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Compactness interpolation results for bilinear operators of convolution type and for operators of product type

Fernando Cobos[a,*], Luz M. Fernández-Cabrera[b], Antón Martínez[c]

[a] *Departamento de Análisis Matemático y Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain*

[b] *Sección Departamental del Departamento de Análisis Matemático y Matemática Aplicada, Facultad de Estudios Estadísticos, Universidad Complutense de Madrid, 28040 Madrid, Spain*

[c] *Departamento de Matemática Aplicada I, Escuela de Ingeniería Industrial, Universidad de Vigo, 36200 Vigo, Spain*

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Abstract

We establish compactness interpolation results for bilinear operators of convolution type and for operators of product type among quasi-Banach spaces. We do not assume any auxiliary condition on the spaces.

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1. Introduction

Interpolation of bilinear operators is a classical question that was already considered by Lions and Peetre [26] and Calderón [6] in their seminal papers on the real interpolation method and the complex method, respectively. Since then this question has attracted the attention of many authors and the results have found interesting applications in analysis and in operator theory (see, for example, in addition to [6,26], the contributions by O´Neil [29], König [22,23], Peetre [30] and Janson [20]).

* Corresponding author.
E-mail addresses: cobos@mat.ucm.es (F. Cobos), luz_fernandez-c@mat.ucm.es (L.M. Fernández-Cabrera), antonmar@uvigo.es (A. Martínez).

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As for the real method, if the couple in the target of the bilinear operator T is formed by p-Banach spaces, then the result reads that T may be uniquely extended to a continuous bilinear mapping

$$T : (A_0, A_1)^{\circ}_{\theta,q_1} \times (B_0, B_1)^{\circ}_{\theta,q_2} \longrightarrow (E_0, E_1)^{\circ}_{\theta,q}$$

provided that $0 < \theta < 1$, $p \le q_1, q_2 \le \infty$, $0 < q \le \infty$ and

$$1/q \le 1/q_1 + 1/q_2 - 1/p \tag{1.1}$$

(see [18,21,22]). Notation is explained in Section 2.

An improvement of this result was established by O´Neil [29] for convolution operators among Lorentz spaces. The counterpart for bilinear operators among quasi-Banach spaces was proved by Janson [20] with the outcome that for operators of convolution type the condition (1.1) can be replaced by

$$1/q \le 1/q_1 + 1/q_2.$$

Besides boundedness, another useful property that a bilinear operator may have is compactness. Complex interpolation of compact bilinear operators was considered by Calderón [6]. For the real method, the problem has been studied more recently by Fernandez and Silva [16], Fernández-Cabrera and Martínez [17,18], Cobos, Fernández-Cabrera and Martínez [8,9] and Mastyło and Silva [28] (see also the papers by Mastyło and Silva [27] and Besoy and Cobos [5] on interpolation of the measure of non-compactness of bilinear operators). A motivation for this research has been that compact bilinear operators occur rather naturally in harmonic analysis. In fact, commutators of bilinear Calderón–Zygmund operators and multiplication by functions in the subspace CMO of BMO are compact bilinear operators from $L_p(\mathbb{R}^n) \times L_q(\mathbb{R}^n) \longrightarrow L_r(\mathbb{R}^n)$ for $1 < p, q < \infty$, $1/2 < r < \infty$ and $1/r = 1/p+1/q$ (see the papers by Bényi and Torres [3], Cobos, Fernández-Cabrera and Martínez [8] and Torres, Xue and Yan [33]).

A compactness interpolation result of the type proved by Janson [20] for bounded bilinear operators has been established by the present authors [10]. However, in addition to the assumptions used by Janson and to the compactness in one of the restrictions, we assumed that the couple of spaces in the target satisfies a certain approximation condition which is essential for the arguments (see [10, Theorem 3.7]). The result applies to bilinear operators of convolution type. Our aim in the present paper is to show that for this kind of operators one can get rid of the approximation condition.

We work in the same setting as Janson [20], that is, we suppose that (A_0, A_1), (B_0, B_1), (E_0, E_1) are quasi-Banach couples and that the bilinear operator T is defined on the vectors of $(A_0 \cap A_1) \times (B_0 \cap B_1)$ with values into $E_0 \cap E_1$ and satisfies suitable boundedness conditions. Both assumptions are motivated by applications where one needs to work with Lebesgue spaces L_p and Lorentz spaces $L_{p,q}$ with $p < 1$. Under these weak boundedness conditions, the operator might not be bounded acting on the sums $T : (A_0 + A_1) \times (B_0 + B_1) \longrightarrow E_0 + E_1$. This produces a number of serious obstructions, with the result that some of the basic ideas of [8,16,17] do not work. Due to these difficulties, techniques of the papers dealing with compactness of operators satisfying weak boundedness assumptions are based on auxiliary approximation conditions (see [6,10,18]) or duality and the counterpart results for linear operators (see [9,11,27]). In this last case, results apply only to Banach spaces. In order to cover quasi-Banach spaces without auxiliary approximation conditions, we follow a different strategy, relying on the vector-valued sequence spaces that come up with the definition of the real method and on the properties of certain families of projections on the sequence spaces (see [7,12,13]). An estimate of [10] will be very important for our considerations.

We also establish a compactness result for bilinear operators of product type (see [29,32]) without assuming any approximation condition on the couples. As an application, we derive a compactness result in the line of the bilinear Marcinkiewicz theorem for multiplication type operators of Gilbert and Nahmod [19].

2. Preliminaries

We work with quasi-Banach spaces $(A, \|\cdot\|_A)$. We write $c_A \geq 1$ for the constant in the quasi-triangle inequality. Let $0 < p \leq 1$ be such that $c_A = 2^{1/p-1}$, then by the Aoki–Rolewicz theorem there is another quasi-norm $|||\cdot|||$ on A which is equivalent to $\|\cdot\|_A$ and such that $|||\cdot|||^p$ satisfies the triangle inequality (see [1,31] or [24, §15.10]). We say that $|||\cdot|||$ is a p-norm and that $(A, |||\cdot|||)$ is a p-Banach space. We remark that if $0 < s < p$ then A is also an s-Banach space.

We put $U_A = \{x \in A : \|x\|_A \leq 1\}$ and we designate by $I = I_A$ the identity operator.

For $0 < q \leq \infty$, we let ℓ_q be the usual space of q-summable sequences with $\mathbb{Z}$ as index set. Given any sequence of quasi-Banach spaces (W_m) with the same constant in the quasi-triangle inequality for all W_m and given any sequence of positive numbers (λ_m), we denote by $\ell_q(\lambda_m W_m)$ the vector-valued ℓ_q-space defined by

$$\ell_q(\lambda_m W_m) = \{w = (w_m) : w_m \in W_m \text{ and } \|w\|_{\ell_q(\lambda_m W_m)} = \|(\lambda_m \|w_m\|_{W_m})\|_{\ell_q} < \infty\}.$$

When $\lambda_m = 1$ for all $m \in \mathbb{Z}$, we write simply $\ell_q(W_m)$.

Let B be another quasi-Banach space. We put $\mathcal{L}(A, B)$ for the space of all bounded linear operators from A into B. As usual, for $R \in \mathcal{L}(A, B)$ we write $\|R\|_{A,B} = \sup\{\|Rx\|_B : \|x\|_A \leq 1\}$.

If E is another quasi-Banach space and $T : A \times B \longrightarrow E$ is a bilinear operator, we say that T is *bounded* if

$$\|T\|_{A\times B,E} = \sup\{\|T(a, b)\|_E : \|a\|_A \leq 1, \|b\|_B \leq 1\} < \infty.$$

We designate by $\mathcal{L}(A \times B, E)$ the space of all bounded bilinear operators from $A \times B$ into E.

We say that $T \in \mathcal{L}(A \times B, E)$ is *compact* if for any bounded sets $V \subseteq A, W \subseteq B$ we have that the closure of $T(V, W) = \{T(a, b) : a \in V, b \in W\}$ is compact in E. This condition is equivalent to the fact that for any bounded sequences $(a_n) \subseteq A, (b_n) \subseteq B$, the sequence $(T(a_n, b_n))$ has a convergent subsequence (see [3, Proposition 1]). We write $\mathcal{K}(A \times B, E)$ for the set of all compact bilinear operators from $A \times B$ into E. As in the linear case, the space $\mathcal{K}(A \times B, E)$ is closed in $\mathcal{L}(A \times B, E)$. That is to say, if $(T_n) \subseteq \mathcal{K}(A \times B, E)$ is a convergent sequence in $\mathcal{L}(A \times B, E)$ to the operator T, then $T \in \mathcal{K}(A \times B, E)$ (see [3, Proposition 3]).

Let E_1, A_1, B_1 be quasi-Banach spaces. If $R \in \mathcal{L}(E, E_1)$ is compact and $S \in \mathcal{L}(A \times B, E)$ then $RS = R \circ S$ belongs to $\mathcal{K}(A \times B, E_1)$. On the other hand, if $T \in \mathcal{K}(A \times B, E)$ and $R \in \mathcal{L}(E, E_1)$ then $RT \in \mathcal{K}(A \times B, E_1)$. Furthermore, if $S_1 \in \mathcal{L}(A_1, A)$ and $S_2 \in \mathcal{L}(B_1, B)$ then the operator

$$T \circ (S_1, S_2)(a, b) = T(S_1, S_2)(a, b) = T(S_1 a, S_2 b)$$

belongs to $\mathcal{K}(A_1 \times B_1, E)$ if $T \in \mathcal{K}(A \times B, E)$.

