

# SINGULARITY CONFINEMENT FOR MATRIX DISCRETE PAINLEVÉ EQUATIONS

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**ABSTRACT.** We study the analytic properties of a matrix discrete system introduced in [7]. The singularity confinement for this system is shown to hold generically, i.e. in the whole space of parameters except possibly for algebraic subvarieties. This paves the way to a generalization of Painlevé analysis to discrete matrix models.

## 1. INTRODUCTION

Since the discovery of the *Painlevé property* for ordinary differential equations at the end of the XIX century [19], the notion of *integrability* has been related to the local analysis of movable isolated singularities of solutions of dynamical systems [8]. This approach to integrability has opened an alternative perspective compared to the standard algebraic approach *à la Liouville*, based on the existence of a suitable number of functionally independent integrals of motion. Both points of view have been extended to the study of evolution equations on a discrete background.

Integrable discrete systems, for several aspects more fundamental objects than the continuous ones, are ubiquitous both in pure and applied mathematics, and in theoretical physics as well. They possess rich algebraic–geometric properties [3], [5], [16], [9], [23] and are relevant, for instance, in the regularization of quantum field theories in a lattice and in discrete quantum gravity [10], [14].

In particular, the problem of integrability preserving discretizations of partial differential equations has become a very active research area [21], and has been widely investigated with both geometrical and algebraic methods [5], [6], [18], [22].

The approach known as *singularity confinement*, introduced in [12], is the equivalent for discrete systems of the singularity analysis for continuous dynamical systems. It essentially relies on the observation that for integrable discrete models, if a singularity appears in some specific point of the lattice of the independent variable, then it would disappear after making evolve the system via a finite number of iterations. Alternative, related approaches are based on the notion of algebraic entropy [4], [15] or on Nevalinna theory [1], [20]. A large class of difference equations coming from unitary integrals and combinatorics possess the confinement property [2]. However, observe that singularity

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confinement, in despite of being extremely useful in isolating integrability, it might not be a sufficient condition for integrability as was noticed by Hietarinta and Viallet [13].

The purpose of this paper is to start a theoretical study of the singularity confinement property for *matrix integrable systems*. Indeed, we hypothesize that the singularity analysis has for matrix systems the same relevance that possesses for both discrete and continuous scalar models.

Apart its intrinsic mathematical interest, the study of matrix discrete dynamical systems can also be related, from an applicative point of view, to the theory of complex networks [17]. Indeed, given a random graph with  $N$  vertices, one associates with it the adjacency matrix, which is a  $N \times N$  matrix, whose entries  $a_{ij}$  represent the number of links associated with the nodes  $i$  and  $j$  ( $i, j = 1, \dots, N$ ). The discrete time evolution of the topology of the network would provide a difference equation for the adjacency matrix, defining a discrete matrix model.

Hereafter, we shall focus on the singularity confinement of the following discrete matrix equation

$$(1) \quad \beta_{n+1} = n\beta_n^{-1} - \beta_{n-1} - \beta_n - \alpha, \quad n = 1, 2, \dots$$

where  $\beta_n \in \mathbb{C}^{N \times N}$  is a  $N \times N$  complex matrix.

Equation (1) can be considered a kind of non Abelian matrix version of the discrete Painlevé equation (dPI). It has been introduced in [7] and describes the recursion relation for the matrix coefficients of a class of Freud matrix orthogonal polynomials with a quartic potential [11]. It is obtained by solving the related Riemann–Hilbert problem. In that paper we also proved the singularity confinement in a simple situation, when the initial data are triangular matrices up to similarity transformations. The aim of this paper is to extend this result to the general case. This extension have required much more effort than in the simple triangularizable situation but finally we succeeded in getting the desired proof. The difficulty mainly resides in the analysis of the genericness of the result given in Theorem 2.

**1.1. Preliminary discussion.** Let us present here the simplest case of singularity analysis for the matrix model (1) which parallels the results for the standard discrete Painlevé I equation. We assume that

$$(2) \quad \beta_{m-1} = \beta_{m-1,0} + \beta_{m-1,1}\epsilon + O(\epsilon^2), \quad \beta_m = \beta_{m,1}\epsilon + \beta_{m,2}\epsilon^2 + O(\epsilon^3), \quad \epsilon \rightarrow 0,$$

with  $\det \beta_{m,1} \neq 0$ . If we introduce conditions (2) into (1), we have that

$$(3) \quad \beta_{m+1} = m\beta_{m,1}^{-1}\epsilon^{-1} + \beta_{m+1,0} + \beta_{m+1,1}\epsilon + \beta_{m+1,2}\epsilon^2 + O(\epsilon^3),$$

where

$$\begin{aligned} \beta_{m+1,0} &= -m\beta_{m,1}^{-1}\beta_{m,2}\beta_{m,1}^{-1} - \beta_{m-1,0} - \alpha, \\ \beta_{m+1,1} &= m\beta_{m,1}^{-1}(\beta_{m,2}\beta_{m,1}^{-1}\beta_{m,2} - \beta_{m,3})\beta_{m,1}^{-1} - \beta_{m,1} - \beta_{m-1,1}, \\ \beta_{m+1,2} &= m\left(\beta_{m,2}\beta_{m,1}^{-1}(\beta_{m,3} - \beta_{m,2}\beta_{m,1}^{-1}\beta_{m,2}) + \beta_{m,3}\beta_{m,1}^{-1}\beta_{m,2} - \beta_{m,4}\right)\beta_{m,1}^{-2} - \beta_{m,2} - \beta_{m-1,2}. \end{aligned}$$

Thus, the pole singularity has shown up, and it will survive still for another step in the sequence. Indeed, we have that

$$(4) \quad \beta_{m+2} = -m\beta_{m,1}^{-1}\epsilon^{-1} + \beta_{m+2,0} + \beta_{m+2,1}\epsilon + \beta_{m+2,2}\epsilon^2 + O(\epsilon^3),$$

where

$$\begin{aligned}\beta_{m+2,0} &= m\beta_{m,1}^{-1}\beta_{m,2}\beta_{m,1}^{-1} + \beta_{m-1,0}, \\ \beta_{m+2,1} &= \frac{(m+1)}{m}\beta_{m,1} - m\beta_{m,1}^{-1}\beta_{m,2}\beta_{m,1}^{-1}\beta_{m,2}\beta_{m,1}^{-1} + m\beta_{m,1}^{-1}\beta_{m,3}\beta_{m,1}^{-1} + \beta_{m-1,1}, \\ \beta_{m+2,2} &= \frac{(m+1)}{m}\beta_{m,2} + \frac{(m+1)}{m^2}\beta_{m,1}(\beta_{m-1,0} + \alpha)\beta_{m,1} + m\beta_{m,2}\beta_{m,1}^{-1}(\beta_{m,2}\beta_{m,1}^{-1}\beta_{m,2}\beta_{m,1}^{-2} - \beta_{m,3}\beta_{m,1}^{-2}) \\ &\quad - m\beta_{m,3}\beta_{m,1}^{-1}\beta_{m,2}\beta_{m,1}^{-2} + \beta_{m-1,2} + m\beta_{m,4}\beta_{m,1}^{-2}.\end{aligned}$$

