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Interpolation of compact multilinear operators between quasi-Banach spaces

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Abstract

We investigate the interpolation properties of compact multilinear operators by the real method between quasi-Banach spaces. As an application we establish a reinforced version of a multilinear Marcinkiewicz theorem.

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1. Introduction

The theory of interpolation of operators is a branch of functional analysis with important applications in approximation theory, partial differential equations, operator theory and harmonic analysis. Details can be found in the books by Butzer and Berens [8], Bergh and Löfström [4], Triebel [36,37], König [24], Bennett and Sharpley [2], Brudnyĭ and Kruglyak [7] and Amrein,

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Boutet de Monvel and Georgescu [1]. Inside this theory, interpolation of compact operators is a classical problem that has attracted the attention of a number of authors. It was already considered in the pioneering papers by Lions and Peetre [28] and Calderón [9].

As for compact linear operators and the real interpolation method, in 1992 solving a long standing problem, Cwikel [18] and Cobos, Kühn and Schonbek [15] proved that if (A_0, A_1), (B_0, B_1) are Banach couples and T is a linear operator such that $T : A_i \longrightarrow B_i$ is bounded for $i = 0, 1$ and one of these restrictions is compact, then the operator $T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}$ is also compact for $0 < \theta < 1$ and $1 \leq q \leq \infty$. Previous contributions to this question are due to Cobos, Edmunds and Potter [11] and Cobos and Peetre [16] among other authors. A result of this generality for the complex interpolation method is still an open problem. The extension of the compactness result for the real method to the case of couples of quasi-Banach spaces and $0 < q \leq \infty$ was done by Cobos and Persson [17]. Let us also mention that Brudnyĭ [6] has studied the interpolation of compact linear operators by general interpolation methods when the last couple is formed by Banach lattices.

Recently the corresponding problem for bilinear and, more generally, multilinear operators has been extensively studied. We refer, for example, to the papers by Fernandez and Silva [19], Fernández-Cabrera and Martínez [20,21], Cobos, Fernández-Cabrera and Martínez [12–14], Mastyło and Silva [31,32], Besoy and Cobos [5] and Manzano, Rueda and Sánchez-Pérez [29, 30]. A motivation for these researches is the fact that compact bilinear operators occur rather naturally in harmonic analysis. See the papers by Bényi and Torres [3], Torres, Xue and Yan [35], Tao, Xue, Yang and Yuan [33] and Tao, Yang and Yang [34]. In particular, commutators of Calderón–Zygmund bilinear operators acting on L_p-spaces are compact (see [3,12,35]).

Our aim in the present paper is to establish a compactness theorem for multilinear operators interpolated by the real method between quasi-Banach spaces. Such a result cannot be derived using the techniques of [32] because they require the use of duality. In particular, our results apply to the case when the target of the multilineal operator is the couple (L_{r_0}, L_{r_1}) with $0 < r_0, r_1 \leq \infty$.

As an application of our results we establish a multilinear version of the classical result of Krasnosel'skiĭ [26], as well as a reinforced version of a multilinear Marcinkiewicz theorem established by Zafran [38].

2. Preliminaries

Let $(A, \|\cdot\|_A)$ be a *quasi-Banach space* and let $c_A \geq 1$ be the constant in the quasi-triangle inequality. Let $0 < p \leq 1$ be such that $2^{1/p-1} = c_A$. By the Aoki–Rolewicz theorem (see [25, § 15.10] or [24, Proposition 1.c.5]), there is another quasi-norm $|||\cdot|||$ on A which is equivalent to $\|\cdot\|_A$ and such that $|||\cdot|||^p$ satisfies the triangle inequality. We say that $|||\cdot|||$ is a $p-$norm and that $(A, |||\cdot|||)$ is a $p-$normed quasi-Banach space. Clearly, if $0 < r < p$ then $(A, |||\cdot|||)$ is also an $r-$normed quasi-Banach space.

The usual spaces l_q of $q-$summable scalar sequences are examples of quasi-Banach spaces. They are Banach spaces if $1 \leq q < \infty$.

Let $0 < q \leq \infty$, let $(\lambda_m)_{m\in\mathbb{Z}}$ be a sequence of positive numbers and let (W_m) be a sequence of quasi-Banach spaces with the same constant in the quasi-triangle inequality. We write

$$l_q(\lambda_m W_m) = \left\{ w = (w_m) : w_m \in W_m \text{ and } (\lambda_m \|w_m\|_{W_m}) \in l_q \right\}.$$

The quasi-norm in $l_q(\lambda_m W_m)$ is defined by

$$\|w\|_{l_q(\lambda_m W_m)} = \|(\lambda_m \|w_m\|_{W_m})\|_{l_q}.$$

By a *quasi-Banach couple* $\bar{A} = (A_0, A_1)$ we mean two quasi-Banach spaces A_0, A_1 which are continuously embedded in the same Hausdorff topological vector space $\mathcal{A}$. If A_0 and A_1 are p-normed, we refer to (A_0, A_1) as a *p-normed quasi-Banach couple*, and if $p = 1$ then we say that (A_0, A_1) is a *Banach couple*.

A quasi-Banach space A is said to be an *intermediate space* with respect to the quasi-Banach couple (A_0, A_1) if $A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1$, where $\hookrightarrow$ means continuous embedding.

For $t > 0$ and $a \in A_0 + A_1$, *Peetre's K-functional* is given by

$$K(t,a) = K(t,a; A_0, A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1,\ a_j \in A_j\}.$$

Peetre's J-functional is defined by

$$J(t,a) = J(t,a; A_0, A_1) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\},\ a \in A_0 \cap A_1.$$

Note that $K(t,\cdot)$ is a quasi-norm in $A_0 + A_1$ and $J(t,\cdot)$ is a quasi-norm in $A_0 \cap A_1$. Triangle inequality for $K(t,\cdot)$ and $J(t,\cdot)$ holds with the constant $\max\{c_{A_0}, c_{A_1}\}$. The usual quasi-norms of $A_0 + A_1$ and $A_0 \cap A_1$ coincide with the functionals $K(1,\cdot)$ and $J(1,\cdot)$ respectively.

Let $0 < p \le 1$. We are also going to use the functional

$$K_p(t,a) = K_p(t,a; A_0, A_1) = \inf\left\{\left(\|a_0\|_{A_0}^p + t^p\|a_1\|_{A_1}^p\right)^{1/p} : a = a_0 + a_1,\ a_j \in A_j\right\}$$

which is equivalent to $K(t,.)$. Namely

$$K(t,a) \le K_p(t,a) \le 2^{(1/p)-1}K(t,a),\ a \in A_0 + A_1,\ t > 0. \tag{2.1}$$

Observe that if $\|\cdot\|_{A_i}$ is a p-norm for $i = 0, 1$, then $J(t,\cdot)$ and $K_p(t,\cdot)$ are also p-norms. This observation will be useful for us later.

For $0 < \theta < 1$ and $0 < q \le \infty$, the *real interpolation space* $(A_0, A_1)_{\theta,q}$ realized as a K-space in the discrete form consists of all $a \in A_0 + A_1$ having a finite quasi-norm

$$\|a\|_{(A_0,A_1)_{\theta,q}} = \|a\|_{\theta,q;K} = \begin{cases} \left(\sum_{m=-\infty}^{\infty}[2^{-\theta m}K(2^m,a)]^q\right)^{1/q} & \text{if } 0 < q < \infty, \\ \sup_{m\in\mathbb{Z}}\{2^{-\theta m}K(2^m,a)\} & \text{if } q = \infty. \end{cases}$$

If (A_0, A_1) is a Banach couple and $1 \le q \le \infty$, then $(A_0, A_1)_{\theta,q}$ is a Banach space. Otherwise, $(A_0, A_1)_{\theta,q}$ is a quasi-Banach space. We refer to the books [2,4,7,36] for properties of these spaces.

The spaces $(A_0, A_1)_{\theta,q}$ can be also described through the J-functional as the collection of all sums $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$), where $(u_m) \subseteq A_0 \cap A_1$ and $(2^{-\theta m}J(2^m, u_m)) \in \ell_q$. Moreover, the quasi-norm $\|\cdot\|_{\theta,q;K}$ is equivalent to

$$\|a\|_{\theta,q;J} = \inf\left\{\left\|\left(2^{-\theta m}J(2^m,a)\right)\right\|_{\ell_q} : a = \sum_{m=-\infty}^{\infty} u_m\right\}$$

(see [4, Theorem 3.11.3]).

We have

$$A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{\theta,q} \hookrightarrow A_0 + A_1.$$

Furthermore, if $0 < q < r \le \infty$ then $(A_0, A_1)_{\theta,q} \hookrightarrow (A_0, A_1)_{\theta,r}$.

Another property of the space $(A_0, A_1)_{\theta,q}$ is that there is a constant $C > 0$ such that

$$\|a\|_{(A_0,A_1)_{\theta,q}} \le C\|a\|_{A_0}^{1-\theta}\|a\|_{A_1}^{\theta} \text{ for any } a \in A_0 \cap A_1. \tag{2.2}$$

We write A_i° for the closure of $A_0 \cap A_1$ in A_i. Using the description of $(A_0, A_1)_{\theta,q}$ by means of the J-functional, it follows that $(A_0^\circ, A_1^\circ)_{\theta,q} = (A_0, A_1)_{\theta,q}$.

