text{SL}(U)$ with $\mu(\lambda, y) = 0$.

If one looks carefully at the arguments given in the proofs of Theorems 4.5.5 and 4.5.8, one sees that, for a point $y = ([q : U \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{A}], \varphi) \in \mathfrak{D}^{\delta\text{-ss}}$, the following observations hold true:

- If $\lambda : \mathbb{G}_m(k) \rightarrow \text{SL}(U)$ verifies $\mu(\lambda, y) = 0$, its weighted flag $(U_\bullet(\lambda), \alpha_\bullet(\lambda))$ has the property that the weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet(\lambda))$ with $\mathcal{A}_j := q(U_j \otimes \mathcal{O}_X(-n))$, $j = 1, \dots, t$, satisfies

$$M(\mathcal{A}_\bullet, \alpha_\bullet(\lambda)) + \delta \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet(\lambda); \varphi) \equiv 0.$$

- Given a weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of \mathcal{A} with

$$M(\mathcal{A}_\bullet, \alpha_\bullet) + \delta \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) \equiv 0,$$

one can assume $h^i(\mathcal{A}_j(n)) = 0$, $i > 0$, and that $\mathcal{A}_j(n)$ is globally generated, $j = 1, \dots, t$. Hence, there is a unique flag U_\bullet in U , such that $H^0(q(U_j))$ maps U_j onto $H^0(\mathcal{A}_j(n))$, $j = 1, \dots, t$. Then, any one parameter subgroup $\lambda : \mathbb{G}_m(k) \rightarrow \text{SL}(U)$ with weighted flag $(U_\bullet, \alpha_\bullet)$ satisfies $\mu(\lambda, y) = 0$.

- For a one parameter subgroup λ with $\mu(\lambda, y) = 0$, $y' := \lim_{z \rightarrow \infty} \lambda(z) \cdot y$, and induced weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ on \mathcal{A} , the dispo sheaf $(\mathcal{A}_{y'}, \varphi_{y'})$ is isomorphic to $\text{df}_{(\mathcal{A}_\bullet, \alpha_\bullet)}(\mathcal{A}, \varphi)$.

This shows that the equivalence relation induced by the GIT process on the closed points of $\mathfrak{D}^{\delta\text{-ss}}$ is just S-equivalence of dispo sheaves as introduced in Section 4.3.

Remark 4.5.11 (*Decorated vector bundles on curves*). In [46], given a homogeneous representation $\kappa : \text{GL}_r(k) \rightarrow \text{GL}(V)$ and a smooth projective curve X over the complex numbers, the moduli problem of classifying triples (E, L, φ) consisting of a vector bundle of rank r on X , a line bundle L on X , and a non-trivial homomorphism $\varphi : E_\kappa \rightarrow L$, E_κ being associated to E via κ , was solved by a GIT procedure similar to the one presented above. Write $V = \tilde{V} \otimes \det(V)^{\otimes -w}$ where \tilde{V} is a homogeneous polynomial $\text{GL}_r(k)$ -module, say, of degree u . The only characteristic zero issue that is necessary for the construction in [46] is the fact that \tilde{V} can be written as the quotient of $(V^{\otimes u})^{\oplus v}$ for an appropriate positive integer v . In characteristic $p > 0$, this is only true when $p > u$. However, we may use the results of Section 2.4. They imply that \tilde{V} is a quotient of $\mathbb{D}^{u,v}(V)$ for an appropriate integer $v > 0$. Now, $\mathbb{D}^{u,v}(V)$ is a subrepresentation of $(V^{\otimes u})^{\oplus N}$. This shows that the arguments given in the present paper may be used to deal with decorated vector bundles on smooth projective curves in any characteristic (see [24] for the analogous case of decorated parabolic vector bundles).

5. The proof of the main theorem

This section is devoted to the proof of the main theorem. In fact, we will prove a slightly stronger theorem which is the exact analog to the main result of [45] in arbitrary characteristic. To do so, we recall the necessary notions of $\tilde{\delta}$ -semistable pseudo G -bundles and so on.

5.1. Associated dispo sheaves

The notion of a “pseudo G -bundle” has been recalled in Section 3.1. Now, we relate pseudo G -bundles to dispo sheaves.

Let S be a scheme, and (\mathcal{A}_S, τ_S) a family of pseudo G -bundles parameterized by S . Let $\iota : U \subseteq S \times X$ be the maximal open subset where \mathcal{A}_S is locally free. The locally free sheaf $\mathcal{A}_{S|U}$ and the $\text{GL}_r(k)$ -module \mathbb{V}_s give rise to the vector bundle $\mathbb{V}_s(\mathcal{A}_{S|U})$, and there is a surjection

$$\mathcal{S}ym^*(\mathbb{V}_s(\mathcal{A}_{S|U})) \rightarrow \mathcal{S}ym^{(s!)}(\mathcal{A}_{S|U} \otimes V)^G.$$

Define $\tilde{\tau}_s$ as the restriction of $\tau_{S|U}$ to the subalgebra $\mathcal{S}ym^{(s!)}(\mathcal{A}_{S|U} \otimes V)^G$. Then, $\tilde{\tau}_s$ is determined by a homomorphism

$$\varphi' : \mathbb{V}_s(\mathcal{A}_{S|U}) \rightarrow \mathcal{O}_U.$$

Thus, $\tau_{S|U}$ gives rise to the homomorphism

$$\varphi_S : \mathbb{V}_s(\mathcal{A}_S) \rightarrow \iota_*(\mathbb{V}_s(\mathcal{A}_{S|U})) \rightarrow \iota_*(\mathcal{O}_U) = \mathcal{O}_{S \times X},$$

by Corollary 2.6.2. Therefore, we can associate to the family (\mathcal{A}_S, τ_S) of pseudo G -bundles the family $(\mathcal{A}_S, \varphi_S)$ of dispo sheaves of type (ϱ, s) .

