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Theory of the Rarita-Schwinger field without superluminality

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It is shown that the noncausality of the Rarita-Schwinger equation in an external electromagnetic field can be avoided by the substitution of a subsidiary constraint by a subsidiary differential equation. As the number of degrees of freedom increases, the theory represents two kinds of particles with spin 1/2 and spin 3/2. Some consequences are studied.

I. INTRODUCTION

During the last few years the theory of invariant wave equations has received the attention of many physicists and mathematicians. The main reason is probably the acute difficulties for higher-spin equations which appear under the form of inconsistencies (e.g., noncausality, loss of constraints,...), which have not been solved so far.

It is usually believed that these inconsistencies are inherent to any theory and that a consistent theory cannot be built for higher-spin particles. Wightman has even suggested that this impossibility could be proved explicitly.[1] In the present paper we try a new approach to the noncausality problem which arises for the spin-$\frac{3}{2}$ field.

In order to describe spin-$\frac{3}{2}$ particles, we adopt the Rarita-Schwinger (RS) formalism[2] by setting the following equations for the free-field case:

$$(\gamma\cdot P-m)\psi_\mu=0\,, \tag{1}$$

$$\gamma^\mu\psi_\mu=0\,, \tag{2}$$

where ψ_μ is the RS vector-spinor and $P_\mu=i\partial_\mu$.

The interaction with an external electromagnetic potential cannot be accounted for by the minimal-coupling principle because it produces algebraic inconsistencies which have been known for a very long time. To avoid this difficulty Fierz and Pauli[3] proposed a method in which both the equations of motion and the constraints are derived from a convenient Lagrangian, whose free solutions correspond to free particles of unique mass and spin. The interaction is then introduced by the minimal-coupling substitution. The algebraic inconsistencies are avoided, but a more subtle type of problem appears as was shown by Velo and Zwanziger (VZ).[4] When an external electromagnetic potential is turned on, the solutions of the equations propagate faster than light, which would imply a noncausal behavior. In a previous paper[5] it was proved that the necessary and sufficient condition for causal propagation is the invariance of the hyperbolicity conditions.

In Sec. II we will show that noncausality for the RS field arises when it is imposed as a subsidiary condition on $\gamma\cdot\psi$ (an equation which does not contain time derivatives and which permits the elimination of some components in terms of others). This constraint, together with a primary constraint, due to the fact that the original RS equation is singular, indicates that the number of independent components is adequate to describe spin-$\frac{3}{2}$ particles, namely 8.

As no causal theory of spin-$\frac{3}{2}$ particles in interaction with an external electromagnetic potential seems to be possible, we will try instead to construct a causal theory where the RS field represents not only spin-$\frac{3}{2}$ particles but also spin-$\frac{1}{2}$ particles, which interact with the external electromagnetic potential. This is done in Sec. III.

The study of the classical field equation that we propose leads to a theory of particles with two different spins, which should be interpreted by the quantum field theory.

Finally, in Sec. IV we state the conclusions.

II. RARITA-SCHWINGER EQUATION FOR SPIN-$\frac{3}{2}$ PARTICLES

The RS Lagrangian densities must satisfy the following properties.[6]

(a) Relativistic and gauge invariance.

(b) If we want to describe spin-$\frac{3}{2}$ particles, when the external field vanishes they should imply (1) and (2).

(c) They should imply as many subsidiary conditions as necessary to describe particles with definite mass and spin.

The most general Lagrangian density satisfying (a) is

$$\mathcal{L}=\bar\psi(\Gamma\cdot\pi-B)\psi\,, \tag{3}$$

where $\pi_\mu=i\partial_\mu-eA_\mu$, and A_μ is a given classical four-vector potential. The matrices Γ^μ and B are given by

$$(\Gamma\cdot\pi)_\mu{}^\lambda=\gamma\cdot\pi g_\mu^\lambda-\lambda_1(\gamma_\mu\pi^\lambda+\gamma^\lambda\pi_\mu)+\lambda_2\gamma_\mu\gamma\cdot\pi\gamma^\lambda\,, \tag{4}$$

$$B_\mu{}^\lambda=m(g_\mu^\lambda-\lambda_3\gamma_\mu\gamma^\lambda)\,, \tag{5}$$

where λ_1, λ_2, and λ_3 are undetermined parameters.