The real interpolation method and its variants allow to construct new quasi-Banach spaces from a given quasi-Banach couple, as well as to study properties of operators acting on them (see, for example, the books [4,15,34,35]). Next we recall the relevant definitions.

By a (p-*Banach*) *quasi-Banach couple* $\bar{A} = (A_0, A_1)$ we mean two (p-Banach) quasi-Banach spaces A_j which are continuously embedded in the same Hausdorff topological vector space. Let $A_0 + A_1$ be their sum and $A_0 \cap A_1$ be their intersection. Given $t > 0$, *Peetre's K-and J-functionals* are defined by

$$K(t,a) = K(t,a;A_0,A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1,\ a_j \in A_j\}$$

where $a \in A_0 + A_1$, and

$$J(t,a) = J(t,a;A_0,A_1) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\},\ a \in A_0 \cap A_1.$$

Note that $K(1,\cdot)$ coincides with the quasi-norm of A_0+A_1 and $J(1,\cdot)$ with the quasi-norm of $A_0 \cap A_1$. Functionals $K(t,\cdot)$ and $J(t,\cdot)$ are quasi-norms in A_0+A_1 and $A_0\cap A_1$, respectively. We can take $c_{\bar{A}} = \max\{c_{A_0}, c_{A_1}\}$ as the constant in the quasi-triangle inequality for all $t > 0$.

If $\|\cdot\|_{A_j}$ is a p-norm for $j = 0, 1$, then $J(t,\cdot)$ is a p-norm in $A_0 \cap A_1$, and the functional

$$K_p(t,a) = \inf\{\big(\|a_0\|_{A_0}^p + t^p\|a_1\|_{A_1}^p\big)^{1/p} : a = a_0 + a_1,\ a_j \in A_j\}$$

is a p-norm on $A_0 + A_1$. Note that we have

$$K(t,a) \le K_p(t,a) \le 2^{(1/p)-1}K(t,a),\ a \in A_0 + A_1. \tag{2.1}$$

For $0 < \theta < 1$ and $0 < q \le \infty$. The *real interpolation space* $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ consists of all $a \in A_0 + A_1$ having a finite quasi-norm

$$\|a\|_{\bar{A}_{\theta,q}} = \|(2^{-\theta m}K(2^m,a))\|_{\ell_q}.$$

For later use, we also define $\bar{A}_{\theta,q}$ for $\theta = 0, 1$ by putting $\bar{A}_{0,q} = A_0$ and $\bar{A}_{1,q} = A_1$ for any $0 < q \le \infty$.

We refer to the books by Bergh and Löfström [4], Triebel [34] and Bennett and Sharpley [2] for properties of the real interpolation spaces.

Real interpolation spaces have the interpolation property for bounded linear operators. That is to say, if $\bar{B} = (B_0, B_1)$ is another quasi-Banach couple and R is a linear operator which maps A_j continuously into B_j for $j = 0, 1$, then R maps $(A_0, A_1)_{\theta,q}$ continuously into $(B_0, B_1)_{\theta,q}$ and

$$\|R\|_{\bar{A}_{\theta,q},\bar{B}_{\theta,q}} \le C\|R\|_{A_0,B_0}^{1-\theta}\|R\|_{A_1,B_1}^{\theta}. \tag{2.2}$$

Here C is a constant independent of T.

Next we give an example. Let (Ω, μ) be a σ-finite measure space, let A be a quasi-Banach space and let $0 < p \le \infty$. We put $L_p(A) = L_p(\Omega; A)$ for the usual vector-valued L_p-space in the sense of the Bochner integral. For $0 < r \le \infty$ and $0 < p < \infty$, the *Lorentz space* $L_{p,r}(A) = L_{p,r}(\Omega; A)$ is formed by all (equivalence classes of) strongly measurable functions f with values in A which have a finite quasi-norm

$$\|f\|_{L_{p,r}(A)} = \Big(\int_0^\infty [t^{1/p}f^*(t)]^r \frac{dt}{t}\Big)^{1/r}$$

(the integral should be replaced by the supremum if $r = \infty$). Here f^* is the non-increasing rearrangement of f

$$f^*(t) = \inf\{s > 0 : \mu(\{x \in \Omega : \|f(x)\|_A > s\}) \le t\}.$$

When A is the scalar field $\mathbb{K}$, we write $L_p(\Omega)$ and $L_{p,r}(\Omega)$. We refer to [2,4,14,34] for details on Lorentz spaces. Note that if $p = r$, then $L_{p,p}(A) = L_p(A)$.

The K-functional of the couple $(L_p(A), L_\infty(A))$ satisfies that

$$K(t, f) \sim \Big(\int_0^{t^p} \big(f^*(s)\big)^p ds\Big)^{1/p} \qquad \text{(see [25])}.$$

Then, for $0 < q \leq \infty$, $0 < p < r < \infty$ and $0 < \theta < 1$ with $1/r = (1-\theta)/p$, we get that

$$(L_p(A), L_\infty(A))_{\theta,q} = L_{r,q}(A) \quad \text{(equivalent quasi-norms)}.$$

Applying the reiteration theorem, we obtain

$$(L_{p_1,r_1}(A), L_{p_2,r_2}(A))_{\theta,q} = L_{p,q}(A) \quad \text{(equivalent quasi-norms)} \tag{2.3}$$

where $0 < r_1, r_2, q \leq \infty$, $0 < p_1 \neq p_2 < \infty$, $0 < \theta < 1$ and $1/p = (1-\theta)/p_1 + \theta/p_2$ (see [34, formula (16), p. 134 and Remark 5, p. 135] or [4, Sections 5.2 and 5.3, pp. 108–114]).

The space $\bar{A}_{\theta,q}$ can be also realized by means of the J-functional as the collection of all $a \in A_0 + A_1$ for which there is a sequence $(u_m) \subseteq A_0 \cap A_1$ with $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in A_0+A_1) and $\|(2^{-\theta m} J(2^m, u_m))\|_{\ell_q} < \infty$. Moreover, $\|\cdot\|_{\bar{A}_{\theta,q}}$ is equivalent to the quasi-norm

$$\|a\|^J_{\theta,q} = \inf\{\|(2^{-\theta m} J(2^m, u_m))\|_{\ell_q} : a = \sum_{m=-\infty}^{\infty} u_m\}.$$

Given any quasi-Banach space A with $A_0 \cap A_1 \subseteq A$, we put A° for the closure of $A_0 \cap A_1$ in A. We write $\bar{A}^\circ = (A_0^\circ, A_1^\circ)$ which is also a (p-Banach) quasi-Banach couple if A_j is p-Banach for $j = 0, 1$. Since $A_0 \cap A_1 = A_0^\circ \cap A_1^\circ$, the J-description yields that $(A_0, A_1)_{\theta,q} = (A_0^\circ, A_1^\circ)_{\theta,q}$. Another consequence of the J-description is that $A_0 \cap A_1$ is dense in $(A_0, A_1)_{\theta,q}$ if $q < \infty$. Hence, $(A_0, A_1)^\circ_{\theta,q} = (A_0, A_1)_{\theta,q}$ if $q < \infty$.

Assume that $\|\cdot\|_{A_j}$ is a p-norm for $j = 0, 1$. Then $F_m = (A_0^\circ \cap A_1^\circ, J(2^m, \cdot))$ is a p-Banach space for each $m \in \mathbb{Z}$. Let $\bar{F}_p = (\ell_p(F_m), \ell_p(2^{-m} F_m))$ and let π be the linear operator assigning to any sequence (u_m) its sum $\pi(u_m) = \sum_{m=-\infty}^{\infty} u_m$ in $A_0^\circ + A_1^\circ$. The J-description of $\bar{A}_{\theta,q}$ yields that $\pi : \ell_q(2^{-\theta m} F_m) \longrightarrow (A_0^\circ, A_1^\circ)_{\theta,q} = (A_0, A_1)_{\theta,q}$ is bounded with norm less than or equal to 1. The operator π is surjective and

$$\|a\|^J_{\bar{A}_{\theta,q}} = \inf\{\|(u_m)\|_{\ell_q(2^{-\theta m} F_m)} : a = \pi(u_m)\}. \tag{2.4}$$

If $q < \infty$, then sequences with only a finite number of coordinates different from zero are dense in $\ell_q(2^{-\theta m} F_m)$ and so $\ell_q(2^{-\theta m} F_m) = \ell_q(2^{-\theta m} F_m)^\circ$. Note also that for any $0 < q \leq \infty$, the operator $\pi : \ell_q(2^{-\theta m} F_m)^\circ \longrightarrow (A_0, A_1)^\circ_{\theta,q}$ is bounded with norm less than or equal to 1. Furthermore, the restriction $\pi : \ell_p(2^{-jm} F_m) \longrightarrow A_j^\circ$ is bounded with norm less than or equal 1 for $j = 0, 1$.