We easily check that in the third step the zero appears again

$$(5) \quad \beta_{m+3} = \frac{-(m+3)}{m}\beta_{m,1}\epsilon + \beta_{m+3,2}\epsilon^2 + O(\epsilon^3),$$

where

$$\beta_{m+3,2} := -\frac{(m+3)}{m}\beta_{m,2} - \frac{(2m+3)}{m^2}\beta_{m,1}\beta_{m-1,0}\beta_{m,1} - \frac{(m+1)}{m^2}\beta_{m,1}\alpha\beta_{m,1}.$$

Finally, if we substitute (4) and (5) into (1) we get no singularity at all:

$$\beta_{m+4} = \frac{m}{(m+3)}\beta_{m-1,0} - \frac{2}{(m+3)}\alpha + O(\epsilon).$$

Observe that  $\beta_{m+3} = O(\epsilon)$ ,  $\beta_{m+4} = O(1)$  and  $\det \beta_{m+4} = O(1)$  for  $\epsilon \rightarrow 0$ . Thus, unless

$$(6) \quad \det(m\beta_{m-1,0} - 2\alpha) = 0,$$

we obtain singularities in the step just after the appearance of a *zero* in  $\beta_m$ , with the poles appearing in the sites  $m+1, m+2$ . Then we have a *zero* for  $m+3$  while we recover the standard behaviour for  $m+4$ . A crucial point is that this singularity confinement holds whenever (6) is *not* satisfied. This observation motivates the definitions proposed in the following discussion.

**Definition 1.** *Whenever the singularity confinement property is satisfied in the whole space  $\mathcal{S}$  of parameters except possibly for a set of algebraic subvarieties  $\mathcal{W}_i \in \mathcal{S}, i = 1, \dots, j \in \mathbb{N}$ , we shall say that the property is satisfied generically.*

In this case we will speak about the genericness of the singularity confinement.

**Definition 2.** *We shall define the confinement time as the minimum number  $\ell \in \mathbb{N}$  of iterations or steps in the lattice, after the appearance of a zero, necessary to recover the form without poles or zeros.*

Thus, in the above case we have generically a singularity confinement with a confinement time  $l = 4$ .

A simple but fundamental observation for the sequel of the paper is the following one.

**Lemma 1.** *The matrix system (1) is invariant under similarity transformations.*

*Proof.* Observe that

$$M\beta_{n+1}M^{-1} = nM\beta_n^{-1}M^{-1} - M\beta_{n-1}M^{-1} - M\beta_nM^{-1} - M\alpha M^{-1}.$$

Therefore, we obtain

$$\phi_{n+1} = n\phi_n^{-1} - \phi_{n-1} - \phi_n - \delta,$$

where  $\phi_n := M\beta_nM^{-1}$  and  $\delta := M\alpha M^{-1}$ . □

**1.2. Main result.** The ideas developed within this example will be used in the subsequent considerations to study the confinement of the singularities of the matrix dPI model (1). We shall assume that

$$(7) \quad \beta_{m-1} = \beta_{m-1,0} + \beta_{m-1,1}\epsilon + O(\epsilon^2), \quad \epsilon \rightarrow 0,$$

$$(8) \quad \beta_m = \beta_{m,0} + \beta_{m,1}\epsilon + O(\epsilon^2), \quad \det \beta_m = O(\epsilon^r), \quad \epsilon \rightarrow 0,$$

where  $\beta_{m-1,i}, \beta_{m,i} \in \mathbb{C}^{N \times N}$  and  $r = 1, \dots, N$ . Consequently, we can distinguish two cases.

- $r = N$ . This is the maximal rank case; for it we have that

$$\beta_{m,0} = 0, \quad \det \beta_{m,1} \neq 0.$$

It presents singularity confinement generically.

- $r \leq N - 1$ . For the non-maximal rank case we instead have

$$(9) \quad \dim \text{Ran } \beta_{m,0} = N - r, \quad \det \beta_m = O(\epsilon^r), \quad \epsilon \rightarrow 0.$$

As will be proven later, by using the invariance under a similarity transformation, one can assume that the matrices  $\beta$  will have the form expressed by eq. (14). So said, we can state the main result of the paper as follows.

**Theorem 1.** *If  $\beta_{m-1}$  and  $\beta_m$  are of the form (7), (8) and (14), and the following conditions for  $\epsilon \rightarrow 0$  are satisfied*

$$(10) \quad \det \beta_{m+1} = O(\epsilon^{-r}),$$

$$(11) \quad \det \beta_{m+2} = O(\epsilon^{-r}),$$

$$(12) \quad \det \beta_{m+3} = O(\epsilon^r),$$

$$(13) \quad \det \beta_{m+4} = O(1),$$

*then, there is singularity confinement for the dPI model (1) with confinement time  $l = 4$ .*

## 2. $N \times N$ MATRIX ASYMPTOTIC EXPANSIONS AND SINGULARITY CONFINEMENT

In this section we will consider the set of matrix asymptotic expansions

$$\mathcal{A} = \mathbb{C}^{N \times N}((\epsilon)) := \{M_0 + M_1\epsilon + O(\epsilon^2), \epsilon \rightarrow 0, M_i \in \mathbb{C}^{N \times N}\}.$$

This set is a ring with identity, given by the matrix  $\mathbb{I}_N$ . For each possible rank (see eq. (14))  $r = 1, \dots, N - 1$  we will use the block notation

$$M := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A \in \mathbb{C}^{r \times r}, B \in \mathbb{C}^{r \times (N-r)}, C \in \mathbb{C}^{(N-r) \times r}, D \in \mathbb{C}^{(N-r) \times (N-r)}.$$

We also introduce two subalgebras of the algebra  $\mathbb{C}^{N \times N}$

$$\mathfrak{K} := \left\{ K = \begin{pmatrix} 0 & 0 \\ K_{21} & K_{22} \end{pmatrix}, K_{21} \in \mathbb{C}^{(N-r) \times r}, K_{22} \in \mathbb{C}^{(N-r) \times (N-r)} \right\},$$

$$\mathfrak{L} := \left\{ L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & 0 \end{pmatrix}, L_{11} \in \mathbb{C}^{r \times r}, L_{21} \in \mathbb{C}^{(N-r) \times r} \right\},$$

and the related subsets of matrix asymptotic expansions

$$\mathcal{A}_{\mathfrak{R}} := \{K \in \mathcal{A}, K|_{\epsilon=0} \in \mathfrak{R}\}, \quad \mathcal{A}_{\mathfrak{L}} := \{L \in \mathcal{A}, L|_{\epsilon=0} \in \mathfrak{L}\},$$

which satisfy several important properties.