The variation of the Lagrangian with respect to the 16 components of ψ and $\bar{\psi}$ independently implies the field equations

$$(\Gamma\cdot\pi - m)_\mu{}^\lambda \psi_\lambda = 0\,, \tag{6}$$

$$\bar{\psi}^\mu(\Gamma\cdot\pi - m)_\mu{}^\lambda = 0\,. \tag{7}$$

As we have too many independent components, we need to impose some constraints. This can be done by requiring that every surface be characteristic. This implies

$$Q(n) = |(\Gamma^\kappa)_\mu{}^\lambda n_\kappa| = 0 \;\forall\, n_\kappa\,. \tag{8}$$

From this condition we get a relation between λ_1 and λ_2

$$\lambda_2 = \tfrac{3}{2}\lambda_1{}^2 - \lambda_1 + \tfrac{1}{2}\,. \tag{9}$$

Before obtaining the primary constraint, we multiply equation (6) successively by γ^μ and π^μ which yields, respectively,

$$2(1-2\lambda_1)\pi\cdot\psi + (4\lambda_2 - \lambda_1 - 1)\gamma\cdot\pi\gamma\cdot\psi + (4\lambda_3 - 1)m\gamma\cdot\psi = 0\,, \tag{10}$$

$$\left[(\lambda_2-\lambda_1)\pi^2 + \lambda_3 m\gamma\cdot\pi - \lambda_2\frac{e}{2}F\cdot\sigma\right]\gamma\cdot\psi + [(1-\lambda_1)\gamma\cdot\pi - m]\pi\cdot\psi + ie\gamma\cdot F\cdot\psi = 0\,. \tag{11}$$

A primary constraint is obtained from Eq. (6) for $\mu = 0$ and Eq. (10), making use of (9):

$$(\vec{\pi} - h\vec{\alpha})\cdot\vec{\psi} + \left[(1-\lambda_1)\vec{\gamma}\cdot\vec{\pi} + \frac{1-3\lambda_1+2\lambda_3}{2(2\lambda_1-1)}\,m\right]\gamma\cdot\psi = 0\,, \tag{12}$$

where $h = \vec{\alpha}\cdot\vec{\pi} + \beta m$ and $\lambda_1 \neq \frac{1}{2}$.

From Eqs. (10) and (11), we deduce the following equation:

$$\left[a\gamma\cdot\pi - (4\lambda_3 - 1)m - \lambda_1(2\lambda_1 - 1)\frac{e}{m}\sigma\cdot F\right]\gamma\cdot\psi + 2(2\lambda_1 - 1)\frac{ie}{m}\gamma\cdot F\cdot\psi = 0\,, \tag{13}$$

where $a = -6\lambda_1{}^2 + 6\lambda_1 + 2\lambda_3 - 2$.

This equation becomes a secondary constraint independent of the primary constraint (12) when $a = 0$. In this case conditions (b) and (c) are verified for spin-$\frac{3}{2}$ particles.

In order to determine under which conditions a solution to the RS equation exists and to find the velocity of propagation of signals, we substitute $\gamma\cdot\psi$ and $\pi\cdot\psi$, as obtained from (11) and (13):

$$\gamma\cdot\psi = \frac{2(\lambda_1-1)}{3(1-2\lambda_1)}\frac{ie}{m^2}\gamma\cdot F\cdot\psi + \frac{2\lambda_1}{3(2\lambda_1-1)}\frac{e}{m^2}\gamma^5\gamma\cdot\tilde{F}\cdot\psi\,, \tag{14}$$

$$\pi\cdot\psi = \frac{2e/m^2}{2\lambda_1-1}\left[\tfrac{1}{6}(3\lambda_1-1)\gamma\cdot\pi - (\lambda_1 - \tfrac{1}{2})m\right] \times [(1-\lambda_1)i\gamma\cdot F\cdot\psi + \lambda_1\gamma^5\gamma\cdot\tilde{F}\cdot\psi] \tag{15}$$

into the original RS equation (6). The resulting equation is

$$(\gamma\cdot\pi - m)\psi_\mu + \frac{2e/3m^2}{2\lambda_1-1}\left(\lambda_1\pi_\mu + \frac{\lambda_1-1}{2}\gamma_\mu\gamma\cdot\pi + \frac{3\lambda_1-2}{2}m\gamma_\mu\right) \times [i(\lambda_1-1)\gamma\cdot F\cdot\psi - \lambda_1\gamma^5\gamma\cdot\tilde{F}\cdot\psi] = 0\,. \tag{16}$$