The quasi-Banach couple $\bar{F}_p = (\ell_p(F_m), \ell_p(2^{-m} F_m))$ will be important for our later considerations. The following families of projections are useful to work with $\bar{F}_p$: For $n \in \mathbb{N}$, let

$$\begin{aligned} P_n(u_m) &= (\ldots 0, 0, u_{-n}, u_{-n+1}, \ldots, u_{n-1}, u_n, 0, 0, \ldots), \\ P_n^+(u_m) &= (\ldots, 0, 0, u_{n+1}, u_{n+2}, u_{n+3}, \ldots), \\ P_n^-(u_m) &= (\ldots, u_{-n-3}, u_{-n-2}, u_{-n-1}, 0, 0, \ldots). \end{aligned}$$

Clearly, the identity operator I on $\ell_p(F_m) + \ell_p(2^{-m} F_m)$ can be decomposed as $I = P_n + P_n^+ + P_n^-$, $n \in \mathbb{N}$. Any of these projections is bounded from $\ell_p(2^{-mj} F_m)$ into $\ell_p(2^{-mj} F_m)$ with norm less than or equal to 1 for $j = 0, 1$, and the same happens on $\ell_q(2^{-\theta m} F_m)$. In addition, we have that $P_n : \ell_p(F_m) + \ell_p(2^{-m}F_m) \longrightarrow \ell_p(F_m) \cap \ell_p(2^{-m}F_m)$, $P_n^+ : \ell_p(F_m) \longrightarrow \ell_p(2^{-m}F_m)$ and $P_n^- : \ell_p(2^{-m}F_m) \longrightarrow \ell_p(F_m)$ are bounded with

$$\|P_n\|_{\ell_p(F_m)+\ell_p(2^{-m}F_m),\ell_p(F_m)\cap\ell_p(2^{-m}F_m)} \leq c\, 2^n,$$

$$\|P_n^+\|_{\ell_p(F_m),\ell_p(2^{-m}F_m)} = 2^{-(n+1)} = \|P_n^-\|_{\ell_p(2^{-m}F_m),\ell_p(F_m)}. \tag{2.5}$$

Next we recall a result on interpolation of vector-valued sequence spaces. Let $0 < q_0, q_1, q \leq \infty, 0 < \theta < 1$ and let (X_m) be any sequence of quasi-Banach spaces with the same constant in the quasi-triangle inequality. The following equality holds with equivalent quasi-norms

$$(\ell_{q_0}(X_m), \ell_{q_1}(2^{-m}X_m))_{\theta,q} = \ell_q(2^{-\theta m}X_m) \tag{2.6}$$

(see [8, Lemma 2.4]).

Let $\bar{E} = (E_0, E_1)$ be another quasi-Banach couple. Now we realize $(E_0, E_1)_{\theta,q} = (E_0^\circ, E_1^\circ)_{\theta,q}$ by means of the K-functional and we put $W_m = (E_0^\circ + E_1^\circ, K(2^m, \cdot)), m \in \mathbb{Z}$. Let τ be the linear operator assigning to each $w \in E_0^\circ + E_1^\circ$ the constant sequence $\tau w = (\cdots, w, w, w, \cdots)$. Then τ is injective and $\tau : \bar{E}_{\theta,q} \longrightarrow \ell_q(2^{-\theta m}W_m)$ satisfies that

$$\|\tau w\|_{\ell_q(2^{-\theta m}W_m)} = \|w\|_{\bar{E}_{\theta,q}}, \quad w \in \bar{E}_{\theta,q}. \tag{2.7}$$

Moreover, for $j = 0, 1$, the restriction $\tau : E_j^\circ \longrightarrow \ell_\infty(2^{-mj}W_m)$ is bounded with norm less than or equal to 1. Let $\bar{W}_\infty = (\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))$. It follows from (2.6) that

$$(\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))_{\theta,q} = \ell_q(2^{-\theta m}W_m).$$

We consider on $\bar{W}_\infty$ similar projections to those on $\bar{F}_p$. We denote them by R_n, R_n^+, R_n^-. They have analogous properties to the projections on $\bar{F}_p$. In particular, the corresponding versions of (2.5) hold for R_n, R_n^+, R_n^-.

3. Interpolation of bounded bilinear operators

In what follows we work with bilinear operators. We put $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$ if T is a bilinear operator defined on $(A_0 \cap A_1) \times (B_0 \cap B_1)$ with values in $E_0 \cap E_1$ such that there are constants $M_j > 0$ with

$$\|T(a, b)\|_{E_j} \leq M_j \|a\|_{A_j} \|b\|_{B_j}, \ a \in A_0 \cap A_1, \ b \in B_0 \cap B_1, \ j = 0, 1. \tag{3.1}$$

It is a consequence of (3.1) that T may be uniquely extended to a bilinear operator $T : A_j^\circ \times B_j^\circ \longrightarrow E_j^\circ$ with $\|T\|_{A_j^\circ \times B_j^\circ, E_j^\circ} \leq M_j$.

If $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$ then T may be also uniquely extended to a bounded bilinear operator

$$T : (A_0 + A_1)^\circ \times (B_0 \cap B_1) \longrightarrow (E_0 + E_1)^\circ. \tag{3.2}$$

Indeed, if $a \in A_0 \cap A_1$ and $a = a_0 + a_1$ with $a_j \in A_j$, then $a_j \in A_0 \cap A_1$ for $j = 0, 1$. For any $b \in B_0 \cap B_1$, we have

$$\begin{aligned}\|T(a, b)\|_{E_0+E_1} &\leq c_{\bar{E}}(\|T(a_0, b)\|_{E_0+E_1} + \|T(a_1, b)\|_{E_0+E_1}) \\ &\leq c_{\bar{E}}(M_0\|a_0\|_{A_0}\|b\|_{B_0} + M_1\|a_1\|_{A_1}\|b\|_{B_1}) \\ &\leq c_{\bar{E}} \max\{M_0, M_1\}(\|a_0\|_{A_0} + \|a_1\|_{A_1})\|b\|_{B_0\cap B_1}.\end{aligned}$$

So,

$$\|T(a,b)\|_{E_0+E_1} \le c_{\bar{E}} \max\{M_0, M_1\}\|a\|_{A_0+A_1}\|b\|_{B_0\cap B_1}$$

which implies (3.2).

Similarly, if $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$ then T may be uniquely extended to a bounded bilinear operator

$$T : (A_0 \cap A_1) \times (B_0 + B_1)^\circ \longrightarrow (E_0 + E_1)^\circ.$$

Interpolation properties of bounded bilinear operators under the real method have been investigated by Lions and Peetre [26], Karadzhov [21], Janson [20] and many other authors. We shall need here the following result which shows a convexity inequality for the norm of the interpolated operator.

Theorem 3.1. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples, let $\bar{E} = (E_0, E_1)$ be a p-Banach couple ($0 < p \le 1$). Assume that $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$ and let M_j be the constants in (3.1). Let $0 < \theta < 1$ and $0 < r_1, r_2, r \le \infty$ with*

$$\frac{1}{r} = \begin{cases} \frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{p} & \text{if} \quad r_1, r_2 \ge p, \\ \frac{1}{\max(r_1, r_2)} & \text{if} \quad r_1 < p \text{ or } r_2 < p. \end{cases}$$

Then there is a constant $C > 0$ independent of T such that

$$\|T(a,b)\|_{\bar{E}_{\theta,r}} \le C\, M_0^{1-\theta} M_1^{\theta}\, \|a\|_{\bar{A}_{\theta,r_1}} \|b\|_{\bar{B}_{\theta,r_2}}, \quad a \in A_0 \cap A_1,\ b \in B_0 \cap B_1.$$

Consequently, T may be uniquely extended to a bounded bilinear operator

$$T : (A_0, A_1)^\circ_{\theta,r_1} \times (B_0, B_1)^\circ_{\theta,r_2} \longrightarrow (E_0, E_1)^\circ_{\theta,r}.$$

Proof. Take any $a \in A_0 \cap A_1$, $b \in B_0 \cap B_1$. Given any decomposition $a = a_0 + a_1$ with $a_j \in A_j$, we have that $a_j \in A_0 \cap A_1$ for $j = 0, 1$. Let $n \in \mathbb{Z}$ such that $2^n \le M_1/M_0 < 2^{n+1}$. For $m \in \mathbb{Z}$, we have

$$\begin{aligned} K(2^m, T(a,b)) &\le \|T(a_0,b)\|_{E_0} + 2^m\|T(a_1,b)\|_{E_1} \\ &\le M_0\|a_0\|_{A_0}\|b\|_{B_0} + 2^m M_1\|a_1\|_{A_1}\|b\|_{B_1} \\ &\le \max\{M_0, 2^{-n}M_1\}(\|a_0\|_{A_0} + 2^{m-k}\|a_1\|_{A_1})J(2^{k+n}, b). \end{aligned}$$

Thus,

$$K(2^m, T(a,b)) \le 2M_0 K(2^{m-k}, a)J(2^{k+n}, b).$$

According to [20, Lemma 1], there is a J-representation $b = \sum_{k=-\infty}^{\infty} u_k$ with only a finite number of terms u_k different from 0 and such that $\|(2^{-\theta k}J(2^k, u_k))\|_{\ell_{r_2}} \le c_1\|b\|_{\bar{B}_{\theta,r_2}}$. Since $b = \sum_{k=-\infty}^{\infty} u_{k+n}$, using (2.1) we obtain

$$\begin{aligned} K(2^m, T(a,b)) &\le \Big(\sum_{k=-\infty}^{\infty} K_p(2^m, T(a, u_{k+n}))^p\Big)^{1/p} \\ &\le 2^{(1/p)-1}\Big(\sum_{k=-\infty}^{\infty} K(2^m, T(a, u_{k+n}))^p\Big)^{1/p} \\ &\le 2^{1/p} M_0\Big(\sum_{k=-\infty}^{\infty} K(2^{m-k}, a)^p J(2^{k+n}, u_{k+n})^p\Big)^{1/p}. \end{aligned}$$

