TORUS RATIONAL FIBRATIONS

VICENTE MUÑOZ

ABSTRACT. We study rational fibrations where the fibre is an r-dimensional torus and the base is a formal space. We make use of the Eilenberg-Moore Spectral Sequence to prove the Toral Rank Conjecture in some cases.

1. INTRODUCTION

The purpose of this note is to present a class of manifolds for which the Toral Rank Conjecture holds. Recall that for a finite dimensional connected smooth manifold E we call rank of E, and denote it by $\mathrm{rk}(E)$, the maximum integer r such that there is an almost free action of the r-dimensional torus $\mathbb{T}^r$ on E (see [4, chapter 5] [8]). Then the Toral Rank Conjecture is the following

Conjecture 1. [4, section 5.2] *Let E be a finite dimensional smooth simply connected manifold and let $r = rk(E)$. Then the (rational) cohomology of E has dimension at least 2^r.*

Recall that any connected CW-complex M of finite type has a (minimal) Sullivan model $(\Lambda X_M, d)$ which computes its rational cohomology, $H^*(\Lambda X_M, d) = H^*(M)$ (when M is simply-connected, X_M gives also the homotopy of M, see [1]). Then we define rational fibration as in [7].

Definition 1. A rational fibration is a couple of maps $T \xrightarrow{i} E \xrightarrow{p} B$ between connected spaces with

- $p \circ i$ homotopically trivial,
- if we consider the KS model of p and the induced map ψ,

$$\begin{array}{ccccc} \Lambda X_B & \longrightarrow & \Lambda X_B \otimes \Lambda V & \longrightarrow & \Lambda V \\ \| & & \downarrow \simeq & & \downarrow \psi \\ \Lambda X_B & \longrightarrow & \Lambda X_E & \longrightarrow & \Lambda X_T \end{array}$$

  then ψ is a quasi-isomorphism.

Date: July, 1997.
*Supported by a grant from Ministerio de Educación y Cultura of Spain.
Mathematics Subject Classification. Primary: 55P62. Secondary: 55T20.

Morally, $T \to E \to B$ is a rational fibration if it has a KS model. We remark that if $T \to E \to B$ is a fibration with B 1-connected, then it is a rational fibration [5, section 6]. We shall henceforth assume that B is always 1-connected.

Now suppose that $T = \mathbb{T}^r$ acts almost freely on E. Then $B = E/T$ is a finite CW-complex and $T \to E \to B$ turns out to be a rational fibration [1, section 5]. This allows us to express conjecture 1 in more natural homotopy terms as

Conjecture 2. [8, problem 1.4] *Let $T \to E \to B$ be a rational fibration of finite connected CW-complexes with B 1-connected, in which $T = \mathbb{T}^r$. Then the rational cohomology of E has dimension at least 2^r.*

One might say that conjecture 1 is the geometric version and conjecture 2 is the rational homotopy version. Conjecture 2 implies conjecture 1 but there is no reason for the converse to hold. The Toral Rank Conjecture 1 is proved in many cases, for example when E is a product of spheres, a homogeneous space or a homology Kähler manifold (see [4, chapter 5]). Let us state our main two results.

Definition 2. For any finite CW-complex B define $\chi_{\text{even}}(B) = \sum_{i\geq 0}(-1)^i \dim H^{2i}(B)$ and $\chi_{\text{odd}}(B) = \sum_{i\geq 0}(-1)^i \dim H^{2i+1}(B)$.

Theorem 3. *Suppose B is formal. If either $\chi_{\text{even}}(B) \neq 0$ or $\chi_{\text{odd}}(B) \neq 0$, then conjecture 2 is true for $T \to E \to B$.*

Theorem 4. *Suppose B is formal. Write $H^{\text{even}}(B) = \mathbb{Q}[t_1, \dots, t_n]/(f_1, \dots, f_m)$ for the even dimensional part of the (rational) cohomology algebra of B. Then $m \geq n$. If either $m = n$ or $m = n + 1$ then conjecture 2 holds for $T \to E \to B$.*

Theorem 4 is a consequence of propositions 10 and 12 together with lemma 8. The paper is organised as follows. In section 2 we give a suitable model for E when B is formal and $T \to E \to B$ is a rational fibration. We use it to prove theorem 3. In section 3 we recall the Eilenberg-Moore Spectral Sequence and use it to prove theorem 4. We will assume throughout that all spaces are connected, of finite type and with finite dimensional rational cohomology. Basic references for rational homotopy theory and Sullivan models are [4] [3] [10], rational fibrations are introduced in [7].

