

SMOOTH EXTENSION OF FUNCTIONS ON A CERTAIN CLASS OF NON-SEPARABLE BANACH SPACES

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ABSTRACT. Let us consider a Banach space X with the property that every real-valued Lipschitz function f can be uniformly approximated by a Lipschitz, \mathcal{C}^1 smooth function g with $\text{Lip}(g) \leq C \text{Lip}(f)$ (with C depending only on the space X). This is the case for a Banach space X bi-Lipschitz homeomorphic to a subset of $c_0(\Gamma)$, for some set Γ , such that the coordinate functions of the homeomorphism are \mathcal{C}^1 -smooth ([11]). Then, we prove that for every closed subspace $Y \subset X$ and every \mathcal{C}^1 -smooth (Lipschitz) function $f : Y \rightarrow \mathbb{R}$, there is a \mathcal{C}^1 -smooth (Lipschitz, respectively) extension of f to X . We also study \mathcal{C}^1 -smooth extensions of real-valued functions defined on closed subsets of X . These results extend those given in [5] to the class of non-separable Banach spaces satisfying the above property.

1. INTRODUCTION AND MAIN RESULTS

In this note we consider the problem of the extension of a smooth function from a subspace of an infinite-dimensional Banach space to a smooth function on the whole space. More precisely, if X is an infinite-dimensional Banach space, Y is a closed subspace of X and $f : Y \rightarrow \mathbb{R}$ is a \mathcal{C}^k -smooth function, under what conditions does there exist a \mathcal{C}^k -smooth function $F : X \rightarrow \mathbb{R}$ such that $F|_Y = f$? Under the assumption that Y is a complemented subspace of a Banach space, an extension of a smooth function $f : Y \rightarrow \mathbb{R}$ is easily found taking the function $F(x) = f(P(x))$, where $P : X \rightarrow Y$ is a continuous linear projection. But this extension does not solve the problem since if a Banach space X is not isomorphic to a Hilbert space, then it has a closed subspace which is not complemented in X [14].

C. J. Atkin in [2] extends every smooth function f defined on a finite union of open convex sets in a separable Banach space which does not admit smooth bump functions, provided that for every point in the domain of f , the restriction of f to a suitable neighbourhood of the point can be extended to the whole space. The most fundamental result has been given by D. Azagra, R. Fry and L. Keener ([5] [6]). They have shown that if X is a Banach space with separable dual X^* , $Y \subset X$ is a closed subspace and $f : Y \rightarrow \mathbb{R}$ is a \mathcal{C}^1 -smooth function, then there exists a \mathcal{C}^1 -smooth extension $F : X \rightarrow \mathbb{R}$ of f . They proved a similar result when Y is a closed convex subset, f is defined on an open set U containing Y and f is \mathcal{C}^1 -smooth on Y as a function on X (i.e., $f : U \rightarrow \mathbb{R}$ is differentiable at every point $y \in Y$ and the function $Y \mapsto X^*$ defined as $y \mapsto f'(y)$ is continuous on Y). For a detailed account

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of the related theory of (smooth) extensions to \mathbb{R}^n of smooth functions defined in closed subsets of \mathbb{R}^n see [5]. Let us point out that the case of analytic maps is quite different (see [1]).

The aim of this note is to extend the results in [5] to the general setting of Banach spaces where every Lipschitz function can be approximated by a \mathcal{C}^1 -smooth, Lipschitz function. By using the results of Lipschitz and smooth approximation of Lipschitz mappings given by P. Hájek and M. Johanis ([10] [11]) we shall extend the results in [5] to a larger class of Banach spaces. We proceed along the same lines as the proof of the separable case [5]. Additionally, we shall use the open coverings given by M. E. Rudin, and the ideas of M. Moulis [15], P. Hájek and M. Johanis [11].

The notation we use is standard. In addition, we shall follow, whenever possible, the notations given in [5] and [11]. We denote by $\|\cdot\|$ the norm considered in X and by $B(x, r)$ the open ball with center $x \in X$ and radius $r > 0$. If Y is a subspace of X we denote the restriction of a function $f : X \rightarrow \mathbb{R}$ to Y by $f|_Y$ and we say that $F : X \rightarrow \mathbb{R}$ is an extension of $f : Y \rightarrow \mathbb{R}$ if $F|_Y = f$. Recall that $\text{Lip}(h)$ denotes the Lipschitz constant of a Lipschitz function $h : Y \rightarrow \mathbb{R}$, where Y is a subset of a Banach space X . We refer to [7] or [8] for any other definition.

Before stating the main results, let us define the property $(*)$ as the following Lipschitz and \mathcal{C}^1 -smooth approximation property for Lipschitz mappings.

Definition 1.1. *A Banach space X satisfies property $(*)$ if there is a constant C_0 , which only depends on the space X , such that, for any Lipschitz function $f : X \rightarrow \mathbb{R}$ and any $\varepsilon > 0$ there is a Lipschitz, \mathcal{C}^1 -smooth function $K : X \rightarrow \mathbb{R}$ such that*

$$|f(x) - K(x)| < \varepsilon \text{ for all } x \in X \text{ and } \text{Lip}(K) \leq C_0 \text{Lip}(f).$$

We may equivalently say that X satisfies property $(*)$ if there is a constant C_0 , which only depends on X , such that for any subset $Y \subset X$, any Lipschitz function $f : Y \rightarrow \mathbb{R}$ and any $\varepsilon > 0$ there is a \mathcal{C}^1 -smooth, Lipschitz function $K : X \rightarrow \mathbb{R}$ such that

$$|f(y) - K(y)| < \varepsilon \text{ for all } y \in Y \text{ and } \text{Lip}(K) \leq C_0 \text{Lip}(f).$$

Indeed, every real-valued Lipschitz function f defined on Y can be extended to a Lipschitz function on X with the same Lipschitz constant (for instance $F(x) = \inf_{y \in Y} \{f(y) + \text{Lip}(f)\|x - y\|\}$).

Let us recall that every Banach space with separable dual satisfies property $(*)$ [9, 11] (see also [5]). J.M. Lasry and P.L. Lions proved in [13] that in a Hilbert space H and for every Lipschitz function $f : H \rightarrow \mathbb{R}$ and every $\varepsilon > 0$ there exists a \mathcal{C}^1 -smooth and Lipschitz function $g : H \rightarrow \mathbb{R}$ such that $|f(x) - g(x)| < \varepsilon$ for every $x \in X$ and $\text{Lip}(g) = \text{Lip}(f)$ (see also [4]). Let us notice that the approximation in [13] is for bounded functions. However it also works for unbounded Lipschitz functions. P. Hájek and M. Johanis have proven in [10] that for any set Γ , $c_0(\Gamma)$ satisfies property $(*)$. Also, they give a sufficient condition for X to have property $(*)$: if there is a bi-Lipschitz homeomorphism φ embedding X into $c_0(\Gamma)$ whose coordinates $e_\gamma^* \circ \varphi$ are \mathcal{C}^1 -smooth, then X satisfies property $(*)$ ([11]). More specifically, they showed the following characterization: a Banach space X has property $(*)$ and is uniformly

homeomorphic to a subset of $c_0(\Gamma)$ (for some set Γ) if and only if there is a bi-Lipschitz homeomorphism φ embedding X into $c_0(\Gamma)$ whose coordinates $e_\gamma^* \circ \varphi$ are \mathcal{C}^1 -smooth.

We shall prove that, if X satisfies property $(*)$ and Y is a closed subspace of X , then for every \mathcal{C}^1 -smooth real-valued function f defined on Y , there is a \mathcal{C}^1 -extension of f to X .

Theorem 1.2. *Let X be a Banach space with property $(*)$. Let $Y \subset X$ be a closed subspace and $f : Y \rightarrow \mathbb{R}$ a \mathcal{C}^1 -smooth function. Then there is a \mathcal{C}^1 -smooth extension of f to X .*

Furthermore, if the given \mathcal{C}^1 -smooth function f is Lipschitz on Y , then there is a \mathcal{C}^1 -smooth and Lipschitz extension $H : X \rightarrow \mathbb{R}$ of f to X such that $\text{Lip}(H) \leq C \text{Lip}(f)$, where C is a constant depending only on X .

