# Multiple orthogonal polynomials, string equations and the large-$\boldsymbol{n}$ limit *

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**Abstract**

The Riemann-Hilbert problems for multiple orthogonal polynomials of types I and II are used to derive string equations associated to pairs of Lax-Orlov operators. A method for determining the quasiclassical limit of string equations in the phase space of the Whitham hierarchy of dispersionless integrable systems is provided. Applications to the analysis of the large-$\boldsymbol{n}$ limit of multiple orthogonal polynomials and their associated random matrix ensembles and models of non-intersecting Brownian motions are given.

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# 1 Introduction

The set of orthogonal polynomials $P_n(x) = x^n + \cdots$ , with respect to an exponential weight

$$\int_{-\infty}^{\infty} P_n(x)\, P_m(x)\, e^{V(\boldsymbol{c},x)}\, dx = h_n \delta_{nm}, \quad V(\boldsymbol{c}, x) := \sum_{k\geq 1} c_k\, x^k,$$

is an essential ingredient of the methods [1]-[2] for studying the large-$n$ limit of the Hermitian matrix model

$$Z_n = \int \mathrm{d}M \exp\Big(\mathrm{Tr}\, V(\boldsymbol{c}, M)\Big). \tag{1}$$

One of the main tools used in these methods is the pair of equations

$$z\, P_n(z) = \mathcal{Z}\, P_n(z), \quad \partial_z\, P_n(z) = \mathcal{M}\, P_n(z), \quad n \geq 0, \tag{2}$$

where $(\mathcal{Z}, \mathcal{M})$ is a pair of Lax-Orlov operators of the form

$$\mathcal{Z} = \Lambda + u_n + v_n\, \Lambda^*, \quad \mathcal{M} = -\sum_{k\geq 1} k\, c_k\, (\mathcal{Z}^{k-1})_+. \tag{3}$$

Here $\Lambda$ is the shift matrix acting in the linear space of sequences, $\Lambda^*$ is its transposed matrix and $(\quad)_+$ denotes the lower part (below the main diagonal) of semi-infinite matrices.

The first equation in (2) represents the standard three-term relation for orthogonal polynomials. Both equations are referred to as the *string equations* in the matrix models of 2D quantum gravity [1] and provide the starting point of several techniques to characterize the large-$n$ limit of (1). A deeper mathematical insight of these methods was achieved after the introduction by Fokas, Its and Kitaev [3] of a matrix valued Riemann-Hilbert (RH) problem which characterizes orthogonal polynomials on the real line, and the formulation by Deift and Zhou [2],[4] of steepest descent methods for studying asymptotic properties of RH problems .

The RH problem of Fokas-Its-Kitaev was generalized by Van Assche, Geronimo and Kuijlaars [5] to characterize multiple orthogonal polynomials. Moreover, it was found [6]-[10] that these families of polynomials are closely connected to important statistical models such as Gaussian ensembles with external sources and one-dimensional non-intersecting Brownian motions.

In this paper we generalize the string equations (2) to multiple orthogonal polynomials of types I and II, and show how these equations can be applied to analyze the large-$\boldsymbol{n}$ limit of multiple orthogonal polynomials and their associated statistical models. Section 2 introduces the basic strategy of our approach to derive string equations, which is inspired by standard methods used in the theory of multi-component integrable systems [11]-[15]. As it was proved in [5] the multiple orthogonal polynomials of types I and II are elements of the first row of the fundamental solution $f$ of the corresponding RH problem. Then, in Sections 3 and 4 we formulate systems of string equations for the elements of the first row of the fundamental solution $f$. In both cases the function $f$ depends on a set of discrete variables

$$\boldsymbol{s} = (s_1, s_2, \ldots, s_q) \in \mathbb{Z}^q, \quad \text{where} \quad \begin{cases} s_i \geq 0 \text{ for type I polynomials} \\ s_i \leq 0 \text{ for type II polynomials.} \end{cases}$$

Therefore, special care is required to determine the form of the string equations on the boundary of the domain of the discrete variables. Thus, we obtain closed-form expressions, free of boundary terms, for the string equations satisfied by these types of multiple orthogonal polynomials. These string equations are associated to pairs $(\mathcal{Z}_i, \mathcal{M}_i)$ of Lax-Orlov operators. In particular those involving the Lax operators $\mathcal{Z}_i$ lead to the well-known recurrence relations for multiple orthogonal polynomials [5].

We take advantage of an interesting observation due to Takasaki and Takebe [16] who showed that the dispersionless limit of a row of a matrix-valued KP wave function is a solution of the zero genus Whitham hierarchy [11]. This is an additional incentive for using Lax-Orlov operators [12]-[15] in order to characterize the large-$\boldsymbol{n}$ limit in terms of quasiclassical (dispersionless limit) expansions. Thus, in Section 5 we show how the leading term of the expansion of the first row of $f$ is determined by a system of dispersionless string equations for $q+1$ Lax-Orlov functions $(z_\alpha, m_\alpha)$ in the phase space of the Whitham hierarchy. The unknowns of this system reduce to a set of $q$ pairs of functions $(u_k, v_k)$, which are determined by means of a system of hodograph type equations. Finally, Section 6 is devoted to illustrate the applications of our results to models of random matrix ensembles and non-intersecting Brownian motions.

The present work deals with multiple orthogonal polynomials of types I and II only, but the same considerations apply to the study of multiple orthogonal polynomials of mixed type [17]. On the other hand, we concentrate on the description of the leading terms of the asymptotic solutions in the dispersionless limit. However, as it was showed in [18]-[19] for the case of the Toda hierarchy and the Hermitian matrix model, the scheme used in the present paper can be further elaborated for determining the general terms of these expansions, as well as their critical points and their corresponding double scaling limit regularizations.

## 2 Riemann-Hilbert problems

In this work we will consider $(q+1)\times(q+1)$ matrix valued functions. Unless otherwise stated Greek $\alpha, \beta, \dots$ and Latin $i, j, \dots$ suffixes will label indices of the sets $\{0, 1, \dots, q\}$ and $\{1, 2, \dots, q\}$, respectively. We will denote by $E_{\alpha\beta}$ the matrices $(E_{\alpha\beta})_{\alpha'\beta'} = \delta_{\alpha\alpha'}\,\delta_{\beta\beta'}$ of the canonical basis and, in particular, its diagonal members will be denoted by $E_\alpha := E_{\alpha\alpha}$. Some useful relations which will be frequently used in the subsequent discussion are

$$E_{\alpha\beta}\,E_{\gamma\lambda} = \delta_{\beta\gamma}\,E_{\alpha\lambda}; \qquad E_\alpha\,a\,E_\beta = a_{\alpha\beta}\,E_{\alpha\beta}, \quad \forall \text{ matrix } a.$$

We will also denote by $V(\boldsymbol{c}, z)$ the scalar function

$$V(\boldsymbol{c}, z) := \sum_{n\geq 1} c_n\, z^n, \quad \boldsymbol{c} = (c_1, c_2, \dots) \in \mathbb{C}^\infty, \tag{4}$$

and will assume that only a finite number of the coefficients $c_n$ are different from zero.

Given a matrix function $g = g(z)$ $(z \in \mathbb{R})$ such that $\det g(z) \equiv 1$, we will consider the RH problem

$$m_-(z)\, g(z) = m_+(z), \quad z \in \mathbb{R}, \tag{5}$$

where $m(z)$ is a sectionally holomorphic function and $m_\pm(z) := \lim_{\epsilon\to 0+} m(z \pm i\epsilon)$. We are interested in solutions $f = f(\boldsymbol{s}, z)$ of (5) depending on $q$ discrete variables $\boldsymbol{s} = (s_1, \dots, s_q) \in \mathbb{Z}^q$

such that

$$f(\boldsymbol{s},z)=\Big(I+\mathcal{O}\Big(\frac{1}{z}\Big)\Big)\,f_0(\boldsymbol{s},z),\quad z\to\infty, \tag{6}$$

where

$$f_0(\boldsymbol{s},z):=\sum_{\alpha=0}^{q} z^{s_\alpha}\,E_\alpha,\quad (s_0:=-\sum_{i=1}^{q} s_i).$$

The set of points $\boldsymbol{s}\in\mathbb{Z}^q$ for which (5) admits a solution $f(\boldsymbol{s},z)$ satisfying (6) will be denoted by $\Gamma$. The solution $f(\boldsymbol{s},z)$, $(\boldsymbol{s}\in\Gamma)$ is unique and will be referred to as the *fundamental solution* of the RH problem (5).

We will apply (5) and (6) to derive certain difference-differential equations for $f$. These equations contain two basic ingredients: the coefficients of the asymptotic expansion of $f(\boldsymbol{s},z)$ as $z\to\infty$

$$f(\boldsymbol{s},z)=\Big(I+\sum_{n\geq 1}\frac{a_n(\boldsymbol{s})}{z^n}\Big)\,f_0(\boldsymbol{s},z), \tag{7}$$

and the $q$ pairs of shift operators $T_i$, $T_i^*$ acting on functions $h(\boldsymbol{s})$ $(\boldsymbol{s}\in\Gamma)$ defined as

$$(T_i\,h)(\boldsymbol{s}):=\begin{cases}h(\boldsymbol{s}-\boldsymbol{e}_i)\ \text{ if } \boldsymbol{s}-\boldsymbol{e}_i\in\Gamma\\[2ex] 0\ \text{ if } \boldsymbol{s}-\boldsymbol{e}_i\notin\Gamma\end{cases},\quad (T_i^*\,h)(\boldsymbol{s}):=\begin{cases}h(\boldsymbol{s}+\boldsymbol{e}_i)\ \text{ if } \boldsymbol{s}+\boldsymbol{e}_i\in\Gamma\\[2ex] 0\ \text{ if } \boldsymbol{s}+\boldsymbol{e}_i\notin\Gamma,\end{cases}$$

where $\boldsymbol{e}_i$ are the elements of the canonical basis of $\mathbb{C}^q$.

We will often consider series of the form

$$\mathcal{A}:=\sum_{n=1}^{\infty} c_n(\boldsymbol{s})\,(T_i^*)^n+c_0'+\sum_{n=1}^{\infty} c_n'(\boldsymbol{s})\,T_i^n,$$

and will denote

$$(\mathcal{A})_{(i,+)}:=\sum_{n=1}^{\infty} c_n(\boldsymbol{s})\,(T_i^*)^n,\quad (\mathcal{A})_{(i,-)}:=c_0'+\sum_{n=1}^{\infty} c_n'(\boldsymbol{s})\,T_i^n. \tag{8}$$

The RH problem (5) admits the following symmetries.

**Proposition 1.** *1. If $h(\boldsymbol{s},z)\,(\boldsymbol{s}\in\Gamma)$ is an entire function of $z$, then $h(\boldsymbol{s},z)\,f(\boldsymbol{s},z)$ satisfies* (5) *for all $\boldsymbol{s}\in\Gamma$.*

*2. The functions $(T_i\,f)(\boldsymbol{s},z)$ and $(T_i^*\,f)(\boldsymbol{s},z)$ satisfy* (5) *for all $\boldsymbol{s}\in\Gamma$.*

*3. If $g(z)$ is an entire function, then for any entire function $\phi(z)$ verifying*

$$g^{-1}\,\phi\,g=\phi-g^{-1}\,\partial_z\,g, \tag{9}$$

*the* covariant derivative

$$D_z\,f:=\partial_z\,f-f\,\phi, \tag{10}$$

*satisfies* (5) *for all $\boldsymbol{s}\in\Gamma$.*

Our strategy to obtain difference-differential equations for $f$ is based on applying the next simple statement to the symmetries of (5).

**Proposition 2.** *Let $\tilde{f}(\boldsymbol{s},z)$ be a a solution of* (5) *defined for $\boldsymbol{s}$ in a certain subset $\Gamma_0 \subset \Gamma$. If $\tilde{f}(\boldsymbol{s},z)\, f(\boldsymbol{s},z)^{-1} - P(\boldsymbol{s},z) \to 0$ as $z \to \infty$, where $P(\boldsymbol{s},z)$ is a polynomial in $z$, then*

$$\tilde{f}(\boldsymbol{s},z) = P(\boldsymbol{s},z)\, f(\boldsymbol{s},z).$$

*Proof.* Since $\det g(z) \equiv 1$ it follows from (5) and (6) that $\det f(\boldsymbol{s},z) \equiv 1$ so that the inverse matrix $f(\boldsymbol{s},z)^{-1}$ is analytic for $z \in \mathbb{C} - \mathbb{R}$ and satisfies the jump condition

$$g(z)^{-1}\, f_-(\boldsymbol{s},z)^{-1} = f_+(\boldsymbol{s},z)^{-1}, \quad z \in \mathbb{R}.$$

As a consequence $\tilde{f}\, f^{-1}$ is an entire function of $z$ and the statements follow at once. □

# 3 Multiple orthogonal polynomials of type I

Given $q$ exponential weights $w_i$ on the real line

$$w_i(x) := e^{-V(\boldsymbol{c}_i,x)}, \quad \boldsymbol{c}_i = (c_{i1}, c_{i2}, \dots) \in \mathbb{C}^\infty,$$

and $\boldsymbol{n} = (n_1, \dots, n_q) \in \mathbb{N}^q$ with $|\boldsymbol{n}| \geq 1$, the type I orthogonal polynomials

$$\boldsymbol{A}(\boldsymbol{n},x) = (A_1(\boldsymbol{n},x), \dots, A_q(\boldsymbol{n},x))$$

are determined by means of the following conditions:

**i)** If $n_j \geq 1$ then the polynomial $A_j(\boldsymbol{n},x)$ has degree $n_j - 1$. If $n_j = 0$ then $A_j(\boldsymbol{n},z) \equiv 0$

**ii)** The following orthogonality relations are satisfied

$$\int_{\mathbb{R}} \frac{dx}{2\pi i} x^l \left( \sum_{j=1}^{q} A_j(\boldsymbol{n},x) w_j(x) \right) = \begin{cases} 0 & l = 0,1,\dots,|\boldsymbol{n}|-2, \\ 1 & l = |\boldsymbol{n}|-1. \end{cases}$$

We assume that all the multi-indices $\boldsymbol{n}$ are strongly normal [17] so that $A_j(\boldsymbol{n},z)$ are unique.