Acknowledgements. I am indebted to Aniceto Murillo, Greg Lupton and Antonio Viruel for very stimulating conversations. Special thanks to Aniceto Murillo for carefully reading first versions of this paper and pointing out many improvements. The author is grateful to the referee for pointing out a mistake in a previous version of lemma 8.

2. A suitable model for E

Fix a rational fibration $T \to E \to B$ with $T = \mathbb{T}^r$. The minimal model of T is $(\Lambda X_T, 0)$, where $\Lambda X_T = \Lambda(y_1, \dots, y_r)$, $|y_i| = 1$, $1 \leq i \leq r$. Let $(\Lambda X_B, d)$ be the minimal model of B. By the definition of rational fibration, the KS-extension corresponding to $T \to E \to B$ is

$$(\Lambda X_B, d) \to (\Lambda X_B \otimes \Lambda X_T, D) \to (\Lambda X_T, 0), \tag{1}$$

where $(\Lambda X_B \otimes \Lambda X_T, D)$ is a model (not minimal in general) of E. The KS-extension (1) is determined by

$$Dy_i = x_i \in (\Lambda X_B)^2.$$

Now let $R = \mathbb{Q}[z_1, \dots, z_r]$ with $|z_i| = 2$, $1 \le i \le r$. The algebra morphism $R \to H^*(\Lambda X_B)$, $z_i \mapsto x_i$ makes $H^*(B) = H^*(\Lambda X_B)$ into an R-graded module. Geometrically, this corresponds to the following. As B is 1-connected, the rational fibration $T \to E \to B$ is determined by a (rational) classifying map $B \to BT$, where BT is the classifying space for the torus T. This gives a morphism of rings $R = H^*(BT) \to H^*(B)$, which is the one defined above.

Lemma 5. *Suppose B is formal. Then a model of E is given by $(H^*(B) \otimes H^*(T), d)$, $d(h \otimes y_i) = x_i \cdot h \otimes 1$. In particular, $H^*(E) = H(H^*(B) \otimes H^*(T), d)$.*

Proof. Consider the model $(\Lambda X_B \otimes \Lambda X_T, D)$ of E given by the KS-extension (1). As B is formal, there is a quasi-isomorphism $\psi : (\Lambda X_B, d) \xrightarrow{\simeq} (H^*(B), 0)$. Then $\psi \otimes \mathrm{id} : (\Lambda X_B \otimes \Lambda X_T, D) \to (H^*(B) \otimes \Lambda X_T, \bar{D})$ is also a quasi-isomorphism, where $\bar{D} = d$. As $\Lambda X_T = H^*(T)$, this means that $(H^*(B) \otimes H^*(T), d)$ is a model of E. □

For any graded R-module M we have defined a differential complex $(M \otimes H^*(T), d)$, $d(m \otimes y_i) = x_i \cdot m \otimes 1$. In general, we can ask whether $\dim(M \otimes H^*(T), d) \ge 2^r$ for any finite dimensional R-module M. This would give an affirmative answer to conjecture 2 for any formal space B.

Note that for an R-module M, we have $M = M^{\mathrm{even}} \oplus M^{\mathrm{odd}}$ and then $(M \otimes H^*(T), d) = (M^{\mathrm{even}} \otimes H^*(T), d) \oplus (M^{\mathrm{odd}} \otimes H^*(T), d)$.

