

# On the minimum genus problem on bordered Klein surfaces

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## ABSTRACT

The minimum genus problem consists in determining the minimum algebraic genus of a surface on which a given group  $G$  acts. For cyclic groups  $G$  this problem on bordered Klein surfaces was solved in 1989. The next step is to fix the number of boundary components of the surface and to obtain the minimum algebraic genus, and so the minimum topological genus. It was achieved for cyclic groups of prime and prime-power order in the nineties.

In this work the corresponding results for cyclic groups of order  $N = pq$ , where  $p$  and  $q$  are different odd primes, is obtained. There appear different results depending on the orientability of the surface.

Finally we obtain general results when the number of boundary components is small, which are valid for any odd  $N$ .

*Key words:* Klein surfaces, algebraic genus, automorphisms of surfaces.

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## 1. Introduction

A bordered Klein surface of algebraic genus  $p \geq 2$  has at most  $12(p-1)$  automorphisms ([6], [7]). This upper bound is a particular case of the problem of finding the minimum algebraic genus of a surface whose group of automorphisms is a given finite group  $G$ . For cyclic groups this value was obtained in [3]. Earlier, Harvey had obtained the latter result for Riemann surfaces [5].

A further step is to minimize the topological genus when the number of connected components of the boundary is fixed, say  $k$ . The interest in this problem comes from the field of real algebraic geometry. In fact, under the equivalence between compact bordered Klein surfaces and real algebraic curves, the boundary components

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of the surfaces correspond to the ovals of the curve. Therefore, in terms of real algebraic geometry, the above problem consists of computing the minimum genus of the projective, smooth, irreducible, real algebraic curves with  $k$  connected components admitting  $G$  as a group of birational automorphisms. If  $G$  is a cyclic group of order  $N$ , this problem was solved in [2] for  $N$  prime and in [4] for  $N$  a prime-power. For this and other related questions see [1] where in particular (page 166) the pending question of solving the problem for an arbitrary  $N$  is recalled.

In this work we solve this question for  $N = pq$ , a product of two different odd primes. The analysis of this case makes it clear that it is quite impossible to obtain an explicit general solution for arbitrary  $N$ . On the other hand, we obtain information enough to have it solved for “small” values of  $k$ .

We shall use the notation of [2] which we now describe. Let  $\mathcal{K}$  be the class of compact bordered Klein surfaces. Given  $S \in \mathcal{K}$  we call  $g$  its topological genus,  $k$  the number of connected components of its boundary, and take  $\alpha = 2$  if  $S$  is orientable, and  $\alpha = 1$  if  $S$  is non-orientable. Then the algebraic genus of  $S$  is  $\mathfrak{p} = \alpha g + k - 1$ . So for a given  $k$ , minimizing  $g$  is equivalent to minimizing  $\mathfrak{p}$ . Now fix  $N \geq 2$ ,  $k \geq 1$ . We denote by  $\mathcal{K}_+^+(N, k)$  (resp.  $\mathcal{K}_+^-(N, k)$ ) the set of orientable surfaces  $S \in \mathcal{K}$  with  $\mathfrak{p} \geq 2$ , having  $k$  connected boundary components, which admit an orientation-preserving (resp. orientation-reversing) automorphism of order  $N$ . In the same way  $\mathcal{K}_-(N, k)$  is the set of non-orientable surfaces  $S \in \mathcal{K}$  with  $\mathfrak{p} \geq 2$  and  $k$  boundary components which admit an automorphism of order  $N$ . Then define

$$\begin{aligned} p_+^+(N, k) &= \min\{\mathfrak{p} \mid \text{there exists } S \in \mathcal{K}_+^+(N, k) \text{ with algebraic genus } \mathfrak{p}\} \\ p_+^-(N, k) &= \min\{\mathfrak{p} \mid \text{there exists } S \in \mathcal{K}_+^-(N, k) \text{ with algebraic genus } \mathfrak{p}\} \\ p_-(N, k) &= \min\{\mathfrak{p} \mid \text{there exists } S \in \mathcal{K}_-(N, k) \text{ with algebraic genus } \mathfrak{p}\} \end{aligned}$$

In our case,  $N$  is odd, and so  $\mathcal{K}_+^-(N, k)$  is void, and we shall obtain  $p_+^+(N, k)$  and  $p_-(N, k)$ . Of course, these results determine the corresponding values of the topological genus, namely

$$\begin{aligned} g_+^+(N, k) &= \frac{1}{2}(p_+^+(N, k) + 1 - k) \\ g_-(N, k) &= p_-(N, k) + 1 - k \end{aligned}$$

From now on we fix an odd number  $N = pq$  where  $p < q$  are odd primes, and  $k \geq 1$ . We shall devote Section 2 to compute  $p_+^+(N, k)$ , and Section 3 to compute  $p_-(N, k)$ . In the final Section 4, we use those results for small values of  $k$  with arbitrary odd  $N$ .