**Proposition 1.** *The following statements hold.*

- (1) Both  $\mathcal{A}_{\mathfrak{R}}$  and  $\mathcal{A}_{\mathfrak{L}}$  are subrings without identity of the ring  $\mathcal{A}$ .
- (2) For  $K \in \mathcal{A}_{\mathfrak{R}}$  such that  $\det K = O(\epsilon^r)$ ,  $\epsilon \rightarrow 0$ , then  $K^{-1} \in \epsilon^{-1}\mathcal{A}_{\mathfrak{L}}$ , and reciprocally if  $L \in \epsilon^{-1}\mathcal{A}_{\mathfrak{L}}$  with  $\det L = O(\epsilon^{-r})$ ,  $\epsilon \rightarrow 0$ , then  $L^{-1} \in \mathcal{A}_{\mathfrak{R}}$ .
- (3) If  $K \in \mathcal{A}_{\mathfrak{R}}$ , that is  $K = \begin{pmatrix} 0 & 0 \\ C_0 & D_0 \end{pmatrix} + \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\epsilon + O(\epsilon^2)$  then

$$\det K = \epsilon^r \det \begin{pmatrix} A_1 & B_1 \\ C_0 & D_0 \end{pmatrix} + O(\epsilon^{r+1}), \quad \epsilon \rightarrow 0.$$

- (4) If  $L \in \epsilon^{-1}\mathcal{A}_{\mathfrak{L}}$ , that is  $L = \begin{pmatrix} A_0 & 0 \\ C_0 & 0 \end{pmatrix}\epsilon^{-1} + \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} + O(\epsilon)$  then

$$\det L = \epsilon^{-r} \det \begin{pmatrix} A_0 & B_1 \\ C_0 & D_1 \end{pmatrix} + O(\epsilon^{-r+1}), \quad \epsilon \rightarrow 0.$$

- (5) The subrings  $\mathcal{A}_{\mathfrak{R}}$  and  $\mathcal{A}_{\mathfrak{L}}$  are right and left ideals of  $\mathcal{A}$  respectively, i.e.  $\mathcal{A}_{\mathfrak{R}} \cdot \mathcal{A} \subset \mathcal{A}_{\mathfrak{R}}$  and  $\mathcal{A} \cdot \mathcal{A}_{\mathfrak{L}} \subset \mathcal{A}_{\mathfrak{L}}$ .
- (6) The following inclusion holds:  $\epsilon^{-1}\mathcal{A}_{\mathfrak{L}} \cdot \mathcal{A}_{\mathfrak{R}} \subset \mathcal{A}$ .

The proof of the previous statements is direct and left to the reader.

To study the singularity confinement of the matrix equation (1) when  $\beta_n$  satisfies conditions (9), we shall use expressions (7) and (8), having applied a similarity transformation to  $\beta$  such that  $\beta_{m,0} \in \mathfrak{R}$ ,  $\beta_m \in \mathcal{A}_{\mathfrak{R}}$ . In other words

$$(14) \quad \beta_{m,0} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \beta_{m,0;r+1,1} & \beta_{m,0;r+1,2} & \cdots & \beta_{m,0;r+1,r+1} & \beta_{m,0;r+1,r+2} & \cdots & \beta_{m,0;r+1,N} \\ \beta_{m,0;r+2,1} & \beta_{m,0;r+2,2} & \cdots & \beta_{m,0;r+2,r+1} & \beta_{m,0;r+2,r+2} & \cdots & \beta_{m,0;r+2,N} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \beta_{m,0;N,1} & \beta_{m,0;N,2} & \cdots & \beta_{m,0;N,r+1} & \beta_{m,0;N,r+2} & \cdots & \beta_{m,0;N,N} \end{pmatrix},$$

where  $m \geq 2$ , and all the entries that are above the  $r+1$ -th row of  $\beta_m$  are zero. Notice that  $\beta_{m-1}$  and  $\beta_m$  belong to the rings  $\mathcal{A}$  and  $\mathcal{A}_{\mathfrak{R}}$ , respectively.

### 2.1. Proof of the Theorem 1.

*Proof.* As  $\beta_{m,0} \in \mathfrak{K}$ , i.e.  $\beta_m \in \mathcal{A}_{\mathfrak{K}}$ , and by hypothesis  $\det \beta_m = O(\epsilon^r)$ ,  $\epsilon \rightarrow 0$ , Proposition 1 implies

$$(15) \quad \beta_m^{-1} = (\beta_m^{-1})_{-1}\epsilon^{-1} + (\beta_m^{-1})_0 + O(\epsilon), \quad \epsilon \rightarrow 0, \quad (\beta_m^{-1})_{-1} \in \mathfrak{L}.$$

If we replace eqs. (8) and (14) into eq. (1), and take into account eq. (7), we deduce

$$\beta_{m+1} = m\beta_m^{-1} + O(1), \quad \epsilon \rightarrow 0.$$

By using the relations (15), (7), (8) and (14), this expression is reduced to

$$(16) \quad \beta_{m+1} = m(\beta_m^{-1})_{-1}\epsilon^{-1} + O(1), \quad \epsilon \rightarrow 0.$$

Since  $(\beta_m^{-1})_{-1} \in \mathfrak{L}$ , from (16) we conclude that  $\beta_{m+1} \in \epsilon^{-1}\mathcal{A}_{\mathfrak{L}}$ , showing a simple pole singularity. Due to the fact that by hypothesis eq. (10) holds, Proposition 1 implies

$$(17) \quad \beta_{m+1}^{-1} \in \mathcal{A}_{\mathfrak{K}}.$$

Then we deduce

$$\beta_{m+2} = -m(\beta_m^{-1})_{-1}\epsilon^{-1} + O(1), \quad \epsilon \rightarrow 0, \quad \beta_{m+2} \in \mathcal{A}_{\mathfrak{L}}.$$

As before, by using condition (11), Proposition 1 gives

$$\beta_{m+2}^{-1} \in \mathcal{A}_{\mathfrak{K}}.$$

Now,

$$(18) \quad \beta_{m+3} = \beta_m - (m+1)\beta_{m+1}^{-1} + (m+2)\beta_{m+2}^{-1},$$

where in the r.h.s. we have used twice eq. (1) to write  $\beta_{m+2}$  as a function of  $\beta_{m+1}$  and  $\beta_m$ . As we have proven that  $\beta_m, \beta_{m+1}^{-1}, \beta_{m+2}^{-1} \in \mathcal{A}_{\mathfrak{K}}$ , we deduce that

$$\beta_{m+3} \in \mathcal{A}_{\mathfrak{K}}.$$

As a consequence of eq. (12) and Proposition 1, we obtain

$$(19) \quad \beta_{m+3}^{-1} \in \epsilon^{-1}\mathcal{A}_{\mathfrak{L}}.$$

Our matrix discrete Painlevé equation (1) gives

$$\beta_{m+4} = (m+3)\beta_{m+3}^{-1} - \beta_{m+2} - \beta_{m+3} - \alpha,$$

which implies

$$(20) \quad \beta_{m+4} = \beta_{m+3}^{-1}A + O(1), \quad \epsilon \rightarrow 0, \quad A := (m+3)\mathbb{I}_N - \beta_{m+3}\beta_{m+2},$$

where we have taken into account that  $\beta_{m+3}$  and  $\alpha$  are  $O(1)$ . We study the matrix  $A$ , by applying eq. (1) once. We get

$$\begin{aligned} A &= \mathbb{I}_N + [(m+1)\beta_{m+1}^{-1} - \beta_m]\beta_{m+2} \\ &= [(m+1)\beta_{m+1}^{-1} - \beta_m][(m+1)\beta_{m+1}^{-1} - \beta_m - \alpha] - m\mathbb{I}_N + \beta_m\beta_{m+1} \\ (21) \quad &= [(m+1)\beta_{m+1}^{-1} - \beta_m][(m+1)\beta_{m+1}^{-1} - \beta_m - \alpha] - \beta_m(\beta_m + \beta_{m-1} + \alpha). \end{aligned}$$

Now, recalling that  $\beta_{m-1} = O(1)$ ,  $\beta_m, \beta_{m+1}^{-1} \in \mathcal{A}_{\mathfrak{K}}$ , and by virtue of Proposition 1 we conclude that

$$(22) \quad A \in \mathcal{A}_{\mathfrak{K}}.$$

Finally, from eqs. (19), (20) and (22) we deduce that

$$\beta_{m+4} \in \mathcal{A}.$$

By taking into account that  $\det \beta_{m+4} = O(1)$ , we have proven that the singularity has disappeared. Thus, the singularity confinement is ensured with a confinement time  $l = 4$ .  $\square$