Corollary 1.3. *Let M be a paracompact \mathcal{C}^1 -smooth Banach manifold modeled on a Banach space X with property $(*)$ (in particular, any Riemannian manifold), and let N be a closed \mathcal{C}^1 -smooth submanifold of M . Then, every \mathcal{C}^1 -smooth function $f : N \rightarrow \mathbb{R}$ has a \mathcal{C}^1 -smooth extension to M .*

A similar result can be stated, as in the separable case [5], if Y is a closed convex subset of X , f is defined on an open subset U of X such that $Y \subset U$ and $f : U \rightarrow \mathbb{R}$ is \mathcal{C}^1 -smooth on Y as a function on U , i.e. $f : U \rightarrow \mathbb{R}$ is differentiable at every point $y \in Y$ and the mapping $Y \mapsto X^*$, $y \mapsto f'(y)$ is continuous on Y .

Theorem 1.4. *Let X be a Banach space with property $(*)$, $Y \subset X$ a closed convex subset, $U \subset X$ an open set containing Y and $f : U \rightarrow \mathbb{R}$ a \mathcal{C}^1 -smooth function on Y as a function on U . Then, there is a \mathcal{C}^1 -smooth extension of $f|_Y$ to X .*

Furthermore, if the given \mathcal{C}^1 -smooth function f is Lipschitz on Y , then there is a \mathcal{C}^1 -smooth and Lipschitz extension $H : X \rightarrow \mathbb{R}$ of $f|_Y$ to X such that $\text{Lip}(H) \leq C \text{Lip}(f|_Y)$, where C is a constant depending only on X .

Finally, we can conclude with the following corollary.

Corollary 1.5. *Let X be a Banach space such that there is a bi-Lipschitz homeomorphism between X and a subset of $c_0(\Gamma)$, for some set Γ , whose coordinate functions are \mathcal{C}^1 -smooth. Let $Y \subset X$ be a closed subspace and $f : Y \rightarrow \mathbb{R}$ a \mathcal{C}^1 -smooth function (respectively, \mathcal{C}^1 -smooth and Lipschitz function). Then there is a \mathcal{C}^1 -smooth extension H of f to X (respectively, a \mathcal{C}^1 -smooth and Lipschitz extension H of f to X with $\text{Lip}(H) \leq C \text{Lip}(f)$, where C is a constant depending only on X).*

Corollary 1.6. *Let X be one of the following Banach spaces:*

- (i) *a Banach space such that there is a bi-Lipschitz homeomorphism between X and a subset of $c_0(\Gamma)$, for some set Γ , whose coordinate functions are \mathcal{C}^1 -smooth,*
- (ii) *a Hilbert space.*

Let $Y \subset X$ be a closed convex subset, $U \subset X$ an open set containing Y and $f : U \rightarrow \mathbb{R}$ be a \mathcal{C}^1 -smooth function on Y as a function on U (respectively, \mathcal{C}^1 -smooth on Y as a function on U and Lipschitz on Y). Then, there is a \mathcal{C}^1 -smooth extension H of $f|_Y$ to X (respectively, \mathcal{C}^1 -smooth and Lipschitz extension H of $f|_Y$ to X with $\text{Lip}(H) \leq C \text{Lip}(f|_Y)$, where C is a constant depending only on X).

In the last section, we study under what conditions a real-valued function defined on a closed, non-convex subset Y of a Banach space X with property $(*)$ can be extended to a \mathcal{C}^1 -smooth function on X .

2. THE PROOFS

The first result we shall need is the existence of \mathcal{C}^1 -smooth and Lipschitz partitions of unity on Banach spaces satisfying property $(*)$. Recall that a Banach space X admits \mathcal{C}^1 -smooth and Lipschitz partitions of unity when for every open cover $\mathcal{U} = \{U_r\}_{r \in \Omega}$ of X there is a collection of \mathcal{C}^1 -smooth, Lipschitz functions $\{\psi_i\}_{i \in I}$ such that (1) $\psi_i \geq 0$ on X for every $i \in I$, (2) the family $\{\text{supp}(\psi_i)\}_{i \in I}$ is locally finite, where $\text{supp}(\psi_i) = \overline{\{x \in X : \psi_i(x) \neq 0\}}$, (3) $\{\psi_i\}_{i \in I}$ is subordinated to $\mathcal{U} = \{U_r\}_{r \in \Omega}$, i.e. for each $i \in I$ there is $r \in \Omega$ such that $\text{supp}(\psi_i) \subset U_r$ and (4) $\sum_{i \in I} \psi_i(x) = 1$ for every $x \in X$. Also let us denote by $\text{dist}(A, B)$ the distance between two sets A and B , that is to say $\inf\{\|a - b\| : a \in A, b \in B\}$.

The following lemma gives us the tool to generalize the construction of suitable open coverings on a Banach space, which will be key to obtain a generalization of the smooth extension result given in [5].

Lemma 2.1. (See M.E. Rudin, [16]) *Let E be a metric space, $\mathcal{U} = \{U_r\}_{r \in \Omega}$ be an open covering of E . Then, there are open refinements $\{V_{n,r}\}_{n \in \mathbb{N}, r \in \Omega}$ and $\{W_{n,r}\}_{n \in \mathbb{N}, r \in \Omega}$ of \mathcal{U} satisfying the following properties:*

- (i) $V_{n,r} \subset W_{n,r} \subset U_r$ for all $n \in \mathbb{N}$ and $r \in \Omega$,
- (ii) $\text{dist}(V_{n,r}, E \setminus W_{n,r}) \geq 1/2^{n+1}$ for all $n \in \mathbb{N}$ and $r \in \Omega$,
- (iii) $\text{dist}(W_{n,r}, W_{n,r'}) \geq 1/2^{n+1}$ for any $n \in \mathbb{N}$ and $r, r' \in \Omega$, $r \neq r'$,
- (iv) for every $x \in E$ there is an open ball $B(x, s_x)$ of E and a natural number n_x such that
 - (a) if $i > n_x$, then $B(x, s_x) \cap W_{i,r} = \emptyset$ for any $r \in \Omega$,
 - (b) if $i \leq n_x$, then $B(x, s_x) \cap W_{i,r} \neq \emptyset$ for at most one $r \in \Omega$.

P. Hájek and M. Johannis [11] proved that if a Banach space X satisfies property $(*)$ then X admits \mathcal{C}^1 -smooth and Lipschitz partitions of unity, which is, in turn, equivalent to the existence of a σ -discrete basis \mathcal{B} of the topology of X such that for every $B \in \mathcal{B}$ there is a \mathcal{C}^1 smooth and Lipschitz function $\psi_B : X \rightarrow [0, 1]$ with $B = \psi_B^{-1}(0, \infty)$ ([11], see also [12]). It is worth noting that given an open covering $\{U_r\}_{r \in \Omega}$ of X , it is not always possible to obtain a \mathcal{C}^1 -smooth and Lipschitz partition of unity $\{\psi_r\}_{r \in \Omega}$ with the same set of indexes such that $\text{supp}(\psi_r) \subset U_r$. For example, if A is a non-empty, closed subset of X , W is an open subset of X such that $A \subset W$ with $\text{dist}(A, X \setminus W) = 0$, and $\{\psi_1, \psi_2\}$ is a \mathcal{C}^1 -smooth partition of unity subordinated to $\{W, X \setminus A\}$, then $\psi_1(A) = 1$ and $\psi_1(X \setminus W) = 0$ and thus ψ_1 is not Lipschitz. Nevertheless, in order to prove Theorem 2.4 we only need the following statement.

Lemma 2.2. *Let X be a Banach space with property $(*)$. Then, for every $\{U_r\}_{r \in \Omega}$ open covering of X , there is an open refinement $\{W_{n,r}\}_{n \in \mathbb{N}, r \in \Omega}$ of $\{U_r\}_{r \in \Omega}$ satisfying the properties of Lemma 2.1, and there is a Lipschitz and \mathcal{C}^1 -smooth partition of unity $\{\psi_{n,r}\}_{n \in \mathbb{N}, r \in \Omega}$ such that $\text{supp}(\psi_{n,r}) \subset W_{n,r} \subset U_r$ and $\text{Lip}(\psi_{n,r}) \leq C_0 2^5 (2^n - 1)$ for every $n \in \mathbb{N}$ and $r \in \Omega$.*

Proof. Let us consider an open covering $\{U_r\}_{r \in \Omega}$ of X . By Lemma 2.1, there are open refinements $\{V_{n,r}\}_{n \in \mathbb{N}, r \in \Omega}$ and $\{W_{n,r}\}_{n \in \mathbb{N}, r \in \Omega}$ of $\{U_r\}_{r \in \Omega}$ satisfying the properties