The map which associates to a pseudo G -bundle a dispo sheaf is injective on isomorphism classes. More precisely, we find

Lemma 5.1.1. *Suppose that (\mathcal{A}, τ) and (\mathcal{A}, τ') are two pseudo G -bundles, such that the associated dispo sheaves are equal. Then, there is a root of unity $\zeta \in k$, such that $\zeta \cdot \text{id}_{\mathcal{A}}$ yields an isomorphism between (\mathcal{A}, τ) and (\mathcal{A}, τ') .*

Proof. For $d > 0$, let

$$\tau_d, \tau'_d : \mathcal{S}ym^d(\mathcal{A} \otimes V)^G \rightarrow \mathcal{O}_X$$

be the degree d component of τ and τ' , respectively. Note that τ is determined by $\bigoplus_{d=1}^s \tau_d$. Let

$$\widehat{\tau}_s : \bigoplus_{\substack{(d_1, \dots, d_s): \\ d_i \geq 0, \sum d_i = s!}} (\mathcal{S}ym^{d_1}((\mathcal{A} \otimes V)^G) \otimes \dots \otimes \mathcal{S}ym^{d_s}(\mathcal{S}ym^s(\mathcal{A} \otimes V)^G)) \rightarrow \mathcal{O}_X$$

be the map induced by τ_1, \dots, τ_s , and define $\widehat{\tau}'_s$ in a similar way. By definition, $\widehat{\tau}_{s|U} = \tilde{\tau}_s$. Our assumption thus grants that $(\mathcal{A}, \widehat{\tau}_s)$ and $(\mathcal{A}, \widehat{\tau}'_s)$ are equal. This implies that, for $1 \leq d \leq s$,

$$\mathcal{S}ym^{\frac{s!}{d}}(\tau_d) = \mathcal{S}ym^{\frac{s!}{d}}(\tau'_d).$$

Restricting this equality to the generic point, it follows that there is an $(s!/d)$ th root of unity ζ_d with

$$\tau'_d = \zeta_d \cdot \tau_d, \quad d = 1, \dots, s.$$

It remains to show that there is an $s!$ th root of unity ζ , such that $\zeta_d = \zeta^d$. To see this, let \mathbb{A} be the restriction of \mathcal{A} to the generic point. Then, $\widehat{\tau}_s$ and $\widehat{\tau}'_s$, restricted to the generic point, define the same point

$$x \in \mathbb{P} := \mathbb{P} \left(\bigoplus_{\substack{(d_1, \dots, d_s); \\ d_i \geq 0, \sum d_i = s!}} \text{Sym}^{d_1}((\mathbb{A} \otimes V)^G) \otimes \dots \otimes \text{Sym}^{d_s}(\text{Sym}^s(\mathbb{A} \otimes V)^G) \right).$$

On the other hand, $\bigoplus_{d=1}^s \tau_d$ and $\bigoplus_{d=1}^s \tau'_d$ define points

$$y, y' \in \mathbb{B} := \left(\bigoplus_{d=1}^s \text{Sym}^d(\mathbb{A} \otimes V)^G \right)^\vee.$$

By our assumption, y and y' map both to x under the quotient map followed by the Veronese embedding

$$\mathbb{B} \setminus \{0\} \rightarrow (\mathbb{B} \setminus \{0\}) // \mathbb{G}_m(K) \hookrightarrow \mathbb{P}.$$

Putting all the information we have gathered so far together, we find the claim about the ζ_i and from that the one of the lemma. \square

Let $\widetilde{\delta} \in \mathbb{Q}[x]$ be a positive polynomial of degree at most $\dim(X) - 1$. We choose an s as before and define $\delta := \widetilde{\delta}/s!$. A pseudo G -bundle is said to be $\widetilde{\delta}$ -(semi)stable, if the associated dispo sheaf (\mathcal{A}, φ) of type (ϱ, s) is δ -(semi)stable. Similarly, given a non-negative rational number δ^* , we define the pseudo G -bundle (\mathcal{A}, τ) to be δ^* -slope (semi)stable, if the associated dispo sheaf (\mathcal{A}, φ) is $(\delta^*/s!)$ -slope (semi)stable.

Remark 5.1.2. (i) The definition of $\widetilde{\delta}$ -(semi)stability is the same as the one given in [45, p. 1192].
 (ii) Using (7), it follows that the notion of $\widetilde{\delta}$ -(semi)stability does not depend on the choice of s . This is why we threw in the factor $1/s!$.

5.2. S -equivalence

We fix a stability parameter $\widetilde{\delta}$ i.e., a positive rational polynomial of degree at most $\dim(X) - 1$. Suppose (\mathcal{A}, τ) is a $\widetilde{\delta}$ -semistable pseudo G -bundle with associated dispo sheaf (\mathcal{A}, φ) and $(\mathcal{A}_\bullet, \alpha_\bullet)$ is a weighted filtration with

$$M(\mathcal{A}_\bullet, \alpha_\bullet) + \frac{\widetilde{\delta}}{s!} \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) \equiv 0.$$

The construction used for defining an associated admissible deformation of a dispo sheaf can be easily extended to give the construction of the associated admissible deformation $\text{df}_{(\mathcal{A}_\bullet, \alpha_\bullet)}(\mathcal{A}, \tau)$. As before, we let S -equivalence be the equivalence relation “ \sim_S ” on $\tilde{\delta}$ -semistable pseudo G -bundles (\mathcal{A}, τ) generated by

$$(\mathcal{A}, \tau) \sim_S \text{df}_{(\mathcal{A}_\bullet, \alpha_\bullet)}(\mathcal{A}, \tau).$$

The injectivity of the map which assigns to the isomorphism class of a pseudo G -bundle the isomorphism class of the associated dispo sheaf (Lemma 5.1.1) and the definitions of semistability for the respective objects easily imply that for two pseudo G -bundles (\mathcal{A}, τ) and (\mathcal{A}', τ') with associated dispo sheaves (\mathcal{A}, φ) and (\mathcal{A}', φ') one has:

$$(\mathcal{A}, \tau) \sim_S (\mathcal{A}', \tau') \iff (\mathcal{A}, \varphi) \sim_S (\mathcal{A}', \varphi'). \tag{36}$$

In Remark 5.4.3 below, we will give a nice description of S -equivalence on semistable singular principal G -bundles.

5.3. Moduli spaces for $\tilde{\delta}$ -semistable pseudo G -bundles

An immediate consequence of the definition of semistability of pseudo G -bundles and Theorem 4.2.1 is that, for a given Hilbert polynomial P , the set of torsion free sheaves \mathcal{A} with Hilbert polynomial P for which there exists a $\tilde{\delta}$ -(semi)stable pseudo G -bundle (\mathcal{A}, τ) is bounded. Finally, the construction carried out in Section 5.1 and Corollary 2.6.2 give a natural transformation

$$\text{AD}: \underline{\mathbf{M}}_P^{\tilde{\delta}\text{-(s)}s}(\varrho) \rightarrow \underline{\mathbf{M}}_P^{\delta\text{-(s)}s}(\varrho, s)$$

of the functor $\underline{\mathbf{M}}_P^{\tilde{\delta}\text{-(s)}s}(\varrho)$ which assigns to a scheme S the set of isomorphism classes of families of $\tilde{\delta}$ -(semi)stable pseudo G -bundles with Hilbert polynomial P parameterized by S into the functor $\underline{\mathbf{M}}_P^{\delta\text{-(s)}s}(\varrho, s)$ which assigns to a scheme S the set of isomorphism classes of families of δ -(semi)stable dispo sheaves of type (ϱ, s) with Hilbert polynomial P parameterized by S .