## 2. The computation of $p_+^+(N, k)$

Chapter 3 in [2] is essentially devoted to determine necessary and sufficient conditions for the existence of a surface  $S \in \mathcal{K}_+^+(N, k)$  (and also in  $\mathcal{K}_+^-(N, k)$  and  $\mathcal{K}_-(N, k)$ ) with topological genus  $g$ .

More precisely, from Theorems 3.1.2 and 3.2.3 of [2], the existence of  $S \in \mathcal{K}_+^+(N, k)$  with topological genus  $g$  is equivalent to the existence of non-negative integers  $g'$ ,  $m_1, \dots, m_r, t_1, \dots, t_{k'}$ , with  $m_i \geq 2$ ,  $m_i$  and  $t_j$  divisors of  $N$ , such that

$$(1) \quad \mu = 2g' + k' - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) > 0.$$

$$(2) k = \sum_{j=1}^{k'} \frac{N}{t_j}.$$

$$(3) \text{ If } g' = 0, \text{ then } \text{lcm}(m_1, \dots, m_r, t_1, \dots, t_{k'}) = N.$$

(4) The set  $X = \{m_1, \dots, m_r, t_1, \dots, t_{k'}\}$  has the *elimination property*, that is to say, the lcm does not change if we delete one of the numbers in  $X$ .

If the conditions (1)-(4) are satisfied, then  $N\mu = 2(g - 1) + k = \mathfrak{p} - 1$ . Hence in order to minimize  $g$  and so  $\mathfrak{p}$ , we shall minimize  $\mu$  constrained to conditions (1)-(4).

In our strategy to minimize  $\mu$ , observe that the conditions (3) and (4) are satisfied, for any choice of  $k'$ , of the  $t_j$  and of the other  $m_i$ , if there exist two values  $m_1 = m_2 = N$ . Also, the addition of a term  $m_i$  adds less than a unity to  $\mu$ , and since  $1 - 1/m_i \geq 2/3$ , it is better having any two than any three values  $m_i$ . On the other hand, from condition (2),  $k$  must be obtained as a sum of divisors of  $N$ . Dividing a number by a non-trivial divisor of  $N$  one has the euclidean division  $x = ay + b$ ; for any other non-negative integers  $a', b'$ , such that  $x = a'y + b'$ , we have  $a + b \leq a' + b' - 2$ . So in any case the best is to take the euclidean division,  $k'$  as small as possible, and add as much (at most two)  $m_i$  as necessary. Hence in all what follows any division will be the euclidean division. Finally observe that  $g'$  is only useful in order to assure that  $\mu > 0$ . For each unity in  $g'$  the value of  $\mu$  increases in 2. Since the same goal can be obtained by adding one or two  $m_i = N$ , which increase  $\mu$  in  $1 - 1/N$  each, the minimum is always attained for  $g' = 0$ . Hence throughout this section we take  $g' = 0$  without indicating it explicitly.

We distinguish four cases according to the relationship between  $k$  and  $N = pq$ , namely i)  $k < q$ ; ii)  $q \leq k < N$ ; iii)  $k = aN$ ; and iv)  $k > N$ ,  $k$  not a multiple of  $N$ .

### 2.1. $k < q$

(2.1.1)  $k = 1$ . By (2)  $k' = 1, t_1 = N$ . Then (3) is satisfied and in order to satisfy (4), either  $m_1 = N$  or  $r \geq 2$ . Hence for having  $\mu > 0$  (condition (1)), the minimum value is obtained for  $k' = 1, t_1 = N, r = 2, m_1 = p, m_2 = q$ . Then  $\mu = 1 - 1/p - 1/q$ , and so  $p_+^+(N, 1) = 1 + N(1 - 1/p - 1/q) = N - p - q + 1$ .

(2.1.2)  $k = 2$ . By (2)  $k' = 2, t_1 = t_2 = N$ . So (3) and (4) are already satisfied. Hence the minimum  $\mu$  is attained for  $k' = 2, t_1 = t_2 = N, r = 1, m_1 = p$ . So  $\mu = 1 - 1/p$ , and  $p_+^+(N, 2) = 1 + N(1 - 1/p) = N - q + 1$ .