3. Real interpolation of multilinear operators

Next let $A^1, \dots, A^N, E$ be quasi-Banach spaces and let $T : A^1 \times \cdots \times A^N \longrightarrow E$ be an N-linear operator. We say that T is *bounded* if there is $M > 0$ such that

$$\|T(a^1, \dots, a^N)\|_E \le M \|a^1\|_{A^1} \cdots \|a^N\|_{A^N} \text{ for any } a^j \in A^j,\ j = 1, \dots, N. \tag{3.1}$$

We put

$$\|T\| = \|T\|_{A^1 \times \cdots \times A^N, E} = \inf\{M > 0 : M \text{ satisfies (3.1)}\}.$$

Note that

$$\|T\| = \sup\{\|T(a^1, \dots, a^N)\|_E : a^j \in U_{A^j},\ j = 1, \dots, N\}$$

where U_{A^j} is the closed unit ball of A^j.

It is not hard to check that T is bounded if and only if T is continuous in $A^1 \times \cdots \times A^N$. The product space $\prod_{j=1}^N A_j := A^1 \times \cdots \times A^N$ is quasi-normed, say, by $\|(a^1, \dots, a^N)\| = \sum_{j=1}^N \|a^j\|_{A^j}$.

We designate by $\mathcal{L}(A^1 \times \cdots \times A^N, E)$ the space of all bounded N-linear operators T from $A^1 \times \cdots \times A^N$ to E.

Interpolation of multilinear operators by the real method between Banach couples was studied by Zafran [38]. The case of bilinear operators was already considered by Lions and Peetre [28]. Here we are interested in the case of quasi-Banach couples. The extension of the bilinear interpolation theorem to quasi-Banach couples has been done by Karadzhov [22] (see also [23]). In order to extend this result to multilinear operators with a handy estimate for the norm of the interpolated operators we will use some ideas of [12] about bilinear operators.

Subsequently, given two sequences $\xi = (\xi_m)$, $\eta = (\eta_m)$ of non-negative real numbers with indices on $\mathbb{Z}$, we define their convolution by the sequence

$$\xi * \eta = \Big(\sum_{k=-\infty}^{\infty} \xi_k \eta_{m-k}\Big)_{m\in\mathbb{Z}} = \Big(\sum_{k_1+k_2=m} \xi_{k_1} \eta_{k_2}\Big)_{m\in\mathbb{Z}}.$$

It is clear that $\xi * \eta = \eta * \xi$. Moreover, if $\mu = (\mu_m)$ is another sequence of non-negative real numbers, then we have

$$\begin{aligned}(\xi * \eta) * \mu &= \Big(\sum_{n+k_3=m} \Big(\sum_{k_1+k_2=n} \xi_{k_1}\eta_{k_2}\Big)\mu_{k_3}\Big)\\ &= \Big(\sum_{k_1+k_2+k_3=m} \xi_{k_1}\eta_{k_2}\mu_{k_3}\Big) = \xi * (\eta * \mu).\end{aligned}$$

So we can get rid of brackets and denote the sequence by $\xi * \eta * \mu$.

According to Young's inequality, if $1 \le q_1, q_2 \le \infty$ and $1/q = 1/q_1 + 1/q_2 - 1 \ge 0$ then

$$\|\xi * \eta\|_{\ell_q} \le \|\xi\|_{\ell_{q_1}} \|\eta\|_{\ell_{q_2}}.$$

If $1 \le q_1, q_2, q_3 \le \infty$ and $1/q = 1/q_1 + 1/q_2 + 1/q_3 - 2 \ge 0$, then $1/s = 1/q_1 + 1/q_2 - 1 \ge 0$ and $1/q = 1/s + 1/q_3 - 1$. Using two times Young's inequality, we obtain

$$\|\xi * \eta * \mu\|_{\ell_q} \le \|\xi * \eta\|_{\ell_s} \|\mu\|_{\ell_{q_3}} \le \|\xi\|_{\ell_{q_1}} \|\eta\|_{\ell_{q_2}} \|\mu\|_{\ell_{q_3}}.$$

By induction, we derive that if $\xi_1, \dots, \xi_N$ are sequences of non-negative real numbers and $1 \le q_1, \dots, q_N \le \infty$ with $1/q = 1/q_1 + \cdots + 1/q_N - (N-1) \ge 0$ then

$$\|\xi_1 * \xi_2 * \cdots * \xi_N\|_{\ell_q} \le \prod_{j=1}^N \|\xi_j\|_{\ell_{q_j}}. \tag{3.2}$$

Theorem 3.1. *Let* (A_0^j, A_1^j) *be a quasi-Banach couple for* $j = 1, \dots, N$ *and let* (E_0, E_1) *be an* r*-normed quasi-Banach couple. Assume that*

$$T : (A_0^1 + A_1^1) \times \cdots \times (A_0^N + A_1^N) \longrightarrow E_0 + E_1$$

is a bounded N*-linear operator such that the restrictions*

$$T : A_i^1 \times \cdots \times A_i^N \longrightarrow E_i\,, \quad i = 0, 1,$$

are bounded with norms M_i.

Let $0 < \theta < 1, r \le q_1, \dots, q_N \le \infty$ *and suppose that*

$$\frac{1}{q} = \sum_{j=1}^{N} \frac{1}{q_j} - \frac{N-1}{r} \ge 0.$$

Then

$$T : (A_0^1, A_1^1)_{\theta,q_1} \times \cdots \times (A_0^N, A_1^N)_{\theta,q_N} \longrightarrow (E_0, E_1)_{\theta,q}$$

is bounded with norm $\|T\| \le C M_0^{1-\theta} M_1^{\theta}$ *where* $C > 0$ *is a constant independent of* T.

Proof. Take $\sigma_i > M_i$, $i = 0, 1$, and pick $n \in \mathbb{Z}$ such that $2^n \le \sigma_1/\sigma_0 < 2^{n+1}$. Let $a \in (A_0^1, A_1^1)_{\theta,q_1}$ and $v_j \in A_0^j \cap A_1^j$ for $j = 2, \dots, N$. If $a = a_0 + a_1$ is any representation of a with $a_0 \in A_0^1$ and $a_1 \in A_1^1$, for any $m \in \mathbb{Z}$ we have

$$\begin{aligned}
&K(2^m, T(a, v_2, \dots, v_N)) \\
&\quad \le \|T(a_0, v_2, \dots, v_N)\|_{E_0} + 2^m \|T(a_1, v_2, \dots, v_N)\|_{E_1} \\
&\quad \le \sigma_0 \|a_0\|_{A_0^1} \|v_2\|_{A_0^2} \cdots \|v_N\|_{A_0^N} + 2^{-n}\sigma_1 2^{m-k_2} 2^{k_2+n} \|a_1\|_{A_1^1} \|v_2\|_{A_1^2} \cdots \|v_N\|_{A_1^N} \\
&\quad \le \max\{\sigma_0, 2^{-n}\sigma_1\}(\|a_0\|_{A_0^1} + 2^{m-k_2}\|a_1\|_{A_1^1}) \\
&\qquad\qquad \times \max\Big\{\prod_{j=2}^{N} \|v_j\|_{A_0^j}, \Big(\prod_{j=2}^{N-1} 2^{k_j-k_{j+1}} \|v_j\|_{A_1^j}\Big) 2^{k_N+n} \|v_N\|_{A_1^N}\Big\} \\
&\quad \le 2\sigma_0(\|a_0\|_{A_0^1} + 2^{m-k_2}\|a_1\|_{A_1^1}) \Big(\prod_{j=2}^{N-1} J(2^{k_j-k_{j+1}}, v_j)\Big) J(2^{k_N+n}, v_N).
\end{aligned}$$

Taking the infimum over all possible representations $a = a_0 + a_1$ with $a_i \in A_i^1$, we get that

$$K(2^m, T(a, v_2, \dots, v_N)) \le 2\sigma_0 K(2^{m-k_2}, a)\Big(\prod_{j=2}^{N-1} J(2^{k_j-k_{j+1}}, v_j)\Big) J(2^{k_N+n}, v_N).$$

Next we consider the spaces $(A_0^j, A_1^j)_{\theta,q_j}$ realized by the J-functional for $j = 2, \dots, N$. Take any $u_j \in (A_0^j, A_1^j)_{\theta,q_j}$ and let $u_j = \sum_{m=-\infty}^{\infty} v_m^j$ be any J-representation of u_j. Then we also have that

$$u_j = \sum_{k_j \in \mathbb{Z}} v_{k_j - k_{j+1}}^j \quad \text{and} \quad u_N = \sum_{k_N \in \mathbb{Z}} v_{k_N+n}^N.$$

Using the K_r-functional, which is an r-norm in $E_0 + E_1$, and using (2.1) we obtain that

$$\begin{aligned}
&\|T(a, u_2, \dots, u_N)\|_{(E_0,E_1)_{\theta,q}} \le \|(2^{-\theta m} K_r(2^m, T(a, u_2, \dots, u_N)))\|_{\ell_q} \\
&\le 2^{1/r}\sigma_0 \Big\| \Big[\sum_{k_2\in\mathbb{Z}} \cdots \sum_{k_N\in\mathbb{Z}} 2^{-\theta(m-k_2)r} K(2^{m-k_2}, a)^r \\
&\qquad \times \Big(\prod_{j=2}^{N-1} 2^{-\theta(k_j-k_{j+1})r} J(2^{k_j-k_{j+1}}, v^j_{k_j-k_{j+1}})^r \Big) \\
&\qquad\quad \times 2^{-\theta k_N r} J(2^{k_N+n}, v^N_{k_N+n})^r \Big]^{1/r} \Big\|_{\ell_q} \\
&= 2^{1/r}\sigma_0 \Big\| \big(2^{-\theta m r} K(2^m, a)^r\big) * \big(2^{-\theta m r} J(2^m, v^2_m)^r\big) * \cdots * \big(2^{-\theta m r} J(2^m, v^{N-1}_m)^r\big) \\
&\qquad * \big(2^{-\theta m} J(2^{m+n}, v^N_{m+n})^r\big) \Big\|^{1/r}_{\ell_{q/r}}.
\end{aligned}$$

