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Departamento de Matemática Aplicada

TESIS DOCTORAL

Escalas de espacios y técnicas de semigrupos para el estudio de ecuaciones de evolución

(Scales of spaces and semigroup techniques for the study of evolution equations)

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Carlos Quesada González

Director

Aníbal Rodríguez Bernal

Madrid, 2016

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Los sabios son sabios porque cuando habló alguien más sabio supieron estar escuchando
- El Chojin, *Para mi es como...*

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Completar una tesis es un camino largo, a veces duro, pero *cualquier cosa a la que el autor le ponga pasión es algo que merece ser disfrutado*[1]. Y yo lo he disfrutado mucho porque he tenido la suerte de estar rodeado de un montón de personas a las que os quiero agradecer. Evidentemente, los más importantes a la hora de lograrlo son aquellos que se relacionan directamente con la investigación que he realizado y por eso tienen un lugar destacado al final de estos agradecimientos. Pero *en la carrera la fatiga es normal*[2], uno no puede estar las 24h dedicado a ello. Me encanta enfrentarme a un problema nuevo, no entenderlo, seguir, no entenderlo, seguir... y al final entenderlo, pero *hay que encontrar el punto justo entre perseguir más triunfos y centrarse en disfrutar los frutos*[3] y es por eso que también son fundamentales todas las personas que en algunos casos ni siquiera entienden qué haces, pero están en tu vida y comparten momentos de distracción, ocio y descanso. Durante estos años realizando la tesis he sido muy feliz gracias a mucha gente y, por serlo, he mantenido la ilusión y las ganas de trabajar. Podría escribir una página de cada uno de vosotros, pero perdonadme si os dedico sólo unas breves líneas, por no hacer unos agradecimientos más grandes que la propia tesis.

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[1] Siéntense y disfruten, *Cosas que pasan, que no pasan y que deberían pasar*, El Chojin
[2] Vivir para contarlo, *Vivir para contarlo*, Violadores del Verso
[3] Siéntense y disfruten, *Cosas que pasan, que no pasan y que deberían pasar*, El Chojin

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[4]Gracias, *Poesía Difusa*, Nach

y Espe y yo ya decidimos cambiar la disposición de las mesas, para ponerlas en círculo. Así, lo difícil hubiera sido no hacernos amigos o hablar. Divertidas conversaciones con Andrea y Espe, sobre básicamente cualquier tema. Por suerte alternadas con momentos de silencios, pero silencios agradables, en un ambiente de familiaridad perfecto para trabajar "a tu bola". Cuando se fue Espe, la sustituyó Manu. También hijo de Arrieta y en la misma mesa. Tú no tan hablador, pero cuando abres la boca... no das puntada sin hilo. No os cambio por ningún otro despacho, gracias a los tres.

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[5]El Chojin, *Sólo para adultos*, Sólo para adultos.
[6]Kase-O, *Mierda*, Mierda
[7]El Chojin, *Pelea*, 100% = 10.000 (mi estilo).

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Más importante aún, creo, es la manera que tienes de concebir las matemáticas, y la manera en la que hay que contarlas. Tienes la idea, o por lo menos conmigo, de hacer cosas de una cierta complicación a base de darle una y mil vueltas a una serie de herramientas no muy numerosas. Lejos de mandar al lector a artículos ilegibles o resolver una duda con un "esto es sabido" o "clásico", prefieres re-explicar algo ya hecho si eso va a ayudar al lector a entenderlo, prefieres re-escribir algo que ya hemos redactado, si es que con eso mejora. A veces incluso prefieres volver a la versión anterior. Y me parece que así es como debería de ser siempre. Uno puede pensar que esta opinión la tengo precisamente porque, como director de mi trabajo, me has influido. Y por supuesto es así y además me encanta que lo sea, pero no sólo. De siempre he visto la ciencia como la búsqueda de la verdad. Y las matemáticas como el máximo exponente de ello ya que no dependes de nada extra, solo de razonamientos, de preguntarte *¿verdad o no verdad? esa es la cuestión*[8]. Y en ese sentido, cómo podemos aspirar a probar cosas y transmitirlas si no somos absolutamente claros, si no revisamos una y otra vez si está bien y si está bien explicado. Si de Cholewa he dicho que es el ejemplo de cómo debería ser una estancia, tú sin duda eres un ejemplo de cómo deben de ser las matemáticas y los directores de tesis. Desde lo más sincero te agradezco todo lo que has hecho por mi estos años.

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[8]Frank-t, *Verdad*, 90 kilos

ramente interesante, pero el caso es que en aquellas clases de matemáticas recreativas lo eran. No puedo ni imaginar cómo serían las matemáticas que de verdad son bonitas contigo. Arriba hay muchas condiciones necesarias; mucha gente sin la cual no podría haber hecho el doctorado, o no igual. Pero no hay ninguna condición suficiente; alguien que una vez conocido garantizase que iba a hacer una tesis en matemáticas. Y esa eres tú. Ha tenido que pasar el tiempo, y ha habido decisiones intermedias, muchas otras cosas que me han gustado, pero creo que la decisión estaba ya tomada en cierto sentido cuando salí de tus clases y que de hecho "tú decidiste por mi" al hacerme que me gustasen tanto las mates. Deseo a todos los niños que tengan en algún momento un maestro que les enseñe a disfrutar así de alguna de las ramas del conocimiento. Gracias.

Contents

Introduction

0.1 Motivation

Partial differential equations involving time are usually referred to as evolution equations, where the underlying idea is that the solution evolves in time from a given initial data. Among them, we find the parabolic differential equations which will be the main topic of discussion in this thesis.

The archetypical parabolic equation is of course the heat equation

$$u_t - \Delta u = 0 \qquad u(0) = u_0 \tag{0.1.1}$$

for $t > 0$, where $-\Delta$ denotes the Laplacian, u_t stands for $\frac{\partial u}{\partial t}$ and $x \in \mathbb{R}^N$ or in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with, say, Dirichlet boundary conditions. We can also consider other parabolic problems by replacing the Laplacian $-\Delta$ by some other differential operator A with similar properties to $-\Delta$, that is, an elliptic operator. The solution evolves from a certain initial data u_0 in a suitable chosen Banach space X, thus it is reasonable to think that the solution is unique and then given by an operator applied to the initial data. That is, the solution to

$$u_t + Au = 0, \quad u(0) = u_0, t > 0,$$

is given by $u(t) = S(t)u_0$ since $u(t)$ is uniquely determined by u_0, where for all $t \geq 0$, $S(t) : X \to X$ is a C^0 semigroup, that is, a family $\{S(t)\}_{t\geq 0}$ of continuous linear operators in the Banach space X such that

$S(0) = I$;

$S(t)S(s) = S(t+s)$ for $t \geq 0$, $s \geq 0$;

$S(t)u \to u$ as $t \to 0^+$ for all $u \in X$.

The first and last property reflect the fact that the initial data is attained, while the second one is because of the uniqueness of the solution. The main operator A in the equation and the semigroup $S(t)$ are related by $-Au = \lim_{t\to 0^+} \frac{1}{t}(S(t)u - u)$, and its domain $D(A)$ consists on all $u \in X$ such that the limit exists. In this situation, $-A$ is called the infinitesimal generator of the semigroup. In general $D(A) \subsetneq X$.

It can be considered that this approach to the study of parabolic differential equations has its origin the milestone paper by E. Hille [33] in 1942, that would inspire many others into developing this theory in the following decades. Prior to that, Gelfand [27] had studied one parametrical groups in normed spaces, giving some properties that semigroups preserve, such as a representation for a group by means of the exponential of $-A$ or the relation between a group and its generator. In his paper, Hille continued and extended the work in [27], particularizing for semigroups. Many representations for the semigroup were given, some of them involving the resolvent of the generator, which proved very useful when applying to partial differential equations.

Later on, Hille [34], Yosida [54], Phillips [45] and [35] (a revision and extension of [34]) set the foundations of what is frequently called the Hille-Yosida theory, that is, the study of parabolic differential equations by means of semigroup theory. This functional setting enabled the study of particular PDE problems from an abstract integral equation given by the Variation of Constants Formula. Similarly to what is done in ODE's, the problem

$$u_t + Au = f(u), \qquad u(0) = u_0$$

is studied through its abstract counterpart

$$u(t;u_0) = S(t)u_0 + \int_0^t S(t-\tau)f(u(\tau;u_0))\,d\tau, \qquad 0 < t \leq T.$$

Not much after that, Balakrishnan [7], Kato [36] and Yosida [55] among others started considering fractional powers of an operator A because it proved very useful to associate a family of spaces to the operator which later on was used to solve particular parabolic problems. It was proved that the fractional power A^s, $0 \leq s \leq 1$ can be constructed whenever $-A$ is the infinitesimal generator of a semigroup. The study of such power A^s in a Banach space X in turn lead to the construction of a family of spaces between $X^0 := X$ and $X^1 := D(A)$ associated to A^s, satisfying $X^\alpha \hookrightarrow X^\beta$ for $\alpha \geq \beta$. The scale X^s, $0 \leq s \leq 1$ constructed from fractional powers of operators is called the fractional power scale. The definition of the operators A^s and the spaces X^s can be extended to any $s \geq 0$. All these results together with some other by Sobolevskii [38] were somehow gathered by Komatsu in [37] where he studied the fractional powers of operators and the associated spaces under an unified point of view.

In the following years, many other authors continued and developed the topic. We shall mention among others Ladyzhenskaya [39], Friedman [23], Ball [8], Weissler [52], Henry [31], Pazy [44], [11] as they stand out in the study of abstract evolution equations. In particular, the books [23, 31, 44] became main references in parabolic equations, as they systematically use the scale of fractional power spaces described above to the study of particular PDE problems. Results on local existence, uniqueness, regularity and smoothing effect for both linear and non-linear problems were given. Blow-up was also studied in the non-linear case.

More recently, the study of parabolic problems in scales of spaces has been continued by authors such as Lunardi [40], Triebel [50] and Amann [1, 2]. We want to highlight [2] as it continues and extends the study of general procedures for associating a scale of spaces to an operator A. There are different ways of constructing such scales from an operator. A particular case is the already mentioned fractional power scale X^s. Another scale considered in [2] is the interpolation scale E^α. In this case, in order to construct intermediate spaces E^α, $0 \leq \alpha \leq 1$ between $E^1 := D(A)$ and $E^0 := X$ an interpolation method (such as complex or real interpolation, see [50]) is used. As for the case of fractional power scale, these spaces satisfy $E^\alpha \subset E^\beta$, $\alpha \geq \beta$ and the construction can be extended to $\alpha \geq 0$.

Both the interpolation and fractional power scales can be considered for indexes $s \geq 0$, $\alpha \geq 0$. In [2] moreover both scales are not only considered for positive indexes, they are also extended to spaces with negative indexes that contain X, in other words the scale is *extrapolated* to negative indexes, obtaining the negative side of the scales. When X is reflexive, the negative side of the scale is furthermore described in terms of duals of certain spaces. This provides a nice representation of what elements are contained in those spaces, and it is also a tool to solve problems in weaker spaces by means of duality, which, when applied to particular problems, unifies and recovers the concept of weak or very weak solutions of PDE problems.

The fractional power scale and the interpolation scale do not coincide in general, but in many particular examples (when the operator from which they are constructed has bounded imaginary powers) these two scales coincide.

Considering parabolic problems in each element of a scale of spaces proved to be very convenient because when applying the theory to concrete problems, it is very common that, depending on the regularity, the initial data can be taken from a space, which is chosen among many spaces in the same scale, and thus it is natural to consider the problem in such scale. The advantage of the interpolation and fractional power scales above is that, for any given a problem, they can always be constructed, and even more, they frequently turn out to be well known spaces such as Sobolev or Bessel spaces (spaces constructed from the Lebesgue spaces and endowed with distributional derivatives, see [31, p. 35] for details). However, when dealing with particular examples, sometimes we can consider other scales specific for that problem. One example of this is the above mentioned linear heat equation (0.1.1) in $\mathbb{R}^N$ which can be solved with initial data in any of the Lebesgue spaces $L^q(\mathbb{R}^N)$, for $1 \leq q \leq \infty$. Furthermore, the solution remains in the same space and in fact enters in any other $L^p(\mathbb{R}^N)$ space with $p > q$ satisfying, for some constants $M_{p,q}, \mu_0 > 0$, that

$$\|S_{-\Delta}(t)u_0\|_{L^p(\mathbb{R}^N)} \leq \frac{M_{p,q}e^{\mu_0 t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}}\|u_0\|_{L^q(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in L^q(\mathbb{R}^N).$$

These are the classical estimates for the heat equation, see for example [14].

The fact that a problem can be set in a family of spaces in a scale is an advantage since, among many things, we can fit initial data in adequate spaces of the scale and study at the same time well-posedness for problems with either linear or nonlinear terms. Not only initial data can be considered in different spaces for the same problem, but also many different problems can be treated at once with the same approach, as these problems share many properties that lay in the same abstract functional framework. Furthermore, other properties can also obtained from this abstract framework, such as regularization effect of the solution or the behaviour of the solution with respect to some norm. More precisely, the solutions to nonlinear problems are local in time, so for some t^* it might happen that as $t \to t^*$ some norm of the solution goes to infinity, this is called blow-up. The rate at which the solution goes to infinity can be studied through blow-up estimates. It is thus natural and valuable to study parabolic equations from this point of view.

In that spirit, all the scales considered above can be described under the following unified abstract setting. Let $\{X^\gamma\}_{\gamma\in\mathcal{J}}$ be family of Banach spaces where $\mathcal{J}$ is an interval of real indexes. The norm of the space X^γ is denoted by $\|\cdot\|_\gamma$. Observe that so far we have used the symbol X^α to denote the fractional power scale however, from now on, we are now denote in a generic scale of spaces.

We assume that we have $\{S(t) : t \geq 0\}$ a C^0 semigroup in each of the spaces of the family $\{X^\gamma\}_{\gamma\in\mathcal{J}}$ (in other words we say we have a C^0 semigroup in the scale) such that for all $\gamma, \gamma' \in \mathcal{J}$, $\gamma' \geq \gamma$ and $T > 0$ we have

$$\|S(t)\|_{\mathcal{L}(X^\gamma, X^{\gamma'})} \leq \frac{M_0}{t^{\gamma'-\gamma}}, \qquad 0 < t \leq T, \tag{0.1.2}$$

where $M_0 := M_0(\gamma, \gamma', T)$ is a positive constant which can be chosen uniformly for T in bounded time intervals.

Observe that no further relation is assumed among the spaces of the scale unless explicitly stated. For example, some times we may assume that the spaces are "nested", that is, for all $\alpha, \beta \in \mathcal{J}$ with $\alpha \geq \beta$ we have

$$X^\alpha \subset X^\beta$$

with continuous inclusion. Both the fractional power scale and the interpolation scale are nested scales, whereas the Lebesgue scale is not nested in $\mathbb{R}^N$.

In this setting lay the results from [47], where linear parabolic problems of the form

$$\begin{cases} u_t + Au = Pu, & t > 0 \\ u(0) = u_0, \end{cases} \tag{0.1.3}$$

are studied using the unified abstract setting described above. For it, the following abstract integral given by the Variation of Constants Formula

$$u(t; u_0) = S_P(t)u_0 = S(t)u_0 + \int_0^t S(t-\tau)Pu(\tau; u_0)\, d\tau, \qquad 0 < t \leq T \tag{0.1.4}$$

is considered, where P is a linear operator defined on some spaces of the scale. In order to obtain existence and uniqueness of solutions the problem is studied as a perturbation of the problem when $P = 0$. Existence and uniqueness of solutions of (0.1.4) is obtained for a *range of spaces* where the initial data can be taken, that is, the initial data is taken from X^γ where γ is in a range of indexes. The regularity of the solution as well as continuity of solutions with respect to the perturbation are also studied. Then, the results are applied to solving second order linear parabolic problems (see (0.2.1) below) in scales of spaces such as the Lebesgue or Bessel scale. This paper can be regarded as the starting point and motivation for the present thesis.

0.2 Aim

In this thesis we will go on with the study of parabolic problems using similar ideas to the ones in [47]. Firstly, we will use the perturbation techniques therein in order to study linear parabolic equations. On one hand we will continue the study of linear second order problems for initial data in a scale with low regularity properties called the Locally Uniform Spaces which will be defined later on. We will extend the results in [47] for such spaces. On the other hand, we will study fourth order linear equations in bounded and unbounded domains, as a natural way of expanding the applications of the perturbation theory.

Secondly, we jump to the study of nonlinear problems. We start by constructing the abstract perturbation results for parabolic equations in scales of spaces which will resemble the spirit of those in [47] for the linear case. Then we apply them to nonlinear PDE problems.

To be more precise, we will first use perturbation results from [47] to extend previous results (see e.g. [3], [5], [47]) for second order problems of the form

$$\begin{cases} u_t - \sum_{k,l=1}^{N} a_{kl}(x)\partial_k\partial_l u + \sum_{j=1}^{N} b_j(x)\partial_j u + c(x)u = 0 & x \in \mathbb{R}^N, \quad t>0 \\ u(0,x) = u_0(x) \qquad x \in \mathbb{R}^N, \end{cases} \tag{0.2.1}$$

with low regularity initial data and coefficients. Existence of solution will be proved for less restrictive conditions than the ones in previous results and for initial data to be chosen from a larger range of spaces.

On the other hand, we will focus on studying fourth order linear parabolic problems

$$\begin{cases} u_t + \Delta^2 u + Pu = 0, & x \in \Omega, t>0 \\ u(0) = u_0 \end{cases} \tag{0.2.2}$$

where P is a space dependent linear perturbation and Ω is either $\mathbb{R}^N$ or a bounded domain (in such case, boundary conditions appear in the problem as well). We will give results about existence, uniqueness, smoothing estimates and robustness with respect to changes in the perturbation P. These results will then be extended naturally with analogous arguments to higher order equations.

We later study the case of nonlinear perturbations. Given a semigroup in a scale as above, we will consider a nonlinear mapping satisfying

$$f : X^\alpha \to X^\beta \text{ for some } \alpha, \beta \in \mathcal{J} \text{ with } 0 \le \alpha - \beta < 1. \tag{0.2.3}$$

and the following growth condition

$$\|f(u)\|_\beta \le L\big(1 + \|u\|_\alpha^\rho\big), \quad u \in X^\alpha.$$

Then, our main goal is the analysis of the abstract integral equation

$$u(t;u_0) = S(t)u_0 + \int_0^t S(t-\tau) f(u(\tau;u_0))\, d\tau, \qquad 0 < t \le T, \tag{0.2.4}$$

where u_0 is taken from some space X^γ in the scale. Notice that (0.2.4) is the corresponding variation of constants formula for mild solutions of the following PDE problem

$$u_t + Au = f(u) \quad u(0) = u_0. \tag{0.2.5}$$

The main questions to analyse are:

(1) The range of γ for which (0.2.4) has a solution (in a suitable class to be defined) for any $u_0 \in X^\gamma$ and for some $T = T(u_0)$.

(2) Uniqueness and continuous dependence with respect to initial data of the solutions in (1).

(3) Smoothing effect: in which other spaces $X^{\gamma'}$ does the solution enter for $t > 0$ and estimates of the norm $\|\cdot\|_{\gamma'}$ of the solution.

(4) Estimate the blow-up rate and existence time if the solution ceases to exist in a finite time.

After developing these abstract techniques and results, we apply the results obtained in this abstract approach to particular equations. In particular, we consider the following general problem

$$\begin{cases} u_t + (-\Delta)^m u = f(u), & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N \end{cases} \tag{0.2.6}$$

where f is a nonlinear function of the form $D^b(h(x, D^a u))$ for some function h and for $m \in \mathbb{N}$. Certain choices of f will lead to well known problems such as the Cahn-Hilliard equation

$$\begin{cases} u_t + \Delta^2 u + \Delta h(x, u) = 0, \ t > 0, \ x \in \mathbb{R}^N, \\ u(0, x) = u_0(x), \ x \in \mathbb{R}^N. \end{cases}$$

which are solved as a particular case of (0.2.6).

The last application of the abstracts results is to the strongly damped wave equation

$$\begin{cases} w_{tt} - \Delta w_t + w_t - \Delta w = h(x, w), \ t > 0, \ x \in \mathbb{R}^N, \\ w(0, x) = w_0(x), \quad w_t(0, x) = z_0(x), \ x \in \mathbb{R}^N. \end{cases} \tag{0.2.7}$$

0.3 Content

The thesis is divided in two parts. Part I is focused on studying the linear problem (0.1.3), focusing on applying the abstract perturbation techniques from [47] to solve (0.2.1) and (0.2.2). Part II is devoted to the study of (0.2.5) and the application of these results to nonlinear PDE problems such as (0.2.6) and (0.2.7).

In Part I we start by recalling the results from [47] which we will use throughout this part and also revisit some results from [2] about the construction of the above mentioned fractional power scale and interpolation scale from a given operator A such that $-A$ generates a semigroup. Both scales will prove very useful as when particularized for many operators, such as the Laplacian, both scales yield well known spaces such as Bessel spaces.

Once the problem abstract framework is set, we are ready to study parabolic equations. We want to consider the abstract perturbed problem

$$u_t + Au = Pu$$

by means of the Variation of Constants Formula (0.1.4) and apply the results in [47] to particular problems.

We now introduce the scale in which the problems will be set, namely the Lebesgue scale, the Bessel scale and in uniform scales. Lebesgue spaces $L^p(\mathbb{R}^N)$, $1 \le p < \infty$ are the set of p^{th}-power integrable functions.

From the Lebesgue spaces one can construct for $\alpha \in \mathbb{R}$, $1 \le q \le \infty$ the Bessel space $H^{\alpha,q}(\mathbb{R}^N)$, see [31, p. 35] for details. When $\alpha = k \in \mathbb{N}$, the Bessel spaces coincide with the Sobolev spaces $W^{k,p}(\mathbb{R}^N)$, $1 \le p \le \infty$, $k \in \mathbb{N}$, composed of functions $f \in L^p(\mathbb{R}^N)$ which have distributional derivatives of order less or equal to k and all of them are p^{th}-power integrable.

We now consider some low regularity spaces called the locally uniform spaces, which have some local integrability properties and no asymptotic decay as $|x| \to \infty$ whatsoever. To be more precise, for $1 \le p < \infty$ let $L^p_U(\mathbb{R}^N)$ denote the locally uniform space composed of the functions $f \in L^p_{loc}(\mathbb{R}^N)$ such that there exists $C > 0$ such that for all $x_0 \in \mathbb{R}^N$

$$\int_{B(x_0,1)} |f|^p \le C \tag{0.3.1}$$

endowed with the norm

$$\|f\|_{L^p_U(\mathbb{R}^N)} = \sup_{x_0 \in \mathbb{R}^N} \|f\|_{L^p(B(x_0,1))}$$

(for $p = \infty$, $L^\infty_U(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$). Also define $\dot{L}^p_U(\mathbb{R}^N)$ as the closed subspace of $L^p_U(\mathbb{R}^N)$ consisting of elements which are translation continuous with respect to $\|\cdot\|_{L^p_U(\mathbb{R}^N)}$ (for $p = \infty$, $\dot{L}^\infty_U(\mathbb{R}^N) = BUC(\mathbb{R}^N)$).

Then, from the locally uniform spaces one can construct for $\alpha \in \mathbb{R}$, $1 \le q < \infty$ the Bessel uniform spaces $\dot{H}^{\alpha,q}_U(\mathbb{R}^N)$ in a similar way as the standard Bessel spaces are constructed from the Lebesgue spaces.

In [47] the abstract results were already applied to problems where the main operator is of the form $Au = -div(a(x)\nabla u)$ in Lebesgue and Bessel spaces for bounded and unbounded domains. We will extend the results in [47] in two ways. On one hand we will extend and improve the results concerning second order problems in locally uniform spaces, on the other hand, we will consider fourth order operators in both bounded and unbounded domains.

We first apply the results to second order problems in locally uniform spaces. In particular, we study the problem

$$\begin{cases} u_t - \sum_{k,l=1}^N a_{kl}(x)\partial_k\partial_l u + \sum_{j=1}^N b_j(x)\partial_j u + c(x)u = 0 & x \in \mathbb{R}^N, \quad t > 0 \\ u(0,x) = u_0(x) \qquad x \in \mathbb{R}^N, \end{cases} \tag{0.3.2}$$

where the real coefficients of the elliptic principal part of the equation are assumed to be bounded and uniformly continuous, that is, $a_{kl} \in BUC(\mathbb{R}^N)$. The lower order coefficients are assumed to belong to locally uniform Lebesgue spaces. In particular we will assume that for $j = 1, ..., N$, $\|b_j\|_{\dot{L}_U^{p_j}(\mathbb{R}^N)} \leq R_j$ and $\|c\|_{\dot{L}_U^{p_0}(\mathbb{R}^N)} \leq R_0$, where $p_j > N$ and $p_0 > \frac{N}{2}$.

Parabolic problems like (0.3.2) with coefficients in uniform spaces have been considered before; see e.g. [3], [5], [47] and references therein. For example, the results in [3] allow to solve (0.3.2) in Lebesgue spaces $L^q(\mathbb{R}^N)$ assuming additionally that

$$p_j \geq q > 1, \quad \text{for } j = 0, \dots, N. \tag{0.3.3}$$

These results were later used in [5] to solve (0.3.2) in uniform spaces $L_U^q(\mathbb{R}^N)$, under the same restrictions, see [5, Section 5], and later on, in [47, Section 6.2] with different techniques. Because of the restrictions above in the coefficients the result in [5, 47] just allowed to take initial data in $\dot{H}_U^{2\gamma,q}(\mathbb{R}^N)$ for some $\gamma \geq 0$.

Here, we remove restriction (0.3.3), allowing a larger class of initial data, in particular γ can be even negative. When the additional assumptions (0.3.3) above are imposed, the results from Theorem 5.3 in [5] and Theorem 30 in [47] are recovered. Finally we will study the continuity of the semigroups with respect to small changes in the lower order coefficients of (0.3.2).

Continuing with application of the theory to particular examples, we then focus on fourth order equations in $\mathbb{R}^N$. We will consider

$$\begin{cases} u_t + \Delta^2 u = Pu, & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N \end{cases} \tag{0.3.4}$$

with u_0 a suitable initial data defined in $\mathbb{R}^N$ and P a linear space dependent perturbation of the form $Pu := \sum_{a,b} P_{a,b}u$ with

$$P_{a,b}u := D^b(d(x)D^a u) \qquad x \in \mathbb{R}^N \tag{0.3.5}$$

for some $a, b \in \{0, 1, 2, 3\}$ such that $a+b \leq 3$, where D^a, D^b denote any partial derivatives of order a, b, and $d \in L_U^p(\mathbb{R}^N)$, $1 \leq p < \infty$ defined as in (0.3.1).

The main goal is to solve the problem (0.3.4) for large classes of initial data u_0. In particular, we will consider for initial data the standard Lebesgue space, $L^q(\mathbb{R}^N)$, $1 < q < \infty$, or Bessel-Lebesgue spaces $H^{\alpha,q}(\mathbb{R}^N)$, with $1 < q < \infty$, $\alpha \in \mathbb{R}$ and even uniform Bessel spaces $\dot{H}_U^{\alpha,q}(\mathbb{R}^N)$ introduced above. Given such classes of initial data and perturbations we also find suitable smoothing estimates on the solutions of (0.3.4).

Previous results on the topic when $P = 0$ can be found e.g. in [21, 22] and [20, 10]. There, the solution of problem (0.3.4) is described as the convolution of the initial data with the self-similar fundamental kernel for the bi–Laplacian operator, which satisfies suitable Gaussian bounds.

We start by analysing the case when $P = 0$. In order to use the approach described so far we start by finding an adequate scale of spaces associated to the main operator, that is Δ^2. As explained above, we could construct the fractional power and the interpolation scales from Δ^2 from scratch, however we are going to use the scales associated to the Laplacian $-\Delta$. More precisely we are going to prove that the fractional power and interpolation scale associated to Δ^2 is in fact the same as the one associated to $-\Delta$. In order to do it, we first use some information about powers of operators from [37] to prove that (under some conditions) the power of an operator that generates an analytic semigroup in a scale also generates an analytic semigroup in the same scale and satisfies some smoothing estimates between spaces of that scale.

Then, we particularize this procedure for two settings. On one hand, we consider the bi-Laplacian Δ^2 in $L^p(\mathbb{R}^N)$. The scale will be proved to be the same as the one associated to $-\Delta$ in $L^p(\mathbb{R}^N)$, that is the Bessel-Lebesgue scale. To prove that the same scale of spaces available for $-\Delta$ can be used for Δ^2 we use resolvent estimates and specific information about the spectrum set of both operators. Since we are dealing with $-\Delta$ in the Lebesgue-Bessel scale, the resolvent estimate is known already, see for example [31, Section 1.3].

On the other hand we consider the bi-Laplacian Δ^2 in uniform spaces, and now some extra work is needed. Now, even though is was known that $-\Delta$ generates an analytic semigroup (see Proposition 2.1, Theorem 2.1 and Theorem 5.3 in [5]), the adequate resolvent estimate was not known. We prove it following some of the ideas in [31] for the standard (non-uniform) Lebesgue-Bessel scale. After doing so, we are able to establish that the problem (0.3.4) can be set in the uniform Bessel-Lebesgue scale.

Once the problem is set in either of the scales above, we can apply the abstract results to obtain existence of solution, regularity and the rest of the properties for problem (0.3.4) with $P = 0$.

We can now use these results to consider the problem when $P \neq 0$. In [16] results were proved for $P \neq 0$ in Bessel-Lebesgue spaces. By means of resolvent estimates for $\Delta^2 + P$, the authors proved the well posedness of (0.3.4) with $Pu = d(x)u$, that is, a perturbation as in (0.3.5) with $a, b = 0$. They also found suitable smoothing estimates on the solutions.