Theorem 5.3.1. *Fix the stability parameter $\tilde{\delta}$ and the Hilbert polynomial P . Then, there is a projective scheme $\mathbf{M}_P^{\tilde{\delta}\text{-ss}}(\varrho)$ which is a coarse moduli space for the functors $\underline{\mathbf{M}}_P^{\tilde{\delta}\text{-ss}}(\varrho)$.*

Remark 5.3.2. This theorem generalizes the main theorem of [45] to arbitrary characteristic.

5.3.1. Construction of the parameter space

There is a constant C , such that $\mu_{\max}(\mathcal{A}) \leq C$ for every $\tilde{\delta}$ -semistable pseudo G -bundle (\mathcal{A}, τ) with $P(\mathcal{A}) = P$, i.e., \mathcal{A} lives in a bounded family. Thus, we may choose the integer s in such a way that $\mathcal{S}ym^*(\mathcal{A} \otimes V)^G$ is generated by elements in degree at most s for all such \mathcal{A} . We choose an $n_0 \gg 0$ with the following properties: For every sheaf \mathcal{A} with Hilbert polynomial P and $\mu_{\max}(\mathcal{A}) \leq C$ and every $n \geq n_0$, one has

- $H^i(\mathcal{A}(n)) = \{0\}$ for $i > 0$;
- $\mathcal{A}(n)$ is globally generated;

- The construction of the moduli space of $(\tilde{\delta}/s!)$ -semistable dispo sheaves of type (ϱ, s) can be performed with respect to n .

We choose a k -vector space U of dimension $P(n)$. Let Ω be the quasi-projective scheme which parameterizes quotients $q: U \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{A}$ where \mathcal{A} is a torsion free sheaf with Hilbert polynomial P and $\mu_{\max}(\mathcal{A}) \leq C$ (so that $\mathcal{S}ym^*(\mathcal{A} \otimes V)^G$ is generated by elements of degree at most s) and $H^0(q(n))$ is an isomorphism. Let

$$q_\Omega : U \otimes \pi_X^*(\mathcal{O}_X(-n)) \rightarrow \mathcal{A}_\Omega$$

be the universal quotient. By Lemma 2.3.2, there is a homomorphism

$$\mathcal{S}ym^*(U \otimes \pi_X^*(\mathcal{O}_X(-n)) \otimes V)^G \rightarrow \mathcal{S}ym^*(\mathcal{A}_\Omega \otimes V)^G$$

which is surjective where \mathcal{A}_Ω is locally free. For a point $[q: U \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{A}] \in \Omega$, any homomorphism $\tau: \mathcal{S}ym^*(\mathcal{A} \otimes V)^G \rightarrow \mathcal{O}_X$ of \mathcal{O}_X -algebras is determined by the composite homomorphism

$$\bigoplus_{i=1}^s \mathcal{S}ym^i(U \otimes \pi_X^*(\mathcal{O}_X(-n)) \otimes V)^G \rightarrow \mathcal{O}_X$$

of \mathcal{O}_X -modules. Noting that

$$\mathcal{S}ym^i(U \otimes \pi_X^*(\mathcal{O}_X(-n)) \otimes V)^G \cong \text{Sym}^i(U \otimes V)^G \otimes \pi_X^*(\mathcal{O}_X(-in)),$$

τ is determined by a collection of homomorphisms

$$\varphi_i : \text{Sym}^i(U \otimes V)^G \otimes \mathcal{O}_X \rightarrow \mathcal{O}_X(in), \quad i = 1, \dots, s.$$

Since φ_i is determined by the induced linear map on global sections, we will construct the parameter space inside

$$\mathfrak{Y}^0 := \Omega \times \bigoplus_{i=1}^s \text{Hom}(\text{Sym}^i(U \otimes V)^G, H^0(\mathcal{O}_X(in))).$$

Note that, over $\mathfrak{Y}^0 \times X$, there are universal homomorphisms

$$\tilde{\varphi}^i : \text{Sym}^i(U \otimes V)^G \otimes \mathcal{O}_{\mathfrak{Y}^0 \times X} \rightarrow H^0(\mathcal{O}_X(in)) \otimes \mathcal{O}_{\mathfrak{Y}^0 \times X}, \quad i = 1, \dots, s.$$

Let $\varphi^i = \text{ev} \circ \tilde{\varphi}^i$ be the composition of $\tilde{\varphi}^i$ with the evaluation map $\text{ev}: H^0(\mathcal{O}_X(in)) \otimes \mathcal{O}_{\mathfrak{Y}^0 \times X} \rightarrow \pi_X^*(\mathcal{O}_X(in))$, $i = 1, \dots, s$. We twist φ^i by $\text{id}_{\pi_X^*(\mathcal{O}_X(-in))}$ and put the resulting maps together to the homomorphism

$$\varphi : \mathcal{V}_{\mathfrak{Y}^0} := \bigoplus_{i=1}^s \mathcal{S}ym^i(U \otimes \pi_X^*(\mathcal{O}_X(-n)) \otimes V)^G \rightarrow \mathcal{O}_{\mathfrak{Y}^0 \times X}.$$

Next, φ yields a homomorphism of $\mathcal{O}_{\mathfrak{Y}^0 \times X}$ -algebras

$$\tilde{\tau}_{\mathfrak{Y}^0} : \mathcal{S}ym^*(\mathcal{V}_{\mathfrak{Y}^0}) \rightarrow \mathcal{O}_{\mathfrak{Y}^0 \times X}.$$