Remark 6. Suppose B is 1-connected. Then the Serre Spectral Sequence for $T \to E \to B$ is the same as the spectral sequence obtained by filtering $\Lambda X_B \otimes \Lambda X_T$ with $\mathcal{F}^p = (\Lambda X_B)^{\ge p} \otimes \Lambda X_T$, from the term E_2 onwards (see [5]). For this spectral sequence, $E_2^{*,*} = H^*(B) \otimes H^*(T)$ and d_2 is the differential d given in lemma 5. E_∞ is isomorphic to the cohomology of E (as vector spaces), so when B is formal $E_3 = E_\infty$ and the Serre Spectral Sequence collapses at the third stage.

Remark 7. In general, for a rational fibration $T \to E \to B$ with B 1-connected, finiteness of $H^*(B)$ implies the convergence of the Serre Spectral Sequence at a finite stage. Lemma 5 guarantees convergence at the third stage under the condition of the formality of B. To see that this condition is necessary, take for instance $T = \mathbb{T}^2$, B to have minimal model $\Lambda X_B = \Lambda(x_1, x_2, u_1, u_2) \otimes \Lambda W^{\ge 5}$, where $|x_i| = 2$, $dx_i = 0$, for $i = 1, 2$, $du_1 = x_1^2$, $du_2 = x_1 x_2$, and W and d on W are defined in such a way that $H^{\ge 6}(B) = 0$. Then there is a non-zero homology class $[z] \in H^5(B)$, $z = x_2 u_1 - x_1 u_2$.

Put $M = H^*(B) = M^{\text{even}} \oplus M^{\text{odd}}$, where

$$M^{\text{even}} = \mathbb{Q} < 1, x_1, x_2, x_2^2 >, \qquad\qquad M^{\text{odd}} = \mathbb{Q} < z > .$$

Then $0 \neq [z] \in H(M^{\text{odd}} \otimes \Lambda X_T, d) \subset H(M \otimes \Lambda X_T, d)$, but the following computation

$$d(y_1 y_2 x_1) = x_1^2 y_2 - x_1 x_2 y_1 = (du_1) y_2 - (du_2) y_1 = d(u_1 y_2 - u_2 y_1) + z$$

shows that $0 = [z] \in H^*(E)$. This implies $H^*(E) \neq H(H^*(B) \otimes H^*(T), d)$ and the Serre Spectral Sequence does not collapse at E_3^{**}.

Proof of theorem 3. Put $M = H^*(B)$. Lemma 5 tells us that the cohomology of E is $H^*(E) = H(M \otimes \Lambda X_T, d)$. As above, we write $M = M^{\text{even}} \oplus M^{\text{odd}}$ so that $H^*(E) = H(M^{\text{even}} \otimes \Lambda X_T, d) \oplus H(M^{\text{odd}} \otimes \Lambda X_T, d)$. We are going to check that if $\chi_{\text{even}}(B) \neq 0$ then $\dim H(\Lambda X_T \otimes M^{\text{even}}, d) \geq 2^r$ (the other case being analogous). So we can suppose that $M = M^{\text{even}}$. Give $V = \Lambda X_T \otimes M$ the following bigradation: $V^{k,l} = (\Lambda X_T)^{k-l} \otimes M^{2l}$, $k, l \in \mathbb{Z}$. Then d has bidegree $(0,1)$, so it restricts to $V^{k,*}$. The Euler characteristic of $V^{k,*}$ is $\chi(V^{k,*}) = \sum_l (-1)^l \binom{r}{k-l} \dim M^{2l}$, so

$$\dim H^*(E) = \dim H(V, d) = \sum_k \dim H(V^{k,*}, d) \geq \sum_k |\chi(H(V^{k,*}, d))| =$$

$$= \sum_k |\chi(V^{k,*})| \geq |\sum_k \chi(V^{k,*})| = |\sum_{k,l} (-1)^l \binom{r}{k-l} \dim M^{2l}| =$$

$$= |\sum_l (-1)^l \dim M^{2l} \sum_{k \in \mathbb{Z}} \binom{r}{k-l}| = 2^r |\chi_{\text{even}}(B)| \geq 2^r. \ \square$$