(2.1.3)  $2 < k < p$ . By (2)  $k' = k, t_1 = t_2 = \dots = t_k = N$ , and (3) and (4) are satisfied. The minimum  $\mu$  is given by  $k' = k, t_1 = t_2 = \dots = t_k = N, r = 0$ . Then  $\mu = k - 2$ , and  $p_+^+(N, k) = N(k - 2) + 1$ .

(2.1.4)  $k = p$ . Then  $k' = 1, t_1 = q, r = 2, m_1 = p, m_2 = N$ . So  $\mu = 1 - 1/p - 1/N$ , and  $p_+^+(N, k) = 1 + N(1 - 1/p - 1/N) = N - q$ .

(2.1.5)  $p < k$ . We divide  $k$  by  $p, k = ap + b$ . The minimum value for  $\mu$  depends on  $b$  being 0, 1 or greater than 1.

$k = ap$ . In this case  $k' = a, t_1 = \dots = t_a = q, r = 2, m_1 = p, m_2 = p$ . Observe that  $a \geq 2$ . Then  $\mu = a - 2/p$ , and so  $p_+^+(N, k) = 1 + N(a - 2/p) = Na - 2q + 1$ .

$k = ap + 1$ . Now  $k' = a + 1, t_1 = \dots = t_a = q, t_{a+1} = N, r = 1, m_1 = p$ . Then  $\mu = a - 1/p$ , and so  $p_+^+(N, k) = 1 + N(a - 1/p) = Na - q + 1$ .

$k = ap + b$ ,  $b > 1$ . We have  $k' = a + b$ ,  $t_1 = \dots = t_a = q$ ,  $t_{a+1} = \dots = t_{a+b} = N$ ,  $r = 0$ . Then  $\mu = a + b - 2$  and so  $p_+^+(N, k) = 1 + N(a + b - 2)$ .

Hence we have proved the following

**Proposition 2.1.** *Let  $N = pq$ ,  $p < q$  odd primes, and  $1 \leq k < q$ . Then*

$$\begin{array}{ll}
 p_+^+(N, k) = & N - p - q + 1 & k = 1 \\
 & N - q + 1 & k = 2 \\
 & N(k - 2) + 1 & 2 < k < p \\
 & N - q & k = p \\
 & Na - 2q + 1 & p < k, k = ap \\
 & Na - q + 1 & p < k, k = ap + 1 \\
 & N(a + b - 2) + 1 & p < k, k = ap + b, b > 1
 \end{array}$$

**2.2.**  $q \leq k < N$

This case is the most involved since it depends on the relationship between  $p$  and  $q$ .

In order to obtain  $k$  as a sum of divisors of  $N$ , there are two types of expressions, and if  $k \geq p + q$ , there is even a third one.

First, consider the euclidean divisions of  $k$  by  $q$  and by  $p$ .

- (I)  $k = a_1p + b_1$ ,  $a_1 \geq 1$ .
- (II)  $k = a_2q + b_2$ ,  $a_2 \geq 1$ .

According to the analysis in (2.1.4) the minimum value of  $\mu$  for each of them is as follows:

- (I) if  $b_1 = 0$ ,  $a_1 = 1$ ,  $\mu_1 = 1 - \frac{1}{p} - \frac{1}{N}$   
 if  $b_1 = 0$ ,  $a_1 > 1$ ,  $\mu_1 = a_1 - \frac{2}{p}$   
 if  $b_1 = 1$ ,  $\mu_1 = a_1 - \frac{1}{p}$   
 if  $b_1 > 1$ ,  $\mu_1 = a_1 + b_1 - 2$
- (II) if  $b_2 = 0$ ,  $a_2 = 1$ ,  $\mu_2 = 1 - \frac{1}{q} - \frac{1}{N}$   
 if  $b_2 = 0$ ,  $a_2 > 1$ ,  $\mu_2 = a_2 - \frac{2}{q}$   
 if  $b_2 = 1$ ,  $\mu_2 = a_2 - \frac{1}{q}$   
 if  $b_2 > 1$ ,  $\mu_2 = a_2 + b_2 - 2$

Consider now the third type of expression, which only occurs if  $k \geq q + p$ . Namely  $k = a_3q + b_3p + c_3$ , where  $a_3, b_3 \geq 1$ .

For such an expression,  $a_3$  runs from 1 to  $\left[ \frac{k}{q} \right]$ , the integer part of  $\frac{k}{q}$ . For each of the  $a_3$  in that range, call  $r = k - a_3q$ . Then  $b_3p + c_3$  is the euclidean division of  $r$  by  $p$ .

We are going to determine the minimal value of  $\mu$  for this expression.