On the other hand, there is a surjective homomorphism

$$\beta : \mathcal{S}ym^*(\mathcal{V}_{\mathfrak{Y}^0}) \rightarrow \mathcal{S}ym^*(\pi^*(\mathcal{A}_{\Omega}) \otimes V)^G$$

of graded algebras where the left-hand algebra is graded by assigning the weight i to the elements in $\mathcal{S}ym^i(U \otimes \pi_X^*(\mathcal{O}_X(-n)) \otimes V)^G$. Here, $\pi : \mathfrak{Y}^0 \times X \rightarrow \Omega \times X$ is the natural projection. The parameter space \mathfrak{Y} is defined by the condition that $\tilde{\tau}_{\mathfrak{Y}^0}$ factorizes over β , i.e., setting $\mathcal{A}_{\mathfrak{Y}} := (\pi^*(\mathcal{A}_{\Omega}))|_{\mathfrak{Y} \times X}$, there is a homomorphism

$$\tau_{\mathfrak{Y}} : \mathcal{S}ym^*(\mathcal{A}_{\mathfrak{Y}} \otimes V)^G \rightarrow \mathcal{O}_{\mathfrak{Y} \times X}$$

with $\tilde{\tau}_{\mathfrak{Y}^0}|_{\mathfrak{Y} \times X} = \tau_{\mathfrak{Y}} \circ \beta$. Formally, \mathfrak{Y} is defined as the scheme theoretic intersection of the closed subschemes

$$\mathfrak{Y}_d := \{y \in \mathfrak{Y}^0 \mid \tilde{\tau}_{\mathfrak{Y}^0}|_{\{y\} \times X} : \ker(\beta|_{\{y\} \times X}^d) \rightarrow \mathcal{O}_X \text{ is trivial}\}, \quad d \geq 0.$$

The family $(\mathcal{A}_{\mathfrak{Y}}, \tau_{\mathfrak{Y}})$ is the *universal family of pseudo G -bundles parameterized by \mathfrak{Y}* . (In all these constructions, one needs to use Lemma 3.1.2.)

Proposition 5.3.3 (Local universal property). *Let S be a scheme and (\mathcal{A}_S, τ_S) a family of $\tilde{\delta}$ -semistable pseudo G -bundles with Hilbert polynomial P parameterized by S . Then, there exist a covering of S by open subschemes S_i , $i \in I$, and morphisms $\beta_i : S_i \rightarrow \mathfrak{Y}$, $i \in I$, such that the family $(\mathcal{A}_{S|S_i}, \tau_{S|S_i})$ is isomorphic to the pullback of the universal family on $\mathfrak{Y} \times X$ by $\beta_i \times \text{id}_X$ for all $i \in I$.*

5.3.2. The group action

There is a natural action of $\text{GL}(U)$ on the quot scheme Ω and on \mathfrak{Y}^0 . This action leaves the closed subscheme \mathfrak{Y} invariant, and therefore yields an action

$$\Gamma : \text{GL}(U) \times \mathfrak{Y} \rightarrow \mathfrak{Y}.$$

Proposition 5.3.4 (Gluing property). *Let S be a scheme and $\beta_i : S \rightarrow \mathfrak{Y}$, $i = 1, 2$, two morphisms, such that the pullback of the universal family via $\beta_1 \times \text{id}_X$ is isomorphic to its pullback via $\beta_2 \times \text{id}_X$. Then, there is a morphism $\mathcal{E} : S \rightarrow \text{GL}(U)$, such that β_2 equals the morphism*

$$S \xrightarrow{\mathcal{E} \times \beta_1} \text{GL}(U) \times \mathfrak{Y} \xrightarrow{\Gamma} \mathfrak{Y}.$$

Remark 5.3.5. The universal family is equipped with a $\text{GL}(U)$ -linearization. If one fixes, in the above proposition, an isomorphism between its pullbacks via $\beta_1 \times \text{id}_X$ and $\beta_2 \times \text{id}_X$, then there is a unique morphism $\mathcal{E} : S \rightarrow \text{GL}(U)$ which satisfies the stated properties and, in addition, that the given isomorphism is induced by pullback via $(\mathcal{E} \times \beta_1 \times \text{id}_X)$ from the linearization of $(\mathcal{A}_{\mathfrak{Y}}, \tau_{\mathfrak{Y}})$. This fact simply expresses that the moduli stack for $\tilde{\delta}$ -(semi)stable pseudo G -bundles will be the quotient stack of an appropriate open subscheme of the parameter space \mathfrak{Y} .

5.3.3. Conclusion of the proof

Suppose we knew that the points $([q : U \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{A}], \tau)$ in the parameter space \mathfrak{Y} for which (\mathcal{A}, τ) is $\tilde{\delta}$ -semistable form an open subscheme $\mathfrak{Y}^{\tilde{\delta}\text{-ss}}$. Then, it suffices to show that $\mathfrak{Y}^{\tilde{\delta}\text{-ss}}$ possesses a (good, uniform) categorical quotient by the action of $\text{GL}(U)$. Indeed, Propositions 5.3.3 and 5.3.4 and the universal property of the categorical quotient then imply that $M_P^{\tilde{\delta}\text{-ss}}(\varrho) := \mathfrak{Y}^{\tilde{\delta}\text{-ss}} // \text{GL}(U)$ has the desired properties. We have the natural surjection $\mathbb{G}_m(k) \times \text{SL}(U) \rightarrow \text{GL}(U)$, $(z, m) \mapsto z \cdot m$, and obviously

$$\mathfrak{Y}^{\tilde{\delta}\text{-ss}} // \text{GL}(U) = \mathfrak{Y}^{\tilde{\delta}\text{-ss}} // (\mathbb{G}_m(k) \times \text{SL}(U)).$$

By Example 2.1.6(ii), we may first form

$$\overline{\mathfrak{Y}}^{\tilde{\delta}\text{-ss}} := \mathfrak{Y}^{\tilde{\delta}\text{-ss}} // \mathbb{G}_m(k)$$

and then

$$\overline{\mathfrak{Y}}^{\tilde{\delta}\text{-ss}} // \text{SL}(U).$$

We can easily form the quotient $\overline{\mathfrak{Y}} := \mathfrak{Y} // \mathbb{G}_m(k)$. Since $\mathbb{G}_m(k)$ is linearly reductive, $\overline{\mathfrak{Y}}$ is a closed subscheme of

$$\Omega \times \left(\bigoplus_{i=1}^s \text{Hom}(\text{Sym}^i(U \otimes V)^G, H^0(\mathcal{O}_X(in))) // \mathbb{G}_m(k) \right).$$