In order to show the genericness of conditions (10)-(13) we shall perform an asymptotic analysis, by introducing the expansions

$$\begin{aligned} \beta_{m-1} &= \sum_{i=0}^{\infty} \begin{pmatrix} A_{m-1,i} & B_{m-1,i} \\ C_{m-1,i} & D_{m-1,i} \end{pmatrix} \epsilon^i, \\ \beta_m &= \begin{pmatrix} 0 & 0 \\ C_{m,0} & D_{m,0} \end{pmatrix} + \sum_{i=1}^{\infty} \begin{pmatrix} A_{m,i} & B_{m,i} \\ C_{m,i} & D_{m,i} \end{pmatrix} \epsilon^i, \end{aligned}$$

whereas  $\alpha$  is written simply as

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}.$$

We introduce

**Definition 3.** *The following matrices will be useful*

$$\begin{aligned} Z_1 &:= D_{m+1,0} + D_{m,0}^{-1} C_{m,0} B_{m+1,0}, \\ Z_2 &:= D_{m+2,0} + D_{m,0}^{-1} C_{m,0} B_{m+2,0}, \\ Z_3 &:= D_{m+3,0}. \end{aligned}$$

The genericness of the singularity confinement is described by

**Theorem 2.** (1) *If  $\det D_{m,0} \neq 0$ , for  $\epsilon \rightarrow 0$  we have*

$$\det \beta_{m+1} = O(\epsilon^{-r}) \Leftrightarrow \det(Z_1) \neq 0.$$

(2) *If  $\det D_{m,0} \neq 0$ ,  $\det Z_1 \neq 0$ , we have that for  $\epsilon \rightarrow 0$*

$$\det \beta_{m+2} = O(\epsilon^{-r}) \Leftrightarrow \det(Z_2) \neq 0.$$

(3) *If  $\det D_{m,0} \neq 0$ ,  $\det Z_1 \neq 0$  and  $\det Z_2 \neq 0$ , we have that for  $\epsilon \rightarrow 0$*

$$\det \beta_{m+3} = O(\epsilon^r) \Leftrightarrow \det Z_3 \neq 0.$$

(4) *If  $\det D_{m,0} \neq 0$ ,  $\det Z_1 \neq 0$ ,  $\det Z_2 \neq 0$  and  $\det Z_3 \neq 0$  we have that*

$$\det \beta_{m+4} = O(1), \quad \epsilon \rightarrow 0,$$

*generically.*

*Proof.* See Appendix B  $\square$

The matrices  $Z_1, Z_2$  and  $Z_3$  can be expressed in terms of initial conditions as follows

**Proposition 2.** *The following expressions in terms of initial conditions hold*

$$\begin{aligned} Z_1 &= mD_{m,0}^{-1} - D_{m-1,0} - D_{m,0} - \alpha_{22} - D_{m,0}^{-1}C_{m,0}(B_{m-1,0} + \alpha_{12}), \\ Z_2 &= (m+1)(mD_{m,0}^{-1} - D_{m,0}^{-1}C_{m,0}(B_{m-1,0} + \alpha_{12}) - D_{m-1,0} - D_{m,0} - \alpha_{22})^{-1} + D_{m,0}^{-1}C_{m,0}B_{m-1,0} - mD_{m,0}^{-1} + D_{m-1,0}, \\ Z_3 &= D_{m,0} - (m+1)Z_1^{-1} + (m+2)Z_2^{-1}. \end{aligned}$$

*Proof.* Is a byproduct of the proof of Theorem 2.  $\square$

## APPENDIX A. SCHUR COMPLEMENTS

To show the genericness of the confinement phenomenon in the non Abelian scenario it is very convenient to introduce Schur complements.

**Definition 4.** *Given  $M$  in block form as in (2), the Schur complements with respect to  $D$  (if  $\det D \neq 0$ ), and to  $A$  (if  $\det A \neq 0$ ) are defined to be*

$$S_D(M) := A - BD^{-1}C, \quad S_A(M) := D - CA^{-1}B,$$

*respectively.*

In terms of the Schur complements we have the following well known expressions for the inverse matrices

$$(23) \quad M^{-1} = \begin{cases} \begin{pmatrix} S_D(M)^{-1} & -S_D(M)^{-1}BD^{-1} \\ -D^{-1}CS_D(M)^{-1} & D^{-1}(\mathbb{I}_{N-r} + CS_D(M)^{-1}BD^{-1}) \end{pmatrix}, & \text{for } \det D, \det S_D(M) \neq 0, \\ \begin{pmatrix} A^{-1} + A^{-1}BS_A(M)^{-1}CA^{-1} & -A^{-1}BS_A(M)^{-1} \\ -S_A(M)^{-1}CA^{-1} & S_A(M)^{-1} \end{pmatrix}, & \text{for } \det A, \det S_A(M) \neq 0, \\ \begin{pmatrix} S_D(M)^{-1} & -S_D(M)^{-1}BD^{-1} \\ -D^{-1}CS_D(M)^{-1} & S_A(M)^{-1} \end{pmatrix}, & \text{for } \det A, \det D, \det S_D(M), \det S_A(M) \neq 0, \end{cases}$$

and for the determinant of  $M$

$$(24) \quad \begin{aligned} \det M &= \det A \det S_A(M) \\ &= \det D \det S_D(M). \end{aligned}$$

Now, if  $K = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ C_0 & D_0 \end{pmatrix} + \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\epsilon + O(\epsilon^2) \in \mathcal{A}_{\mathbb{R}}$  then we can write the Schur complements in the form

$$(25) \quad \begin{aligned} S_D(K) &= A - BD^{-1}C =: S_D(K)_1\epsilon + S_D(K)_2\epsilon^2 + O(\epsilon^3), & \epsilon \rightarrow 0, \\ S_A(K) &= D - CA^{-1}B =: S_A(K)_0 + S_A(K)_1\epsilon + O(\epsilon^2), & \epsilon \rightarrow 0, \end{aligned}$$



where

$$\begin{aligned}
S_D(K)_1 &= A_1 - B_1 D_0^{-1} C_0, \\
S_D(K)_2 &= A_2 - B_1 D_0^{-1} C_1 - B_2 D_0^{-1} C_0 + B_1 D_0^{-1} D_1 D_0^{-1} C_0, \\
S_D(K)_3 &= A_3 - B_1 D_0^{-1} C_2 + (B_1 D_0^{-1} D_1 D_0^{-1} - B_2 D_0^{-1}) C_1 + B_1 (D_0^{-1} D_2 D_0^{-1} - D_0^{-1} D_1 D_0^{-1} D_1 D_0^{-1}) C_0 \\
&\quad + B_2 D_0^{-1} D_1 D_0^{-1} C_0 - B_3 D_0^{-1} C_0, \\
S_D(K)_4 &= A_4 - B_1 D_0^{-1} C_3 + B_1 D_0^{-1} D_1 D_0^{-1} C_2 - B_1 D_0^{-1} (D_1 D_0^{-1} D_1 D_0^{-1} - D_2 D_0^{-1}) C_1 - B_1 D_0^{-1} D_2 D_0^{-1} D_1 D_0^{-1} C_0 \\
&\quad + B_1 D_0^{-1} D_1 (D_0^{-1} D_1 D_0^{-1} D_1 D_0^{-1} - D_0^{-1} D_2 D_0^{-1}) C_0 + B_1 D_0^{-1} D_3 D_0^{-1} C_0 - B_2 D_0^{-1} C_2 \\
&\quad + B_2 D_0^{-1} D_1 D_0^{-1} C_1 - B_2 D_0^{-1} (D_1 D_0^{-1} D_1 D_0^{-1} - D_2 D_0^{-1}) C_0 - B_3 D_0^{-1} (C_1 - D_1 D_0^{-1} C_0) - B_4 D_0^{-1} C_0, \\
S_A(K)_0 &= D_0 - C_0 A_1^{-1} B_1, \\
S_A(K)_1 &= D_1 - C_0 A_1^{-1} B_2 - C_1 A_1^{-1} B_1 + C_0 A_1^{-1} A_2 A_1^{-1} B_1.
\end{aligned}$$