If  $c_3 \geq 1$ , then  $k' = a_3 + b_3 + c_3$ ,  $t_1 = \dots = t_{a_3} = p$ ,  $t_{a_3+1} = \dots = t_{a_3+b_3} = q$ ,  $t_{a_3+b_3+1} = \dots = t_{a_3+b_3+c_3} = N$ . Hence  $r = 0$ ,  $\mu = a_3 + b_3 + c_3 - 2$ .

If  $c_3 = 0$ , the values depend on  $a_3$  and  $b_3$  be greater than or equal to 1:

(a) If  $a_3 = b_3 = 1$ ,  $k' = 2$ ,  $t_1 = p$ ,  $t_2 = q$ . Then  $r = 1$ ,  $m_1 = N$ ,  $\mu = 1 - 1/N$ .

(b) If  $a_3 = 1$ ,  $b_3 > 1$ ,  $k' = b_3 + 1$ ,  $t_1 = p$ ,  $t_2 = \dots = t_{b_3+1} = q$ ,  $r = 1$  again,  $m_1 = p$ ,  $\mu = b_3 - 1/p$ .

(c) If  $a_3 > 1$ ,  $b_3 = 1$ , conversely  $k' = a_3 + 1$ ,  $t_1 = \dots = t_{a_3} = p$ ,  $t_{a_3+1} = q$ ,  $r = 1$ ,  $m_1 = q$ ,  $\mu = a_3 - 1/q$ .

(d) Finally if both  $a_3, b_3 > 1$ , then  $k' = a_3 + b_3$ ,  $r = 0$ , and so  $\mu = a_3 + b_3 - 2$ .

Call then  $\mu_3$  the minimal of the corresponding  $\mu$  for each of the  $a_3$  running from 1 to  $\lfloor \frac{k}{q} \rfloor$  and giving  $b_3 \geq 1$ . We have the following

**Proposition 2.2.** *Let  $N = pq$ ,  $p < q$ , odd primes, and  $q \leq k < N$ . Then*

$$p_+^+(N, k) = 1 + N(\min(\mu_1, \mu_2, \mu_3)),$$

for  $\mu_1, \mu_2$  and  $\mu_3$  described above.

Before to deal with the two remaining cases we are going to see several numerical examples.

*Example 2.3.* Consider  $N = 15 = 3 \cdot 5$ , and  $5 \leq k < 15$  as in Proposition 2.2. Then for each  $k$  we calculate the values of  $\mu_1, \mu_2$  and  $\mu_3$  and then  $p_+^+(15, k)$  as follows, where we write in boldface the minimum value of  $\mu_i$  for each  $k$ .

$k$	$a_1 \cdot 3 + b_1$	$a_2 \cdot 5 + b_2$	$a_3 \cdot 5 + b_3 \cdot 3 + c_3$	$\mu_1$	$\mu_2$	$\mu_3$	$p_+^+(15, k)$
5	$1 \cdot 3 + 2$	$1 \cdot 5 + 0$	—	1	$\frac{11}{15}$		12
6	$2 \cdot 3 + 0$	$1 \cdot 5 + 1$	—	$\frac{4}{3}$	$\frac{4}{5}$		13
7	$2 \cdot 3 + 1$	$1 \cdot 5 + 2$	—	$\frac{5}{3}$	<b>1</b>		16
8	$2 \cdot 3 + 2$	$1 \cdot 5 + 3$	$1 \cdot 5 + 1 \cdot 3 + 0$	2	2	$\frac{14}{15}$	15
9	$3 \cdot 3 + 0$	$1 \cdot 5 + 4$	$1 \cdot 5 + 1 \cdot 3 + 1$	$\frac{7}{3}$	3	<b>1</b>	16
10	$3 \cdot 3 + 1$	$2 \cdot 5 + 0$	$1 \cdot 5 + 1 \cdot 3 + 2$	$\frac{8}{3}$	$\frac{8}{5}$	2	25
11	$3 \cdot 3 + 2$	$2 \cdot 5 + 1$	$1 \cdot 5 + 2 \cdot 3 + 0$	3	$\frac{9}{5}$	$\frac{5}{3}$	26
12	$4 \cdot 3 + 0$	$2 \cdot 5 + 2$	$1 \cdot 5 + 2 \cdot 3 + 1$	$\frac{10}{3}$	<b>2</b>	<b>2</b>	31
13	$4 \cdot 3 + 1$	$2 \cdot 5 + 3$	$1 \cdot 5 + 2 \cdot 3 + 2$	$\frac{11}{3}$	3		
			$2 \cdot 5 + 1 \cdot 3 + 0$			$\frac{9}{5}$	28
14	$4 \cdot 3 + 2$	$2 \cdot 5 + 4$	$1 \cdot 5 + 3 \cdot 3 + 0$	4	4		
			$2 \cdot 5 + 1 \cdot 3 + 1$			<b>2</b>	31