Here, instead of relying on elliptic resolvent estimates for the operators $\Delta^2 + P$, with P as in (0.3.5) that acts between two spaces of the scale, we use the perturbation techniques from [47]. With these ingredients we obtain a perturbed semigroup which gives the solution to the equation (0.3.4) with $P \neq 0$. Such perturbed semigroup inherits some of the smoothing estimates of the original one in some of the spaces of the scale which are determined by the perturbation P itself, that is by a, b and the regularity of $d(x)$.

After considering the different types of possible perturbations, we will also study how to combine more than one perturbation, establishing all possible ways of combining perturbations and giving an existence and regularity result for it.

Even though our focus is on fourth order equations, the same techniques can be extended to higher order operators to obtain analogous results, so we will briefly explain this as well.

Finally, the problem is considered in bounded domains. We first study the Bessel scale with Neumann boundary conditions. This scale incorporates information about the boundary conditions which varies depending on the regularity (in other words, the index in the scale). Then we apply the perturbation results into this setting in a similar way as we did in the unbounded case and solve the problem in the integral sense of the Variation of Constants Formula. Then, given a perturbation, we want to re-read the integral equation in terms of an actual PDE problem. When the perturbation is defined in the interior of the domain, the problem is handled in the same way as it was done in $\mathbb{R}^N$ and the results we obtain are analogous to the ones for $\mathbb{R}^N$. However, being now in a bounded domain allows to consider perturbations defined on the boundary in a similar way to what was done in [47] for second order problems. However, dealing now with fourth order problems allows new kinds of boundary perturbations. In this situation, it is often the case that the perturbation in the abstract integral equation corresponds to some boundary condition in the associated PDE problem. In fact, it might happen that one of such perturbations correspond to terms in the boundary conditions and also in the equation. We illustrate some examples in detail, giving the full correspondence of the integral equation and the PDE in those cases. However, a full description of the corresponding PDE problem for all possible perturbations remains for future work.

We now exhibit Theorem 5.2.10 in Part I as an example of the kind of results that we obtain. This is the result for a single perturbation with $a \neq 0 \neq b$ in the Bessel scale.

Theorem 0.3.1 *Let $P_{a,b} = D^b(d(x)D^a u)$ with $k, a, b \in \{0,1,2,3\}$, $k = a + b$. Assume that $\|d\|_{L^p_U(\mathbb{R}^N)} \leq R_0$ with $p > \frac{N}{4-k}$, then for any $1 < q < \infty$ and such $P_{a,b}$, there exists an interval $I(q,a,b) \subset (-1 + \frac{a}{4}, 1 - \frac{b}{4})$ containing $(-1 + \frac{a}{4} + \frac{N}{4p}, 1 - \frac{b}{4} - \frac{N}{4p})$, such that for any $\gamma \in I(q,a,b)$, we have a strongly continuous analytic semigroup, $S_{P_{a,b}}(t)$, in the space $H^{4\gamma,q}(\mathbb{R}^N)$, for the problem*

$$\begin{cases} u_t + \Delta^2 u + D^b(d(x)D^a u) = 0, & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimates

$$\|S_{P_{a,b}}(t)u_0\|_{H^{4\gamma',q}(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{4\gamma,q}(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in H^{4\gamma,q}(\mathbb{R}^N)$$

for every $\gamma, \gamma' \in I(q,a,b)$ with $\gamma' \geq \gamma$, and

$$\|S_{P_{a,b}}(t)u_0\|_{L^r(\mathbb{R}^N)} \leq \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in L^q(\mathbb{R}^N)$$

with $1 < q \leq r \leq \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q,a,b)$ is given by

$$I(q,a,b) = (-1 + \frac{a}{4} + \frac{N}{4}(\frac{1}{p} - \frac{1}{q'})_+, 1 - \frac{b}{4} - \frac{N}{4}(\frac{1}{p} - \frac{1}{q})_+).$$

Finally, if

$$d_\varepsilon \to d \quad \text{in } L^p_U(\mathbb{R}^N), \quad p > \frac{N}{4-k}$$

then for every $1 < q < \infty$ *and* $T > 0$ *there exists* $C(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*, such that*

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N), H^{4\gamma',q}(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \leq T$$

for all $\gamma, \gamma' \in I(q, a, b)$*,* $\gamma' \geq \gamma$ *and for any* $1 < q \leq r \leq \infty$

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \leq T.$$

In regard of all this, Part I is structured as follows. The abstract results from [47] are revisited in Chapter 1, as we will use them extensively. Then, in Chapter 2 we recall some results about abstract scales of spaces from [2, Chapter V] and extend them, using perturbation techniques from [47], so that they can be used for any sectorial operator. Uniform spaces as in (0.3.1) are of great interest to us when applying to particular problems, as low regular initial data can be considered such as constant functions in $\mathbb{R}^N$, and thus we will repeatedly work with them through the thesis. Chapter 3 compiles the detailed definition and construction of both Lebesgue and Bessel uniform spaces, reviewing their properties. It is worth highlighting Proposition 3.0.1 inside this chapter, which provides with an embedding for the negative side of the scale, needed for most of the results related with uniform spaces in Part I and II.

Results including the negative side of the uniform scale for second order operator can be found in Chapter 4. Chapter 5 deals with fourth order problems in $\mathbb{R}^N$. Its first section constructs the abstract scale for squares of operators, the second one uses them to fourth order equations in Bessel-Lebesgue spaces and the third one studies the already mentioned resolvent estimates for $-\Delta$ and Δ^2 in uniform spaces. Chapter 6 extends these results to higher order problems. Finally, in Chapter 7 the problems in bounded domains are studied.

As for Part II, in the setting of (0.1.2) we will study problem (0.2.4) when f is nonlinear. More precisely, we assume that there exist $\rho \geq 1$, $L > 0$ such that for some $\alpha, \beta \in \mathcal{J}$, $\alpha \geq \beta$

$$\|f(u) - f(v)\|_\beta \leq L\big(1 + \|u\|_\alpha^{\rho-1} + \|v\|_\alpha^{\rho-1}\big)\|u - v\|_\alpha, \quad u, v \in X^\alpha. \tag{0.3.6}$$

Thus f is continuous and

$$\|f(u)\|_\beta \leq L\big(1 + \|u\|_\alpha^\rho\big), \quad u \in X^\alpha \tag{0.3.7}$$

where the constants in (0.3.6) and (0.3.7) can be chosen the same.

We start with the analysis of the abstract nonlinear integral equation (0.2.4), which is the corresponding variation of constants formula for mild solutions of the nonlinear problem

$$u_t + Au = f(u), \quad u(0) = u_0.$$

As stated above, we search the range of γ for which (0.2.4) has a solution (in a suitable class to be defined) for any $u_0 \in X^\gamma$ and for some $T = T(u_0)$. The answer will be determined by α, β and ρ.

Additionally, uniqueness, continuous dependence with respect to initial data, smoothing effect and blow up estimates are also studied in this abstract framework.

In order to do this, the first step before attempting to solve (0.2.4) is to define a suitable notion of solution and for this many options could be available. In any case, to make sense of (0.2.4), any definition of solution has to include the minimal requirements that $u : (0, T] \to X^\alpha$ and, that for any $0 < \tau < T$ and for all $\tau \leq t \leq T$, $u(t)$ satisfies

$$u(t) = S(t-\tau)u(\tau) + \int_\tau^t S(t-s)f(u(s))\, ds.$$

Additionally, it is also natural to require that for any $\tau > 0$, $u \in L^\infty([\tau, T], X^\alpha)$. Also, any suitable notion of solution must incorporate information on the initial data and the behaviour of the solution near $t = 0$. In particular, we define

Definition 0.3.2 *If $u_0 \in X^\gamma$, then $u \in L^\infty_{loc}((0, T], X^\alpha)$ that satisfes $t^{\alpha-\gamma}\|u(t)\|_\alpha \leq M$, $t \in (0, T]$ for some $M > 0$, $u(0) = u_0$ and (0.2.4) for $0 < t \leq T$ is called a γ–solution of (0.2.4) in $[0, T]$.*

Notice that, from (0.1.2), the behaviour of the γ–solution at $t = 0$ is the same as that of the linear semigroup $S(t)u_0$.

For this class of solutions we show existence, uniqueness and continuous dependence with respect to the initial data, for the following ranges of γ:

$$\gamma \in E(\alpha, \beta, \rho) = \begin{cases} (\alpha - \frac{1}{\rho}, \alpha], & \text{if } \ 0 \leq \alpha - \beta \leq \frac{1}{\rho} \\ [\frac{\alpha\rho-\beta-1}{\rho-1}, \alpha], & \text{if } \ \frac{1}{\rho} < \alpha - \beta < 1. \end{cases}$$

The case $\gamma = \frac{\alpha\rho-\beta-1}{\rho-1}$ is called critical and subcritical otherwise. In particular, we will prove that given $\gamma \in E(\alpha, \beta, \rho)$ as above, then there exists $r > 0$ such for any $v_0 \in X^\gamma$ there exists $T > 0$ such that for any u_0 such that $\|u_0 - v_0\|_\gamma < r$ there exits a γ–solution of (0.2.4) with initial data u_0 defined in $[0, T]$. In the subcritical case r can be taken arbitrarily large.

The γ-solutions will be shown to regularize, that is, to enter continuously in other spaces of the scale of larger index. We will also give estimates on the existence time and, when the existence time is finite, the blow–up rate for different norms.

It will be shown that the conditions of the theorems are essentially optimal. More precisely, when $\gamma < \inf E(\alpha, \beta, \rho)$ then uniqueness or continuous dependence cannot be expected for the solutions, and thus the problem is in general not well posed. For the critical case, that is when $\inf E(\alpha, \beta, \rho) = \frac{\rho\alpha-\beta-1}{\rho-1}$ we show that when $\gamma < \inf E(\alpha, \beta, \rho)$ continuous dependence cannot be expected.

Finally, uniqueness is extended for functions that, satisfying the minimum requirements described above and (0.2.4), are bounded at $t = 0$ in X^γ in the subcritical case, or continuous at $t = 0$ in X^γ in the critical one.

When applying these techniques to concrete problems one finds that there typically exist many admissible pairs (α, β) such that the term f satisfies (0.2.3) and (0.3.6). Such pairs make up an "admissible region" associated to the problem considered. In this situation we develop a general bootstrapping argument which can be used for all concrete problems, that leads to the largest range of γ for which the solution can be constructed as well as to the largest range of spaces into which the solution regularizes. It will be shown that this ranges can be computed from the admissible region, so applying the abstract results into particular problems can be done by minimizing a continuous function in a convex region.

Then, we apply to particular PDE problems in concrete scales of spaces. Firstly we consider parabolic problems of the type

$$u_t + (-\Delta)^m u = f(x,u) := D^b(h(x, D^a u)), \ t > 0, \ x \in \mathbb{R}^N, \tag{0.3.8}$$

with $m \in \mathbb{N}$, where D^c represents any partial derivative of order $c \in \mathbb{N}$, $h(\cdot, 0) = 0$ and for some $\rho > 1$, $L > 0$ we have

$$|h(x,u) - h(x,v)| \leq L|u - v|(|u|^{\rho-1} + |v|^{\rho-1}), \ u, v \in \mathbb{R}, \ x \in \mathbb{R}^N.$$

We want to find the range of spaces for which the problem is well posed and we will also study uniqueness in several classes, continuous dependence with respect to the initial data, blow up estimates when blow up occurs and smoothing estimates.

Then problem (0.3.8) is studied in several different scales of spaces. We first choose the Lebesgue scale of spaces to set the problem in and $a = b = 0$. The results there recover and slightly improve the results in [52], [53], [11], [4] and [46, Chapter 2] when $m = 1$, see the discussion in Section 11.1 for details. Afterwards, the problem is considered in the Bessel scale of spaces with $a \neq 0 \neq b$. A particular case of fourth order parabolic equation with such a nonlinearity with $a = 0$, $b = 2$ is the Cahn-Hilliard equation

$$\begin{cases} u_t + \Delta^2 u + \Delta h(x,u) = 0, \ t > 0, \ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^N. \end{cases}$$

We give a result in the Bessel scale for this problem, which shows the power of the abstract tools (and bootstrap) we develop, since the results for the Cahn-Hilliard problem appear here just a corollary but in fact recover recent results on the topic (see [17, 49, 56]).

We finally dive into solving (0.3.8) in uniform spaces. Such spaces are not reflexive, and therefore duality cannot be used to give a representation of the perturbation P when $b \neq 0$. Therefore, results in this case are for $b = 0$.

The last application we present here is the study of the following strongly damped wave equation

$$\begin{cases} w_{tt} - \Delta w_t + w_t - \Delta w = h(x,w), \ t > 0, \ x \in \mathbb{R}^N, \\ w(0,x) = w_0(x), \quad w_t(0,x) = z_0(x), \ x \in \mathbb{R}^N. \end{cases}$$

with h as in (0.2.6). This problem can be rewritten as

$$\dot{u} + Au = f(u) := \left[\begin{smallmatrix} 0 \\ h(w)+w \end{smallmatrix} \right], \quad u(0) = u_0 := \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix} \right]$$

in a suitable scale of spaces to be specified below, so it is under the requirements to apply the theory, and we obtain again similar results about existence, uniqueness, regularity and blow up.

We now show Theorem 11.2.11 from Part II as an example of the kind of results that we obtain. This is the result for a single perturbation in Bessel spaces.

Theorem 0.3.3 *Assume h is as in (0.3.8) for some $\rho > 1$, $L > 0$. Denote $k = a+b < 2m$ and assume $p_0 < \rho p(1 - \frac{k}{2m})$ with $p_0 := \frac{N}{2m}(\rho - 1)$. Then for*

$$\gamma_c = \max\left\{ \frac{N}{2m}\left(\frac{1}{p} - \frac{1}{\rho}\right)_+ + \frac{a}{2m} - \frac{1}{\rho}, \frac{N}{2mp} + \frac{a\rho + b}{2m(\rho-1)} - \frac{1}{\rho - 1} \right\} < \gamma < 1 - \frac{b}{2m}$$

there exist $r > 0$ and $T > 0$, such that for any $v_0 \in H^{2m\gamma,p}(\mathbb{R}^N)$ and any u_0 satisfying $\|u_0 - v_0\|_{H^{2m\gamma,p}(\mathbb{R}^N)} < r$, there exists a function $u(\cdot, u_0)$ such that for all $\gamma \leq \gamma' < 1 - \frac{b}{2m}$, $u(\cdot, u_0) \in C((0,T], H^{2m\gamma',p}(\mathbb{R}^N)) \cap C([0,T], H^{2m\gamma,p}(\mathbb{R}^N))$ and

$$t^{\gamma'-\gamma}\|u(t,u_0)\|_{H^{2m\gamma',p}(\mathbb{R}^N)} \leq M(u_0,\gamma') \qquad \text{for } 0 < t < T$$

$$t^{\gamma'-\gamma}\|u(t,u_0)\|_{H^{2m\gamma',p}(\mathbb{R}^N)} \to 0 \qquad \text{as } t \to 0, \gamma' \neq \gamma$$

and satisfies

$$u(t,u_0) = S(t)u_0 + \int_0^t S(t-s)h(u(s,u_0))ds \quad t \in [0,T].$$

Also, there exists $M > 0$ such that for all $u_0^i \in H^{2m\gamma,p}(\mathbb{R}^N)$, $i = 1,2$ such that $\|u_0^i - v_0\|_{H^{2m\gamma,p}(\mathbb{R}^N)} < r$, we have for $\gamma' \in [\gamma, 1 - \frac{b}{2m})$

$$\|u(t,u_0^1) - u(t,u_0^2)\|_{H^{2m\gamma',p}(\mathbb{R}^N)} \leq \frac{M}{t^{\gamma'-\gamma}}\|u_0^1 - u_0^2\|_{H^{2m\gamma,p}(\mathbb{R}^N)}, \quad t \in (0,T].$$

When $\frac{N}{2mp} + \frac{a\rho+b}{2m(\rho-1)} - \frac{1}{\rho-1} > \frac{N}{2m}\left(\frac{1}{p} - \frac{1}{\rho}\right)_+ + \frac{a}{2m} - \frac{1}{\rho}$ then the above hold also for $\gamma = \gamma_c = \frac{N}{2mp} + \frac{a\rho+b}{2m(\rho-1)} - \frac{1}{\rho-1}$.

If $\gamma_c < \gamma < 1 - \frac{b}{2m}$ then r can be taken arbitrarily large, that is, the existence time is uniform in bounded sets in $H^{2m\gamma,p}(\mathbb{R}^N)$.

There are several previous results where the abstract framework of the Variation of Constants Formula is used to solve PDE problems that have been inspiring for us.

In [52] the abstract problem (0.2.4) is studied for a class of solutions similar to our γ-solutions. Then, the results are applied to (0.2.6) with $a = b = 0$ in a bounded domain in Lebesgue spaces with Dirichlet boundary conditions (which makes the Lebesgue scale to be nested), obtaining the same range of existence as we do for that case. The uniqueness result in Theorem 3 in [52] was stated in a smaller class than the one in our result (Theorem 11.1.6), see also Theorem 2.a).i) in [52].

In [53] blow–up estimates are obtained for (0.2.6) with $a = b = 0$, $m = 1$. Our result (11.1.11) below, particularized for $m = 1$, $q = p$ and $p > \frac{N}{2m}(\rho - 1)$, is the same as the one in [53].

In [29] a result on non-uniqueness for (0.2.6) with $a = b = 0$, $m = 1$, $u(0) = 0$ was stated for $A = -\Delta$ in $\mathbb{R}^N$ in a very similar fashion to our Proposition 9.3.1.

Later on, [9] showed non-uniqueness without assuming $u(0) = 0$, for positive, radial, decreasing solutions in bounded domains. Similar results can also be found in [43], Theorems 3 and 4.

In [28] problem (0.2.6) with $a = b = 0$, $m = 1$ is considered in a bounded domain or in $\mathbb{R}^N$. There, uniqueness was stated for $p = \frac{N}{2m}(\rho - 1)$ in the class $L^r((0, T], L^q(\mathbb{R}^N))$ with $\frac{1}{r} = \frac{N}{2}(\frac{1}{p} - \frac{1}{q})$, $q, r > \rho$, $q > p$. The class of uniqueness in our Theorem 11.1.6 ii) is a subclass of $u \in L^r((0, T], L^q(\mathbb{R}^N))$ with $\frac{1}{r} > \frac{N}{2}(\frac{1}{p} - \frac{1}{q})$. Blow estimates are studied, and again coincide with our result.

In [11] problem (0.2.6) with $a = b = 0$, $m = 1$ is studied in a bounded domain in Lebesgue spaces with Dirichlet boundary conditions. In this paper, they focus on improving the uniqueness result from [52], extending it to a larger class of functions. More precisely, it was stated in the class of classical solutions of (0.2.6) (with $a = b = 0$) such that $u \in C([0, T], L^p(\mathbb{R}^N))$ which is a particular case of our Theorem 11.1.6 below.

Also, [4] made use of the abstract scale of fractional power spaces associated to a sectorial operator, as developed in [31], and applied their results to the Navier-Stokes equation and to problems similar to (0.2.6) with $a = b = 0$ in similar scales to the Bessel scale. More precisely, they consider the fractional power scale associated to A, $\{Y^\alpha\}_{\alpha \geq 0}$ and prove existence of ε-regular solution for the problem (0.2.4) with initial data $u_0 \in Y^1$. Our setting includes that of [4] and in their case, we can construct solutions for more spaces of initial data and not only for Y^1. Conversely, studying our abstract framework in their setting yields an extra assumption compared to our case, see Remark 9.1.16.

Therefore, our setting extends these previous results in the following ways. On one hand, we use scales which are not necessarily the Lebesgue scale or the fractional power scale, in fact, the scale does not need to be nested. Furthermore, the previous works choose the initial data in a fixed space, whereas we fit initial data in adequate spaces of the scale and study well-posedness at the same time. On the other hand, even when using the same setting as in the previous works, the range of spaces from which we can choose the initial data for a given problem is larger than those in the results in [31] or [4], see again Remark 9.1.16.

Taking all this into account, Part II is structured in the following way. Chapter 8 compiles some preparatory material about the Variation of Constants Formula along with some definition to be used later on. Then, Chapter 9 develops all the abstract theory. First the existence, then the improved uniqueness and finally the study of the optimality of the results. Next, in Chapter 10 we develop in detail the bootstrap argument that will serve as a bridge between these abstract arguments and the application to particular problems. Afterwards the application to problems like (0.3.8) is done in Chapter 11. Finally, in Chapter 12 we deal with the strongly damped wave equation.

0.4 Conclusions

The main results in the thesis can be summarised as follows

- The second order linear problem (0.2.1) with $a_{kl} \in BUC(\mathbb{R}^N)$, $\|b_j\|_{\dot{L}^{p_j}_U(\mathbb{R}^N)} \leq R_j$, $j = 1, ..., N$ and $\|c\|_{\dot{L}^{p_0}_U(\mathbb{R}^N)} \leq R_0$, where $p_j > N$ and $p_0 > \frac{N}{2}$ is solved in uniform Bessel spaces, see Theorem 4.0.6. These conditions are less restrictive than the ones in previous results for (0.2.1) in uniform spaces.
- The scale of spaces associated to an operator that generates an analytic semigroup can also be used for the square of that operator which also generates an analytic semigroup in that same scale, see Propositions 5.1.3, 5.1.4.
- The fourth order linear problem (0.2.2) is solved in the Bessel scale, Theorem 5.2.10.
- The Laplacian in the uniform Lebesgue spaces $\dot{L}^q_U(\mathbb{R}^N)$ satisfies the estimate
$$\|(-\Delta - \lambda)^{-1}\|_{\mathcal{L}(E^0)} \leq M|\lambda|^{-1}$$
for all λ in a sector $S_{0,\phi}$ as in (5.2.3) for $\phi > 0$ arbitrarily small, see Proposition 5.3.1. This implies that the bi-Laplacian Δ^2 generates an analytic semigroup in the uniform Bessel scale.
- The fourth order linear problem (0.2.2) is solved in the uniform Bessel scale, Theorem 5.3.5.
- The nonlinear abstract problem (0.2.4) is well posed for γ-solutions as in Definition 0.3.2 for $\gamma \in E(\alpha, \beta, \rho)$, see Theorems 9.1.7,9.1.8.
- Uniqueness results are extended, see Theorem 9.2.2.
- The nonlinear problem (0.3.8) is solved in Lebesgue, Bessel, uniform Lebesgue and uniform Bessel spaces, see Chapter 11.
- The strongly damped wave equation (0.2.7) is solved, see Theorem 12.0.3.

All these results have been communicated in the following way. The results in Chapters 2,3, 5 and 6 are published in the paper "Smoothing and perturbation for some fourth order linear parabolic equations in $\mathbb{R}^N$", Journal of Mathematical Analysis and Applications, volume 412, pages 1105-1134, 2014.

The results from Chapter 4 are sent to publish and also partially gathered in the paper "Perturbation of analytic semigroups in uniform spaces in $\mathbb{R}^N$", Advances in differential equations and applications, volume 4 of SEMA SIMAI Springer Ser., pages 41–52. Springer, Cham, 2014.

The results in Chapter 7 are being prepared to send to publication.

As for the nonlinear problems, Chapters 8,9,10,11 and 12 are compiled altogether and sent to publish.

The results have also been presented in conferences and talks in a variety of congresses, seminars and workshops, such as the XXII CEDYA congress 2011 in Mallorca, XXIII CEDYA congress 2013 in Castellon, the ICMC Summer Meeting on Differential Equations 2013 in Brazil, the 10th AIMS Conference 2014 in Madrid and the XXIV CEDYA congress 2015 in Cadiz.

Finally, we mention possible paths for future work.

- One possible way of continuing our approach in the linear case when the domain is $\mathbb{R}^N$ is constructing scales for operators of order $2m$ which are not obtained as the square (or powers) of the Laplacian. Also, the fractional Laplacian might be considered for the scale associated to the Laplacian under our setting.

- One of the main topics that remains to be fully understood is that of the linear problems in bounded domains. Even though the abstract part and many particular cases are fully described in the present work, when dealing with particular cases, there are some perturbations in the boundary for which we do not give a correspondence with a PDE problem. In those cases we know that we can solve an abstract integral problem, but it remains to show what particular PDE problem corresponds to the integral problem.

- Also, we could construct the scales with boundary conditions for operators that do not appear as the square of the Laplacian and study perturbations for those cases.

- Another possible continuation to the work is considering many nonlinear perturbations at the same time. More than one perturbation are already considered in some of the examples below, however considering more than one perturbation already in the abstract part might yield to an improvement in the range of initial data.

- Finally, the approach in the thesis can also be applied to nonlinear problems in bounded domains, where perturbations in the interior and in the boundary will arise.

Introducción

0.5 Motivación

Las ecuaciones diferenciales en las que está involucrado el tiempo se conocen habitualmente como ecuaciones de evolución, donde la idea subyacente es que la solución evoluciona en tiempo desde un dato inicial. Entre ellas se hallan las ecuaciones diferenciales parabólicas, que serán el principal tema de discusión en esta tesis.

La ecuación parabólica por excelencia es por supuesto la ecuación del calor

$$u_t - \Delta u = 0 \qquad u(0) = u_0 \tag{0.5.1}$$

para $t > 0$, donde $-\Delta$ denota el Laplaciano, $u_t := \frac{\partial u}{\partial t}$ y $x \in \mathbb{R}^N$ o en un dominio acotado $\Omega \subset \mathbb{R}^N$ con, digamos, condiciones de frontera Dirichlet. Podemos considerar otros problemas parabólicos reemplazando el Laplaciano $-\Delta$ por otro operador diferencial A con propiedades similares, es decir, un operador elíptico. La solución evoluciona desde un cierto dato inicial u_0 en un espacio de Banach X adecuado, por tanto es razonable pensar que la solución es única y viene determinada por un operador que se aplica al dato inicial. Es decir, la solución al problema

$$u_t + Au = 0, \quad u(0) = u_0, t > 0$$

viene dada por $u(t) = S(t)u_0$ ya que $u(t)$ está determinado de forma única por u_0, donde para todo $t \geq 0$, $S(t) : X \to X$ es un semigrupo C^0, esto es, una familia $\{S(t)\}_{t\geq 0}$ de operadores lineales y continuos en el espacio de Banach X tal que

$S(0) = I$;

$S(t)S(s) = S(t+s)$ para $t \geq 0$, $s \geq 0$;

$S(t)u \to u$ cuando $t \to 0^+$ para todo $u \in X$.

La primera y última propiedad reflejan que el dato inicial se alcanza, mientras que la segunda se debe a la unicidad de la solución. El operador principal A de la ecuación y el semigrupo $S(t)$ están relacionados por $-Au = \lim_{t\to 0^+} \frac{1}{t}(S(t)u - u)$, y su dominio $D(A)$ se compone de todos los $u \in X$ tales que el límite existe. En este marco, se denomina a $-A$ el generador infinitesimal del semigrupo. En general $D(A) \subsetneq X$.

Se puede considerar que este planteamiento para estudiar ecuaciones diferenciales tuvo su origen en el artículo de E. Hille [33] en 1942, que inspiraría a muchos otros para desarrollar esta teoría en las décadas posteriores. Previamente, Gelfand [27] había estudiado grupos uniparamétricos en espacios normados, dando algunas propiedades que los semigrupos conservan, como la representación del grupo mediante la exponencial de A o la relación entre un grupo y su generador. En su artículo, Hille continua y extiende el trabajo de [27], particularizándolo para semigrupos. Aporta gran cantidad de formas de representar el semigrupo, algunas de ellas involucrando a la resolvente del generador, que como demostraría, es de gran utilidad al aplicar a ecuaciones diferenciales.

Más adelante, Hille [34], Yosida [54], Phillips [45] y [35] (una revisión y extensión de [34]) sentaron las bases de lo que habitualmente se comoce como teoría de Hille-Yosida, esto es, el estudio de ecuaciones diferenciales parabólicas mediante la teoría de semigrupos. Este marco funcional permitió el estudio de de problemas particulares de EDPs a partir de una ecuación integral abstracta dada por la Fórmula de Variación de las Constantes. De forma similar a lo que se hace en EDO, el problema

$$u_t + Au = f(u), \quad u(0) = u_0$$

se estudia a través de su versión abstracta

$$u(t; u_0) = S(t)u_0 + \int_0^t S(t-\tau)f(u(\tau; u_0))\, d\tau, \qquad 0 < t \leq T.$$

No mucho después, Balakrishnan [7], Kato [36] y Yosida [55] entre otros consideraron las potencias fraccionarias de un operador A ya que resultó ser muy útil para asociar una familia de espacios a ese operador, que más adelante se utilizaría para resolver problemas parabólicos concretos. Demostraron que la potencia fraccionaria A^s, $0 \leq s \leq 1$ se puede construir siempre que $-A$ sea el generador infinitesimal de un semigrupo. El estudio de la potencia A^s en un espacio de Banach X llevó a su vez a la construcción de una familia de espacios intermedios entre $X^0 := X$ y $X^1 := D(A)$ asociados a A^s, tales que $X^\alpha \hookrightarrow X^\beta$ for $\alpha \geq \beta$. La escala X^s, $0 \leq s \leq 1$ construida a partir de potencias fraccionarias de operadores es lo que se conoce como la escala de potencias fraccionarias. La definición de los operadores A^s y los espacios X^s puede ser extendida para $s \geq 0$. Todos estos resultados junto con algunos otros de Sobolevskii [38] fueron recopilados por Komatsu en [37] donde estudió las potencias fraccionarias de operadores y los espacios asociados desde un punto de vista unificado.