Since $r/q = \sum_{j=1}^N r/q_j - (N-1) \ge 0$, we can use Young's inequality (3.2) deriving that

$$\begin{aligned}
&\|T(a, u_2, \dots, u_N)\|_{(E_0,E_1)_{\theta,q}} \\
&\le 2^{1/r}\sigma_0 \|(2^{-\theta m} K(2^m, a))\|_{\ell_{q_1}} \Big(\prod_{j=2}^{N-1} \|(2^{-\theta m} J(2^m, v^j_m))\|_{\ell_{q_j}} \Big) 2^{\theta n} \|(2^{-\theta m} J(2^m, v^N_m))\|_{\ell_{q_N}}.
\end{aligned}$$

By the choice of n, we have $\sigma_0 2^{\theta n} \le \sigma_0^{1-\theta}\sigma_1^{\theta}$. Hence, taking the infimum over all possible J-representations of $u_2, \dots, u_N$ and the infimum on the values $\sigma_i > M_i$, we conclude that

$$\begin{aligned}
&\|T(a, u_2, \dots, u_N)\|_{(E_0,E_1)_{\theta,q}} \\
&\qquad \le 2^{1/r} M_0^{1-\theta} M_1^{\theta} \|a\|_{(A_0^1,A_1^1)_{\theta,q_1}} \|u_2\|_{(A_0^2,A_1^2)_{\theta,q_2}} \cdots \|u_N\|_{(A_0^N,A_1^N)_{\theta,q_N}}.
\end{aligned}$$

The proof is complete. □

Next we give two results which cover the other possible positions of the q_j with respect to r.

Corollary 3.2. *Let (A_0^j, A_1^j) be quasi-Banach couples for $j = 1, \dots, N$ and let (E_0, E_1) be an r-normed quasi-Banach couple. Assume that*

$$T : (A_0^1 + A_1^1) \times \cdots \times (A_0^N + A_1^N) \longrightarrow E_0 + E_1$$

is a bounded N-linear operator such that the restrictions

$$T : A_i^1 \times \cdots \times A_i^N \longrightarrow E_i\,, \quad i = 0, 1,$$

are bounded with norms M_i.

Let $0 < \theta < 1$ and $0 < q_1, \dots, q_N \le \infty$ such that

(a) *$0 < q_1, \dots, q_N \le r$, or*

(b) *only one q_j is greater than r.*

Put

$$q = \max\{q_1, \dots, q_N\}.$$

Then

$$T : (A_0^1, A_1^1)_{\theta,q_1} \times \cdots \times (A_0^N, A_1^N)_{\theta,q_N} \longrightarrow (E_0, E_1)_{\theta,q}$$

is bounded with norm $\|T\| \le C M_0^{1-\theta} M_1^{\theta}$ where $C > 0$ is a constant independent of T.

Proof. If **(a)** holds, then

$$\frac{1}{q}=\frac{1}{q}+\Big(\sum_{j=1}^{N-1}\frac{1}{q}\Big)-\frac{N-1}{q}\geq 0$$

and the couple (E_0,E_1) is q-normed because $q\leq r$. Applying Theorem 3.1 we obtain that

$$T:(A_0^1,A_1^1)_{\theta,q}\times\cdots\times(A_0^N,A_1^N)_{\theta,q}\longrightarrow(E_0,E_1)_{\theta,q}\quad\text{boundedly.}$$

Besides, since $q_j\leq q$ for $j=1,\ldots,N$, we have that $(A_0^j,A_1^j)_{\theta,q_j}\hookrightarrow(A_0^j,A_1^j)_{\theta,q}$. So

$$T:(A_0^1,A_1^1)_{\theta,q_1}\times\cdots\times(A_0^N,A_1^N)_{\theta,q_N}\longrightarrow(E_0,E_1)_{\theta,q}$$

is bounded and it satisfies the required estimate for the norm.

Assume now that **(b)** holds. For simplicity of notation, we suppose that $q_1>r$. Then $q=\max\{q_1,\ldots,q_N\}=q_1$. Since

$$\frac{1}{q}=\frac{1}{q_1}+\Big(\sum_{j=1}^{N-1}\frac{1}{r}\Big)-\frac{N-1}{r}\geq 0,$$

applying Theorem 3.1 we derive that

$$T:(A_0^1,A_1^1)_{\theta,q_1}\times(A_0^2,A_1^2)_{\theta,r}\times\cdots\times(A_0^N,A_1^N)_{\theta,r}\longrightarrow(E_0,E_1)_{\theta,q}$$

is bounded. Moreover, $(A_0^j,A_1^j)_{\theta,q_j}\hookrightarrow(A_0^j,A_1^j)_{\theta,r}$ because $q_j\leq r$ for $j=2,\ldots,N$. Whence

$$T:(A_0^1,A_1^1)_{\theta,q_1}\times\cdots\times(A_0^N,A_1^N)_{\theta,q_N}\longrightarrow(E_0,E_1)_{\theta,q}\quad\text{boundedly}$$

and its norm satisfies the required estimate. □

Corollary 3.3. *Let (A_0^j,A_1^j) be quasi-Banach couples for $j=1,\ldots,N$ and let (E_0,E_1) be an r-normed quasi-Banach couple. Assume that*

$$T:(A_0^1+A_1^1)\times\cdots\times(A_0^N+A_1^N)\longrightarrow E_0+E_1$$

is a bounded N-linear operator such that the restrictions

$$T:A_i^1\times\cdots\times A_i^N\longrightarrow E_i\,,\quad i=0,1,$$

are bounded with norms M_i.

Let $0<\theta<1$, $0<q_1,\ldots,q_N\leq\infty$ with $\min\{q_1,\ldots,q_N\}\leq r$ and suppose that

$$\frac{1}{q}=\sum_{j=1}^{N}\frac{1}{q_j}-\frac{N-1}{min\{q_1,\ldots q_N\}}\geq 0.$$

Then

$$T:(A_0^1,A_1^1)_{\theta,q_1}\times\cdots\times(A_0^N,A_1^N)_{\theta,q_N}\longrightarrow(E_0,E_1)_{\theta,q}$$

is bounded with norm $\|T\|\leq CM_0^{1-\theta}M_1^{\theta}$ where $C>0$ is a constant independent of T.

Proof. Since $\min\{q_1,\ldots,q_N\}\leq r$, the couple (E_0,E_1) is $\min\{q_1,\ldots,q_N\}$-normed. Therefore, Theorem 3.1 yields that

$$T:(A_0^1,A_1^1)_{\theta,q_1}\times\cdots\times(A_0^N,A_1^N)_{\theta,q_N}\longrightarrow(E_0,E_1)_{\theta,q}\quad\text{boundedly}$$

with the desired estimate for the norm. □

Remark 3.4. For some choices of $q_1, \dots, q_N$ we can apply any of Corollaries 3.2 and 3.3. In such case, the result given by Corollary 3.2 is better than the one produced by Corollary 3.3 because if $q = \max\{q_1, \dots, q_N\}$ and $1/s = \sum_{j=1}^{N} 1/q_j - (N-1)/\min\{q_1, \dots q_N\} \geq 0$, then $q \leq s$ and so $(E_0, E_1)_{\theta,q} \hookrightarrow (E_0, E_1)_{\theta,s}$.

4. Interpolation of compact N-linear operators

Let $A^1, \dots, A^N, E$ be quasi-Banach spaces and let $T \in \mathcal{L}(A^1 \times \cdots \times A^N, E)$. We say that T is *compact* if the set $T(U_{A_1} \times \cdots \times U_{A_N})$ is relatively compact in E. Here

$$U_{A_1} \times \cdots \times U_{A_N} = \{(a^1, \dots, a^N) : \|a^j\|_{A^j} \leq 1, \quad 1 \leq j \leq N\}.$$

It is not hard to check that compactness of T is equivalent to any of the three following conditions:

(a) For all bounded sequences $\big((a_n^1, a_n^2, \dots, a_n^N)\big) \subseteq \prod_{j=1}^{N} A^j$, the sequence $\big(T(a_n^1, a_n^2, \dots, a_n^N)\big)$ has a convergent subsequence in E.

(b) For all bounded sets $B \subseteq \prod_{j=1}^{N} A^j$, $T(B)$ is relatively compact in E.

(c) For any $\varepsilon > 0$ there is a finite set $\{z_1, \dots, z_r\} \subseteq E$ such that

$$T(U_{A_1} \times \cdots \times U_{A_N}) \subseteq \bigcup_{k=1}^{r} \{z_k + \varepsilon U_E\}$$

(that is to say, $T(U_{A_1} \times \cdots \times U_{A_N})$ is precompact in E).

Using **(a)** it is not difficult to show that if $(T_n) \subseteq \mathcal{L}(A^1 \times \cdots A^N, E)$ is a sequence of compact N-linear operators such that (T_n) converge to T in $\mathcal{L}(A^1 \times \cdots \times A^N, E)$, then T is compact (see [3, Proposition 3] for the case of bilinear operators).