The RH problem which characterizes these polynomials [5] is determined by

$$g(z) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -w_1(z) & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ -w_q(z) & 0 & \dots & 0 & 1 \end{pmatrix}. \tag{11}$$

The corresponding fundamental solution $f(\boldsymbol{s},z)$ exists on the domain

$$\Gamma_I = \{\boldsymbol{s} \in \mathbb{Z}^q \,:\, s_i \geq 0,\, \forall i = 1,\dots,q\}. \tag{12}$$

For $\boldsymbol{s}\neq\boldsymbol{0}$ it is given by

$$
\begin{aligned}
f(\boldsymbol{s},z) &= \begin{pmatrix} R(\boldsymbol{s},z) & \boldsymbol{A}(\boldsymbol{s},z) \\ d_1^{-1}R(\boldsymbol{s}+\boldsymbol{e}_1,z) & d_1^{-1}\boldsymbol{A}(\boldsymbol{s}+\boldsymbol{e}_1,z) \\ \vdots & \vdots \\ d_q^{-1}R(\boldsymbol{s}+\boldsymbol{e}_q,z) & d_q^{-1}\boldsymbol{A}(\boldsymbol{s}+\boldsymbol{e}_q,z) \end{pmatrix}, \\
R(\boldsymbol{s},z) &:= \int_{\mathbb{R}} \frac{\mathrm{d}x}{2\pi i}\frac{\sum_{j=1}^{q} A_j(\boldsymbol{s},x)\,w_j(x)}{z-x},
\end{aligned}
\tag{13}
$$

where $d_j$ is the leading coefficient of $A_j(\boldsymbol{s}+e_j,z)$. Furthermore, for $\boldsymbol{s}=\boldsymbol{0}$

$$
f(\boldsymbol{0},z) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ R_1(z) & 1 & 0 & \cdots & 0 \\ R_2(z) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ R_q(z) & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad R_j(z) := \int_{\mathbb{R}} \frac{\mathrm{d}x}{2\pi i}\frac{w_j(x)}{z-x}. \tag{14}
$$

Because of the form of $\Gamma_I$ we have that

$$
(T_i^*\,h)(\boldsymbol{s}) = h(\boldsymbol{s}+\boldsymbol{e}_i), \quad (T_i\,h)(\boldsymbol{s}) := \begin{cases} h(\boldsymbol{s}-\boldsymbol{e}_i) & \text{if } s_i \leq 1 \\ \\ 0 & \text{if } s_i = 0, \end{cases}
$$

for functions $h(\boldsymbol{s})$ $(\boldsymbol{s}\in\Gamma_I)$. It is clear that

$$
T_i^*\,T_i = \mathbb{I}, \quad T_i\,T_i^* = (1-\delta_{s_i,0})\,\mathbb{I},
$$

where $\mathbb{I}$ stands for the identity operator. Sometimes it is helpful to think of the functions $h(\boldsymbol{s})$ as column vectors $(h|_{s_i=0}, h|_{s_i=1}, h|_{s_i=2},\ldots)^T$. Thus, in this representation, $T_i$, $T_i^*$ become the infinite-dimensional matrices

$$
T_i^* = \begin{pmatrix} 0 & 1 & 0 & \ldots & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad T_i = \begin{pmatrix} 0 & 0 & 0 & \ldots & \ldots \\ 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.
$$

## 3.1 The first system of string equations

From the asymptotic expansion (7) we have that as $z\to\infty$

$$
\begin{aligned}
(T_i\,f)\,f^{-1} &= \Big(I+\frac{a(\boldsymbol{s}-\boldsymbol{e}_i)}{z}+\mathcal{O}\Big(\frac{1}{z^2}\Big)\Big)\Big(z\,E_0+\frac{E_i}{z}+I-E_0-E_i\Big)\Big(I-\frac{a(\boldsymbol{s})}{z}+\mathcal{O}\Big(\frac{1}{z^2}\Big)\Big) \\
&= z\,E_0 + a(\boldsymbol{s}-\boldsymbol{e}_i)\,E_0 - E_0\,a(\boldsymbol{s}) + I - E_0 - E_i + \mathcal{O}\Big(\frac{1}{z}\Big), \quad \forall \boldsymbol{s}\in\Gamma_I+\boldsymbol{e}_i,
\end{aligned}
$$

where we are denoting

$$a(\boldsymbol{s}) := a_1(\boldsymbol{s}).$$

Hence by applying Proposition 2 it follows that

$$(T_i\, f)(\boldsymbol{s}, z) = \Big(z\, E_0 + a(\boldsymbol{s} - \boldsymbol{e}_i)\, E_0 - E_0\, a(\boldsymbol{s}) + I - E_0 - E_i\Big)\, f(\boldsymbol{s}, z), \quad \boldsymbol{s} \in \Gamma_I + \boldsymbol{e}_i$$

which implies

$$(T_i E_0 f)(\boldsymbol{s}, z) = \Big((z - u_i(\boldsymbol{s}))\, E_0 - \sum_j a_{0j}(\boldsymbol{s})\, E_{0j}\Big) f(\boldsymbol{s}, z), \quad \forall \boldsymbol{s} \in \Gamma_I + \boldsymbol{e}_i, \tag{15}$$

where

$$u_i(\boldsymbol{s}) := a_{00}(\boldsymbol{s}) - a_{00}(\boldsymbol{s} - \boldsymbol{e}_i).$$

Similarly one finds

$$(T_j^* E_0 f)(\boldsymbol{s}, z) = a_{0j}(\boldsymbol{s} + \boldsymbol{e}_j)\, E_{0j}\, f(\boldsymbol{s}, z), \quad \forall \boldsymbol{s} \in \Gamma_I. \tag{16}$$

Note that as $\det f(\boldsymbol{s}, z) \equiv 1$ for all $(\boldsymbol{s}, z) \in \Gamma_I \times \mathbb{C}$ then , as a consequence of (16) we deduce that

$$a_{0j}(\boldsymbol{s} + \boldsymbol{e}_j) \neq 0, \quad \forall \boldsymbol{s} \in \Gamma_I.$$

If we now define

$$v_j(\boldsymbol{s}) := \frac{a_{0j}(\boldsymbol{s})}{a_{0j}(\boldsymbol{s} + \boldsymbol{e}_j)}, \quad \boldsymbol{s} \in \Gamma_I, \tag{17}$$

then from (15) it follows that

**Proposition 3.** *The function $f$ satisfies the equations*

$$z\, (E_0\, f)(\boldsymbol{s}, z) = \Big(T_i + u_i(\boldsymbol{s}) + \sum_j v_j(\boldsymbol{s})\, T_j^*\Big)\, (E_0\, f)(\boldsymbol{s}, z), \tag{18}$$

*for all $\boldsymbol{s} \in \Gamma_I + \boldsymbol{e}_i$ and $i = 1, \ldots, q$.*

As a consequence we get the following system of string equations

**Theorem 1.** *The multiple orthogonal polynomials of type I verify*

$$z\, \boldsymbol{A}(\boldsymbol{n}, z) = \Big(T_i + u_i(\boldsymbol{n}) + \sum_j v_j(\boldsymbol{n})\, T_j^*\Big)\, \boldsymbol{A}(\boldsymbol{n}, z), \tag{19}$$

*for all $\boldsymbol{n} \in \Gamma_I + e_i$ and $i = 1, \ldots, q$.*

For $q = 1$ Eq.(19) reduces to the classical three-term recurrence relation for systems of orthogonal polynomials on the real line.

On the other hand Eq.(19) implies

$$\boldsymbol{A}(\boldsymbol{n} - \boldsymbol{e}_i, z) - \boldsymbol{A}(\boldsymbol{n} - \boldsymbol{e}_j, z) = (a_{00}(\boldsymbol{n} - \boldsymbol{e}_i) - a_{00}(\boldsymbol{n} - \boldsymbol{e}_j))\, \boldsymbol{A}(\boldsymbol{n}, z), \quad \forall \boldsymbol{s} \in \Gamma_I + \boldsymbol{e}_i + \boldsymbol{e}_j,\ i \neq j. \tag{20}$$

The relations (19) and (20) lead to a recursive method to construct the multiple orthogonal polynomials of type I. Indeed, it is clear that for $\boldsymbol{n} = n_i \boldsymbol{e}_i$ we have

$$\boldsymbol{A}(n_i \boldsymbol{e}_i, z) = A_1^{(i)}(n_i, z)\, \boldsymbol{e}_i = (0, \ldots, 0, A_1^{(i)}(n_i, z), 0, \ldots, 0),$$

where $A_1^{(i)}(n_i, z)$ are the orthogonal polynomials with respect to the weight $w_i(x)$. Then starting from $A_1^{(i)}(n_i, z)$ and using (20) we can generate the multiple orthogonal polynomials of type I for higher $q$.

**Example**

Let us denote by $I_{j,n}$ the moments with respect to the weight $w_j$

$$I_{j,n} := \int_{\mathbb{R}} \frac{dx}{2\pi i} x^n w_j(x). \tag{21}$$

We have that

$$A_1(1,z) = \frac{1}{I_{1,0}}, \qquad A_1(2,z) = \frac{I_{1,0}z - I_{1,1}}{I_{1,0}I_{1,2} - I_{1,1}^2}.$$

The recurrence relation (19) for $q = 1$ is

$$A_1(n+1,z) = \frac{a_{01}(n+1)}{a_{01}(n)}[(z + a_{00}(n-1) - a_{00}(n))A_1(n,z) - A_1(n-1,z)], \quad \forall n \geq 2, \tag{22}$$

where according to (13)

$$a_{00}(n) = \int_{\mathbb{R}} \frac{dx}{2\pi i} x^n A(n,x) w_1(x). \tag{23}$$

Moreover, the normalization condition gives us

$$\frac{a_{01}(n)}{a_{01}(n+1)} = \int \frac{dx}{2\pi i} x^n [(x + a_{00}(n-1) - a_{00}(n))A(n,x) - A(n-1,x)] w_1(x). \tag{24}$$

The system (22)-(23)-(24) allows us to construct the polynomials $A(n,z)$ for $n \geq 3$. For example one gets

$$A_1(3,z) = \frac{\left(I_{1,1}^2 - I_{1,0}I_{1,2}\right) z^2 - I_{1,1}I_{1,3} + (I_{1,0}I_{1,3} - I_{1,1}I_{1,2})z + I_{1,2}^2}{I_{1,2}^3 - (2I_{1,1}I_{1,3} + I_{1,0}I_{1,4})I_{1,2} + I_{1,0}I_{1,3}^2 + I_{1,1}^2 I_{1,4}}.$$

If we write (20) in the form

$$\boldsymbol{A}(\boldsymbol{n},z) = \frac{\boldsymbol{A}(\boldsymbol{n}-\boldsymbol{e}_i,z) - \boldsymbol{A}(\boldsymbol{n}-\boldsymbol{e}_j,z)}{a_{00}(\boldsymbol{n}-\boldsymbol{e}_i) - a_{00}(\boldsymbol{n}-\boldsymbol{e}_j)}, \quad \boldsymbol{n} \in \Gamma_I + \boldsymbol{e}_i + \boldsymbol{e}_j, \tag{25}$$

and take into account that

$$a_{00}(\boldsymbol{n}) = \int_{\mathbb{R}} \frac{dx}{2\pi i} x^{|\boldsymbol{n}|} \sum_{k=1}^{q} A_k(\boldsymbol{n},x) w_k(x). \tag{26}$$

we can construct all the multiple orthogonal polynomials of type I. Thus, for example for $q = 2$ we obtain

$$\boldsymbol{A}(1,1,z) = \frac{1}{C_1}\Big(I_{2,0}, -I_{1,0}\Big),$$

$$\boldsymbol{A}(2,1,z) = \frac{1}{C_2}\Big(I_{1,2}I_{2,0} - I_{1,1}I_{2,1} + z(I_{1,0}I_{2,1} - I_{1,1}I_{2,0})\,,\, I_{1,1}^2 - I_{1,0}I_{2,0}\Big),$$

where

$$C_1 := I_{1,1}I_{2,0} - I_{1,0}I_{2,1},$$

$$C_2 := I_{2,2}I_{1,1}^2 - I_{1,3}I_{2,0}I_{1,1} - I_{1,2}I_{2,1}I_{1,1} + I_{1,2}^2 I_{2,0} + I_{1,0}I_{1,3}I_{2,1} - I_{1,0}I_{1,2}I_{2,2}.$$