This theorem covers many examples. For instance, let us recall example 3 in [4, section 5.3]. Consider $B = \underbrace{\mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2}_{n}$, $f_i : B \to \mathbb{CP}^2$ given by contracting every $\mathbb{CP}^2$ expect the i-th one. Then pull back the universal fibration $\mathbb{T}^n = (\mathbb{S}^1)^n \to E\mathbb{T}^n \to (\mathbb{CP}^\infty)^n$ under the map $f = f_1 \times \cdots \times f_n : B \to (\mathbb{CP}^2)^n \hookrightarrow (\mathbb{CP}^\infty)^n$ to get a rational fibration $T \to E \to B$, with $T = \mathbb{T}^n$. As B is formal and $\chi_{\text{even}}(B) = 2 - n$, conjecture 2 holds for these fibrations when $n \neq 2$. The case $n = 2$ can be worked out explicitly. In this case, $H^*(B) = \mathbb{Q}[x_1, x_2]/(x_1 x_2, x_1^2 - x_2^2)$, with $|x_1| = |x_2| = 2$. Then the E_2 term of the Serre Spectral Sequence of remark 6 is (the numbers denote the dimensions)

$$\begin{array}{|ccccc}
1 & & 2 & & 1 \\
& \searrow & & \searrow & \\
2 & & 4 & & 2 \\
& \searrow & & \searrow & \\
1 & & 2 & & 1 \\
\hline
\end{array}$$

so $\dim E_3^{00} = 1$, $\dim E_3^{21} \geq 2$, $\dim E_3^{42} = 1$. As $E_\infty = E_3$, we have $\dim H^*(E) \geq 4 = 2^n$ (actually we do have equality).

3. Use of Eilenberg-Moore Spectral Sequence

Let $T \to E \to B$ be a rational fibration with $T = \mathbb{T}^r$, whose associated KS-extension is (1). Consider the Koszul resolution of $\mathbb{Q}$ given by

$$K^* = R \otimes \Lambda X_T = \mathbb{Q}[z_1, \dots, z_r] \otimes \Lambda(y_1, \dots, y_r),\ dy_i = z_i,\ |y_i| = 1,\ |z_i| = 2,\ 1 \leq i \leq r.$$

Filter the model of E given by $(\Lambda X_B \otimes \Lambda X_T, D)$ with $\mathcal{F}^p = \Lambda X_B \otimes \Lambda^{\leq p} X_T$. Then we get a spectral sequence with

$$E_2^{**} = H(H^*(B) \otimes \Lambda X_T, d) = H(H^*(B) \otimes_R K^*, \bar{D}) = \mathrm{Tor}_R^*(H^*(B), \mathbb{Q}), \qquad (2)$$

$$E_\infty^{**} = H^*(E) = H(\Lambda X_B \otimes \Lambda X_T, D) = H(\Lambda X_B \otimes_R K^*, D) = \mathrm{Tor}_R^*(\Lambda X_B, \mathbb{Q}).$$

Again, by lemma 5, if B is formal E_r^{**} degenerates at the second stage, i.e. $H^*(E) = \mathrm{Tor}_R^*(H^*(B), \mathbb{Q})$. To understand this spectral sequence, consider $(R \otimes \Lambda X_B \otimes \Lambda X_T, \mathcal{D})$, $\mathcal{D}|_{X_B} = d$, $\mathcal{D} z_i = 0$, $\mathcal{D} y_i = 1 \otimes x_i - z_i \otimes 1$. Then

$$(\Lambda X_B, d) \xrightarrow{\simeq} (R \otimes \Lambda X_B \otimes \Lambda X_T, \mathcal{D}) \cong (\Lambda X_B, d) \otimes (R \otimes \Lambda X_T, d)$$

is a quasi-isomorphism. So we have a KS-extension

$$(R, 0) \to (R \otimes \Lambda X_B \otimes \Lambda X_T, \mathcal{D}) \to (\Lambda X_B \otimes \Lambda X_T, D) \qquad (3)$$

where the term in the middle is a model for B and the term in the right a model for E. Then E_r^{**} is the usual Eilenberg-Moore Spectral Sequence associated to (3).