Now we distinguish the different cases. If $r_1, r_2 \geq p$, using Young's inequality with parameters $p/r_1 + p/r_2 = 1 + p/r$, we derive

$$\begin{aligned}
\|T(a,b)\|_{\bar{E}_{\theta,r}} &\leq 2^{1/p} M_0 \Big\| \Big(\sum_{k=-\infty}^{\infty} 2^{-\theta(m-k)p} K(2^{m-k}, a)^p 2^{-\theta k p} J(2^{k+n}, u_{k+n})^p \Big) \Big\|_{\ell_{r/p}}^{1/p} \\
&\leq 2^{1/p} M_0 \big\| (2^{-\theta m} K(2^m, a)) \big\|_{\ell_{r_1}} \big\| (2^{-\theta k} J(2^{k+n}, u_{k+n})) \big\|_{\ell_{r_2}} \\
&\leq 2^{1/p} M_0 2^{\theta n} \|a\|_{\bar{A}_{\theta,r_1}} \big\| (2^{-\theta k} J(2^k, u_k)) \big\|_{\ell_{r_2}} \\
&\leq C M_0^{1-\theta} M_1^{\theta} \|a\|_{\bar{A}_{\theta,r_1}} \|b\|_{\bar{B}_{\theta,r_2}}.
\end{aligned}$$

If $r_2 \leq r_1 < p$, then $r = r_1$. Applying Young's inequality now with parameters $r_1/p, 1, r_1/p$ we obtain

$$\begin{aligned}
\|T(a,b)\|_{\bar{E}_{\theta,r}} &\leq 2^{1/p} M_0 \Big\| \Big(\sum_{k=-\infty}^{\infty} 2^{-\theta(m-k)p} K(2^{m-k}, a)^p 2^{-\theta k p} J(2^{k+n}, u_{k+n})^p \Big) \Big\|_{\ell_{r_1/p}}^{1/p} \\
&\leq 2^{1/p} M_0 \big\| (2^{-\theta m} K(2^m, a)) \big\|_{\ell_{r_1}} \big\| (2^{-\theta k p} J(2^{k+n}, u_{k+n})^p) \big\|_{\ell_1}^{1/p} \\
&\leq 2^{1/p} M_0 2^{\theta n} \|a\|_{\bar{A}_{\theta,r_1}} \big\| (2^{-\theta k} J(2^k, u_k)) \big\|_{\ell_p} \\
&\leq 2^{1/p} M_0^{1-\theta} M_1^{\theta} \|a\|_{\bar{A}_{\theta,r_1}} \big\| (2^{-\theta k} J(2^k, u_k)) \big\|_{\ell_{r_2}} \\
&\leq C M_0^{1-\theta} M_1^{\theta} \|a\|_{\bar{A}_{\theta,r_1}} \|b\|_{\bar{B}_{\theta,r_2}}.
\end{aligned}$$

The remaining cases can be treated similarly. □

Next we introduce the set Ω following [20]. Recall that $\bar{A}_{j,q} = A_j$ for $j = 0, 1$ and any $0 < q \leq \infty$.

Definition 3.2. Given $\bar{A}$, $\bar{B}$, $\bar{E}$ quasi-Banach couples, given a bilinear operator T and α_0, α_1, $\alpha_2 \in \mathbb{R}$ with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, the set $\Omega = \Omega(\bar{A}, \bar{B}, \bar{E}, T, \alpha_0, \alpha_1, \alpha_2)$ is formed by all couples $\bar{\theta} = (\theta_1, \theta_2) \in [0,1]^2$ satisfying that

$$\text{the number } \theta := \alpha_0 + \alpha_1\theta_1 + \alpha_2\theta_2 \text{ belongs to } [0,1] \tag{3.3}$$

and there is $\bar{r} = (r_1, r_2, r) \in (0, \infty]^3$ and $M > 0$ such that

$$\|T(a,b)\|_{\bar{E}_{\theta,r}} \leq M \|a\|_{\bar{A}_{\theta_1,r_1}} \|b\|_{\bar{B}_{\theta_2,r_2}} \quad \text{for any } a \in A_0 \cap A_1,\ b \in B_0 \cap B_1.$$

It turns out that Ω is a convex set (see [20, Theorem 1]) and if $\bar{\theta} = (\theta_1, \theta_2)$ belongs to the interior of Ω then for any parameters $0 < q, q_1, q_2 \leq \infty$ with $1/q_1 + 1/q_2 = 1/q$, the operator T may be uniquely extended to a bounded bilinear operator

$$T : (A_0, A_1)^{\circ}_{\theta_1,q_1} \times (B_0, B_1)^{\circ}_{\theta_2,q_2} \longrightarrow (E_0, E_1)^{\circ}_{\theta,q}$$

(see [20, Theorem 2]).

The arguments of the proof of [10, Theorem 3.6] yield the following result which will be important in our later considerations.

Lemma 3.3. *Let $\bar{A}, \bar{B}, \bar{E}$ be quasi-Banach couples, let Q_n be a sequence of bilinear operators, let $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \neq 0, \alpha_2 \neq 0$, and let Ω_0 be a convex set in the plane such that $\Omega(\bar{A}, \bar{B}, \bar{E}, Q_n, \alpha_0, \alpha_1, \alpha_2) = \Omega_0$ for any $n \in \mathbb{N}$. Assume that there is a* *triple* $\bar{r} = (r_1, r_2, r) \in (0, \infty]^3$ *such that for any* $\bar{\eta} = (\eta_1, \eta_2)$ *in the interior of* Ω_0 *we have* $\lim_{n\to\infty} \|Q_n\|_{\bar{A}^\circ_{\eta_1,r_1}\times\bar{B}^\circ_{\eta_2,r_2},\bar{E}^\circ_{\eta,r}} = 0$, *where* $\eta = \alpha_0 + \alpha_1\eta_1 + \alpha_2\eta_2$.

If $\bar{\theta} = (\theta_1, \theta_2)$ *belongs to the interior of* Ω_0 *and* $\theta = \alpha_0 + \alpha_1\theta_1 + \alpha_2\theta_2$, *then for any* $0 < q_1, q_2, q \le \infty$ *with* $1/q = 1/q_1 + 1/q_2$, *we have that*

$$\lim_{n\to\infty} \|Q_n\|_{\bar{A}^\circ_{\theta_1,q_1}\times\bar{B}^\circ_{\theta_2,q_2},\bar{E}^\circ_{\theta,q}} = 0.$$

In the next section we deal with *operators of convolution type*, that is, operators T satisfying that the following restrictions are bounded

$$\begin{cases} T : A_0^\circ \times B_0^\circ \longrightarrow E_0^\circ, \\ T : A_1^\circ \times B_0^\circ \longrightarrow E_1^\circ, \\ T : A_0^\circ \times B_1^\circ \longrightarrow E_1^\circ, \end{cases}$$

(see [29, Definition 1.1], [2, p. 258] or [18, p. 1200] for the case $\bar{A} = \bar{B} = \bar{E} = (L_1, L_\infty)$). Choose $\alpha_0 = 0, \alpha_1 = \alpha_2 = 1$ and let $\Delta = \Omega(\bar{A}, \bar{B}, \bar{E}, T, 0, 1, 1)$. By the assumption on T, we know that $(0, 0), (1, 0), (0, 1) \in \Delta$. Moreover, if $(\theta_1, \theta_2) \in \Delta$ then $0 \le \theta_1, \theta_2 \le 1$ and $0 \le \theta = \theta_1 + \theta_2 \le 1$ because of (3.3). Hence,

$$\Delta \text{ is the triangle with vertices } (0, 0), (1, 0), (0, 1). \tag{3.4}$$

Closely connected with convolution operators are *product type operators* which are also of interest for us. These are those bilinear operators satisfying that the following restrictions are bounded

$$\begin{cases} T : A_1^\circ \times B_1^\circ \longrightarrow E_1^\circ, \\ T : A_0^\circ \times B_1^\circ \longrightarrow E_0^\circ, \\ T : A_1^\circ \times B_0^\circ \longrightarrow E_0^\circ, \end{cases}$$

(see [29, Definition 3.2] or [32, p. 192] for the case $\bar{A} = \bar{B} = \bar{E} = (L_1, L_\infty)$). Choose $\alpha_0 = -1, \alpha_1 = \alpha_2 = 1$ and let $\Gamma = \Omega(\bar{A}, \bar{B}, \bar{E}, T, -1, 1, 1)$. This time we have that $(1, 1), (0, 1), (1, 0) \in \Gamma$ and if $(\theta_1, \theta_2) \in \Gamma$ then $0 \le \theta_1, \theta_2 \le 1$ and $0 \le \theta = -1 + \theta_1 + \theta_2 \le 1$. Therefore,

$$\Gamma \text{ is the triangle in the plane with vertices } (1, 1), (0, 1), (1, 0).$$

4. Interpolation of compact bilinear operators

We start with two auxiliary results for the case when the couple in the target reduces to a single quasi-Banach space.