(i)-(iv) of Lemma 2.1. Consider the distance function $D_n(x) = \text{dist}(x, X \setminus \bigcup_{r \in \Omega} W_{n,r})$ which is 1-Lipschitz. By applying property (*), there is a \mathcal{C}^1 -smooth, C_0 -Lipschitz function $g_n : X \rightarrow \mathbb{R}$ such that $|g_n(x) - D_n(x)| < \frac{1}{2^{n+3}}$ for every $x \in X$. Thus $g_n(x) > \frac{1}{2^{n+2}}$ whenever $x \in \bigcup_{r \in \Omega} V_{n,r}$ and $g_n(x) < \frac{1}{2^{n+3}}$ whenever $x \in X \setminus \bigcup_{r \in \Omega} W_{n,r}$. By composing g_n with a suitable C^∞ smooth function $\varphi_n : \mathbb{R} \rightarrow [0, 1]$ with $\text{Lip}(\varphi_n) \leq 2^{n+4}$ we obtain a \mathcal{C}^1 -smooth function $h_n := \varphi_n(g_n)$ that is zero on an open set including $X \setminus \bigcup_{r \in \Omega} W_{n,r}$, $h_n|_{\bigcup_{r \in \Omega} V_{n,r}} \equiv 1$ and $\text{Lip}(h_n) \leq C_0 2^{n+4}$. Now, let us define

$$H_1 = h_1, \text{ and } H_n = h_n(1 - h_1) \cdots (1 - h_{n-1}) \text{ for } n \geq 2.$$

It is clear that $\sum_n H_n(x) = 1$ for all $x \in X$. Since $\text{supp}(h_n) \subset \bigcup_{r \in \Omega} W_{n,r}$ and $\overline{W_{n,r}} \cap \overline{W_{n,r'}} = \emptyset$ for every $n \in \mathbb{N}$ and $r \neq r'$, we can write $h_n = \sum_{r \in \Omega} h_{n,r}$, where $h_{n,r}(x) = h_n(x)$ on $W_{n,r}$ and $\text{supp}(h_{n,r}) \subset W_{n,r}$. Notice that $\text{Lip}(h_{n,r}) \leq \text{Lip}(h_n) \leq C_0 2^{n+4}$. Let us define, for every $r \in \Omega$,

$$\psi_{1,r} = h_{1,r}, \text{ and } \psi_{n,r} = h_{n,r}(1 - h_1) \cdots (1 - h_{n-1}) \text{ for each } n \geq 2.$$

The functions $\{\psi_{n,r}\}_{n \in \mathbb{N}, r \in \Omega}$ satisfy that

- (i) they are \mathcal{C}^1 -smooth and Lipschitz, with $\text{Lip}(\psi_{n,r}) \leq C_0 \sum_{i=5}^{n+4} 2^i = C_0 2^5 (2^n - 1)$,
- (ii) $\text{supp}(\psi_{n,r}) \subset \text{supp}(h_{n,r}) \subset W_{n,r}$ and
- (iii) for every $x \in X$,

$$\sum_{n \in \mathbb{N}, r \in \Omega} \psi_{n,r}(x) = \sum_{r \in \Omega} \psi_{1,r}(x) + \sum_{n \geq 2} \left(\sum_{r \in \Omega} h_{n,r}(x) \right) \prod_{i=1}^{n-1} (1 - h_i(x)) = \sum_{n \in \mathbb{N}} H_n(x) = 1.$$

□

An alternative proof of Theorem 2.4 uses the existence of the σ -discrete basis for X previously mentioned and the construction, as above, of suitable \mathcal{C}^1 smooth and Lipschitz partitions of unity subordinated to any subfamily of this basis.

The following lemma is a necessary modification of property (*) to show the main results.

Lemma 2.3. *Let X be a Banach space with the property (*). Then for every subset $Y \subset X$, every continuous function $F : X \rightarrow \mathbb{R}$ such that $F|_Y$ is Lipschitz, and every $\varepsilon > 0$, there exists a \mathcal{C}^1 -smooth function $G : X \rightarrow \mathbb{R}$ such that*

- (i) $|F(x) - G(x)| \leq \varepsilon$ for all $x \in X$, and
- (ii) $\text{Lip}(G|_Y) \leq C_0 \text{Lip}(F|_Y)$. Moreover, $\|G'(y)\|_{X^*} \leq C_0 \text{Lip}(F|_Y)$ for all $y \in Y$, where C_0 is the constant given by property (*).
- (iii) In addition, if F is Lipschitz on X , there exists a constant $C_1 \geq C_0$ that depends only on X , such that the function G can be chosen to be Lipschitz on X and $\text{Lip}(G) \leq C_1 \text{Lip}(F)$.

Proof. Assume that the function $F : X \rightarrow \mathbb{R}$ is continuous on X and $F|_Y$ is Lipschitz. Since X admits \mathcal{C}^1 -smooth partitions of unity, there is (by [7, Theorem VIII 3.2]) a \mathcal{C}^1 -smooth function $h : X \rightarrow \mathbb{R}$, such that $|F(x) - h(x)| < \varepsilon$ for all $x \in X$. Let $\tilde{F} : X \rightarrow \mathbb{R}$ be a Lipschitz extension of $F|_Y$ to X with $\text{Lip}(\tilde{F}) = \text{Lip}(F|_Y)$. Let us apply property (*) to \tilde{F} to obtain a \mathcal{C}^1 -smooth, Lipschitz function $g : X \rightarrow \mathbb{R}$ such that $|\tilde{F}(x) - g(x)| < \varepsilon/4$ for every $x \in X$, and $\text{Lip}(g) \leq C_0 \text{Lip}(F|_Y)$. Consider the

open sets $A = \{x \in X : |F(x) - \tilde{F}(x)| < \varepsilon/4\}$, $B = \{x \in X : |F(x) - \tilde{F}(x)| < \varepsilon/2\}$ in X and the closed set $C = \{x \in X : |F(x) - \tilde{F}(x)| \leq \varepsilon/4\}$ in X . Then $Y \subset A \subset C \subset B$. By [7, Proposition VIII 3.7] there is a \mathcal{C}^1 -smooth function $u : X \rightarrow [0, 1]$ such that

$$u(x) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{if } x \in X \setminus B. \end{cases}$$

Let us define $G : X \rightarrow \mathbb{R}$ as

$$G(x) := u(x)g(x) + (1 - u(x))h(x).$$

It is clear that G is a \mathcal{C}^1 -smooth function. Since $u(x) = 0$ for all $x \in X \setminus B$, we deduce that $|F(x) - G(x)| = |F(x) - h(x)| < \varepsilon$ for all $x \in X \setminus B$. Now, if $x \in B$, then $|F(x) - G(x)| \leq u(x)|F(x) - g(x)| + (1 - u(x))|F(x) - h(x)| \leq u(x)(|F(x) - \tilde{F}(x)| + |\tilde{F}(x) - g(x)|) + (1 - u(x))|F(x) - h(x)| \leq \varepsilon$. Finally, since $u(x) = 1$ and $G(x) = g(x)$ for every $x \in A$, we obtain that $\text{Lip}(G|_Y) = \text{Lip}(g|_Y) \leq C_0 \text{Lip}(F|_Y)$ and $\|G'(y)\|_{X^*} = \|g'(y)\|_{X^*} \leq C_0 \text{Lip}(F|_Y)$ for all $y \in Y$.

Let us now assume that F is Lipschitz on X . Let us apply property (*) to F and \tilde{F} (where \tilde{F} is a Lipschitz extension of $F|_Y$ to X with $\text{Lip}(\tilde{F}) = \text{Lip}(F|_Y)$). Thus, we obtain \mathcal{C}^1 -smooth and Lipschitz functions $g, h : X \rightarrow \mathbb{R}$ such that

- (a) $|\tilde{F}(x) - g(x)| < \varepsilon/4$, for all $x \in X$,
- (b) $|F(x) - h(x)| < \varepsilon$, for every $x \in X$ and
- (c) $\text{Lip}(g) \leq C_0 \text{Lip}(F|_Y)$ and $\text{Lip}(h) \leq C_0 \text{Lip}(F)$.