Geometrically, this corresponds to the following. The fibration $T \to E \to B$ is determined by a (rational) classifying map $B \to BT$ which yields a rational fibration $E \to B \to BT$ with KS-extension (3) (recall that $(R, 0)$ is a minimal model for BT). The Eilenberg-Moore Spectral Sequence associated to this fibration is E_r^{**} (see [9]).

With this understood, we aim to prove theorem 4. First a technical lemma.

Lemma 8. *Let $S = \mathbb{Q}[t_1, \dots, t_n]$ be a polynomial ring, $\mathfrak{m} = (t_1, \dots, t_n)$ maximal ideal, and $f_1, \dots, f_m \in \mathfrak{m}$ non-zero elements such that for $I = (f_1, \dots, f_m)$, S/I is finite dimensional. Then $m \geq n$. If $m = n$, $f_1, \dots, f_n$ form a regular sequence for S. If $m > n$ then we can choose $g_1, \dots, g_m$ generators of I such that $g_1, \dots, g_n$ are a regular sequence for S.*

Proof. Let S_0 be the localisation of S at $\mathfrak{m}$. Its Krull dimension is $\mathrm{Kd}(S_0) = n$, so by [2, theorem 11.14], $m \geq n$. Now suppose $m = n$. Since for any local noetherian ring A and $f \in \mathfrak{m}_A$ it is $\mathrm{Kd}(A) - 1 \leq \mathrm{Kd}(A/f) \leq \mathrm{Kd}(A)$, we must have $\mathrm{Kd}(S_i) = n - i$, where $S_i = S_0/(f_1, \dots, f_i)$, $1 \leq i \leq n$. To prove that $f_1, \dots, f_n$ is a regular sequence we have to prove that f_{i+1} is not a zero divisor in S_i, $0 \leq i \leq n-1$. Suppose f_{i+1} is a zero divisor. Then there must be a minimal prime $\mathfrak{p} \supset (f_1, \dots, f_i)$ with $f_{i+1} \in \mathfrak{p}$. By [2, corollary 11.16], $\mathrm{ht}\, \mathfrak{p} \leq i$, so $\mathrm{Kd}(S/\mathfrak{p}) \geq n - i$, hence $\mathrm{Kd}(S_{i+1}) \geq n - i$, which is a contradiction.

Now suppose $m > n$. We shall construct $g_1, \dots, g_n$ inductively such that they are a regular sequence and (up to reordering f_i) $I = (g_1, \dots, g_{i-1}, f_i, \dots, f_m)$. Let

$g_1 = f_1$. Suppose $g_1, \dots, g_{i-1}$ constructed. Then $\mathrm{Kd}(S_0/(g_1, \dots, g_{i-1})) = n - i + 1$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be the minimal primes containing $(g_1, \dots, g_{i-1})$. Define

$$H_j = \{\mu = (\mu_1, \dots, \mu_{m-i+1})/\mu_1 f_i + \cdots + \mu_{m-i+1} f_m \in \mathfrak{p}_j\} \subset \mathbb{Q}^{m-i+1},$$

for $j = 1, \dots, k$. As $i \leq n$, $\mathrm{Kd}(S_0/\mathfrak{p}_j) \neq 0$, so H_j is a proper linear subvariety of $\mathbb{Q}^{m-i+1}$. As a conclusion, there is an element μ not lying in any H_j, so $g_i = \mu_1 f_i + \cdots + \mu_{m-i+1} f_m \notin \cup \mathfrak{p}_j$. This means that g_i is not a zero divisor in $S_0/(g_1, \dots, g_{i-1})$. We reorder $f_i, \dots, f_m$ suitably and repeat the process. □

Remark 9. The elements g_i obtained in the proof of the previous lemma are not homogeneous in general, even when the elements f_i are so. It is probably the case that we cannot arrange them to be homogeneous.