For the determinant  $\det M$  we just take into account eqs. (24) and (25) to get

$$\begin{aligned}
\det K &= \epsilon^r \det(A_1 - B_1 D_0^{-1} C_0 + O(\epsilon)) \det(D_0 + O(\epsilon)) \\
&= \det(A_1 - B_1 D_0^{-1} C_0) \det(D_0) \epsilon^r + O(\epsilon^{r+1}).
\end{aligned}$$

## APPENDIX B. PROOF OF THEOREM 2

**Lemma 2.** (1) *Assuming that  $\det D_{m,0} \neq 0$  the following asymptotic holds*

$$\det \beta_{m+1} = \epsilon^{-r} \left| \begin{array}{cc} m S_D(\beta_m)_1^{-1} & -m S_D(\beta_m)_1^{-1} B_{m,1} D_{m,0}^{-1} - B_{m-1,0} - \alpha_{12} \\ -m D_{m,0}^{-1} C_{m,0} S_D(\beta_m)_1^{-1} & m D_{m,0}^{-1} + m D_{m,0}^{-1} C_{m,0} S_D(\beta_m)_1^{-1} B_{m,1} D_{m,0}^{-1} - D_{m-1,0} - D_{m,0} - \alpha_{22} \end{array} \right| + O(\epsilon^{-r+1})$$

for  $\epsilon \rightarrow 0$ , where  $S_D(\beta_m)_1 := A_{m,1} - B_{m,1} D_{m,0}^{-1} C_{m,0} \in \mathbb{C}^{r \times r}$ .

*Proof.* From eq. (8) we know that

$$\det \begin{pmatrix} A_{m,1} & B_{m,1} \\ C_{m,0} & D_{m,0} \end{pmatrix} \neq 0,$$

hence  $S_D(\beta_m)_1$  is invertible. Then, from (23) and (25) we deduce

$$\beta_m^{-1} = \begin{pmatrix} (\beta_m^{-1})_{11,-1} & 0 \\ (\beta_m^{-1})_{21,-1} & 0 \end{pmatrix} \epsilon^{-1} + \begin{pmatrix} (\beta_m^{-1})_{11,0} & (\beta_m^{-1})_{12,0} \\ (\beta_m^{-1})_{21,0} & (\beta_m^{-1})_{22,0} \end{pmatrix} + \begin{pmatrix} (\beta_m^{-1})_{11,1} & (\beta_m^{-1})_{12,1} \\ (\beta_m^{-1})_{21,1} & (\beta_m^{-1})_{22,1} \end{pmatrix} \epsilon + O(\epsilon^2), \quad \epsilon \rightarrow 0,$$

where the pole coefficients are

$$(26) \quad (\beta_m^{-1})_{11,-1} := S_D(\beta_m)_1^{-1}, \quad (\beta_m^{-1})_{21,-1} := -D_{m,0}^{-1} C_{m,0} S_D(\beta_m)_1^{-1},$$

while the regular part coefficients are

$$\begin{aligned}
(\beta_m^{-1})_{11,0} &:= -S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1}, \\
(\beta_m^{-1})_{12,0} &:= -S_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1}, \\
(\beta_m^{-1})_{21,0} &:= D_{m,0}^{-1}(C_{m,0}S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1} - (C_{m,1} - D_{m,1}D_{m,0}^{-1}C_{m,0})S_D(\beta_m)_1^{-1}), \\
(\beta_m^{-1})_{22,0} &:= D_{m,0}^{-1}(\mathbb{I}_{N-r} + C_{m,0}S_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1}), \\
(\beta_m^{-1})_{11,1} &:= S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1} - S_D(\beta_m)_1^{-1}S_D(\beta_m)_3S_D(\beta_m)_1^{-1}, \\
(\beta_m^{-1})_{12,1} &:= (S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1}B_{m,1} - S_D(\beta_m)_1^{-1}(B_{m,2} - B_{m,1}D_{m,0}^{-1}D_{m,1}))D_{m,0}^{-1}, \\
(\beta_m^{-1})_{21,1} &:= -D_{m,0}^{-1}\left(C_{m,0}[S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1} - S_D(\beta_m)_1^{-1}S_D(\beta_m)_3S_D(\beta_m)_1^{-1}] \right. \\
&\quad \left. - (C_{m,1} - D_{m,1}D_{m,0}^{-1}C_{m,0})S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1} + \right. \\
&\quad \left. - ((D_{m,1}D_{m,0}^{-1}D_{m,1} - D_{m,2})D_{m,0}^{-1}C_{m,0} + C_{m,2} - D_{m,1}D_{m,0}^{-1}C_{m,1})S_D(\beta_m)_1^{-1}\right), \\
(\beta_m^{-1})_{22,1} &:= D_{m,0}^{-1}\left(-D_{m,1} + (C_{m,1} - D_{m,1}D_{m,0}^{-1}C_{m,0} - C_{m,0}S_D(\beta_m)_1^{-1}S_D(\beta_m)_2)S_D(\beta_m)_1^{-1}B_{m,1} \right. \\
&\quad \left. + C_{m,0}S_D(\beta_m)_1^{-1}(B_{m,2} - B_{m,1}D_{m,0}^{-1}D_{m,1})\right)D_{m,0}^{-1}, \\
(\beta_m^{-1})_{11,2} &:= S_D(\beta_m)_1^{-1}S_D(\beta_m)_3S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1} \\
&\quad - S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1}(S_D(\beta_m)_2S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1} \\
&\quad - S_D(\beta_m)_3S_D(\beta_m)_1^{-1}) - S_D(\beta_m)_1^{-1}S_D(\beta_m)_4S_D(\beta_m)_1^{-1}, \\
(\beta_m^{-1})_{12,2} &:= S_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1}(D_{m,2}D_{m,0}^{-1} - D_{m,1}D_{m,0}^{-1}D_{m,1}D_{m,0}^{-1}) + S_D(\beta_m)_1^{-1}(B_{m,2} \\
&\quad - S_D(\beta_m)_2S_D(\beta_m)_1^{-1}B_{m,1})D_{m,0}^{-1}D_{m,1}D_{m,0}^{-1} - S_D(\beta_m)_1^{-1}(B_{m,3} - S_D(\beta_m)_2S_D(\beta_m)_1^{-1}B_{m,2} \\
&\quad + S_D(\beta_m)_2S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1}B_{m,1} - S_D(\beta_m)_3S_D(\beta_m)_1^{-1}B_{m,1})D_{m,0}^{-1}.
\end{aligned}$$