The minimum value can also be obtained by  $\mu_1$  as it is shown in the following

*Example 2.4.* Let  $N = 35 = 5 \cdot 7$ ,  $k = 10$ . Then

$$\begin{aligned} 10 &= 2 \cdot 5 + 0 & \mu_1 &= 2 - \frac{2}{5} = \frac{8}{5} \\ 10 &= 1 \cdot 7 + 3 & \mu_2 &= 2 \end{aligned}$$

Even when  $k \geq p + q$ , and so  $\mu_1, \mu_2$  and  $\mu_3$  intervene:

*Example 2.5.* Let  $N = 55 = 5 \cdot 11$ ,  $k = 20$ . Then

$$\begin{aligned} 20 &= 4 \cdot 5 + 0 & \mu_1 &= 4 - \frac{2}{5} = \frac{18}{5} \\ 20 &= 1 \cdot 11 + 9 & \mu_2 &= 8 \\ 20 &= 1 \cdot 11 + 1 \cdot 5 + 4 & \mu_3 &= 4 \end{aligned}$$

Finally,  $\mu_3$  can be obtained by an expression  $a_3q + b_3p + c_3$ , where  $a_3$  is not  $\left[\frac{k}{q}\right]$ :

*Example 2.6.* Let  $N = 143 = 11 \cdot 13$ ,  $k = 99$ . Then

$$\begin{aligned} 99 &= 7 \cdot 13 + 8 & \mu_1 &= 13 \\ 99 &= 9 \cdot 11 + 0 & \mu_2 &= 9 - \frac{2}{11} = \frac{97}{11} \\ 99 &= 6 \cdot 13 + 1 \cdot 11 + 10 & \mu &= 15 \\ 99 &= 5 \cdot 13 + 3 \cdot 11 + 1 & \mu &= 7 \\ 99 &= 4 \cdot 13 + 4 \cdot 11 + 3 & \mu &= 9 \\ 99 &= 3 \cdot 13 + 5 \cdot 11 + 5 & \mu &= 11 \\ 99 &= 2 \cdot 13 + 6 \cdot 11 + 7 & \mu &= 13 \\ 99 &= 1 \cdot 13 + 7 \cdot 11 + 9 & \mu &= 15 \end{aligned}$$

Hence  $\mu_3 = 7$  coming from  $99 = 5 \cdot 13 + 3 \cdot 11 + 1$ .

We now return to study the two remaining cases.

### 2.3. $k = aN$

This is the easiest case. It is clear that the minimum value of  $\mu$  is attained for  $k' = a$ ,  $t_1 = \dots = t_a = 1$ ,  $r = 2$ ,  $m_1 = m_2 = N$ . Then  $\mu = a - \frac{2}{N}$ , and so  $p_+^+(N, aN) = 1 + N(a - \frac{2}{N}) = aN - 1 = k - 1$ .

**Proposition 2.7.** *Let  $N = pq$ ,  $p < q$  odd primes. Then  $p_+^+(N, aN) = aN - 1$ .*

### 2.4. $k > N$ , $k$ not a multiple of $N$

It is also clear that in this case the value of  $p_+^+(N, k)$  depends on  $\left[\frac{k}{N}\right]$  and  $k - \left[\frac{k}{N}\right]$ . Let us divide  $k$  by  $N$  and write  $k = aN + N'$ . Since  $N' < N$ , we had already obtained the minimum  $\mu$  corresponding to  $k = N'$ . Let us call it  $\mu_0$  corresponding to certain  $k'_0, t_1, \dots, t_{k'_0}, r_0, m_1, \dots, m_{r_0}$ . Then in order to get the minimum  $\mu$  for  $N$  it suffices to take  $k' = k'_0 + a$ ,  $t_1, \dots, t_{k'_0}$ , and  $t_{k'_0+1} = \dots = t_{k'_0+a} = 1$ ,  $r_0, m_1, \dots, m_{r_0}$ . So the minimum  $\mu$  for  $k = N$  is  $\mu_0 + a$ , and hence

$$p_+^+(N, k) = 1 + N(\mu_0 + a) = 1 + N\mu_0 + Na.$$

Since  $p_+^+(N, N') = 1 + N\mu_0$ , we have  $p_+^+(N, k) = p_+^+(N, N') + Na$ .