We have, thus, a one-parameter family of equations which can be reduced to that considered by Velo and Zwanziger[4] (corresponding to the value $\lambda_1 = 1$) by transforming the Lagrangian density (3) as follows:

$$\psi_\mu \to \psi'_\mu = \psi_\mu + \frac{1-\lambda_1}{2(2\lambda_1-1)}\gamma_\mu\gamma\cdot\psi\,. \tag{17}$$

This linear transformation does not modify the propagation properties of the field which are studied by the method of the characteristic surfaces. The solution of the characteristic equation provides us with the normals n_μ to the characteristic surfaces in every point of the space-time:

$$Q(n) = (n^2)^6\left[n^2 + \left(\frac{2e}{3m^2}n\cdot\tilde{F}\right)^2\right]^2 = 0\,. \tag{18}$$

In the weak-field case $[((2e/3m^2)\vec{B})^2 < 1]$ these equations are hyperbolic and noncausal. The solutions propagate faster than light, as Velo and Zwanziger showed.

III. RARITA-SCHWINGER EQUATION FOR SPIN $\frac{3}{2}$ AND SPIN $\frac{1}{2}$

So far we have tried to find an equation which describes particles with definite spin and mass. This seems not to be possible, because the resulting equations are noncausal. We stress that the crucial point of the derivation of Eq. (16) is to impose Eq. (13) to be a secondary constraint rather than an equation for $(\gamma\cdot\psi)$. The latter occurs when $a \neq 0$. In this case there are 12 independent components, rather than 8 as in the spin-$\frac{3}{2}$ case. The four new independent components are $(\gamma\cdot\psi)$ which satisfy the equation

$$\gamma\cdot\pi(\gamma\cdot\psi)=\frac{4\lambda_3-1}{a}m(\gamma\cdot\psi)+\frac{2(2\lambda_1-1)}{a}\frac{e}{m}[(\lambda_1-1)i\gamma\cdot F\cdot\psi-\lambda_1\gamma^5\gamma\cdot\tilde{F}\cdot\psi]. \tag{19}$$

From Eqs. (10) and (19) we obtain

$$\pi\cdot\psi=\frac{(4\lambda_3-1)(3\lambda_1+2\lambda_3-1)}{2a(2\lambda_1-1)}m\gamma\cdot\psi+\frac{3\lambda_1-6\lambda_1^2-1}{a}\frac{e}{m}[(1-\lambda_1)i\gamma\cdot F\cdot\psi+\lambda_1\gamma^5\gamma\cdot\tilde{F}\cdot\psi]. \tag{20}$$

We substitute both Eqs. (19) and (20) for $\gamma\cdot\pi(\gamma\cdot\psi)$ and $\pi\cdot\psi$ in the original RS equation, as was done for the case $a=0$. The resulting equation is

$$(\gamma\cdot\pi-m)\psi_\mu-\lambda_1\pi_\mu(\gamma\cdot\psi)+mA\gamma_\mu\gamma\cdot\psi+B\frac{e}{m}\gamma_\mu[(\lambda_1-1)i\gamma\cdot F\cdot\psi-\lambda_1\gamma^5\gamma\cdot\tilde{F}\cdot\psi]=0\,, \tag{21}$$

where

$$A=\frac{-6\lambda_1^3-4\lambda_1^2\lambda_3+10\lambda_1^2+2\lambda_1\lambda_3-4\lambda_3^2-5\lambda_1+1}{2a(2\lambda_1-1)}\,,$$

$$B=\frac{-4\lambda_1^2+3\lambda_1-1}{a}\,.$$

Equation (21) constitutes a hyperbolic system of partial differential equations if $\lambda_1\neq 1$ and $\frac{1}{2}$ for arbitrary external fields. From the calculation of the characteristic polynomial we discover that the equation is causal:

$$Q(n)=(1-\lambda_1)^2(n^2)^8\,, \tag{22}$$

so that the interaction does not modify the hyperbolicity of Eq. (21).