## 3.2 Lax operators

The functions $f_{0i}(\boldsymbol{s},z)=A_i(\boldsymbol{s},z)$ can be written as series expansions of the form

$$f_{0i}(\boldsymbol{s},z)=\left(\frac{\alpha_{i1}(\boldsymbol{s})}{z}+\frac{\alpha_{i2}(\boldsymbol{s})}{z^2}+\cdots\right)z^{s_i},$$

where

$$\alpha_{in}(\boldsymbol{s})=0,\ \forall n\geq s_i+1. \tag{27}$$

On the other hand, it is easy to see that

$$T_i^n\, z^{s_i}=\frac{1}{z^n}\,(z^{s_i}-\sum_{k=0}^{n-1} z^k\,\delta_{s_i-k,0}), \tag{28}$$

Hence, from (27) and (28) it is clear that

$$\frac{\alpha_{i,n+1}(\boldsymbol{s}+\boldsymbol{e}_i)}{z^n}\,z^{s_i}=\alpha_{i,n+1}(\boldsymbol{s}+\boldsymbol{e}_i)T_i^n\,z^{s_i},\quad \forall n\geq 1,$$

so that we may write

$$f_{0i}(\boldsymbol{s}+\boldsymbol{e}_i,z)=(G_i\,\xi_i)(\boldsymbol{s},z),\quad \xi_i(\boldsymbol{s},z):=z^{s_i},\quad \boldsymbol{s}\in\Gamma_I,$$

where the symbols $G_i$ are dressing operators defined by the expansions

$$G_i=\sum_{n\geq 1}\alpha_{in}(\boldsymbol{s}+\boldsymbol{e}_i)\,T_i^{n-1},\quad \alpha_{in}(\boldsymbol{s}+\boldsymbol{e}_i):=(a_n)_{0i}(\boldsymbol{s}+\boldsymbol{e}_i), \tag{29}$$

or, equivalently, by the triangular matrices

$$G_i=\begin{pmatrix} G_{00} & 0 & 0 & \dots & \dots \\ G_{10} & G_{11} & 0 & 0 & \dots \\ G_{20} & G_{21} & G_{22} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},\quad G_{nm}=\alpha_{i,n-m+1}(\boldsymbol{s}+\boldsymbol{e}_i)\Big|_{s_i=m}.$$

The inverse operators can be written as

$$G_i^{-1}:=\sum_{n\geq 1}\beta_{in}(\boldsymbol{s})\,T_i^{n-1},\quad \beta_{i1}(\boldsymbol{s})=\frac{1}{\alpha_{i1}(\boldsymbol{s}+\boldsymbol{e}_i)}=\frac{1}{a_{0i}(\boldsymbol{s}+\boldsymbol{e}_i)}.$$

We define the Lax operators $\mathcal{Z}_i$ by

$$\mathcal{Z}_i:=G_i\,T_i^*\,G_i^{-1}. \tag{30}$$

It follows at once that they can be expanded as

$$\mathcal{Z}_i=\gamma_i(\boldsymbol{s})\,T_i^*+\sum_{n\geq 0}\gamma_{in}(\boldsymbol{s})\,T_i^n, \tag{31}$$

where

$$\gamma_i(\boldsymbol{s})=\alpha_{i1}(\boldsymbol{s}+\boldsymbol{e}_i)\,(T_i^*\,\beta_{i1})(\boldsymbol{s})=\frac{\alpha_{i1}(\boldsymbol{s}+\boldsymbol{e}_i)}{\alpha_{i1}(\boldsymbol{s}+2\,\boldsymbol{e}_i)}=v_i(\boldsymbol{s}+\boldsymbol{e}_i). \tag{32}$$

**Proposition 4.** *The functions $f_{0i}$ satisfy the equations*

$$z\, f_{0i}(\boldsymbol{s}+\boldsymbol{e}_i, z) = (\mathcal{Z}_i\, f_{0i})(\boldsymbol{s}+\boldsymbol{e}_i, z), \quad \forall \boldsymbol{s} \in \Gamma_I. \tag{33}$$

*Proof.* From the definition of $G_i$ we have

$$z\, f_{0i}(\boldsymbol{s}+\boldsymbol{e}_i, z) = G_i\,(z\,\xi_i) = (G_i\, T_i^*)\,(\xi_i) = (\mathcal{Z}_i\, f_{0i})(\boldsymbol{s}+\boldsymbol{e}_i, z).$$

□

### 3.3 The second system of string equations

Let us consider diagonal solutions

$$\Phi(z) = \sum_\alpha \phi_\alpha(z)\, E_\alpha$$

of the condition (9) corresponding to the function $g(z)$ of (11). They are characterized by

$$\partial_z\, w_i - \phi_0\, w_i + \phi_i\, w_i = 0, \quad i = 1, \ldots, q.$$

In this way, by setting $\phi_0 \equiv 0$ we get

$$\Phi(z) = \sum_i V'(\boldsymbol{c}_i, z)\, E_i.$$

The corresponding covariant derivative is

$$D_z\, f := \partial_z f - \sum_i V'(\boldsymbol{c}_i, z)\, f\, E_i. \tag{34}$$

Hence we have

$$D_z(\,E_0\, f) = \partial_z f_{00}\, E_0 + \sum_i (\partial_z f_{0i} - V'(\boldsymbol{c}_i, z)\, f_{0i})\, E_{0i}. \tag{35}$$

It is clear that (33) implies

$$z^n\, f_{0i}(\boldsymbol{s}+\boldsymbol{e}_i, z) = (\mathcal{Z}_i^n\, f_{0i})(\boldsymbol{s}+\boldsymbol{e}_i, z), \quad \forall \boldsymbol{s} \in \Gamma_I. \tag{36}$$

On the other hand, as $z \to \infty$

$$\begin{aligned}
((T_j^*)^n\, f_{0\alpha})(\boldsymbol{s}, z) &= \begin{cases} \mathcal{O}\Big(\dfrac{1}{z^n}\Big)\, z^{s_0}, \text{ for } \alpha = 0, \\[2ex] \mathcal{O}\Big(\dfrac{1}{z}\Big)\, z^{s_i}, \text{ for } \alpha = i \neq j\ , \end{cases} \quad , \quad n \geq 1, \\[3ex]
(T_i^n\, f_{0i})(\boldsymbol{s}, z) &= \begin{cases} \mathcal{O}\Big(\dfrac{1}{z^{n+1}}\Big)\, z^{s_i}, \text{ for } s_i \geq n, \\[2ex] 0, \text{ for } s_i < n \end{cases} \quad , \quad n \geq 0.
\end{aligned} \tag{37}$$

**Proposition 5.** *The function $f$ satisfies the equation*

$$(D_z + \mathcal{H})\,(E_0\, f)(\boldsymbol{s}, z) = 0, \quad \forall \boldsymbol{s} \in \Gamma_I + \sum_j \boldsymbol{e}_j, \tag{38}$$

*where $\mathcal{H}$ is the operator*

$$\mathcal{H} := \sum_{j=1}^{q} V'(\boldsymbol{c}_j, \mathcal{Z}_j)_{(j,+)} \tag{39}$$

*Proof.* Given $\boldsymbol{s} \in \Gamma_I + \sum_j \boldsymbol{e}_j$ let us denote

$$\boldsymbol{s}^{(i)} := \boldsymbol{s} - \boldsymbol{e}_i \in \Gamma_I + \sum_{k \neq i} \boldsymbol{e}_k.$$

From (35) it follows that

$$\begin{aligned}(D_z + \mathcal{H})\,(E_0\, f) \;&=\; \left[\partial_z f_{00} + \sum_{j=1}^{q} V'(\boldsymbol{c}_j, \mathcal{Z}_j)_{(j,+)}\, f_{00}\right] E_0 \\ &+ \sum_{i=1}^{q} \left[\partial_z f_{0i} + \sum_{j=1}^{q} V'(\boldsymbol{c}_j, \mathcal{Z}_j)_{(j,+)}\, f_{0i} - V'(\boldsymbol{c}_i, z)\, f_{0i}\right] E_{0i}\end{aligned} \tag{40}$$

Now from (37) we have

$$\partial_z f_{00} + \sum_{j=1}^{q} V'(\boldsymbol{c}_j, \mathcal{Z}_j)_{(j,+)}\, f_{00} = \mathcal{O}\Big(\frac{1}{z}\Big)\, z^{s_0},$$

$$\partial_z f_{0i} + \sum_{j \neq i} V'(\boldsymbol{c}_j, \mathcal{Z}_j)_{(j,+)}\, f_{0i} = \mathcal{O}\Big(\frac{1}{z}\Big)\, z^{s_i}.$$

$$\Big(V'(\boldsymbol{c}_i, \mathcal{Z}_i)_{(i,+)} - V'(\boldsymbol{c}_i, z)\Big)\, f_{0i}(\boldsymbol{s}, z) = \Big(V'(\boldsymbol{c}_i, \mathcal{Z}_i) - V'(\boldsymbol{c}_i, z)\Big)\, f_{0i}(\boldsymbol{s}, z) + \mathcal{O}\Big(\frac{1}{z}\Big)\, z^{s_i},$$

Moreover, from (36) it is clear that

$$\Big(V'(\boldsymbol{c}_i, \mathcal{Z}_i) - V'(\boldsymbol{c}_i, z)\Big)\, f_{0i}(\boldsymbol{s}, z) = \Big(V'(\boldsymbol{c}_i, \mathcal{Z}_i) - V'(\boldsymbol{c}_i, z)\Big)\, f_{0i}(\boldsymbol{s}^{(i)} + \boldsymbol{e}_i, z)$$

$$= \sum_{n \geq 1} n\, c_{in}\, \Big(\mathcal{Z}_i^{n-1} - z^{n-1}\Big)\, f_{0i}(\boldsymbol{s}^{(i)} + \boldsymbol{e}_i, z) = 0$$

Therefore we find

$$(D_z + \mathcal{H})\,(E_0\, f)(\boldsymbol{s}, z) = \mathcal{O}(\frac{1}{z})\, f_0(\boldsymbol{s}, z), \quad z \to \infty.$$

The first member $\tilde{f} := (D_z + \mathcal{H})\,(E_0\, f)$ of this equation is a solution of (5) for all $\boldsymbol{s} \in \Gamma_I + \sum_j \boldsymbol{e}_j$ and $\tilde{f}(\boldsymbol{s}, z)\, f(\boldsymbol{s}, z)^{-1} \to 0$ as $z \to \infty$. Therefore, the statement of Proposition 2 implies $\tilde{f} \equiv 0$ . □

As a consequence we deduce the following system of string equations

**Theorem 2.** *The multiple orthogonal polynomials of type I verify*

$$\partial_z A_i(\boldsymbol{n}, z) = V'(\boldsymbol{c}_i, \mathcal{Z}_i)\, A_i(\boldsymbol{n}, z) - \sum_{j=1}^{q} V'(\boldsymbol{c}_j, \mathcal{Z}_j)_{(j,+)}\, A_i(\boldsymbol{n}, z), \tag{41}$$

*for all* $\boldsymbol{n} \in \Gamma_I + \sum_k \boldsymbol{e}_k$ *and* $i = 1, \dots, q$.

## 3.4 Orlov operators

We define the Orlov operators $\mathcal{M}_i$ by

$$\mathcal{M}_i := G_i \cdot s_i \cdot T_i \cdot G_i^{-1}. \tag{42}$$

They satisfy $[\mathcal{Z}_i, \mathcal{M}_i] = \mathbb{I}$ and can be expanded as

$$\mathcal{M}_i = \sum_{n \geq 1} \mu_{in}(\boldsymbol{s})\, T_i^n, \tag{43}$$

where

$$\mu_{i1}(\boldsymbol{s}) = \frac{s_i}{v_i(\boldsymbol{s})}. \tag{44}$$

**Proposition 6.** *The functions* $f_{0i}$ *satisfy the equations*

$$\partial_z f_{0i}(\boldsymbol{s} + \boldsymbol{e}_i, z) = (\mathcal{M}_i\, f_{0i})(\boldsymbol{s}, z + \boldsymbol{e}_i), \quad \forall \boldsymbol{s} \in \Gamma_I. \tag{45}$$

*Proof.* From the definition of $G_i$ we have

$$\partial_z f_{0i}(\boldsymbol{s} + \boldsymbol{e}_i, z) = G_i\,(s_i\, z^{-1}\, \xi_i) = G \cdot s_i(T_i\, \xi_i) = G_i \cdot s_i \cdot T_i \cdot G_i^{-1} f_{0i} = \mathcal{M}_i\, f_{0i}(\boldsymbol{s} + \boldsymbol{e}_i, z).$$

□

# 4 Multiple orthogonal polynomials of type II

We consider now $q$ exponential weights $w_i$ on the real line

$$w_i(x) := e^{V(\boldsymbol{c}_i, x)}, \quad \boldsymbol{c}_i = (c_{i1}, c_{i2}, \dots) \in \mathbb{C}^\infty.$$

Note the difference in the sign of the exponents with respect to the weights for multiple orthogonal polynomials of type I. Given $\boldsymbol{n} = (n_1, \dots, n_q) \in \mathbb{N}^q$, the associated type II monic orthogonal polynomial $P(\boldsymbol{n}, x) = x^{|\boldsymbol{n}|} + \cdots$ is determined by the conditions

$$\int_{\mathbb{R}} P(\boldsymbol{n}, x)\, w_i(x)\, x^j\, \mathrm{d}x = 0, \quad j = 0, \dots, n_i - 1.$$

We assume that all the multi-indices $\boldsymbol{n}$ are strongly normal [17] so that $P(\boldsymbol{n}, z)$ is unique.