In particular, $\overline{\mathfrak{Y}}$ is projective over Ω . Let $\mathfrak{D} \rightarrow \Omega$ be the parameter space for dispo sheaves of type (ϱ, s) constructed above. If we apply the construction described in Section 5.1 to the universal family $(\mathcal{A}_{\mathfrak{Y}}, \tau_{\mathfrak{Y}})$, we get an $\text{SL}(U)$ -equivariant and $\mathbb{G}_m(k)$ -invariant morphism

$$\tilde{\psi} : \mathfrak{Y} \rightarrow \mathfrak{D}$$

and, thus, a proper $\text{SL}(U)$ -equivariant morphism

$$\psi : \overline{\mathfrak{Y}} \rightarrow \mathfrak{D}.$$

By Lemma 5.1.1, ψ is injective, so that it is even *finite*. Now, there are open subsets $\mathfrak{D}^{\delta\text{-(s)}s}$, $\delta := \tilde{\delta}/s!$, which parameterize the δ -(semi)stable dispo sheaves of type (ϱ, s) , such that the good, uniform categorical quotient

$$M_P^{\delta\text{-ss}}(\varrho, s) = \mathfrak{D}^{\delta\text{-ss}} // \text{SL}(U)$$

exists as a projective scheme and the geometric, uniform categorical quotient

$$M_P^{\delta\text{-s}}(\varrho, s) = \mathfrak{D}^{\delta\text{-s}} / \text{SL}(U)$$

as an open subscheme of $M_P^{\delta\text{-ss}}(\varrho, s)$. By definition of semistability,

$$\tilde{\psi}^{-1}(\mathfrak{D}^{\delta\text{-ss}}) = \mathfrak{Y}^{\tilde{\delta}\text{-ss}},$$

whence

$$\psi^{-1}(\mathfrak{D}^{\delta\text{-ss}}) = \mathfrak{Y}^{\tilde{\delta}\text{-ss}} // \mathbb{G}_m(k).$$

Now, Lemma 2.1.7 implies that the quotient

$$M_P^{\tilde{\delta}\text{-ss}}(\varrho) := \mathfrak{Y}^{\tilde{\delta}\text{-ss}} // \text{GL}(U) = \psi^{-1}(\mathfrak{D}^{\delta\text{-ss}}) // \text{SL}(U)$$

exists as a projective scheme. Likewise, the open subscheme

$$M_P^{\tilde{\delta}\text{-s}}(\varrho) := \mathfrak{Y}^{\tilde{\delta}\text{-s}} / \text{GL}(U) = \psi^{-1}(\mathfrak{D}^{\delta\text{-s}}) / \text{SL}(U)$$

is a uniform (universal) geometric quotient and an open subscheme of $M_P^{\tilde{\delta}\text{-ss}}(\varrho)$.

5.4. Semistable singular principal bundles

Theorem 5.4.1. Fix a Hilbert polynomial P , and let δ_∞ be as in Corollary 4.2.2. For every polynomial $\tilde{\delta} \succ s! \cdot \delta_\infty$ and every singular principal G -bundle (\mathcal{A}, τ) with $P(\mathcal{A}) = P$, the following properties are equivalent:

- (i) (\mathcal{A}, τ) is (semi)stable.
- (ii) (\mathcal{A}, τ) is $\tilde{\delta}$ -(semi)stable.

Taking into account Corollary 4.2.2, the theorem reduces to:

Lemma 5.4.2. Let (\mathcal{A}, τ) be a singular principal G -bundle with associated dispo sheaf (\mathcal{A}, φ) . Then, for a weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of \mathcal{A} , the condition

$$\mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) = 0$$

is satisfied if and only if

$$(\mathcal{A}_\bullet, \alpha_\bullet) = (\mathcal{A}_\bullet(\beta), \alpha_\bullet(\beta))$$

for some reduction β of (\mathcal{A}, τ) to a one parameter subgroup λ of G .

Proof. We show that the first condition implies the second one, the converse being an easy exercise. Let $\lambda' : \mathbb{G}_m(k) \rightarrow \text{SL}_r(V)$ be a one parameter subgroup, such that the associated weighted flag

$$(V_\bullet(\lambda') : 0 \subsetneq V_1 \subsetneq \dots \subsetneq V_{t'} \subsetneq V, \alpha_\bullet(\lambda'))$$

satisfies $t' = t$, $\dim_k(V_i) = \text{rk } \mathcal{A}'_i$, $\mathcal{A}'_i = \ker(\mathcal{A}^\vee \rightarrow \mathcal{A}^\vee_{t+1-i})$, $i = 1, \dots, t$, and $\alpha_\bullet(\lambda') = (\alpha_t, \dots, \alpha_1)$, if $\alpha_\bullet = (\alpha_1, \dots, \alpha_t)$. Then, the weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ is associated to a reduction β' of the principal $\text{GL}(V)$ -bundle $\mathcal{I} \text{som}(V \otimes \mathcal{O}_{U'} / \mathcal{A}^\vee_{|U'})$ to λ' with U' the maximal

open subset where \mathcal{A} is locally free and all the \mathcal{A}'_i are subbundles. We may choose an open subset $\tilde{U} \subseteq X$, such that there is a trivialization $\psi: \mathcal{A}'_{|\tilde{U}} \rightarrow V \otimes \mathcal{O}_{\tilde{U}}$ with $\psi(\mathcal{A}'_i) = V_i \otimes \mathcal{O}_{\tilde{U}}$, $i = 1, \dots, t$. By definition of the number $\mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi)$, (8), and Proposition 2.1.4, we see that there is a one parameter subgroup $\lambda: \mathbb{G}_m(k) \rightarrow G$, such that

$$(V_\bullet(\lambda), \alpha_\bullet(\lambda)) = (V_\bullet(\lambda'), \alpha_\bullet(\lambda')).$$

To the principal bundles $\mathcal{P}(\mathcal{A}, \tau)$ and $\mathcal{I} \text{som}(V \otimes \mathcal{O}_U, \mathcal{A}'_{|U})$, we may associate group schemes $\mathcal{G} \subset \mathcal{G}\mathcal{L}(V)$ over U . Now, $\mathcal{G}\mathcal{L}(V)$ acts on $\mathcal{I} \text{som}(V \otimes \mathcal{O}_{U'}, \mathcal{A}'_{|U'})/\mathcal{Q}_{\text{GL}(V)}(\lambda)$, and the stabilizer of the section $\beta': U' \rightarrow \mathcal{I} \text{som}(V \otimes \mathcal{O}_{U'}, \mathcal{A}'_{|U'})/\mathcal{Q}_{\text{GL}(V)}(\lambda)$ is a parabolic subgroup $\mathcal{Q} \subset \mathcal{G}\mathcal{L}(V)_{|U'}$, such that

$$\mathcal{G}\mathcal{L}(V)_{|U'}/\mathcal{Q} = \mathcal{I} \text{som}(V \otimes \mathcal{O}_{U'}, \mathcal{A}'_{|U'})/\mathcal{Q}_{\text{GL}(V)}(\lambda).$$