En los siguintes años, muchos otros autores extendieron y desarrollaron el tema. Debemos mencionar entre otros a Ladyzhenskaya [39], Friedman [23], Ball [8], Weissler [52], Henry [31], Pazy [44], [11] ya que destacan en el estudio abstracto de las ecuaciones de evolución. En concreto, los libros [23, 31, 44] se han convertido en referencias principales en el estudio de ecuaciones parabólicas, ya que hacen uso sistemático de la escala de potencias fraccionarias arriba descrita para resolver problemas de EDPs. Se obtuvieron resultados de existencia local, unicidad, regularidad, efecto regularizante tanto para problemas lineales como no lineales. Para los no lineales se estudia también el blow-up.

Más recientemente, el estudio de problemas parabólicos en escalas de espacios ha sido continuado por otros autores como Lunardi [40], Triebel [50] y Amann [1, 2]. Queremos destacar [2] ya que continua y extiende el estudio de procedimientos generales para asociar una escala de espacios a un operador A. Hay distintas formas de construir una escala de espacios a partir de dicho operador. Un caso particular es la ya mencionada escala de potencias fraccionarias X^s. Otra escala que se considera en [2] es la escala de interpolación E^α. En este caso, para construir espacios intermedios entre $E^1 := D(A)$ y $E^0 := X$ se utiliza un método de interpolación (como interpolación real o compleja, ver [50]). Como en el caso de la escala de potencias fraccionarias, esto se puede extender para $\alpha \geq 0$.

Las escalas de interpolación y potencias fraccionarias se pueden considerar para índices $s \geq 0$, $\alpha \geq 0$. Además, en [2] ambas escalas no se consideran sólo para índices positivos, sino que se extienden a espacios con índices negativos que contienen a X, es decir se *extrapola* la escala a espacios negativos creando así la parte negativa de la escala. Cuando X es reflexivo, la parte negativa de la escala se puede describir en términos de los duales de ciertos espacios, obteniendo así una buena representación de los elementos que se encuentran en dichos espacios. Esto sirve además como herramienta para interpretar las soluciones al resolver problemas en espacios más débiles, que al aplicar a problemas concretos, unifica y recupera los conceptos de solución débil y muy débil para problemas de EDP.

En general la escala de interpolación y de potencias fraccionarias no coinciden, pero en muchos casos particulares sí lo hacen (por ejemplo cuando el operador a partir del cual se construyen tiene potencias imaginarias acotadas).

Considerar los problemas parabólicos en cada elemento de la escala de espacios resulta muy conveniente ya que, cuando se aplica la teoría a casos concretos, es muy común que el dato inicial se pueda tomar en un espacio, elegido de entre varios de la misma escala, y por tanto es natural considerar el problema en la escala. La ventaja de las escalas de interpolación y fraccionaria arriba descritas es que dado un problema cualquiera siempre se pueden construir y, de hecho, frecuentemente resultan ser espacios bien conocidos como los espacios de Bessel (espacios que se construyen a partir de los de Lebesgue dotándolos de derivadas distribucionales, ver [31, p. 35] para más detalles). Sin embargo, al tratar con ejemplos concretos, a veces se pueden considerar escalas particulares para el problema dado. Un ejemplo es la arriba mencionada ecuación del calor (0.5.1) en $\mathbb{R}^N$ que se puede resolver con dato inicial en cualquiera de los espacios de Lebesgue $L^q(\mathbb{R}^N)$, para $1 \leq q \leq \infty$. La solución permanece en el mismo espacio y de hecho entra en cualquier otro $L^p(\mathbb{R}^N)$ con $p > q$ satisfaciendo además, para constantes $M_{p,q}, \mu_0 > 0$, que

$$\|S_{-\Delta}(t)u_0\|_{L^p(\mathbb{R}^N)} \leq \frac{M_{p,q}e^{\mu_0 t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}}\|u_0\|_{L^q(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in L^q(\mathbb{R}^N).$$

Estas son las estimaciones clśicas de la ecuación del calor, ver por ejemplo [14].

El hecho de que un problema se pueda estudiar en una escala de espacios es una ventaja ya que, entre otras cosas, podemos encajar el dato inicial en espacios adecuados de la escala de forma que se puede estudiar al mismo tiempo si un problema, tanto con términos lineales como no lineales, está bien propuesto. No sólo el dato inicial se puede considerar en diferentes espacios de las escala para el mimso problema, también se pueden tratar distintos problemas al mismo tiempo con la misma estrategia, ya que estos problemas comparten muchas propiedades en el mismo marco funcional abstracto.

Además, otras propiedades, como efecto regularizante de la solución o estimaciones de blow-up también se obtienen mediante estos procedimientos. Es por tanto natural y valioso estudiar problemas parabólicos desde este punto de vista.

En ese espíritu, las escalas consideras arriba se pueden describir desde el siguiente enfoque abstracto unificado. Sea $\{X^\gamma\}_{\gamma\in\mathcal{J}}$ una familia de espacios de Banach donde $\mathcal{J}$ es un intervalo de índices reales. La norma de X^γ se denota por $\|\cdot\|_\gamma$. Obsérvese que a pesar de haber usado la notación X^α para referirnos previamente a la escala de potencias fraccionarias, la usamos ahora para referirnos a una escala abstracta genérica.

Asumimos que $\{S(t) : t \geq 0\}$ es un semigrupo C^0 en cada uno de los espacios de la familia $\{X^\gamma\}_{\gamma\in\mathcal{J}}$ (en otras palabras decimos que es un semigrupo C^0 en la escala) tal que para todo $\gamma, \gamma' \in \mathcal{J}$, $\gamma' \geq \gamma$ y $T > 0$ tenemos

$$\|S(t)\|_{\mathcal{L}(X^\gamma, X^{\gamma'})} \leq \frac{M_0}{t^{\gamma'-\gamma}}, \qquad 0 < t \leq T, \tag{0.5.2}$$

donde $M_0 := M_0(\gamma, \gamma', T)$ es una constante positiva que se puede tomar uniformemente en T para intervalos de tiempo acotados.

Obsérvese que no se asume ninguna otra relación en los espacios de la escala, salvo que se especifique. Por ejemplo, a veces asumiremos que la escala es anidada, esto es, para todo $\alpha, \beta \in \mathcal{J}$ con $\alpha \geq \beta$ se tiene

$$X^\alpha \subset X^\beta$$

con inclusión continua. Tanto la escala de potencias fraccionarias como la de interpolación son escalas anidadas, mientras que la de Lebesgue no lo es en $\mathbb{R}^N$.

En este contexto, en [47] se consideran problemas parabólicos de la forma

$$\begin{cases} u_t + Au = Pu, & t > 0 \\ u(0) = u_0, \end{cases} \tag{0.5.3}$$

usando el enfoque unificado descrito arriba. Para ello, se estudia la siguiente ecuación integral abstracta que viene dado por la Fórmula de Variación de las Constantes

$$u(t; u_0) = S_P(t)u_0 = S(t)u_0 + \int_0^t S(t-\tau)Pu(\tau; u_0)\, d\tau, \qquad 0 < t \leq T \tag{0.5.4}$$

donde P es un operador lineal definido en algún espacio de la escala. Se obtiene existencia y unicidad de soluciones de (0.5.4) mediante argumentos de perturbación para un *rango de espacios* en el que se puede tomar el dato inicial, es decir el dato incial se toma en X^γ donde γ está en un rango de índices. También se estudia la regularidad de la solución así como la continuidad de las soluciones con respecto a la perturbación. Estos resultados se aplican entonces para resolver problemas lineales paraólicos de segundo orden en escalas de espacios, como por ejemplo los de Lebesgue o Bessel. El artículo [47] se puede ver como el punto de partida y motivación de la presente tesis.

0.6 Objetivo

En esta tesis continuaremos con el estudio de problemas parabólicos usando ideas similares a las de [47]. En primer lugar, usaremos las técnicas de perturbación de dicho artículo para estudiar ecuaciones parabólicas lineales. Por un lado, continuaremos con el estudio de problemas lineales de segundo orden para datos iniciales en una escala de espacios con propiedades de poca regularidad llamados Espacios Localmente Uniformes que definiremos más adelante. Extenderemos los resultados de [47] para esos espacios. Por otro lado, estudiaremos ecuaciones lineales de cuarto orden en dominios acotados y no acotados, expandiendo de forma natural las aplicaciones de la teoría de perturbación.

En segundo lugar, pasamos a la Then we jump to the study of nonlinear problems. Empezamos construyendo los resultados abstractos de perturbación para ecuaciones parabólicas en escalas de espacios que se asemajan a los de [47] para el caso lineal. Después los aplicamos a problemas de EDPs no lineales.

Siendo más específicos, primero utilizaremos los resultados de perturbación de [47] para extender resultados previos (ver por ejemplo [3], [5], [47]) para problemas de segundo orden de la forma

$$\begin{cases} u_t - \sum_{k,l=1}^N a_{kl}(x)\partial_k\partial_l u + \sum_{j=1}^N b_j(x)\partial_j u + c(x)u = 0 & x \in \mathbb{R}^N, \quad t > 0 \\ u(0,x) = u_0(x) & x \in \mathbb{R}^N, \end{cases} \tag{0.6.1}$$

con coeficientes y dato inicial de baja regularidad. La existencia de solución se probará para condiciones menos restrictivas que las de resultados previos y para datos iniciales que podrán ser escogidos de un rango de espacios mayor.

Por otro lado, nos centraremos en estudiar problemas parabólicos lineales de cuarto orden

$$\begin{cases} u_t + \Delta^2 u + Pu = 0, & x \in \Omega, t > 0 \\ u(0) = u_0 \end{cases} \tag{0.6.2}$$

donde P es una perturbación con dependencia espacial y Ω es o bien $\mathbb{R}^N$ o bien un dominio acotado (en cuyo caso, condiciones de frontera aparecen también en el problema). Daremos resultados de existencia, unicidad, regularización y robustez respecto a variaciones en la perturbación P. Estos resultados se extenderán de forma natural con argumentos análogos a ecuaciones de orden superior.

Estudiamos posteriormente el caso de perturbaciones no lineales. Dado un semigrupo en una escala de espacios como la de arriba, consideramos una aplicación no lineal que satisface

$$f : X^\alpha \to X^\beta \quad \text{para algún } \alpha, \beta \in \mathcal{J} \text{ con } 0 \le \alpha - \beta < 1. \tag{0.6.3}$$

y la siguiente condición de crecimiento

$$\|f(u)\|_\beta \le L\big(1 + \|u\|_\alpha^\rho\big), \quad u \in X^\alpha.$$

Entonces, el objetivo principal es el análisis de la ecuación integral abstracta

$$u(t; u_0) = S(t)u_0 + \int_0^t S(t-\tau)f(u(\tau; u_0))\, d\tau, \qquad 0 < t \le T, \tag{0.6.4}$$

donde u_0 se toma en un espacio X^γ de la escala. Nótese que (0.6.4) es la correspondiente fórmula de variación de las constantes para soluciones débiles del siguiente problema de EDP

$$u_t + Au = f(u) \quad u(0) = u_0. \tag{0.6.5}$$

Las principales cuestiones a analizar son:

(1) El rango de γ para el que (0.6.4) tiene solución (en una clase adecuada a definir) para algún $u_0 \in X^\gamma$ y para algún $T = T(u_0)$.

(2) Unicidad y dependencia continua con respecto al dato inicial de las soluciones en (1).

(3) Efecto regularizante: en qué otros espacios $X^{\gamma'}$ dentra la solución para $t > 0$ y estimaciones de la norma $\|\cdot\|_{\gamma'}$ de la solución.

(4) Estimar la tasa de blow-up y el tiempo de existencia si la solución deja de existir en tiempo finito.

Tras desarrollar las técnicas abstractas, aplicamos los resultados obtenidos gracias a este enfoque abstracto a problemas particulares. En concreto, consideramos el siguiente problema general

$$\begin{cases} u_t + (-\Delta)^m u = f(u), & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0 & \text{en } \mathbb{R}^N \end{cases} \tag{0.6.6}$$

donde f es una función no lineal de la forma $D^b(h(x, D^a u))$ para una cierta funciíon h y $m \in \mathbb{N}$. En función de la elección de f se obtienen algunos problemas famosos, como la ecuación de Cahn-Hilliard

$$\begin{cases} u_t + \Delta^2 u + \Delta h(x,u) = 0, \ t > 0, \ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^N. \end{cases}$$

que resolvemos como un caso particular de (0.6.6).

La última aplicación de los resultados abstractos es a la ecuación de ondas fuertemente amortiguada

$$\begin{cases} w_{tt} - \Delta w_t + w_t - \Delta w = h(x,w), \ t > 0, \ x \in \mathbb{R}^N, \\ w(0,x) = w_0(x), \quad w_t(0,x) = z_0(x), \ x \in \mathbb{R}^N. \end{cases} \tag{0.6.7}$$

0.7 Contenido

La tesis se divide en dos partes. La Parte I se concentra en el estudio del problema (0.5.3), aplicando las técnicas abstractas de perturbación de [47] para resolver los problemas (0.6.1) y (0.6.2). La Parte II se dedica al estudio de (0.6.5) y la aplicación de estos resultados a los problemas de EDP no lineales (0.6.6) y (0.6.7).

En la Parte I empezamos encontrando un marco funcional abstracto. Para ello, retomamos los resultados de [47] que usaremos a lo largo de esta parte y también revisamos algunos resultados de [2] acerca de la construcción de las arriba mencionadas escala de interpolación y potencias fraccionaria a partir de un operador A dado, tal que $-A$ genera un semigrupo. Ambas escalas probarán su valía al usarlas para muchos operadores, como por ejemplo el Laplaciano, ya que ambas producen espacios bien conocidos, como los espacioes de Bessel.

Una vez fijado el marco abstracto del problema, estudiamos problemas parabólicos. Queremos considerar el problema perturbado

$$u_t + Au = Pu$$

mediante la Fórmula de Variación de las Constantes (0.5.4) y usar los resultados en [47] para problemas particulares.

Plantearemos los problemas en las escalas de Lebesgue, Bessel y uniforme que introducimos a continuación. Los espacios de Lebesgue $L^p(\mathbb{R}^N)$, $1 \leq p < \infty$ son el conjuntos de funciones cuya potencia p-ésima es integrable.

A partir de los espacios de Lebesgue se construye, para $\alpha \in \mathbb{R}$, $1 \leq q \leq \infty$ el espacio de Bessel $H^{\alpha,q}(\mathbb{R}^N)$, ver [31, p. 35] para más detalles. Cuando $\alpha = k \in \mathbb{N}$, los espacios de Bessel coinciden con los espacios de Sobolev $W^{k,p}(\mathbb{R}^N)$, $1 \leq p \leq \infty$, $k \in \mathbb{N}$, compuestos de funciones $f \in L^p(\mathbb{R}^N)$ que tienen derivada distribucional de orden igual o menor que k y todas ellas tienen su potencia p-ésima integrable.

Finalmente, tenemos unos espacios de baja regularidad llamados espacios localmente uniformes, que tienen propiedades de integrabilidad local pero sin ningún tipo de decaimiento asintótico cuando $|x| \to \infty$. Más concretamente, para $1 \leq p < \infty$ se denota $L^p_U(\mathbb{R}^N)$ al espacio localmente uniforme compuesto por funciones $f \in L^p_{loc}(\mathbb{R}^N)$ tal que existe $C > 0$ tal que para todo $x_0 \in \mathbb{R}^N$

$$\int_{B(x_0,1)} |f|^p \leq C \tag{0.7.1}$$

dotado de la norma

$$\|f\|_{L^p_U(\mathbb{R}^N)} = \sup_{x_0 \in \mathbb{R}^N} \|f\|_{L^p(B(x_0,1))}$$

(para $p = \infty$, $L^\infty_U(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$). Se define $\dot{L}^p_U(\mathbb{R}^N)$ como el subespacio cerrado de $L^p_U(\mathbb{R}^N)$ que se compone de elementos que son continuos con respecto a la traslación $\|\cdot\|_{L^p_U(\mathbb{R}^N)}$ (para $p = \infty$, $\dot{L}^\infty_U(\mathbb{R}^N) = BUC(\mathbb{R}^N)$).

Entonces, a partir de los espacios localmente uniformes podemos construir para $\alpha \in \mathbb{R}$, $1 \leq q < \infty$ los espacios de Bessel uniformes $\dot{H}^{\alpha,q}_U(\mathbb{R}^N)$ de forma similar a como se hace para los espacios de Bessel estandar a partir de los espacios de Lebesgue.

En [47] los resultados se aplicaban a problemas en los que el operador principal es de la forma $Au = -div(a(x)\nabla u)$ en espacios de Lebesgue y Bessel para dominios acotados y no acotados. Extenderemos los resultados de [47] de dos formas. Por un lado, extenderemos y mejoraremos los resultados que conciernen a problemas de segundo orden en espacios localmente uniformes. Por otro lado, consideraremos ecuaciones de cuarto orden tanto en dominios acotados como no acotados.

Primero aplicamos los resultados a problemas de segundo orden en espacios localmente uniformes. En concreto estudiamos el problema

$$\begin{cases} u_t - \sum_{k,l=1}^N a_{kl}(x)\partial_k\partial_l u + \sum_{j=1}^N b_j(x)\partial_j u + c(x)u = 0 & x \in \mathbb{R}^N, \quad t > 0 \\ u(0,x) = u_0(x) & x \in \mathbb{R}^N, \end{cases} \tag{0.7.2}$$

donde los coeficientes reales de la parte principal elíptica del problema se asumen acotados y uniformemente continuos, es decir, $a_{kl} \in BUC(\mathbb{R}^N)$. Se asume que los coeficientes de grado inferior pertenecen a espacios localmente uniformes. En concreto asumimos que para $j = 1, ..., N$, $\|b_j\|_{\dot{L}_U^{p_j}(\mathbb{R}^N)} \le R_j$ y $\|c\|_{\dot{L}_U^{p_0}(\mathbb{R}^N)} \le R_0$, donde $p_j > N$ y $p_0 > \frac{N}{2}$.

Problemas parabólicos como (0.7.2) con coeficientes en espacios uniformes se han considerado con anterioridad; ver por ejemplo[3], [5], [47] y las referencias de estos artículos. Por ejemplo, los resultados de [3] permiten resolver (0.7.2) en espacios de Lebesgue $L^q(\mathbb{R}^N)$ asumiendo adicionalmente que

$$p_j \ge q > 1, \quad \text{para } j = 0, \dots, N. \tag{0.7.3}$$

Estos resultados se usaron luego en [5] para resolver (0.7.2) en espacios uniformes $L^q_U(\mathbb{R}^N)$, bajo las mismas restricciones, ver [5, Section 5], y más tarde, en [47, Section 6.2] con técnicas diferentes. Las restricciones en los coeficientes provocan que en [5, 47] sólo se pueda tomar dato inicial en $\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)$ para $\gamma \ge 0$.

Aquí, removemos la restriccióm (0.7.3), permitiendo una clase de datos iniciales más grande, en particular γ puede ser negativo. Cuando las restricciones adicionales (0.7.3) se imponen, se recuperan los resultados en [5, Theorem 5.3] y [47, Theorem 30]. Finalmente estudiaremos la continuidad del semigrupo con respecto a pequeños cambios en los coeficientes de orden bajo de (0.7.2).

Continuando con la aplicación de la teoría a ejemplos particulares, estudiamos después ecuaciones de cuarto orden en $\mathbb{R}^N$. Consideramos

$$\begin{cases} u_t + \Delta^2 u = Pu, & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0 & \text{en } \mathbb{R}^N \end{cases} \tag{0.7.4}$$

con u_0 un dato inicial adecuado definido en $\mathbb{R}^N$ y P una perturbación lineal con dependencia espacial de la forma $Pu := \sum_{a,b} P_{a,b} u$ con

$$P_{a,b} u := D^b(d(x) D^a u) \qquad x \in \mathbb{R}^N \tag{0.7.5}$$

para algún $a, b \in \{0, 1, 2, 3\}$ tal que $a + b \le 3$, donde D^a, D^b denota cualquier derivada parcial de orden a, b, y $d \in L^p_U(\mathbb{R}^N)$, $1 \le p < \infty$ definido como en (0.7.1).

El objetivo principal es resolver el problema (0.7.4) para una clase grande de datos iniciales u_0. En particular, tomamos datos iniciales en espacios de Lebesgue, $L^q(\mathbb{R}^N)$, $1 < q < \infty$, o en espacios de Bessel $H^{\alpha,q}(\mathbb{R}^N)$, con $1 < q < \infty$, $\alpha \in \mathbb{R}$ e incluso en los espacios de Bessel uniformes $\dot{H}^{\alpha,q}_U(\mathbb{R}^N)$ introducidos arriba. Para dichas clases de datos iniciales y perturbaciones encontramos también estimaciones de perturbación para las soluciones de (0.7.4).

Se pueden encontrar resultados previos sobre el tema cuando $P = 0$ en [21, 22] y [20, 10]. En estos artículos, la solución del problema (0.7.4) se describe como la convolución del dato inicial contra el núcleo fundamental del bi–Laplaciano, que satisface cotas Gaussian adecuadas.

Empezamos analizando el caso $P = 0$. Para usar las técnicas descritas hasta el momento, empezamos encontrando una escala de espacios asociada al operador principal, esto es, al Δ^2. Como se ha explicado, podríamos construir las escalas de interpolación y potencias fraccionarias a partir del Δ^2, sin embargo vamos a usar las escalas asociadas al Laplaciano $-\Delta$. Más concretamente, vamos a demostrar que la escala de interpolación y potencias fraccionarias asociada al Δ^2 es en realidad la misma que la asociada al $-\Delta$. Para ello, utilizamos primero resultados de [37] sobre potencias de operadores para probar que (bajo ciertas condiciones) la potencia de un operador que genera un semigrupo analítico en una escala también genera un semigrupo analítico en la misma escala y satisface estimaciones de regularización entre espacios de dicha escala.

Después, particularizamos este procedimiento para dos casos. Por un lado, consideramos el bi-Laplaciano Δ^2 en $L^p(\mathbb{R}^N)$. Se prueba que la escala es la misma que la asociada al $-\Delta$ en $L^p(\mathbb{R}^N)$, es decir, la escala de Bessel-Lebesgue. Para probar que la misma escala asociada a $-\Delta$ se puede usar para Δ^2 utilizamos estimaciones de la resolvente e información específica sobre el espectro de ambos operadores. Como estamos tratando con el $-\Delta$ en la escala de Lebesgue-Bessel, la estimación de la resolvente es ya conocida, ver por ejemplo [31, Sección 1.3].

Por otro lado consideramos el bi-Laplaciano Δ^2 en espacios uniformes, y en este caso se necesita trabajo extra. Ahora, a pesar de que era conocido que el $-\Delta$ genera un semigrupo analítico (ver Proposition 2.1, Theorem 2.1 y Theorem 5.3 en [5]), la estimación de la resolvente adecuada no se conocía. La demostramos siguiendo ideas de [31] para la escala de Lebesgue-Bessel estandar (no uniforme). Tras hacerlo, podemos establecer que el problema (0.7.4) se puede plantear en la escala de Bessel-Lebesgue uniforme.

Una vez que se situa el problema en cualquiera de las dos escalas de arriba, podemos aplicar los resultados abstractos para obtener existencia de solución, regularidad y el resto de propiedades para el problema (0.7.4) con $P = 0$.

Ahora podemos usar estos resultados para considerar el problema cuando $P \neq 0$. En [16] se probaron resultados para $P \neq 0$ en la escala de Bessel-Lebesgue. Mediante estimaciones de la resolvente para $\Delta^2 + P$, los autores probaron que (0.7.4) está bien propusto para $Pu = d(x)u$, esto es, una perturbación con $a, b = 0$. Encuentran además estimaciones de regularización para las soluciones.

Aquí, en vez de confiar en estimaciones elípticas para la resolvente para los operadores $\Delta^2 + P$, con P como en (0.7.5), utilizamos las técnicas de perturbación de [47]. La perturbación P actua entre dos espacios de la escala. Con estos ingredientes obtenemos un semigrupo perturbado que da la solución de la ecuación (0.7.4) con $P \neq 0$. Dicho semigrupo perturbado hereda algunas la las estimaciones del semigrupo original en algunos espacios de la escala que vienen determinados por la propia perturbación P, es decir, por a, b y la regularidad de $d(x)$.

Tras considerar los distintos tipos posibles de perturbación, estudiamos también todas las posibles combinaciones de perturbaciones.

A pesar de que principalmente abordamos ecuaciones de cuarto orden, el mismo tipo de técnicas se pueden extender a operadores de orden superior, así que explicaremos esto brevemente.

Finalmente, el problema se considera en dominios acotados. Primero estudiamos la escala de Bessel con condiciones de contorno de tipo Neumann. Esta escala incorpora información sobre las condiciones de frontera que varía dependiendo de la regularidad (es decir, de índice de la escala). Aplicamos entonces los resultados de perturbación a este contexto de la misma forma que lo hicimos en el caso no acotado y resolvemos el problema en el sentido integral de la Fórmula de Variación de las Constantes. Dada una perturbación, queremos releer la ecuación integral en términos de un problema de EPDs. Cuando la perturbación se define en el interior, el problema se estudia como en el caso de $\mathbb{R}^N$ y los resultados obtenidos son análogos a los de $\mathbb{R}^N$. Sin embargo, al estar ahora en un dominio acotado, podemos considerar perturbaciones definidas en la frontera de forma similar a lo que se hace en [47] para problemas de segundo orden, pero al tratar ahora con problemas de cuarto orden, la variedad de perturbaciones es mucho mayor. En este caso, es frecuente que la perturbación en la integral abstracta se corresponda a alguna condición de frontera del problema de EDP asociado. De hecho, puede ocurrir que una de tales perturbaciones se corresponda a términos en la frontera y en la ecuación. Ilustramos algunos ejemplos en detalle, dando la coreespondencia total entre la ecuación integral y el problema de EDP en esos casos. Sin embargo, una descripción de esta correspondencia para todas las posibles perturbaciones queda para futuros trabajos.

Mostramos ahora el Teorema 5.2.10 de la Parte I como un ejemplo del tipo de resultados que obtenemos. Este es el resultado para una única perturbación con $a \neq 0 \neq b$ en la escala de Bessel.

Theorem 0.7.1 *Let $P_{a,b} = D^b(d(x)D^a u)$ with $k, a, b \in \{0, 1, 2, 3\}$, $k = a + b$. Assume that $\|d\|_{L^p_U(\mathbb{R}^N)} \leq R_0$ with $p > \frac{N}{4-k}$, then for any $1 < q < \infty$ and such $P_{a,b}$, there exists an interval $I(q, a, b) \subset (-1 + \frac{a}{4}, 1 - \frac{b}{4})$ containing $(-1 + \frac{a}{4} + \frac{N}{4p}, 1 - \frac{b}{4} - \frac{N}{4p})$, such that for any $\gamma \in I(q, a, b)$, we have a strongly continuous analytic semigroup, $S_{P_{a,b}}(t)$, in the space $H^{4\gamma,q}(\mathbb{R}^N)$, for the problem*

$$\begin{cases} u_t + \Delta^2 u + D^b(d(x)D^a u) = 0, & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & \textit{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimates

$$\|S_{P_{a,b}}(t)u_0\|_{H^{4\gamma',q}(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{4\gamma,q}(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in H^{4\gamma,q}(\mathbb{R}^N)$$

for every $\gamma, \gamma' \in I(q, a, b)$ with $\gamma' \geq \gamma$, and

$$\|S_{P_{a,b}}(t)u_0\|_{L^r(\mathbb{R}^N)} \leq \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in L^q(\mathbb{R}^N)$$

with $1 < q \leq r \leq \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q, a, b)$ is given by

$$I(q, a, b) = (-1 + \frac{a}{4} + \frac{N}{4}(\frac{1}{p} - \frac{1}{q'})_+, 1 - \frac{b}{4} - \frac{N}{4}(\frac{1}{p} - \frac{1}{q})_+).$$

Finally, if

$$d_\varepsilon \to d \quad \textit{in } L^p_U(\mathbb{R}^N), \quad p > \frac{N}{4-k}$$

then for every $1 < q < \infty$ *and* $T > 0$ *there exists* $C(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*, such that*

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N), H^{4\gamma',q}(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \leq T$$

for all $\gamma, \gamma' \in I(q,a,b)$*,* $\gamma' \geq \gamma$ *and for any* $1 < q \leq r \leq \infty$

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \leq T.$$

En vista de todo esto la Parte I se estructura como sigue. Los resultados abstractos de [47] se recuperan en el Capítulo 1, ya que se utilizarán con frecuencia. Después, en el Capítulo 2 recogemos algunos resultados de [2, Chapter V] sobre escalas abstractas de espacios y los extendemos, usando las técnicas de perturbación de [47], para que sean válidos para cualquier operador sectorial. Los espacios uniformes (0.7.1) son de gran interés al aplicar a problemas concretos, ya que se pueden considerar datos iniciales de baja regularidad y por tanto los usaremos repetidamente en la tesis. el Capítulo 3 compila la definición detallada y la construcción de las escalas de Lebesgue y Bessel uniformes, repasando algunas de sus propiedades. Cabe destacar la Proposición 3.0.1 en este capítulo, que exhibe un embadding para la parte negativa de la escala, necesario para la mayoría de los resultados sobre espacios uniform en la Parte I y II.

Los resultados para la parte negativa de la escala uniforme para ecuaciones de segundo orden se pueden encontrar en el Capítulo 4. El Capítulo 5 se ocupa de problemas de cuarto orden en $\mathbb{R}^N$. En su primera sección, se construyen las escalas abstractas para cuadrados de operadores. En la segunda se usan estas escalas para ecuaciones de cuarto orden en espacios de Bessel-Lebesgue. La tercera estudia las ya mencionadas estimaciones de la resolvente para el $-\Delta$ y el Δ^2 en espacios uniformes. El capítulo 6 extiende estos resultados a problemas de orden superior. Finalmente, el Capítulo 7 estudia los problemas en dominios acotados.