The RH problem for the multiple orthogonal polynomials of type II is determined by [17]

$$g(z) = \begin{pmatrix} 1 & w_1(z) & w_2(z) & \dots & w_q(z) \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \tag{46}$$

Its fundamental solution $f(\boldsymbol{s},z)$ exists on the domain

$$\Gamma_{II} = \{\boldsymbol{s} \in \mathbb{Z}^q \,:\, s_i \leq 0,\, \forall i = 1,\dots,q\}, \tag{47}$$

For $s_i \leq -1\, \forall i$, it is given by

$$f(\boldsymbol{s},z) = \begin{pmatrix} P(\boldsymbol{n},z) & \boldsymbol{R}(\boldsymbol{n},z) \\ d_1\, P(\boldsymbol{n}-\boldsymbol{e}_1,z) & d_1\, \boldsymbol{R}(\boldsymbol{n}-\boldsymbol{e}_1,z) \\ \vdots & \vdots \\ d_q\, P(\boldsymbol{n}-\boldsymbol{e}_q,z) & d_q\, \boldsymbol{R}(\boldsymbol{n}-\boldsymbol{e}_q,z) \end{pmatrix}, \quad \boldsymbol{s} = -\boldsymbol{n},$$

$$R_j(\boldsymbol{n},z) := \int_{\mathbb{R}} \frac{\mathrm{d}x}{2\pi i} \frac{P(\boldsymbol{n},x)\, w_j(x)}{x-z}, \qquad \frac{1}{d_j} := -\int_{\mathbb{R}} \frac{\mathrm{d}x}{2\pi i} P(\boldsymbol{n}-\boldsymbol{e}_j,x)\, w_j(x)\, x^{n_j-1}. \tag{48}$$

For the remaining cases, in which one or several $s_i$ vanish, one must insert the following corresponding row substitutions in (48)

$$(d_i\, P(\boldsymbol{n}-\boldsymbol{e}_i,z) \quad d_i\, \boldsymbol{R}(\boldsymbol{n}-\boldsymbol{e}_i,z)) \longrightarrow (0 \quad \boldsymbol{e}_i). \tag{49}$$

In particular

$$f(\boldsymbol{0},z) = \begin{pmatrix} 1 & R_1(z) & R_2(z) & \cdots & R_q(z) \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad R_j(z) := \int_{\mathbb{R}} \frac{\mathrm{d}x}{2\pi i} \frac{w_j(x)}{x-z}.$$

In view of (47) we have that

$$(T_i\, h)(\boldsymbol{s}) = h(\boldsymbol{s}-\boldsymbol{e}_i), \quad (T_i^*\, h)(\boldsymbol{s}) := \begin{cases} h(\boldsymbol{s}+\boldsymbol{e}_i) & \text{if } s_i \leq -1 \\ \\ 0 & \text{if } s_i = 0, \end{cases}$$

for functions $h(\boldsymbol{s})\,(\boldsymbol{s} \in \Gamma_{II})$. Note also that

$$T_i\, T_i^* = \mathbb{I}, \quad T_i^*\, T_i = (1-\delta_{s_i,0})\, \mathbb{I}$$

where $\mathbb{I}$ stands for the identity operator. If we think of $h(\boldsymbol{s})$ as a column vector $(h|_{s_i=0}, h|_{s_i=-1}, h|_{s_i=-2}, \dots)^T$, then $T_i$, $T_i^*$ are represented by the infinite-dimensional matrices

$$T_i = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad T_i^* = \begin{pmatrix} 0 & 0 & 0 & \dots & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

### 4.1 The first system of string equations

The same analysis as in the subsection 3.1 leads now to the equations

$$(T_i E_0 f)(\boldsymbol{s}, z) = \Big((z - u_i(\boldsymbol{s}))\, E_0 - \sum_j a_{0j}(\boldsymbol{s})\, E_{0j}\Big) f(\boldsymbol{s}, z), \quad \forall \boldsymbol{s} \in \Gamma_{II}, \tag{50}$$

where

$$u_i(\boldsymbol{s}) := a_{00}(\boldsymbol{s}) - a_{00}(\boldsymbol{s} - \boldsymbol{e}_i).$$

Similarly one finds

$$(T_j^* E_0 f)(\boldsymbol{s}, z) = a_{0j}(\boldsymbol{s} + \boldsymbol{e}_j)\, E_{0j}\, f(\boldsymbol{s}, z), \quad \forall \boldsymbol{s} \in \Gamma_{II} - \boldsymbol{e}_i, \tag{51}$$

and taking into account that $\det f(\boldsymbol{s}, z) \equiv 1$ for all $(\boldsymbol{s}, z) \in \Gamma_{II} \times \mathbb{C}$, from (51) we obtain

$$a_{0j}(\boldsymbol{s}) \neq 0, \quad \forall \boldsymbol{s} \in \Gamma_{II}.$$

Now we define

$$v_j(\boldsymbol{s}) := \begin{cases} \dfrac{a_{0j}(\boldsymbol{s})}{a_{0j}(\boldsymbol{s} + \boldsymbol{e}_j)}, & \boldsymbol{s} \in \Gamma_{II} - \boldsymbol{e}_i, \\ 0, \quad \text{for } s_j = 0. \end{cases} \tag{52}$$

Notice that the functions $v_j(\boldsymbol{s})\,(\boldsymbol{s} \in \Gamma_I)$ for multiple orthogonal polynomials of type I defined in (17) also satisfies $v_j(\boldsymbol{s}) = 0$ for $s_j = 0$.

If we now recall that according to (49)

$$E_{0j}\, f(\boldsymbol{s}, z) = E_{0j}, \quad \text{for } s_j = 0,$$

from (50) it follows that

**Proposition 7.** *The function $f$ satisfies the equations*

$$z\, E_0\, f(\boldsymbol{s}, z) = \Big(T_i + u_i(\boldsymbol{s}) + \sum_j v_j(\boldsymbol{s})\, T_j^*\Big)\, (E_0\, f)(\boldsymbol{s}, z) + \sum_j \delta_{s_j,0}\, a_{0j}(\boldsymbol{s})\, E_{0j}, \tag{53}$$

*for all $\boldsymbol{s} \in \Gamma_{II}$ and $i = 1 \dots q$.*

As a consequence we get the string equations

**Theorem 3.** *The multiple orthogonal polynomials of type II verify*

$$z\, P(\boldsymbol{n}, z) = \Big(T_i + u_i(-\boldsymbol{n}) + \sum_j v_j(-\boldsymbol{n})\, T_j^*\Big)\, P(\boldsymbol{n}, z), \tag{54}$$

*for all $\boldsymbol{n}$ and $i = 1, \dots, q$.*

These equation provide a recursive method to construct multiple orthogonal polynomials of type II. We may write (54) as

$$P(\boldsymbol{n} + \boldsymbol{e}_j, z) - a_{00}(\boldsymbol{n} + \boldsymbol{e}_j) P(\boldsymbol{n}, z) = (z - a_{00}(\boldsymbol{n})) P(\boldsymbol{n}, z) - \sum_{k=1, n_k \geq 1}^{q} \frac{a_{0k}(\boldsymbol{n})}{a_{0k}(\boldsymbol{n} - e_k)} P(\boldsymbol{n} - e_k, z), \tag{55}$$

where, according to(48), we have that

$$
\begin{aligned}
a_{00}(\boldsymbol{n}) &= \text{coeff}[P(\boldsymbol{n},z), z^{|\boldsymbol{n}|-1}],\\
a_{0k}(\boldsymbol{n}) &= -\int_{\mathbb{R}} \frac{dx}{2\pi i} P(\boldsymbol{n},x) x^{n_k} w_k(x) dx.
\end{aligned}
\tag{56}
$$

On the other hand, multiplying the equation (55) by $z^{n_j} w_j(z)$, integrating on $\mathbb{R}$ and using the orthogonality condition for $P(\boldsymbol{n}+\boldsymbol{e}_j, z)$, we obtain

$$
a_{00}(\boldsymbol{n}+e_j)\left[-\int_{\mathbb{R}} \frac{dx}{2\pi i} P(\boldsymbol{n},x) x^{n_j} w_j(x)\right] =
$$

$$
\int_{\mathbb{R}} \left[(x - a_{00}(\boldsymbol{n}))P(\boldsymbol{n},x) - \sum_{k=1, n_k \geq 1}^{q} \frac{a_{0k}(\boldsymbol{n})}{a_{0k}(\boldsymbol{n}-e_k)} P(\boldsymbol{n}-e_k, x)\right] x^{n_j} w_j(x) dx,
$$

so that

$$
a_{00}(\boldsymbol{n}+e_j) = \frac{1}{a_{0j}(\boldsymbol{n})} \int_{\mathbb{R}} \left[(x - a_{00}(\boldsymbol{n}))P(\boldsymbol{n},x) - \sum_{k=1, n_k \geq 1}^{q} \frac{a_{0k}(\boldsymbol{n})}{a_{0k}(\boldsymbol{n}-e_k)} P(\boldsymbol{n}-e_k, x)\right] x^{n_j} w_j(x) dx.
\tag{57}
$$

The system (55)-(56)-(57) determines the multiple orthogonal polynomials of type II in terms of the moments $I_{j,n}$ .

## Example

For $q = 1$ is clear that

$$
P(0,z) = 1, \qquad P(1,z) = z - \frac{I_{1,1}}{I_{1,0}}.
$$

From (55)-(56)-(57) we easily obtain that

$$
\begin{aligned}
P(2,z) &= z^2 + \frac{(I_{1,0}I_{1,3} - I_{1,1}I_{1,2})z}{I_{1,1}^2 - I_{1,0}I_{1,2}} + \frac{I_{1,2}^2 - I_{1,1}I_{1,3}}{I_{1,1}^2 - I_{1,0}I_{1,2}},\\
P(3,z) &= z^3 + \frac{\left(-I_{1,5}I_{1,1}^2 + I_{1,3}^2 I_{1,1} + I_{1,2}I_{1,4}I_{1,1} - I_{1,2}^2 I_{1,3} - I_{1,0}I_{1,3}I_{1,4} + I_{1,0}I_{1,2}I_{1,5}\right) z^2}{I_{1,2}^3 - (2I_{1,1}I_{1,3} + I_{1,0}I_{1,4})I_{1,2} + I_{1,0}I_{1,3}^2 + I_{1,1}^2 I_{1,4}}\\
&+ \frac{\left(-I_{1,4}I_{1,2}^2 + I_{1,3}^2 I_{1,2} + I_{1,1}I_{1,5}I_{1,2} + I_{1,0}I_{1,4}^2 - I_{1,1}I_{1,3}I_{1,4} - I_{1,0}I_{1,3}I_{1,5}\right) z}{I_{1,2}^3 - 2I_{1,1}I_{1,3}I_{1,2} - I_{1,0}I_{1,4}I_{1,2} + I_{1,0}I_{1,3}^2 + I_{1,1}^2 I_{1,4}}\\
&- \frac{I_{1,3}^3 - 2I_{1,2}I_{1,4}I_{1,3} - I_{1,1}I_{1,5}I_{1,3} + I_{1,1}I_{1,4}^2 + I_{1,2}^2 I_{1,5}}{I_{1,2}^3 - 2I_{1,1}I_{1,3}I_{1,2} - I_{1,0}I_{1,4}I_{1,2} + I_{1,0}I_{1,3}^2 + I_{1,1}^2 I_{1,4}}.
\end{aligned}
$$

To determine the orthogonal polynomials for $q \geq 2$ we use the property

$$
P(n_i \boldsymbol{e}_i, z) = P^{(i)}(n_i, z),
$$

where $P^{(i)}(n_i, z)$ are the orthogonal polynomials for $q = 1$ with respect to the weight $w_i(x)$. For example for $q = 2$ and $j = 2$, Eq.(55) yields

$$
\begin{aligned}
P(1,1,z) &= z^2 + \frac{(I_{1,2}I_{2,0} - I_{1,0}I_{2,2})z}{I_{1,0}I_{2,1} - I_{1,1}I_{2,0}} + \frac{I_{1,2}I_{2,1} - I_{1,1}I_{2,2}}{I_{1,1}I_{2,0} - I_{1,0}I_{2,1}}, \\
P(2,1,z) &= z^3 + \frac{\left(-I_{2,3}I_{1,1}^2 + I_{1,4}I_{2,0}I_{1,1} + I_{1,3}I_{2,1}I_{1,1} - I_{1,2}I_{1,3}I_{2,0} - I_{1,0}I_{1,4}I_{2,1} + I_{1,0}I_{1,2}I_{2,3}\right) z^2}{I_{2,2}I_{1,1}^2 - I_{1,3}I_{2,0}I_{1,1} + I_{1,2}^2 I_{2,0} + I_{1,0}I_{1,3}I_{2,1} - I_{1,2}(I_{1,1}I_{2,1} + I_{1,0}I_{2,2})} \\
&\quad + \frac{\left(I_{2,0}I_{1,3}^2 - I_{1,1}I_{2,2}I_{1,3} - I_{1,0}I_{2,3}I_{1,3} - I_{1,2}I_{1,4}I_{2,0} + I_{1,0}I_{1,4}I_{2,2} + I_{1,1}I_{1,2}I_{2,3}\right) z}{I_{2,2}I_{1,1}^2 - I_{1,3}I_{2,0}I_{1,1} - I_{1,2}I_{2,1}I_{1,1} + I_{1,2}^2 I_{2,0} + I_{1,0}I_{1,3}I_{2,1} - I_{1,0}I_{1,2}I_{2,2}} \\
&\quad + \frac{I_{2,3}I_{1,2}^2 - I_{1,4}I_{2,1}I_{1,2} + I_{1,3}^2 I_{2,1} + I_{1,1}I_{1,4}I_{2,2} - I_{1,3}(I_{1,2}I_{2,2} + I_{1,1}I_{2,3})}{-I_{2,2}I_{1,1}^2 + I_{1,3}I_{2,0}I_{1,1} + I_{1,2}I_{2,1}I_{1,1} - I_{1,2}^2 I_{2,0} - I_{1,0}I_{1,3}I_{2,1} + I_{1,0}I_{1,2}I_{2,2}}.
\end{aligned}
$$