As for compositions, note that if $B^1, \dots B^N, F$ are quasi-Banach spaces, and we have operators $Q_j \in \mathcal{L}(B^j, A^j)$ for $j = 1, \dots, N$, $P \in \mathcal{L}(E, F)$, $T \in \mathcal{L}(A^1 \times \cdots \times A^N, E)$ with T being compact, then the operator

$$PT(Q_1, \dots, Q_N) : B^1 \times \cdots \times B^N \longrightarrow F$$

defined by

$$PT(Q_1, \dots, Q_N)(b^1, \dots, b^N) = P[T(Q_1 b^1, \dots, Q_N b^N)]$$

is also compact.

The next two propositions extend results of [20] to N-linear operators among quasi-Banach spaces.

Proposition 4.1. *Let $A^1, \dots, A^N$ be quasi-Banach spaces and let (E_0, E_1) be a quasi-Banach couple. If $T : A^1 \times \cdots \times A^N \longrightarrow E_0 \cap E_1$ is a bounded N-linear operator such that $T : A^1 \times \cdots \times A^N \longrightarrow E_i$ is compact for $i = 0$ or 1, then for any $0 < \theta < 1$ and $0 < q \leq \infty$, we have that $T : A^1 \times \cdots \times A^N \longrightarrow (E_0, E_1)_{\theta,q}$ is compact.*

Proof. Assume that $T : A^1 \times \cdots \times A^N \longrightarrow E_0$ is compact. Take any bounded sequence (z_n) in $\prod_{j=1}^{N} A^j$. Then $(Tz_n) \subseteq E_0 \cap E_1$ and there is $M > 0$ such that $\|Tz_n\|_{E_1} \leq M$ for any

$n \in \mathbb{N}$. By compactness of $T : A^1 \times \cdots \times A^N \longrightarrow E_0$, there exists a subsequence $(Tz_{n'})$ of (Tz_n) such that $(Tz_{n'})$ is a Cauchy sequence in E_0. Using (2.2), we obtain

$$\begin{aligned}\|Tz_{n'} - Tz_{m'}\|_{(E_0,E_1)_{\theta,q}} &\le C\|Tz_{n'} - Tz_{m'}\|_{E_0}^{1-\theta}\|Tz_{n'} - Tz_{m'}\|_{E_1}^{\theta}\\ &\le C(c_{E_1}2M)^{\theta}\|Tz_{n'} - Tz_{m'}\|_{E_0}^{1-\theta} \to 0 \text{ as } n', m' \to 0.\end{aligned}$$

So, $(Tz_{n'})$ is a Cauchy sequence in the complete space $(E_0, E_1)_{\theta,q}$ and compactness of $T : A^1 \times \cdots \times A^N \longrightarrow (E_0, E_1)_{\theta,q}$ follows.

The case when $T : A^1 \times \cdots \times A^N \longrightarrow E_1$ is compact can be treated analogously. □

Proposition 4.2. *Let (A_0^j, A_1^j) be quasi-Banach couples for $j = 1, \ldots, N$ and let E be a quasi-Banach space. If $T : (A_0^1 + A_1^1) \times \cdots \times (A_0^N + A_1^N) \longrightarrow E$ is a bounded N-linear operator such that the restriction $T : A_i^1 \times \cdots \times A_i^N \longrightarrow E$ is compact for $i = 0$ or 1, then for any $0 < \theta_j < 1$ and $0 < q_j \le \infty$ we have that*

$$T : (A_0^1, A_1^1)_{\theta_1,q_1} \times \cdots \times (A_0^N, A_1^N)_{\theta_N,q_N} \longrightarrow E$$

is compact.

Proof. Since $(A_0, A_1)_{\theta_j,q_j} \hookrightarrow (A_0, A_1)_{\theta_j,\infty}$, it is enough to establish the result with all q_j equal to ∞. We work with the equivalent quasi-norm in $(A_0, A_1)_{\theta_j,\infty}$ given by $\|a\|_{\theta_j,\infty} = \sup_{t>0}\{t^{-\theta_j}K(t,a)\}$. Suppose that $T : A_0^1 \times \cdots \times A_0^N \longrightarrow E$ is compact. The case when $T : A_1^1 \times \cdots \times A_1^N \longrightarrow E$ is compact can be carried out in the same way. Take any bounded set W in $(A_0^1, A_1^1)_{\theta_1,\infty} \times \cdots \times (A_0^N, A_1^N)_{\theta_N,\infty}$, then there is $M > 0$ such that for any $a = (a^j) \in W$ we have $\sum_{j=1}^N \|a^j\|_{\theta_j,\infty} < M$. We may assume that $M > 1$.

Take any $\nu = (\nu_1, \ldots, \nu_N) \in \{0,1\}^N$ and put

$$D^{\nu_j} = \begin{cases}(A_0^j, A_1^j)_{\theta_j,\infty} & \text{if } \nu_j = 0,\\ A_1^j & \text{if } \nu_j = 1.\end{cases}$$

The assumption on T yields that $T : D^{\nu_1} \times \cdots \times D^{\nu_N} \longrightarrow E$ is bounded. Let $L_\nu = \|T : D^{\nu_1} \times \cdots \times D^{\nu_N} \longrightarrow E\|$ and write $L = \max\{L_\nu : \nu \in \{0,1\}^N\}$.

In order to check that $T(W)$ is precompact, take any $\varepsilon > 0$ and let us find an ε-net for $T(W)$. We can pick t big enough so that

$$Mt^{\theta_j-1} < \min\Big\{\frac{\varepsilon}{2^N L M^{N-1} c_E^{2N}}, 1\Big\}, \quad j = 1, \ldots, N. \tag{4.1}$$

Given any $a = (a^j) \in W$, we can find decompositions $a^j = a_0^j + a_1^j$ with $a_i^j \in A_i^j$, $i = 0, 1$, and

$$\|a_1^j\|_{A_1^j} \le Mt^{\theta_j-1}, \quad \|a_0^j\|_{A_0^j} \le Mt^{\theta_j}. \tag{4.2}$$

Let $V \subseteq A_0^1 \times \cdots \times A_0^N$ be the set of all (v^j) where v^j is any of the elements of A_0^j which have appeared in the decompositions (4.2) for $a \in W$. Then V is bounded because, by (4.2), we have

$$\|(v^j)\|_{A_0^1 \times \cdots \times A_0^N} \le \sum_{j=1}^N Mt^{\theta_j} \text{ for any } (v^j) \in V.$$

Compactness of $T : A_0^1 \times \cdots \times A_0^N \longrightarrow E$ yields that there is a finite set $Z = \{z_1, \dots, z_r\} \subseteq E$ such that for any $(v^j) \in V$ there exists $z_s \in Z$ such that

$$\|T(v^j) - z_s\| \leq \frac{\varepsilon}{2^N c_E}. \tag{4.3}$$

We are going to show that Z is an ε-net for $T(W)$.

Take any $a = (a^j) \in W$ and let $a^j = a_0^j + a_1^j$ be the decomposition given by (4.2). Let $z_s \in Z$ satisfying (4.3) for $(v^j) = (a_0^j)$. Given $\nu = (\nu_1, \dots, \nu_N) \in \{0, 1\}^N$, $\nu \neq 0$, put

$$a^{\nu,j} = \begin{cases} a^j & \text{if } \nu_j = 0, \\ a_1^j & \text{if } \nu_j = 1. \end{cases}$$

Using [32, Proposition 4.1], we derive

$$\begin{aligned}\|Ta - z_s\|_E &= \|T(a_0^j) + \sum_{\nu\in\{0,1\}^N,\nu\neq 0} (-1)^{\nu_1+\cdots+\nu_N+1} T(a^{\nu,1}, \dots, a^{\nu,N}) - z_s\|_E \\ &\leq c_E\big(\|T(a_0^j) - z_s\|_E + c_E^{2^N-2} \sum_{\nu\in\{0,1\}^N,\nu\neq 0} \|T(a^{\nu,1}, \dots, a^{\nu,N})\|_E\big).\end{aligned}$$

Moreover, for any $\nu \in \{0, 1\}^N$, $\nu \neq 0$, if r is the number of 1 in ν, so $1 \leq r \leq N$, we obtain

$$\|T(a^{\nu,1}, \dots, a^{\nu,N})\|_E \leq L_\nu \prod_{j=1}^N \|a^{\nu,j}\|_{D^{\nu_j}} \leq L_\nu M^{N-r} \frac{\varepsilon}{2^N L M^{N-1} c_E^{2^N}} \leq \frac{\varepsilon}{2^N c_E^{2^N}},$$

where we have used (4.1) and (4.2) in the estimates. Consequently,

$$\|Ta - z_s\|_E \leq c_E\big(\frac{\varepsilon}{2^N c_E} + c_E^{2^N-2} \sum_{\nu\in\{0,1\}^N,\nu\neq 0} \frac{\varepsilon}{2^N c_E^{2^N}}\big) \leq \varepsilon. \quad \square$$

In order to establish a general result where we have quasi-Banach couples both in the domain and in the target of the N-linear operator, we shall also need the following auxiliary results.