We have given up noncausality by considering the new set of equations (19) and (21), but the problem of its interpretation arises. Before proceeding to this interpretation, we shall put the field equations in a simpler form by means of a transformation of the RS field. The associated currents to Eqs. (19) and (21) are, respectively,

$$\partial_\mu(\bar{\psi}\cdot\gamma\gamma^\mu\gamma\cdot\psi)=-\frac{2(2\lambda_1-1)}{a}\frac{e}{m}(\bar{\psi}\cdot\gamma\gamma\cdot F\cdot\psi-\bar{\psi}\cdot F\cdot\gamma\gamma\cdot\psi)\,, \tag{23}$$

$$\partial_\mu(\bar{\psi}\gamma^\mu\psi-\lambda_1\bar{\psi}^\mu\gamma\cdot\psi-\lambda_1\bar{\psi}\cdot\gamma\psi^\mu)=\frac{2\lambda_2(2\lambda_1-1)}{a}\frac{e}{m}(\bar{\psi}\cdot\gamma\gamma\cdot F\cdot\psi-\bar{\psi}\cdot F\cdot\gamma\gamma\cdot\psi)\,. \tag{24}$$

In the absence of an electromagnetic field these currents are independently conserved. If we impose the condition $\gamma\cdot\psi=0$ for $t=t_0$, then $\gamma\cdot\psi=0$ for any t. In this case Eq. (21) becomes the RS equation in the free-field case. When the electromagnetic external field is turned on, $\gamma\cdot\psi$ ceases in general to be zero; its value depends on the external field $F_{\mu\nu}$ through Eq. (19) which is coupled to Eq. (21). We have obtained for $\gamma\cdot\psi$ not a subsidiary constraint but a subsidiary equation.

From Eqs. (23) and (24) we deduce a total conserved current

$$J^\mu=\bar{\psi}\gamma^\mu\psi-\lambda_1(\bar{\psi}^\mu\gamma\cdot\psi+\bar{\psi}\cdot\gamma\psi^\mu)+\lambda_2\bar{\psi}\cdot\gamma\gamma^\mu\gamma\cdot\psi\,. \tag{25}$$

If we transform the RS field ψ as follows,

$$\psi_\mu=\psi'_\mu+\frac{\lambda_1}{2(1-2\lambda_1)}\gamma_\mu\gamma\cdot\psi'\,, \tag{26}$$

we eliminate the mixed terms $\bar{\psi}^\mu\gamma\cdot\psi$ and $\bar{\psi}\cdot\gamma\psi^\mu$. The total conserved current is now

$$J^\mu=\bar{\psi}'\gamma^\mu\psi'+\tfrac{1}{2}\bar{\psi}'\cdot\gamma\gamma^\mu\gamma\cdot\psi'\,. \tag{27}$$

Note that the coupling constants λ_1 do not appear explicitly in these expressions.

Equations (19) and (21) [note that (20) can be deduced from them] for the new field ψ' adopt a simple form (we shall write in the following ψ)

$$(\gamma\cdot\pi-m)\psi_\mu-m'\gamma_\mu\gamma\cdot\psi-iq\gamma_\mu\gamma\cdot F\cdot\psi=0\,, \tag{28}$$

$$(\gamma\cdot\pi-m_{1/2})\gamma\cdot\psi+2iq\gamma\cdot F\cdot\psi=0\,, \tag{29}$$

where

$$m_{1/2}=\frac{4\lambda_3-1}{a}m\,,\quad m'=\frac{a^2-(4\lambda_3-1)^2}{6a(2\lambda_1-1)^2}m\,,$$

$$q=-\frac{(2\lambda_1-1)^2}{a}\frac{e}{m}\,.$$

The constants appearing in Eqs. (28) and (29) are not independent. They verify the following relations:

$$m^2-m_{1/2}^2+2m'(2m-m_{1/2})=0\,, \tag{30}$$

$$q=\frac{e}{2(m+\frac{1}{2}m_{1/2}+m')}\,, \tag{31}$$

which can be deduced by applying successively γ^μ and π^μ to Eqs. (28) and (29). These relations can be proved also by their substitution in terms of λ_1 and λ_3.