## 4.2 Lax operators

Let us introduce dressing operators $G_i$ according to

$$f_{0i}(\boldsymbol{s}, z) = (G_i\, \xi_i)(\boldsymbol{s}, z), \quad G_i := \sum_{n\geq 0} \alpha_{in}(\boldsymbol{s})\, T_i^n, \quad \alpha_{in}(\boldsymbol{s}) := (a_{n+1})_{0i}(\boldsymbol{s}),$$

where $\boldsymbol{s} \in \Gamma_{II}$ and $\xi_i(\boldsymbol{s}, z) := z^{s_i - 1}$. In the matrix representation they are given by the triangular matrices

$$G_i = \begin{pmatrix} G_{00} & G_{01} & G_{02} & \dots & \dots \\ 0 & G_{11} & G_{12} & G_{13} & \dots \\ 0 & 0 & G_{22} & G_{23} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad G_{nm} = \alpha_{i,m-n}(\boldsymbol{s})\Big|_{s_i=-m}.$$

The corresponding inverse operators are characterized by expansions of the form

$$G_i^{-1} := \sum_{n\geq 0} \beta_{in}(\boldsymbol{s})\, T_i^n, \quad \beta_{i0}(\boldsymbol{s}) = \frac{1}{\alpha_{i0}(\boldsymbol{s})} = \frac{1}{a_{0i}(\boldsymbol{s})}.$$

We define the Lax operators $\mathcal{Z}_i$ by

$$\mathcal{Z}_i := G_i\, T_i^*\, G_i^{-1}. \tag{58}$$

It follows at once that they can be expanded as

$$\mathcal{Z}_i = \gamma_i(\boldsymbol{s})\, T_i^* + \sum_{n\geq 0} \gamma_{in}(\boldsymbol{s})\, T_i^n, \tag{59}$$

where

$$\gamma_i(\boldsymbol{s}) = \alpha_{i0}(\boldsymbol{s})\, (T_i^*\beta_{i0})(\boldsymbol{s}) = v_i(\boldsymbol{s}). \tag{60}$$

**Proposition 8.** *The functions $f_{0i}$ satisfy the equations*

$$z\, f_{0i}(\boldsymbol{s},z) = (\mathcal{Z}_i\, f_{0i})(\boldsymbol{s},z) + a_{0i}(\boldsymbol{s})\, \delta_{s_i 0}, \quad \forall \boldsymbol{s}\in\Gamma_{II}. \tag{61}$$

*Proof.* From the definition of $G_i$ we have

$$z\, f_{0i}(\boldsymbol{s},z) = G_i\,(z\,\xi_i) = G_i\,(T_i^*\,(\xi_i) + \delta_{s_i 0}) = (\mathcal{Z}_i\, f_{0i})(\boldsymbol{s},z) + \alpha_{i0}(\boldsymbol{s})\,\delta_{s_i 0},$$

where we have taken into account that

$$T_i^n(\delta_{s_i 0}) = \delta_{s_i - n, 0} = 0, \quad \forall n\geq 1,\ \boldsymbol{s}\in\Gamma_{II}.$$

□

## 4.3 The second system of string equations

The diagonal solutions

$$\Phi(z) = \sum_\alpha \phi_\alpha(z)\, E_\alpha$$

of the condition (9) corresponding to the function $g(z)$ of (46) are characterized by

$$\partial_z\, w_i + \phi_0\, w_i - \phi_i\, w_i = 0, \quad i = 1,\ldots,q.$$

Hence, setting $\phi_0\equiv 0$ we get

$$\Phi(z) = \sum_i V'(\boldsymbol{c}_i, z)\, E_i. \tag{62}$$

The corresponding covariant derivative is

$$D_z\, f := \frac{\partial f}{\partial z} - \sum_i V'(\boldsymbol{c}_i,z)\, f\, E_i, \tag{63}$$

so that we may write

$$D_z(\,E_0\, f) = \partial_z f_{00}\, E_0 + \sum_i (\partial_z f_{0i} - V'(\boldsymbol{c}_i,z)\, f_{0i})\, E_{0i}. \tag{64}$$

In order to take advantage of the last identity we observe that (61) can be generalized to

$$z^n\, f_{0i}(\boldsymbol{s},z) = (\mathcal{Z}_i^n\, f_{0i})(\boldsymbol{s},z) - \sum_{r=0}^{n-1} p_{(i,r)}^{(n)}(\boldsymbol{s},z)\,\delta_{s_i+r,0}, \quad \forall \boldsymbol{s}\in\Gamma_{II}. \tag{65}$$

where the coefficients $p_{(i,r)}^{(n)}(\boldsymbol{s},z)$ are polynomials in $z$. On the other hand we have that as $z\to\infty$

$$((T_j^*)^n\, f_{0\alpha})(\boldsymbol{s},z) = \begin{cases} \mathcal{O}\Big(\dfrac{1}{z^n}\Big)\, z^{s_0}, \text{ for } \alpha = 0,\ s_j\leq -n, \\[2ex] \mathcal{O}\Big(\dfrac{1}{z}\Big)\, z^{s_i}, \text{ for } \alpha = i\neq j \text{ and } s_j \leq -n, \\[2ex] 0, \text{ for } s_j > -n \end{cases}, \quad n\geq 1, \tag{66}$$

$$(T_i^n\, f_{0i})(\boldsymbol{s},z) = \mathcal{O}\Big(\frac{1}{z^{n+1}}\Big)\, z^{s_i}, \quad n\geq 0.$$

We are now ready to prove the following result.

**Proposition 9.** *The function $f$ satisfies the equation*

$$(D_z + \mathcal{H})\,(E_0\,f)(\boldsymbol{s}, z) = \sum_{i=1}^{q} \Delta_i(\boldsymbol{s}, z)\,E_{0i}, \quad \forall \boldsymbol{s} \in \Gamma_{II}, \tag{67}$$

*where $\mathcal{H}$ is the operator*

$$\mathcal{H} := \sum_{j=1}^{q} V'(\boldsymbol{c}_j, \mathcal{Z}_j)_{(j,+)}, \tag{68}$$

*and $\Delta_i(\boldsymbol{s}, z)$ are functions of the form*

$$\Delta_i(\boldsymbol{s}, z) = \sum_{n=1}^{N_i} p_{(i,n)}(\boldsymbol{s}, z)\,\delta_{s_i+n,0}. \tag{69}$$

*with $p_{(i,n)}(\boldsymbol{s}, z)$ being polynomials in $z$ and $N_i = degree\,V(\boldsymbol{c}_i, z) - 2$.*

*Proof.* From (64) it follows that

$$\begin{aligned} D_z(E_0\,f)(\boldsymbol{s}, z) + \mathcal{H}(E_0\,f)(\boldsymbol{s}, z) \;=&\; \left[\partial_z f_{00} + \sum_{j=1}^{q} V'(\boldsymbol{c}_j, \mathcal{Z}_j)_{(j,+)}\,f_{00}\right] E_0 \\ &+ \sum_{i=1}^{q} \left[\partial_z f_{0i} + \sum_{j=1}^{q} V'(\boldsymbol{c}_j, \mathcal{Z}_j)_{(j,+)}\,f_{0i} - V'(\boldsymbol{c}_i, z)\,f_{0i}\right] E_{0i}. \end{aligned} \tag{70}$$

Using (66) we find

$$\begin{aligned} &\partial_z f_{00} + \sum_{j=1}^{q} V'(\boldsymbol{c}_j, \mathcal{Z}_j)_{(j,+)}\,f_{00} = \mathcal{O}\Big(\frac{1}{z}\Big)\,z^{s_0}, \\ &\partial_z f_{0i} + \sum_{j\neq i} V'(\boldsymbol{c}_j, \mathcal{Z}_j)_{(j,+)}\,f_{0i} = \mathcal{O}\Big(\frac{1}{z}\Big)\,z^{s_i}, \\ &\Big(V'(\boldsymbol{c}_i, \mathcal{Z}_i)_{(i,+)} - V'(\boldsymbol{c}_i, z)\Big)\,f_{0i} = \Big(V'(\boldsymbol{c}_i, \mathcal{Z}_i) - V'(\boldsymbol{c}_i, z)\Big)\,f_{0i} + \mathcal{O}\Big(\frac{1}{z}\Big)\,z^{s_i}. \end{aligned} \tag{71}$$

On the other hand (65) implies

$$\Big(V'(\boldsymbol{c}_i, \mathcal{Z}_i) - V'(\boldsymbol{c}_i, z)\Big)\,f_{0i} = \sum_{n\geq 1} n\,c_{in}\,(\mathcal{Z}_i^{n-1} - z^{n-1})\,f_{0i} = \Delta_i(\boldsymbol{s}, z)\,f_{0i}, \tag{72}$$

where

$$\Delta_i(\boldsymbol{s}, z) := \sum_{n\geq 1} n\,c_{in} \sum_{r=0}^{n-2} p_{(i,r)}^{(n-1)}(\boldsymbol{s}, z)\,\delta_{s_i+r,0}. \tag{73}$$

Hence Eq.(70) says that

$$(D_z + \mathcal{H})\,(E_0\,f)(\boldsymbol{s}, z) - \sum_{i=1}^{q} \Delta_i(\boldsymbol{s}, z)\,E_{0i} = \mathcal{O}(\frac{1}{z})\,f_0(\boldsymbol{s}, z).$$

The first member of this equation is a solution of the Riemann-Hilbert problem for all $\boldsymbol{s} \in \Gamma_{II}$ so that from Proposition 2 the statement follows. □

As a consequence we deduce the string equations

**Theorem 4.** *The multiple orthogonal polynomials of type II verify*

$$\partial_z P(\boldsymbol{n}, z) + \sum_{j=1}^{q} V'(\boldsymbol{c}_j, \mathcal{Z}_j)_{(j,+)} P(\boldsymbol{n}, z) = 0. \tag{74}$$

### 4.4 Orlov operators

We define the Orlov operators $\mathcal{M}_i$ by

$$\mathcal{M}_i := G_i \cdot (s_i - 1) \cdot T_i \cdot G_i^{-1}. \tag{75}$$

They satisfy $[\mathcal{Z}_i, \mathcal{M}_i] = \mathbb{I}$ and can be expanded as

$$\mathcal{M}_i = \sum_{n \geq 1} \mu_{in}(\boldsymbol{s})\, T_i^n, \tag{76}$$

where

$$\mu_{i1}(\boldsymbol{s}) = \frac{s_i - 1}{v_i(\boldsymbol{s} - \boldsymbol{e}_i)}. \tag{77}$$

**Proposition 10.** *The functions $f_{0i}$ satisfy the equations*

$$\partial_z f_{0i}(\boldsymbol{s}, z) = (\mathcal{M}_i\, f_{0i})(\boldsymbol{s}, z), \quad \forall \boldsymbol{s} \in \Gamma_{II}. \tag{78}$$

*Proof.* From the definition of $G_i$ we have

$$\partial_z f_{0i} = G_i\, ((s_i - 1)\, z^{-1}\, \xi_i) = G \cdot (s_i - 1)(T_i\, \xi_i) = G_i \cdot (s_i - 1) \cdot T_i \cdot G_i^{-1} f_{0i} = \mathcal{M}_i\, f_{0i}.$$

□

## 5 The large-$\boldsymbol{n}$ limit

The large-$\boldsymbol{n}$ limit of multiple orthogonal polynomials is closely connected to the *quasiclassical* limit of the functions $f_{0\alpha}(\boldsymbol{s}, z)$. In this section we will consider these functions for large values of the discrete parameters $s_i$

$$s_i >> 1, \forall i \quad \text{(Type I case)}; \quad s_i << -1, \forall i \quad \text{(Type II case)}.$$

Note that in particular the string equations (53) and (67) simplify since all the $\delta$ terms vanish. As a consequence the resulting equations are the same for both types of multiple orthogonal polynomials and can be summarized as follows:

$$\begin{cases} z\, f_{0\alpha} = \Big(T_i + u_i(\boldsymbol{s}) + \sum_j v_j(\boldsymbol{s})\, T_j^*\Big)\, f_{0\alpha}, \quad \forall \alpha, i; \\ \\ \partial_z f_{00} = -\mathcal{H}\, f_{00}, \quad \partial_z f_{0i} = \Big( -\mathcal{H} + V'(\boldsymbol{c}_i, \mathcal{Z}_i)\Big)\, f_{0i}. \end{cases} \tag{79}$$

In order to define the large-$\boldsymbol{n}$ limit we introduce an small parameter $\epsilon$, define *slow variables*

$$t_i := \epsilon\, s_i,; \quad t_0 := -\sum_{i=1}^{q} t_i, \quad \boldsymbol{t} := (t_1, \dots, t_q), \tag{80}$$

and rescale the exponents of the weight functions (11) and (46) as

$$w_i(\epsilon, z) = \exp\Big( \mp \frac{V(\boldsymbol{c}_i, z)}{\epsilon}\Big),$$

where the exponent sign is negative (positive) for polynomials of type I (type II). Moreover, we perform a continuum limit in which as $\epsilon \to 0$, the discrete parameters $s_i$ tend to $+\infty$ $(-\infty)$ for the type I case (type II case) and $t_\alpha$ become continuous variables.