Finally, from eq. (1) we deduce

$$(27) \quad \beta_{m+1} = \begin{pmatrix} mS_D(\beta_m)_1^{-1} & 0 \\ -mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1} & 0 \end{pmatrix} \epsilon^{-1} + \begin{pmatrix} A_{m+1,0} & B_{m+1,0} \\ C_{m+1,0} & D_{m+1,0} \end{pmatrix} + \begin{pmatrix} A_{m+1,1} & B_{m+1,1} \\ C_{m+1,1} & D_{m+1,1} \end{pmatrix} \epsilon + O(\epsilon^2), \epsilon \rightarrow 0,$$

where, in terms of eqs. (26)–(??),

$$\begin{aligned}
(28) \quad A_{m+1,0} &:= m(\beta_m^{-1})_{11,0} - A_{m-1,0} - \alpha_{11}, & B_{m+1,0} &:= m(\beta_m^{-1})_{12,0} - B_{m-1,0} - \alpha_{12}, \\
(29) \quad C_{m+1,0} &:= m(\beta_m^{-1})_{21,0} - C_{m-1,0} - C_{m,0} - \alpha_{21}, & D_{m+1,0} &:= m(\beta_m^{-1})_{22,0} - D_{m-1,0} - D_{m,0} - \alpha_{22}, \\
(30) \quad A_{m+1,1} &:= m(\beta_m^{-1})_{11,1} - A_{m-1,1} - A_{m,1}, & B_{m+1,1} &:= m(\beta_m^{-1})_{12,1} - B_{m-1,1} - B_{m,1}, \\
(31) \quad C_{m+1,1} &:= m(\beta_m^{-1})_{21,1} - C_{m-1,1} - C_{m,1}, & D_{m+1,1} &:= m(\beta_m^{-1})_{22,1} - D_{m-1,1} - D_{m,1}, \\
(32) \quad A_{m+1,2} &:= m(\beta_m^{-1})_{11,2} - A_{m-1,2} - A_{m,2}, & B_{m+1,2} &:= m(\beta_m^{-1})_{12,2} - B_{m-1,2} - B_{m,2}.
\end{aligned}$$

Observing that

$$\det \beta_{m+1} = \begin{vmatrix} mS_D(\beta_m)_1^{-1} & B_{m+1,0} \\ -mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1} & D_{m+1,0} \end{vmatrix} \epsilon^{-r} + O(\epsilon^{-r+1}), \quad \epsilon \rightarrow 0,$$

the result follows.  $\square$

Now observe that

$$\begin{aligned} Z_1 &:= mD_{m,0}^{-1} + mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} - D_{m-1,0} - D_{m,0} - \alpha_{22} \\ &\quad - (-mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1})(mS_D(\beta_m)_1^{-1})^{-1}(-mS_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} - B_{m-1,0} - \alpha_{12}) \\ &= mD_{m,0}^{-1} - D_{m-1,0} - D_{m,0} - \alpha_{22} - D_{m,0}^{-1}C_{m,0}(B_{m-1,0} + \alpha_{12}). \end{aligned}$$

By using the determinant expansion in Schur complements of Lemma 2, one observes that

$$\begin{vmatrix} mS_D(\beta_m)_1^{-1} & -mS_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} - B_{m-1,0} - \alpha_{12} \\ -mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1} & mD_{m,0}^{-1} + mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} - D_{m-1,0} - D_{m,0} - \alpha_{22} \end{vmatrix} = \det \left( mS_D(\beta_m)_1^{-1} \right) \det Z_1.$$

and the first point of the Theorem is proved.

Let us now go one step further in the discrete matrix chain and move to position  $m+2$ .

**Lemma 3.** *Whenever  $\det D_{m,0} \neq 0$  and  $\det Z_1 \neq 0$  the following asymptotic hold*

$$\det \beta_{m+2} = \epsilon^{-r} \begin{vmatrix} -mS_D(\beta_m)_1^{-1} & mS_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} + B_{m-1,0} \\ mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1} & (m+1)Z_1^{-1} - mD_{m,0}^{-1} - mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} + D_{m-1,0} \end{vmatrix} + O(\epsilon^{-r+1})$$

for  $\epsilon \rightarrow 0$ .

*Proof.* As  $\det \beta_{m+1} = O(\epsilon^{-r})$ ,  $\epsilon \rightarrow 0$ , and consequently point (2) of Proposition 1 tells us that  $\beta_{m+1}^{-1} \in \mathcal{A}_{\mathfrak{R}}$ . Therefore, the following asymptotic expansion for the inverse matrix holds

$$(33) \quad \beta_{m+1}^{-1} = \begin{pmatrix} 0 & 0 \\ (\beta_{m+1}^{-1})_{21,0} & (\beta_{m+1}^{-1})_{22,0} \end{pmatrix} + \begin{pmatrix} (\beta_{m+1}^{-1})_{11,1} & (\beta_{m+1}^{-1})_{12,1} \\ (\beta_{m+1}^{-1})_{21,1} & (\beta_{m+1}^{-1})_{22,1} \end{pmatrix} \epsilon + \begin{pmatrix} (\beta_{m+1}^{-1})_{11,2} & (\beta_{m+1}^{-1})_{12,2} \\ (\beta_{m+1}^{-1})_{21,2} & (\beta_{m+1}^{-1})_{22,2} \end{pmatrix} \epsilon^2 + O(\epsilon^3),$$

for  $\epsilon \rightarrow 0$ . Here the blocks  $(\beta_{m+1}^{-1})_{ab,j}$  are to be found from the asymptotic expansion (27). We conclude

$$\begin{aligned} (\beta_{m+1}^{-1})_{21,0} &= Z_1^{-1}D_{m,0}^{-1}C_{m,0}, & (\beta_{m+1}^{-1})_{22,0} &= Z_1^{-1}, \\ (\beta_{m+1}^{-1})_{11,1} &= \frac{1}{m}S_D(\beta_m)_1 - \frac{1}{m}S_D(\beta_m)_1B_{m+1,0}Z_1^{-1}D_{m,0}^{-1}C_{m,0}, & (\beta_{m+1}^{-1})_{12,1} &= -\frac{1}{m}S_D(\beta_m)_1B_{m+1,0}Z_1^{-1}, \end{aligned}$$

$$\begin{aligned}
(\beta_{m+1}^{-1})_{21,1} &= -Z_1^{-1}D_{m,0}^{-1}C_{m,0} - \frac{1}{m}Z_1^{-1}(C_{m+1,0} + D_{m,0}^{-1}C_{m,0}A_{m+1,0})S_D(\beta_m)_1(\mathbb{I}_r - B_{m+1,0}Z_1^{-1}D_{m,0}^{-1}C_{m,0}), \\
(\beta_{m+1}^{-1})_{22,1} &= -Z_1^{-1} + \frac{1}{m}Z_1^{-1}(C_{m+1,0} + D_{m,0}^{-1}C_{m,0}A_{m+1,0})S_D(\beta_m)_1B_{m+1,0}Z_1^{-1}, \\
(\beta_{m+1}^{-1})_{11,2} &= -\frac{1}{m^2}S_D(\beta_m)_1A_{m+1,0}S_D(\beta_m)_1 + \frac{1}{m^2}S_D(\beta_m)_1A_{m+1,0}S_D(\beta_m)_1B_{m+1,0}Z_1^{-1}D_{m,0}^{-1}C_{m,0} \\
&\quad + \frac{1}{m^2}S_D(\beta_m)_1B_{m+1,0}Z_1^{-1}(C_{m+1,0} + D_{m,0}^{-1}C_{m,0}A_{m+1,0})S_D(\beta_m)_1(\mathbb{I}_r - B_{m+1,0}Z_1^{-1}D_{m,0}^{-1}C_{m,0}), \\
(\beta_{m+1}^{-1})_{12,2} &= -\frac{1}{m^2}S_D(\beta_m)_1B_{m+1,0}Z_1^{-1}(C_{m+1,0} + D_{m,0}^{-1}C_{m,0}A_{m+1,0})S_D(\beta_m)_1B_{m+1,0}Z_1^{-1} \\
&\quad + \frac{1}{m^2}S_D(\beta_m)_1A_{m+1,0}S_D(\beta_m)_1B_{m+1,0}Z_1^{-1}.
\end{aligned}$$