The main consequence we deduce from (30) and (31) is that the set of Eqs. (28) and (29) depends only on two parameters m and $m_{1/2}$. On the other hand, the subsidiary equation (29) added to Eq. (28) implies the equation

$$\pi\cdot\psi=\frac{m_{1/2}+m+4m'}{2}\gamma\cdot\psi+iq\gamma\cdot F\cdot\psi \tag{32}$$

and using the component $\mu=0$ of (28), we deduce the constraint

$$h\psi^0 - \vec{\pi}\cdot\vec{\psi} = M\gamma\cdot\psi\,, \tag{33}$$

where $h = \vec{\alpha}\cdot\pi + \beta m$ and $M = \frac{1}{2}(m_{1/2} + m + 2m')$. This constraint is, in fact, a consequence of the singular hyperbolic character of the original RS equation, from which we have deduced Eqs. (28) and (29).

There is no secondary constraint, as in the preceding case $a=0$. The RS field ψ_μ which satisfies Eqs. (28) and (29) has 12 degrees of freedom.

It is clear that Eq. (28) is hyperbolic because the principal part (the derivative-dependent part) is not affected by the coupling to the external field, in the absence of which it is already hyperbolic. The maximum velocity at which the solutions propagate is the velocity of light. It is shown in the Appendix that constraint (33) is conserved in the evolution given by Eq. (28).

In the free-field case Eqs. (28) and (29) can be easily solved by using the Fourier transform method because of their linearity.

The mass spectrum of Eq. (28), which in the absence of an external field is

$$(\gamma\cdot P - m)\psi_\mu - m'\gamma_\mu\gamma\cdot\psi = 0\,, \tag{34}$$

can be obtained from the determinant

$$D(P) = \det(\gamma\cdot P g_{\mu\nu} - m'\gamma_\mu\gamma_\nu)\,.$$

In our case,

$$D(P) = (P^2 - m^2)^4 (P^2 - m_{1/2}{}^2)^2 \times [P^2 - (m_{1/2} + 2m')^2]^2\,. \tag{35}$$

The mass spectrum is $(m, m_{1/2}, m_{1/2} + 2m')$.

Equation (29), which is now

$$(\gamma\cdot P - m_{1/2})(\gamma\cdot\psi) = 0\,, \tag{36}$$

implies that $\gamma\cdot\psi = 0$ off the mass shell $\Omega(m_{1/2})$. This eliminates in Eq. (34) the mass shell $\Omega(m_{1/2} + 2m')$. Consequently, the mass spectrum is only $(m, m_{1/2})$.

The general solution of (34) and (35) is

$$\psi_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int_{\Omega(m)} d\sigma_{3/2}(P)\hat{\psi}_\mu(P) e^{-ip\cdot x} + \frac{1}{(2\pi)^{3/2}} \int_{\Omega(m_{1/2})} d\sigma_{1/2}(P)\hat{\psi}_\mu(P) e^{-ipx}\,, \tag{37}$$

where $d\sigma(P) = d^3P/2|P^0|$ is the Lorentz invariant volume element in $\Omega(m)$ and $\Omega(m), \Omega(m_{1/2})$ are the mass hyperboloids ($p^2 = m^2$, $p^2 = m_{1/2}{}^2$). The RS field can be written as a linear superposition of two fields $\psi_\mu^{(3/2)}(x)$ and $\psi_\mu^{(1/2)}(x)$ corresponding to expansion (37). The former $\psi^{(3/2)}$ verifies the usual RS equation:

$$(\gamma\cdot P - m)\psi_\mu^{(3/2)} = 0\,, \qquad \gamma\cdot\psi^{(3/2)} = 0\,, \tag{38}$$

which permits us to interpret $\psi^{(3/2)}$ as a field with spin $\frac{3}{2}$ and mass m.

The field $\psi^{(1/2)}$ has four independent components which can be expressed as functions of the spinor $\gamma\cdot\psi$:

$$\psi_\mu^{(1/2)} = \frac{1}{2m - m_{1/2}}\left(i\partial_\mu + \frac{m - m_{1/2}}{2}\gamma_\mu\right)(\gamma\cdot\psi)\,. \tag{39}$$

This field has spin $\frac{1}{2}$ and mass $m_{1/2}$.