The problem now is to determine solutions $f_{0\alpha}(\epsilon, \boldsymbol{t}, z)$ of (79) defined for $\boldsymbol{t}$ on some domain $\Omega$ of $\mathbb{R}^q$, that have the *quasiclassical* form [16]

$$f_{0\alpha}(\epsilon, \boldsymbol{t}, z) = z^{\delta_{0\alpha}-1} \exp\Big(\frac{1}{\epsilon}\, \mathbb{S}_\alpha\Big), \quad \mathbb{S}_\alpha = t_\alpha \, \log z + \sum_{n\geq 0} \frac{1}{z^n} \mathbb{S}_{\alpha n}, \tag{81}$$

where

$$\mathbb{S}_{\alpha n} = \sum_{k\geq 0} \epsilon^k\, \mathbb{S}^{(k)}_{\alpha n}(\boldsymbol{t}), \quad n \geq 0; \quad \mathbb{S}_{00} \equiv 0.$$

Note the leading behaviour

$$f_{0\alpha}(\epsilon, \boldsymbol{t}, z) = z^{\delta_{0\alpha}-1} \exp\Big(\frac{1}{\epsilon}\, S_\alpha + \mathcal{O}(1)\Big), \quad \text{as } \epsilon \to 0, \tag{82}$$

where

$$S_\alpha(\boldsymbol{t}, z) := t_\alpha \, \log z + \sum_{n\geq 0} \frac{1}{z^n}\, S_{\alpha n}(\boldsymbol{t}), \quad S_{\alpha n} = \mathbb{S}^{(0)}_{\alpha n}, \quad S_{00} \equiv 0, \tag{83}$$

are the *classical action* functions.

In terms of slow variables the operators $T_i$ and $T_i^*$ become translation operators

$$T_i = \exp(-\epsilon\, \partial_i), \quad T_i^* = T_i^{-1} = \exp(\epsilon\, \partial_i), \quad \partial_i := \frac{\partial}{\partial\, t_i}. \tag{84}$$

Hence, we have the following useful relations

$$T_i^{\pm 1}\, f_{0\alpha} = \exp\Big( \mp\, \partial_i\, S_\alpha + \mathcal{O}(\epsilon)\Big)\, f_{0\alpha}. \tag{85}$$

It is now a simple matter to deal with the corresponding dressing and Lax-Orlov operators. Indeed, expressing the functions (81) in the form

$$f_{0\mu} = \Big(\delta_{0\mu} + \sum_{n\geq 1} \frac{\alpha_{\mu n}(\epsilon, \boldsymbol{t})}{z^n}\Big) \exp\Big(\frac{t_\mu}{\epsilon}\, \log z\Big),$$

we have

$$f_{0i} = G_i \, \exp\Big(\frac{t_i}{\epsilon}\,\log z\Big), \quad G_i := \sum_{n\geq 1} \alpha_{in}(\epsilon,\boldsymbol{t})\, T_i^n$$

$$\mathcal{Z}_i : = G_i\, T_i^{-1}\, G_i^{-1}, \quad \mathcal{M}_i := G_i \cdot t_i \cdot T_i \cdot\, G_i^{-1}$$

We can also introduce Lax-Orlov associated with $f_{00}$. In fact we may do it in $q$ different ways

$$f_{00} = G_0^{(i)} \, \exp\Big(\frac{t_0}{\epsilon}\,\log z\Big), \quad G_0^{(i)} = 1+\sum_{n\geq 1} \alpha_{0n}(\epsilon,\boldsymbol{t})\, T_i^{-n},$$

$$\mathcal{Z}_0^{(i)} : = G_0^{(i)}\, T_i\, (G_0^{(i)})^{-1}, \quad \mathcal{M}_0^{(i)} := G_0^{(i)} \cdot t_0 \cdot T_i^{-1} \cdot\, (G_0^{(i)})^{-1}.$$

In terms of Lax-Orlov operators and taking into account the assumption (81) the system of string equations (79) becomes

$$\begin{cases} z\, f_{0\alpha} = \mathcal{Z}_\alpha\, f_{0\alpha} = \Big(T_j + \mathrm{u}_j(\epsilon,\boldsymbol{t}) + \displaystyle\sum_k \mathrm{v}_k(\epsilon,\boldsymbol{t})\, T_k^*\Big)\, f_{0\alpha}, \quad \forall \alpha, j; \\ \\ \epsilon\, \partial_z\, f_{00} = \mathcal{M}_0\, f_{00} = -\mathcal{H}\, f_{00}, \quad \epsilon\, \partial_z\, f_{0j} = \mathcal{M}_j\, f_{0j} = \Big(-\mathcal{H} + V'(\boldsymbol{c}_j, \mathcal{Z}_j)\Big)\, f_{0i}, \end{cases} \tag{86}$$

for all choices $\mathcal{Z}_0 = \mathcal{Z}_0^{(i)}, \mathcal{M}_0 = \mathcal{M}_0^{(i)}$. It follows from (81) that the recurrence coefficients $\mathrm{u}_j$ and $\mathrm{v}_j$ can be written as quasiclassical expansions of the form

$$\mathrm{u}_i = u_i(\boldsymbol{t}) + \sum_{n=1}^{\infty} \epsilon^n\, u_{i,n}(\boldsymbol{t}), \quad \mathrm{v}_i = v_i(\boldsymbol{t}) + \sum_{n=1}^{\infty} \epsilon^n\, v_{i,n}(\boldsymbol{t}), \tag{87}$$

## 5.1 Leading behaviour and hodograph equations

Our next aim is to characterize the leading behaviour of the solutions $f_{0\alpha}$ of (86). More concretely we are going to see how the leading terms

$$\boldsymbol{u} := (u_1(\boldsymbol{t}), \ldots, u_q(\boldsymbol{t})), \quad \boldsymbol{v} := (v_1(\boldsymbol{t}), \ldots, v_q(\boldsymbol{t})),$$

of the recurrence coefficients (87) are determined by a system of hodograph type equations.

In order to formulate the classical limits $(z_\alpha, m_\alpha)$ of the Lax-Orlov operators $(\mathcal{Z}_\alpha, \mathcal{M}_\alpha)$ we observe that as a consequence of the first group of string equations in (86) we have that

$$\Big(T_i + \mathrm{u}_i(\epsilon,\boldsymbol{t})\Big)\, f_{0\alpha} = \Big(T_j + \mathrm{u}_j(\epsilon,\boldsymbol{t})\Big)\, f_{0\alpha}, \quad \forall i,j,\alpha. \tag{88}$$

Then, using (85) we get

$$\exp\Big(-\partial_i\, S_\alpha(\boldsymbol{t}, z)\Big) + u_i(\boldsymbol{t}) = \exp\Big(-\partial_j\, S_\alpha(\boldsymbol{t}, z)\Big) + u_j(\boldsymbol{t}), \quad \forall i,j. \tag{89}$$

In view of these identities we define $z_\alpha(\boldsymbol{t},p)$ by the implicit equations

$$p=\exp\Big(-\partial_i\, S_\alpha(\boldsymbol{t},z_\alpha(\boldsymbol{t},p))\Big)+u_i(\boldsymbol{t}). \tag{90}$$

Notice that according to (89) these definitions are independent of the value of the index $i$ used in (90). Moreover, (90) imply

$$\partial_i\, S_\alpha(\boldsymbol{t},z_\alpha)=-\log(p-u_i(\boldsymbol{t})). \tag{91}$$

From the asymptotic expansion (83) of the action functions $S_\alpha$ and the defining equations (90) it is straightforward to prove that the Lax functions can be expanded as

$$\begin{cases} z_0=p+\displaystyle\sum_{n=1}^{\infty}\frac{v_{0n}(\boldsymbol{t})}{p^n}, \quad p\to\infty,\\[2ex] z_i=\dfrac{v_i(\boldsymbol{t})}{p-u_i(\boldsymbol{t})}+\displaystyle\sum_{n=0}^{\infty}v_{in}(\boldsymbol{t})\,(p-u_i(\boldsymbol{t}))^n, \quad p\to u_i(\boldsymbol{t}). \end{cases} \tag{92}$$

On the other hand, we define the corresponding Orlov functions $m_\alpha(\boldsymbol{t},z_\alpha)$ by

$$m_\alpha(\boldsymbol{t},z_\alpha):=\partial_z\, S_\alpha(\boldsymbol{t},z_\alpha). \tag{93}$$

The definitions (90) and (93) provide the classical limits of the Lax-Orlov operators. Indeed, from (86) it follows at once that

$$\begin{aligned} (\mathcal{Z}_\alpha\, f_{0\alpha})(\boldsymbol{t},z_\alpha(\boldsymbol{t},p))&=z_\alpha(\boldsymbol{t},p)\, f_{0\alpha}(\boldsymbol{t},z_\alpha(\boldsymbol{t},p)),\\ (\mathcal{M}_\alpha\, f_{0\alpha})(\boldsymbol{t},z_\alpha(\boldsymbol{t},p))&=\Big(m_\alpha+\mathcal{O}(\epsilon)\Big)\, f_{0\alpha}(\boldsymbol{t},z_\alpha(\boldsymbol{t},p)), \end{aligned} \tag{94}$$

for all choices of $\mathcal{Z}_0=\mathcal{Z}_0^{(i)},\mathcal{M}_0=\mathcal{M}_0^{(i)}$. In particular this means that all the pairs of Lax-Orlov operators $(\mathcal{Z}_0^{(i)},\mathcal{M}_0^{(i)})$ have the same classical limit given by $(z_0(\boldsymbol{t},p),m_0(\boldsymbol{t},p))$.

**Theorem 5.** *The Lax-Orlov functions satisfy the classical string equations*

$$\begin{cases} z_0=z_1=\cdots=z_q=E(\boldsymbol{u},\boldsymbol{v},p),\\[2ex] m_0=m_1-V'(\boldsymbol{c}_1,z_1)=\cdots=m_q-V'(\boldsymbol{c}_q,z_q)=-H(\boldsymbol{u},\boldsymbol{v},p), \end{cases} \tag{95}$$

*where*

$$E:=p+\sum_{k=1}^{q}\frac{v_k(\boldsymbol{t})}{p-u_k(\boldsymbol{t})}, \quad H:=\sum_{k=1}^{q}V'(\boldsymbol{c}_k,z_k)_{(k,+)}, \tag{96}$$

*and* $(\quad)_{(k,+)}$ *stand for the projections of power series in* $(p-u_k)^n$, $(n\in\mathbb{Z})$ *on the subspaces generated by* $(p-u_k)^{-n}$ $(n\geq 1)$.

*Proof.* Taking into account that

$$T_j^n\, f_{0\alpha}(\boldsymbol{t},z_\alpha(p,\boldsymbol{t}))=\Big((p-u_j)^n+\mathcal{O}(\epsilon)\Big)\, f_{0\alpha}(\boldsymbol{t},z_\alpha(\boldsymbol{t},p)), \quad n=\pm1,\pm2,\ldots,$$

it is easy to see that the equations (95) are the classical limit ($\epsilon\to 0$) of the system (86). ∎

In view of the first group of equations in (95), it is clear that the functions $\boldsymbol{u}$ and $\boldsymbol{v}$ are the only unknowns for determining the Lax-Orlov functions. However, the Lax-Orlov functions must verify the correct asymptotic expansions. Obviously, the functions $z_\alpha = E$ satisfy (92). Nevertheless, Eq.(83) requires the Orlov functions to satisfy

$$m_\alpha = \frac{t_\alpha}{z_\alpha} - \sum_{n\geq 1} \frac{n\, S_{\alpha n}(\boldsymbol{t})}{z_\alpha^{n+1}}, \quad \text{as } z_\alpha \to \infty, \tag{97}$$

and this behaviour must be compatible with the second group of equations in (95)

$$m_0 = -H(\boldsymbol{u},\boldsymbol{v},p), \quad m_i = V'(\boldsymbol{c}_i, E) - H(\boldsymbol{u},\boldsymbol{v},p), \tag{98}$$

where we have already inserted the substitutions $z_i = E$. Let us analyze the equations (98) in terms of series expansions as $p\to\infty$ for $m_0$, and as $p \to u_i(\boldsymbol{t})$ for $m_i$. If we take into account that

$$\frac{1}{z_0} = \frac{1}{p} + \mathcal{O}\Big(\frac{1}{p^2}\Big), \quad H = \mathcal{O}\Big(\frac{1}{p}\Big), \quad p \to \infty,$$

$$\frac{1}{z_i} = \mathcal{O}((p-u_i)), \quad \frac{1}{p-u_j} = \mathcal{O}(1); \quad V'(\boldsymbol{c}_i, E) - H = \mathcal{O}(1), \quad j \neq i, \quad p \to u_i(\boldsymbol{t}),$$

then the consistency between (98) and (97) requires

$$\oint_{\gamma_0} \frac{\mathrm{d}p}{2i\pi}\, H(\boldsymbol{u},\boldsymbol{v},p) = -t_0, \tag{99}$$

and

$$\begin{cases} \displaystyle\oint_{\gamma_i} \frac{\mathrm{d}p}{2i\pi}\, \frac{V'(\boldsymbol{c}_i, E(\boldsymbol{u},\boldsymbol{v},p)) - H(\boldsymbol{u},\boldsymbol{v},p)}{p-u_i} = 0 \\ \displaystyle\oint_{\gamma_i} \frac{\mathrm{d}p}{2i\pi}\, \frac{V'(\boldsymbol{c}_i, E(\boldsymbol{u},\boldsymbol{v},p)) - H(\boldsymbol{u},\boldsymbol{v},p)}{(p-u_i)^2} = \frac{t_i}{v_i}. \end{cases} \tag{100}$$

These conditions are obtained by comparing coefficients of $p^{-1}$ and $(p-u_i(\boldsymbol{t}))^n$ with $(n=0,1)$ in the equations (98) for $m_0$ and $m_i$, respectively. Here $\gamma_i$ are positively oriented small circles around $p=u_i$ such that $p=u_j$ is outside $\gamma_i$ for all $j\neq i$, and $\gamma_0$ is a large positively oriented circles which encircles all the $\gamma_i$ (see fig.1).