We take again the subsets A, B and C as in the previous case. Notice that $\text{dist}(C, X \setminus B) \geq \frac{\varepsilon}{4(\text{Lip}(F) + \text{Lip}(F|_Y))} = \varepsilon'$. Let us prove that there is a \mathcal{C}^1 -smooth, Lipschitz function $u : X \rightarrow [0, 1]$ such that $u(x) = 1$ on C and $u(x) = 0$ on $X \setminus B$, with $\text{Lip}(u) \leq \frac{9C_0(\text{Lip}(F) + \text{Lip}(F|_Y))}{\varepsilon}$. Let us consider $0 < r \leq \varepsilon'/4$, and the distance function $D : X \rightarrow \mathbb{R}$, $D(x) = \text{dist}(x, C)$. Since the function D is 1-Lipschitz, we apply property (*) to obtain a \mathcal{C}^1 -smooth, Lipschitz function $R : X \rightarrow \mathbb{R}$ such that $\text{Lip}(R) \leq C_0$ and $|D(x) - R(x)| < r$ for all $x \in X$. Also, let us take a \mathcal{C}^1 -smooth and Lipschitz function $\varphi : \mathbb{R} \rightarrow [0, 1]$ with (i) $\varphi(t) = 1$ whenever $|t| \leq r$, (ii) $\varphi(t) = 0$ whenever $|t| \geq \varepsilon' - r$ and (iii) $\text{Lip}(\varphi) \leq \frac{9}{8(\varepsilon' - 2r)} \leq \frac{9}{4\varepsilon'}$. Next, we define the \mathcal{C}^1 -smooth function $u : X \rightarrow [0, 1]$, $u(x) = \varphi(R(x))$. Notice that $\text{Lip}(u) \leq \frac{9C_0(\text{Lip}(F) + \text{Lip}(F|_Y))}{\varepsilon}$.

Let us now consider $G : X \rightarrow \mathbb{R}$ as

$$G(x) = u(x)g(x) + (1 - u(x))h(x).$$

Clearly G is \mathcal{C}^1 smooth on X . We follow the above proof to obtain that $|F(x) - G(x)| < \varepsilon$ on X , $\text{Lip}(G|_Y) = \text{Lip}(g|_Y) \leq C_0 \text{Lip}(F|_Y)$ and $\|G'(y)\|_{X^*} \leq C_0 \text{Lip}(F|_Y)$ for all $y \in Y$. Additionally, if $x \in X \setminus \bar{B}$, then $u(x) = 0$, $G(x) = h(x)$, and

$\|G'(x)\|_{X^*} = \|h'(x)\|_{X^*} \leq C_0 \text{Lip}(F)$. For $x \in \overline{B}$, we have

$$\begin{aligned} \|G'(x)\|_{X^*} &\leq \\ &\leq \|g(x)u'(x) + h(x)(1-u)'(x)\|_{X^*} + \|u(x)g'(x) + (1-u(x))h'(x)\|_{X^*} \leq \\ &\leq \|(g(x) - F(x))u'(x) + (h(x) - F(x))(1-u)'(x)\|_{X^*} + C_0 \text{Lip}(F) \leq \\ &\leq (|g(x) - \tilde{F}(x)| + |\tilde{F}(x) - F(x)| + |h(x) - F(x)|) \|u'(x)\|_{X^*} + C_0 \text{Lip}(F) \leq \\ &\leq \frac{7\varepsilon}{4} \cdot \frac{9C_0(\text{Lip}(F) + \text{Lip}(F|_Y))}{\varepsilon} + C_0 \text{Lip}(F) \leq 33C_0 \text{Lip}(F). \end{aligned}$$

We define $C_1 := 33C_0$ and obtain that $\text{Lip}(G) \leq C_1 \text{Lip}(F)$.

□

The following approximation result is the key to prove Theorem 1.2. Recall that the separable case was given in [5, Theorem 1].

Theorem 2.4. *Let X be a Banach space with property $(*)$, and $Y \subset X$ a closed subspace. Let $f : Y \rightarrow \mathbb{R}$ be a \mathcal{C}^1 -smooth function, and F a continuous extension of f to X . Then, for every $\varepsilon > 0$ there exists a \mathcal{C}^1 -smooth function $G : X \rightarrow \mathbb{R}$ such that if $g = G|_Y$ then*

- (i) $|F(x) - G(x)| < \varepsilon$ on X , and
- (ii) $\|f'(y) - g'(y)\|_{Y^*} < \varepsilon$ on Y .
- (iii) Furthermore, if f is Lipschitz on Y and F is a Lipschitz extension of f to X , then the function G can be chosen to be Lipschitz on X and $\text{Lip}(G) \leq C_2 \text{Lip}(F)$, where C_2 is a constant only depending on X .

Proof. We follow the steps given in the proof of the separable case with the necessary modifications. Notice that by the Tietze theorem, a continuous extension F of f always exists. Since X is a Banach space, $Y \subset X$ is a closed subspace and f' is a continuous function on Y , there exists $\{B(y_\gamma, r_\gamma)\}_{\gamma \in \Gamma}$ a covering of Y by open balls of X , with centers $y_\gamma \in Y$, $0 \notin \Gamma$, such that

$$(2.1) \quad \|f'(y_\gamma) - f'(y)\|_{Y^*} < \frac{\varepsilon}{8C_0}$$

on $B(y_\gamma, r_\gamma) \cap Y$, where C_0 is the positive constant given by property $(*)$ (which depends only on X). We denote $B_\gamma := B(y_\gamma, r_\gamma)$.

Let us define T_γ an extension of the first order Taylor Polynomial of f at y_γ given by $T_\gamma(x) = f(y_\gamma) + H(f'(y_\gamma))(x - y_\gamma)$, for $x \in X$, where $H(f'(y_\gamma)) \in X^*$ denotes a Hahn-Banach extension of $f'(y_\gamma)$ with the same norm, i.e. $\|H(f'(y_\gamma))\|_{X^*} = \|f'(y_\gamma)\|_{Y^*}$. Notice that T_γ satisfies the following properties:

- (B.1) T_γ is \mathcal{C}^∞ -smooth on X ,
- (B.2) $T'_\gamma(x) = H(f'(y_\gamma))$ for all $x \in X$, $T'_\gamma(y)|_Y = f'(y_\gamma)$ for every $y \in Y$ and
- (B.3) from (B.2), (2.1) and the fact that $B_\gamma \cap Y$ is convex, we deduce $\text{Lip}((T_\gamma - F)|_{B_\gamma \cap Y}) \leq \frac{\varepsilon}{8C_0}$.

Since $F : X \rightarrow \mathbb{R}$ is a continuous function, and X admits \mathcal{C}^1 -smooth partitions of unity, there is a \mathcal{C}^1 -smooth function $F_0 : X \rightarrow \mathbb{R}$ such that $|F(x) - F_0(x)| < \frac{\varepsilon}{2}$ for every $x \in X$.

Let us denote $B_0 := X \setminus Y$, $\Sigma := \Gamma \cup \{0\}$, and $\mathcal{C} := \{B_\beta : \beta \in \Sigma\}$, which is a covering of X . By Lemma 2.2, there is an open refinement $\{W_{n,\beta}\}_{n \in \mathbb{N}, \beta \in \Sigma}$

of $\mathcal{C} = \{B_\beta : \beta \in \Sigma\}$ satisfying the four properties of Lemma 2.1. In particular, $W_{n,\beta} \subset B_\beta$ and for each $x \in X$ there is an open ball $B(x, s_x)$ of X with center x and radius $s_x > 0$, and a natural number n_x such that

- (1) if $i > n_x$, then $B(x, s_x) \cap W_{i,\beta} = \emptyset$ for every $\beta \in \Sigma$,
- (2) if $i \leq n_x$, then $B(x, s_x) \cap W_{i,\beta} \neq \emptyset$ for at most one $\beta \in \Sigma$.

There is, by Lemma 2.2, a \mathcal{C}^1 -smooth and Lipschitz partition of unity $\{\psi_{n,\beta}\}_{n \in \mathbb{N}, \beta \in \Sigma}$ such that $\text{supp}(\psi_{n,\beta}) \subset W_{n,\beta} \subset B_\beta$ and $\text{Lip}(\psi_{n,\beta}) \leq C_0 2^5 (2^n - 1)$ for every $(n, \beta) \in \mathbb{N} \times \Sigma$.