Lemma 4.1. *Let* $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ *be quasi-Banach couples, let* E *be a quasi-Banach space and let* $T \in \mathcal{B}(\bar{A} \times \bar{B}, (E, E))$ *with*

$$T : A_0^\circ \times B_0^\circ \longrightarrow E \quad \textit{compactly}.$$

Let $0 < \theta < 1$ *and* $0 < q \le \infty$. *Then* T *may be uniquely extended to a compact bilinear operators*

$$T : (A_0, A_1)^\circ_{\theta,q} \times (B_0 \cap B_1) \longrightarrow E$$

and

$$T : (A_0 \cap A_1) \times (B_0, B_1)^\circ_{\theta,q} \longrightarrow E.$$

Proof. Using (3.2) and that $(A_0, A_1)_{\theta,q} \hookrightarrow A_0 + A_1$, we have that T may be uniquely extended to a bounded bilinear operator $T : (A_0, A_1)^\circ_{\theta,q} \times (B_0 \cap B_1) \longrightarrow E$. Next we show that it is compact. Let $V \subseteq A_0 \cap A_1$ be any bounded set in $(A_0, A_1)_{\theta,q}$ and let Y be any bounded set in $B_0 \cap B_1$. Then there is $c_1 > 0$ such that $\|b\|_{B_0 \cap B_1} \le c_1$ for any $b \in Y$. Since $(A_0, A_1)_{\theta,q} \hookrightarrow (A_0, A_1)_{\theta,\infty}$, there exists also a $c_2 > 0$ such that any $a \in V$ can be decomposed as

$$a = a_0 + a_1 \text{ with } a_j \in A_0 \cap A_1 \text{ and } \|a_0\|_{A_0} + t\|a_1\|_{A_1} \le c_2 t^\theta. \tag{4.1}$$

We are going to check that $T(V, Y)$ is precompact in E. Given any $\varepsilon > 0$, choose $t_0 > 0$ such that $c_2 t_0^{\theta-1} < \varepsilon/(2c_1 c_E \|T\|_{A_1^\circ \times B_1^\circ, E})$. Let $K \subseteq (A_0 \cap A_1) \times (B_0 \cap B_1)$ be the set of all pairs (a_0, b) where $b \in Y$ and a_0 runs over all vectors of $A_0 \cap A_1$ which have appeared in the decomposition (4.1) of the elements $a \in V$ with the value t_0. Since K is bounded in $A_0^\circ \times B_0^\circ$, compactness of $T : A_0^\circ \times B_0^\circ \longrightarrow E$ yields that there is a finite set $\{z_1, \dots, z_r\} \subseteq E$ such that for any $(a_0, b) \in K$ there is $1 \le j \le r$ with $\|T(a_0, b) - z_j\|_E \le \varepsilon/2c_E$. For this z_j, we obtain

$$\begin{aligned} \|T(a, b) - z_j\|_E &\le c_E(\|T(a_0, b) - z_j\|_E + \|T(a_1, b)\|_E) \\ &\le c_E\Big(\frac{\varepsilon}{2c_E} + \|T\|_{A_1^\circ \times B_1^\circ, E} \|a_1\|_{A_1} \|b\|_{B_1}\Big) \\ &\le \frac{\varepsilon}{2} + c_E\, c_2\, t_0^{\theta-1} c_1 \|T\|_{A_1^\circ \times B_1^\circ, E} \le \varepsilon. \end{aligned}$$

Consequently, $\{z_1, \dots, z_r\}$ is an ε-net for $T(V, Y)$ in E.

The operator $T : (A_0 \cap A_1) \times (B_0, B_1)^\circ_{\theta,q} \longrightarrow E$ can be treated similarly. □

Lemma 4.2. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples, let E be a quasi-Banach space and let T be a bilinear operator satisfying that*

$$T : A_0^\circ \times B_0^\circ \longrightarrow E \text{ compactly,} \tag{4.2}$$

$$T : A_1^\circ \times B_0^\circ \longrightarrow E \text{ boundedly,} \tag{4.3}$$

$$T : A_0^\circ \times B_1^\circ \longrightarrow E \text{ boundedly.} \tag{4.4}$$

Let $0 < \theta_1, \theta_2 < 1$ with $\theta_1 + \theta_2 < 1$ and let $0 < q_1, q_2 \le \infty$. Then T may be uniquely extended to a compact bilinear operator

$$T : (A_0, A_1)^\circ_{\theta_1,q_1} \times (B_0, B_1)^\circ_{\theta_2,q_2} \longrightarrow E.$$

Proof. Let $\bar{E} = (E, E)$. As we have seen in (3.4), if $\alpha_0 = 0$ and $\alpha_1 = \alpha_2 = 1$, the set $\Omega = \Omega(\bar{A}, \bar{B}, \bar{E}, T, 0, 1, 1)$ coincides with the triangle Δ with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Since (θ_1, θ_2) belongs to the interior of Δ, it follows from Janson's theorem [20, Theorem 2], that T may be uniquely extended to a bounded operator $T : (A_0, A_1)^\circ_{\theta_1,q_1} \times (B_0, B_1)^\circ_{\theta_2,q_2} \longrightarrow E$. Next we show that the extension is compact.

Applying Lemma 4.1 to restrictions (4.2) and (4.4), we have that

$$T : A_0^\circ \times (B_0, B_1)^\circ_{\theta_2,q_2} \longrightarrow E \text{ compactly.} \tag{4.5}$$

Let $0 < p \le 1$ such that A_j are p-Banach for $j = 0, 1$. Let $F_m = (A_0^\circ \cap A_1^\circ, J(2^m, \cdot))$, $\bar{F}_p = (\ell_p(F_m), \ell_p(2^{-m} F_m))$ and consider the surjective operator π and the projections P_n, P_n^+, P_n^-

satisfying (2.5). Put $\widetilde{T}((u_m), b) = T(\pi(u_m), b)$. By (2.4), in order to establish the result it suffices to prove that

$$\widetilde{T} : \ell_{q_1}(2^{-\theta_1 m} F_m)^\circ \times (B_0, B_1)^\circ_{\theta_2, q_2} \longrightarrow E \text{ is compact.} \tag{4.6}$$

With this aim, we split $\widetilde{T}$ with the help of the projections. We have

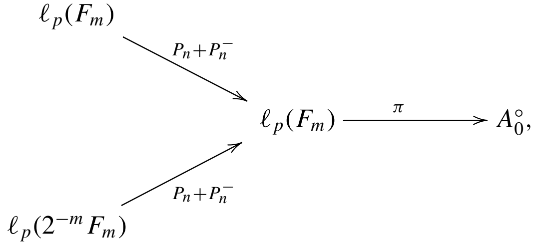
$$\widetilde{T} = \widetilde{T}(P_n + P_n^- + P_n^+, I) = \widetilde{T}(P_n + P_n^-, I) + \widetilde{T}(P_n^+, I).$$

We are going to show that $\widetilde{T}(P_n + P_n^-, I)$ is compact for any $n \in \mathbb{N}$ and that

$$\|\widetilde{T}(P_n^+, I)\|_{\ell_{q_1}(2^{-\theta m} F_m)^\circ \times \bar{B}^\circ_{\theta_2, q_2}, E} \to 0 \text{ as } n \to \infty.$$

These two facts yield (4.6).

Using the diagram



together with the formula (2.6) and the interpolation property for bounded linear operators, we have that

$$\pi(P_n + P_n^-) : \ell_{q_1}(2^{-\theta_1 m} F_m)^\circ = (\ell_p(F_m), \ell_p(2^{-m} F_m))^\circ_{\theta_1, q_1} \longrightarrow A_0^\circ \text{ boundedly.}$$

Then, by (4.5), we obtain that the composition

$$\widetilde{T}(P_n + P_n^-, I) : \ell_{q_1}(2^{-\theta_1 m} F_m)^\circ \times (B_0, B_1)^\circ_{\theta_2, q_2} \longrightarrow E$$

is compact for any $n \in \mathbb{N}$.

As for $\widetilde{T}(P_n^+, I)$, notice that

$$\widetilde{T}(P_n^+, I) : \ell_p(F_m) \times B_0^\circ \longrightarrow E \text{ compactly,}$$

$$\widetilde{T}(P_n^+, I) : \ell_p(2^{-m} F_m) \times B_0^\circ \longrightarrow E \text{ boundedly,}$$

$$\widetilde{T}(P_n^+, I) : \ell_p(F_m) \times B_1^\circ \longrightarrow E \text{ boundedly.} \tag{4.7}$$

Hence $\Omega(\bar{F}_p, \bar{B}, \bar{E}, \widetilde{T}(P_n^+, I), 0, 1, 1) = \Delta$ for any $n \in \mathbb{N}$. According to Lemma 3.3, to show that $\|\widetilde{T}(P_n^+, I)\|_{\ell_{q_1}(2^{-\theta_1 m} F_m)^\circ \times \bar{B}^\circ_{\theta_2, q_2}, E} \to 0$ as $n \to \infty$, it suffices to check that for the triple $(2p, 2p, \infty)$ and for any $0 < \eta_1, \eta_2 < 1$ with $\eta = \eta_1 + \eta_2 < 1$ we have that

$$\|\widetilde{T}(P_n^+, I)\|_{\ell_{2p}(2^{-\eta_1 m} F_m) \times \bar{B}^\circ_{\eta_2, 2p}, E} \to 0 \text{ as } n \to \infty. \tag{4.8}$$