Identifying the coefficients of the remaining powers $p^{-n}$ and $(p-u_i(\boldsymbol{t}))^n$ in (98) determines the Orlov functions in terms of $(\boldsymbol{u},\boldsymbol{v})$.

The equation (99) is a consequence of the second group of equations in (100) and the fact that $t_0 := -\sum_i t_i$. To see this, notice that

$$\oint_{\gamma_0} H(p)\,\mathrm{d}p = \oint_{\gamma_0} H(p)\,\partial_p\, E(p)\,\mathrm{d}p,$$

$$\oint_{\gamma_i} (V'(\boldsymbol{c}_i, E(p)) - H(p))\, \frac{v_i}{(p-u_i)^2}\,\mathrm{d}p = -\oint_{\gamma_i} (V'(\boldsymbol{c}_i, E(p)) - H(p))\,\partial_p\, E(p)\,\mathrm{d}p.$$

Hence

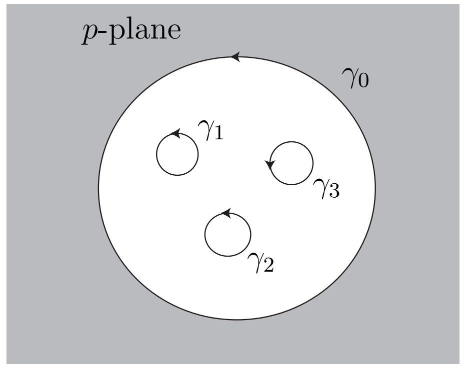


Figure 1:

$$-\oint_{\gamma_0} H(p)\,\mathrm{d}p + \sum_i \oint_{\gamma_i} (V'(\boldsymbol{c}_i, E(p)) - H(p))\,\frac{v_i}{(p-u_i)^2}\,\mathrm{d}p$$

$$= -\oint_{\gamma_0} H(p)\,\partial_p\,E(p)\,\mathrm{d}p - \sum_i \oint_{\gamma_i} \Big(V'(\boldsymbol{c}_i, E(p)) - H(p)\Big)\;\partial_p\,E(p)\,\mathrm{d}p$$

$$= -\oint_{\gamma_0 - \sum_i \gamma_i} H(p)\,\partial_p\,E(p)\,\mathrm{d}p - \sum_i \oint_{\gamma_i} V'(\boldsymbol{c}_i, E(p))\,\partial_p\,E(p)\,\mathrm{d}p = 0,$$

where we have taken into account that $V'(\boldsymbol{c}_i, E(p))\,\partial_p\,E(p) = \partial_p\Big(V(\boldsymbol{c}_i, E(p))\Big)$. Moreover, $H(p)\,\partial_p\,E(p)$ is a rational function of $p$ with poles at the points $p_i = u_i$ only and

$$\gamma_0 - \sum_i \gamma_i \sim 0 \quad \text{in} \quad \mathbb{C}\setminus\{p_1,\dots,p_q\}.$$

Therefore we are finally lead to the system (100) of $2q$ equations for determining the $2q$ functions $u_i, v_i$. These equations are of hodograph type as they depend linearly on the parameters $\boldsymbol{t}$ and $\boldsymbol{c}_i$. For example the first few terms are

$$\begin{cases} c_{i1} + 2\,c_{i2}\,u_i + \displaystyle\sum_{j\neq i} \frac{(c_{i2} - c_{j2})\,v_j}{u_i - u_j} + \cdots = 0 \\[2ex] 2\,c_{i2} - \displaystyle\sum_{j\neq i} \frac{(c_{i2} - c_{j2})\,v_j}{(u_i - u_j)^2} + \cdots = \dfrac{t_i}{v_i}. \end{cases} \tag{101}$$

## 5.2 Connection with the Whitham hierarchy

If we assume that the coefficients $\boldsymbol{c}_i$ of exponents of the weight functions (11) and (46) are free parameters and write them in the form

$$\boldsymbol{c}_i = \boldsymbol{t}_0 - \boldsymbol{t}_i, \quad \boldsymbol{t}_\alpha = (t_{\alpha 1}, \dots, t_{\alpha n}, \dots) \in \mathbb{C}^\infty, \tag{102}$$

then, as we are going to see, the solution of (95) turns out to determine a solution of the Whitham hierarchy of dispersionless integrable system [11].

Let us introduce the modified Orlov functions

$$\widetilde{m}_\alpha = V'(\boldsymbol{t}_\alpha, z_\alpha) + m_\alpha. \tag{103}$$

It is clear that $(z_\alpha, \widetilde{m}_\alpha)$ solve the system

$$\begin{cases} z_0 = z_1 = \cdots = z_q, \\ \\ \widetilde{m}_0 = \widetilde{m}_1 = \cdots = \widetilde{m}_q. \end{cases} \tag{104}$$

Moreover, they are rational functions of $p$ with poles at the points $p_i = u_i$ only. Furthermore, they satisfy the asymptotic properties (92) and

$$\widetilde{m}_\alpha = \sum_{n\geq 1} n\, t_{\alpha n}\, z_\alpha^{n-1} + \frac{t_\alpha}{z_\alpha} - \sum_{n\geq 1} \frac{n\, S_{\alpha n}(\boldsymbol{t})}{z_\alpha^{n+1}}, \quad \text{as } z_\alpha \to \infty.$$

Thus, the functions $(z_\alpha, \widetilde{m}_\alpha)$ satisfy all the conditions of Theorem 1 of [15] and, as a consequence, they verify the equations of the Whitham hierarchy

$$\frac{\partial z_\alpha}{\partial t_{\mu n}} = \{\Omega_{\mu n}, z_\alpha\}, \quad \frac{\partial \widetilde{m}_\alpha}{\partial t_{\mu n}} = \{\Omega_{\mu n}, \widetilde{m}_\alpha\} \tag{105}$$

where the Poisson bracket is given by

$$\{F, G\} := \frac{\partial F}{\partial p}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial p}, \quad x := t_{01}.$$

and the Hamiltonian functions are

$$\Omega_{\mu n} := \begin{cases} (z_\mu^n)_{(\mu,+)}, & n \geq 1, \\ \\ -\log(p - u_i), & n = 0, \quad \mu = i = 1, \ldots, q. \end{cases} \tag{106}$$

Here $(\cdot)_{(0,+)}$ stands for the projector on $\{p^n\}_{n=0}^{\infty}$ .

In this way we conclude that $(z_\alpha, \widetilde{m}_\alpha)$, as functions of the coupling constants $\boldsymbol{c}_i = \boldsymbol{t}_0 - \boldsymbol{t}_i$, determine a reduced solution of the Whitham hierarchy. This property is in complete agreement with the results of recents works [16] which prove that the universal Whitham hierarchy can be obtained as a particular dispersionless limit of the multi-component KP hierarchy. Moreover, as it as been observed in [24], appropriate deformations of the Riemann-Hilbert problems for multiple orthogonal polynomials determine solutions of the multi-component KP hierarchy. In fact these deformations correspond to the flows induced by changes in the parameters $\boldsymbol{t}_\alpha$. Indeed, for both types of multiple orthogonal polynomials, (102) implies

$$\partial_{t_{\alpha n}}\, g = [z^n\, E_\alpha, g].$$

Therefore the covariant derivatives

$$D_{\alpha n}\, f := \partial_{\alpha n}\, f + z^n\, f\, E_\alpha,$$

are symmetries of the corresponding Riemman-Hilbert problems. Hence, using Proposition 2 one concludes that

$$\partial_{\alpha n} f + (z^n f E_\alpha f^{-1})_- f = 0, \tag{107}$$

where $(\quad)_-$ stands for the projections of power series in $z^k$, $(k \in \mathbb{Z})$ on the subspaces generated by $z^{-k}\,(k \geq 1)$. The equations (107) constitute the linear system of the multi-component KP hierarchy.

# 6 Applications: random matrix models and non-intersecting Brownian motions

As we have seen the multiple orthogonal polynomials of type I are the elements $f_{0i}$ of the fundamental solution of their associated RH problem. Thus, in the quasiclassical limit we have

$$A_i(\boldsymbol{n}, z) \sim \frac{1}{z} \exp\Big(\frac{1}{\epsilon} S_i(\boldsymbol{t}, z))\Big), \quad \text{as } \epsilon \to 0,$$

where $S_i = S_i(\boldsymbol{t}, z)$ are the classical action functions defined in (83). Hence

$$\epsilon\, \partial_z \log A_i(\boldsymbol{n}, z) \sim \partial_z S_i(\boldsymbol{t}, z) - \frac{\epsilon}{z} = m_i(\boldsymbol{t}, z) - \frac{\epsilon}{z}. \tag{108}$$

On the other hand, if we denote by $x_i$ the roots of $A_i(\boldsymbol{n}, z)$ we have

$$\partial_z \log A_i(\boldsymbol{n}, z) = \sum_{i=1}^{n_i-1} \frac{1}{z - x_i}.$$

Thus if we assume that in the large-$\boldsymbol{n}$ limit the roots of $A_i$ are distributed with a continuous density $\rho_i = \rho_i(x)$ on some compact (possibly disconnected) support $I_i \subset \mathbb{R}$

$$\epsilon \sum_{i=1}^{n_i-1} \frac{1}{z - x_i} \sim \int_{I_i} \frac{\rho_i(x)}{z - x}\, \mathrm{d}x, \quad \text{as } \epsilon \to 0. \tag{109}$$

from (108) and (109) we deduce the important relation

$$m_i(z) = \int_{I_i} \frac{\rho_i(x)}{z - x}\, \mathrm{d}x, \tag{110}$$

where $m_i$, $I_i$ and $\rho_i$ depend on the slow variables $\boldsymbol{t}$. This means that the Orlov functions $m_i$ are the Cauchy transforms of the root densities $\rho_i$. Hence, they determines the distribution of roots in the large-$\boldsymbol{n}$ limit according to

$$m_{i+}(x) - m_{i-}(x) = -2\, i\pi\, \rho_i(x), \quad x \in I_i. \tag{111}$$

Moreover, from (97) we see that

$$\int_I \rho_i(x)\, \mathrm{d}x = t_i. \tag{112}$$

On the other hand, the multiple orthogonal polynomials of type II represent the element $f_{00}$ of their associated RH problem. Therefore, in the quasiclassical limit we have

$$P(\boldsymbol{n},z)\sim\exp\Big(\frac{1}{\epsilon}\,S_0(\boldsymbol{t},z)\Big),\quad \text{as } \epsilon\to 0,$$

Thus if we assume that in the large-$\boldsymbol{n}$ limit the roots of $P(\boldsymbol{n},z)$ tend to be distributed with a continuous density $\rho_0=\rho_0(x)$ on some compact support $I_0\subset\mathbb{R}$, we deduce

$$m_0(z)=\int_{I_0}\frac{\rho_0(x)}{z-x}\,\mathrm{d}x, \tag{113}$$

where $m_0$, $I_0$ and $\rho_0$ depend on the slow variables $\boldsymbol{t}$. Thus the Orlov function $m_0$ is the Cauchy transform of the density $\rho_0$ and therefore

$$m_{0+}(x)-m_{0-}(x)=-2\,i\pi\,\rho_0(x),\quad x\in I_0. \tag{114}$$

Note also that

$$\int_{I_0}\rho_0(x)\,\mathrm{d}x=t_0. \tag{115}$$

The string equations (95) provide also useful information to determine the limiting supports and the root densities. They imply

$$m_0(z)=-H(p_0(z)),\quad m_i(z)=V'(\boldsymbol{c}_i,z)-H(p_i(z)),$$

where $p_\alpha(z)$ denote the $q+1$ inverses of the map

$$z(p):=E(p)=p+\sum_{k=1}^{q}\frac{v_k(\boldsymbol{t})}{p-u_k(\boldsymbol{t})},$$

verifying

$$p_0(z)=z+\mathcal{O}\Big(\frac{1}{z}\Big),\quad p_i(z)=u_i+\mathcal{O}\Big(\frac{1}{z}\Big);\quad \text{as } z\to\infty.$$

Therefore (111) and (114) reduce to

$$H(p_{\alpha+}(x))-H(p_{\alpha-}(x))=2\,i\pi\,\rho_\alpha(x),\quad x\in I_\alpha. \tag{116}$$

In general the limiting supports $I_\alpha$ may consist of several disconnected segments

$$I_\alpha=\bigcup_{k=1}^{d_\alpha}I_{\alpha k}$$

which, due to (116), constitute the branch cuts of the functions $H(p_\alpha(z))$ . As a consequence the end-points of the segments $I_{\alpha k}$ are the branch points of these functions, which are in turn given by the critical points $x_i$ of the function $z(p)=E(p)$

$$x_i=E(q_i)\in\mathbb{R},\quad \partial_p E(q_i)=0. \tag{117}$$

## 6.1 The Hermitian matrix model

For $q = 1$ the multiple orthogonal polynomials of type II reduce to the orthogonal polynomials on the real line associated to the weight function $w = \exp V(\boldsymbol{c}, z)$. These polynomials are connected to the random matrix model of $n\times n$ Hermitian matrices [1]-[2]

$$Z_n = \int \mathrm{d}M \exp\Big(\mathrm{Tr}\, V(\boldsymbol{c}, M)\Big), \tag{118}$$

through the crucial relation

$$P_n(z) = \mathbb{E}\,[\det(z - M)], \tag{119}$$

where $\mathbb{E}$ denotes the expectation value with respect to the probability measure determined by (118). This means that in the large-$n$ limit the eigenvales of $M$ converge with unit probability to the roots of $P_n$. As a consequence the root density $\rho_0$ of the family of polynomials represents the eigenvalue density of the matrix model.