**Proposition 2.8.** *Let  $N = pq$ ,  $p < q$  odd primes and  $k > N$ ,  $k$  not a multiple of  $N$ . Take the euclidean division  $k = aN + N'$ . Then  $p_+^+(N, k) = p_+^+(N, N') + Na$ .*

### 3. The computation of $p_-(N, k)$

From Theorem 3.1.3 in [2], the existence of a surface  $S \in \mathcal{K}_-(N, k)$  with topological genus  $g$  is equivalent to the existence of non-negative integers  $g', m_1, \dots, m_r, t_1, \dots, t_{k'}$ , with  $g' \geq 1, m_i \geq 2$  and  $t_j$  divisors of  $N$ , such that

- (1)  $\mu = g' + k' - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) > 0$
- (2)  $k = \sum_{j=1}^{k'} \frac{N}{t_j}$ .
- (3) If  $g' = 1$ , then  $\text{lcm}(m_1, \dots, m_r, t_1, \dots, t_{k'}) = N$

If the conditions (1)-(3) are satisfied,  $N\mu = g - 2 + k = p - 1$ . Observe that in this case the condition on the elimination property does not appear, and this fact along with  $g' \geq 1$ , shall make the analysis simpler than in the previous Section. The strategy for minimizing  $\mu$  is as in Section 2,  $k'$  will be determined by  $k$ , always  $g' = 1$ , and since  $g' + k' \geq 2$ , we have  $r = 0$  unless  $r = 1$ , if necessary, in order to get the condition (3) or  $\mu > 0$ .

We distinguish according to the values of  $p$  and  $q$  as in Section 2.

#### 3.1. $k < q$

(3.1.1)  $k = 1$ . Then  $k' = 1, t_1 = N$ . In order to have  $\mu > 0$  we need  $r = 1$ , and we take  $m_1 = p$ . So  $\mu = 1 - 1/p$ , and  $p_-(N, 1) = 1 + N(1 - 1/p) = N - q + 1$ .

(3.1.2)  $1 < k < p$ . Then  $k' = k, t_1 = \dots = t_k = N$ , and so  $r = 0$ . Hence  $\mu = k - 1$ , and  $p_-(N, k) = 1 + N(k - 1)$ .

(3.1.3)  $p \leq k$ . We divide  $k = ap + b$ , and the minimum  $\mu$  depends on  $b$  being 0 or not.

$k = ap$ . Then  $k' = a, t_1 = \dots = t_a = q$ , and by condition (3)  $r = 1, m_1 = p$ . Hence  $\mu = a - 1/p$ , and so  $p_-(N, k) = 1 + N(a - 1/p) = Na - q + 1$ .

$k = ap + b, b > 0$ . Then  $k' = a + b, t_1 = \dots = t_a = q, t_{a+1} = \dots = t_{a+b} = N, r = 0$ . So  $\mu = a + b - 1$ , and  $p_-(N, k) = 1 + N(a + b - 1)$ .

We have proved the following

**Proposition 3.1.** *Let  $N = pq, p < q$  odd primes, and  $1 \leq k < q$ . Then*

$$\begin{aligned}
 p_-(N, k) = & \begin{array}{ll} N - q + 1 & k = 1 \\ N(k - 1) + 1 & 1 < k < p \\ Na - q + 1 & p \leq k, k = ap \\ N(a + b - 1) + 1 & p \leq k, k = ap + b, b > 0. \end{array}
 \end{aligned}$$

#### 3.2. $q \leq k < N$

Now we have the divisions of  $k$  by  $p$  and by  $q$ .

- (I)  $k = a_1p + b_1, a_1 \geq 1$ .
- (II)  $k = a_2q + b_2, a_2 \geq 1$ .

For each of them, by (3.1.3) the minimum value of  $\mu$  is given by

- (I) if  $b_1 = 0$ ,  $\mu_1 = a_1 - \frac{1}{p}$   
     if  $b_1 > 0$ ,  $\mu_1 = a_1 + b_1 - 1$
- (II) if  $b_2 = 0$ ,  $\mu_2 = a_2 - \frac{1}{q}$   
     if  $b_2 > 0$ ,  $\mu_2 = a_2 + b_2 - 1$ .