The intersection $\mathcal{Q}_{\mathcal{G}} := \mathcal{Q} \cap \mathcal{G}_{|U'}$ is a parabolic subgroup. This follows if one applies the above reasoning on weighted flags to the geometric fibers of $\mathcal{G} \subset \mathcal{G}\mathcal{L}(V)$ over U' . Furthermore, $\mathcal{G}_{|U'}/\mathcal{Q}_{\mathcal{G}} = \mathcal{P}(\mathcal{A}, \tau)_{|U'}/\mathcal{Q}_G(\lambda)$. This can be seen as follows: Let C be the set of conjugacy classes of parabolic subgroups of G . There is a scheme $\mathcal{P}ar_p(\mathcal{G}_{|U'}/U')$ over U' , such that giving a parabolic subgroup $\mathcal{Q}_{\mathcal{G}}$ of $\mathcal{G}_{|U'}$ the fibers of which belong to $p \in C$ is the same as giving a section $U' \rightarrow \mathcal{P}ar_p(\mathcal{G}_{|U'}/U')$ [8, p. 443ff]. It is easy to see that $\mathcal{P}ar_p(\mathcal{G}_{|U'}/U') \cong \mathcal{P}(\mathcal{A}, \tau)_{|U'}/\mathcal{Q}_p$, \mathcal{Q}_p being a representative for p (compare [42, p. 281]). Finally, $\mathcal{G}_{|U'}/\mathcal{Q}_{\mathcal{G}} \cong \mathcal{P}ar_p(\mathcal{G}_{|U'}/U')$ [8, Corollaire 3.6, p. 445]. Therefore, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{Q}_{\mathcal{G}} & \hookrightarrow & \mathcal{Q} \\ \downarrow & & \downarrow \\ \mathcal{G}_{|U'} & \hookrightarrow & \mathcal{G}\mathcal{L}(V)_{|U'}. \end{array}$$

Taking $\mathcal{Q}_{\mathcal{G}}$ -quotients in the left-hand column and \mathcal{Q} -quotients in the right-hand column yields the commutative diagram

$$\begin{array}{ccc} U' & \xlongequal{\quad\quad\quad} & U' \\ \beta \downarrow & & \downarrow \beta' \\ \mathcal{P}(\mathcal{A}, \tau)_{|U'}/\mathcal{Q}_G(\lambda) & \hookrightarrow & \mathcal{I} \text{som}(V \otimes \mathcal{O}_{U'}, \mathcal{A}'_{|U'})/\mathcal{Q}_{\text{GL}(V)}(\lambda) \end{array}$$

and settles the claim. \square

Remark 5.4.3 (*S-equivalence for semistable singular principal G-bundles*). Let (\mathcal{A}, τ) be a semistable singular principal G -bundle. By Lemma 5.4.2, an admissible deformation is associated to a reduction $\beta: U' \rightarrow \mathcal{P}(\mathcal{A}, \tau)_{|U'}/\mathcal{Q}_G(\lambda)$ to a one parameter subgroup, such that $M(\mathcal{A}_\bullet(\beta), \alpha_\bullet(\beta)) \equiv 0$. The structure of the rational principal G -bundle $\mathcal{P}(\text{df}_{(\mathcal{A}_\bullet, \alpha_\bullet)}(\mathcal{A}, \tau))$ may be described in the following way: The reduction β defines a principal $\mathcal{Q}_G(\lambda)$ -bundle \mathcal{Q} over U' , such that $\mathcal{P}(\mathcal{A}, \tau)_{|U'}$ is obtained from \mathcal{Q} by means of extending the structure group

via $Q_G(\lambda) \subset G$. Extending the structure group of \mathcal{Q} via $Q_G(\lambda) \rightarrow L_G(\lambda) \subset G$ yields the principal bundle $\mathcal{P}(\text{df}_{(\mathcal{A}_\bullet, \alpha_\bullet)}(\mathcal{A}, \tau))|_{U'}$. Thus, our notion of S-equivalence naturally extends the one considered by Ramanathan (see, e.g., [44]).

Fix a Hilbert polynomial P and a stability parameter $\tilde{\delta}$. The most important basic fact which has to be kept in mind is that a $\tilde{\delta}$ -semistable pseudo G -bundle (\mathcal{A}, τ) with $P(\mathcal{A}) = P$ which is S-equivalent to a semistable singular principal G -bundle (\mathcal{A}', τ') is itself a semistable singular principal G -bundle. In other words, the class of semistable singular principal G -bundles with Hilbert polynomial P is closed under S-equivalence inside the class of $\tilde{\delta}$ -semistable pseudo G -bundles with Hilbert polynomial P .

We now come to the statement which grants semistable reduction theorem and, in particular, projectivity of the moduli spaces of semistable singular principal G -bundles.

Theorem 5.4.4. *Assume that either $\varrho : G \rightarrow \text{GL}(V)$ is of low separable index or G is an adjoint group, ϱ is the adjoint representation and it is of low height. Then, for every polynomial $\tilde{\delta}$ with $\tilde{\delta} > s! \cdot \delta_\infty$, a $\tilde{\delta}$ -semistable pseudo G -bundle (\mathcal{A}, τ) with $P(\mathcal{A}) = P$ is a singular principal G -bundle.*

Proof. Let (\mathcal{A}, τ) be a pseudo G -bundle with associated dispo sheaf (\mathcal{A}, φ) . Write \mathbb{A} for the restriction of \mathcal{A} to the generic point of X . As in Section 2.1.2, we set

$$\mathbb{V}_s(\mathbb{A}) := \bigoplus_{\substack{(d_1, \dots, d_s): \\ d_i \geq 0, \sum i d_i = s!}} (\text{Sym}^{d_1}((\mathbb{A} \otimes_k V)^G) \otimes \dots \otimes \text{Sym}^{d_s}(\text{Sym}^s(\mathbb{A} \otimes_k V)^G)),$$

$$\mathbb{W}_s(\mathbb{A}) := \bigoplus_{i=1}^s (\text{Sym}^i(\mathbb{A} \otimes_k V)^G)^\vee.$$

The restriction of φ to the generic point yields an element $v \in \mathbb{P}(\mathbb{V}_s(\mathbb{A}))$ and the restriction of τ to the generic point an element $w \in \mathbb{W}_s(\mathbb{A})$. Note that there is the surjection

$$\mathbb{W}_s(\mathbb{A}) \setminus \{0\} \rightarrow \mathbb{P}(\mathbb{V}_s(\mathbb{A})),$$

such that the point w maps to v .