If we substitute equations (28)-(32) into eq. (1), we have that for  $\epsilon \rightarrow 0$

$$(34) \quad \beta_{m+2} = \begin{pmatrix} -mS_D(\beta_m)_1^{-1} & 0 \\ mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1} & 0 \end{pmatrix} \epsilon^{-1} + \begin{pmatrix} A_{m+2,0} & B_{m+2,0} \\ C_{m+2,0} & D_{m+2,0} \end{pmatrix} + \begin{pmatrix} A_{m+2,1} & B_{m+2,1} \\ C_{m+2,1} & D_{m+2,1} \end{pmatrix} \epsilon + O(\epsilon^2),$$

where

$$\begin{aligned}
A_{m+2,0} &:= -A_{m+1,0} - \alpha_{11}, & B_{m+2,0} &:= -B_{m+1,0} - \alpha_{12}, \\
C_{m+2,0} &:= (m+1)(\beta_{m+1}^{-1})_{21,0} - C_{m+1,0} - C_{m,0} - \alpha_{21}, & D_{m+2,0} &:= (m+1)(\beta_{m+1}^{-1})_{22,0} - D_{m+1,0} - D_{m,0} - \alpha_{22}, \\
A_{m+2,1} &:= (m+1)(\beta_{m+1}^{-1})_{11,1} - A_{m+1,1} - A_{m,1}, & B_{m+2,1} &:= (m+1)(\beta_{m+1}^{-1})_{12,1} - B_{m+1,1} - B_{m,1}, \\
C_{m+2,1} &:= (m+1)(\beta_{m+1}^{-1})_{21,1} - C_{m+1,1} - C_{m,1}, & D_{m+2,1} &:= (m+1)(\beta_{m+1}^{-1})_{22,1} - D_{m+1,1} - D_{m,1}, \\
A_{m+2,2} &:= (m+1)(\beta_{m+1}^{-1})_{11,2} - A_{m+1,2} - A_{m,2}, & B_{m+2,2} &:= (m+1)(\beta_{m+1}^{-1})_{12,2} - B_{m+1,2} - B_{m,2}.
\end{aligned}$$

Now, observing that

$$\det \beta_{m+2} = \begin{vmatrix} -mS_D(\beta_m)_1^{-1} & B_{m+2,0} \\ mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1} & D_{m+2,0} \end{vmatrix} \epsilon^{-r} + O(\epsilon^{-r+1}), \quad \epsilon \rightarrow 0,$$

the result follows.  $\square$

Notice that

$$\begin{aligned}
Z_2 &:= (m+1)(mD_{m,0}^{-1} - D_{m,0}^{-1}C_{m,0}(B_{m-1,0} + \alpha_{12}) - D_{m-1,0} - D_{m,0} - \alpha_{22})^{-1} \\
&\quad + D_{m,0}^{-1}C_{m,0}B_{m-1,0} - mD_{m,0}^{-1} + D_{m-1,0}.
\end{aligned}$$

We expand the determinant according to Schur complements, obtaining

$$\begin{aligned}
&\begin{vmatrix} -mS_D(\beta_m)_1^{-1} & mS_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} + B_{m-1,0} \\ mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1} & (m+1)Z_1^{-1} - mD_{m,0}^{-1} - mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} + D_{m-1,0} \end{vmatrix} \\
&= \det \left( -mS_D(\beta_m)_1^{-1} \right) \det Z_2
\end{aligned}$$

from which the second point of the Theorem follows immediately.

**Lemma 4.** *Assuming that  $\det D_{m,0} \neq 0$ ,  $\det Z_1 \neq 0$  and  $\det Z_2 \neq 0$  the following asymptotic expansion for  $\epsilon \rightarrow 0$  holds*

$$\det \beta_{m+3} = \epsilon^r \begin{vmatrix} (m+2)(\beta_{m+2}^{-1})_{11,1} - (m+1)(\beta_{m+1}^{-1})_{11,1} + A_{m,1} & (m+2)(\beta_{m+2}^{-1})_{12,1} - (m+1)(\beta_{m+1}^{-1})_{12,1} + B_{m,1} \\ (m+2)(\beta_{m+2}^{-1})_{21,0} - (m+1)(\beta_{m+1}^{-1})_{21,0} + C_{m,0} & (m+2)(\beta_{m+2}^{-1})_{22,0} - (m+1)(\beta_{m+1}^{-1})_{22,0} + D_{m,0} \end{vmatrix} + O(\epsilon^{r+1}),$$

where

$$\begin{aligned} (\beta_{m+2}^{-1})_{21,0} &:= Z_2^{-1} D_{m,0}^{-1} C_{m,0}, & (\beta_{m+2}^{-1})_{22,0} &:= Z_2^{-1}, \\ (\beta_{m+2}^{-1})_{11,1} &:= -\frac{1}{m} S_D(\beta_m)_1 (\mathbb{I}_r - B_{m+2,0} Z_2^{-1} D_{m,0}^{-1} C_{m,0}), & (\beta_{m+2}^{-1})_{12,1} &:= \frac{1}{m} S_D(\beta_m)_1 B_{m+2,0} Z_2^{-1}, \\ (\beta_{m+2}^{-1})_{21,1} &:= -Z_2^{-1} D_{m,0}^{-1} C_{m,0} + \frac{1}{m} Z_2^{-1} (C_{m+2,0} + D_{m,0}^{-1} C_{m,0} A_{m+2,0}) S_D(\beta_m)_1 (\mathbb{I}_r - B_{m+2,0} Z_2^{-1} D_{m,0}^{-1} C_{m,0}), \\ (\beta_{m+2}^{-1})_{22,1} &:= -Z_2^{-1} - \frac{1}{m} Z_2^{-1} (C_{m+2,0} + D_{m,0}^{-1} C_{m,0} A_{m+2,0}) S_D(\beta_m)_1 B_{m+2,0} Z_2^{-1}, \\ (\beta_{m+2}^{-1})_{11,2} &:= \frac{1}{m^2} S_D(\beta_m)_1 B_{m+2,0} Z_2^{-1} (C_{m+2,0} + D_{m,0}^{-1} C_{m,0} A_{m+2,0}) S_D(\beta_m)_1 (\mathbb{I}_r - B_{m+2,0} Z_2^{-1} D_{m,0}^{-1} C_{m,0}) \\ &\quad - \frac{1}{m^2} S_D(\beta_m)_1 A_{m+2,0} S_D(\beta_m)_1 (\mathbb{I}_r - B_{m+2,0} Z_2^{-1} D_{m,0}^{-1} C_{m,0}), \\ (\beta_{m+2}^{-1})_{12,2} &:= \frac{1}{m^2} S_D(\beta_m)_1 A_{m+2,0} S_D(\beta_m)_1 B_{m+2,0} Z_2^{-1} \\ &\quad - \frac{1}{m^2} S_D(\beta_m)_1 B_{m+2,0} Z_2^{-1} (C_{m+2,0} + D_{m,0}^{-1} C_{m,0} A_{m+2,0}) S_D(\beta_m)_1 B_{m+2,0} Z_2^{-1}. \end{aligned}$$