The decomposition of any solution of Eqs. (34) and (36) in its spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ parts implies, also, a decomposition of the physical magnitudes such as energy-momentum, angular momentum, and charge of the field. To show this, let us return to the Lagrangian density from which we deduce Eqs. (28) and (29):

$$\begin{aligned}\mathcal{L} = {} & \frac{i}{2}[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi}\psi - e\bar{\psi}\gamma^\mu\psi A_\mu \\ & + \frac{i}{4}[\bar{\psi}\cdot\gamma\gamma^\mu\partial_\mu(\gamma\cdot\psi) - \partial_\mu(\bar{\psi}\cdot\gamma)\gamma^\mu\gamma\cdot\psi] \\ & - \left(m' + \frac{m_{1/2}}{2}\right)\bar{\psi}\cdot\gamma\gamma\cdot\psi - \frac{e}{2}\bar{\psi}\cdot\gamma\gamma^\mu\gamma\cdot\psi A_\mu\,.\end{aligned} \tag{40}$$

Standard methods can be used to achieve the expressions for the dynamical variables.

For the energy-momentum tensor

$$\theta_{\mu\nu}^{\rm sym} = \theta_{\mu\nu}^{\rm can} + \partial^\lambda f_{\lambda\mu\nu}\,, \tag{41}$$

where

$$\theta_{\mu\nu}^{\rm can} = \frac{i}{2}\bar{\psi}\gamma_\mu\partial_\nu\psi + \frac{i}{4}(\bar{\psi}\cdot\gamma)\gamma_\mu\partial_\nu(\gamma\cdot\psi) + \text{conj} - g_{\mu\nu}\mathcal{L}\,, \tag{42}$$

$$f_{\lambda\mu\nu} = \tfrac{1}{2}(R_{\lambda\mu\nu} + R_{\mu\nu\lambda} + R_{\nu\mu\lambda})\,, \tag{43}$$

where $R_{\mu\nu\lambda}$ is defined by (45).

For the angular momentum density tensor

$$M_{\mu\nu\lambda} = L_{\mu\nu\lambda} + R_{\mu\nu\lambda}\,, \tag{44}$$

where $L_{\mu\nu\lambda}$ and $R_{\mu\nu\lambda}$ are the orbital and spin angular momentum density tensors,

$$R_{\lambda\mu\nu} = \frac{i}{2}(\bar{\psi}_\mu\gamma_\lambda\psi_\nu - \bar{\psi}_\nu\gamma_\lambda\psi_\mu) + \tfrac{1}{4}\bar{\psi}\gamma_\lambda\sigma_{\mu\nu}\psi + \tfrac{1}{8}\bar{\psi}\cdot\gamma\gamma_\lambda\sigma_{\mu\nu}\gamma\cdot\psi + \text{conj}\,. \tag{45}$$

Finally, the vector current, given by

$$J_\mu = \bar{\psi}\gamma_\mu\psi + \tfrac{1}{2}\bar{\gamma}\cdot\gamma\gamma_\mu\gamma\cdot\psi\,, \tag{46}$$

is conserved.

In the free-field case the decomposition $\psi_\mu = \psi_\mu^{(3/2)} + \psi_\mu^{(1/2)}$ permits us to write the canonical energy-momentum tensor in the form

$$\theta_{\mu\nu}^{\rm can} = \theta_{\mu\nu}^{(3/2)} + \theta_{\mu\nu}^{(1/2)} + \theta_{\mu\nu}^{(\rm interf)}\,, \tag{47}$$

where

$$\theta_{\mu\nu}^{(3/2)}=\frac{i}{2}\bar{\psi}^{(3/2)}\gamma_\mu\partial_\nu\psi^{(3/2)}+\text{conj}\,, \tag{48}$$

$$\theta_{\mu\nu}^{(1/2)}=\frac{i(\partial_\lambda\partial^\lambda+3m^2)}{4(2m-m_{1/2})^2}(\bar{\psi}\cdot\gamma)\gamma_\mu\partial_\nu(\gamma\cdot\psi)+\frac{i(m-m_{1/2})}{2(2m-m_{1/2})^2}\partial^\lambda(\bar{\psi}\cdot\gamma\sigma_{\lambda\mu}\partial_\nu\gamma\cdot\psi)+\text{conj}\,, \tag{49}$$

$$\theta_{\mu\nu}^{(\text{interf})}=\frac{i}{2(2m-m_{1/2})}\{i\partial^\lambda[\bar{\psi}_\lambda^{(3/2)}\gamma_\mu\partial_\nu(\gamma\cdot\psi)-(\bar{\psi}\cdot\gamma)\gamma_\mu\partial_\nu\psi_\lambda^{(3/2)}]+(m-m_{1/2})[\bar{\psi}_\mu^{(3/2)}\partial_\nu(\gamma\cdot\psi)+(\bar{\psi}\cdot\gamma)\partial_\nu\psi_\mu^{(3/2)}]\}+\text{conj}\,. \tag{50}$$