Let us define $L_{n,\beta} := \max\{\text{Lip}(\psi_{n,\beta}), 1\}$ for every $n \in \mathbb{N}$ and $\beta \in \Sigma$. Now, for every $n \in \mathbb{N}$ and $\gamma \in \Gamma$, we apply Lemma 2.3 to $T_\gamma - F$ on $B_\gamma \cap Y$ to obtain a \mathcal{C}^1 -smooth map $\delta_{n,\gamma} : X \rightarrow \mathbb{R}$ so that

$$(C.1) \quad |T_\gamma(x) - F(x) - \delta_{n,\gamma}(x)| < \frac{\varepsilon}{2^{n+2} L_{n,\gamma}} \quad \text{for every } x \in X$$

and

$$(C.2) \quad \|\delta'_{n,\gamma}(y)\|_{X^*} \leq \frac{\varepsilon}{8} \quad \text{for every } y \in B_\gamma \cap Y.$$

From inequality (2.1), (B.2) and (C.2), we have for all $y \in B_\gamma \cap Y$,

$$\|T'_\gamma(y) - f'(y) - \delta'_{n,\gamma}(y)\|_{Y^*} \leq \|T'_\gamma(y) - f'(y)\|_{Y^*} + \|\delta'_{n,\gamma}(y)\|_{Y^*} < \frac{\varepsilon}{4}.$$

Notice that in the above inequality we consider the norm on Y^* (i.e. the norm of the functional restricted to Y). Let us define

$$(2.2) \quad \Delta_\beta^n(x) = \begin{cases} F_0(x), & \text{if } \beta = 0, \\ T_\beta(x) - \delta_{n,\beta}(x), & \text{if } \beta \in \Gamma. \end{cases}$$

Thus, $|\Delta_\beta^n(x) - F(x)| < \frac{\varepsilon}{2}$ whenever $n \in \mathbb{N}$, $\beta \in \Sigma$ and $x \in B_\beta$. We now define

$$G(x) = \sum_{(n,\beta) \in \mathbb{N} \times \Sigma} \psi_{n,\beta}(x) \Delta_\beta^n(x).$$

Since $\{\psi_{n,\beta}\}_{n \in \mathbb{N}, \beta \in \Sigma}$ is locally finitely nonzero, then G is \mathcal{C}^1 -smooth. Now, if $x \in X$ and $\psi_{n,\beta}(x) \neq 0$, then $x \in B_\beta$ and thus

$$|G(x) - F(x)| \leq \sum_{(n,\beta) \in \mathbb{N} \times \Sigma} \psi_{n,\beta}(x) |\Delta_\beta^n(x) - F(x)| \leq \sum_{(n,\beta) \in \mathbb{N} \times \Sigma} \psi_{n,\beta}(x) \frac{\varepsilon}{2} < \varepsilon.$$

Let us now estimate the distance between the derivatives. From the definitions given above, notice that

- (D.1) since $\sum_{\mathbb{N} \times \Sigma} \psi_{n,\beta}(x) = 1$ for all $x \in X$, we have that $\sum_{\mathbb{N} \times \Sigma} \psi'_{n,\beta}(x) = 0$ for all $x \in X$.
- (D.2) Thus, we can write $f'(y) = \sum_{\mathbb{N} \times \Sigma} (\psi'_{n,\beta}(y))|_Y f(y) + \sum_{\mathbb{N} \times \Sigma} \psi_{n,\beta}(y) f'(y)$, for every $y \in Y$.
- (D.3) $\text{supp}(\psi_{n,0}) \subset B_0 = X \setminus Y$, for all n .
- (D.4) $G'(x) = \sum_{\mathbb{N} \times \Sigma} \psi'_{n,\beta}(x) \Delta_\beta^n(x) + \sum_{\mathbb{N} \times \Sigma} \psi_{n,\beta}(x) (\Delta_\beta^n)'(x)$, for all $x \in X$.
- (D.5) If $g = G|_Y$ then $g'(y) = \sum_{\mathbb{N} \times \Sigma} \psi'_{n,\beta}(y)|_Y \Delta_\beta^n(y) + \sum_{\mathbb{N} \times \Sigma} \psi_{n,\beta}(y) (\Delta_\beta^n)'(y)|_Y$, for every $y \in Y$.

(D.6) Properties (1) and (2) of the open refinement $\{W_{n,\beta}\}$ imply that for every $x \in X$ and $n \in \mathbb{N}$, there is at most one $\beta \in \Sigma$, which we shall denote by $\beta_x(n)$, such that $x \in \text{supp}(\psi_{n,\beta})$. In the case that $y \in Y$, then $\beta_y(n) \in \Gamma$. We define $F_x := \{(n, \beta) \in \mathbb{N} \times \Sigma : x \in \text{supp}(\psi_{n,\beta})\}$. In particular, $F_y \subset \mathbb{N} \times \Gamma$, whenever $y \in Y$.

We obtain, for $y \in Y$,

(2.3)

$$\begin{aligned}
& \|g'(y) - f'(y)\|_{Y^*} \leq \\
& \leq \sum_{(n,\beta) \in F_y} \|\psi'_{n,\beta}(y)\|_{Y^*} |T_\beta(y) - f(y) - \delta_{n,\beta}(y)| \\
& \quad + \sum_{(n,\beta) \in F_y} \psi_{n,\beta}(y) \|T'_\beta(y) - f'(y) - \delta'_{n,\beta}(y)\|_{Y^*} \leq \\
& \leq \sum_{\{n: (n, \beta_y(n)) \in F_y\}} L_{n, \beta_y(n)} |T_{\beta_y(n)}(y) - f(y) - \delta_{n, \beta_y(n)}(y)| + \\
& \quad + \sum_{\{n: (n, \beta_y(n)) \in F_y\}} \psi_{n, \beta_y(n)}(y) \|T'_{\beta_y(n)}(y) - f'(y) - \delta'_{n, \beta_y(n)}(y)\|_{Y^*} \leq \\
& \leq \sum_{\{n: (n, \beta_y(n)) \in F_y\}} (L_{n, \beta_y(n)} \frac{\varepsilon}{2^{n+2} L_{n, \beta_y(n)}} + \psi_{n, \beta_y(n)}(y) \frac{\varepsilon}{4}) \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon,
\end{aligned}$$

where all the functionals involved are considered restricted to Y .

Let us now consider the case when f is \mathcal{C}^1 -smooth and Lipschitz on Y and F is a Lipschitz extension of f on X . In this case, we can assume that f is not constant (otherwise the assertion is trivial) and thus $\text{Lip}(F) \geq \text{Lip}(f) > 0$. Let us fix $0 < \varepsilon < \text{Lip}(F)$. If we follow the above construction for the open covering $\{B_\beta\}_{\beta \in \Sigma}$ of X satisfying the conditions (2.1), (B.1), (B.2) and (B.3), we additionally obtain

(B.4) $T_\beta - F$ is Lipschitz on X and $\text{Lip}(T_\beta - F) \leq \text{Lip}(f) + \text{Lip}(F)$ for every $\beta \in \Gamma$.

Also, the construction of the open refinement $\{W_{n,\beta}\}_{n \in \mathbb{N}, \beta \in \Sigma}$ of $\mathcal{C} = \{B_\beta\}_{\beta \in \Sigma}$, the Lipschitz partition of unity $\{\psi_{n,\beta}\}_{n \in \mathbb{N}, \beta \in \Sigma}$ and the definition of $L_{n,\beta}$ are similar to the previous case (recall that $\Sigma = \Gamma \cup \{0\}$ and $B_0 = X \setminus Y$).

Now, for any $n \in \mathbb{N}$ and $\beta \in \Gamma$, we apply Lemma 2.3 to $T_\beta - F$ on $B_\beta \cap Y$ to obtain a \mathcal{C}^1 -smooth map $\delta_{n,\beta} : X \rightarrow \mathbb{R}$ satisfying conditions (C.1), (C.2) and

$$(C.3) \quad \text{Lip}(\delta_{n,\beta}) \leq C_1 \text{Lip}(T_\beta - F) \leq C_1(\text{Lip}(f) + \text{Lip}(F)).$$

Besides, for all $n \in \mathbb{N}$ and $\beta = 0$, by applying property (*), we select a \mathcal{C}^1 -smooth function $F_0^n : X \rightarrow \mathbb{R}$, such that

$$|F_0^n(x) - F(x)| < \frac{\varepsilon}{2^{n+2} L_{n,0}} \text{ for every } x \in X \text{ and } \text{Lip}(F_0^n) \leq C_0 \text{Lip}(F).$$

Thus, if we define $\Delta_\beta^n : X \rightarrow \mathbb{R}$

$$(2.4) \quad \Delta_\beta^n(x) = \begin{cases} F_0^n(x) & \text{if } \beta = 0, \\ T_\beta(x) - \delta_{n,\beta}(x) & \text{if } \beta \in \Gamma, \end{cases}$$

we obtain for every $\beta \in \Sigma$,

$$|\Delta_\beta^n(x) - F(x)| < \frac{\varepsilon}{2^{n+2}L_{n,\beta}} \quad \text{whenever } x \in X$$

and $\text{Lip}(\Delta_\beta^n) \leq \max\{(1 + C_1)\text{Lip}(f) + C_1\text{Lip}(F), C_0\text{Lip}(F)\} \leq R\text{Lip}(F)$ where $R := 1 + 2C_1$ is a constant depending only on X . Similarly to the first case, the definition of G is

$$G(x) = \sum_{(n,\beta) \in \mathbb{N} \times \Sigma} \psi_{n,\beta}(x) \Delta_\beta^n(x).$$