*Proof.* From equation (34) we get that  $\beta_{m+2} \in \mathbb{L}$ . Therefore, since  $\det Z_2 \neq 0$ , we have

(35)

$$\beta_{m+2}^{-1} = \begin{pmatrix} 0 & 0 \\ (\beta_{m+2}^{-1})_{21,0} & (\beta_{m+2}^{-1})_{22,0} \end{pmatrix} + \begin{pmatrix} (\beta_{m+2}^{-1})_{11,1} & (\beta_{m+2}^{-1})_{12,1} \\ (\beta_{m+2}^{-1})_{21,1} & (\beta_{m+2}^{-1})_{22,1} \end{pmatrix} \epsilon + \begin{pmatrix} (\beta_{m+2}^{-1})_{11,2} & (\beta_{m+2}^{-1})_{12,2} \\ (\beta_{m+2}^{-1})_{21,2} & (\beta_{m+2}^{-1})_{22,2} \end{pmatrix} \epsilon^2 + O(\epsilon^3),$$

where the blocks  $(\beta_{m+2}^{-1})_{ab,j}$ , are determined by the asymptotic expansion (34). If we substitute (27), (34) and (35) into the matrix equation (1), we have that

$$\beta_{m+3} = \begin{pmatrix} 0 & 0 \\ C_{m+3,0} & D_{m+3,0} \end{pmatrix} + \begin{pmatrix} A_{m+3,1} & B_{m+3,1} \\ C_{m+3,1} & D_{m+3,1} \end{pmatrix} \epsilon + \begin{pmatrix} A_{m+3,2} & B_{m+3,2} \\ C_{m+3,2} & D_{m+3,2} \end{pmatrix} \epsilon^2 + O(\epsilon^3),$$

where

$$\begin{aligned} C_{m+3,0} &:= (m+2)(\beta_{m+2}^{-1})_{21,0} - (m+1)(\beta_{m+1}^{-1})_{21,0} + C_{m,0}, & D_{m+3,0} &:= (m+2)(\beta_{m+2}^{-1})_{22,0} - (m+1)(\beta_{m+1}^{-1})_{22,0} + D_{m,0}, \\ A_{m+3,1} &:= (m+2)(\beta_{m+2}^{-1})_{11,1} - (m+1)(\beta_{m+1}^{-1})_{11,1} + A_{m,1}, & B_{m+3,1} &:= (m+2)(\beta_{m+2}^{-1})_{12,1} - (m+1)(\beta_{m+1}^{-1})_{12,1} + B_{m,1}, \\ C_{m+3,1} &:= (m+2)(\beta_{m+2}^{-1})_{21,1} - (m+1)(\beta_{m+1}^{-1})_{21,1} + C_{m,1}, & D_{m+3,1} &:= (m+2)(\beta_{m+2}^{-1})_{22,1} - (m+1)(\beta_{m+1}^{-1})_{22,1} + D_{m,1}, \\ A_{m+3,2} &:= (m+2)(\beta_{m+2}^{-1})_{11,2} - (m+1)(\beta_{m+1}^{-1})_{11,2} + A_{m,2}, & B_{m+3,2} &:= (m+2)(\beta_{m+2}^{-1})_{12,2} - (m+1)(\beta_{m+1}^{-1})_{12,2} + B_{m,2}. \end{aligned}$$

Then, if we use again Proposition 1, we deduce

$$(36) \quad \det \beta_{m+3} = \epsilon^r \begin{vmatrix} A_{m+3,1} & B_{m+3,1} \\ C_{m+3,0} & D_{m+3,0} \end{vmatrix} + O(\epsilon^{r+1}), \quad \epsilon \rightarrow 0,$$

and the result follows.  $\square$

Notice that

$$Z_3 = D_{m,0} - (m+1)Z_1^{-1} + (m+2)Z_2^{-1}.$$

Notice the similarity with eq. (18).

Taking into account that

$$(37) \quad C_{m+3,0} = Z_3 D_{m,0}^{-1} C_{m,0}, \quad D_{m+3,0} = Z_3,$$

we express the determinant in equation (36) as follows

$$(38) \quad \begin{vmatrix} A_{m+3,1} & B_{m+3,1} \\ C_{m+3,0} & D_{m+3,0} \end{vmatrix} = \det Z_3 \det(A_{m+3,1} - B_{m+3,1} D_{m,0}^{-1} C_{m,0}),$$

where

$$A_{m+3,1} - B_{m+3,1} D_{m,0}^{-1} C_{m,0} = -\frac{(m+3)}{m} S_D(\beta_m)_1.$$

This implies that the determinant in equation (36) vanishes if and only if

$$\det Z_3 = 0.$$

Finally, under the previous hypotheses, eqs. (10)-(12) hold. As a by product of the proof of Theorem 1, we get that

$$\beta_{m+4} = \beta_{m+3}^{-1} A - \beta_{m+3} - \alpha,$$

where  $\beta_{m+3}, A \in \mathcal{A}_{\mathfrak{R}}$  and  $(\beta_{m+3})^{-1} \in \epsilon^{-1} \mathcal{A}_{\mathfrak{L}}$ . According to Proposition 1 (6),  $\beta_{m+3}^{-1} A \in \mathcal{A}$ , so that we can write

$$\beta_{m+4} = O(1), \quad \epsilon \rightarrow 0.$$

We can write the matrix dynamical system (1) as

$$(39) \quad \beta_{n-1} = n\beta_n^{-1} - \beta_{n+1} - \beta_n - \alpha,$$

which can be seen as the application of a *time reversal symmetry*. From  $\beta_{m+4} \in \mathcal{A}$  and  $\beta_{m+3} \in \mathcal{A}_{\mathfrak{R}}$ , understood now as initial conditions, we get the quantities  $\beta_{m+2}$ ,  $\beta_{m+1}$ ,  $\beta_m$  and  $\beta_{m-1}$ . Observe that our initial assumption was precisely that  $\beta_{m-1} \in \mathcal{A}$  and  $\beta_m \in \mathcal{A}_{\mathfrak{R}}$ , see (7) and (8). Hence, the whole forward process, and its conclusions about the asymptotic behaviours, can be reversed backwards. Consequently, since the assumption that  $\det \beta_{m+4,0} = 0$  reduces the number of free parameters from  $N^2$  to  $N^2 - 1$ , we conclude that  $\beta_{m-1,0}$  involves at most  $N^2 - 1$  free parameters (if no further constraint is requested). This is in contradiction to our departing hypothesis that  $\beta_{m-1,0}$  has  $N^2$  free parameters. Therefore  $\det \beta_{m+4} = O(1)$  as  $\epsilon \rightarrow 0$  generically.

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