With this aim, put $\delta = \eta_1/(1 - \eta_2)$. We know by (2.5) that the restrictions $P_n^+ : \ell_p(F_m) \longrightarrow \ell_p(2^{-m} F_m)$ and $P_n^+ : \ell_p(2^{-m} F_m) \longrightarrow \ell_p(2^{-m} F_m)$ are bounded with

$$\|P_n^+\|_{\ell_p(F_m), \ell_p(2^{-m} F_m)} \leq 2^{-(n+1)} \quad , \quad \|P_n^+\|_{\ell_p(2^{-m} F_m), \ell_p(2^{-m} F_m)} \leq 1.$$

The interpolation property (2.2) and formula (2.6) yield that $P_n^+ : \ell_p(2^{-\delta m} F_m) \longrightarrow \ell_p(2^{-m} F_m)$ is bounded with $\|P_n^+\|_{\ell_p(2^{-\delta m} F_m), \ell_p(2^{-m} F_m)} \le C\, 2^{-(1-\delta)(n+1)}$. Combining this with (4.3) we get that

$$\widetilde{T}(P_n^+, I) : \ell_p(2^{-\delta m} F_m) \times B_0^\circ \longrightarrow E \tag{4.9}$$

is bounded with $\|\widetilde{T}(P_n^+, I)\|_{\ell_p(2^{-\delta m} F_m) \times B_0^\circ, E} \to 0$ as $n \to \infty$. Besides, we know that restriction (4.7) is bounded and that, by (2.6), $(\ell_p(2^{-\delta m} F_m), \ell_p(F_m))_{\eta_2, 2p} = \ell_{2p}(2^{-\eta_1 m} F_m)$. Therefore, using the restrictions (4.9),(4.7) and applying the bilinear interpolation Theorem 3.1 with parameter η_2 and the triple $(2p, 2p, \infty)$, we obtain that

$$\widetilde{T}(P_n^+, I) : \ell_{2p}(2^{-\eta_1 m} F_m) \times (B_0, B_1)_{\eta_2, 2p}^\circ \longrightarrow E$$

is bounded with

$$\|\widetilde{T}(P_n^+, I)\|_{\ell_{2p}(2^{-\eta_1 m} F_m) \times \bar{B}_{\eta_2, 2p}^\circ, E} \to 0 \text{ as } n \to \infty.$$

This establishes (4.8) and completes the proof. □

Next we prove the main result of the paper: the compactness theorem for convolution type operators without any assumption on the quasi-Banach couples. For the proof we proceed as in Lemma 4.2 but now the situation is more involved because we have a couple in the target of the operator and so a more refined splitting of the operator is needed.

Theorem 4.3. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be quasi-Banach couples and let T be a bilinear operator satisfying that*

$$\begin{cases} T : A_0^\circ \times B_0^\circ \longrightarrow E_0^\circ \text{ compactly,} \\ T : A_1^\circ \times B_0^\circ \longrightarrow E_1^\circ \text{ boundedly,} \\ T : A_0^\circ \times B_1^\circ \longrightarrow E_1^\circ \text{ boundedly.} \end{cases}$$

Let $0 < \theta_1, \theta_2 < 1$ with $\theta = \theta_1 + \theta_2 < 1$ and let $0 < q_1, q_2, q \le \infty$ with $1/q = 1/q_1 + 1/q_2$. Then T may be uniquely extended to a compact bilinear operator

$$T : (A_0, A_1)_{\theta_1, q_1}^\circ \times (B_0, B_1)_{\theta_2, q_2}^\circ \longrightarrow (E_0, E_1)_{\theta, q}^\circ.$$

Proof. We know by Janson's theorem [20, Theorem 2] that T may be uniquely extended to a bounded operator

$$T : (A_0, A_1)_{\theta_1, q_1}^\circ \times (B_0, B_1)_{\theta_2, q_2}^\circ \longrightarrow (E_0, E_1)_{\theta, q}^\circ. \tag{4.10}$$

In order to show that it is compact, we pick $0 < p \le 1$ such that A_j, B_j, E_j are p-Banach spaces for $j = 0, 1$, and we realize $(A_0, A_1)_{\theta_1, q_1}$ and $(B_0, B_1)_{\theta_2, q_2}$ by means of the J-functional. We put $F_m = (A_0^\circ \cap A_1^\circ, J(2^m, \cdot; A_0^\circ, A_1^\circ))$, $\bar{F}_p = (\ell_p(F_m), \ell_p(2^{-m} F_m))$ and we consider the operators π, P_n, P_n^+, P_n^- (see (2.4) and (2.5)). Let $G_m = (B_0^\circ \cap B_1^\circ, J(2^m, \cdot; B_0^\circ, B_1^\circ))$, $\bar{G}_p = (\ell_p(G_m), \ell_p(2^{-m} G_m))$ and let S_n, S_n^+, S_n^- be the corresponding sequences of projections on the couple $\bar{G}_p$. They satisfy analogous properties to (2.5). As for $(E_0, E_1)_{\theta, q}$, we realize it by means of the K-functional. Let $W_m = (E_0^\circ + E_1^\circ, K(2^m, \cdot; E_0^\circ, E_1^\circ))$, $\bar{W}_\infty = (\ell_\infty(W_m), \ell_\infty(2^{-m} W_m))$, let τ be the operator in (2.7) and let R_n, R_n^+, R_n^- be the projections on the couple $\bar{W}_\infty$. The following diagram holds

$$\ell_{q_1}(2^{-\theta_1 m}F_m)^\circ \times \ell_{q_2}(2^{-\theta_2 m}G_m)^\circ \xrightarrow{(\pi,\pi)} \bar{A}^\circ_{\theta_1,q_1} \times \bar{B}^\circ_{\theta_2,q_2} \xrightarrow{T} \bar{E}^\circ_{\theta,q} \xrightarrow{\tau} \ell_q(2^{-\theta m}W_m)^\circ \tag{4.11}$$

Put $\hat{T} = \tau T(\pi,\pi)$. Since τ is a metric injection and π a metric surjection, it turns out that T in (4.10) is compact if and only if $\hat{T}$ in (4.11) is compact.

To establish compactness of $\hat{T}$, notice that by the assumption on T and the properties of π and τ we have that

$$\hat{T} : \ell_p(F_m) \times \ell_p(G_m) \longrightarrow \ell_\infty(W_m)^\circ \text{ compactly,} \tag{4.12}$$

$$\hat{T} : \ell_p(2^{-m}F_m) \times \ell_p(G_m) \longrightarrow \ell_\infty(2^{-m}W_m)^\circ \text{ boundedly,} \tag{4.13}$$

$$\hat{T} : \ell_p(F_m) \times \ell_p(2^{-m}G_m) \longrightarrow \ell_\infty(2^{-m}W_m)^\circ \text{ boundedly.} \tag{4.14}$$

Therefore $\Omega(\bar{F}_p, \bar{G}_p, \bar{W}_\infty, \hat{T}, 0, 1, 1)$ coincides with the triangle Δ with vertices $(0,0), (1,0)$ and $(0,1)$. Next we split the operator $\hat{T}$ as follows

$$\begin{aligned}
\hat{T} &= R_n\hat{T} + R_n^+\hat{T} + R_n^-\hat{T} \\
&= R_n\hat{T} + R_n^+\hat{T}(P_n + P_n^+ + P_n^-, S_n + S_n^+ + S_n^-) + R_n^-\hat{T} \\
&= R_n\hat{T} + R_n^+\hat{T}(P_n + P_n^-, S_n + S_n^-) \\
&\quad + R_n^-\hat{T} + R_n^+\hat{T}(P_n^-, S_n^+) + R_n^+\hat{T}(P_n, S_n^+) \\
&\quad + R_n^+\hat{T}(P_n^+, S_n^+) + R_n^+\hat{T}(P_n^+, S_n^-) + R_n^+\hat{T}(P_n^+, S_n).
\end{aligned}$$

We remark that if Q_n is any of the operators of this decomposition, then we have

$$\Omega(\bar{F}_p, \bar{G}_p, \bar{W}_\infty, Q_n, 0, 1, 1) = \Delta \text{ for any } n \in \mathbb{N}.$$

We claim that the first two sequences of bilinear operators are compact. Indeed, since

$$R_n : \ell_\infty(W_m) + \ell_\infty(2^{-m}W_m) \longrightarrow \ell_\infty(W_m) \cap \ell_\infty(2^{-m}W_m) \hookrightarrow \ell_q(2^{-\theta m}W_m)^\circ$$

is bounded, it follows from (4.12), (4.13), (4.14) that

$$\begin{cases}
R_n\hat{T} : \ell_p(F_m) \times \ell_p(G_m) \longrightarrow \ell_q(2^{-\theta m}W_m)^\circ \text{ compactly,} \\
R_n\hat{T} : \ell_p(2^{-m}F_m) \times \ell_p(G_m) \longrightarrow \ell_q(2^{-\theta m}W_m)^\circ \text{ boundedly,} \\
R_n\hat{T} : \ell_p(F_m) \times \ell_p(2^{-m}G_m) \longrightarrow \ell_q(2^{-\theta m}W_m)^\circ \text{ boundedly.}
\end{cases}$$

Moreover, by (2.6), $(\ell_p(F_m), \ell_p(2^{-m}F_m))_{\theta_1,q_1} = \ell_{q_1}(2^{-\theta_1 m}F_m)$ and $\ell_{q_2}(2^{-\theta_2 m}G_m) = (\ell_p(G_m), \ell_p(2^{-m}G_m))_{\theta_2,q_2}$. Therefore, Lemma 4.2 yields that

$$R_n\hat{T} : \ell_{q_1}(2^{-\theta_1 m}F_m)^\circ \times \ell_{q_2}(2^{-\theta_2 m}G_m)^\circ \longrightarrow \ell_q(2^{-\theta m}W_m)^\circ$$

is compact for each $n \in \mathbb{N}$.