Besides, if  $k \geq q + p$ , we can write  $k = a_3q + b_3p + c_3$ ,  $a_3, b_3 \geq 1$ , where  $a_3$  runs from 1 to  $\left[\frac{k}{q}\right]$ , and  $b_3p + c_3$  is the euclidean division of  $k - a_3q$  by  $p$ . The minimum value of  $\mu$  for this expression is obtained by  $k' = a_3 + b_3 + c_3$ ,  $t_1 = \dots = t_{a_3} = p$ ,  $t_{a_3+1} = \dots = t_{a_3+b_3} = q$ ,  $t_{a_3+b_3+1} = \dots = t_{a_3+b_3+c_3} = N$ . Since  $a_3, b_3 \geq 1$ ,  $k' \geq 2$ , and so  $r = 0$ ,  $\mu = a_3 + b_3 + c_3 - 1$ . We call  $\mu_3$  the minimum value of  $a_3 + b_3 + c_3 - 1$  when  $a_3$  runs from 1 to  $\left[\frac{k}{q}\right]$  with  $b_3 \geq 1$  and so we have

**Proposition 3.2.** *Let  $N = pq$ ,  $p < q$  odd primes, and  $q \leq k < N$ . Then*

$$p_-(N, k) = 1 + N(\min(\mu_1, \mu_2, \mu_3)),$$

for  $\mu_1$ ,  $\mu_2$  and  $\mu_3$  described above.

**3.3.  $k = aN$**

In this case  $k' = a$ ,  $t_1 = \dots = t_a = 1$ ,  $r = 1$ ,  $m_1 = N$ . Then  $\mu = a - 1/N$  and so  $p_-(N, k) = 1 + N(a - 1/N) = Na = k$ .

We have

**Proposition 3.3.** *Let  $N = pq$ ,  $p < q$  odd primes. Then*

$$p_-(N, aN) = aN$$

**3.4.  $k > N$ ,  $k$  not a multiple of  $N$**

Take  $k = aN + N'$ . Let  $\mu_0$  be the minimum  $\mu$  corresponding to  $k = N'$ , obtained by means of certain  $k'_0, t_1, \dots, t_{k'_0}, r_0, m_1, \dots, m_{r_0}$ . Then we take  $k' = k'_0 + a$ ,  $t_1, \dots, t_{k'_0}, t_{k'_0+1} = \dots = t_{k'_0+a} = 1$ ,  $r_0, m_1, \dots, m_{r_0}$ , and we have  $\mu = \mu_0 + a$ . Hence  $p_-(N, k) = 1 + N(\mu_0 + a) = 1 + N\mu_0 + Na = p_-(N, N') + Na$ .

**Proposition 3.4.** *Let  $N = pq$ ,  $p < q$  odd primes,  $k > N$  not a multiple of  $N$ , and take the euclidean division  $k = aN + N'$ . Then*

$$p_-(N, k) = p_-(N, N') + Na$$

**4. Final remarks**

The study in Sections 2 and 3, specially the case  $k \geq p + q$ , makes it clear that given an arbitrary odd  $N$ , there is no hope of having an explicit formula for  $p_+^+(N, k)$  or  $p_-(N, k)$ . Of course, for a given pair  $(N, k)$  these values can be obtained because all numbers appearing in the respective set of conditions (1)-(4) ((1)-(3), respectively) are bounded above. This fact was already noted in Example 3.1.10 of [2].

However, if  $k$  is “small” in terms of  $N$ , we can deduce the values of  $p_+^+(N, k)$  and  $p_-(N, k)$  from the results in Sections 2 and 3 as follows. Let  $N = p_1^{\alpha_1} \dots p_t^{\alpha_t}$ ,  $p_1 < \dots < p_t$  odd primes,  $t > 1$ ,  $\alpha_i \geq 1$ . We will deal with  $k < p_2$ ,  $k < p_1^2$  if  $\alpha_1 \geq 2$ . Then we have

**Proposition 4.1.** *Let  $N = p_1^{\alpha_1} \dots p_t^{\alpha_t}$ ,  $p_1 < \dots < p_t$  odd primes,  $t > 1$ ,  $\alpha_i \geq 1$ ,  $k < p_2$ ,  $k < p_1^2$  if  $\alpha_1 \geq 2$ . Then*

$$\begin{aligned}
 p_+^+(N, k) = & \begin{array}{ll} N - p_1 - N/p_1 + 1 & k = 1, \alpha_1 = 1 \\ N - N/p_1 & k = 1, \alpha_1 > 1 \\ N - N/p_1 + 1 & k = 2 \\ N(k - 2) + 1 & 2 < k < p_1 \\ N - N/p_1^{\alpha_1} & k = p_1 \\ Na - 2N/p_1^{\alpha_1} + 1 & p_1 < k, k = ap_1 \\ Na - N/p_1^{\alpha_1} + 1 & p_1 < k, k = ap_1 + 1 \\ N(a + b - 2) + 1 & p_1 < k, k = ap_1 + b, b > 1 \end{array}
 \end{aligned}$$

*Proof.* It suffices to follow the same argument as in subsection 2.1, substituting the value  $N/p = q$  by  $N/p_1$ .