En cuanto a la Parte II, en el contexto de (0.5.2) estudiamos el problema (0.6.4) cuando f es no lineal. Más concretamente, asumimos que existe $\rho \geq 1$, $L > 0$ tal que para $\alpha, \beta \in \mathcal{J}$, $\alpha \geq \beta$

$$\|f(u) - f(v)\|_\beta \leq L\big(1 + \|u\|_\alpha^{\rho-1} + \|v\|_\alpha^{\rho-1}\big)\|u - v\|_\alpha, \quad u, v \in X^\alpha. \tag{0.7.6}$$

Por tanto f es continua y

$$\|f(u)\|_\beta \leq L\big(1 + \|u\|_\alpha^\rho\big), \quad u \in X^\alpha \tag{0.7.7}$$

donde las constantes (0.7.6) y (0.7.7) se pueden elegir la misma.

Empezamos con el análisis de la ecuación integral no lineal abstracta (0.6.4), que es la fórmula de variación de las constantes correspondiente a soluciones débiles del problema no lineal

$$u_t + Au = f(u), \quad u(0) = u_0.$$

Buscamos el rango de γ para el cual (0.6.4) tiene una solución (en una clase que definiremos de forma adecuada) para cualquier $u_0 \in X^\gamma$ y para algún $T = T(u_0)$. La solución viene determinada por α, β y ρ.

Además, se estudian también la unicidad, dependencia continua con respecto al dato inicial, efecto regularizante y estimaciones de blow-up en este contexto abstracto.

Para ello, el primer paso antes de intentar resolver (0.6.4) es definir una noción adecuada de solución y para ello hay diversas opciones. En cualquier caso, para que (0.6.4) tenga sentido, cualquier definición de solución debe incluir los siguientes requisitos mínimos; $u : (0, T] \to X^\alpha$ y para todo $0 < \tau < T$ y para todo $\tau \le t \le T$, $u(t)$ satisfaces

$$u(t) = S(t-\tau)u(\tau) + \int_\tau^t S(t-s) f(u(s))\, ds.$$

Además, es natural requerir $\tau > 0$, $u \in L^\infty([\tau, T], X^\alpha)$. Finalmente, cualquier noción de solución debe incluir información acerca del dato inicial y de su comportamiento cerca de $t = 0$. Así pues, definimos

Definition 0.7.2 *Si $u_0 \in X^\gamma$, entonces $u \in L^\infty_{loc}((0, T], X^\alpha)$ tal que $t^{\alpha-\gamma}\|u(t)\|_\alpha \le M$, $t \in (0, T]$ para algún $M > 0$, $u(0) = u_0$ y (0.6.4) para $0 < t \le T$ es una γ–solución de (0.6.4) en $[0, T]$.*

Nótese que de (0.5.2), se obtiene que el comportamiento de la γ–solución en $t = 0$ es el mismo que el del semigrupo lineal $S(t)u_0$.

Para esta clase de soluciones probamos existencia, unicidad y dependencia continua con respecto al dato inicial para los siguientes rango de γ:

$$\gamma \in E(\alpha, \beta, \rho) = \begin{cases} (\alpha - \frac{1}{\rho}, \alpha], & \text{si } \ 0 \le \alpha - \beta \le \frac{1}{\rho} \\ [\frac{\alpha\rho-\beta-1}{\rho-1}, \alpha], & \text{si } \ \frac{1}{\rho} < \alpha - \beta < 1. \end{cases}$$

Cuando $\gamma = \frac{\alpha\rho-\beta-1}{\rho-1}$ se llama caso crírtico, el resto subcrítico. En particular, probaremos que dado $\gamma \in E(\alpha, \beta, \rho)$ como arriba, existe $r > 0$ tal que para todo $v_0 \in X^\gamma$ existe $T > 0$ tal que para todo u_0 tal que $\|u_0 - v_0\|_\gamma < r$ existe una γ–solución de (0.6.4) con dato inicial u_0 definido en $[0, T]$. En el caso subcrítico r se puede tomar arbitrariamente grande.

Se estudiará la regularización de γ-soluciones esto es, cómo entran en otros espacios de la escala de índice mayor. También daremos estimaciones del tiempo de existencia y, cuando éste es finito, estudiamos la tasa de blow–up para distintas normas.

Se mostrará que las condiciones de los teoremas son esencialmente óptimas. Más concretamente, cuando $\gamma < \inf E(\alpha, \beta, \rho)$ entonces no se puede esperar unicidad o dependencia continua para las soluciones. y por tanto el problema no está bien propuesto en general. En el caso crítico, es decir, cuando $\inf E(\alpha, \beta, \rho) = \frac{\rho\alpha-\beta-1}{\rho-1}$ mostramos que para $\gamma < \inf E(\alpha, \beta, \rho)$ no se puede esperar dependencia continua.

Finalmente, se extiende el resultado de unicidad a funciones que, satisfaciendo los requisitos mínimos comentados arriba y (0.6.4), son acotadas para $t = 0$ en X^γ en el caso subcrítico, o continuas para $t = 0$ en X^γ en el crítico.

Al aplicar estas técnicas a problemas concretos ocurre con frecuencia que hay muchas parejas (α, β) tales que el término f satisface (0.6.3) y (0.7.6). Dichas parejas conforman la "región admisible" asociada al problema considerado. En esta situación desarrollamos un método general de bootstrapp que podemos usar para todos los problemas concretos, que sirve para encontrar el mayor rango de γ posible para el que la solución se puede construir, así como el rango más grande de espacios para el que la solución regulariza. Se mostrará que estos rangos se pueden calcular a partir de la región admisible, con lo que se puede aplicar resultados abstractos a problemas concretos mediante la minimización de una función continua en una región convexa.

Lo aplicamos a problemas concretos de EDP en escalas de espacios concretos. Consideramos primero problemas del tipo

$$u_t + (-\Delta)^m u = f(x, u) := D^b(h(x, D^a u)), \ t > 0, \ x \in \mathbb{R}^N, \tag{0.7.8}$$

con $m \in \mathbb{N}$, donde D^c representa cualquier derivada parcial de orden $c \in \mathbb{N}$, $h(\cdot, 0) = 0$ y para algún $\rho > 1$, $L > 0$ tenemos

$$|h(x, u) - h(x, v)| \leq L|u - v|(|u|^{\rho-1} + |v|^{\rho-1}), \ u, v \in \mathbb{R}, \ x \in \mathbb{R}^N.$$

Queremos encontrar el rango de espacios para el que el problema está bien propuesto y estudiaremos también la unicidad en varias clases, dependencia continua con respecto al dato inicial, estimaciones de blow up cuando lo hay y estimaciones de regularización.

El problema (0.7.8) se estudia en varias escalas de espacios diferentes. Primero fijamos el problema en la escala de Lebesgue con $a = b = 0$. Los resultados aquí presentados, cuando $m = 1$ recuperan y mejoran ligeramente los resultados de [52], [53], [11], [4] y [46, Chapter 2], ver la discusión en la Sección 11.1 para los detalles. Después, se considera el problema en la escala de Bessel con $a \neq 0 \neq b$. Un caso particular de problema parabólico de cuarto orden con tal función no lineal para $a = 0$, $b = 2$ es la ecuación de Cahn-Hilliard

$$\begin{cases} u_t + \Delta^2 u + \Delta h(x, u) = 0, \ t > 0, \ x \in \mathbb{R}^N, \\ u(0, x) = u_0(x), \ x \in \mathbb{R}^N. \end{cases}$$

Resolvemos este problema en la escala de Bessel, lo que muestra la potencia de las herramientas abstractas (junto con el bootstrap) que hemos desarrollado, ya que los resultados para el problema de Cahn-Hilliard aparecen como un mero corolario, pero en realidad recupera resultados recientes en el tema (ver [17, 49, 56]).

Finalmente resolvemos (0.7.8) en espacios uniformes. Dichos espacios no son reflexivos, y por tanto no se puede utilizar la dualidad para dar una representación de P cuando $b \neq 0$. Por tanto, los resultados en este caso son para $b = 0$.

La última aplicación que presentamos es el estudio de la siguiente ecuación de ondas fuertemente amortiguada

$$\begin{cases} w_{tt} - \Delta w_t + w_t - \Delta w = h(x, w), \ t > 0, \ x \in \mathbb{R}^N, \\ w(0, x) = w_0(x), \quad w_t(0, x) = z_0(x), \ x \in \mathbb{R}^N. \end{cases}$$

con h como en (0.6.6). Este problema se puede reescribir como

$$\dot{u} + Au = f(u) := \left[\begin{smallmatrix} 0 \\ h(w)+w \end{smallmatrix}\right], \quad u(0) = u_0 := \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right]$$

en una escala de espacios adecuada que será descrita, y por tanto está dentro del marco para aplicar la teoría, y obtenemos de nuevo resultados similares acerca de existencia, unicidad, regularidad y blow up.

Mostramos ahora el Teorema 11.2.11 de la Parte II como ejemplo del tipo de resultados que probamos.

Theorem 0.7.3 *Assume h is as in (0.3.8) for some $\rho > 1$, $L > 0$. Denote $k = a+b < 2m$ and assume $p_0 < \rho p(1 - \frac{k}{2m})$ with $p_0 := \frac{N}{2m}(\rho - 1)$. Then for*

$$\gamma_c = \max\left\{ \frac{N}{2m}\left(\frac{1}{p} - \frac{1}{\rho}\right)_+ + \frac{a}{2m} - \frac{1}{\rho}, \ \frac{N}{2mp} + \frac{a\rho + b}{2m(\rho-1)} - \frac{1}{\rho - 1} \right\} < \gamma < 1 - \frac{b}{2m}$$

there exist $r > 0$ and $T > 0$, such that for any $v_0 \in H^{2m\gamma,p}(\mathbb{R}^N)$ and any u_0 satisfying $\|u_0 - v_0\|_{H^{2m\gamma,p}(\mathbb{R}^N)} < r$, there exists a function $u(\cdot, u_0)$ such that for all $\gamma \leq \gamma' < 1 - \frac{b}{2m}$, $u(\cdot, u_0) \in C((0,T], H^{2m\gamma',p}(\mathbb{R}^N)) \cap C([0,T], H^{2m\gamma,p}(\mathbb{R}^N))$ and

$$t^{\gamma'-\gamma}\|u(t,u_0)\|_{H^{2m\gamma',p}(\mathbb{R}^N)} \leq M(u_0,\gamma') \qquad \textit{for } 0 < t < T$$

$$t^{\gamma'-\gamma}\|u(t,u_0)\|_{H^{2m\gamma',p}(\mathbb{R}^N)} \to 0 \qquad \textit{as } t \to 0, \gamma' \neq \gamma$$

and satisfies

$$u(t,u_0) = S(t)u_0 + \int_0^t S(t-s)h(u(s,u_0))ds \quad t \in [0,T].$$

Also, there exists $M > 0$ such that for all $u_0^i \in H^{2m\gamma,p}(\mathbb{R}^N)$, $i = 1,2$ such that $\|u_0^i - v_0\|_{H^{2m\gamma,p}(\mathbb{R}^N)} < r$, we have for $\gamma' \in [\gamma, 1 - \frac{b}{2m})$

$$\|u(t,u_0^1) - u(t,u_0^2)\|_{H^{2m\gamma',p}(\mathbb{R}^N)} \leq \frac{M}{t^{\gamma'-\gamma}}\|u_0^1 - u_0^2\|_{H^{2m\gamma,p}(\mathbb{R}^N)}, \quad t \in (0,T].$$

When $\frac{N}{2mp} + \frac{a\rho+b}{2m(\rho-1)} - \frac{1}{\rho-1} > \frac{N}{2m}\left(\frac{1}{p} - \frac{1}{\rho}\right)_+ + \frac{a}{2m} - \frac{1}{\rho}$ then the above hold also for $\gamma = \gamma_c = \frac{N}{2mp} + \frac{a\rho+b}{2m(\rho-1)} - \frac{1}{\rho-1}$.

If $\gamma_c < \gamma < 1 - \frac{b}{2m}$ then r can be taken arbitrarily large, that is, the existence time is uniform in bounded sets in $H^{2m\gamma,p}(\mathbb{R}^N)$.

Hay una gran variedad de resultados previos donde el marco abstracto de la Fórmula de Variación de las Constantes se usa para resolver problemas de EDP que nos han servido de inspiración.

En [52] se estudia el problema abstracto (0.6.4) para una clase de soluciones similar a nuestras γ-soluciones. Después, los resultados se aplican a (0.6.6) con $a = b = 0$ en un dominio acotado en espacios de Lebesgue con condiciones Dirichlet (que hacen que la escala de Lebesgue sea anidada), obteniendo el mismo rango de existencia que nosotros para ese caso. El resultado de unicidad del Teorema 3 en [52] es para una clase de soluciones más pequeña que la que obtenemos en nuestro resultado (Teorema 11.1.6), ver también Teorema 2.a).i) en [52].

En [53] se obtienen estimaciones de blow–up para (0.6.6) con $a = b = 0$, $m = 1$. Nuestro resultado (11.1.11), particularizado para $m = 1$, $q = p$ y $p > \frac{N}{2m}(\rho - 1)$, recupera las mencionadas estimaciones en [53].

En [29] se obtiene un resultado de no unicidad para (0.6.6) con $a = b = 0$, $m = 1$, $u(0) = 0$ para $A = -\Delta$ in $\mathbb{R}^N$ de forma muy similar a lo que se hace en la Proposición 9.3.1.

Después, [9] estableció no unicidad sin asumir que $u(0) = 0$, para soluciones positivas, radiales, decrecientes en dominios acotados. Resultados similares se pueden encontrar en [43], Teoremas 3 y 4.

En [28] se considera el (0.6.6) con $a = b = 0$, $m = 1$ en un dominio acotado o en $\mathbb{R}^N$. Se obtiene unicidad para $p = \frac{N}{2m}(\rho - 1)$ en la clase $L^r((0,T], L^q(\mathbb{R}^N))$ con $\frac{1}{r} = \frac{N}{2}(\frac{1}{p} - \frac{1}{q})$, $q, r > \rho$, $q > p$. La clase de unicidad en nuestro Teorema 11.1.6 ii) es una subclase de $u \in L^r((0,T], L^q(\mathbb{R}^N))$ con $\frac{1}{r} > \frac{N}{2}(\frac{1}{p} - \frac{1}{q})$. Se estudian también estimaciones de blow–up, y coinciden de nuevo con las que obtenemos aquí.

En [11] el problema (0.6.6) con $a = b = 0$, $m = 1$ se estudia en un dominio acotado en espacios de Lebesgue con condiciones Dirichlets. En dicho artículo, se centran en mejorar el resultado de unicidad de [52], extendiéndolo a clases de funciones más grandes. Más concretamente, se enuncia para la clase de soluciones clásicas de (0.6.6) (con $a = b = 0$) tales que $u \in C([0,T], L^p(\mathbb{R}^N))$ que es un caso particular de nuestro Teorema 11.1.6.

En [4] se estudia el problema abstracto en la escala de potencias fraccionarias asociada a un operador sectorial y aplican sus resultados a la ecuación de Navier-Stokes y a problemas de la forma (0.6.6) con $a = b = 0$ en escalas similares a la escala de Bessel. Más concretamente, consideran la escala de potencias fraccionarias asociada a A, $\{Y^\alpha\}_{\alpha \geq 0}$ y prueban existencia de soluciones ε-regulares para el problema (0.6.4) con dato inicial $u_0 \in Y^1$. Nuestro enfoque contiene al de [4] y en su caso, podemos construir soluciones para más espacios de datos iniciales , no sólo para Y^1. Recíprocamente, estudiar nuestro problema con su enfoque requiere asumir hipótesis adicionales, ver el Remark 9.1.16.

Por tanto, nuestro ectiende los resultados previos de la siguiente forma. Por un lado, utilizamos escalas que no son necesariamente la escala de Lebesgue o de potencias fraccionarias, de hecho, usamos escalas que ni siquiera tienen por qué ser anidadas. Además, los resultados previos escogen el dato incial en un espacio fijado, mientras que nosotros encajamos el dato inicial en espacios de la escala y estudiamos el problema para todos esos espacios al mismo tiempo. Por otro lado, incluso cuando nos restringimos al mismo contexto que los trabajos previos, el rango de espacios para el que podemos escoger un dato inicial dado un problema es más grande que el que se obtiene en los resultados de [31] o [4], ver de nuevo el Remark 9.1.16.

Teniendo todo esto en cuenta, la Parte II se estructura de la siguiente manera. El

Capítulo 8 compila algún material preparatorio sobre la Fórmula de Variación de las Constantesjunto con alguna definición para usar más adelante. Después, en el Capítulo 9 se desarrolla toda la teoría abstracta. Primero la existencia, luego la unicidad mejorada y finalmente el estudio de la optimalidad de los resultados. A continuación, en el Capítulo 10 desarrollamos en detalle el argumento de bootstrap que sirve como puente entre los resultados abstractos y la aplicación a problemas concretos. Posteriormente, la aplicación al problema (0.7.8) se realiza en el Capítulo 11. Finalmente, en el Capítulo 12 tratamos la ecuación de ondas fuertemente amortiguada.

0.8 Conclusiones

Los resultados más importantes de la tesis se pueden resumir de la siguiente manera

- Resolvemos el problema lineal de segundo orden (0.6.1) con $a_{kl} \in BUC(\mathbb{R}^N)$, $\|b_j\|_{\dot{L}^{p_j}_U(\mathbb{R}^N)} \leq R_j$, $j = 1, ..., N$ y $\|c\|_{\dot{L}^{p_0}_U(\mathbb{R}^N)} \leq R_0$, donde $p_j > N$ y $p_0 > \frac{N}{2}$ en espacios de Bessel uniformes, ver Teorema 4.0.6. Estas condiciones son menos restrictivas que las que se conocían en resultados previos para el problema (0.2.1) en espacios uniformes.

- La escala de espacios asociada a un operador que genera un semigrupo analítico se puede usar también para el cuadrado de dicho operador, que además también genera un semigrupo en esa escala, ver Proposiciones 5.1.3, 5.1.4.

- Se resuelve el problema lineal de cuarto orden (0.6.2) en la escala de Bessel, Teorema 5.2.10.

- El Laplaciano en el espacio uniforme de Lebesgue $\dot{L}^q_U(\mathbb{R}^N)$ satisface la estimación

$$\|(-\Delta - \lambda)^{-1}\|_{\mathcal{L}(E^0)} \leq M|\lambda|^{-1}$$

  para todo λ en un sector $S_{0,\phi}$ definido como en (5.2.3) para $\phi > 0$ arbitrariamente pequeño, ver la Proposición 5.3.1. Esto implica que el bi-Laplaciano Δ^2 genera un analytic semigrupo la escala de Bessel uniforme.

- Se resuelve el problema lineal de cuarto orden (0.6.2) en la escala de Bessel uniforme, ver Teorema 5.3.5.

- Se prueba que el problema no lineal abstracto (0.6.4) está bien propuesto para γ-soluciones como las de la Definición 0.7.2 para $\gamma \in E(\alpha, \beta, \rho)$, ver los Teoremas 9.1.7,9.1.8.

- Se extienden los resultados de unicidad, ver Teorema 9.2.2.

- Se resuelve el problema no lineal (0.7.8) en espacios de Lebesgue, Bessel, Lebesgue uniforme y Bessel uniforme, ver Capítulo 11.

- Se resuleve la ecuación de ondas fuertemente amortiguada (0.6.7), ver Teorema 12.0.3.

Los resultados de la tesis han sido expuestos de la siguiente manera. Los resultados de los Capítulos 2,3, 5 y 6 están publicados en el artículo "Smoothing and perturbation for some fourth order linear parabolic equations in $\mathbb{R}^N$", Journal of Mathematical Analysis and Applications, volumen 412, páginas 1105-1134, 2014.

Los resultados del Capítulo 4 se han enviado a publicar y además están parcialmente recogidos en el artículo "Perturbation of analytic semigroups in uniform spaces in $\mathbb{R}^N$", Advances in differential equations and applications, volumen 4 de SEMA SIMAI Springer Ser., páginas 41–52. Springer, Cham, 2014.

Los resultados en el Capítulo 7 se están preparando para enviar a publicar.

Los problemas no lineales de los Capítulos 8,9,10,11 y 12 están enviados a publicar.

Los resultados también se han presentado en conferencias y charlas en variedad de congresos, seminarios y workshops, algunos de carácter internacional, como el XXII CEDYA de 2011 en Mallorca, el XXIII CEDYA de 2013 en Castellon, el ICMC Summer Meeting on Differential Equations 2013 en Brazil, el 10th AIMS Conference 2014 en Madrid y el XXIV CEDYA de 2015 en Cádiz.

Finalmente, comentamos posibles cuestiones para futuros trabajos.

- Una posible forma de continuar con el enfoque que realizamos en el caso lineal cuando el dominio es $\mathbb{R}^N$ podría ser construir escalas para operadores de orden $2m$ que no se obtengan como el cuadrado (u otra potencia) del Laplaciano. Además, se podría considerar el Laplaciano fraccionario para la escala asociada al Laplaciano.

- Uno de los principales temas que queda por entender en su totalidad es el de los problemas lineales en dominios acotados. A pesar de tener una descripción completa de la parte abstracta y de algunos casos particulares, al tratar algunos problemas concretos, hay perturbaciones en la frontera para las cuales no podemos establecer una correspondencia con un problema de EDP. En esos casos sabemos resolver el problema integral abstracto pero queda por establecer a qué problema particular de EDP se corresponde.

- También se podría estudiar la construcción de escalas con condiciones de contorno para operadores que no aparecen como el cuadrado del Laplaciano y estudiar perturbaciones en esos casos.

- Otra posible continuación del trabajo es considerar varias perturbaciones no lineales al mismo tiempo. En algunos ejemplos de la tesis se considera más de una perturbación, sin embargo tratar el problema ya en la parte abstracta para más de una perturbación puede llevar a una mejora en el rango de datos iniciales.

- Finalmente, el enfoque de esta tesis se puede aplicar también a problemas no lineales en dominios acotados, donde aparecerán perturbaciones en el interior y en la frontera.

Part I

Linear parabolic problems

This part is devoted to the study of parabolic problems with linear perturbations. In particular we consider abstract problems of the form

$$\begin{cases} u_t + Au = Pu, & t > 0 \\ u(0,x) = u_0(x) \end{cases} \tag{i.0.1}$$

where the main operator, A, is some differential operator, P is a linear perturbation and u_0 is the initial data in some space to be considered. The abstract techniques will be applied to solving second and fourth order problems in Lebesgue, Bessel and uniform scales in bounded and unbounded domains. We will briefly explain how to extend these results to higher order problems.

Throughout this part we focus on choosing suitable spaces for initial data, so that problem (i.0.1) is well posed. We are also interested on properties of the solutions, such as smoothing estimates and robustness with respect to small changes in P.

We will start this part by recalling some abstract results that we will use extensively throughout the thesis. Firstly, in Chapter 1, we recover the main abstract results from the above mentioned [47]. Then, Chapter 2 revisits some results from [2] about scales of spaces which will be used again in Part II.

As stated in the introduction, when dealing with examples we will use the uniform spaces in (0.3.1) in many cases. Thus, before studying any particular problem, in Chapter 3 we recall the definition and known properties of this spaces. There, we also prove Proposition 3.0.1 which provides a description of the negative side of this scale.

We then focus on the application of the abstract theory to particular problems. We start by studying second order problems in $\mathbb{R}^N$ have been for initial data in uniform spaces. Proposition 3.0.1 allows to extend and improve some previous results in second order parabolic equations in uniform spaces. Being more specific, the problem

$$\begin{cases} u_t - \sum_{k,l=1}^N a_{kl}(x)\partial_k\partial_l u + \sum_{j=1}^N b_j(x)\partial_j u + c(x)u = 0 & x \in \mathbb{R}^N, \quad t > 0 \\ u(0,x) = u_0(x) \qquad x \in \mathbb{R}^N, \end{cases} \tag{i.0.2}$$

where the real coefficients of the elliptic principal part of the equation are assumed to be bounded and uniformly continuous, that is, $a_{kl} \in BUC(\mathbb{R}^N)$. The lower order coefficients are assumed to belong to locally uniform Lebesgue spaces. In particular we will assume that for $j = 1, ..., N$, $\|b_j\|_{\dot{L}_U^{p_j}(\mathbb{R}^N)} \leq R_j$ and $\|c\|_{\dot{L}_U^{p_0}(\mathbb{R}^N)} \leq R_0$, where $p_j > N$ and $p_0 > \frac{N}{2}$.

Parabolic problems like (i.0.2) with coefficients in uniform spaces have been considered before; see e.g. [3], [5], [47] and references therein. For example, the results in [3] allow to solve (i.0.2) in Lebesgue spaces $L^q(\mathbb{R}^N)$ assuming additionally that

$$p_j \geq q > 1, \quad \text{for } j = 0, \dots, N. \tag{i.0.3}$$

These results were later used in [5] to solve (i.0.2) in uniform spaces $L^q_U(\mathbb{R}^N)$, under the same restrictions, see [5, Section 5], and later on, in [47, Section 6.2] with different techniques. Because of the restrictions above in the coefficients the result in [5, 47] just allowed to take initial data in $\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)$ for some $\gamma \geq 0$.

Here, we remove restriction (i.0.3), allowing a larger class of initial data, in particular γ can be even negative. When the additional assumptions (i.0.3) above are imposed, the results from Theorem 5.3 in [5] and Theorem 30 in [47] are recovered. Finally we will study the continuity of the semigroups with respect to small changes in the lower order coefficients of (i.0.2).

Continuing with application of the theory to particular examples, Chapter 5 deals fourth order equations in $\mathbb{R}^N$. We will consider

$$\begin{cases} u_t + \Delta^2 u = Pu, & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N \end{cases} \tag{i.0.4}$$

with u_0 a suitable initial data defined in $\mathbb{R}^N$ and P a linear space dependent perturbation of the form $Pu := \sum_{a,b} P_{a,b}u$ with

$$P_{a,b}u := D^b(d(x)D^a u) \qquad x \in \mathbb{R}^N$$

for some $a, b \in \{0, 1, 2, 3\}$ such that $a+b \leq 3$, where D^a, D^b denote any partial derivatives of order a, b, and $d \in L^p_U(\mathbb{R}^N)$, $1 \leq p < \infty$.

The main goal is to solve the problem (i.0.4) for large classes of initial data u_0. In particular, we will consider for initial data the standard Lebesgue space, $L^q(\mathbb{R}^N)$, $1 < q < \infty$, or Bessel-Lebesgue spaces $H^{\alpha,q}(\mathbb{R}^N)$, with $1 < q < \infty$, $\alpha \in \mathbb{R}$ and even uniform Bessel spaces $\dot{H}^{\alpha,q}_U(\mathbb{R}^N)$ introduced above. Given such classes of initial data and perturbations we also find suitable smoothing estimates on the solutions of (i.0.4).

In order to do this, Chapter 5 is divided in three sections. The first one deals with the construction of a suitable scale for the bi-Laplacian and even more, shows that this scale is in fact the same as the one associated to the Laplacian. This, together with the results from [47] is applied to the Lebesgue and Bessel scales of space in Section 5.2 and to the uniform Lebesgue-Bessel scale in Section 5.3.

Then, in Chapter 6 similar results are obtained for higher order operators.

Finally, in Chapter 7 we study the problem when the main operator is again the bi-Laplacian, but the domain is now bounded. As above, the approach is the same so we start by considering a scale to set the problem, in this case the Bessel scale with Neumann boundary conditions. This scale incorporates information about the boundary conditions which varies depending on the regularity (in other words, the index in the scale). Then we consider a linear perturbation and apply the perturbation results from [47] into this setting in the same way as we did in the unbounded case and obtain analogous theorems, solving the problem in the integral sense of the Variation of Constants Formula. In particular, we obtain the range of spaces in which the initial data can be taken and the problem is well posed. Then, given a perturbation, we re-read the integral equation in terms of an actual PDE problem. When the perturbation is defined in the interior of the domain, the problem is handled in the same way as it was done in $\mathbb{R}^N$ and the results we obtain are analogous to the ones for $\mathbb{R}^N$. However, being now in a bounded domain allows to consider perturbations defined on the boundary. We illustrate some examples in detail, giving the full correspondence of the integral equation and the PDE in those cases.

Chapter 1

Some results on perturbation of semigroups

We start by recalling some results from [47] that will be used extensively throughout Part I. Some results in the same spirit will be proved at the beginning or Part II in order to deal with nonlinear terms.

Let $\{X^\alpha\}_{\alpha\in I}$ be a family of Banach spaces, with α in an interval I, endowed with a norm $\|\cdot\|_\alpha$. Let $S(t)$ be a semigroup on a scale $\{X^\alpha\}_{\alpha\in I}$, such that

$$\|S(t)\|_{\beta,\alpha} := \|S(t)\|_{\mathcal{L}(X^\beta,X^\alpha)} \leq \frac{M_0(\beta,\alpha)}{t^{\alpha-\beta}}, \qquad \forall \quad 0 < t \leq 1 \tag{1.0.1}$$

for all $\alpha, \beta \in I$, $\alpha \geq \beta$ for some constant $M_0(\beta,\alpha) > 0$.