Lemma 4.3. *Let A^j for $j = 1, \dots, N$, E, Y be quasi-Banach spaces and let D^j be a dense subspace of A^j. Assume that $T : A^1 \times \cdots \times A^N \longrightarrow E$ is a compact N-linear operator and let $(S_n) \subseteq \mathcal{L}(E, Y)$ with $\sup_{n\in\mathbb{N}} \|S_n\|_{E,Y} = M < \infty$. If $\lim_{n\to\infty} \|S_n T u\|_Y = 0$ for all $u \in D^1 \times \cdots \times D^N$, then $\lim_{n\to\infty} \|S_n T\|_{\prod_{j=1}^N A^j, Y} = 0$.*

Proof. Take any $\varepsilon > 0$. Since $\prod_{j=1}^N U_{A^j}$ is a bounded set in $A^1 \times \cdots \times A^N$, by compactness of T, there exists $\{z_1, \dots, z_r\} \subseteq \prod_{j=1}^N D^j$ with $z_k = (z_k^j)$, $\|z_k^j\|_{A^j} \leq 2$ and

$$T\Big(\prod_{j=1}^N U_{A^j}\Big) \subseteq \bigcup_{k=1}^r \{T z_k + \frac{\varepsilon}{2M c_Y} U_E\}.$$

Using the assumption on (S_n), we can find $N_0 \in \mathbb{N}$ such that for any $n \geq N_0$ we have

$$\|S_n T z_k\|_Y \leq \frac{\varepsilon}{2c_Y}, \qquad k = 1, \dots, r.$$

Whence, for any $n \geq N_0$ and any $a = (a^j) \in \prod_{j=1}^N U_{A^j}$, if $\|Ta - Tz_k\|_E \leq \varepsilon/2Mc_Y$, then

$$\begin{aligned}\|S_n T a\|_Y &\leq c_Y\big(\|S_n(Ta - Tz_k)\|_Y + \|S_n T z_k\|_Y\big) \\ &\leq c_Y M \|Ta - Tz_k\|_E + \frac{\varepsilon}{2} \leq \varepsilon. \quad \square\end{aligned}$$

Lemma 4.4. *Let* $(A_0^1, A_1^1), \dots, (A_0^N, A_1^N), (E_0, E_1)$ *be quasi-Banach couples and let* $A^1, \dots, A^N, E$ *be intermediate spaces with respect to* $(A_0^1, A_1^1), \dots, (A_0^N, A_1^N)$, (E_0, E_1) *respectively. Assume that* $T : (A_0^1 + A_1^1) \times \cdots \times (A_0^N + A_1^N) \longrightarrow E_0 + E_1$ *is a bounded* N*-linear operator and that* $T : A^1 \times \cdots \times A^N \longrightarrow E$ *is compact. Let* X^j, $j = 1, \dots N$, *be quasi-Banach spaces and let* $(R_n^j)_{n\in\mathbb{N}} \subseteq \mathcal{L}(X^j, A^j)$ *such that* $\sup_{n\in\mathbb{N}}\{\|R_n^j\|_{X^j,A^j}\} = M_j < \infty$ *and* $\lim_{n\to\infty} \|T(R_n^1, \dots, R_n^N)\|_{X^1\times\cdots\times X^N, E_0+E_1} = 0$. *Then* $\lim_{n\to\infty} \|T(R_n^1, \dots, R_n^N)\|_{X^1\times\cdots\times X^N, E} = 0$.

Proof. We proceed by contradiction. If this were not the case, there would exist $\varepsilon > 0$ such that for any $N_0 \in \mathbb{N}$ there would be $n \geq N_0$ satisfying that $\|T(R_n^1, \dots, R_n^N)\|_{X^1\times\cdots\times X^N, E} > \varepsilon$. Since

$$\sup_{n\in\mathbb{N}} \|T(R_n^1, \dots, R_n^N)\|_{X^1\times\cdots\times X^N, E} \leq \|T\|_{A^1\times\cdots\times A^N, E} \prod_{j=1}^{N} M_j,$$

there exist a subsequence (n') and vectors $x_{n'} = (x_{n'}^j) \in \prod_{j=1}^N U_{X^j}$ such that

$$\lim_{n'\to\infty} \|T(R_{n'}^1 x_{n'}^1, \dots, R_{n'}^N x_{n'}^N)\|_E \geq \varepsilon.$$

The sequence $\big((R_{n'}^1 x_{n'}^1, \dots, R_{n'}^N x_{n'}^N)\big)$ is bounded in $A^1 \times \cdots \times A^N$. Using the compactness of $T : A^1 \times \cdots \times A^N \longrightarrow E$ and passing to another subsequence (n'') if needed, we get that there is $w \in E$ such that

$$\big(T(R_{n''}^1 x_{n''}^1, \dots, R_{n''}^N x_{n''}^N)\big) \longrightarrow w \text{ in } E.$$

Hence $\|w\|_E \geq \varepsilon > 0$. However, since

$$\lim_{n\longrightarrow\infty} \|T(R_n^1, \dots, R_n^N)\|_{X^1\times\cdots\times X^N, E_0+E_1} = 0$$

and

$$\lim_{n''\to\infty} \big(T(R_{n''}^1 x_{n''}^1, \dots, R_{n''}^N x_{n''}^N)\big) = w \text{ in } E_0 + E_1,$$

we derive that $w = 0$, which contradicts that $w \neq 0$. □

Next we establish the main result of the paper.

Theorem 4.5. *Let* (A_0^j, A_1^j) *be quasi-Banach couples for* $j = 1, \dots, N$ *and let* (E_0, E_1) *be an* r*-normed quasi-Banach couple. Let* $T : (A_0^1 + A_1^1) \times \cdots \times (A_0^N + A_1^N) \longrightarrow E_0 + E_1$ *be a bounded* N*-linear operator such that the restrictions* $T : A_i^1 \times \cdots \times A_i^N \longrightarrow E_i$ *are bounded for* $i = 0, 1$ *and one of them is compact.*

Let $0 < \theta < 1, r \leq q_1, \dots, q_N \leq \infty$ *and suppose that*

$$\frac{1}{q} = \sum_{j=1}^{N} \frac{1}{q_j} - \frac{N-1}{r} \geq 0.$$

Then

$$T : (A_0^1, A_1^1)_{\theta,q_1} \times \cdots \times (A_0^N, A_1^N)_{\theta,q_N} \longrightarrow (E_0, E_1)_{\theta,q}$$

is compact.

Proof. We can find $0 < p \leq 1$ such that (A_0^j, A_1^j) is a p-normed quasi-Banach couple for $j = 1, \dots, N$. Then $((A_0^j)^\circ, (A_1^j)^\circ)$ is also a p-normed quasi-Banach couple. We have $((A_0^j)^\circ, (A_1^j)^\circ)_{\theta,q_j} = (A_0^j, A_1^j)_{\theta,q_j}$, with

$$T : ((A_0^1)^\circ + (A_1^1)^\circ) \times \cdots \times ((A_0^N)^\circ + (A_1^N)^\circ) \longrightarrow E_0 + E_1$$

and

$$T : (A_i^1)^\circ \times \cdots \times (A_i^N)^\circ \longrightarrow E_i, \quad i = 0, 1$$

bounded, and the last restriction being compact provided that $T : A_i^1 \times \cdots \times A_i^N \longrightarrow E_i$ is compact for $i = 0$ or $i = 1$. So, without loss of generality we may assume in the following that (A_0^j, A_1^j) is a p-normed quasi-Banach couple with $A_0^j \cap A_1^j$ dense in A_0^j and A_1^j for $j = 1, \dots, N$.

We assume that $T : A_1^1 \times \cdots \times A_1^N \longrightarrow E_1$ is compact. The case when $T : A_0^1 \times \cdots \times A_0^N \longrightarrow E_0$ is compact can be treated similarly.

For $m \in \mathbb{Z}$, consider the p-normed spaces

$$F_m^j = \big(A_0^j \cap A_1^j, J(2^m, \cdot; A_0^j, A_1^j)\big), \quad j = 1, \dots, N$$

and the r-normed spaces

$$W_m = \big(E_0 + E_1, K_r(2^m, \cdot; E_0, E_1)\big).$$

Realizing $(A_0^j, A_1^j)_{\theta,q_j}$ by means of the J-functional, the map $\pi(u_m) = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0^j + A_1^j$) is surjective from $\ell_p(2^{-\theta m} F_m^j)$ into $(A_0^j, A_1^j)_{\theta,q_j}$ and it induces the quasi-norm of $(A_0^j, A_1^j)_{\theta,q_j}$. We also have that $\pi : \ell_p(2^{-im} F_m^j) \longrightarrow A_i^j$ is bounded for $i = 0, 1$. On the other hand, the map $\tau w = (\dots, w, w, w, \dots)$ is a metric injection from $(E_0, E_1)_{\theta,q}$ into $\ell_q(2^{-\theta m} W_m)$ provided that we realize $(E_0, E_1)_{\theta,q}$ as a K-space but replacing the K-functional by the equivalent K_r-functional. Moreover, $\tau : E_i \longrightarrow \ell_\infty(2^{-im} W_m)$ is bounded for $i = 0, 1$.