The unique question to be noted comes from the elimination property. When  $k = 1$ , then  $k' = 1$ ,  $t_1 = N$ , and necessarily  $r = 2$  in such a way that  $\{N, m_1, m_2\}$  has the elimination property and  $1/m_1 + 1/m_2$  is as big as possible. If  $\alpha_1 = 1$ , it is clear that  $m_1 = p_1$ ,  $m_2 = N/p_1$ , satisfy these conditions. On the other hand, if  $\alpha_1 > 1$ , the maximum is obtained from  $m_1 = p_1$ ,  $m_2 = N$ . Then  $\mu$  is respectively  $1 - 1/p_1 - 1/(N/p_1)$  and  $1 - 1/p_1 - 1/N$ . From here we obtain the value of  $p_+^+(N, 1) = 1 + N\mu$ .

In the same way, when  $k = p_1$ , the values of  $m_i$  are  $m_1 = p_1^{\alpha_1}$ ,  $m_2 = N$ ; for  $k = ap_1$  ( $a \geq 2$ ),  $m_1 = m_2 = p_1^{\alpha_1}$ ; and finally for  $k = ap_1 + 1$ , we have  $m_1 = p_1^{\alpha_1}$ .  $\square$

The computation of  $p_-(N, k)$  does not require to keep in mind the elimination property, and so Proposition 3.1 applies directly by changing  $N/p = q$  by  $N/p_1$ . Only for  $k = ap_1$ , since  $t_1 = \dots = t_a = N/p_1$ , we need  $m_1 = p_1^{\alpha_1}$ , and so  $\mu = a - 1/p_1^{\alpha_1}$ , and  $p_-(N, K) = 1 + N(a - 1/p_1^{\alpha_1})$ . Thus we have

**Proposition 4.2.** *Let  $N = p_1^{\alpha_1} \dots p_t^{\alpha_t}$ ,  $p_1 < \dots < p_t$  odd primes,  $t > 1$ ,  $\alpha_i \geq 1$ ,  $k < p_2$ ,  $k < p_1^2$  if  $\alpha_1 \geq 2$ . Then*

$$\begin{aligned}
 p_-(N, k) = & \begin{array}{ll} N - N/p_1 + 1 & k = 1 \\ N(k - 1) + 1 & 1 < k < p_1 \\ Na - N/p_1^{\alpha_1} + 1 & p_1 \leq k, k = ap_1 \\ N(a + b - 1) + 1 & p_1 \leq k, k = ap_1 + b, b > 0. \end{array}
 \end{aligned}$$

Finally, we obtain the corresponding result for low values of  $k$ . Since for any odd  $N$  with the above expression,  $p_2 \geq 5$ ,  $p_1^2 \geq 9$ , the former results hold for  $k \leq 4$ . Hence we have

**Proposition 4.3.** *Let  $N = p_1^{\alpha_1} \dots p_t^{\alpha_t}$ ,  $p_1 < \dots < p_t$  odd primes,  $t > 1$ ,  $\alpha_i \geq 1$ . For  $k = 1, 2, 3, 4$  the values of  $p_+^+(N, k)$  and  $p_-(N, k)$  are given in the following table:*

$k$	$p_+^+(N, k)$	$p_-(N, k)$
1	$N - p_1 - N/p_1 + 1$ $\alpha_1 = 1$ $N - N/p_1$ $\alpha_1 > 1$	$N - N/p_1 + 1$
2	$N - N/p_1 + 1$	$N + 1$
3	$N + 1$ $p_1 > 3$ $N - N/3^{\alpha_1}$ $p_1 = 3$	$2N + 1$ $p_1 > 3$ $N - N/3^{\alpha_1} + 1$ $p_1 = 3$
4	$2N + 1$ $p_1 > 3$ $N - N/3^{\alpha_1} + 1$ $p_1 = 3$	$3N + 1$ $p_1 > 3$ $N + 1$ $p_1 = 3$

*Proof.* The result is obtained directly by substituting the value of  $k$  in Propositions 4.1 and 4.2. □

Recall that for the remaining odd numbers  $N$ , namely prime or prime-power  $N$ , the corresponding results were obtained in [2] and [4]. So that, the computation is complete for all odd numbers  $N$ .

In a forthcoming paper we complete the computation of  $p_+^+(N, k)$ ,  $p_+^-(N, k)$  and  $p_-(N, k)$  when  $N = 2q$ , for  $q$  an odd prime, what will clarify the global case  $N$  even.

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