Proposition 10. *Let B be formal and with finite-dimensional cohomology. Suppose that $H^{\mathrm{even}}(B) = \mathbb{Q}[t_1, \dots, t_n]/(f_1, \dots, f_n)$. Then conjecture 2 holds for $T \to E \to B$.*

Proof. By the discussion above, we only need to bound below the dimension of $\mathrm{Tor}_R(H^*(B), \mathbb{Q})$. As in the proof of theorem 3, this splits as $\mathrm{Tor}_R(H^{\mathrm{even}}(B), \mathbb{Q}) \oplus \mathrm{Tor}_R(H^{\mathrm{odd}}(B), \mathbb{Q})$, so it suffices to prove $\dim \mathrm{Tor}_R(H^{\mathrm{even}}(B), \mathbb{Q}) \geq 2^r$. Put $M = H^{\mathrm{even}}(B)$. As M is an $R = \mathbb{Q}[z_1, \dots, z_r]$-algebra, we can suppose that

$$M = \mathbb{Q}[z_1, \dots, z_r, t_{r+1}, \dots, t_{r+k}]/(f_1, \dots, f_{r+k}),$$

where $k \geq 0$ (it is possible that we have added some algebra generator z_j together with a relation $f_i = z_j$, but still we have the same number of generators and relations). To compute $\mathrm{Tor}_R(M, \mathbb{Q})$ this time we will resolve M. By lemma 8, $f_1, \dots, f_{r+k}$ is a regular sequence for the polynomial ring $S = \mathbb{Q}[z_1, \dots, z_r, t_{r+1}, \dots, t_{r+k}]$. Then the Koszul complex, given by $(S \otimes \Lambda(e_1, \dots, e_{r+k}), d)$, $de_i = f_i$, $|e_i| = |f_i| - 1$, is a free S-resolution of M. Now we distinguish between the two cases:

1. If $k = 0$, the Koszul complex is a free R-resolution and then $\mathrm{Tor}_R(M, \mathbb{Q}) = H((R \otimes \Lambda(e_1, \dots, e_r)) \otimes_R \mathbb{Q}, d \otimes_R \mathbb{Q}) = \Lambda(e_1, \dots, e_r)$ has dimension 2^r.
2. If $k > 0$, the same argument yields that $\mathrm{Tor}_S(M, \mathbb{Q})$ has dimension 2^{r+k}. Now $S = R \otimes T$, where $T = \mathbb{Q}[t_{r+1}, \dots, t_{r+k}]$. There is a spectral sequence with $E_2 = \mathrm{Tor}_T(\mathrm{Tor}_R(M, \mathbb{Q}), \mathbb{Q})$ converging to $\mathrm{Tor}_S(M, \mathbb{Q})$. This is given as follows: resolve $\mathbb{Q}$ as R-module $K_R \xrightarrow{\simeq} \mathbb{Q}$ and as T-module $K_T \xrightarrow{\simeq} \mathbb{Q}$. Then $K_R \otimes K_T \xrightarrow{\simeq} \mathbb{Q}$ is an S-resolution of $\mathbb{Q}$. The spectral sequence is obtained from

$$M \otimes_{R \otimes T} (K_R \otimes K_T) = M \otimes_{R \otimes T} ((K_R \otimes T) \otimes_T K_T) =$$

$$= (M \otimes_{R \otimes T} (K_R \otimes T)) \otimes_T K_T = (M \otimes_R K_R) \otimes_T K_T.$$

We conclude $\dim \mathrm{Tor}_T(\mathrm{Tor}_R(M, \mathbb{Q}), \mathbb{Q}) \geq 2^{r+k}$. But $\dim \mathrm{Tor}_T(N, \mathbb{Q}) \leq 2^k \dim N$, for any finite dimensional T-module N. Thus $\dim \mathrm{Tor}_R(M, \mathbb{Q}) \geq 2^r$.

□

Remark 11. By a result of Halperin [6] (see also [4, section 2.6]), if B is a formal 1-connected rational space with $H^*(B) = H^{\mathrm{even}}(B) = \mathbb{Q}[t_1, \dots, t_n]/(f_1, \dots, f_n)$, then it has finite dimensional rational homotopy and $\chi_\pi(B) = 0$. Many properties are known of these elliptic spaces. Note for instance that such an algebra is always a Poincaré duality algebra. However proposition 10 is valid also for spaces B with some odd dimensional cohomology.