Put $\widehat{T} = \tau T(\pi, \dots, \pi)$. Then

$$\widehat{T} : \big(\ell_p(F_m^1) + \ell_p(2^{-m} F_m^1)\big) \times \cdots \times \big(\ell_p(F_m^N) + \ell_p(2^{-m} F_m^N)\big) \longrightarrow \ell_\infty(W_m) + \ell_\infty(2^{-m} W_m)$$

is bounded, with the restrictions

$$\widehat{T} : \ell_p(2^{-im} F_m^1) \times \cdots \times \ell_p(2^{-im} F_m^N) \longrightarrow \ell_\infty(2^{-im} W_m)$$

being also bounded and compact if $i = 1$. Note that the quasi-Banach couples $\big(\ell_p(F_m^j), \ell_p(2^{-m} F_m^j)\big)$ are p-normed for $j = 1, \dots, N$ and $\big(\ell_\infty(W_m), \ell_\infty(2^{-m} W_m)\big)$ is r-normed. According to [17, p. 155], the following interpolation formulae hold with equivalence of quasi-norms

$$\big(\ell_p(F_m^j), \ell_p(2^{-m} F_m^j)\big)_{\theta,q_j} = \ell_{q_j}(2^{-\theta m} F_m^j), \quad j = 1, \dots, N, \tag{4.4}$$

$$\big(\ell_\infty(W_m), \ell_\infty(2^{-m} W_m)\big)_{\theta,q} = \ell_q(2^{-\theta m} W_m). \tag{4.5}$$

By the properties of π and τ, we have that

$$T : (A_0^1, A_1^1)_{\theta,q_1} \times \cdots \times (A_0^N, A_1^N)_{\theta,q_N} \longrightarrow (E_0, E_1)_{\theta,q} \text{ is compact}$$

if and only if

$$\widehat{T} : \ell_{q_1}(2^{-\theta m} F_m^1) \times \cdots \times \ell_{q_N}(2^{-\theta m} F_m^N) \longrightarrow \ell_q(2^{-\theta m} W_m) \text{ is compact.} \tag{4.6}$$

Since the vector valued spaces will allow us a more easy splitting of the operator, in what follows we focus our attention in proving the compactness of $\widehat{T}$ in (4.6). The following families of projections on $\big(\ell_\infty(W_m), \ell_\infty(2^{-m}W_m)\big)$ will be useful. For $n \in \mathbb{N}$, we put

$$P_n(w_m) = (\ldots 0, 0, w_{-n}, \ldots, w_n, 0, 0, \ldots),$$

$$P_n^+(w_m) = (\ldots 0, 0, w_{n+1}, w_{n+2}, w_{n+3}, \ldots),$$

$$P_n^-(w_m) = (\ldots, w_{-n-3}, w_{-n-2}, w_{-n-1}, 0, 0, \ldots).$$

Note that

$$P_n : \ell_\infty(W_m) + \ell_\infty(2^{-m}W_m) \longrightarrow \ell_\infty(W_m) \cap \ell_\infty(2^{-m}W_m) \text{ is bounded.} \tag{4.7}$$

Moreover

$$\|P_n^+\|_{\ell_\infty(W_m),\ell_\infty(2^{-m}W_m)} = 2^{-(n+1)} = \|P_n^-\|_{\ell_\infty(2^{-m}W_m),\ell_\infty(W_m)} \tag{4.8}$$

and for any $n \in \mathbb{N}$, we have $I = P_n + P_n^+ + P_n^-$, where I is the identity operator on $\ell_\infty(W_m) + \ell_\infty(2^{-m}W_m)$. The corresponding projections on $\big(\ell_p(F_m^j), \ell_p(2^{-m}F_m^j)\big)$ are denoted by Q_n, Q_n^+, Q_n^-. They have analogous properties. In particular, we have

$$\|Q_n^+\|_{\ell_p(F_m^j),\ell_p(2^{-m}F_m^j)} = 2^{-(n+1)} = \|Q_n^-\|_{\ell_p(2^{-m}F_m^j),\ell_p(F_m^j)}. \tag{4.9}$$

Using these projections we are going to split the operator $\widehat{T}$ in (4.6) into several pieces. Then we will show that some of them are compact and the others have norms tending to zero, which will yield the compactness of $\widehat{T}$.

We have $\widehat{T} = P_n\widehat{T} + P_n^+\widehat{T} + P_n^-\widehat{T}$. Consider the operator $P_n\widehat{T}$. Using (4.7), we obtain the following diagram

$$\begin{tikzcd}
\prod_{j=1}^N \ell_p(F_m^j) \arrow[r, "\widehat{T}"] & \ell_\infty(W_m) \arrow[dr, "P_n"] & \\
 & & \ell_q(2^{-\theta m}W_m). \\
\prod_{j=1}^N \ell_p(2^{-m}F_m^j) \arrow[r, "\widehat{T}"] & \ell_\infty(2^{-m}W_m) \arrow[ur, "P_n"'] &
\end{tikzcd}$$

Since $\widehat{T} : \prod_{j=1}^N \ell_p(2^{-m}F_m^j) \longrightarrow \ell_\infty(2^{-m}W_m)$ is compact, applying Proposition 4.2 and having in mind (4.4), we derive that

$$P_n\widehat{T} : \prod_{j=1}^N \ell_{q_j}(2^{-\theta m}F_m^j) \longrightarrow \ell_q(2^{-\theta m}W_m) \text{ is compact.}$$

Next we consider $P_n^+\widehat{T} = P_n^+\tau T(\pi, \ldots, \pi)$. We know that

$$\tau T : A_1^1 \times \cdots \times A_1^N \longrightarrow \ell_\infty(2^{-m}W_m) \text{ is compact.}$$

In addition, by (4.8), for any $a = (a^j) \in (A_0^1 \cap A_1^1) \times \cdots \times (A_0^N \cap A_1^N)$ we get

$$\|P_n^+\tau Ta\|_{\ell_\infty(2^{-m}W_m)} \le 2^{-(n+1)}\|\tau Ta\|_{\ell_\infty(W_m)} \to 0 \text{ as } n \to \infty.$$

Hence, using Lemma 4.3, we derive that

$$\lim_{n\to\infty} \|P_n^+\tau T\|_{\prod_{j=1}^N A_1^j,\ell_\infty(2^{-m}W_m)} = 0 \text{ and so } \lim_{n\to\infty} \|P_n^+\widehat{T}\|_{\prod_{j=1}^N \ell_p(2^{-m}F_m^j),\ell_\infty(2^{-m}W_m)} = 0.$$

Since

$$\|P_n^+\widehat{T}\|_{\prod_{j=1}^N \ell_p(F_m^j),\ell_\infty(W_m)} \le \|T\|_{\prod_{j=1}^N A_0^j,E_0},$$

applying Theorem 3.1 and having in mind (4.4) and (4.5), we conclude that

$$\|P_n^+\widehat{T}\|_{\prod_{j=1}^N \ell_{q_j}(2^{-\theta m}F_m^j),\ell_q(2^{-\theta m}W_m)} \to 0 \text{ as } n\to\infty.$$

As for $P_n^- T$, we start by splitting it with the help of projections Q. Since $I = Q_n + Q_n^+ + Q_n^-$, we have

$$P_n^-\widehat{T} = P_n^-\widehat{T}(Q_n,\dots,Q_n) + P_n^-\widehat{T}(Q_n^-,\dots,Q_n^-) + \sum P_n^-\widehat{T}(\widetilde{Q_n},\dots,\widetilde{Q_n})$$

where $\widetilde{Q_n}$ is any of the operators Q_n, Q_n^+, Q_n^- but with Q_n^+ appearing at least once in $(\widetilde{Q_n},\dots,\widetilde{Q_n})$. The last term means the finite sum of all operators of this type composed with $P_n^-\widehat{T}$.

For the operator $P_n^-\widehat{T}(Q_n,\dots,Q_n)$, according to the corresponding property to (4.7), we have the diagram

$$\begin{tikzcd}
 & \prod_{j=1}^N \ell_p(F_m^j) \arrow[r, "P_n^-\widehat{T}"] & \ell_\infty(W_m) \\
\prod_{j=1}^N \ell_{q_j}(2^{-\theta m}F_m^j) \arrow[ur, "\overline{Q}_n"] \arrow[dr, "\overline{Q}_n"'] & & \\
 & \prod_{j=1}^N \ell_p(2^{-m}F_m^j) \arrow[r, "P_n^-\widehat{T}"] & \ell_\infty(2^{-m}W_m)
\end{tikzcd}$$

where $\overline{Q}_n = (Q_n,\dots,Q_n)$. Since

$$P_n^-\widehat{T}(Q_n,\dots,Q_n) : \prod_{j=1}^N \ell_{q_j}(2^{-\theta m}F_m^j) \longrightarrow \ell_\infty(2^{-m}W_m)$$

is compact, according to Proposition 4.1 and (4.5), we obtain that

$$P_n^-\widehat{T}(Q_n,\dots,Q_n) : \prod_{j=1}^N \ell_{q_j}(2^{-\theta m}F_m^j) \longrightarrow \ell_q(2^{-\theta m}W_m)$$

is compact.