Now we consider $R_n^+\hat{T}(P_n + P_n^-, S_n + S_n^-)$. Since restrictions

$$P_n^- : \ell_p(F_m) \longrightarrow \ell_p(F_m)\ ,\ \ P_n^- : \ell_p(2^{-m}F_m) \longrightarrow \ell_p(F_m)$$

are bounded, the interpolation property gives that

$$P_n^- : \ell_{q_1}(2^{-\theta_1 m}F_m)^\circ = (\ell_p(F_m), \ell_p(2^{-m}F_m))^\circ_{\theta_1,q_1} \longrightarrow \ell_p(F_m)$$

is bounded. Moreover, $P_n : \ell_{q_1}(2^{-\theta_1 m} F_m)^\circ \longrightarrow \ell_p(F_m)$ is also bounded. Hence, we get that

$$P_n + P_n^- : \ell_{q_1}(2^{-\theta_1 m} F_m)^\circ \longrightarrow \ell_p(F_m) \text{ is bounded.}$$

Similarly, one can check that

$$S_n + S_n^- : \ell_{q_2}(2^{-\theta_2 m} G_m)^\circ \longrightarrow \ell_p(G_m) \text{ boundedly,}$$

and that

$$R_n^+ : \ell_\infty(W_m)^\circ \longrightarrow \ell_q(2^{-\theta m} W_m)^\circ \text{ boundedly.}$$

Consequently, compactness of (4.12) yields that the composition

$$R_n^+ \hat{T}(P_n + P_n^-, S_n + S_n^-) : \ell_{q_1}(2^{-\theta_1 m} F_m)^\circ \times \ell_{q_2}(2^{-\theta_2 m} G_m)^\circ \longrightarrow \ell_q(2^{-\theta m} W_m)^\circ$$

is compact for each $n \in \mathbb{N}$.

Next we are going to show that the norms of the other sequences of operators in the splitting of $\hat{T}$ tend to 0 as $n \to \infty$. This will imply that $\hat{T}$ in (4.11) is the limit of a sequence of compact bilinear operators and therefore it is compact.

We start with $(R_n^- \hat{T})$. We claim that

$$\|R_n^- \hat{T}\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)^\circ} \to 0 \text{ as } n \to \infty. \tag{4.15}$$

Indeed, we can pick sequences $(x_n) \subseteq U_{\ell_p(G_m)}$, $(y_n) \subseteq U_{\ell_p(F_m)}$ where each x_n and y_n have only a finite number of coordinates different from 0 and such that

$$\|R_n^- \hat{T}\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)^\circ} \le \frac{1}{n} + \|R_n^- \hat{T}(x_n, y_n)\|_{\ell_\infty(W_m)^\circ}.$$

Compactness of $\hat{T}$ in (4.12) implies that there is $w \in \ell_\infty(W_m)^\circ$ and a subsequence (n_1) such that $\hat{T}(x_{n_1}, y_{n_1}) \to w$ in $\ell_\infty(W_m)^\circ$. Given any $\varepsilon > 0$, let $\delta = \varepsilon c_{\bar{E}}^{-2}$. Since $\hat{T}(x_{n_1}, y_{n_1}) \in \ell_\infty(W_m) \cap \ell_\infty(2^{-m} W_m)$, we can find $k_1 \in \mathbb{N}$ such that $z = \hat{T}(x_{k_1}, y_{k_1}) \in \ell_\infty(W_m) \cap \ell_\infty(2^{-m} W_m)$ satisfies that $\|w - z\|_{\ell_\infty(W_m)^\circ} < \delta/4$. Let $N \in \mathbb{N}$ such that for any $n_1 \ge N$ we have

$$\frac{1}{n_1} \le \frac{\delta}{4},\ 2^{-n_1}\|z\|_{\ell_\infty(2^{-m} W_m)^\circ} \le \frac{\delta}{4} \text{ and } \|\hat{T}(x_{n_1}, y_{n_1}) - w\|_{\ell_\infty(W_m)^\circ} \le \frac{\delta}{4}.$$

Then, if $n_1 \ge N$, we obtain

$$\begin{aligned}
&\|R_{n_1}^- \hat{T}\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)^\circ} \le \frac{1}{n_1} + \|R_{n_1}^- \hat{T}(x_{n_1}, y_{n_1})\|_{\ell_\infty(W_m)^\circ} \\
&\le \frac{\delta}{4} + c_{\bar{E}}(\|R_{n_1}^- \hat{T}(x_{n_1}, y_{n_1}) - R_{n_1}^- w\|_{\ell_\infty(W_m)^\circ} + \|R_{n_1}^- w\|_{\ell_\infty(W_m)^\circ}) \\
&\le \frac{\delta}{4} + c_{\bar{E}}(\frac{\delta}{4} + c_{\bar{E}}(\|R_{n_1}^- w - R_{n_1}^- z\|_{\ell_\infty(W_m)^\circ} + \|R_{n_1}^- z\|_{\ell_\infty(W_m)^\circ})) \\
&\le \frac{\delta}{4} + c_{\bar{E}}(\frac{\delta}{4} + c_{\bar{E}}(\frac{\delta}{4} + 2^{-n_1}\|z\|_{\ell_\infty(2^{-m} W_m)^\circ})) \\
&\le \frac{\delta}{4} + c_{\bar{E}}(\frac{\delta}{4} + c_{\bar{E}}\frac{\delta}{2}) \le \varepsilon.
\end{aligned}$$

Since $\|R_1^- \hat{T}\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)^\circ} \ge \|R_2^- \hat{T}\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)^\circ} \ge \cdots$, we conclude that (4.15) holds.

Now, to derive that

$$\|R_n^- \hat{T}\|_{\ell_{q_1}(2^{-\theta_1 m} F_m)^\circ \times \ell_{q_2}(2^{-\theta_2 m} G_m)^\circ, \ell_q(2^{-\theta m} W_m)^\circ} \to 0 \text{ as } n \to \infty, \tag{4.16}$$

we base the arguments on Lemma 3.3 and the norm estimate of Theorem 3.1. Consider the triple (p, p, p) and take any $0 < \eta_1, \eta_2 < 1$ con $\eta = \eta_1 + \eta_2 < 1$. Let $\mu = \eta_2/\eta < 1$. By (2.6), we have that $(\ell_p(2^{-m}F_m), \ell_p(F_m))_{\mu,2p} = \ell_{2p}(2^{-(1-\mu)m}F_m)$ and $(\ell_p(G_m), \ell_p(2^{-m}G_m))_{\mu,2p} = \ell_{2p}(2^{-\mu m}G_m)$. Hence, applying Theorem 3.1 with the restrictions (4.13), (4.14) and parameters μ and $(2p, 2p, \infty)$, we obtain that

$$R_n^- \hat{T} : \ell_{2p}(2^{-(1-\mu)m}F_m) \times \ell_{2p}(2^{-\mu m}G_m) \longrightarrow \ell_\infty(2^{-m}W_m)^\circ \text{ boundedly.} \tag{4.17}$$

On the other hand, we have that $(\ell_\infty(W_m)^\circ, \ell_\infty(2^{-m}W_m)^\circ)_{\eta,p} = \ell_p(2^{-\eta m}W_m)$, $(\ell_p(F_m), \ell_{2p}(2^{-(1-\mu)m}F_m))_{\eta,p} = \ell_p(2^{-\eta_1 m}F_m)$ and $(\ell_p(G_m), \ell_{2p}(2^{-\mu m}G_m))_{\eta,p} = \ell_p(2^{-\eta_2 m}G_m)$. So, applying Theorem 3.1 with parameters η and (p, p, p), it follows from (4.15) and (4.17), that

$$R_n^- \hat{T} : \ell_p(2^{-\eta_1 m}F_m) \times \ell_p(2^{-\eta_2 m}G_m) \longrightarrow \ell_p(2^{-\eta m}W_m)$$

is bounded with norm tending to 0 as $n \to \infty$. Then Lemma 3.3 yields (4.16).

Next we proceed with $R_n^+ \hat{T}(P_n^-, S_n^+)$ following the same strategy. We have

$$\|R_n^+ \hat{T}(P_n^-, S_n^+)\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)^\circ} \le \|\hat{T}(P_n^-, S_n^+)\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)^\circ}.$$

We claim that

$$\|\hat{T}(P_n^-, S_n^+)\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)^\circ} \to 0 \text{ as } n \to \infty. \tag{4.18}$$

Indeed, if this would not be the case, there would exist $\delta > 0$ and a subsequence (n_1) such that

$$\delta < \|\hat{T}(P_{n_1}^-, S_{n_1}^+)\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)^\circ} \text{ for all } n_1.$$

Then, there are sequences $(x_{n_1}) \subseteq U_{\ell_p(F_m)}, (y_{n_1}) \subseteq U_{\ell_p(G_m)}$ such that $\delta < \|\hat{T}(P_{n_1}^- x_{n_1}, S_{n_1}^+ y_{n_1})\|_{\ell_\infty(W_m)^\circ}$. Since $(P_{n_1}^- x_{n_1}) \subseteq U_{\ell_p(F_m)}$, $(S_{n_1}^+ y_{n_1}) \subseteq U_{\ell_p(G_m)}$, compactness of (4.12) yields, passing to another subsequence if necessary, that there is $w \in \ell_\infty(W_m)^\circ$ such that $\hat{T}(P_{n_2}^- x_{n_2}, S_{n_2}^+ y_{n_2}) \to w$ in $\ell_\infty(W_m)^\circ$. So $\delta \le \|w\|_{\ell_\infty(W_m)^\circ}$ and therefore $w \neq 0$. But

$$\|S_{n_2}^+ y_{n_2}\|_{\ell_p(2^{-m}G_m)} \le 2^{-n_2} \to 0 \text{ as } n_2 \to \infty.$$

Using boundedness of (4.14), we derive that

$$\hat{T}(P_{n_2}^- x_{n_2}, S_{n_2}^+ y_{n_2}) \to 0 \text{ in } \ell_\infty(2^{-m}W_m)^\circ.$$

By compatibility we get that $w = 0$, contradicting that $w \neq 0$. This establishes (4.18).