Now, assume that for some fixed $\alpha \geq \beta$, with $0 \leq \alpha - \beta < 1$ we have a linear perturbation satisfying

$$P \in \mathcal{L}(X^\alpha, X^\beta). \tag{1.0.2}$$

$$0 \leq \alpha - \beta < 1. \tag{1.0.3}$$

We will sometimes use "nested" spaces, that is, for all $\alpha, \beta \in I$ with $\alpha \geq \beta$ we have

$$X^\alpha \subset X^\beta \tag{1.0.4}$$

with continuous inclusion and the norm of the inclusion will be denoted $\|i\|_{\alpha,\beta}$. This will be explicitly stated when used.

Consider the perturbed problem

$$u(t;u_0) = S(t)u_0 + \int_0^t S(t-\tau)Pu(\tau;u_0)\,d\tau, \qquad t > 0, \tag{1.0.5}$$

which corresponds to solving the problem $u_t + Au = Pu$, where $-A$ is the infinitesimal generator of the semigroup $S(t)$.

The following result is taken from [47, Proposition 10] and states the existence of a perturbed semigroup defined by (1.0.5).

Theorem 1.0.1 *Assume (1.0.1), (1.0.2), and (1.0.3). Then for every $R_0 > 0$ and every*

$$P \in \mathcal{L}(X^\alpha, X^\beta) \quad \textit{with} \quad \|P\|_{\mathcal{L}(X^\alpha, X^\beta)} \leq R_0$$

and for every $\gamma, \gamma' \in I$ such that

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta + 1), \qquad \gamma' \geq \gamma, \tag{1.0.6}$$

there exist constants $\omega = \omega(\gamma, \gamma', R_0) \geq 0$ and $M_0 = M_0(\gamma, \gamma', R_0)$ such that, for $t > 0$, there exists a unique solution of (1.0.5), which defines a mapping from X^γ into $X^{\gamma'}$ as

$$S_P(t)u_0 := u(t; u_0), \quad \textit{for all} \quad t > 0$$

such that

$$\|S_P(t)u_0\|_{\gamma'} \leq M_0 e^{\omega t} t^{-(\gamma' - \gamma)} \|u_0\|_\gamma, \quad \gamma' \geq \gamma. \tag{1.0.7}$$

In particular for any $\gamma \in [\beta, \alpha]$, $S_P(t) \in \mathcal{L}(X^\gamma)$ and it is a semigroup of linear continuous operators in X^γ.

The same is true for any $\gamma \in E(\alpha)$, if the scale is nested.

Now we turn into the continuity of the perturbed semigroup with respect to the perturbation. With the setting above, assume that we have two perturbations

$$P_i \in \mathcal{L}(X^\alpha, X^\beta), \quad i = 1, 2, \quad 0 \leq \alpha - \beta < 1.$$

Our goal is then to compare semigroups $S_{P_i}(t)$, $i = 1, 2$. Hence assume

$$\|P_i\|_{\mathcal{L}(X^\alpha, X^\beta)} \leq R_0 \quad i = 1, 2$$

for some $R_0 > 0$. Also, consider the existence and regularity intervals as in (1.0.6)

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta + 1), \qquad \gamma' \geq \gamma.$$

Consider then an initial data $u_0 \in X^\gamma$, and the corresponding solutions of the perturbed problem

$$u^i(t; u_0) = S_{P_i}(t)u_0 = S(t)u_0 + \int_0^t S(t - \tau) P_i u^i(\tau; u_0)\, d\tau, \qquad t > 0.$$

Then we have the following continuity result, see [47, Theorem 14].

Theorem 1.0.2 *With the notations above, for any $R_0 > 0$, there exists a sufficiently small T_0 such that for all perturbations P_i, $i = 1, 2$, such that $\|P_i\|_{\mathcal{L}(X^\alpha, X^\beta)} \leq R_0$,*

$$\|S_{P_1}(t) - S_{P_2}(t)\|_{\mathcal{L}(X^\gamma, X^{\gamma'})} \leq \frac{L(T_0, R_0)}{t^{\gamma' - \gamma}} \|P_1 - P_2\|_{\mathcal{L}(X^\alpha, X^\beta)}, \qquad \textit{for all} \quad 0 < t \leq T_0$$

and for every $T > T_0$

$$\|S_{P_1}(t) - S_{P_2}(t)\|_{\mathcal{L}(X^\gamma, X^{\gamma'})} \leq L(T, T_0, R_0) \|P_1 - P_2\|_{\mathcal{L}(X^\alpha, X^\beta)}, \qquad \textit{for all} \quad T_0 < t \leq T.$$

Finally we will also need the following result about the analyticity of the semigroup defined by (1.0.5). Note that the statement assuming the first interpolation property in the Theorem below is taken from [47, Theorem 12]. The statement assuming (1.0.9) is introduced here and will also be needed further below.

Theorem 1.0.3 *Assume the scale is nested, that is, (1.0.4), and that for any $\gamma \in I$, if $-A$ denotes the infinitesimal generator of $S(t)$ in X^γ, then its domain is given by $D(A) = X^{\gamma+1}$.*

Also assume the scale satisfies either one of the following interpolation properties:
i) If Y is a Banach space and $T \in \mathcal{L}(X^\gamma, Y)$ and $T \in \mathcal{L}(X^{\gamma'}, Y)$ then $T \in \mathcal{L}(X^{\theta\gamma+(1-\theta)\gamma'}, Y)$ for $\theta \in [0,1]$ and

$$\|T\|_{\mathcal{L}(X^{\theta\gamma+(1-\theta)\gamma'},Y)} \leq \|T\|^{\theta}_{\mathcal{L}(X^\gamma,Y)} \|T\|^{1-\theta}_{\mathcal{L}(X^{\gamma'},Y)}. \tag{1.0.8}$$

ii) The following condition is satisfied for any $\gamma, \gamma' \in I$ and $0 < \theta < 1$

$$\|u\|_{X^{\theta\gamma+(1-\theta)\gamma}} \leq C \|u\|^{\theta}_{X^\gamma} \|u\|^{1-\theta}_{X^{\gamma'}}. \tag{1.0.9}$$

Finally, as in Theorem 1.0.1, assume that for some fixed $\alpha \geq \beta$, with $0 \leq \alpha - \beta < 1$ we have a linear perturbation satisfying

$$P \in \mathcal{L}(X^\alpha, X^\beta) \quad \textit{with} \quad \|P\|_{\mathcal{L}(X^\alpha,X^\beta)} \leq R_0.$$

Then, there exists some $0 < \omega_0 = \omega_0(R_0)$ such that for any $Re(\lambda) \geq \omega_0$ and any $\gamma \in (\alpha - 1, \beta)$ the operator $A + \lambda I - P$, between $X^{\gamma+1}$ and X^γ, is invertible and

$$\|\big(A + \lambda I - P\big)^{-1}\|_{\mathcal{L}(X^\gamma,X^\gamma)} \leq \frac{C}{|\lambda|}, \qquad Re(\lambda) \geq \omega_0$$

and

$$\|\big(A + \lambda I - P\big)^{-1}\|_{\mathcal{L}(X^\gamma,X^{\gamma+1})} \leq C, \qquad Re(\lambda) \geq \omega_0$$

where C is independent of P and λ.

In particular, for every $\gamma \in (\alpha - 1, \beta)$, the semigroup $S_P(t)$ in X^γ in Theorem 1.0.1 is analytic.

Proof. The proof of part i) can be found in [47, Theorem 12].

Under the assumption in ii) the same proof remains unchanged up to the point where for all $\gamma \in I$ the following inequalities are obtained

$$\|(A + \lambda)^{-1}\|_{\mathcal{L}(X^\gamma,X^\gamma)} \leq \frac{C}{|\lambda|}, \qquad \mathrm{Re}(\lambda) \geq \omega$$

$$\|(A + \lambda)^{-1}\|_{\mathcal{L}(X^{\gamma+1},X^{\gamma+1})} \leq \frac{C}{|\lambda|}, \qquad \mathrm{Re}(\lambda) \geq \omega \tag{1.0.10}$$

$$\|(A + \lambda)^{-1}\|_{\mathcal{L}(X^\gamma,X^{\gamma+1})} \leq C, \qquad \mathrm{Re}(\lambda) \geq \omega.$$

At this point we proceed as follows. For any $\gamma \in I$ and $\tilde{\gamma} \in (\gamma, \gamma+1)$ we have that $\gamma+1 \in (\tilde{\gamma}, \tilde{\gamma}+1)$ and thus, using (1.0.10) and (1.0.9), we get for $\mathrm{Re}(\lambda) \geq \omega$

$$\|(A+\lambda)^{-1}u\|_{\gamma+1} \leq \|(A+\lambda)^{-1}u\|_{\tilde{\gamma}}^{\theta}\|(A+\lambda)^{-1}u\|_{\tilde{\gamma}+1}^{1-\theta} \leq \frac{C}{|\lambda|^{\theta}}\|u\|_{\tilde{\gamma}}^{\theta}\|u\|_{\tilde{\gamma}}^{1-\theta} = \frac{C}{|\lambda|^{\theta}}\|u\|_{\tilde{\gamma}}$$

for θ such that $\gamma+1 = \theta\tilde{\gamma} + (1-\theta)(\tilde{\gamma}+1)$, that is, $\theta = \tilde{\gamma} - \gamma$. Hence we get

$$\|(A+\lambda)^{-1}\|_{\mathcal{L}(X^{\tilde{\gamma}}, X^{\gamma+1})} \leq \frac{C}{|\lambda|^{\tilde{\gamma}-\gamma}}, \qquad \mathrm{Re}(\lambda) \geq \omega.$$

Now the proof concludes as in [47, Theorem 12]. ■

Chapter 2

Scales of spaces for sectorial operators

We now construct suitable scales of spaces for sectorial operators in Banach spaces. Sectorial operators generate analytic semigroups, so the results from the previous Chapter apply to the operators considered here. The constructions for the scales of spaces follow [2] and, in view of the applications later on, we particularize for the scale of complex interpolation–extrapolation spaces and the scale of fractional power spaces.

Following [2], let E^0, E^1 be Banach spaces with continuous inclusion $E^1 \subset E^0$ and consider the class $\mathcal{H}(E^1, E^0)$ of linear operators in E^0, with domain E^1 such that if $A_0 \in \mathcal{H}(E^1, E^0)$, then $-A_0$ generates a strongly continuous analytic semigroup in E^0, $\{e^{-A_0 t}; t \geq 0\}$.

For generators of analytic semigroups we have the following well known definitions.

Definition 2.0.1
i) [31, Definition 1.3.1 pg 18]. A closed operator in a Banach space E^0, A_0, with domain $D(A_0)$, is sectorial if there exists a sector

$$S_{a,\phi} = \{z \in \mathbb{C} : \phi \leq |arg(z-a)| \leq \pi,\ z \neq a\} \subset \rho(A_0) \tag{2.0.1}$$

for some $a \in \mathbb{R}$ and $\phi \in (0, \pi/2)$, such that

$$\|(A_0 - \lambda)^{-1}\|_{E^0} \leq M|\lambda - a|^{-1} \qquad \text{for all } \lambda \in S_{a,\phi}. \tag{2.0.2}$$

ii)[2, Section 1.2]. $\mathcal{H}(E^1, E^0) = \bigcup_{\substack{\kappa \geq 1 \\ \omega > 0}} \mathcal{H}(E^1, E^0, \kappa, \omega)$, where $A_0 \in \mathcal{H}(E^1, E^0, \kappa, \omega)$ if $-\omega + A_0 \in \mathcal{L}is(E^1, E^0)$ and

$$\kappa^{-1} \leq \frac{\|(A_0 - \lambda)x\|_{E^0}}{|\lambda|\|x\|_{E^0} + \|x\|_{E^1}} \leq \kappa, \qquad Re(\lambda) \leq -\omega \qquad x \in E^1. \tag{2.0.3}$$

The following result establishes the equivalence between both definitions.

Proposition 2.0.2 *Both definitions i) and ii) in Definition 2.0.1 are equivalent.*

Proof. i)⇒ii)

Define $E^1 := D(A_0)$ with the graph norm, that is

$$\|\cdot\|_{E^1} := \|\cdot\|_{G(A_0)} := \|\cdot\|_{E^0} + \|A_0(\cdot)\|_{E^0}.$$

Note that [2, Remark 1.2.1 pg 11] proves (2.0.3) provided we prove

$$|\lambda|\|x\|_{E^0} \leq \kappa\|(A_0-\lambda)x\|_{E^0} \qquad \mathrm{Re}(\lambda) \leq -\omega \qquad x \in E^1.$$

Thus from (2.0.2) we get

$$M|\lambda - a|\|x\|_{E^0} \leq \|(A_0-\lambda)x\|_{E^0} \qquad \text{for all } \lambda \in S_{a,\phi},\ x \in D(A_0) = E^1.$$

Now, if we take $\omega > 0$ such that $-\omega <$Re(a), then $-\omega \in \rho(A_0)$, thus $-\omega + A_0 \in \mathcal{L}is(E^1, E^0)$ and $\frac{|\lambda|}{|\lambda-a|} \leq \tilde{M}$ for all $\mathrm{Re}(\lambda) \leq -\omega$. Hence

$$\tilde{M}M|\lambda|\|x\|_{E^0} \leq \|(A_0-\lambda)x\|_{E^0} \qquad \mathrm{Re}(\lambda) \leq -\omega \qquad x \in E^1.$$

ii)⇒i)

For proving this, we first use [2, Proposition I.1.4.1, pg 15], which read in terms of our notation, states that if $A_0 \in \mathcal{H}(E^1, E^0, \kappa, \omega)$ then there exist $\kappa \geq 1$, $\omega > 0$, $-\omega_0 \in (-\omega, 0)$ and $\theta \in (0, \pi/2)$ such that we have that

$$\frac{1}{5\kappa} \leq \frac{\|(A_0-\lambda)x\|_{E^0}}{|\lambda|\|x\|_{E^0} + \|x\|_{E^1}} \leq 5\kappa \qquad x \in E^1$$

for $\lambda \in \Sigma_{-\omega_0,\theta} := \{|\arg(z-\omega_0) \leq \theta + \pi/2|\} \subset \rho(A_0)$.

Note that taking $a = -\omega_0$ and $\phi = \frac{\pi}{2} - \theta$ we define $S_{a,\phi} = \Sigma_{-\omega_0,\theta}$ and we just need to check that

$$\|(A_0-\lambda)^{-1}\|_{E^0} \leq M|\lambda - a|^{-1} \qquad \lambda \in S_{a,\phi}$$

From $\frac{1}{5\kappa} \leq \frac{\|(A_0-\lambda)x\|_{E^0}}{|\lambda|\|x\|_{E^0}+\|x\|_{E^1}}$ we get for $\lambda \in S_{a,\phi}$

$$C|\lambda|\|x\|_0 \leq \|(A_0-\lambda)x\|_0 \qquad x \in E^1$$

which, taking $y = (A_0-\lambda)x$, reads

$$\|(A_0-\lambda)^{-1}y\|_{E^0} \leq \frac{C}{|\lambda|}\|y\|_{E^0}$$

and since $\frac{|\lambda+\omega_0|}{|\lambda|} \leq \tilde{C}$ for all $\lambda \in S_{a,\phi}$, we get

$$\|(A_0-\lambda)^{-1}y\|_{E^0} \leq \frac{C\tilde{C}}{|\lambda+\omega_0|}\|y\|_{E^0} := \frac{M}{|\lambda - a|}\|y\|_{E^0}.$$

■

Note that for $A_0 \in \mathcal{H}(E^1, E^0)$, we define

$$\mathrm{type}(A_0) = -\inf\{\mathrm{Re}(\sigma(A_0))\}$$

and observe that this quantity will play an important role in the estimates for semigroups below, see e.g. (2.1.5). For details on this definition see [2, pg. 17, pg. 34 and II.5.1.2, pg. 69], noting that there, the notation is slightly different.

In order to follow [2] we will momentarily assume that

$$0 \in \rho(A_0). \tag{2.0.4}$$

With this it can be proved that the norm $\|\cdot\|_{E^1}$ is equivalent $\|A_0 \cdot\|_{E^0}$, and we can start a recurring construction as follows. This condition will be later removed, see Proposition 2.1.4.

Consider $E^2 := D(A_1) = \{u \in E^1,\ A_1 u \in E^1\}$ where $A_1 : E^2 \hookrightarrow E^1$ is the realization (and also the closure) of A_0 in E^1 and endowed with the norm $\|\cdot\|_{E^2} = \|A_1 \cdot\|_{E^1}$.

We can iterate this process to get a discrete scale of Banach spaces $\{E^n, n \in \mathbb{N}\}$ and the realizations of A_0 in E^n, which we denote by A_n, satisfy $A_n \in \mathcal{H}(E^{n+1}, E^n)$ and are isometric isomorphisms from $E^{n+1} \to E^n$, see [2, V.1.2.1, pg. 256].

We can also consider negative indexes for the spaces, that is the negative side of the scale. Define E^{-1} as the completion of E^0 relatively to the norm $\|\cdot\|_{E^{-1}} := \|A_0^{-1} \cdot\|_{E^0}$, which is a Banach space such that $E^0 \hookrightarrow E^{-1}$ densely and A_{-1} is the unique continuous extension of A_0, which is an isometric isomorphism from $E^0 \to E^{-1}$. This extension is called again the realization of A_0 in E^{-1}.

Again, we iterate the process of completion with the norm generated by the new operator and we get a negative discrete scale $\{E^{-n}, n \in \mathbb{N}\}$ and $A_{-n} \in \mathcal{H}(E^{-n+1}, E^{-n})$, where A_{-n} denotes the realization of A_0, the closure of A_{-n+1} in E^{-n} and is an isometric isomorphism from $E^{-n+1} \to E^{-n}$ see [2, V.1.3.2, pg. 263] and the comments on [2, pg. 264].

So we have a two-sided discrete nested scale ([2, V.1.3.4, pg 264]):

$$\{E^k,\ k \in \mathbb{Z}\}, \qquad A_k \in \mathcal{H}(E^{k+1}, E^k) \tag{2.0.5}$$

where A_k denotes the realization of A_0, the closure of A_{k+1} in E^k and is an isometric isomorphism from $E^{k+1} \to E^k$ which satisfies

$$\rho(A_k) = \rho(A_0) \quad k \in \mathbb{Z}. \tag{2.0.6}$$

In order to have a better description of the negative scale we can use dual spaces as follows, provided E^0 is reflexive.

Assume E^0 is reflexive and let $E^{0\sharp} := (E^0)'$ the dual space and $E^{1\sharp} := D(A_0^\sharp)$, where $A_0^\sharp : E^{1\sharp} \subset E^{0\sharp} \hookrightarrow E^{0\sharp}$ is the adjoint operator of A_0, which satisfies $A_0^\sharp \in \mathcal{H}(E^{1\sharp}, E^{0\sharp})$, see [2, I.1.2.3, pg. 13].

Then, we repeat the process above and construct a discrete scale $\{E^{n\sharp};\ n \in \mathbb{N}\}$, which can be identified with the original one by

$$E^{-n} = (E^{n\sharp})' \quad \textit{and} \quad A_{-n} = (A_n^\sharp)' \qquad n \in \mathbb{N}, \tag{2.0.7}$$

where the dashes denote the duals, see [2, V.1.4.9, pg. 272].

Now we construct intermediate spaces between the discrete scale $\{E^k,\ k \in \mathbb{Z}\}$ following two different procedures.

2.1 Construction of the interpolation-extrapolation scale for A_0

Recall from [50] that if a Banach space, say G, is densely included in other Banach space, H, they are said to be an *interpolation couple*. Also, an *interpolation functor* of exponent $0 < \theta < 1$, $[\cdot,\cdot]_\theta$, is a map such that for two given interpolation couples G_0, G_1 and H_0, H_1, we have Banach spaces $G_\theta = [G_1, G_0]_\theta$ and $H_\theta = [H_1, H_0]_\theta$ such that $G_1 \subset G_\theta \subset G_0$, $H_1 \subset H_\theta \subset H_0$ and for $A \in \mathcal{L}(G_0, H_0) \cap \mathcal{L}(G_1, H_1)$, then $A \in \mathcal{L}(G_\theta, H_\theta)$ and

$$\|A\|_{\mathcal{L}(G_\theta,H_\theta)} \leq \|A\|^{1-\theta}_{\mathcal{L}(G_0,H_0)} \|A\|^{\theta}_{\mathcal{L}(G_1,H_1)} \tag{2.1.1}$$

Remark 2.1.1 *There are many interpolation functors that can be used here, but in particular we choose complex interpolation for simplicity and because in the applications below it leads to a very convenient scale of spaces.*

Starting with the discrete scale (2.0.5) and taking the complex interpolation method, we proceed as in [2, V.1.5.1, pg. 275] to obtain the spaces

$$E^\alpha := E^{k+\theta} := [E^{k+1}, E^k]_\theta, \quad \theta \in (0,1) \quad k \in \mathbb{Z}, \tag{2.1.2}$$

and the operator A_α as the interpolation of A_{k+1} and A_k, as in (2.1.1). Thus we obtain the continuous nested interpolation scale

$$\{E^\alpha,\ \alpha \in \mathbb{R}\}, \qquad A_\alpha \in \mathcal{H}(E^{\alpha+1}, E^\alpha) \tag{2.1.3}$$

and A_α is an isometry from $E^{\alpha+1}$ into E^α. Note that if $\alpha > \beta$, E^α is densely included in E^β and A_α is the realization of A_0 in E^α. Moreover, for every $\alpha \in \mathbb{R}$

$$\rho(A_\alpha) = \rho(A_0), \tag{2.1.4}$$

see [2, V.1.1.2.e), pg. 252].

Now, since $A_\beta \in \mathcal{H}(E^{\beta+1}, E^\beta)$, $-A_\beta$ generates an analytic semigroup in E^β with the property [2, V.2.1.3, pg. 289]:

$$\|e^{-A_\beta t}\|_{\mathcal{L}(E^\beta,E^\alpha)} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\sigma t} \qquad t > 0,\ \alpha, \beta \in \mathbb{R}, \alpha \geq \beta \tag{2.1.5}$$

for any $\sigma > \operatorname{type}(A_0)$ and $C(\alpha - \beta)$ is bounded for α, β in bounded sets of $\mathbb{R}$.

If E^0 is reflexive, we can interpolate in the dual scale $\{E^{n\sharp}, n \in \mathbb{Z}\}$ as well. We take again the complex interpolation $[\cdot,\cdot]_\theta$, and the negative intermediate spaces can be identified with the dual of the positive ones as

$$E^{-\alpha} = (E^{\alpha\sharp})' \quad \textit{and} \quad A_{-\alpha} = (A_\alpha^\sharp)' \quad \text{for } \alpha > 0, \tag{2.1.6}$$

see [2, V.1.5.12, pg. 282]. Also, the semigroup in the spaces of the negative side can be identified with the duals by [2, V.2.3.2, pg. 298]:

$$e^{-A_{-\alpha}t} = (e^{-A_\alpha^\sharp t})' \quad \alpha > 0. \tag{2.1.7}$$

Note that the semigroups in (2.1.5) are extensions or restrictions of each other one, that is, given $\alpha \geq \beta$, then

$$e^{-A_\beta t}|_{E^\alpha} = e^{-A_\alpha t}, \quad t \geq 0.$$

For details see [2, Lemma V.2.1.2]. Hence, we have the following.

Definition 2.1.2 *Under the assumptions above we say that the operator A_0 defines an analytic semigroup $S_{A_0}(t)$ in the interpolation scale $\{E^\alpha\}_{\alpha\in\mathbb{R}}$ in the sense that*

$$S_{A_0}(t)|_{E^\alpha} = e^{-A_\alpha t}, \qquad \forall \alpha \in \mathbb{R}.$$

Observe that

$$\|S_{A_0}(t)\|_{\mathcal{L}(E^\beta, E^\alpha)} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\sigma t} \qquad t > 0,\ \alpha, \beta \in \mathbb{R}, \alpha \geq \beta$$

for any $\sigma > \text{type}(A_0)$ and $C(\alpha - \beta)$ is bounded for α, β in bounded sets of $\mathbb{R}$.

Remark 2.1.3 *Note that we could have taken any other interpolation functor as long as it has the reiteration property (as the complex interpolation does)*

$$[E^\alpha, E^\beta]_\eta = E^{(1-\eta)\alpha + \eta\beta} \quad 0 < \eta < 1, \quad \alpha, \beta \in \mathbb{R}$$

such as real interpolation, and the scale would have the same properties (2.1.2), (2.1.3), (2.1.4) and (2.1.5). But then we would need to use the associated dual interpolation functor of it for the negative part of the scale, to obtain (2.1.6) and (2.1.7). For more information see [2, V.1.5.11 pg. 282].

Now we construct the interpolation scale and the semigroup in the scale, as in Definition 2.1.2, without assuming (2.0.4).

Proposition 2.1.4 *Let $A_0 \in \mathcal{H}(E^1, E^0)$ and take c such that $0 \in \rho(A_0 + cI)$.*

Then the scale $\{E^\alpha\}_{\alpha\in\mathbb{R}}$ generated by $A_0 + cI$, as above, is independent of c and for any $\alpha \in \mathbb{R}$, the realization of A_0 in E^α, denoted as A_α, satisfies

$$A_\alpha \in \mathcal{H}(E^{\alpha+1}, E^\alpha)$$

and for all $\alpha \in \mathbb{R}$

$$\rho(A_\alpha) = \rho(A_0).$$

Hence we have an analytic semigroup $S_{A_0}(t)$ defined in the scale $\{E^\alpha\}_{\alpha\in\mathbb{R}}$ such that $S_{A_0}(t)|_{E^\alpha} = e^{-A_\alpha t}$, $\alpha \in \mathbb{R}$, and satisfies

$$\|S_{A_0}(t)\|_{\mathcal{L}(E^\beta,E^\alpha)} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}}e^{\sigma t} \qquad t>0,\ \alpha \geq \beta \in \mathbb{R}$$

for any $\sigma > type(A_0)$.

Furthermore if E^0 is reflexive, then $E^{-\alpha} = (E^{\alpha\sharp})'$, $A_{-\alpha} = (A^\sharp_\alpha)'$ for $\alpha > 0$, and

$$e^{-A_{-\alpha}t} = (e^{-A^\sharp_\alpha t})'.$$

Proof. If $0 \in \rho(A_0)$, the construction has been carried above.

If $0 \notin \rho(A_0)$, there exists $c \in \mathbb{R}$ such that $\tilde{A}_0 = A_0 + cI$ satisfies $0 \in \rho(\tilde{A}_0)$, so we can perform the construction above for the operator $\tilde{A}_0$. Note that the corresponding scale of spaces is independent of c because the interpolation scale is only determined by the spaces $\{E^k\}_{k\in\mathbb{Z}}$, and these spaces have equivalent norms independently of the c chosen.

Thus, with $\tilde{A}_\alpha = A_\alpha + cI$ in E^α and applying standard arguments in [44] or [31] we obtain that

$$e^{-A_\alpha t} = e^{-ct}e^{-\tilde{A}_\alpha t}$$

and the result follows. ■

2.2 Construction of the fractional power scale for A_0

Now, starting again with the discrete scale (2.0.5), we construct a fractional power scale $\{F^\alpha\}_{\alpha\in\mathbb{R}}$ following [2]. See also [31] and [37]. For this we will also assume for a moment that

$$(-\infty, 0] \subset \rho(A_0). \tag{2.2.1}$$

Since the intermediate spaces between the integer scale (2.0.5) might be different to the ones in the previous section, see Remark 2.2.2 below, we denote now

$$F^k = E^k \quad \text{for } k \in \mathbb{Z}.$$

We first construct the positive fractional power scale. Using (2.2.1), the resolvent estimate (2.0.2), see Definition 2.0.1, and integrating on a curve which surrounds the sector (2.0.1), one can give a suitable integral expression for the operator $A_0^{-\alpha}$ for $\alpha > 0$, which is bijective from $E^0 \to R(A_0^{-\alpha}) \subset E^0$; for more details see [2, III.4.6, pg. 147], [31], [37]. This implies that $A_0^\alpha = (A_0^{-\alpha})^{-1}$ is well defined, and therefore we can define

$$F^\alpha = D(A_0^\alpha) = R(A_0^{-\alpha}), \quad \alpha \geq 0 \tag{2.2.2}$$

with the norm $\|\cdot\|_\alpha = \|A_0^\alpha \cdot\|_0$. Note that this construction for $\alpha = n \in \mathbb{N}$ coincides with A_0^n and $F^n = E^n$.

So we get the positive fractional power scale

$$\{F^\alpha,\ \alpha \geq 0\}, \qquad A_\alpha \in \mathcal{H}(F^{\alpha+1}, F^\alpha), \qquad \alpha \geq 0, \tag{2.2.3}$$

where A_α is the realization of A_0 on F^α and is an isometry, see [2, V.1.2.4, pg. 258] and [2, V.1.2.6, pg. 260]. Moreover, for every $\alpha \geq 0$

$$\rho(A_\alpha) = \rho(A_0)$$

again because of [2, V.1.1.2.e), pg. 252].

For the negative scale, note that (2.2.1) together with (2.0.6) implies $(-\infty, 0] \subset \rho(A_n)$ for any $n \in \mathbb{Z}$. Fix now $N \in \mathbb{N}$ and take $A_{-N} \in \mathcal{H}(F^{-N+1}, F^{-N})$. With the construction above as in (2.2.2) but with the operator A_{-N} in F^{-N}, we get the extrapolated fractional power scale of order N,

$$F^{\alpha-N} = D(A^\alpha_{-N}) \qquad \alpha \geq 0, \tag{2.2.4}$$

see [2, V.1.3.8, pg. 266] and [2, V.1.3.9, pg. 267]. Then we have

$$\{F^\alpha,\ \alpha \geq -N\}, \qquad A_\alpha \in \mathcal{H}(F^{\alpha+1}, F^\alpha), \qquad \rho(A_\alpha) = \rho(A_0) \qquad \alpha \geq -N$$

and A_α is an isometry from $E^{\alpha+1}$ into E^α.