Consider now the operator $P_n^-\widehat{T}(Q_n^-,\dots,Q_n^-) = P_n^-\tau T(\pi Q_n^-,\dots,\pi Q_n^-)$. The norm of $T(\pi Q_n^-,\dots,\pi Q_n^-)$ from $\prod_{j=1}^N \ell_p(2^{-m}F_m^j)$ into E_0+E_1 can be estimated with the help of (4.9) as follows

$$\begin{aligned}
&\|T(\pi Q_n^-,\dots,\pi Q_n^-)\|_{\prod_{j=1}^N \ell_p(2^{-m}F_m^j),E_0+E_1} \\
&\qquad \le \|T\|_{\prod_{j=1}^N (A_0^j+A_1^j),E_0+E_1} \prod_{j=1}^N \|\pi\|_{\ell_p(F_m^j),A_0^j+A_1^j} \|Q_n^-\|_{\ell_p(2^{-m}F_m^j),\ell_p(F_m^j)} \\
&\qquad \le 2^{-(n+1)N}\|T\|_{\prod_{j=1}^N (A_0^j+A_1^j),E_0+E_1} \longrightarrow 0 \text{ as } n\to\infty.
\end{aligned}$$

Having in mind that $T : \prod_{j=1}^N A_1^j \longrightarrow E_1$ is compact and applying Lemma 4.4, we get that

$$\lim_{n\to\infty} \|T(\pi Q_n^-,\dots,\pi Q_n^-)\|_{\prod_{j=1}^N \ell_p(2^{-m}F_m^j),E_1} = 0.$$

This yields that

$$\lim_{n\to\infty} \|P_n^- \tau T(\pi Q_n^-, \ldots, \pi Q_n^-)\|_{\prod_{j=1}^N \ell_p(2^{-m}F_m^j), \ell_\infty(2^{-m}W_m)} = 0.$$

On the other hand,

$$\|P_n^- \widehat{T}(Q_n^-, \ldots, Q_n^-)\|_{\prod_{j=1}^N \ell_p(F_m^j), \ell_\infty(W_m)} \le \|\widehat{T}\|_{\prod_{j=1}^N A_0^j, E_0}.$$

Consequently, by Theorem 3.1, (4.4) and (4.5), we conclude that

$$\lim_{n\to\infty} \|P_n^- \widehat{T}(Q_n^-, \ldots, Q_n^-)\|_{\prod_{j=1}^N \ell_{q_j}(2^{-\theta m}F_m^j), \ell_q(2^{-\theta m}W_m)} = 0.$$

Finally, consider any of the operators in the sum $\sum P_n^- \widehat{T}(Q_n^{\sim}, \ldots, Q_n^{\sim})$. To fix ideas, we assume that Q_n^+ appears in the first position. So the operator is $P_n^- \widehat{T}(Q_n^+, Q_n^{\sim}, \ldots, Q_n^{\sim})$. Factorization

$$\begin{tikzcd}
\prod_{j=1}^N \ell_p(F_m^j) \arrow[r, "(Q_n^+, Q_n^{\sim}, \ldots, Q_n^{\sim})"] & \ell_p(2^{-m}F_m) \times \prod_{j=2}^N \ell_p(F_m^j) \arrow[d, "I"] \\
 & \prod_{j=1}^N \left(\ell_p(F_m^j) + \ell_p(2^{-m}F_m^j)\right) \arrow[d, "\widehat{T}"] \\
\ell_\infty(W_m) & \ell_\infty(W_m) + \ell_\infty(2^{-m}W_m) \arrow[l, "P_n^-"']
\end{tikzcd}$$

yields that

$$\begin{aligned}
&\| \; P_n^- \widehat{T}(Q_n^+, Q_n^{\sim}, \ldots, Q_n^{\sim}) \; \|_{\prod_{j=1}^N \ell_p(F_m^j), \ell_\infty(W_m)} \\
&\qquad \le 2^{-(n+1)} \|\widehat{T}\|_{\prod_{j=1}^N (\ell_p(F_m^j) + \ell_p(2^{-m}F_m^j)), \ell_\infty(W_m) + \ell_\infty(2^{-m}W_m)}.
\end{aligned}$$

Therefore, using Theorem 3.1, we derive that

$$\|P_n^- \widehat{T}(Q_n^+, Q_n^{\sim}, \ldots, Q_n^{\sim})\|_{\prod_{j=1}^N \ell_{q_j}(2^{-\theta m}F_m^j), \ell_q(2^{-\theta m}W_m)} \longrightarrow 0 \text{ as } n \to \infty.$$

Collecting all these estimates, we obtain that the operator $\widehat{T}$ in (4.6) is the limit of the sequence of compact operators $(P_n \widehat{T} + P_n^- \widehat{T}(Q_n, \ldots, Q_n))$. Therefore $\widehat{T}$ is compact and so

$$T : (A_0^1, A_1^1)_{\theta, q_1} \times \cdots \times (A_0^N, A_1^N)_{\theta, q_N} \longrightarrow (E_0, E_1)_{\theta, q}$$

is also compact. This finishes the proof. □

Proceeding as in Corollaries 3.2 and 3.3, we can complement Theorem 4.5 with the following results.

Corollary 4.6. *Let (A_0^j, A_1^j) be quasi-Banach couples for $j = 1, \ldots, N$ and let (E_0, E_1) be an r-normed quasi-Banach couple. Assume that*

$$T : (A_0^1 + A_1^1) \times \cdots \times (A_0^N + A_1^N) \longrightarrow E_0 + E_1$$

is a bounded N-linear operator such that the restrictions $T : A_i^1 \times \cdots \times A_i^N \longrightarrow E_i$ are bounded for $i = 0, 1$ and one of them is compact.

Let $0 < \theta < 1, 0 < q_1, \ldots, q_N \leq \infty$ such that

(a) $0 < q_1, \ldots, q_N \leq r$, *or*

(b) *only one q_j is greater than r.*

Put

$$q = \max\{q_1, \ldots, q_N\}.$$

Then

$$T : (A_0^1, A_1^1)_{\theta,q_1} \times \cdots \times (A_0^N, A_1^N)_{\theta,q_N} \longrightarrow (E_0, E_1)_{\theta,q} \textit{ is compact.}$$

Corollary 4.7. *Let (A_0^j, A_1^j) be quasi-Banach couples for $j = 1, \ldots, N$ and let (E_0, E_1) be an r-normed quasi-Banach couple. Assume that*

$$T : (A_0^1 + A_1^1) \times \cdots \times (A_0^N + A_1^N) \longrightarrow E_0 + E_1$$

is a bounded N-linear operator such that the restrictions $T : A_i^1 \times \cdots \times A_i^N \longrightarrow E_i$ are bounded for $i = 0, 1$ and one of them is compact.

Let $0 < \theta < 1, 0 < q_1, \ldots, q_N \leq \infty$ with $\min\{q_1, \ldots, q_N\} \leq r$ and suppose that

$$\frac{1}{q} = \sum_{j=1}^{N} \frac{1}{q_j} - \frac{N-1}{min\{q_1, \ldots q_N\}} \geq 0.$$

Then

$$T : (A_0^1, A_1^1)_{\theta,q_1} \times \cdots \times (A_0^N, A_1^N)_{\theta,q_N} \longrightarrow (E_0, E_1)_{\theta,q} \textit{ is compact.}$$

5. Applications to compact multilinear operators between L_p spaces

Given a $\sigma-$finite measure space (Ω, μ), we write $\mathfrak{M}(\mu)$ for the collection of all (equivalence classes of) measurable functions f on Ω which are finite almost everywhere. The space $\mathfrak{M}(\mu)$ is endowed with the topology of convergence in measure on each set of finite measure. We write χ_D for the characteristic function of a measurable set D.

For $0 < p \leq \infty$, we denote by $L_p = L_p(\Omega)$ the usual *Lebesgue space*. For $0 < p < \infty$ and $0 < q \leq \infty$, we put $L_{p,q} = L_{p,q}(\Omega)$ for the *Lorentz space* formed by all (equivalence classes of) measurable functions f on Ω which have a finite quasi-norm

$$\|f\|_{L_{p,q}(\Omega)} = \left(\int_0^{\mu(\Omega)} (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q}$$

(the integral should be replaced by the supremum if $q = \infty$). Here f^* is the non-increasing rearrangement of f. If $p = q$, then $L_p(\Omega) = L_{p,p}(\Omega)$. Spaces $L_p(\Omega)$ and $L_{p,q}(\Omega)$ are continuously embedded in $\mathfrak{M}(\mu)$.

If $1 < p < \infty$ and $1 \leq q \leq \infty$ then $L_{p,q}(\Omega)$ is a Banach space (with an equivalent norm) (see [8, Theorem 3.3.8], [36, Theorem 1.18.6/1] or [2, Theorem IV.4.6]).

According to [4, Theorem 5.3.1] or [36, 1.18.6], we have

$$\big(L_{p_0,r_0}(\Omega), L_{p_1,r_1}(\Omega)\big)_{\theta,q} = L_{p,q}(\Omega) \text{ (equivalent quasi-norms)} \tag{5.1}$$

provided that $0 < \theta < 1$, $0 < r_0, r_1, q \leq \infty$, $0 < p_0 \neq p_1 < \infty$ and $1/p = (1-\theta)/p_0 + \theta/p_1$.

Next we establish an interpolation theorem for compact $N-$linear operators between L_p spaces.

Theorem 5.1. *Let* (Ω, μ) *and* (Ω_j, μ_j) *be* $\sigma-$*finite measure spaces for* $1 \leq j \leq N$. *Suppose* $1 \leq p_0^j, p_1^j \leq \infty$ *and* $0 < r_0, r_1 \leq \infty$. *Let* $0 < \theta < 1$ *and put* $1/p_j = (1-\theta)/p_0^j + \theta/p_1^j$ *and* $1/r = (1-\theta)/r_0 + \theta/r_1$. *Suppose that* $p_j \neq \infty$ *for* $1 \leq j \leq N$.