Proposition 12. *Let B be formal and with finite-dimensional cohomology. Suppose that $H^{\mathrm{even}}(B) = \mathbb{Q}[t_1, \dots, t_n]/(f_1, \dots, f_{n+1})$. Then conjecture 2 holds for $T \to E \to B$.*

Proof. Again we want to prove that $\dim \mathrm{Tor}_R(M, \mathbb{Q}) \geq 2^r$, with $M = H^{\mathrm{even}}(B)$. Write $M = \mathbb{Q}[z_1, \dots, z_r, t_{r+1}, \dots, t_{r+k}]/(f_1, \dots, f_{r+k+1})$, $k \geq 0$, as in the proof of proposition 10. Suppose first that $f_1, \dots, f_{r+k}$ is a regular sequence for $S = \mathbb{Q}[z_1, \dots, z_r, t_{r+1}, \dots, t_{r+k}]$. Put $\tilde{M} = S/(f_1, \dots, f_{r+k})$ and $f = f_{r+k+1}$, so that $M = \tilde{M}/f\tilde{M}$. The proof of proposition 10 ensures us that $\dim \mathrm{Tor}_S(\tilde{M}, \mathbb{Q}) = 2^{r+k}$. Let us consider the two cases separately:

(1) If $k = 0$, take the Koszul complex for the given presentation of M, i.e. $L^* = R \otimes \Lambda(e_1, \dots, e_{r+1}) \to M$, $de_i = f_i$, $|e_i| = |f_i| - 1$. The main point is that this is not a resolution (i.e. it is not a quasi-isomorphism). In fact, $\tilde{L}^* = R \otimes \Lambda(e_1, \dots, e_r)$ is an R-resolution of $\tilde{M}$ and $L^* = \tilde{L}^* \otimes \Lambda(e)$ where $e = e_{r+1}$, $de = f$. Filter L^* by powers of e. So we get an spectral sequence with $E_1^{**} = \tilde{M} \otimes \Lambda(e)$ and there is only one non-trivial differential $\tilde{M} \otimes e \to \tilde{M} \otimes 1$, $m \otimes e \mapsto f \cdot m \otimes 1$. Then

$$E_\infty^{**} = (\tilde{M}/f\tilde{M} \otimes 1) \oplus (\mathrm{Ann}_{\tilde{M}}(f) \otimes e)$$

By remark 11, $\tilde{M}$ is a Poincaré duality space. This implies that $\tilde{M}/f\tilde{M} \otimes \mathrm{Ann}_{\tilde{M}}(f) \to \mathbb{Q}$ is a perfect pairing, so it gives an isomorphism $\mathrm{Ann}_{\tilde{M}}(f) \cong (\tilde{M}/f\tilde{M})^\vee = M^\vee$. Thus $H^*(L) \cong M \oplus M^\vee$.

Now consider the standard Koszul resolution $K^* \xrightarrow{\simeq} \mathbb{Q}$. The bicomplex $L^* \otimes_R K^*$ gives two spectral sequences, E_r^{**} and $\bar{E}_r^{**}$,

$$\bar{E}_2^{**} = \bar{E}_\infty^{**} = L^* \otimes_R \mathbb{Q} = \Lambda(e_1, \dots, e_{r+1}),$$

$$E_2^{**} = \mathrm{Tor}_R^*(H^*(L), \mathbb{Q}) = \mathrm{Tor}_R^*(M, \mathbb{Q}) \oplus \mathrm{Tor}_R^*(M, \mathbb{Q})^\vee.$$

$\bar{E}_\infty$ has dimension 2^{r+1} and we know that $E_\infty = \bar{E}_\infty$ (as vector spaces), so as E_2 converges to E_∞,

$$2 \dim \mathrm{Tor}_R^*(M, \mathbb{Q}) = \dim E_2 \geq 2^{r+1},$$

whence the result.

(2) If $k > 0$, the same argument yields that $\dim \mathrm{Tor}_S(M, \mathbb{Q}) \geq 2^{r+k}$. Now we use the argument in the second case of proposition 10 to get $\dim \mathrm{Tor}_R(M, \mathbb{Q}) \geq 2^r$.