Again, $F^k = E^k$ for $k \in \mathbb{Z}$, $k \geq -N$, and for $\alpha \geq 0$, F^α and A_α above coincide with the ones in (2.2.3).

Now fix $A_\beta : F^{\beta+1} \to F^\beta$ for any $\beta \geq -N$. Renaming $F^\beta = Z$, $F^{\beta+1} = Z^1$ we have the following reiteration property (see [2, V.1.2.6, pg. 260] or [37, Proposition 10.6])

$$Z^\varepsilon = D(A^\varepsilon_\beta) = F^{\beta+\varepsilon} \tag{2.2.5}$$

for $\varepsilon \in [0, 1]$, and A_β is sectorial in Z, thus we can apply [31, I.1.4.3, pg. 26], to get

$$\|e^{-A_\beta t}\|_{\mathcal{L}(F^\beta, F^\alpha)} \leq \frac{C(\alpha - \beta)}{t^{\alpha-\beta}}, \qquad t > 0, \quad \alpha \geq \beta \geq -N \tag{2.2.6}$$

for any $\sigma > \text{type}(A_0)$.

As above, if E^0 is reflexive, we can identify the negative side of the scale with some dual spaces by means of [2, V.1.4.12, pg. 274] getting

$$F^{-\alpha} = (F^{\alpha\sharp})' \quad \textit{and} \quad A_{-\alpha} = (A^\sharp_\alpha)', \qquad \alpha > 0 \tag{2.2.7}$$

with

$$e^{-A_{-\alpha}t} = (e^{-A^\sharp_\alpha t})'. \tag{2.2.8}$$

Therefore analogously to Definition 2.1.2 we say that A_0 defines an analytic semigroup $S_{A_0}(t)$ in the fractional power scale $\{F^\alpha\}_{\alpha \geq -N}$ in the sense that

$$S_{A_0}(t)|_{F^\alpha} = e^{-A_\alpha t}, \qquad \forall\ \alpha \geq -N$$

and

$$\|S_{A_0}(t)\|_{\mathcal{L}(F^\beta, F^\alpha)} \leq \frac{C(\alpha - \beta)}{t^{\alpha-\beta}}, \qquad t > 0, \quad \alpha \geq \beta \geq -N.$$

Now we construct the fractional power scale and the semigroup without assuming (2.2.1).

Proposition 2.2.1 *Let $A_0 \in \mathcal{H}(E^1, E^0)$ and take c such that $(-\infty, 0] \in \rho(A_0 + cI)$.*

Then given $N \in \mathbb{N}$, the scale $\{F^\alpha\}_{\alpha \geq -N}$ generated by $A_0 + cI$, as above, is independent of c and the realizations of A_0 in F^α, denoted by A_α, satisfy

$$A_\alpha \in \mathcal{H}(F^{\alpha+1}, F^\alpha) \qquad \rho(A_\alpha) = \rho(A_0) \qquad \alpha \geq -N.$$

Hence we have an analytic semigroup $S_{A_0}(t)$ defined in the scale $\{F^\alpha\}_{\alpha \geq -N}$ such that $S_{A_0}(t)|_{F^\alpha} = e^{-A_\alpha t}$, $\alpha \geq -N$, satisfies

$$\|S_{A_0}(t)\|_{\mathcal{L}(F^\beta, F^\alpha)} \leq \frac{C(\alpha - \beta)}{t^{\alpha - \beta}} e^{\sigma t} \qquad t > 0, \quad \alpha \geq \beta \geq -N$$

for any $\sigma > type(A_0)$.

Furthermore if E^0 is reflexive, then $F^{-\alpha} = (F^{\alpha\sharp})'$, $A_{-\alpha} = (A^\sharp_\alpha)'$ and $e^{-A_{-\alpha}t} = (e^{-A^\sharp_\alpha t})'$ for $0 < \alpha \leq N$.

Proof. The case $(-\infty, 0] \in \rho(A_0)$ has been discussed before.

If $(-\infty, 0] \notin \rho(A_0)$, there exists $c \in \mathbb{R}$ such that $\tilde{A}_0 = A_0 + cI$ satisfies $(-\infty, 0] \in \rho(\tilde{A}_0)$. Then the corresponding scale of spaces is independent of c, see the comments on Definition 1.4.7 in [31]. Thus, with $\tilde{A}_\alpha = A_\alpha + cI$ in F^α and applying standard arguments in [44] or [31] we obtain that

$$e^{-A_\alpha t} = e^{-ct} e^{-\tilde{A}_\alpha t}$$

and the result follows. ■

Remark 2.2.2 *Note that after Propositions 2.1.4 and 2.2.1, for $A_0 \in \mathcal{H}(E^1, E^0)$ we have a discrete scale (2.0.5) and with the notations of these propositions, we have*

$$F^k = E^k \quad \textit{for } k \in \mathbb{Z}, \ k \geq -N.$$

However, the intermediate spaces, F^α and E^α, for $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, $\alpha \geq -N$, do not need to coincide in general. But, if A_0 has bounded imaginary powers, that is, there exist $\varepsilon > 0$ and $M \geq 1$ such that

$$\|A_0^{it}\|_{\mathcal{L}(E^1, E^0)} \leq M \quad \textit{for } t \in [-\varepsilon, \varepsilon], \tag{2.2.9}$$

then E^α and the scale of fractional powers F^α coincide, see [2, V.1.5.13, pg. 283].

An important case when this happens is when E^0 is a Hilbert space and A_0 is selfadjoint.

Finally observe that abusing of the notations we have used the same notations A_α and $e^{-A_\alpha t}$ for both the interpolation and fractional power scales. This should produce no confusion since it will be always clear from the context what scale are we working with.

Chapter 3

Some properties of uniform Lebesgue and Bessel spaces

As stated in the introduction, we are interested in using low regularity spaces for initial data in the applications below. For it we now recover the definition of uniform spaces and some properties about them and we prove Proposition 3.0.1 which shows some complementary information to [5] concerning uniform spaces with negative index in the scale.

For $1 \leq p < \infty$ let $L^p_U(\mathbb{R}^N)$ denote the locally uniform space composed of the functions $f \in L^p_{loc}(\mathbb{R}^N)$ such that there exists $C > 0$ such that for all $x_0 \in \mathbb{R}^N$

$$\int_{B(x_0,1)} |f|^p \leq C$$

endowed with the norm

$$\|f\|_{L^p_U(\mathbb{R}^N)} = \sup_{x_0 \in \mathbb{R}^N} \|f\|_{L^p(B(x_0,1))}$$

(for $p = \infty$, $L^\infty_U(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$). Also, denote by $\dot{L}^q_U(\mathbb{R}^N)$ the closed subspace of $L^q_U(\mathbb{R}^N)$ consisting of all elements which are translation continuous with respect to $\|\cdot\|_{L^q_U(\mathbb{R}^N)}$. That is

$$\|\tau_y \phi - \phi\|_{L^q_U(\mathbb{R}^N)} \to 0 \quad \text{as} \quad |y| \to 0,$$

where $\{\tau_y, y \in \mathbb{R}^N\}$ denotes the group of translations in $\mathbb{R}^N$. With this we get $L^q(\mathbb{R}^N) \subset \dot{L}^q_U(\mathbb{R}^N)$ for $1 \leq q < \infty$ and for $q = \infty$ we get $L^\infty_U(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$ and $\dot{L}^\infty_U(\mathbb{R}^N) = BUC(\mathbb{R}^N)$.

Thus we introduce the *uniform Bessel-Sobolev spaces* $H^{k,q}_U(\mathbb{R}^N)$, with $k \in \mathbb{N} \cup \{0\}$, as the set of functions $\phi \in H^{k,q}_{loc}(\mathbb{R}^N)$ such that

$$\|\phi\|_{H^{k,q}_U(\mathbb{R}^N)} = \sup_{x \in \mathbb{R}^N} \|\phi\|_{H^{k,q}(B(x,1))} < \infty$$

for $k \in \mathbb{N}$, where $H^{k,q}(B(x,1))$ is the standard Bessel space and $\dot{H}^{0,q}_U(\mathbb{R}^N) = \dot{L}^q_U(\mathbb{R}^N)$. Then denote by $\dot{H}^{k,q}_U(\mathbb{R}^N)$ a subspace of $H^{k,q}_U(\mathbb{R}^N)$ consisting of all elements which are translation continuous with respect to $\|\cdot\|_{H^{k,q}_U(\mathbb{R}^N)}$, that is

$$\|\tau_y \phi - \phi\|_{H^{k,q}_U(\mathbb{R}^N)} \to 0 \quad \text{as} \quad |y| \to 0$$

where $\{\tau_y, y \in \mathbb{R}^N\}$ denotes the group of translations.

To construct intermediate spaces of noninteger order, consider the complex interpolation functor denoted by $[\cdot,\cdot]_\theta$, for $\theta \in (0,1)$, see [50] for details. Then for $1 \leq q < \infty$, $k \in \mathbb{N} \cup \{0\}$ and $s \in (k, k+1)$ we define $\theta \in (0,1)$ such that $s = \theta(1+k) + (1-\theta)k$, that is $\theta = s - k$. Then one can define the intermediate spaces by interpolation as

$$H_U^{s,q}(\mathbb{R}^N) = [H_U^{k+1,q}(\mathbb{R}^N), H_U^{k,q}(\mathbb{R}^N)]_\theta,$$

and

$$\dot{H}_U^{s,q}(\mathbb{R}^N) = [\dot{H}_U^{k+1,q}(\mathbb{R}^N), \dot{H}_U^{k,q}(\mathbb{R}^N)]_\theta. \tag{3.0.1}$$

For details on this construction, see [2, 5].

Using [5, Proposition 4.2] it is easy to see that the sharp embeddings of Bessel spaces translate into

$$\dot{H}_U^{s,q}(\mathbb{R}^N) \subset \begin{cases} \dot{L}_U^r(\mathbb{R}^N),\ s - \frac{N}{q} \geq -\frac{N}{r},\ 1 \leq r < \infty & if\ s - \frac{N}{q} < 0 \\ \dot{L}_U^r(\mathbb{R}^N),\ 1 \leq r < \infty & if\ s - \frac{N}{q} = 0 \\ C_b^\eta(\mathbb{R}^N) & if\ s - \frac{N}{q} > \eta \geq 0. \end{cases} \tag{3.0.2}$$

In order to show why we consider (and will use) the "dotted" scale, observe that in [5], the Laplace operator was considered in the scale of spaces $H_U^{s,q}(\mathbb{R}^N)$ and $\dot{H}_U^{s,q}(\mathbb{R}^N)$, $s \geq 0$, and it was proved that $-\Delta$ defines an analytic semigroup. However in the "undotted" spaces the semigroup generated by $-\Delta$ is analytic but not strongly continuous. Also, these spaces are less convenient to use because smooth functions are not dense in them; whereas in "dotted" spaces, smooth functions are dense, see [5, Lemma 4.2].

It was moreover proved in [5, Theorem 5.3, pg. 290], that $-\Delta$ has bounded imaginary powers, and therefore this scale coincides with the fractional power one; see [2, V.1.5.13, pg. 283] or Remark 2.2.2. Note that from the results in [5] we have in particular that $\dot{H}_U^{1,q}(\mathbb{R}^N) = [\dot{H}_U^{2,q}(\mathbb{R}^N), \dot{L}_U^q(\mathbb{R}^N)]_{1/2}$; see Remark 5.7, page 291 in that reference. From this reiteration property of the interpolation, we obtain that $\dot{H}_U^{2\theta,q}(\mathbb{R}^N) = [\dot{H}_U^{2,q}(\mathbb{R}^N), \dot{L}_U^q(\mathbb{R}^N)]_\theta$ for $\theta \in [0,1]$.

The scale above can be extended to negative indexes by a general extrapolation procedure as in [2], see Chapter 2 and (2.1.3). In this way one can define the extrapolated space $\dot{H}_U^{-k,q}(\mathbb{R}^N)$ as the completion of $\dot{L}_U^q(\mathbb{R}^N)$ with the norm $\|(-\Delta + I)^{-k/2}u\|_{\dot{L}_U^q(\mathbb{R}^N)}$. Again, by complex interpolation, for $0 < s < k$, $k \in \mathbb{N}$, the intermediate spaces are given by

$$\dot{H}_U^{-s,q}(\mathbb{R}^N) = [\dot{L}_U^q(\mathbb{R}^N), \dot{H}_U^{-k,q}(\mathbb{R}^N)]_\theta, \qquad \text{with } \theta = \frac{s}{k}.$$

Note that because of the reiteration property of the complex interpolation (see [2, (2.8.4), pg. 31] and [2, Theorem 1.5.4, pg. 278]) this definition of $\dot{H}_U^{-s,q}(\mathbb{R}^N)$ does not depend on k. Also the operator $-\Delta$ and the analytic semigroup it generates extend to the spaces with negative index above.

For the standard (not uniform) Bessel spaces, there is a simple characterization for the spaces with negative indexes using duality and reflexivity, see [2, V.1.5.12, pg. 282] and

(5.2.1) below. However, since the uniform spaces are not reflexive, even for $q=2$, there is no simple characterization of the uniform spaces with negative indexes, see (2.1.6).

Therefore, we start proving the following result which gives some description of the spaces with negative indexes and complements the results in [5].

Proposition 3.0.1 *We have that*

$$\dot{L}^p_U(\mathbb{R}^N) \hookrightarrow \dot{H}^{-s,q}_U(\mathbb{R}^N) \qquad \textit{if} \quad s-\frac{N}{q'} \geq -\frac{N}{p'}, \qquad s>0.$$

Proof. We first assume that $0 \leq s \leq 2$.
i) First note that $\dot{H}^{-s,q}_U(\mathbb{R}^N)$ is the completion of $\dot{H}^{2-s,q}_U(\mathbb{R}^N)$ with the norm $\|(-\Delta+I)^{-1}\cdot\|_{\dot{H}^{2-s,q}_U(\mathbb{R}^N)}$ (see Chapter 2). This means that $f\in \dot{H}^{-s,q}_U(\mathbb{R}^N)$ if and only if there exists an approximating sequence $\{f_n\}\in \dot{H}^{2-s,q}_U(\mathbb{R}^N)$ that converges to f in $\dot{H}^{-s,q}(\mathbb{R}^N)$.

Since $(-\Delta+I)^{-1}$ is an isometry from $\dot{H}^{2-s,q}_U(\mathbb{R}^N)$ to $\dot{H}^{-s,q}_U(\mathbb{R}^N)$, see Chapter 2, this is equivalent to

$$(-\Delta+I)^{-1}f_n \longrightarrow (-\Delta+I)^{-1}f \quad \text{in } \dot{H}^{2-s,q}(\mathbb{R}^N),$$

and observe that since $f_n\in \dot{H}^{2-s,q}_U(\mathbb{R}^N)$ then $(-\Delta+I)^{-1}f_n\in \dot{H}^{4-s,q}_U(\mathbb{R}^N)$. Thus, we get that $f\in \dot{H}^{-s,q}_U(\mathbb{R}^N)$ if and only if there exists $\{u_n\}\in \dot{H}^{4-s,q}_U(\mathbb{R}^N)$ such that $u_n\to (-\Delta+I)^{-1}f$ in $\dot{H}^{2-s,q}_U(\mathbb{R}^N)$.
ii) Now, take $f\in \dot{L}^p_U(\mathbb{R}^N)$, then from the results in [5] we have $u=(-\Delta+I)^{-1}f\in \dot{H}^{2,p}_U(\mathbb{R}^N)$ and since $s-\frac{N}{q'}\geq -\frac{N}{p'}$ holds by assumption, we have $\dot{H}^{2,p}_U(\mathbb{R}^N)\hookrightarrow \dot{H}^{2-s,q}_U(\mathbb{R}^N)$, and $2-s\geq 0$. Therefore $u\in \dot{H}^{2-s,q}_U(\mathbb{R}^N)$.

Since $\dot{H}^{4-s,q}_U(\mathbb{R}^N)$ is dense in $\dot{H}^{2-s,q}_U(\mathbb{R}^N)$, there exist $u_n\in \dot{H}^{4-s,q}_U(\mathbb{R}^N)$ such that $\|u_n-u\|_{\dot{H}^{2-s,q}_U(\mathbb{R}^N)} \overset{n\to\infty}{\to} 0$ and therefore by i), $f\in \dot{H}^{-s,q}_U(\mathbb{R}^N)$. Note that the inclusion is continuous, since $(-\Delta+I)^{-1}$ is an isometry on the scale and then

$$\|f\|_{\dot{H}^{-s,q}_U(\mathbb{R}^N)} = \|(-\Delta+I)^{-1}f\|_{\dot{H}^{2-s,q}_U(\mathbb{R}^N)} \leq C\|(-\Delta+I)^{-1}f\|_{\dot{H}^{2,p}_U(\mathbb{R}^N)} = C\|f\|_{\dot{L}^p_U(\mathbb{R}^N)}.$$

In order to prove the result for $s\geq 0$, we can repeat the whole argument above, using $(-\Delta+I)^{-n}$, which is an isometry on the scale, for a suitable n. If $2\leq s\leq 4$ we use $n=2$, thus in part i) we obtain that $f\in \dot{H}^{-s,q}_U(\mathbb{R}^N)$ if there exists a sequence $\{u_n\}\in \dot{H}^{6-s,q}_U(\mathbb{R}^N)$ converging to $u=(-\Delta+I)^{-2}f$ in $\dot{H}^{4-s,q}_U(\mathbb{R}^N)$. In part ii) we now have $u\in \dot{H}^{4,p}_U(\mathbb{R}^N)\hookrightarrow \dot{H}^{4-s,q}_U(\mathbb{R}^N)$ since now $4-s\geq 0$ and the result follows as before.

In the same way, for $2(k-1)\leq s\leq 2k$, we use $n=k$ and repeat the argument above.
■

Remark 3.0.2 *Note that the embedding in Proposition 3.0.1 is precisely the one that could be expected from (3.0.2) if the spaces were reflexive. Also, this is the embedding that holds for the standard Bessel scale, see Section 5.2 below. Needless to say the conditions for the embeddings read also* $s\geq \frac{N}{p}-\frac{N}{q}$.

Chapter 4

Second order problems in uniform Lebesgue-Bessel spaces in $\mathbb{R}^N$

In this chapter we study the solvability of some second order linear parabolic equations in $\mathbb{R}^N$. In particular, we study the problem

$$\begin{cases} u_t - \sum_{k,l=1}^N a_{kl}(x)\partial_k\partial_l u + \sum_{j=1}^N b_j(x)\partial_j u + c(x)u = 0 \qquad x \in \mathbb{R}^N, \quad t > 0 \\ u(0,x) = u_0(x) \qquad x \in \mathbb{R}^N, \end{cases} \tag{4.0.1}$$

where the real coefficients of the elliptic principal part of the equation are assumed to be bounded and uniformly continuous, that is, $a_{kl} \in BUC(\mathbb{R}^N)$. The lower order coefficients are assumed to belong to locally uniform Lebesgue spaces, described in Chapter 3. In particular we will assume that for $j = 1, ..., N$, $\|b_j\|_{\dot{L}^{p_j}_U(\mathbb{R}^N)} \leq R_j$ and $\|c\|_{\dot{L}^{p_0}_U(\mathbb{R}^N)} \leq R_0$, where $p_j > N$ and $p_0 > \frac{N}{2}$.

Parabolic problems like (4.0.1) with coefficients in uniform spaces have been considered before; see e.g. [3], [5], [47] and references therein. For example, the results in [3] allow to solve (4.0.1) in Lebesgue spaces $L^q(\mathbb{R}^N)$ assuming additionally that

$$p_j \geq q > 1, \quad \text{for } j = 0, \dots, N.$$

These results were later used in [5] to solve (4.0.1) in uniform spaces $L^q_U(\mathbb{R}^N)$, under the same restrictions, see [5, Section 5], and later on, in [47, Section 6.2] with different techniques. Because of the restrictions above in the coefficients the result in [5, 47] just allowed $\gamma \geq 0$ for the smoothing estimates.

Here, we remove such restrictions allowing a larger class of initial data, in particular γ can be even negative. When the additional assumptions above are imposed, Theorem 4.0.6 below recovers Theorem 5.3 in [5] and Theorem 30 in [47]. Finally note that the results in [3], [5] do not include the continuity of solutions with respect to perturbations in the coefficients like (4.0.18), (4.0.19).

Note that the additional assumption in Theorem 4.0.6 which says that "if $q' < \tilde{p}$ and $q > p_0$, we will also assume $p_0 > \frac{Nq}{N+q}$" applies only when q is large relative to the exponents p_j, $j = 0, \dots, N$. For $1 < q \leq N$ this imposes no additional restriction since in this range $\frac{Nq}{N+q} \leq \frac{N}{2}$. Furthermore, since $\frac{Nq}{N+q} < N$ for all q, if $p_0 \geq N$ no additional assumption is imposed either.

It is also worth mentioning that the estimates (4.0.15), (4.0.16) on the semigroups of solutions of (4.0.14) are uniform with respect to bounded families of coefficients. Finally Theorem 4.0.6 gives the continuity of the semigroups with respect to perturbations in the lower order coefficients of (4.0.14).

We start by considering the operator

$$A_0 := -\sum_{k,l=1}^{N} a_{kl}(x)\partial_k\partial_l$$

where we assume $a_{kl} \in BUC(\mathbb{R}^N)$. Hence, for some modulus of continuity ω, we have the norm

$$\|a_{kj}\|_{BUC(\mathbb{R}^N,\omega)} = \sup_{x\in\mathbb{R}^N} |a_{kj}(x)| + \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|a_{kj}(x) - a_{kj}(y)|}{\omega(|x-y|)}.$$

We also assume the following ellipticity condition: for some constants $M > 0$ and $\theta_0 \in (0, \frac{\pi}{2})$, the following holds for all $x, \xi \in \mathbb{R}^N$ with $|\xi| = 1$

$$A_0(x,\xi) := \sum_{k,l=1}^{N} a_{kl}(x)\xi_k\xi_l \geq \frac{1}{M} > 0, |arg(A_0(x,\xi))| \leq \theta_0.$$

Note that M can be chosen such that $\|a_{kj}\|_{BUC(\mathbb{R}^N,\omega)} < M$. Finally, we will assume

$$\int_0^1 \frac{\omega^{1/3}(t)}{t}\,dt < \infty.$$

Note that these assumptions are satisfied for the case $a_{kl} = \delta_{kl}$, i.e. when $A_0 = -\Delta$.

Proposition 4.0.1 *Under the above assumptions, for any $1 < q < \infty$ and $\beta \in [-1,1]$, the problem*

$$\begin{cases} u_t - \sum_{k,l=1}^{N} a_{kl}(x)\partial_k\partial_l u = 0, & x \in \mathbb{R}^N, \quad t > 0 \\ u(0) = u_0 \end{cases} \tag{4.0.2}$$

where $u_0 \in \dot{H}^{2\beta,q}_U(\mathbb{R}^N)$, has a unique solution $u(t;u_0)$ that satisfies the smoothing estimates

$$\|u(t;u_0)\|_{\dot{H}^{2\alpha,q}_U(\mathbb{R}^N)} \leq \frac{M_{\alpha,\beta}e^{\mu_0 t}}{t^{\alpha-\beta}}\|u_0\|_{\dot{H}^{2\beta,q}_U(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in \dot{H}^{2\beta,q}_U(\mathbb{R}^N) \tag{4.0.3}$$

for any $1 \geq \alpha \geq \beta$, with some $\mu_0 > 0$ which depends only on q, M and θ_0.

In particular, setting $u(t;u_0) = S_0(t)u_0$ for $t \geq 0$, defines an order preserving, C^0 analytic semigroup $S_0(t)$ in $\dot{H}^{2\beta,q}_U(\mathbb{R}^N)$.

Proof. Because of [5, Theorem 5.3], for any $1 < q < \infty$, A_0 generates an analytic semigroup in $\dot{L}^q_U(\mathbb{R}^N)$ with domain $\dot{H}^{2,q}_U(\mathbb{R}^N)$ which, for some $\mu \in \mathbb{R}$, $A_0 - \mu I$ has bounded imaginary powers. Thus, we can construct an interpolation scale as in Chapter 2 and the complex interpolation spaces (3.0.1) coincide with the fractionary power ones; see [2, Theorem V.1.5.13, pg. 283] or Remark 2.2.2.

Then as a consequence of [2, Theorem 2.1.3, page 289] (with $m = n = 1$), we have that for $\beta \in [-1, 1]$ a suitable extension of A_0 generates an analytic semigroup in $\dot{H}^{2\beta,q}_U(\mathbb{R}^N)$, and the solutions of (4.0.2) satisfy (4.0.3). Hence, $S_0(t)$ is a C^0 analytic semigroup in $\dot{H}^{2\beta,q}_U(\mathbb{R}^N)$.

The fact that μ_0 depends only on q, M and θ_0, follows from [5] and the results in [2] quoted above.

For the order preserving property, recall that for coefficients $a_{kl}(x)$ as above and regular initial data, if $u_0 \geq 0$ then $S_0(t)u_0 \geq 0$ for all $t \geq 0$; see, e.g. [24, Section 2.4] and also [5, Proposition 5.3 and Remark 5.3] or [1, Theorem 11.6]. Now, for $u_0 \in \dot{H}^{2\beta,q}_U(\mathbb{R}^N)$ take $\{u_0^n\}_{n\in\mathbb{N}}$ regular such that $u_0^n \to u_0$ then $S_0(t)u_0^n \to S_0(t)u_0$ and since $S_0(t)u_0^n \geq 0$ for all $n \in \mathbb{N}$ then $S_0(t)u_0 \geq 0$. Note that this can be done because we are using the "dotted" spaces, where regular functions are dense, see [5, Lemma 4.2]. ■

We now want to extend this result stated for A_0 to the full equation in (4.0.1). For it we consider (4.0.1) as a perturbation of (4.0.2). First, from [47, Lemma 26], we have

Lemma 4.0.2 *i) Assume that $m \in L^p_U(\mathbb{R}^N)$, $1 \leq p \leq \infty$, then the multiplication operator*

$$Pu(x) = m(x)u(x)$$

satisfies, for $r \geq p'$ and $\frac{1}{s} = \frac{1}{r} + \frac{1}{p}$, that

$$P \in \mathcal{L}(L^r_U(\mathbb{R}^N), L^s_U(\mathbb{R}^N)), \qquad \|P\|_{\mathcal{L}(L^r_U(\mathbb{R}^N), L^s_U(\mathbb{R}^N))} \leq C\|m\|_{L^p_U(\mathbb{R}^N)}.$$

ii) If moreover $m \in \dot{L}^p_U(\mathbb{R}^N)$, $1 \leq p \leq \infty$, we have for $r \geq p'$ and $\frac{1}{s} = \frac{1}{r} + \frac{1}{p}$, that

$$P \in \mathcal{L}(\dot{L}^r_U(\mathbb{R}^N), \dot{L}^s_U(\mathbb{R}^N)), \qquad \|P\|_{\mathcal{L}(\dot{L}^r_U(\mathbb{R}^N), \dot{L}^s_U(\mathbb{R}^N))} \leq C\|m\|_{L^p_U(\mathbb{R}^N)}.$$

Combining this with Proposition 3.0.1 we get the following result. From now on, we denote $(x)_- = \min\{0, x\}$ and $(x)_+ = \max\{0, x\}$, respectively, the negative and positive parts of $x \in \mathbb{R}$.

Proposition 4.0.3 *Let $Pu = d(x)D^a u$, with $d \in \dot{L}^p_U(\mathbb{R}^N)$, $1 \leq p \leq \infty$, $a \in \mathbb{N} \cup \{0\}$ and let $s \geq a$, $\sigma \geq 0$. Assume also that $1 < q < \infty$ and*

$$(s - a - \frac{N}{q})_- + (\sigma - \frac{N}{q'})_- \geq -\frac{N}{p'}. \tag{4.0.4}$$

Then, we have

$$P \in \mathcal{L}(\dot{H}^{s,q}_U(\mathbb{R}^N), \dot{H}^{-\sigma,q}_U(\mathbb{R}^N)), \qquad \|P\|_{\mathcal{L}(\dot{H}^{s,q}_U(\mathbb{R}^N), \dot{H}^{-\sigma,q}_U(\mathbb{R}^N))} \leq C\|d\|_{\dot{L}^p_U(\mathbb{R}^N)}.$$

Proof. First note that $u \in \dot{H}^{s,q}_U(\mathbb{R}^N)$, thus $D^a u \in \dot{H}^{s-a,q}_U(\mathbb{R}^N) \hookrightarrow \dot{L}^r_U(\mathbb{R}^N)$, see (3.0.2), for $r \geq 1$ such that $s - a - \frac{N}{q} \geq -\frac{N}{r}$. Then from Lemma 4.0.2 we get $P_a u \in \dot{L}^\rho_U(\mathbb{R}^N)$ provided $r \geq p'$ and $\frac{1}{\rho} = \frac{1}{r} + \frac{1}{p}$. Now we use the inclusion $\dot{L}^\rho_U(\mathbb{R}^N) \hookrightarrow \dot{H}^{-\sigma,q}_U(\mathbb{R}^N)$ from Proposition 3.0.1 provided $\sigma - \frac{N}{q'} \geq -\frac{N}{\rho'}$.