The Hermitian matrix model provides an appropriate example to illustrate all the aspects of our method for characterizing the quasiclassical limit. In this case we set $\epsilon := 1/n$, $t_0 = 1$ and we have

$$z(p) = E(u, v, p) = p + \frac{v}{p - u}.$$

Here $u$ and $v$ depend on the coupling constants $\boldsymbol{c} = (c_1, c_2, \dots)$ and can be determined by means of the hodograph equations (99)-(100)

$$\oint_{\gamma_0} \frac{\mathrm{d}p}{2i\pi}\, H(p) = -1, \quad \oint_{\gamma_1} \frac{\mathrm{d}p}{2i\pi}\, \frac{V'(\boldsymbol{c}, E(p)) - H(p)}{p - u} = 0.$$

By introducing the change of variable $p - u \to p$ these equations are equivalent to the well-known system [1]

$$\oint_{\gamma} \frac{\mathrm{d}p}{2i\pi}\, V'(\boldsymbol{c},\, p + u + \frac{v}{p}) = -1, \quad \oint_{\gamma} \frac{\mathrm{d}p}{2i\pi\, p}\, V'(\boldsymbol{c},\, p + u + \frac{v}{p}) = 0, \tag{120}$$

which characterizes the *spherical limit* in the Hermitian matrix model of 2D gravity. Here $\gamma$ is a large positively oriented circle around the origin.

The critical points of $E$ are $q_\pm = u \pm \sqrt{v}$, so that the support of eigenvalues is

$$I = [x_-, x_+], \quad x_\pm := u \pm 2\sqrt{v}. \tag{121}$$

We use (116) to determine the density of eigenvalues according to

$$H(p_{0+}(x)) - H(p_{0-}(x)) = 2\, i\pi\, \rho_0(x), \quad x \in [x_-, x_+]. \tag{122}$$

Furthermore, the two inverses of $z(p)$ are

$$p_0(z) := \frac{1}{2}(z + u + \sqrt{(z - x_-)(z - x_+)}\,), \quad p_1(z) := \frac{1}{2}(z + u - \sqrt{(z - x_-)(z - x_+)}\,) \tag{123}$$

and we have

$$H(p_0(z)) = V'(\boldsymbol{c}, z(p))_{(1,+)}\Big|_{p = p_0(z)}.$$

Now, using the identities

$$\frac{v}{p-u} = z - p, \quad p^2 = (u+z)\,p - z\,u - v,$$

it is clear that there exist polynomials $\alpha_k(z)$ and $\beta_k(z)$ satisfying

$$\Big(z(p)^k\Big)_{(1,+)}\Big|_{p=p_0(z)} = \alpha_k(z) + \beta_k(z)\,p_0(z), \quad \Big(z(p)^k\Big)_{(1,+)}\Big|_{p=p_1(z)} = \alpha_k(z) + \beta_k(z)\,p_1(z). \tag{124}$$

In particular, taking into account that

$$p_0(z) = z + \mathcal{O}\Big(\frac{1}{z}\Big), \quad p_1(z) = u + \mathcal{O}\Big(\frac{1}{z}\Big); \quad \text{as } z \to \infty,$$

from (124) we deduce

$$\beta_k(z) = -\Big(\frac{z^k}{p_0 - p_1}\Big)_\oplus = -\Big(\frac{z^k}{\sqrt{(z-x_-)(z-x_+)}}\Big)_\oplus, \tag{125}$$

where $(\quad)_\oplus$ means the projection of power series in $z^n$, $(n \in \mathbb{Z})$ on the subspace generated by $z^n$, $(n \geq 0)$. Hence it follows that

$$H(p_0(z)) = \sum_{k\geq 1} k\,c_k\,\Big(\alpha_{k-1}(z) + \beta_{k-1}(z)\,p_0(z)\Big),$$

and therefore we get

$$\rho(x) = \frac{1}{2\,i\pi}\sum_{k\geq 1} k c_k \beta_{k-1}(x)(p_{0+}(x) - p_{0-}(x)) = -\frac{1}{2\,\pi}\Big(\frac{V'(\boldsymbol{c},x)}{\sqrt{(x-x_-)(x-x_+)}}\Big)_\oplus \sqrt{(x-x_-)(x_+-x)},$$

which represents the well-known eigenvalue density for the Hermitian model in the one-cut case.

### 6.2 Gaussian models with an external source and non-intersecting Brownian motions

For $q > 1$ the multiple orthogonal polynomials of type II are connected to the Gaussian Hermitian matrix model with an external source term $A\,M$ [6]-[9], where $A$ is a fixed diagonal $n{\times}n$ real matrix. The partition function of this model is given by

$$Z_n = \int \mathrm{d}M \exp\Big( - \mathrm{Tr}\,\Big(\frac{1}{2}M^2 - A\,M\Big)\Big). \tag{126}$$

It turns out that if the eigenvalues of $A$ are given by $a_j$, $(j = 1,\ldots,q)$ with multiplicities $n_j$, then the expectation values

$$P(\boldsymbol{n}, z) = \mathbb{E}\,[\det(z - M)], \quad \boldsymbol{n} := (n_1,\ldots,n_q), \tag{127}$$

are multiple orthogonal polynomials with respect to the Gaussian weights

$$w_j(x) = \exp(a_j\,x - \frac{1}{2}\,x^2).$$

These matrix models are deeply connected to one-dimensional non-intersecting Brownian motion [20]-[22]. More concretely, the joint probability density for the eigenvalues $(\lambda_1,\dots,\lambda_n)$ of M is the same as the probability density at time $t\in(0,1)$ for the positions $(x_1,\dots,x_n)$ of $n$ non-intersecting Brownian motions starting at the origin at $t=0$ and forming $q$ groups ending at $q$ fixed points $b_i$, $(i=1,\dots,q)$ at $t=1$. The corresponding dictionary for this duality is

$$\lambda_j=\frac{x_j}{\sqrt{t(1-t)}},\quad a_k=b_k\sqrt{\frac{t}{1-t}}.$$

We discuss next an example of application to the large-$\boldsymbol{n}$ limit of non-intersecting Brownian motions. Let us consider an even number $n$ non-intersecting Brownian motions ending at two points $\pm b$ with $n_1=n_2=n/2$ [9]. In this case the slow variables take the values $t_1=t_2=-1/2$. Moreover, we have

$$V(\boldsymbol{c}_1,z)=a\,z-\frac{z^2}{2},\quad V(\boldsymbol{c}_2,z)=-a\,z-\frac{z^2}{2},\quad a:=b\sqrt{\frac{t}{1-t}},$$

and

$$z(p)=E(p)=p+\frac{v_1}{p-u_1}+\frac{v_2}{p-u_2},\quad H(p)=-\frac{v_1}{p-u_1}-\frac{v_2}{p-u_2}=p-z(p). \tag{128}$$

Using the hodograph equations (101) one finds

$$u_1=a,\quad u_2=-a,\quad v_1=v_2=\frac{1}{2},$$

so that

$$z(p)=\frac{p^3+(1-a^2)p}{p^2-a^2}.$$

The corresponding algebraic function $p=p(z)$ satisfies the Pastur equation [23]

$$p^3-z\,p^2+(1-a^2)\,p+a^2\,z=0,$$

which defines a three-sheeted Riemann surface. The restrictions of $p(z)$ to the three sheets are the functions $p_\alpha(z)$ characterized by the asymptotic behaviour

$$p_0(z)=z+\mathcal{O}\Big(\frac{1}{z}\Big),\quad p_i(z)=u_i+\mathcal{O}\Big(\frac{1}{z}\Big),\ i=1,2;\quad \text{as } z\to\infty.$$

There are four critical points of $z(p)$ which give rise to four branch points $\pm x_1$, $\pm x_2$ in the $z$-plane where

$$x_1=q_1\,\frac{\sqrt{1+8\,a^2}+3}{\sqrt{1+8\,a^2}+1},\quad x_2=q_2\,\frac{\sqrt{1+8\,a^2}-3}{\sqrt{1+8\,a^2}-1},$$
$$q_{1,2}=\sqrt{\frac{1}{2}+a^2\pm\frac{1}{2}\sqrt{1+8\,a^2}}.$$

It is easy to see that $x_1$ is real for all $a\geq 0$, while $x_2$ is real for $a\geq 1$ $(x_2<x_1)$ and purely imaginary for $0<a<1$. Now, from (122) and taking into account that $H(p)=p-z(p)$ we deduce that the eigenvalue density is given by

$$\rho_0(x)=\frac{1}{2\,i\pi}\,(H(p_{0+}(x))-H(p_{0-}(x))=\frac{1}{2\,i\pi}\,(p_{0+}(x)-p_{0-}(x)),\quad x\in I_0. \tag{129}$$

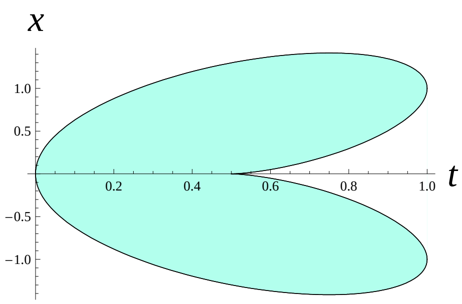


Figure 2: Limit support for Brownian motions with two symmetric endpoints for $b = 1$

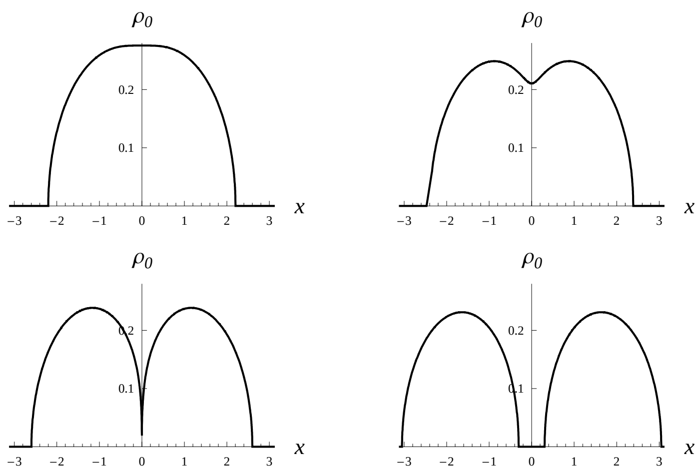


Figure 3: The density of Brownian motions $\rho_0(x)$ for $a = 1/2$, $3/4$, $1$ and $3/2$, respectively

Using Cardano's formula for $p_0$ one finds

$$\rho_0(x) = \frac{2\,x^2 + 6\,(a^2-1) - \sqrt[3]{2}\left(r(x) - \sqrt{r(x)^2 - 4\,s(x)^3}\right)^{2/3}}{2^{5/3}\sqrt{3}\,\pi\,\sqrt[3]{r(x) - \sqrt{r(x)^2 - 4\,s(x)^3}}},$$

where

$$r(x) := -2\,x^3 + 18\,a^2\,x + 9\,x, \quad s(x) := x^2 + 3\,(a^2-1).$$

The form of the support $I_0$ depends on the analytic properties of the function $p_0(z)$ ( see [6]-[9]):

**a)** For $0 < a \leq 1$ the function $p_0$ is analytic in $\mathbb{C} - [-x_1, x_1]$ and $I_0 = [-x_1, x_1]$.

**b)** For $a > 1$ the function $p_0$ is analytic in $\mathbb{C} - ([-x_1, -x_2] \cup [x_2, x_1])$ and $I_0 = [-x_1, -x_2] \cup [x_2, x_1]$.

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