In the general case, lemma 8 ensures us that we can write $M = S/(g_1, \dots, g_{r+k+1})$ where $g_1, \dots, g_{r+k}$ form a regular sequence (these elements g_i are non-homogeneous in general). We can use the same argument that we have used above, but this time forgetting the degree, i.e. we consider S concentrated in degree 0 and $|e_i| = -1$, $1 \leq i \leq r+k+1$. Also the Koszul resolution $K^* \xrightarrow{\simeq} \mathbb{Q}$ has to be graded accordingly. This does not affect to the computation of the dimension of $\mathrm{Tor}_R(M, \mathbb{Q})$ although it gives a completely different grading. $\square$

Remark 13. Let B be a formal 1-connected rational space whose cohomology is $H^*(B) = H^{\text{even}}(B) = \mathbb{Q}[t_1, \dots, t_n]/(f_1, \dots, f_{n+1})$. Then B is always hyperbolic (i.e. it has infinite dimensional rational homotopy). In fact, since $f_1, \dots, f_{n+1}$ is not a regular sequence, there is a non-trivial relation $a_1 f_1 + \cdots a_{n+1} f_{n+1} = 0$. Take one of minimal degree. In the bigraded model of $H^*(B)$, $Z_0 =< t_1, \dots, t_n >$, $Z_1 =< u_1, \dots, u_{n+1} >$, $du_i = f_i$ and then $a_1 u_1 + \cdots + a_{n+1} u_{n+1} = dv$, for some non-zero $v \in Z_2$. So $Z_2 \neq 0$, which implies the hyperbolicity of B (see [4, section 7.4]). The author wants to thank Greg Lupton for pointing out this to him.

One can hope of proving conjecture 2 for $T \to E \to B$, where $H^{\text{even}}(B) = \mathbb{Q}[t_1, \dots, t_n]/(f_1, \dots, f_{n+s})$, inductively on s, but the argument above does not seem to generalise.

Remark 14. Let $T \to E \to B$ be a rational fibration with $T = \mathbb{T}^r$, but this time we will not suppose that E and B are finite CW-complexes but only finite type CW-complexes. Let a stand for the Krull dimension of $H^{\text{even}}(B)$. Then the arguments of this section carry out to prove that $\dim H^*(E) \geq 2^{r-a}$ whenever B is formal with $H^{\text{even}}(B) = \mathbb{Q}[t_1, \dots, t_n]/(f_1, \dots, f_m)$, $m = n-a$, $n-a+1$.

References

[1] C. Allday and S. Halperin, Lie group actions on spaces of finite rank, *Quart. Jour. Math. Oxford,* 28 (1978) 63-76.

[2] M. Atiyah and I. Macdonald, *Introduction to commutative algebra,* Addison-Wesley Publishing Company, Massachusetts, 1969.

[3] Y. Félix, *La dichotomie elliptique-hyperbolique en homotopie rationnelle,* Astérisque, **176**, 1989.

[4] Y. Félix, D. Tanré, J-C. Thomas, *Minimal Models and Geometry,* Preprint, 1993.

[5] P-P. Grivel, Formes differentielles et suites spectrales, *Ann. Inst. Fourier,* 29 (1979) 17-37.

[6] S. Halperin, Finiteness in the minimal models of Sullivan, *Trans. Amer. Math. Soc.,* 230 (1977) 173-199.

[7] S. Halperin, Rational fibrations, minimal models, and fiberings of homogeneous spaces, *Trans. Amer. Math. Soc.,* 244 (1978) 199-224.

[8] S. Halperin, Rational homotopy and torus actions, *Aspects of Topology, In memory of Hugh Dowker,* Lecture Notes Series, **93**, (1985) 293-306.

[9] J. McCleary, *User's guide to spectral sequences,* Mathematics Lecture Series, **12**, Publish or Perish, 1985.

[10] D. Tanré, *Homotopie rationnelle: Modèles de Chen, Quillen, Sullivan,* Lecture Notes in Maths, **1025**, 1983.

Departamento de Álgegra, Geometría y Topología, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain
E-mail address: vmunoz@agt.cie.uma.es