Now we show that we can chose ρ and r as above. For this we write all conditions above in terms of $z = -\frac{N}{\rho} \in [-N, 0]$ as

$$-\sigma - \frac{N}{q} \leq z \leq s - a - \frac{N}{q} - \frac{N}{p}$$

and, because of (4.0.4), this condition defines a nonempty interval. To check that we can chose $z = -\frac{N}{\rho} \in [-N, 0]$ satisfying the inequalities above, just note that $s - a - \frac{N}{q} - \frac{N}{p} \geq -N$ again by (4.0.4). Thus we can choose $r, \rho \geq 1$ such that $(s - a - \frac{N}{q})_- \geq -\frac{N}{r}$ and $(\sigma - \frac{N}{q'})_- \geq -\frac{N}{\rho'}$ with $\frac{1}{\rho} = \frac{1}{r} + \frac{1}{p}$ (and so $r \geq p'$). ■

Thus we get the following result for second order operators, where A_0 is perturbed by some lower order term.

Theorem 4.0.4 *Let $a \in \{0, 1\}$, $d \in \dot{L}^p_U(\mathbb{R}^N)$ be such that $\|d\|_{\dot{L}^p_U(\mathbb{R}^N)} \leq R_0$ with $p > \frac{N}{2-a}$. Then for any $1 < q < \infty$ and any P as above there exists an interval $I(q, a) \subset (-1 + \frac{a}{2}, 1)$ containing $(-1 + \frac{a}{2} + \frac{N}{2p}, 1 - \frac{N}{2p})$, such that for any $\gamma \in I(q, a)$, we have an order preserving, strongly continuous, analytic semigroup $S_P(t)$ in the space $\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)$, for the problem*

$$\begin{cases} u_t + A_0 u + d(x) D^a u = 0, & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & in\ \mathbb{R}^N, \end{cases} \tag{4.0.5}$$

with $u(t; u_0) = S_P(t) u_0$, $t \geq 0$.

Moreover the semigroup has the smoothing estimate

$$\|S_P(t) u_0\|_{\dot{H}^{2\gamma',q}_U(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma} e^{\mu t}}{t^{\gamma' - \gamma}} \|u_0\|_{\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{H}^{2\gamma}_U(\mathbb{R}^N) \tag{4.0.6}$$

for every $\gamma, \gamma' \in I(q, a)$ with $\gamma' \geq \gamma$, and

$$\|S_P(t) u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \leq \frac{M_{q,r} e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{q} - \frac{1}{r})}} \|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{L}^q_U(\mathbb{R}^N) \tag{4.0.7}$$

for $1 < q \leq r \leq \infty$ with some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on M, θ_0 and R_0.

The interval $I(q, a)$ is given by

$$I(q, a) = (-1 + \frac{a}{2} + \frac{N}{2}(\frac{1}{p} - \frac{1}{q'})_+, 1 - \frac{N}{2}(\frac{1}{p} - \frac{1}{q})_+) \subset (-1 + \frac{a}{2}, 1).$$

Finally, if, as $\varepsilon \to 0$

$$d_\varepsilon \to d \quad in\ \dot{L}^p_U(\mathbb{R}^N), \quad p > \frac{N}{2 - a}$$

then for every $T>0$ there exists $C(\varepsilon)\to 0$ as $\varepsilon\to 0$, such that

$$\|S_{P_\varepsilon}(t)-S_P(t)\|_{\mathcal{L}(\dot H^{2\gamma,q}_U(\mathbb{R}^N),\dot H^{2\gamma',q}_U(\mathbb{R}^N))}\le \frac{C(\varepsilon)}{t^{\gamma'-\gamma}},\qquad \forall\, 0<t\le T$$

for all $\gamma,\gamma'\in I(q,a)$, $\gamma'\ge\gamma$ and for all $1<q\le r\le\infty$,

$$\|S_{P_\varepsilon}(t)-S_P(t)\|_{\mathcal{L}(\dot L^q_U(\mathbb{R}^N),\dot L^r_U(\mathbb{R}^N))}\le \frac{C(\varepsilon)}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}},\qquad \forall\, 0<t\le T.$$

Proof. We first prove (4.0.6) and for this we follow several steps.
Step 1. Denote $X^\alpha:=\dot H^{2\alpha,q}_U(\mathbb{R}^N)$, $\alpha\in[-1,1]$. If we assume for a moment that (4.0.4) is satisfied for some $s_0\ge a$ and $\sigma_0\ge 0$, then, by Proposition 4.0.3, we would have

$$P\in\mathcal{L}(X^{s_0/2},X^{-\sigma_0/2}),\qquad \|P\|_{\mathcal{L}(X^{s_0/2},X^{-\sigma_0/2})}\le C\|d\|_{\dot L^p_U(\mathbb{R}^N)}.$$

Hence we can apply [47, Proposition 10] (see Theorem 1.0.1) with $\alpha=s_0/2$ and $\beta=-\sigma_0/2$ provided $0\le\alpha-\beta<1$, that is, $s_0+\sigma_0<2$. This result gives a solution of (4.0.5), $u(t;u_0)=S_P(t)u_0$, $t\ge 0$, satisfying (4.0.6) for any $\gamma\in E(\alpha):=(\alpha-1,\alpha]$ and $\gamma'\in R(\beta):=[\beta,\beta+1)$ with $\gamma'\ge\gamma$. Note that we can always take at least $\gamma\in[\beta,\alpha]$, $\gamma'\in[\gamma,\beta+1)$.
Step 2. To determine the set of pairs (s,σ) satisfying (4.0.4) and $s+\sigma<2$, we define

$$\tilde s=s-a-\frac{N}{q}\quad\text{and}\quad \tilde\sigma=\sigma-\frac{N}{q'},\tag{4.0.8}$$

so $\tilde s\ge -\frac{N}{q}$, $\tilde\sigma\ge-\frac{N}{q'}$ since $s\ge a$, $\sigma\ge 0$. Then (4.0.4) and $s+\sigma<2$ read

$$\tilde s\ge-\frac{N}{q},\quad \tilde\sigma\ge-\frac{N}{q'},\qquad -\frac{N}{p'}\le\tilde s_-+\tilde\sigma_-,\qquad \tilde s+\tilde\sigma<2-a-N.\tag{4.0.9}$$

Note that this region is nonempty since $-N\le-\frac{N}{p'}<2-a-N$ because $p>\frac{N}{2-a}$.

The set of admissible parameters $(\tilde s,\tilde\sigma)$ given by (4.0.9) depends on the relationship between q, q' and p. Note that (4.0.9) defines a planar trapezium–shaped polygon, $\tilde{\mathcal{P}}$, whose long base is on the line $\tilde s+\tilde\sigma=2-a-N$ and the short base is on the line $\tilde s+\tilde\sigma=-\frac{N}{p'}$ in the third quadrant. As for the lateral sides note that the restriction $-\frac{N}{p'}\le\tilde s_-+\tilde\sigma_-$ adds the condition that $\tilde s\ge-\frac{N}{p'}$ in the second quadrant and $\tilde\sigma\ge-\frac{N}{p'}$ in the fourth. These have to be combined with $\tilde s\ge-\frac{N}{q}$ and $\tilde\sigma\ge-\frac{N}{q'}$. Therefore the lateral sides are given by the lines $\tilde s=\max\{-\frac{N}{p'},-\frac{N}{q}\}$ and $\tilde\sigma=\max\{-\frac{N}{p'},-\frac{N}{q'}\}$. One of the possible cases is depicted in Figure 4.1.

In any case, projecting $\tilde{\mathcal{P}}$ onto the axes gives the following ranges for $\tilde s$ and $\tilde\sigma$

$$\tilde s\in[\max\{-\frac{N}{p'},-\frac{N}{q}\},2-a-N-\max\{-\frac{N}{p'},-\frac{N}{q'}\})$$

$$\tilde\sigma\in[\max\{-\frac{N}{p'},-\frac{N}{q'}\},2-a-N-\max\{-\frac{N}{p'},-\frac{N}{q}\}).$$

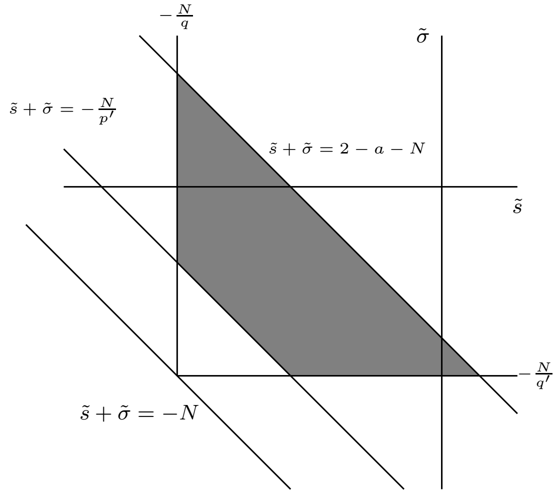

Figure 4.1: Admissible $\tilde{s}$ and $\tilde{\sigma}$ with $p > q, q'$

Note that, by (4.0.8), the polygon $\tilde{\mathcal{P}}$ transforms into a similar shaped polygon $\mathcal{P}$ which determines the region of admissible pairs (s, σ). Thus the projection ranges for s and σ are given by

$$s \in J_1 = [a + (\frac{N}{q} - \frac{N}{p'})_+, 2 - (\frac{N}{q'} - \frac{N}{p'})_+) \tag{4.0.10}$$

$$\sigma \in J_2 = [(\frac{N}{q'} - \frac{N}{p'})_+, 2 - a - (\frac{N}{q} - \frac{N}{p'})_+). \tag{4.0.11}$$

Step 3. Now we perform a bootstrap argument with the solutions of (4.0.5). For any $(s_0, \sigma_0) \in \mathcal{P}$, by Step 1 above, we have a solution of (4.0.5) satisfying (4.0.6) for any $\gamma \in [-\frac{\sigma_0}{2}, \frac{s_0}{2}]$ and $\gamma' \in [\gamma, \frac{s_0}{2} + 1)$. Now the line $s + \sigma = s_0 + \sigma_0 := k_0 < 2$ intersects $\mathcal{P}$ along a segment $\mathcal{S}(s_0, \sigma_0)$ which, using (4.0.10), (4.0.11), can be parametrized in terms of $s \in J_1(k_0) = [a + (\frac{N}{q} - \frac{N}{p'})_+, k_0 - (\frac{N}{q'} - \frac{N}{p'})_+]$.

Then take $(s, \sigma) \in \mathcal{S}(s_0, \sigma_0)$ with $s \geq s_0$, hence $\sigma \leq \sigma_0$, and such that $s_0 \leq s < 4 - \sigma_0$ which implies that $R(-\frac{\sigma_0}{2}) \cap E(\frac{s}{2}) \neq \emptyset$. Then, using $S_P(t) = S_P(t/2) \circ S_P(t/2)$ and taking $\gamma' \in R(-\frac{\sigma_0}{2}) \cap E(\frac{s}{2}) \neq \emptyset$ with $\gamma' \geq \gamma$ we get

$$\|S_P(t)u_0\|_{\dot{H}_U^{\gamma'',q}(\mathbb{R}^N)} \leq \frac{\tilde{M}e^{\mu(t/2)}}{(t/2)^{\gamma''-\gamma'}} \|S_P(t/2)u_0\|_{\dot{H}_U^{\gamma',q}(\mathbb{R}^N)} \leq \tag{4.0.12}$$

$$\frac{\tilde{M}e^{\mu(t/2)}}{(t/2)^{\gamma''-\gamma'}} \frac{\tilde{M}e^{\mu(t/2)}}{(t/2)^{\gamma'-\gamma}} \|u_0\|_{\dot{H}_U^{\gamma,q}(\mathbb{R}^N)} = \frac{Me^{\mu t}}{t^{\gamma''-\gamma}} \|u_0\|_{\dot{H}_U^{\gamma,q}(\mathbb{R}^N)}$$

that is, (4.0.6) for any $\gamma \in [-\frac{\sigma_0}{2}, \frac{s_0}{2}]$ and any $\gamma'' \in R(-\frac{\sigma}{2}) = [-\frac{\sigma}{2}, -\frac{\sigma}{2} + 1)$, $\gamma'' \geq \gamma' \geq \gamma$, and M depending on γ and γ''.

Note that $s_0 \leq s \leq k_0 - (\frac{N}{q'} - \frac{N}{p'})_+ \leq k_0 < 2 < 4 - k_0 \leq 4 - \sigma_0$ so all conditions above are met. Also, as we take $s \in [s_0, k_0 - (\frac{N}{q'} - \frac{N}{p'})_+]$ and $\sigma = k_0 - s$, we get (4.0.12) for any

$\gamma \in [-\frac{\sigma_0}{2}, \frac{s_0}{2}]$

$$\gamma'' \in \bigcup_{\{\sigma=k_0-s,\ s\in[s_0,k_0-(\frac{N}{q'}-\frac{N}{p'})_+]\}} R(-\frac{\sigma}{2}) = [-\frac{\sigma_0}{2}, 1-\frac{1}{2}(\frac{N}{q'}-\frac{N}{p'})_+) \tag{4.0.13}$$

with $\gamma'' \geq \gamma$.

So, as (s_0, σ_0) range in the region $\mathcal{P}$, from (4.0.10), (4.0.11) and (4.0.13), we get (4.0.6) for

$$\gamma \in (-\frac{\sup J_2}{2}, \frac{\sup J_1}{2}), \qquad \gamma' \in (-\frac{\sup J_2}{2}, 1-\frac{1}{2}(\frac{N}{q'}-\frac{N}{p'})_+), \qquad \gamma' \geq \gamma$$

which leads to

$$\gamma, \gamma' \in I(q,a) = (-1+\frac{a}{2}+\frac{N}{2}(\frac{1}{q}-\frac{1}{p'})_+, 1-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})_+), \qquad \gamma' \geq \gamma$$

which concludes the proof of (4.0.6).

For the estimates in uniform Lebesgue spaces, (4.0.7), we use the uniform Bessel inclusions (3.0.2). First note that for any $1 < q < \infty$, $I(q,a) \supset (-1+\frac{a}{2}+\frac{N}{2p}, 1-\frac{N}{2p})$ which does not depend on q and is not empty because $p > \frac{N}{2-a}$. Let $\tilde{\gamma} := 1-\frac{N}{2p} > 0$ and take $0 \leq \gamma < \tilde{\gamma}$, then from (3.0.2), $\dot{H}_U^{2\gamma,q}(\mathbb{R}^N) \hookrightarrow \dot{L}_U^{\tilde{q}}(\mathbb{R}^N)$, for $\tilde{q} \geq q$ such that $-\frac{N}{\tilde{q}} \leq 2\gamma - \frac{N}{q}$, i.e. $\frac{1}{q}-\frac{1}{\tilde{q}} \leq \frac{2\gamma}{N}$, and we get

$$\|S_P(t)u_0\|_{\dot{L}_U^{\tilde{q}}(\mathbb{R}^N)} \leq C\|S_P(t)u_0\|_{\dot{H}_U^{2\gamma,q}(\mathbb{R}^N)} \leq \frac{M_\gamma e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})}}\|u_0\|_{\dot{L}_U^q(\mathbb{R}^N)}.$$

In particular we can take $0 \leq \gamma \leq \frac{\tilde{\gamma}}{2}$ and we get the estimate above for all $\tilde{q} \geq q$ such that $\frac{1}{q}-\frac{1}{\tilde{q}} \in [0, \frac{\tilde{\gamma}}{N}]$ and this interval does not depend on q. This range of $\tilde{q}$ can be written as $\left(\frac{1}{q}-\frac{\tilde{\gamma}}{N}\right)_+ \leq \frac{1}{\tilde{q}} \leq \frac{1}{q}$. Hence if $q \geq \frac{N}{\tilde{\gamma}}$ we get $\tilde{q} \in [q, \infty]$. If $q < \frac{N}{\tilde{\gamma}}$, reiterating this argument, starting with $r_0 := q$ and defining the numbers r_i, $i = 1, 2, 3, \ldots$ such that

$$\frac{1}{r_{i+1}} = \left(\frac{1}{r_i}-\frac{\tilde{\gamma}}{N}\right)_+$$

we obtain the estimate above for any $\tilde{q}$ such that $\tilde{q} \in [q, r_{i+1}]$. Hence, in a finite number of steps we can reach $r_i \geq \frac{N}{\tilde{\gamma}}$ and then the estimate for any $\tilde{q} \in [q, \infty]$, which concludes the proof of (4.0.7).

The convergence of the semigroups is a direct consequence of [47, Theorem 14] (see Theorem 1.0.2), since Proposition 4.0.3 gives that if $d_\varepsilon \to d$ in $\dot{L}_U^p(\mathbb{R}^N)$, then $P_\varepsilon \to P$ in $\mathcal{L}(X^{s/2}, X^{-\sigma/2})$ for any pair of admissible $(s, \sigma) \in \mathcal{P}$. The case of Lebesgue spaces follows from this as well.

The order preserving property is obtained by approximation as in Proposition 4.0.1. Indeed, for smooth enough coefficient d and regular initial data, if $u_0 \geq 0$ then $S_P(t)u_0 \geq 0$ for all $t \geq 0$; see e.g. [24, Section 2.4] and also [5, Proposition 5.3, Remark 5.3] and [1, Theorem 11.6]. Now, for $u_0 \in \dot{H}^{\gamma,q}_U(\mathbb{R}^N)$, with $\gamma \in I(q,a)$ and d as in the statement take d_n and $\{u_0^n\}_{n\in\mathbb{N}}$ regular such that $d_n \to d$ in $\dot{L}^p_U(\mathbb{R}^N)$ and $u_0^n \to u_0$ in $\dot{H}^{\gamma,q}_U(\mathbb{R}^N)$. Then $S_{P_n}(t)u_0^n \to S_P(t)u_0$ and therefore $S_P(t)u_0 \geq 0$. Note again that this works because we are working with the "dotted" spaces, where regular functions are dense, see [5, Lemma 4.2].

Finally, the analyticity comes from [47, Theorem 12], see Theorem 1.0.3. ■

Now, we can combine several perturbations simultaneously.

Remark 4.0.5 *For the problem*

$$\begin{cases} u_t - \sum_{k,l=1}^N a_{kl}(x)\partial_k\partial_l u + \sum_{j=1}^N b_j(x)\partial_j u = 0 & x \in \mathbb{R}^N, \quad t > 0 \\ u(0) = u_0 \end{cases}$$

with $b_j \in \dot{L}^{p_j}_U(\mathbb{R}^N)$ with $p_j > N$, since the uniform Lebesgue spaces are nested we have that $b_j \in \dot{L}^p_U(\mathbb{R}^N)$ with $p = \min_{j=1,\dots,N}\{p_j\} > N$ and then $P = \sum_{j=1}^N b_j(x)\partial_j$ satisfies Proposition 4.0.3 with such p and $a = 1$. Thus Theorem 4.0.4 remains valid for the problem above.

When we combine zeroth and first order terms, we get the following result.

Theorem 4.0.6 *Fix $1 < q < \infty$ and assume for $j = 1,\dots,N$,*

$$\|b_j\|_{\dot{L}^{p_j}_U(\mathbb{R}^N)} \leq R_j \quad \text{and} \quad \|c\|_{\dot{L}^{p_0}_U(\mathbb{R}^N)} \leq R_0$$

where $p_j > N$ and $p_0 > \frac{N}{2}$. Define $a_0 = 0$ and $a_j = 1$ for $j = 1,\dots,N$ and, $\tilde{p} = \min\{p_j,\ j = 1,\dots,N\} > N$. If $q' < \tilde{p}$ and $q > p_0$, we will also assume $p_0 > \frac{Nq}{N+q}$.

Then there exists a non-empty interval $I(q) \subset (-\frac{1}{2}, 1)$ containing $(-1 + \max_j\{\frac{a_j}{2} + \frac{N}{2p_j}\}, 1 - \max_j\{\frac{N}{2p_j}\})$, such that for any $\gamma \in I(q)$, we have a strongly continuous, order preserving, analytic semigroup $S(t)$, in the space $\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)$, for the problem

$$\begin{cases} u_t - \sum_{k,l=1}^N a_{kl}(x)\partial_k\partial_l u + \sum_{j=1}^N b_j(x)\partial_j u + c(x)u = 0 & x \in \mathbb{R}^N, \quad t > 0 \\ u(0,x) = u_0(x) & x \in \mathbb{R}^N \end{cases} \tag{4.0.14}$$

with $u(t;u_0) = S(t)u_0$, $t \geq 0$.

Moreover the semigroup has the smoothing estimate

$$\|S(t)u_0\|_{\dot{H}^{2\gamma',q}_U(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{H}^{2\gamma}_U(\mathbb{R}^N) \tag{4.0.15}$$

for every $\gamma, \gamma' \in I(q)$ with $\gamma' \geq \gamma$, and

$$\|S(t)u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \leq \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{L}^q_U(\mathbb{R}^N) \tag{4.0.16}$$

for $1 < q \leq r \leq \infty$ *with some* $M_{\gamma',\gamma}$, $M_{q,r}$ *and* $\mu \in \mathbb{R}$ *depending on* M, θ_0, R_j *and* R_0.

Furthermore,

$$I(q) = (-1 + \max_{j=0,\dots,N}\{\frac{a_j}{2} + \frac{N}{2}(\frac{1}{p_j} - \frac{1}{q})_+\}, 1 - \frac{N}{2}(\frac{1}{\min_{j=0,\dots,N}\{p_j\}} - \frac{1}{q})_+). \quad (4.0.17)$$

Finally, if, as $\varepsilon \to 0$

$$b_j^\varepsilon \to b_j \quad in\ \dot{L}_U^{p_j}(\mathbb{R}^N), \quad p_j > N, \quad j = 1, \dots, N,$$

$$c^\varepsilon \to c \quad in\ \dot{L}_U^{p_0}(\mathbb{R}^N), \quad p_0 > N/2$$

then for every $T > 0$ *there exists* $C(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*, such that*

$$\|S_\varepsilon(t) - S(t)\|_{\mathcal{L}(\dot{H}_U^{2\gamma,q}(\mathbb{R}^N),\dot{H}_U^{2\gamma',q}(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \leq T \quad (4.0.18)$$

for all $\gamma, \gamma' \in I(q)$, $\gamma' \geq \gamma$ *and for all* $1 < q \leq r \leq \infty$,

$$\|S_\varepsilon(t) - S(t)\|_{\mathcal{L}(\dot{L}_U^q(\mathbb{R}^N),\dot{L}_U^r(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \leq T. \quad (4.0.19)$$

Proof. Consider the lower order terms as perturbations $P_j u := b_j \partial_j u$, $P_0 u := cu$. As in the proof of Theorem 4.0.4, for each perturbation P_j there exists a non-empty trapezoidal polygon $\mathcal{P}_j$ of admissible pairs of spaces (s, σ) described in terms of $\tilde{s} = s - a_j - \frac{N}{q}$ and $\tilde{\sigma} = \sigma - \frac{N}{q'}$, see (4.0.9).

According to [47, Lemma 13, iii)], we can consider $P := \sum_{j=0}^N P_j$, that is, all perturbations acting at the same time, if there exists a common region $\mathcal{P}$ of admissible pairs (s, σ), that is if $\mathcal{P} := \cap_{j=0}^N \mathcal{P}_j \neq \emptyset$.

Recall from the proof of Theorem 4.0.4 that the polygon $\mathcal{P}_j$ of the perturbation P_j is given by a planar trapezium whose long base is on the line $s + \sigma = 2$ and the short base is on the line $s + \sigma = a_j + \frac{N}{p_j}$ in the third quadrant. Also, the lateral sides they are given by the lines $s = a_i + (\frac{N}{q} - \frac{N}{p'_j})_+$ and $\sigma = (\frac{N}{q'} - \frac{N}{p'_j})_+$. Thus the projection of $\mathcal{P}_j$ on the axes give the intervals

$$s \in J_1^j = [s_{min}^j, 2 - \sigma_{min}^j) \quad and \quad \sigma \in J_2^j = [\sigma_{min}^j, 2 - s_{min}^j)$$

see (4.0.10) and (4.0.11). Therefore, the set $\mathcal{P}$ is non-empty if and only if

$$\max_j\{\inf J_1^j\} < \min_j\{\sup J_1^j\} \quad i.e. \quad \max_j\{s_{min}^j\} < \min_j\{2 - \sigma_{min}^j\}$$

and

$$\max_j\{\inf J_2^j\} < \min_j\{\sup J_2^j\} \quad i.e. \quad \max_j\{\sigma_{min}^j\} < \min_j\{2 - s_{min}^j\}$$

which are equivalent to $\max_j\{s_{min}^j\} + \max_j\{\sigma_{min}^j\} < 2$, that is,

$$\max_{j=0,\dots,N}\{a_j + (\frac{N}{p_j} - \frac{N}{q'})_+\} + \max_{j=0,\dots,N}\{(\frac{N}{p_j} - \frac{N}{q})_+\} < 2. \quad (4.0.20)$$

We prove below that this condition is always satisfied; see Lemma 4.0.9.

Assuming this for a while, the projection of $\mathcal{P} = \bigcap_{j=0}^{N} \mathcal{P}_j$ on the axes gives the intervals

$$s \in J_1 = [\max_j(\inf J_1^j), \min_j(\sup J_1^j)) = [\max_j\{a_j + (\frac{N}{p_j} - \frac{N}{q'})_+\}, 2 - \max_j\{(\frac{N}{p_j} - \frac{N}{q})_+\})$$

$$\sigma \in J_2 = [\max_j(\inf J_2^j), \min_j(\sup J_2^j)) = [\max_j\{(\frac{N}{p_j} - \frac{N}{q})_+\}, 2 - \max_j\{a_j + (\frac{N}{p_j} - \frac{N}{q})_+\}).$$

For each pair of admissible pairs $(s, \sigma) \in \mathcal{P}$, by [47, Proposition 10], that is Theorem 1.0.1, (see the proof of Theorem 4.0.4) with $\alpha = \frac{s}{2}$ and $\beta = -\frac{\sigma}{2}$, we get a solution of (4.0.14) satisfying (4.0.15) for

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta + 1), \qquad \gamma' \geq \gamma.$$

Hence as (s, σ) range in the region $\mathcal{P}$, a repeated bootstrap argument as in (4.0.12) gives that for $\gamma \in \bigcup_{(s,\sigma)\in\mathcal{P}}[-\sigma/2, s/2]$ and $\gamma' \in \bigcup_{(s,\sigma)\in\mathcal{P}} R(-\sigma/2)$, $\gamma' \geq \gamma$ the smoothing estimates hold. This leads to

$$\gamma \in (-\frac{\sup J_2}{2}, \frac{\sup J_1}{2}), \qquad \gamma' \in (-\frac{\sup J_2}{2}, 1 - \frac{\inf J_2}{2}), \quad \gamma' \geq \gamma$$

which reads

$$\gamma, \gamma' \in I(q) = (-1 + \max_j\{\frac{a_j}{2} + \frac{N}{2}(\frac{1}{p_j} - \frac{1}{q'})_+\}, 1 - \max_j\{\frac{N}{2}(\frac{1}{p_j} - \frac{1}{q})_+\}), \quad \gamma' \geq \gamma,$$

and which gives (4.0.17). Note that this interval is contained in the interval $(-\frac{1}{2}, 1)$ and contains $(-1 + \max_j\{\frac{a_j}{2} + \frac{N}{2p_j}\}, 1 - \max_j\{\frac{N}{2p_j}\})$, which is independent of q and non-empty because $p_j > \frac{N}{2-a_j}$. To see this note that the latter condition gives $\frac{a_j}{2} + \frac{N}{2p_j} < 1$ and $\frac{N}{2p_j} < 1 - \frac{a_j}{2} < 1$.

The estimates in uniform Lebesgue spaces (4.0.16) are obtained using Sobolev embeddings as in Theorem 4.0.4.

The order preserving property is obtained by approximation with smooth coefficients and initial data as in Theorem 4.0.4. Finally, the analyticity comes from [47, Theorem 12], see Theorem 1.0.3. ■

Remark 4.0.7

i) Note that the interval in (4.0.17) is in fact the intersection of the intervals of each separate perturbation as obtained in Theorem 4.0.4.

ii) The additional assumption in Theorem 4.0.6 that "if $q' < \tilde{p}$ and $q > p_0$, we will also assume $p_0 > \frac{Nq}{N+q}$" applies only when q is large relative to the exponents p_j, $j = 0, \dots, N$.

Also, for $1 < q \leq N$ this imposes no additional restriction since in this range $\frac{Nq}{N+q} \leq \frac{N}{2}$. Furthermore, since $\frac{Nq}{N+q} < N$ for all q, if $p_0 \geq N$ no additional assumption is imposed either.

Remark 4.0.8 *If we assume that $p_j \geq q$ for $j = 0, \ldots, N$ as in [5, Theorem 5.3], then Theorem 4.0.6 applies and we get in (4.0.17) an interval*

$$I(q) = (-1 + \max_{j=0,\ldots,N}\{\frac{a_j}{2} + \frac{N}{2}(\frac{1}{p_j} - \frac{1}{q})_+\}, 1).$$

Since this interval contains 0*, then Theorem 4.0.6 recovers [5, Theorem 5.3] and improves it since in (4.0.15) we can even take γ slightly negative.*

This case includes the case in which b_j and c_0 are bounded functions.

Now we prove our claim about (4.0.20).

Lemma 4.0.9 *With the assumptions in Theorem 4.0.6, condition (4.0.20) is satisfied.*

Proof. Observe that denoting $\tilde{p} = \min\{p_j,\ j = 1, \ldots, N\} > N$ and $p = \min\{p_j,\ j = 0, \ldots, N\} = \min\{p_0, \tilde{p}\} > \frac{N}{2}$, then (4.0.20) can be written as

$$\max\{(\frac{N}{p_0} - \frac{N}{q'})_+, 1 + (\frac{N}{\tilde{p}} - \frac{N}{q'})_+\} + \max\{(\frac{N}{p_0} - \frac{N}{q})_+, (\frac{N}{\tilde{p}} - \frac{N}{q})_+\} < 2.$$

To prove the lemma we prove that all possible sums of the terms inside the “max” above are less than 2.