Take again the triple (p, p, p) and take any $0 < \eta_1, \eta_2 < 1$ with $\eta = \eta_1 + \eta_2 < 1$. Using twice Theorem 3.1 as we did in the previous case, we derive that the norm of

$$R_n^+ \hat{T}(P_n^-, S_n^+) : \ell_p(2^{-\eta_1 m}F_m) \times \ell_p(2^{-\eta_2 m}G_m) \longrightarrow \ell_p(2^{-\eta m}W_m)$$

tends to 0 as $n \to \infty$. Then, Lemma 3.3 gives that the norm of

$$R_n^+ \hat{T}(P_n^-, S_n^+) : \ell_{q_1}(2^{-\theta_1 m}F_m)^\circ \times \ell_{q_2}(2^{-\theta_2 m}G_m)^\circ \longrightarrow \ell_q(2^{-\theta m}W_m)^\circ$$

tends to 0 as $n \to \infty$.

The sequences $(R_n^+ \hat{T}(P_n, S_n^+))$ and $(R_n^+ \hat{T}(P_n^+, S_n^+))$ can be treated as $(R_n^+ \hat{T}(P_n^-, S_n^+))$. Concerning $(R_n^+ \hat{T}(P_n^+, S_n^-))$ and $(R_n^+ \hat{T}(P_n^+, S_n))$, we can also proceed similarly but using now that $\|P_n^+\|_{\ell_p(F_m),\ell_p(2^{-m}F_m)} \le 2^{-(n+1)}$ and (4.13).

In conclusion,

$$\hat{T} : \ell_{q_1}(2^{-\theta_1 m}F_m)^\circ \times \ell_{q_2}(2^{-\theta_2 m}G_m)^\circ \longrightarrow \ell_q(2^{-\theta m}W_m)^\circ$$

is the limit of a sequence of compact operators and therefore it is compact. □

The corresponding result for operators of product type reads as follows.

Theorem 4.4. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be quasi-Banach couples and let T be a bilinear operator satisfying that*

$$\begin{cases} T : A_1^\circ \times B_1^\circ \longrightarrow E_1^\circ \text{ compactly,} \\ T : A_0^\circ \times B_1^\circ \longrightarrow E_0^\circ \text{ boundedly,} \\ T : A_1^\circ \times B_0^\circ \longrightarrow E_0^\circ \text{ boundedly.} \end{cases}$$

Let $0 < \theta_1, \theta_2 < 1$ with $0 < \theta = \theta_1 + \theta_2 - 1 < 1$ and let $0 < q_1, q_2, q \leq \infty$ with $1/q = 1/q_1 + 1/q_2$. Then T may be uniquely extended to a compact bilinear operator

$$T : (A_0, A_1)^\circ_{\theta_1,q_1} \times (B_0, B_1)^\circ_{\theta_2,q_2} \longrightarrow (E_0, E_1)^\circ_{\theta,q}.$$

Proof. Let $X_0 = A_1$, $X_1 = A_0$, $Y_0 = B_1$, $Y_1 = B_0$, $Z_0 = E_1$, $Z_1 = E_0$, $\sigma_1 = 1-\theta_1$, $\sigma_2 = 1-\theta_2$ and $\sigma = \sigma_1 + \sigma_2 = 2 - \theta_1 - \theta_2 = 1 - \theta$. With respect to the couples $\bar{X} = (X_0, X_1)$, $\bar{Y} = (Y_0, Y_1)$, $\bar{Z} = (Z_0, Z_1)$ the operator T satisfies the assumptions of Theorem 4.3. Therefore T may be uniquely extended to a compact bilinear operator

$$T : (X_0, X_1)^\circ_{\sigma_1,q_1} \times (Y_0, Y_1)^\circ_{\sigma_2,q_2} \longrightarrow (Z_0, Z_1)^\circ_{\sigma,q}.$$

Since

$$(X_0, X_1)^\circ_{\sigma_1,q_1} = (A_1, A_0)^\circ_{\sigma_1,q_1} = (A_0, A_1)^\circ_{1-\sigma_1,q_1} = (A_0, A_1)^\circ_{\theta_1,q_1}$$

and similar equalities hold for the other couples, we conclude that T may be uniquely extended to a compact bilinear operator

$$T : (A_0, A_1)^\circ_{\theta_1,q_1} \times (B_0, B_1)^\circ_{\theta_2,q_2} \longrightarrow (E_0, E_1)^\circ_{\theta,q}. \quad \square$$

We finish the paper with a compactness result in the line of the bilinear Marcinkiewicz interpolation theorem for multiplication type operators of Gilbert and Nahmod [19, Theorem 6.4]. We work with the Lorentz spaces $L_{p,q}(A) = L_{p,q}(\mathbb{R}^n; A)$ introduced in Section 2, with $\Omega = \mathbb{R}^n$ and μ being the Lebesgue measure. Interpolation formula (2.3) is important for the result.

Theorem 4.5. *Let A be a quasi-Banach space and let T be a bilinear operator which is defined on some space of functions $\mathcal{F}(\mathbb{R}^n; A)$ which is dense in every $L_p(A) = L_p(\mathbb{R}^n; A)$, $0 < p < \infty$.*

Let $1 < p_0 < p_1 < \infty$, $1/r_0 = 1/p_0 + 1/p_1$ and assume that T extends to a compact bilinear operator

$$T : L_{p_1}(A) \times L_{p_1}(A) \longrightarrow L_{p_1/2,\infty}(A)$$

and it also extends to bounded bilinear operators

$$\begin{cases} T : L_{p_0}(A) \times L_{p_1}(A) \longrightarrow L_{r_0,\infty}(A), \\ T : L_{p_1}(A) \times L_{p_0}(A) \longrightarrow L_{r_0,\infty}(A). \end{cases}$$

Then T extends to a bilinear operator

$$T : L_p(A) \times L_q(A) \longrightarrow L_r(A) \text{ with } 1/r = 1/p + 1/q$$

which is compact for all

$$p_0 < p, q < p_1 \text{ with } 1/p + 1/q < 1/r_0.$$

Proof. Consider the quasi-Banach couples $\bar{A} = \bar{B} = (L_{p_0}(A), L_{p_1}(A))$ and $\bar{E} = (L_{r_0,\infty}(A), L_{p_1/2,\infty}(A))$. Applying Theorem 4.4, we have that for any $0 < \theta_1, \theta_2 < 1$ with $0 < \theta = \theta_1 + \theta_2 - 1 < 1$, the operator T may be uniquely extended to a compact bilinear operator

$$T : (L_{p_0}(A), L_{p_1}(A))_{\theta_1,p} \times (L_{p_0}(A), L_{p_1}(A))_{\theta_2,q} \longrightarrow (L_{r_0,\infty}(A), L_{p_1/2,\infty}(A))_{\theta,r} . \quad (4.19)$$

Since $p_0 < p, q < p_1$, we can pick $0 < \theta_1, \theta_2 < 1$ such that $1/p = (1-\theta_1)/p_0 + \theta_1/p_1$ and $1/q = (1-\theta_2)/p_0 + \theta_2/p_1$. Let $\theta = \theta_1 + \theta_2 - 1$. Then $\theta < 1$. We also have that $\theta > 0$ because substituting in the inequality

$$1/r = 1/p + 1/q < 1/r_0 = 1/p_0 + 1/p_1$$

we get

$$\frac{2-\theta_1-\theta_2}{p_0} + \frac{\theta_1+\theta_2}{p_1} < \frac{1}{p_0} + \frac{1}{p_1}.$$

Hence $(\theta_1 + \theta_2 - 1)(1/p_0 - 1/p_1) > 0$ which implies that $\theta > 0$. Since

$$\begin{aligned}\frac{1-\theta}{r_0} + \frac{\theta}{\frac{p_1}{2}} &= \frac{2-\theta_1-\theta_2}{p_0} + \frac{2-\theta_1-\theta_2}{p_1} + \frac{2(\theta_1+\theta_2-1)}{p_1} \\ &= \frac{2-\theta_1-\theta_2}{p_0} + \frac{\theta_1+\theta_2}{p_1} = \frac{1}{p} + \frac{1}{q} = \frac{1}{r},\end{aligned}$$

it follows from (2.3) and (4.19) that

$$T : L_p(A) \times L_q(A) \longrightarrow L_r(A) \text{ compactly.} \quad \square$$

Note that in Theorem 4.5 the parameters r_0 and r might be less than 1.

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