The proofs that G is \mathcal{C}^1 -smooth, $|G(x) - F(x)| < \varepsilon$ for all $x \in X$ and $\|g'(y) - f'(y)\|_{Y^*} < \varepsilon$ for all $y \in Y$ (recall that $g = G|_Y$) follow along the same lines. In addition, let us check that G is Lipschitz. From properties (D.1) to (D.6), in particular from the fact that $\sum_{(n,\beta) \in F_x} \psi'_{n,\beta}(x) = 0$, we deduce that

(2.5)

$$\begin{aligned} \|G'(x)\|_{X^*} &\leq \sum_{(n,\beta) \in F_x} \|\psi'_{n,\beta}(x)\|_{X^*} |\Delta_\beta^n(x) - F(x)| + \sum_{(n,\beta) \in F_x} \psi_{n,\beta}(x) \|(\Delta_\beta^n)'(x)\|_{X^*} \leq \\ &\leq \sum_{\{n:(n,\beta(n)) \in F_x\}} L_{n,\beta(n)} \frac{\varepsilon}{2^{n+2}L_{n,\beta(n)}} + \sum_{\{n:(n,\beta(n)) \in F_x\}} \psi_{n,\beta(n)}(x) R\text{Lip}(F) \leq \\ &\leq \frac{\varepsilon}{4} + R\text{Lip}(F). \end{aligned}$$

Since $\varepsilon < \text{Lip}(F)$, then $\text{Lip}(G) \leq C_2 \text{Lip}(F)$ where $C_2 := R + \frac{1}{4}$ and this finishes the proof. \square

The above theorem provides the tool to prove the extension Theorem 1.2, which states that every \mathcal{C}^1 -smooth function (\mathcal{C}^1 -smooth and Lipschitz function) defined on a closed subspace has a \mathcal{C}^1 -smooth extension (\mathcal{C}^1 -smooth and Lipschitz extension, respectively) to X .

Proof of Theorem 1.2. For every bounded, Lipschitz function, $h : Y \rightarrow \mathbb{R}$, we define $\bar{h}(x) := \min\{\|h\|_\infty, \inf_{y \in Y} \{h(y) + \text{Lip}(h)\|x - y\|\}\}$ for any $x \in X$. The function \bar{h} is a Lipschitz extension of h to X with $\text{Lip}(\bar{h}) = \text{Lip}(h)$ and $\|\bar{h}\|_\infty = \|h\|_\infty = \sup\{|h(y)| : y \in Y\}$.

Let us assume that the function $f : Y \rightarrow \mathbb{R}$ is \mathcal{C}^1 -smooth and consider $F : X \rightarrow \mathbb{R}$ a continuous extension of f to X and $\varepsilon > 0$. We apply Theorem 2.4 to deduce the existence of a \mathcal{C}^1 -smooth function $G_1 : X \rightarrow \mathbb{R}$ such that if $g_1 := G_1|_Y$, we have

- (i) $|F(x) - G_1(x)| < \varepsilon/2$ for $x \in X$, and
- (ii) $\|f'(y) - g_1'(y)\|_{Y^*} < \varepsilon/2C_2$ for $y \in Y$. Since Y is convex, then we have $\text{Lip}(f - g_1) \leq \varepsilon/2C_2$.

The function $f - g_1$ is bounded by $\varepsilon/2$ and $\frac{\varepsilon}{2C_2}$ -Lipschitz on Y . Thus, there exists a bounded, Lipschitz extension to X , $\overline{f - g_1}$, satisfying $|\overline{(f - g_1)}(x)| \leq \varepsilon/2$ on X and $\text{Lip}(\overline{f - g_1}) \leq \varepsilon/2C_2$. Following the construction given for the separable case [5], we apply Theorem 2.4 (Lipschitz case) to $\overline{f - g_1}$ to obtain a \mathcal{C}^1 -smooth function $G_2 : X \rightarrow \mathbb{R}$ such that if $g_2 := G_2|_Y$, we have

- (i) $|\overline{(f - g_1)}(x) - G_2(x)| < \varepsilon/2^2$ for $x \in X$,
- (ii) $\|f'(y) - (g_1'(y) + g_2'(y))\|_{Y^*} < \varepsilon/2^2C_2$ for $y \in Y$, and
- (iii) $\text{Lip}(G_2) \leq C_2 \text{Lip}(f - g_1) \leq \varepsilon/2$.

Thus, we find, by induction, a sequence $\{G_n\}_{n=1}^\infty$ of \mathcal{C}^1 -smooth function such that for $n \geq 2$, the functions $G_n : X \rightarrow \mathbb{R}$ and their restrictions $g_n := G_n|_Y$ satisfy:

- (i) $|(f - \sum_{i=1}^{n-1} g_i)(x) - G_n(x)| < \varepsilon/2^n$ for $x \in X$,
- (ii) $\|f'(y) - \sum_{i=1}^n g'_i(y)\|_{Y^*} < \varepsilon/2^n C_2$ for $y \in Y$, and
- (iii) $\text{Lip}(G_n) \leq C_2 \text{Lip}(f - \sum_{i=1}^{n-1} g_i) \leq \varepsilon/2^{n-1}$.

Let us define the function $H : X \rightarrow \mathbb{R}$ as $H(x) := \sum_{n=1}^\infty G_n(x)$. Since $|G_n(x)| \leq \varepsilon/2^{n-2}$ and $\|G'_n(x)\|_{X^*} \leq \text{Lip}(G_n) \leq \varepsilon/2^{n-1}$ for all $x \in X$ and $n \geq 2$, the series $\sum_{n=1}^\infty G_n$ and $\sum_{n=1}^\infty G'_n$ are absolutely and uniformly convergent on X . Hence, the function H is \mathcal{C}^1 -smooth on X . It follows from (i) that $|f(y) - \sum_{i=1}^n G_i(y)| < \varepsilon/2^n$ for every $y \in Y$ and $n \geq 1$. Thus $H(y) = f(y)$ for all $y \in Y$.

Let us now consider $f : Y \rightarrow \mathbb{R}$, \mathcal{C}^1 -smooth and Lipschitz on Y . Let $F : X \rightarrow \mathbb{R}$ be a Lipschitz extension of f with $\text{Lip}(F) = \text{Lip}(f)$. We may assume $\text{Lip}(f) > 0$ (otherwise the result trivially holds) and take $0 < \varepsilon < \text{Lip}(f)$. Let us apply Theorem 2.4 (Lipschitz case) to obtain a \mathcal{C}^1 -smooth function $G_1 : X \rightarrow \mathbb{R}$ such that if $g_1 := G_1|_Y$, we have

- (i) $|F(x) - G_1(x)| < \varepsilon/2$ for $x \in X$,
- (ii) $\|f'(y) - g'_1(y)\|_{Y^*} < \varepsilon/2 C_2$ for $y \in Y$, and
- (iii) $\text{Lip}(G_1) \leq C_2 \text{Lip}(f)$.

Let us define $G_n : X \rightarrow \mathbb{R}$ for $n \geq 2$ as in the general case and $H(x) := \sum_{n=1}^\infty G_n(x)$. Then, H is \mathcal{C}^1 -smooth, $H|_Y = f$ and

$$\text{Lip}(H) \leq \text{Lip}(G_1) + \sum_{n=2}^\infty \text{Lip}(G_n) \leq C_2 \text{Lip}(f) + \sum_{n=2}^\infty \frac{\varepsilon}{2^{n-1}} \leq (C_2 + 1) \text{Lip}(f).$$

□

The proof of Corollary 1.3 is similar to the separable case [5]. (Recall that a paracompact \mathcal{C}^1 manifold M modelled on a Banach space admits \mathcal{C}^1 -smooth partitions of unity whenever the Banach space where it is modelled does).

An analogous result to Theorem 2.4 can be stated for a smooth function defined on a closed, convex subset Y of X with the required conditions given in Theorem 1.4. Let us sketch the required modifications of the proof: first, in the non-Lipschitz case, we take $H(f'(y_\beta)) = f'(y_\beta)$ and we evaluate the norms of the functionals in X^* rather than in Y^* . Secondly, in the Lipschitz case, there is no loss of generality in assuming that $0 \in Y$. Then, it can be easily checked that $\|f'(y)|_Z\|_{Z^*} \leq \text{Lip}(f)$ for all $y \in Y$, where we define $Z := \overline{\text{span}}(Y)$. Next, we select a continuous linear extension of $f'(y_\beta)|_Z$ to X with the same norm and denote it by $H(f'(y_\beta))$ for every $\beta \in \Gamma$ (i.e. $\|H(f'(y_\beta))\|_{X^*} \leq \text{Lip}(f)$). In this case, assertion (ii) in Theorem 2.4 reads as follows: $\|f'(y) - g'(y)\|_{Z^*} < \varepsilon$ for every $y \in Y$. Finally, the proof of Theorem 1.4 is similar to the proof of Theorem 1.2.