The last term does not contribute to the total energy-momentum of the field, by virtue of the Gaussian theorem, for fields vanishing rapidly enough at infinity,

$$P_\mu=\int_\Sigma d\sigma^\nu\theta_{\nu\mu}^{\text{can}}=P_\mu^{(3/2)}+P_\mu^{(1/2)}\,. \tag{51}$$

If we take a surface $t=$ constant,

$$P_\mu^{(1/2)}=\frac{3m^2}{2(2m-m_{1/2})^2}\int d^3x(\gamma\cdot\psi)^\dagger i\partial_\mu(\gamma\cdot\psi)\,, \tag{52}$$

$E^{(1/2)}$ is positive for positive frequency solutions $\psi^{(1/2)}$ and $E^{(3/2)}$ is negative for positive frequency solutions $\psi^{(3/2)}$.

The intrinsic spin of the fields follows from (45). For the field $\psi_\mu^{(3/2)}$ we obtain

$$\vec{S}_{3/2}=-\int d^3x\,\vec{\psi}^{(3/2)\dagger}\vec{S}^{(3/2)}\vec{\psi}^{(3/2)}\,, \tag{53}$$

where $\vec{S}^{(3/2)}$ are the usual spin-$\frac{3}{2}$ matrices defined by

$$(S_k^{(3/2)})_{ij}=-i\epsilon_{ijk}+\tfrac{1}{2}\delta_{ij}\Sigma_k \tag{54}$$

and

$$\Sigma_k=\tfrac{1}{2}\epsilon_{ijk}\sigma_{ij}\,.$$

$\vec{S}_{3/2}$ appears with the opposite sign as expected.

For stationary solutions of the field $\psi^{(1/2)}$ we obtain

$$R_{0ij}=\frac{3m^2}{2(2m-m_{1/2})^2}\epsilon_{ijk}(\gamma\cdot\psi)^\dagger\tfrac{1}{2}\Sigma_k(\gamma\cdot\psi)\,. \tag{55}$$

Similar to the decomposition (47) of the energy-momentum tensor, we obtain for the current

$$J_\mu=J_\mu^{(3/2)}+J_\mu^{(1/2)}+J_\mu^{(\text{interf})}\,, \tag{56}$$

where

$$J_\mu^{(3/2)}=e\bar{\psi}^{(3/2)}\gamma_\mu\psi^{(3/2)}\,, \tag{57}$$

$$J_\mu^{(1/2)}=e\frac{\partial_\lambda\partial^\lambda+3m^2}{2(2m-m_{1/2})^2}(\bar{\psi}\cdot\gamma)\gamma_\mu(\gamma\cdot\psi)+e\frac{m-m_{1/2}}{(2m-m_{1/2})^2}\partial^\lambda(\bar{\psi}\cdot\gamma\sigma_{\lambda\mu}\gamma\cdot\psi)\,, \tag{58}$$

$$J_\mu^{(\text{interf})}=\frac{e}{2m-m_{1/2}}\times[i\partial^\lambda(\bar{\psi}_\lambda^{(3/2)}\gamma_\mu\gamma\cdot\psi-\bar{\psi}\cdot\gamma\gamma_\mu\psi_\lambda^{(3/2)})+(m-m_{1/2})(\bar{\psi}_\mu^{(3/2)}\gamma\cdot\psi+\bar{\psi}\cdot\gamma\psi_\mu^{(3/2)})]\,. \tag{59}$$

The interference between spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ solutions is reflected in the last term $J^{(\text{int})}$, which does not contribute to the total charge

$$Q=\int d\sigma^\mu J_\mu=Q^{(3/2)}+Q^{(1/2)}\,, \tag{60}$$

where

$$Q^{(3/2)}=-e\int d^3x[\vec{\psi}^{(3/2)\dagger}\vec{\psi}^{(3/2)}-(\vec{\alpha}\cdot\vec{\psi}^{(3/2)})^\dagger(\vec{\alpha}\cdot\vec{\psi}^{(3/2)})]\,, \tag{61}$$

$$Q^{(1/2)}=\frac{3m^2e}{2(2m-m_{1/2})^2}\int d^3x(\gamma\cdot\psi)^\dagger(\gamma\cdot\psi)\,. \tag{62}$$

Both fields have opposite charges ($Q^{(3/2)}/e\leq 0$ and $Q^{(1/2)}/e\geq 0$).