Let $\lambda : \mathbb{G}_m(K) \rightarrow \text{SL}(\mathbb{A})$ be the one parameter subgroup from Corollary 2.2.8 with $\mu(\lambda, w) < 0$. According to Lemma 2.1.5, we also have $\mu(\lambda, v) < 0$. Let $(\mathbb{A}_\bullet, \alpha_\bullet)$ be the weighted flag of λ in \mathbb{A} . We may find a weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of \mathcal{A} whose restriction to the generic point yields $(\mathbb{A}_\bullet, \alpha_\bullet)$. For this weighted filtration, we find

$$\mu(\mathcal{A}_\bullet, \alpha_\bullet; \tau) = \frac{1}{s!} \cdot \mu(\lambda, v) < 0.$$

By Corollary 4.2.2, the weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ contradicts $\tilde{\delta}$ -semistability. \square

Example 5.4.5. The above proof can also be used for classical groups with their standard representations. Assume, for example, that $G = \text{Sp}_n(k) \subset \text{SL}_{2n}(k)$. If \mathcal{A} is a torsion free sheaf, then giving τ is the same as giving $\tau|_U$, U being the maximal open subset where \mathcal{A} is locally

free. Now, giving $\tau|_U$ is equivalent to giving a non-trivial anti-symmetric form $\varphi_U : \mathcal{A}|_U \rightarrow \mathcal{A}|_U^\vee$. Since \mathcal{A}^\vee is reflexive, the datum of φ_U is the same as the datum of an anti-symmetric form $\varphi : \mathcal{A} \rightarrow \mathcal{A}^\vee$. Assume that (\mathcal{A}, φ) is a singular principal $\mathrm{Sp}_n(k)$ -bundle. Then, $\mathcal{B} := \ker(\varphi)$ is a proper saturated subsheaf. If it is non-trivial, then the restriction of the weighted filtration $(0 \subsetneq \mathcal{B} \subsetneq \mathcal{A}, (1))$ to the generic point will come from a one parameter subgroup $\lambda : \mathbb{G}_m(K) \rightarrow \mathrm{SL}(\mathbb{A})$ with $\mu(\lambda, v) < 0$. Therefore, the theorem holds for $\mathrm{Sp}_n(k)$ with its standard representation in any characteristic.

A similar reasoning can be applied to $\mathrm{SO}_n(k)$, if the characteristic of k is not two. It works also for $\mathrm{GO}_n(k)$ and $\mathrm{GSp}_n(k)$, if one uses the moduli construction suggested in the introduction.

5.5. Proof of the semistable reduction theorem

Before going into the proof, we need to recall the following result of Seshadri⁵ which can be thought of as the semistable reduction theorem for GIT quotients.

Theorem 5.5.1. (See Seshadri [53, Theorem 4.1].) *Let (X, L) be a polarized projective scheme over the field k on which the reductive group G acts. Then, given a K -valued point x of $X^{\mathrm{ss}}(L)$, where K is the quotient field of the complete discrete valuation ring R , there exist a finite extension $R \subseteq R'$ and $g \in G(K')$, K' being the fraction field of R' , such that $g \cdot x$ is an R' -valued point of $X^{\mathrm{ss}}(L)$.*

So, to prove the semistable reduction theorem it is sufficient to show that the constructed moduli space is a GIT quotient of a projective scheme.

We fix a stability parameter $\tilde{\delta} \succ \tilde{\delta}_0$ (see Theorem 5.4.1) and use the notation of Section 5.3. By elimination theory, the points in the parameter space \mathfrak{Y} corresponding to singular principal G -bundles form an open subset \mathfrak{H} . By Theorem 5.4.1, $\mathfrak{H}^{(s)s} := \mathfrak{Y}^{\tilde{\delta}\text{-ss}} \cap \mathfrak{H}$ is the open subset corresponding to (semi)stable singular principal G -bundles. It suffices to show that the quotients

$$M_P^{(s)s}(\varrho) := \mathfrak{H}^{(s)s} // \mathrm{GL}(U)$$

exist as open subschemes of $M_P^{\tilde{\delta}\text{-}(s)s}(\varrho)$. This will follow immediately, if we show that $\mathfrak{H}^{\mathrm{ss}}$ is a $\mathrm{GL}(U)$ -saturated open subset of $\mathfrak{Y}^{\tilde{\delta}\text{-ss}}$. This means that for every point $x \in \mathfrak{H}^{\mathrm{ss}}$ the closure of the $\mathrm{GL}(U)$ -orbit in $\mathfrak{Y}^{\tilde{\delta}\text{-ss}}$ is entirely contained in $\mathfrak{H}^{\mathrm{ss}}$. Since the points with closed $\mathrm{GL}(U)$ -orbit in $\mathfrak{Y}^{\tilde{\delta}\text{-ss}}$ are mapped to the points with closed $\mathrm{SL}(U)$ -orbit in $\overline{\mathfrak{Y}}^{\tilde{\delta}\text{-ss}}$ (see [41, proof of Proposition 1.3.2]), it suffices to show that

$$\overline{\mathfrak{H}}^{\mathrm{ss}} := \mathfrak{H}^{\mathrm{ss}} // \mathbb{G}_m(k)$$

is an $\mathrm{SL}(U)$ -saturated open subset of $\overline{\mathfrak{Y}}^{\tilde{\delta}\text{-ss}}$. Let $y, y' \in \overline{\mathfrak{Y}}^{\tilde{\delta}\text{-ss}}$, such that y' lies in the closure of the $\mathrm{SL}(U)$ -orbit of y . Then, $\psi(y')$ lies in the closure of the $\mathrm{SL}(U)$ -orbit of $\psi(y)$. We may assume that the orbit of y' and hence of $\psi(y')$ is closed. By the Hilbert–Mumford criterion, one knows that there exists a one parameter subgroup $\lambda : \mathbb{G}_m(k) \rightarrow \mathrm{SL}(U)$ with $\lim_{x \rightarrow \infty} \lambda(z) \cdot \psi(y) \in \mathrm{SL}(U) \cdot \psi(y')$. Note that the injectivity of ψ thus implies $\lim_{z \rightarrow \infty} \lambda(z) \cdot y \in \mathrm{SL}(U) \cdot y'$. Suppose