3. APPENDIX. C^1 -SMOOTH EXTENSIONS OF CERTAIN C^1 -SMOOTH FUNCTIONS DEFINED ON CLOSED SUBSETS OF A BANACH SPACE.

Let us finish this note by studying the case of the C^1 -smooth extension of a real-valued function f defined on a (possibly non-convex) closed subset Y of a Banach space X with property (*). Unfortunately, we can find examples of a real-valued function f defined on an open neighborhood U of a (non-convex) closed subset Y of \mathbb{R} such that $f : U \rightarrow \mathbb{R}$ is differentiable at every point $y \in Y$, the mapping $Y \mapsto \mathbb{R}$, $y \mapsto f'(y)$ is continuous on Y and $f|_Y$ does not admit any C^1 -smooth extension to \mathbb{R} . Thus, in general, we cannot obtain a similar statement to Theorem 1.4 for a non-convex closed subset Y of X . It is worth pointing out that, by an application of the Mean Value Theorem, if $f : Y \rightarrow \mathbb{R}$ admits a C^1 -smooth extension $H : X \rightarrow \mathbb{R}$, then, setting $D(y) := H'(y) \in X^*$ for $y \in Y$, we have the following mean value condition:

$$(E) \quad \text{for all } y \in Y, \text{ for all } \varepsilon > 0, \text{ there exists } r > 0 \text{ such that,} \\ |f(z) - f(w) - D(y)(z - w)| \leq \varepsilon \|z - w\|, \quad \text{for all } z, w \in Y \cap B(y, r).$$

Definition 3.1. *If Y is a closed subset of a Banach space X , let $Z := \overline{\text{span}}\{y - y' : y, y' \in Y\}$. Let $f : Y \rightarrow \mathbb{R}$ and let $D : Y \rightarrow Z^*$ a continuous mapping. We will then say that $f : Y \rightarrow \mathbb{R}$ satisfies condition (E) for D if the statement (E) is true.*

Thus, condition (E) is necessary to extend $f : Y \rightarrow \mathbb{R}$ to a C^1 -smooth function $H : X \rightarrow \mathbb{R}$. Moreover, the next result states that condition (E) is also sufficient in spaces with property (*).

Theorem 3.2. *Let X be a Banach space with property (*), $Y \subset X$ a closed subset and $f : Y \rightarrow \mathbb{R}$ a function on Y . Then, f satisfies condition (E) if and only if there is a C^1 -smooth extension H of f to X .*

Furthermore, assume that the given function f is Lipschitz on Y . Then, f satisfies condition (E) for some continuous function $D : Y \rightarrow Z^*$ with $\sup\{\|D(y)\|_{Z^*} : y \in Y\} < \infty$ if and only if there is a C^1 -smooth and Lipschitz extension H of f to X . In this case, if f satisfies condition (E) for D with $M := \sup\{\|D(y)\|_{Z^*} : y \in Y\} < \infty$, then we can obtain H with $\text{Lip}(H) \leq (1 + C_1)(M + \text{Lip}(f))$, where C_1 is the constant defined in Lemma 2.3. (Recall that C_1 depends only on X .)

The proof of Theorem 3.2 follows the same lines as Theorem 1.2. It can be deduced from Lemmas 2.2, 2.3 and the following statement, which is similar to Theorem 2.4.

Theorem 3.3. *Let X be a Banach space with property (*), $Y \subset X$ a closed subset and $f : Y \rightarrow \mathbb{R}$ a function satisfying condition (E) for some continuous function $D : Y \rightarrow Z^*$. Let us consider F a continuous extension of f to X . Then, for every $\varepsilon > 0$ there exists a C^1 -smooth function $G : X \rightarrow \mathbb{R}$ such that if $g := G|_Y$ then*

- (i) $|F(x) - G(x)| < \varepsilon$ on X , and
- (ii) $\|D(y) - G'(y)\|_{Z^*} < \varepsilon$ for all $y \in Y$ and $\text{Lip}(f - g) < \varepsilon$.
- (iii) Furthermore, assume that f is Lipschitz on Y , F is a Lipschitz extension of f to X and $M = \sup\{\|D(y)\|_{Z^*} : y \in Y\} < \infty$. Then the function G can be chosen to be Lipschitz on X and $\text{Lip}(G) \leq \frac{\varepsilon}{4} + (1 + C_1)M + C_1 \text{Lip}(F)$.

Let us outline the required modifications of the proof of Theorem 3.3. The covering $\{B(y_\gamma, r_\gamma) := B_\gamma\}_{\gamma \in \Gamma}$ of Y by open balls of X with centers $y_\gamma \in Y$ is selected in such a way that

$$\|D(y) - D(y_\gamma)\|_{Z^*} \leq \frac{\varepsilon}{8C_0} \quad \text{and} \quad |f(z) - f(w) - D(y_\gamma)(z - w)| \leq \frac{\varepsilon}{8C_0} \|z - w\|,$$

for every $y, z, w \in B_\gamma \cap Y$ (this can be done because f satisfies condition (E) for D). In this case, T_γ is defined as $T_\gamma(x) = f(y_\gamma) + \widehat{D}(y_\gamma)(x - y_\gamma)$, for $x \in X$, where $\widehat{D}(y_\gamma)$ is an extension of $D(y_\gamma)$ to X with the same norm. The functions T_γ are \mathcal{C}^∞ -smooth on X , $T'_\gamma(x) = \widehat{D}(y_\gamma)$ for all $x \in X$ and satisfy that for all $z, w \in B_\gamma \cap Y$,

$$|(T_\gamma - F)(z) - (T_\gamma - F)(w)| = |f(w) - f(z) - D(y_\gamma)(w - z)| \leq \frac{\varepsilon}{8C_0} \|z - w\|.$$

Thus, $\text{Lip}((T_\gamma - F)|_{B_\gamma \cap Y}) \leq \frac{\varepsilon}{8C_0}$. We can apply Lemma 2.3 to $T_\gamma - F$ on $B_\gamma \cap Y$ to obtain a \mathcal{C}^1 -smooth map $\delta_{n,\gamma} : X \rightarrow \mathbb{R}$ satisfying (C.1), (C.2) and

$$(C.2') \quad \text{Lip}((\delta_{n,\gamma})|_{B_\gamma \cap Y}) \leq \frac{\varepsilon}{8}.$$

From the above, we deduce that, for all $y \in B_\gamma \cap Y$,

$$\|T'_\gamma(y) - D(y) - \delta'_{n,\gamma}\|_{Z^*} \leq \frac{\varepsilon}{4} \quad \text{and} \quad \text{Lip}((T_\gamma - F - \delta_{n,\gamma})|_{B_\gamma \cap Y}) \leq \frac{\varepsilon}{4}.$$

The inequality $\|D(y) - G'(y)\|_{Z^*} < \varepsilon$ follows as in (2.3) (in the proof of Theorem 2.4). Let us prove that $\text{Lip}(g - f) < \varepsilon$, where $g = G|_Y$. In order to simplify the notation let us write $S_\beta^n(y) := \Delta_\beta^n(y) - f(y)$ for $y \in Y$ (where Δ_β^n is defined as in (2.2)), and $R(y, z) := \sum_{(n,\beta) \in F_y} \psi_{n,\beta}(z) S_\beta^n(y)$ for $y, z \in Y$. Notice that $\psi_{(n,\beta)}(z) = 0$ whenever $(n, \beta) \notin F_z$ and thus, $R(y, z) = \sum_{(n,\beta) \in F_y \cap F_z} \psi_{n,\beta}(z) S_\beta^n(y)$ for $y, z \in Y$. In addition, let us write $M(z, y) := \sum_{(n,\beta) \in F_z \setminus F_y} \psi_{n,\beta}(y) S_\beta^n(z) = 0$ for $y, z \in Y$. Now, from the above and properties (D.1) to (D.6), we obtain