Let T *be a bounded* $N-$*linear operator*

$$T : \left(L_{p_0^1}(\Omega_1) + L_{p_1^1}(\Omega_1)\right) \times \cdots \times \left(L_{p_0^N}(\Omega_N) + L_{p_1^N}(\Omega_N)\right) \longrightarrow L_{r_0}(\Omega) + L_{r_1}(\Omega)$$

such that for $i = 0, 1$ *the restrictions*

$$T : L_{p_i^1}(\Omega_1) \times \cdots \times L_{p_i^N}(\Omega_N) \longrightarrow L_{r_i}(\Omega)$$

are bounded. Moreover, we assume that $r_0 \neq \infty$ *and that*

$$T : L_{p_0^1}(\Omega_1) \times \cdots \times L_{p_0^N}(\Omega_N) \longrightarrow L_{r_0}(\Omega) \quad \textit{is compact}$$

Then

$$T : L_{p_1}(\Omega_1) \times \cdots \times L_{p_N}(\Omega_N) \longrightarrow L_r(\Omega) \quad \textit{is compact.}$$

Proof. We have that $r < \infty$ because $r_0 < \infty$. By [27, Lemma I.1.1] or [2, page 31], in order to check that $T\big(\prod_{j=1}^N U_{L_{p_j}(\Omega_j)}\big)$ is relatively compact in $L_r(\Omega)$ is sufficient to show that the following two conditions hold:

(a) $\lim_{\mu(D)\to 0} \|P_D T\|_{\prod_{j=1}^N L_{p_j}(\Omega_j), L_r(\Omega)} = 0,$ where $P_D f = \chi_D f$.

(b) $T\big(\prod_{j=1}^N U_{L_{p_j}(\Omega_j)}\big)$ is relatively compact in $\mathfrak{M}(\mu)$.

Given any $\mu-$measurable set $D \subseteq \Omega$, since $\|P_D T f\|_{L_{r_i}(\Omega_i)} \leq \|Tf\|_{L_{r_i}(\Omega_i)}$, we have that

$$P_D T : \big(L_{p_0^1}(\Omega_1) + L_{p_1^1}(\Omega_1)\big) \times \cdots \times \big(L_{p_0^N}(\Omega_N) + L_{p_1^N}(\Omega_N)\big) \longrightarrow L_{r_0}(\Omega) + L_{r_1}(\Omega)$$

is a bounded $N-$linear operator such that for $i = 0, 1$

$$P_D T : L_{p_i^1}(\Omega_1) \times \cdots \times L_{p_i^N}(\Omega_N) \longrightarrow L_{r_i}(\Omega)$$

is bounded. Moreover, compactness of $T : \prod_{j=1}^N L_{p_0^j}(\Omega_j) \to L_{r_0}(\Omega)$ yields that $\lim\limits_{\mu(D)\to 0} \|P_D T\|_{\prod_{j=1}^N L_{p_0^j}(\Omega_j), L_{r_0}(\Omega)} = 0$. Therefore, applying [10, Theorem B_1] we obtain that

$$\begin{aligned} \|P_D T\|_{\prod_{j=1}^N L_{p_j}(\Omega_j), L_r(\Omega)} &\leq \|P_D T\|^{1-\theta}_{\prod_{j=1}^N L_{p_0^j}(\Omega_j), L_{r_0}(\Omega)} \|P_D T\|^{\theta}_{\prod_{j=1}^N L_{p_1^j}(\Omega_j), L_{r_1}(\Omega)} \\ &\leq \|P_D T\|^{1-\theta}_{\prod_{j=1}^N L_{p_0^j}(\Omega_j), L_{r_0}(\Omega)} \|T\|^{\theta}_{\prod_{j=1}^N L_{p_1^j}(\Omega_j), L_{r_1}(\Omega)} \to 0 \end{aligned}$$

as $\mu(D) \to 0$. Hence, condition (a) holds.

As for (b), since $L_{r_i}(\Omega) \hookrightarrow L_{r_0}(\Omega) + L_{r_1}(\Omega)$ we have the following diagram

$$\begin{tikzcd}
\prod_{j=1}^N L_{p_j^0}(\Omega_j) \arrow[r, "T"] & L_{r_0}(\Omega) \arrow[dr, hook] & \\
& & L_{r_0}(\Omega) + L_{r_1}(\Omega). \\
\prod_{j=1}^N L_{p_j^1}(\Omega_j) \arrow[r, "T"] & L_{r_1}(\Omega) \arrow[ur, hook] &
\end{tikzcd}$$

Therefore, applying Proposition 4.2 with $\theta_j = \theta$, $q_j = p_j$, $j = 1, \dots, N$, and having in mind (5.1), we get that

$$T : L_{p_1}(\Omega_1) \times \cdots \times L_{p_N}(\Omega_N) \longrightarrow L_{r_0}(\Omega) + L_{r_1}(\Omega)$$

is compact. Since $L_{r_0}(\Omega) + L_{r_1}(\Omega) \hookrightarrow \mathfrak{M}(\Omega)$, we conclude that $T\big(\prod_{j=1}^{N} U_{L_{p_j}(\Omega_j)}\big)$ is relatively compact in $\mathfrak{M}(\Omega)$. This completes the proof. □

Writing down Theorem 5.1 for linear operators and $1 \le r_0, r_1 \le \infty$ we recover a classical result of Krasnosel'skiĭ [26].

We close the paper with a reinforced version of a multilinear Marcinkiewicz theorem due to Zafran [38, Theorem 2.9].

Theorem 5.2. *Let (Ω_j, μ_j), $1 \le j \le N$, and (Ω, μ) be $\sigma-$finite measure spaces. Suppose that $1 \le p_0^j \ne p_1^j \le \infty$, $1 < q_0 \ne q_1 \le \infty$ and $0 < \theta < 1$. Define $1/p_j = (1-\theta)/p_0^j + \theta/p_1^j$, $1/q = (1-\theta)/q_0 + \theta/q_1$ and suppose that $1/q \le \sum_{j=1}^{N} 1/p_j - N + 1$.*

Let T be a bounded $N-$linear operator

$$T : \big(L_{p_0^1,1}(\Omega_1)+L_{p_1^1,1}(\Omega_1)\big)\times\cdots\times\big(L_{p_0^N,1}(\Omega_N)+L_{p_1^N,1}(\Omega_N)\big) \longrightarrow L_{q_0,\infty}(\Omega)+L_{q_1,\infty}(\Omega)$$

such that the restrictions

$$T : L_{p_i^1,1}(\Omega_1) \times \cdots \times L_{p_i^N,1}(\Omega_N) \longrightarrow L_{q_i,\infty}(\Omega), \quad i = 0, 1, \tag{5.2}$$

are bounded and one of them is compact. Then

$$T : L_{p_1}(\Omega_1) \times \cdots \times L_{p_N}(\Omega_N) \longrightarrow L_q(\Omega) \quad \textit{is compact.}$$

Proof. Assume that $1 < q_1 < q_0 \le \infty$. The case $q_0 < q_1$ can be treated similarly. We have that $q_1 < q < q_0$. Pick r_0, r_1 such that $q_1 < r_1 < q < r_0 < q_0$ and pick θ_0, θ_1 such that $0 < \theta_0 < \theta < \theta_1 < 1$ and $1/r_i = (1-\theta_i)/q_0 + \theta_i/q_1$, $i = 0, 1$. For $j = 1, \dots, N$ and $i = 0, 1$, put $1/s_i^j = (1-\theta_i)/p_0^j + \theta_i/p_1^j$. Take also $0 < \eta < 1$ so that $\theta = (1-\eta)\theta_0 + \eta\theta_1$. Then we have that

$$\frac{1}{p_j} = \frac{1-\eta}{s_0^j} + \frac{\eta}{s_1^j} \quad \text{and} \quad \frac{1}{q} = \frac{1-\eta}{r_0} + \frac{\eta}{r_1}.$$

By (5.1), we know that

$$\big(L_{p_0^j,1}(\Omega_j), L_{p_1^j,1}(\Omega_j)\big)_{\theta_i,1} = L_{s_i^j,1}(\Omega_j) \text{ and } \big(L_{q_0,\infty}(\Omega), L_{q_1,\infty}(\Omega)\big)_{\theta_i,1} = L_{r_i,1}(\Omega).$$

Moreover, the couple $\big(L_{q_0,\infty}(\Omega), L_{q_1,\infty}(\Omega)\big)$ is formed by Banach spaces and so it is 1-normed. Applying Theorem 4.5 to restrictions (5.2) with parameters θ_i and $1 = \sum_{k=1}^{N} 1 - (N-1)$, we derive that restrictions

$$T : L_{s_i^1,1}(\Omega_1) \times \cdots \times L_{s_i^N,1}(\Omega_N) \longrightarrow L_{r_i,1}(\Omega),\ i = 0, 1,$$

are compact. Applying again Theorem 4.5 to these two restrictions now with parameters η and $1/u = \sum_{j=1}^{N} 1/p_j - (N-1)$, and using that

$$\big(L_{s_0^j,1}(\Omega_j), L_{s_1^j,1}(\Omega_j)\big)_{\eta,p_j} = L_{p_j}(\Omega_j),\ j = 1, \dots, N,$$

and that, since $u \le q$,

$$\big(L_{r_0,1}(\Omega), L_{r_1,1}(\Omega)\big)_{\eta,u} \hookrightarrow \big(L_{r_0,1}(\Omega), L_{r_1,1}(\Omega)\big)_{\eta,q} = L_q(\Omega),$$

we conclude that

$$T : L_{p_1}(\Omega_1) \times \cdots \times L_{p_N}(\Omega_N) \longrightarrow L_q(\Omega)$$

is compact. □

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No data was used for the research described in the article.

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