⁵ A proof will be given in Appendix A.

that y and $\psi(y)$ represent (\mathcal{A}, τ) and (\mathcal{A}, φ) , respectively. Now, from the GIT constructions in Section 4.5, in particular Remark 4.5.10, and Section 5.3, one infers that λ corresponds to a filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ with

$$M(\mathcal{A}_\bullet, \alpha_\bullet) + \frac{\tilde{\delta}}{s!} \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet; \varphi) \equiv 0$$

and that a point in the orbit of $\psi(y')$ represents $\text{df}_{(\mathcal{A}_\bullet, \alpha_\bullet)}(\mathcal{A}, \varphi)$, so that a point in the orbit of y' represents $\text{df}_{(\mathcal{A}_\bullet, \alpha_\bullet)}(\mathcal{A}, \tau)$, by Section 5.2. Together with Remark 5.4.3, this shows

$$y \in \overline{\mathfrak{H}}^{\text{ss}} \implies y' \in \overline{\mathfrak{H}}^{\text{ss}},$$

and this is what we wanted to prove. \square

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Appendix A. Semistable reduction for good quotients

In this appendix, we provide a short proof of Seshadri’s theorem 5.5.1 (following his ideas) used in the proof of the semistable reduction theorem for singular principal G -bundles. As is well known to the experts (e.g., [4]), Seshadri’s theorem together with the GIT construction

of the moduli spaces gives the respective semistable reduction theorem. As an illustration, we show how we can recover the semistable reduction theorem of Langton for semistable sheaves and the semistable reduction theorem for curves. Even if one has constructed the moduli space as a projective scheme, the semistable reduction theorem remains of interest, because it has implications on the moduli stack or related stacks (see, e.g., [24,57]).

Let X be a scheme over some scheme S and let G be a smooth affine S -group scheme acting on X . As usual, for an S -scheme T , we set $X_T := X \times_S T$. In the whole section, K denotes the quotient field of a discrete valuation ring R . Let us recall that an S -morphism $\pi : X \rightarrow Y$ is called a *good quotient*, if π is an affine G -invariant morphism, such that $\pi_*(\mathcal{O}_X)^G \simeq \mathcal{O}_Y$.

Lemma A.0.1. *Assume that there exists a good quotient $\pi : X \rightarrow Y$, that Y is proper over S , and that there is a commutative diagram*

$$\begin{array}{ccc} \text{Spec}(K) & \xrightarrow{x} & X \\ \downarrow & & \downarrow \\ \text{Spec}(R) & \longrightarrow & S. \end{array}$$

Then, there exist a finite extension $R \subseteq R'$ and $g \in G(K')$, K' being the fraction field of R' , such that $g \cdot x$ is an R' -valued point of X .

Proof. Let Z_K be the closure in X_K of the G_K -orbit of the graph of $x : \text{Spec}(K) \rightarrow X$. Then, there exists a uniquely determined closed subscheme Z_R of X_R , such that $Z_R \rightarrow \text{Spec}(R)$ is flat and the generic fiber is isomorphic to Z_K . It is the closure of the G_R -orbit of the graph of $\text{Spec}(K) \rightarrow X$ in X_R . Let us remark that $Z_R \rightarrow \text{Spec}(R)$ is faithfully flat, i.e., that the fiber T over the closed point of $\text{Spec}(R)$ is non-empty. This follows from the fact that $X_R \rightarrow Y_R$ is a closed surjective map and the K -point $\pi(x)$ of Y can be extended to an R -point of Y . Now, the lemma follows from the existence of quasi-sections of faithfully flat morphisms. \square

The above lemma is a slight strengthening of a reformulation of [7, Lemma 2.9]. It implies a generalization of Theorem 5.5.1 by the following remark: By Seshadri’s generalization of Mumford’s GIT (see [54, Theorem 4]), the assumptions of the lemma are satisfied, if S is of finite type over a universally Japanese ring, G/S is a reductive group scheme, acting on a projective scheme with a linearization in an ample line bundle on it, and X is the open subset of G -semistable points.

Theorem A.0.2 (Stable reduction for curves). *The Deligne–Mumford stack of stable curves is proper over \mathbb{Z} . More precisely, if $X \rightarrow \text{Spec}(K)$ is a stable curve, then there exist a finite extension $K \subset K'$ and a (unique) stable family $X' \rightarrow \text{Spec}(R')$, where R' is the normalization of R in K' , such that the restriction of X' to $\text{Spec}(K')$ is isomorphic to $X \times_K K'$.*

For the history and references concerning this theorem, we refer to [11].

Proof. The moduli scheme of stable curves is constructed as a GIT quotient of the scheme \overline{H}_g that parameterizes stable curves of genus g together with their n -canonical embeddings into some \mathbb{P}^N by an action of $\text{PGL}(N + 1)$ (see [14]). Since the GIT quotient is projective, we can use the above lemma. A curve $X \rightarrow \text{Spec}(K)$, after choosing an embedding into \mathbb{P}^N , gives rise

to a map $\text{Spec}(K) \rightarrow \overline{H}_g$. Then, after possibly changing the map with a group action, we can extend it to a map $\text{Spec}(R') \rightarrow \overline{H}_g$. This gives the required family, because there is a universal family over \overline{H}_g . \square

Theorem A.0.3 (Langton's theorem). (See [29, Theorem 2.B.1].) *Let X be a projective \mathbb{Z} -scheme with geometrically connected fibers and let $\mathcal{O}_X(1)$ be an ample line bundle on X . Let \mathcal{F}_K be a Gieseker semistable sheaf on $X \times \text{Spec}(K)$. Then, there exist a finite extension $K \subset K'$ and a family $\mathcal{F}'_{R'}$ of Gieseker semistable sheaves on X parameterized by $\text{Spec}(R')$, where R' is the normalization of R in K' , such that $\mathcal{F}'_{K'} \simeq \mathcal{F}_K \otimes_K K'$.*

In fact, Langton proved the slightly stronger assertion that in the above theorem one can always take $K' = K$, but we need to start with an R -flat family of sheaves on X . Langton's algorithm works also for slope semistable sheaves for which there is no moduli space in general.

Proof. The theorem follows from the above lemma and the GIT construction of the moduli space of Gieseker semistable sheaves (see [32]). \square

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