$$\begin{aligned} |(g(y) - f(y)) - (g(z) - f(z))| &= \\ &= \left| \sum_{(n,\beta) \in F_y} \psi_{n,\beta}(y) S_\beta^n(y) - \sum_{(n,\beta) \in F_z} \psi_{n,\beta}(z) S_\beta^n(z) - R(y, z) + R(y, z) + M(z, y) \right| = \\ &= \left| \left(\sum_{(n,\beta) \in F_y} \psi_{n,\beta}(y) S_\beta^n(y) - R(y, z) \right) + (R(y, z) - \sum_{(n,\beta) \in F_z \cap F_y} \psi_{n,\beta}(z) S_\beta^n(z)) \right. \\ &\quad \left. + (M(z, y) - \sum_{(n,\beta) \in F_z \setminus F_y} \psi_{n,\beta}(z) S_\beta^n(z)) \right| \leq \\ &\leq \sum_{(n,\beta) \in F_y} |\psi_{n,\beta}(y) - \psi_{n,\beta}(z)| |S_\beta^n(y)| + \sum_{(n,\beta) \in F_z \cap F_y} \psi_{n,\beta}(z) |S_\beta^n(y) - S_\beta^n(z)| + \\ &\quad + \sum_{(n,\beta) \in F_z \setminus F_y} |\psi_{n,\beta}(y) - \psi_{n,\beta}(z)| |S_\beta^n(z)| \leq \sum_{(n,\beta) \in F_y} L_{n,\beta} \|y - z\| \frac{\varepsilon}{2^{n+2} L_{n,\beta}} + \\ &\quad + \sum_{(n,\beta) \in F_z \cap F_y} \psi_{n,\beta}(z) \frac{\varepsilon}{4} \|y - z\| + \sum_{(n,\beta) \in F_z \setminus F_y} L_{n,\beta} \|y - z\| \frac{\varepsilon}{2^{n+2} L_{n,\beta}} < \varepsilon \|y - z\|. \end{aligned}$$

In the Lipschitz case, we additionally obtain that $T_\beta - F$ is Lipschitz on X and $\text{Lip}(T_\beta - F) \leq \text{Lip}(T_\beta) + \text{Lip}(F) \leq M + \text{Lip}(F)$ for every $\beta \in \Gamma$. Thus, $\delta_{n,\beta} : X \rightarrow \mathbb{R}$ satisfies conditions (C.1), (C.2), (C.2') and $\text{Lip}(\delta_{n,\beta}) \leq C_1 \text{Lip}(T_\beta - F) \leq$

$C_1(M + \text{Lip}(F))$. Also, if Δ_β^n is defined as in (2.4), then $\text{Lip}(\Delta_\beta^n) \leq \max\{(1 + C_1)M + C_1 \text{Lip}(F), C_0 \text{Lip}(F)\} = (1 + C_1)M + C_1 \text{Lip}(F)$. Similarly to (2.5), we obtain the following upper bound,

$$\|G'(x)\|_{X^*} \leq \frac{\varepsilon}{4} + (1 + C_1)M + C_1 \text{Lip}(F).$$

(Recall, that here we do not assume $\varepsilon < \text{Lip}(F)$.) This finishes the required changes for the proof of Theorem 3.3.

Let us give also the necessary modifications for the proof of Theorem 3.2. In this case, we apply Theorem 3.3 to deduce the existence of a \mathcal{C}^1 -smooth function $G_1 : X \rightarrow \mathbb{R}$ such that if $g_1 := G_{1|_Y}$, we have

- (i) $|F(x) - G_1(x)| < \frac{\varepsilon}{2(1+C_1)}$ for $x \in X$,
- (ii) $\|D(y) - G'_1(y)\|_{Z^*} < \frac{\varepsilon}{2(1+C_1)}$ for all $y \in Y$ and $\text{Lip}(f - g_1) < \frac{\varepsilon}{2(1+C_1)}$.

The function $f - g_1$ satisfies condition (E) for $D - G'_1$. Thus, we apply again Theorem 3.3 (Lipschitz case) to $\overline{f - g_1}$ to obtain a \mathcal{C}^1 -smooth function $G_2 : X \rightarrow \mathbb{R}$ such that if $g_2 := G_{2|_Y}$, we have

- (i) $|\overline{(f - g_1)}(x) - G_2(x)| < \frac{\varepsilon}{2^2(1+C_1)}$ for $x \in X$,
- (ii) $\|D(y) - (G'_1(y) + G'_2(y))\|_{Z^*} < \frac{\varepsilon}{2^2(1+C_1)}$ for all $y \in Y$, $\text{Lip}(f - (g_1 + g_2)) < \frac{\varepsilon}{2^2(1+C_1)}$, and
- (iii) $\text{Lip}(G_2) \leq \frac{\varepsilon}{2^4} + \varepsilon$.

We find, by induction, a sequence $\{G_n\}_{n=1}^\infty$ of \mathcal{C}^1 -smooth function such that for $n \geq 2$, the functions $G_n : X \rightarrow \mathbb{R}$ and their restrictions $g_n := G_{n|_Y}$ satisfy:

- (i) $|\overline{(f - \sum_{i=1}^{n-1} g_i)}(x) - G_n(x)| < \frac{\varepsilon}{2^n(1+C_1)}$ for $x \in X$,
- (ii) $\|D(y) - \sum_{i=1}^n G'_i(y)\|_{Z^*} < \frac{\varepsilon}{2^n(1+C_1)}$ for $y \in Y$, $\text{Lip}(f - \sum_{i=1}^n g_i) < \frac{\varepsilon}{2^n(1+C_1)}$ and
- (iii) $\text{Lip}(G_n) \leq \frac{\varepsilon}{2^{n+2}} + \frac{\varepsilon}{2^{n-2}}$.

In the Lipschitz case and for $\varepsilon < \frac{4}{9} \text{Lip}(f)$, we obtain the upper bound, $\text{Lip}(H) \leq (1 + C_1)(M + \text{Lip}(f))$.

□

It is worth pointing out the following corollary, where a characterization of property (*) is given.

Corollary 3.4. *Let X be a Banach space. The following statements are equivalent:*

- (i) X satisfies property (*).
- (ii) Assume that $f : Y \rightarrow \mathbb{R}$ is a Lipschitz function satisfying condition (E) for some continuous function $D : Y \rightarrow Z^*$ with $M := \sup\{\|D(y)\|_{Z^*} : y \in Y\} < \infty$. Then, there is a \mathcal{C}^1 -smooth and Lipschitz extension of f to X , $H : X \rightarrow \mathbb{R}$, with $\text{Lip}(H) \leq C_3(M + \text{Lip}(f))$ (where C_3 is a constant depending only on the space X).

Proof. We only need to check (ii) \Rightarrow (i). By [11, Proposition 1] it is enough to prove that there is a number $K \geq 1$ such that for every subset $A \subset X$ there is a \mathcal{C}^1 -smooth and K -Lipschitz function $h_A : X \rightarrow [0, 1]$ such that $h_A(x) = 0$ for $x \in A$ and $h_A(x) = 1$ for every $x \in X$ such that $\text{dist}(x, A) \geq 1$. For every subset A of X ,

we consider the closed subsets $B = \{x \in X : \text{dist}(x, A) \geq 1\}$ and $Y = \overline{A} \cup B$ of X . Let us define the function $f : Y \rightarrow \mathbb{R}$ as

$$f(x) := \begin{cases} 0 & \text{if } x \in \overline{A}, \\ 1 & \text{if } x \in B. \end{cases}$$

It is clear that f is 1-Lipschitz and satisfies condition (E) for the continuous function $D : Y \rightarrow Z^*$, $D(y) = 0$ for every $y \in Y$ (thus, $\sup\{\|D(y)\|_{Z^*} : y \in Y\} = 0$). By assumption, we can find a \mathcal{C}^1 -smooth and C_3 -Lipschitz extension of f to X , $H : X \rightarrow \mathbb{R}$ such that $H(x) = 0$ for all $x \in A$ and $H(x) = 1$ for all $x \in B$. We take a 2-Lipschitz and C^∞ -smooth function $\theta : \mathbb{R} \rightarrow [0, 1]$ such that $\theta(t) = 0$ whenever $t \leq 0$, and $\theta(t) = 1$ whenever $t \geq 1$. Let us define $h_A(x) = \theta(H(x))$. The \mathcal{C}^1 -smooth function $h_A : X \rightarrow [0, 1]$ is $2C_3$ -Lipschitz, $h_A(x) = 0$ for $x \in A$ and $h_A(x) = 1$ for $x \in B$. \square

Notice that the following assertion can be proved along the same lines: if every real-valued function defined on a closed subset of a Banach space X and satisfying condition (E) has a \mathcal{C}^1 -smooth extension to X , then every real-valued, continuous function on X can be uniformly approximated by \mathcal{C}^1 -smooth functions. In fact, both properties are equivalent in the separable case to properties (i) and (ii) of the above corollary.

Finally, let us mention, that after submitting the paper to the journal, we were informed of the final version of [11], where a similar statement to Lemma 2.2 is established. In any case, we have decided to keep this lemma in order to give a more self-contained proof of the main result.

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