A positive-definite total charge Q would be incompatible with the field Eqs. (28) and (29) due to the fact that the charges $Q^{(3/2)}$ and $Q^{(1/2)}$ are not independently conserved in the presence of an external field $F_{\mu\nu}$. From Eq. (29), we deduce the spin-$\frac{1}{2}$ charge created during the whole process interaction,

$$Q_{\text{out}}^{(1/2)}-Q_{\text{in}}^{(1/2)}=-2qeZ\int d^4x(\bar{\psi}\cdot\gamma\gamma\cdot F\cdot\psi-\bar{\psi}\cdot F\cdot\gamma\gamma\cdot\psi)\,, \tag{63}$$

where

$$Z=\frac{3m^2}{2(2m-m_{1/2})^2}\,.$$

There is no limit for $Q_{\text{out}}^{(1/2)}$ (it will be the case if $Q\geq 0$, namely, $Q_{\text{out}}^{(1/2)}\leq Q$). Note that the constant Z can be used to renormalize simultaneously the energy-momentum, spin, and charge of the field $\psi^{(1/2)}$.

We have studied the field equations at the classical level. A particle interpretation must be formulated by quantum field theory. In this sense it is essential to study the stability of this theory.

IV. SUMMARY AND CONCLUSIONS

We have studied the RS equations in their most general form which, with the appropriate constraints, represent spin-$\frac{3}{2}$ particles and present a serious problem, first noted by Velo and Zwanziger. The problem is the appearance of velocities higher than c. We proposed an alternative approach which leads to a consistent formulation of the problem, free from noncausality, by reducing the number of constraints in such a way that the theory has 12 degrees of freedom and accommodates simultaneously spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ particles. To accomplish this formulation, we substituted a subsidiary constraint by a subsidiary differential equation. The currents corresponding to $s=\frac{3}{2}$ and $s=\frac{1}{2}$ are independently conserved in the free-field case. However, in an external electromagnetic field only their sum is conserved. The introduction of $s=\frac{1}{2}$ degrees of freedom seems to be the price to pay for recovering the causality. There is a certain parallelism with the appearence of antiparticles in the case of the Dirac field, necessary to achieve a fully relativistic formulation.

We pointed out before the Wightman suggestion that a relativistic theory with well-defined higher spin must always suffer from inconsistencies as the VZ problem. The results of the present paper suggest that these inconsistencies may be avoided by the use of equations containing several values of the spin. It could even be interesting to study the following conjecture: If a consistent theory contains particles with a value of the spins, it must also include other values of s together with the corresponding antiparticles. This is a generalization of what happens in the cases $s=\frac{1}{2}$ and $s=\frac{3}{2}$. The physical implications would be far-reaching and will be considered in a future paper.

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APPENDIX

The conservation of the constraint (33),

$$\Omega = h\psi^0 - \vec{\pi}\cdot\vec{\psi} - M\gamma\cdot\psi\,, \tag{A1}$$

by the Eq. (28) is a consequence of the equation which satisfies Ω. From Eq. (28) for $\mu=0$ and contracting by γ^0, we obtain

$$\pi_0\psi_0 = h\psi_0 + m'\gamma\cdot\psi + \frac{i}{2}q\gamma\cdot F\cdot\psi\,, \tag{A2}$$

which permits us to express Ω in an invariant form:

$$\Omega = \pi\cdot\psi - \frac{i}{2}q\gamma\cdot F\cdot\psi - (M+m')\gamma\cdot\psi\,. \tag{A3}$$

Contracting Eq. (27) by γ^μ and π^μ, and taking into account formulas (30) and (31), one finally finds an equation for Ω:

$$(\gamma\cdot\pi + m_{1/2} + 2m')\Omega = 0\,. \tag{A4}$$

From which it is clear that if $\Omega=0$ for $t=t_0$, then $\Omega=0$ for all t.

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