1. Let $M = (\frac{N}{p_0} - \frac{N}{q'})_+ + (\frac{N}{p_0} - \frac{N}{q})_+$
   (a) If $q, q' < p_0$ then $M = 0 < 2$.
   (b) If $q < p_0 < q'$ then $M = \frac{N}{p_0} - \frac{N}{q'} < \frac{N}{p_0} < 2$.
   (c) If $q' < p_0 < q$ then $M = \frac{N}{p_0} - \frac{N}{q} < \frac{N}{p_0} < 2$.
   (d) If $p_0 < q, q'$ then $M = \frac{2N}{p_0} - N = \frac{N}{p_0} - \frac{N}{p_0'} < \frac{N}{p_0} < 2$.
2. Let $M = (\frac{N}{p_0} - \frac{N}{q'})_+ + (\frac{N}{\tilde{p}} - \frac{N}{q})_+$
   (a) If $q' < p_0$ and $q < \tilde{p}$ then $M = 0$.
   (b) If $p_0 < q'$ and $q < \tilde{p}$ then $M = \frac{N}{p_0} - \frac{N}{q'} < \frac{N}{p_0} < 2$.
   (c) If $q' < p_0$ and $q > \tilde{p}$ then $M = \frac{N}{\tilde{p}} - \frac{N}{q} < \frac{N}{\tilde{p}} < 1$.
   (d) If $p_0 < q'$ and $q > \tilde{p}$ then $M = \frac{N}{p_0} + \frac{N}{\tilde{p}} - N = \frac{N}{\tilde{p}} - \frac{N}{p_0'} < \frac{N}{\tilde{p}} < 1$.
3. Let $M = 1 + (\frac{N}{\tilde{p}} - \frac{N}{q'})_+ + (\frac{N}{p_0} - \frac{N}{q})_+$
   (a) If $q' < \tilde{p}$ and $q < p_0$ then $M = 1$.
   (b) If $\tilde{p} < q'$ and $q < p_0$ then $M = 1 + \frac{N}{\tilde{p}} - \frac{N}{q'} < 1 + \frac{N}{\tilde{p}} < 2$.
   (c) If $q' < \tilde{p}$ and $q > p_0$ then $M = 1 + \frac{N}{p_0} - \frac{N}{q} < 2$ because $p_0 > \frac{Nq}{N+q}$ by assumption.

(d) If $\tilde{p} < q'$ and $q > p_0$ then $M = 1 + \frac{N}{\tilde{p}} + \frac{N}{p_0} - N = 1 + \frac{N}{\tilde{p}} - \frac{N}{p_0'} < 1 + \frac{N}{\tilde{p}} < 2$.

4. Let $M = 1 + (\frac{N}{\tilde{p}} - \frac{N}{q'})_+ + (\frac{N}{\tilde{p}} - \frac{N}{q})_+$

(a) If $q', q < \tilde{p}$ then $M = 1$.

(b) If $q < \tilde{p} < q'$ then $M = 1 + \frac{N}{\tilde{p}} - \frac{N}{q'} < 1 + \frac{N}{\tilde{p}} < 2$.

(c) If $q' < \tilde{p} < q$ then $M = 1 + \frac{N}{\tilde{p}} - \frac{N}{q} < 1 + \frac{N}{\tilde{p}} < 2$.

(d) If $\tilde{p} < q, q'$ then $M = 1 + \frac{2N}{\tilde{p}} - N = 1 + \frac{N}{\tilde{p}} - \frac{N}{\tilde{p}'} < 1 + \frac{N}{\tilde{p}} < 2$.

■

Chapter 5

Fourth order problems in $\mathbb{R}^N$

We now study the solvability of some fourth order linear parabolic equations in $\mathbb{R}^N$. More precisely, we consider

$$\begin{cases} u_t + \Delta^2 u + Pu = 0, & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N \end{cases} \tag{5.0.1}$$

with u_0 a suitable initial data defined in $\mathbb{R}^N$ and P a linear perturbation. We will consider space dependent perturbations of the form $Pu := \sum_{a,b} P_{a,b} u$ with

$$P_{a,b} u := D^b(d(x) D^a u) \qquad x \in \mathbb{R}^N \tag{5.0.2}$$

for some $a, b \in \{0, 1, 2, 3\}$ such that $a+b \leq 3$, where D^a, D^b denote any partial derivatives of order a, b, and $d(x)$ is a given function with $x \in \mathbb{R}^N$.

The main goal in this chapter is to consider some large classes of initial data u_0 in $\mathbb{R}^N$ for the problem (5.0.1) as well as to consider wide classes of low regularity perturbations. For the latter we will consider classes of coefficients $d(x)$ with weak integrability properties. More precisely, we will assume below that the coefficient $d(x)$ belongs to some locally uniform space $L^p_U(\mathbb{R}^N)$, $1 \leq p < \infty$ defined as in Chapter 3.

As for the initial data we will consider the standard Lebesgue space, $L^q(\mathbb{R}^N)$, $1 < q < \infty$, or Bessel-Lebesgue spaces $H^{\alpha,q}(\mathbb{R}^N)$, with $1 < q < \infty$, $\alpha \in \mathbb{R}$ and even uniform Bessel spaces $\dot{H}^{\alpha,q}_U(\mathbb{R}^N)$ introduced in Chapter 3.

Given such classes of initial data and perturbations we want to find suitable smoothing estimates on the solutions of (5.0.1) as will be explained below.

Note that for $P = 0$ the solution of problem (5.0.1) can be described as the convolution of the initial data with the self-similar fundamental kernel for the bi–Laplacian operator, which satisfies suitable Gaussian bounds; see e.g. [21, 22] and [20, 10].

Recently, results in Bessel-Lebesgue spaces have been proved in [16] for $P \neq 0$. By means of resolvent estimates for $\Delta^2 + P$, the authors proved the well posedness of (5.0.1) with $Pu = d(x)u$, that is, a perturbation with $a, b = 0$. They also found suitable smoothing estimates on the solutions as the ones we will find in (5.0.2).

Here, instead of relying on elliptic resolvent estimates for the operators $\Delta^2 + P$, with P as in (5.0.2), we rely on a more abstract "parabolic" argument developed in [47] and applied there to parabolic equations with second order elliptic operators. With this approach we consider a simpler problem, the one with $P = 0$, that we can solve in several spaces simultaneously. That is, we consider a semigroup of solutions defined on a scale of spaces. For such simpler problem we start by proving suitable smoothing estimates on the spaces of the scale. Then we consider a suitable perturbation, P, that acts between two spaces of the scale. With these ingredients the abstract results in [47] (recovered in Chapter 1) and the results in Chapter 2 (inspired in [2]) allow to obtain a perturbed semigroup that corresponds to the equation (5.0.1) with $P \neq 0$. Such perturbed semigroup inherits some of the smoothing estimates of the original one in some of the spaces of the scale which are determined by the perturbation P itself.

Another important result that we are able to stablish using the tools developed in [47], is that of the robustness with respect to the perturbation. In this direction, we are able to prove two important results. First, we show that all constants involved in the smoothing estimates of the perturbed semigroups, including the exponential bounds on them, are bounded uniformly for bounded families of perturbations (i.e. for families of coefficients $d(x)$ as in (5.0.2) which are bounded in the uniform space $L^p_U(\mathbb{R}^N)$). Second, we prove that the perturbed semigroups obtained as above, continuously depend on the perturbation. That is, if the coefficients $d(x)$ depend on a parameter and converge in the space $L^p_U(\mathbb{R}^N)$, then the corresponding semigroups converge in norm.

As mentioned above this approach was applied in [47] to second order parabolic equations in bounded and unbounded domains, allowing perturbations in the equation and in the boundary conditions.

We now carry out these ideas to fourth order parabolic equations in $\mathbb{R}^N$ such as (5.0.1). For that, we use an existence and regularity theory in suitable scales of spaces for the parabolic bi–Laplacian equation, i.e. (5.0.1) with $P = 0$, in order to later introduce the perturbations. For this we use some available information about the heat equation $u_t - \Delta u = 0$, in $\mathbb{R}^N$ and use that Δ^2 is the square operator of $-\Delta$. In particular, the same scales of spaces available for $-\Delta$ can be used for (5.0.1). In such scales suitable smoothing estimates for (5.0.1) with $P = 0$ are obtained.

5.1 The scales and semigroup for A_0^2

In this section we show how the scale of spaces constructed in Chapter 2 for A_0 can be used for the squared operator $A_0^2 := A_0 \circ A_1$. More precisely, our goal here is to relate the scales of the square of an operator, A_0^2, with the scale of the A_0. We will show that if we perform the constructions in Chapter 2 with A_0^2 we arrive to the same spaces associated for A_0 with a suitable labeling.

As in Chapter 2 we assume

$$A_0 \in \mathcal{H}(E^1, E^0).$$

Observe that by Propositions 2.1.4 and 2.2.1 we can consider the associated interpolation scale $\{E^\alpha\}_{\alpha\in\mathbb{R}}$ or the fractional power scale $\{F^\alpha\}_{\alpha\geq -N}$, $N \in \mathbb{N}$ without assuming $0 \in \rho(A_0)$ or $(-\infty, 0] \in \rho(A_0)$, respectively. Also, note that with the notation from Chapter

2,

$$A_0^2 := A_0 \circ A_1, \qquad A_0^2 : E^2 \to E^0.$$

Hence, we will assume furthermore that

$$A_0^2 \in \mathcal{H}(E^2, E^0).$$

The following result, which is a particular case of [37, Proposition 10.5], gives a criteria for determining when A_0^2 is a sectorial operator.

Proposition 5.1.1 *Let $A_0 \in \mathcal{H}(E^1, E^0)$ with $(-\infty, 0] \subset \rho(A_0)$ and satisfying $\|(A_0 - \lambda)^{-1}\| \leq \frac{K}{|\lambda|}$ for $\lambda \in S_{0,\phi}$ with $\phi \in (0, \frac{\pi}{4})$ where $S_{0,\phi}$ is a sector as (2.0.1) with vertex $a = 0$.*
Then A_0^2 satisfies $S_{0,2\phi} \subset \rho(A_0^2)$ and

$$\|(A_0^2 - \lambda)^{-1}\|_{E^0} \leq \frac{K}{|\lambda|}$$

for $\lambda \in S_{0,2\phi}$, thus $A_0^2 \in \mathcal{H}(E^2, E^0)$.

Remark 5.1.2
i) For the proof we refer to [37, Proposition 10.5]. As an indication for the proof observe that to solve $A_0^2 u - \lambda u = f$, with $\lambda \in \mathbb{C}$ we can rewrite this equation as

$$(A_0 + \omega_2)(A_0 + \omega_1)u = f$$

where ω_1 and $\omega_2 = -\omega_1$ denote the complex square roots of λ. Thus λ will be in $\rho(A_0^2)$ if both ω_1, $\omega_2 \in \rho(A_0)$. In particular, if $\lambda \in S_{0,2\phi}$, with $\phi < \frac{\pi}{4}$, then ω_1, $\omega_2 \in S_{0,\phi} \subset \rho(A_0)$, thus $S_{0,2\phi} \subset \rho(A_0^2)$. For the estimate, just note that

$$\|(A_0^2 - \lambda)^{-1}\|_{E^0} \leq \|(A_0 + \omega_1)^{-1}(A_0 + \omega_2)^{-1}\|_{E^0} \leq \frac{K_1}{|\omega_1|}\|(A_0 + \omega_2)^{-1}\|_{E^0} \leq \frac{K}{|\omega_1||\omega_2|} = \frac{K}{|\lambda|}.$$

ii) $0 \in \rho(A_0)$ implies $0 \in \rho(A_0^2)$.
iii) In general, there is no relationship between type(A_0^2) and type(A_0).

With this, we can construct both interpolation and fractional scales for A_0^2 following the procedures explained in Chapter 2. However, it is not clear how this scale might be related with the one generated by A_0. In the next two results we show that the scales constructed from A_0^2 coincide with the ones from A_0 after a suitable labeling.

Proposition 5.1.3 *Let $A_0 \in \mathcal{H}(E^1, E^0)$ and assume $A_0^2 := A_0 \circ A_1 \in \mathcal{H}(E^2, E^0)$. Let $\{E^\alpha\}_{\alpha \in \mathbb{R}}$ be the interpolation scale for A_0 as in Proposition 2.1.4. Then on the scale $X^\alpha = E^{2\alpha}$ with $\alpha \in \mathbb{R}$ we have $A_\alpha^2 := A_\alpha \circ A_{\alpha+1} \in \mathcal{H}(X^{\alpha+1}, X^\alpha)$ and A_0^2 defines a semigroup $S_{A_0^2}(t)$ in the scale $\{X^\alpha\}_{\alpha \in \mathbb{R}}$ that satisfies $S_{A_0^2}(t)|_{X^\alpha} = e^{-A_\alpha^2 t}$ and*

$$\|S_{A_0^2}(t)\|_{\mathcal{L}(X^\beta, X^\alpha)} \leq \frac{C(\alpha - \beta)}{t^{\beta - \alpha}} e^{\mu t} \qquad t > 0, \alpha, \beta \in \mathbb{R}, \alpha \geq \beta$$

for any $\mu > type(A_0^2)$. The constant $C(\alpha-\beta)$ is bounded for α, β in bounded sets of $\mathbb{R}$.

If E^0 is reflexive, the negative side of the scale can be described as

$$X^{-\alpha} = (X^{\alpha\sharp})' \quad \text{and} \quad A^2_{-\alpha} = (A^{2\sharp}_{\alpha})', \qquad \alpha > 0$$

and it holds that

$$e^{-A^2_{-\alpha}t} = (e^{-A^{2\sharp}_{\alpha}t})'.$$

Furthermore, the problem

$$\begin{cases} u_t + A^2_\alpha u = 0, \quad t > 0 \\ u(0) = u_0 \in X^\alpha \end{cases}$$

for any $\alpha \in \mathbb{R}$ has a unique solution $u(t) = S_{A_0^2}(t)u_0 = e^{-A^2_\alpha t}u_0$.

Proof. Step 1. We start proving the result assuming $0 \in \rho(A_0)$.

Hence, $0 \in \rho(A_0^2)$ and in this case it is easy to see that the construction (2.0.4)–(2.0.7) applied to A_0^2 leads to the discrete scale $\{X^k : k \in \mathbb{Z}\}$ with $X^k = E^{2k}$, $k \in \mathbb{Z}$ and $A^2_k = A_k \circ A_{k+1} \in \mathcal{H}(X^{k+1}, X^k)$.

By means of the complex interpolation, the construction (2.1.2)–(2.1.5) leads for $\alpha = k + \theta$ with $\theta \in (0,1)$, $k \in \mathbb{Z}$, to

$$X^\alpha := X^{k+\theta} := [X^{k+1}, X^k]_\theta = [E^{2(k+1)}, E^{2k}]_\theta = E^{2\alpha}$$

and

$$A^2_\alpha := A_\alpha \circ A_{\alpha+1} \in \mathcal{H}(X^{\alpha+1}, X^\alpha)$$

for any $\alpha \in \mathbb{R}$.

In particular, by (2.1.5) with A^2_α, we have as in Definition 2.1.2 that A_0^2 defines an analytic semigroup $S_{A_0^2}(t)$ in the scale $\{X^\alpha\}_{\alpha\in\mathbb{R}}$ that satisfies $S_{A_0^2}(t)|_{X^\alpha} = e^{-A^2_\alpha t}$ and

$$\|S_{A_0^2}(t)\|_{\mathcal{L}(X^\alpha, X^\beta)} \le \frac{C(\alpha-\beta)}{t^{\beta-\alpha}} e^{\mu t} \quad t > 0,\ \alpha, \beta \in \mathbb{R}, \alpha \ge \beta$$

for any $\mu > \text{type}(A_0^2)$.

If E^0 is reflexive we can identify, as above, the negative side of this scale with some dual spaces. In fact, from (2.0.7) we have $X^{-k} = (X^{k\sharp})'$ and $A^2_{-k} = (A^{\sharp 2}_k)'$ and by interpolation, see (2.1.6), $X^{-\alpha} = (X^{\alpha\sharp})'$ and $A^2_{-\alpha} = (A^{\sharp 2}_\alpha)'$, $\alpha > 0$, with $e^{-A^2_{-\alpha}t} = (e^{-A^{\sharp 2}_\alpha t})'$ and $(A^2_\alpha)^\sharp = (A^\sharp_\alpha)^2$, see (2.1.7).

Step 2. Now, if $0 \notin \rho(A_0)$, there exists $c \in \mathbb{R}$ such that $\tilde{A}_0 = A_0 + cI$ satisfies $0 \in \rho(\tilde{A}_0)$ and $\tilde{A}_0 \in \mathcal{H}(E^1, E^0)$. Now we prove that $\tilde{A}_0^2 \in \mathcal{H}(E^2, E^0)$. For this note that $\tilde{A}_0^2 = A_0^2 + P$, with $P = 2cA_0 + c^2 I$, which satisfies $\|P\|_{\mathcal{L}(E^1,E^0)} \le R_0$. Since $A_0^2 \in \mathcal{H}(E^2, E^0)$, using this and [31, Corollary 1.4.5, page 27] we get $\tilde{A}_0^2 \in \mathcal{H}(E^2, E^0)$.

Therefore we can use Step 1 for $\tilde{A}_0^2$ and observe that from Proposition 2.1.4 the interpolation scale for $\tilde{A}_0$, $\{E^\alpha\}_{\alpha\in\mathbb{R}}$, is independent of c. Denote then $X^\alpha = E^{2\alpha}$.

Then $\tilde{A}_0^2$ defines an analytic semigroup $S_{\tilde{A}_0^2}(t)$ in the scale $\{X^\alpha\}_{\alpha\in\mathbb{R}}$ and as above $S_{\tilde{A}_0^2}(t)|_{X^\alpha} = e^{-\tilde{A}_\alpha^2 t}$ and

$$\|S_{\tilde{A}_0^2}(t)\|_{\mathcal{L}(X^\alpha,X^\beta)} \le \frac{C(\alpha-\beta)}{t^{\beta-\alpha}} e^{\tilde{\mu}t} \quad t>0,\ \alpha,\beta\in\mathbb{R}, \alpha\ge\beta$$

where $\tilde{\mu} > \text{type}(\tilde{A}_0^2)$.

Now we transfer this information to the semigroup defined by A_0^2. For this observe that $A_0^2 = \tilde{A}_0^2 - P$, with $P = 2cA_0 + c^2 I$ as above, and for all $\alpha\in\mathbb{R}$,

$$\|P\|_{\mathcal{L}(X^\alpha, X^{\alpha-\frac{1}{2}})} \le R_0$$

with R_0 independent of α.

Then we can apply Theorem 1.0.1 with $\beta = \alpha - \frac{1}{2}$ and α arbitrary, to obtain the semigroup $S_{A_0^2}(t)$ defined in X^γ for all $\gamma \in E(\alpha) := (\alpha-1, \alpha]$ and satisfying the smoothing estimate (1.0.7) from X^γ to $X^{\gamma'}$ for $\gamma\in E(\alpha)$ and $\gamma' \in R(\beta) := [\alpha-\frac{1}{2}, \alpha+\frac{1}{2})$, $\gamma'\ge\gamma$.

In order to extend (1.0.7) for all $\gamma' > \gamma$, we perform a "jump" argument as follows. Given $\alpha\in\mathbb{R}$, take $\beta = \alpha-\frac{1}{2}$ and $\alpha' > \alpha$ such that $\alpha' < \alpha + \frac{1}{2}$, so $\alpha' \in R(\beta)$. Then we can estimate the semigroup for γ' in $R(\beta')$ through an intermediate "jump", that is

$$\gamma\in E(\alpha) \to \tilde{\gamma} \in R(\beta)\cap E(\alpha') \to \gamma' \in R(\beta')$$

and using $S_{A_0^2}(t) = S_{A_0^2}(t/2)\cdot S_{A_0^2}(t/2)$

$$\|S_{A_0^2}(t)u_0\|_{\gamma'} \le \frac{\tilde{M}e^{\mu(t/2)}}{(t/2)^{\gamma'-\tilde{\gamma}}}\|S_{A_0^2}(t/2)u_0\|_{\tilde{\gamma}} \le \frac{\tilde{M}e^{\mu(t/2)}}{(t/2)^{\gamma'-\tilde{\gamma}}}\frac{\tilde{M}e^{\mu(t/2)}}{(t/2)^{\tilde{\gamma}-\gamma}}\|u_0\|_\gamma = \frac{Me^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_\gamma. \tag{5.1.1}$$

So we get (1.0.7) for $\gamma \in E(\alpha) = (\alpha-1,\alpha]$ and $\gamma'\in R(\beta') = [\alpha'-\frac{1}{2}, \alpha'+\frac{1}{2})$ and M depending on γ and γ'. Iterating this process, we get (1.0.7) for all $\gamma' > \gamma$ with $\mu >$type(A_0^2).

For the analyticity we use Theorem 1.0.3. Since $\{X^\alpha\}_{\alpha\in\mathbb{R}}$ are interpolation spaces, this scale satisfies the assumptions of case i) in Theorem 1.0.3; see (1.0.8). ■

Now we turn to the fractional power scale to obtain

Proposition 5.1.4 *Let $A_0 \in \mathcal{H}(E^1,E^0)$ and assume $A_0^2 := A_0\circ A_1 \in \mathcal{H}(E^2,E^0)$. Let $N\in\mathbb{N}$ and $\{F^\alpha\}_{\alpha\ge -2N}$ be the fractional power scale for A_0 as in Proposition 2.2.1. Then on the fractional power scale $Y^\alpha = F^{2\alpha}$ with $\alpha \ge -N$ we have $A_\alpha^2 := A_\alpha \circ A_{\alpha+1} \in \mathcal{H}(Y^{\alpha+1}, Y^\alpha)$ and A_0^2 defines a semigroup $S_{A_0^2}(t)$ in the scale $\{Y^\alpha\}_{\alpha\ge -N}$ that satisfies $S_{A_0^2}(t)|_{F^\alpha} = e^{-A_\alpha^2 t}$ and*

$$\|S_{A_0^2}(t)\|_{\mathcal{L}(Y^\beta,Y^\alpha)} \le \frac{C(\alpha-\beta)}{t^{\alpha-\beta}}e^{\mu t} \qquad t>0,\ \alpha\ge\beta\ge -N$$

for any $\mu >$ type(A_0^2). The constant $C(\alpha-\beta)$ is bounded for α,β in bounded sets of $\mathbb{R}$.

If E^0 is reflexive, the negative side of the scale can be described as

$$Y^{-\alpha} = (Y^{\alpha\sharp})' \quad \textit{and} \quad A^2_{-\alpha} = (A^{\sharp 2}_{\alpha})' \qquad \alpha > 0,$$

and it holds that

$$e^{-A^2_{-\alpha}t} = (e^{-A^{2\sharp}_{\alpha}t})'.$$

Furthermore, the problem

$$\begin{cases} u_t + A^2_{\alpha}u = 0, & t > 0 \\ u(0) = u_0 \in Y^{\alpha} \end{cases}$$

for any $\alpha \geq -N$ has a unique solution $u(t) = S_{A_0^2}(t)u_0 = e^{-A^2_{\alpha}t}u_0$.

Proof. Step 1. We first assume that $(-\infty, 0] \subset \rho(A_0)$. As before, it is easy to see that the construction in (2.0.4)–(2.0.7) applied to A_0^2 leads to the discrete scale $\{Y^k : k \in \mathbb{Z}\}$ with $Y^k = E^{2k} = F^{2k}$, $k \in \mathbb{Z}$ and $A^2_k = A_k \circ A_{k+1} \in \mathcal{H}(Y^{k+1}, Y^k)$.

Now for $\alpha \geq -N$ the construction in (2.2.4) applied to A^2_{-N}, gives a fractional power scale $\{Y^{\alpha} : \alpha \geq -N\}$

$$Y^{\alpha} = D((A^2_{-N})^{\alpha+N}), \ \alpha \geq -N, \qquad A^2_{\alpha} = A_{\alpha} \circ A_{\alpha+1} \in \mathcal{H}(Y^{\alpha+1}, Y^{\alpha}).$$

We prove now that $Y^{\alpha} = F^{2\alpha}$ for $\alpha \geq -N$. In fact, because of (2.2.4) and (2.2.5), we have

$$Y^{\alpha} = D((A^2_{-N})^{\alpha+N}) = D(A^{2\alpha+2N}_{-N}) = F^{2\alpha}.$$

Hence, as above, A_0^2 defines a semigroup $S_{A_0^2}(t)$ in the scale $\{Y^{\alpha}\}_{\alpha \geq -N}$ that satisfies $S_{A_0^2}(t)|_{F^{\alpha}} = e^{-A^2_{\alpha}t}$ and

$$\|S_{A_0^2}(t)\|_{\mathcal{L}(Y^{\beta}, Y^{\alpha})} \leq \frac{C(\alpha - \beta)}{t^{\alpha-\beta}} e^{\mu t} \qquad t > 0, \ \alpha \geq \beta \geq -N$$

for any $\mu > \operatorname{type}(A_0^2)$, see (2.2.6).

Also, if E^0 is reflexive we can again, by (2.2.7), identify the negative side of this new scale with dual spaces

$$Y^{-\alpha} = (Y^{\alpha\sharp})' \quad \textit{and} \quad A^2_{-\alpha} = (A^{\sharp 2}_{\alpha})' \quad 0 < \alpha \leq N$$

and from (2.2.8) we get $e^{-A^2_{-\alpha}t} = (e^{-A^{\sharp 2}_{\alpha}t})'$.

Step 2. Now, if $(-\infty, 0] \not\subset \rho(A_0)$, there exists $c \in \rho(A_0)$ such that $\tilde{A}_0 = A_0 + cI$ satisfies $(-\infty, 0] \in \rho(\tilde{A}_0)$ and $\tilde{A}_0 \in \mathcal{H}(E^1, E^0)$. Now we prove that $\tilde{A}_0^2 \in \mathcal{H}(E^2, E^0)$. For this note that $\tilde{A}_0^2 = A_0^2 + P$, with $P = 2cA_0 + c^2 I$, which satisfies $\|P\|_{\mathcal{L}(E^1, E^0)} \leq R_0$. Since $A_0^2 \in \mathcal{H}(E^2, E^0)$, using this and [31, Corollary 1.4.5, page 27] we get $\tilde{A}_0^2 \in \mathcal{H}(E^2, E^0)$.

Note that from Proposition 2.2.1 the fractional power scale for $\tilde{A}_0$ is independent of c and by Step 1 we get the fractional power scale $X^{\alpha} = F^{2\alpha}$ and a sectorial operator $\tilde{A}^2_{\alpha} = \tilde{A}_{\alpha} \circ \tilde{A}_{\alpha+1} \in \mathcal{H}(Y^{\alpha+1}, Y^{\alpha})$. Also $\tilde{A}_0^2$ defines an analytic semigroup $S_{A_0^2}(t)$ in the scale $\{Y^{\alpha}\}_{\alpha \geq -N}$ and as above $S_{\tilde{A}_0^2}(t)|_{Y^{\alpha}} = e^{-\tilde{A}^2_{\alpha}t}$ and

$$\|S_{\tilde{A}_0^2}(t)\|_{\mathcal{L}(Y^{\alpha}, Y^{\beta})} \leq \frac{C(\alpha - \beta)}{t^{\beta-\alpha}} e^{\tilde{\mu}t}, \qquad t > 0, \quad \alpha \geq \beta \geq -N$$

where $\tilde{\mu} > \text{type}(\tilde{A}_0^2)$.

To transfer this information to the semigroup defined by A_0^2, observe that $A_0^2 = \tilde{A}_0^2 - P$ with $P = 2cA_0 + c^2 I$, as above and

$$\|P\|_{\mathcal{L}(Y^\alpha, Y^{\alpha-\frac{1}{2}})} \leq R_0, \qquad \alpha \geq -N$$

with R_0 independent of α. Then, we can apply Theorem 1.0.1 to obtain the semigroup $S_{A_0^2}(t)$ in Y^γ and smoothing from Y^γ to $Y^{\gamma'}$ for $\gamma \in E(\alpha) := (\alpha - 1, \alpha]$ and $\gamma' \in R(\beta) := [\alpha - \frac{1}{2}, \alpha + \frac{1}{2})$, $\gamma' \geq \gamma$. A similar jump argument as (5.1.1) concludes the estimate for all $\gamma' > \gamma \geq -N$.

Finally, the analyticity comes again from Theorem 1.0.3, part ii). In fact note that fractional power spaces satisfty (1.0.9), see [2, V.(1.2.12)]. ■

Remark 5.1.5 *According to Remark 2.2.2 if A_0 has bounded imaginary powers, then A_0^2 does as well, see (2.2.9). In such case both scales and semigroups in Propositions 5.1.3 and 5.1.4 coincide, that is, $X^\alpha = Y^\alpha$ for $\alpha \geq -N$, see [2, V.1.5.13, pg. 283].*

5.2 Fourth order equations in the Bessel-Lebesgue spaces in $\mathbb{R}^N$

We take, $A_0 = -\Delta$ in $L^q(\mathbb{R}^N)$, with $1 < q < \infty$ with domain $D(A_0) = H^{2,q}(\mathbb{R}^N)$, where $H^{k,q}(\mathbb{R}^N)$, $k \in \mathbb{N}$ denotes the standard Sobolev spaces (often denoted $W^{k,q}(\mathbb{R}^N)$). In this setting, $-\Delta$ is a sectorial operator, see [31], [3], and $\text{type}(-\Delta) := \inf\{\text{Re}(\sigma(-\Delta))\} = 0$.

Using complex interpolation, these spaces can be extended to non integer indexes, known as Bessel spaces. These spaces are very convenient because they satisfy the sharp Sobolev embeddings

$$H^{s,q}(\mathbb{R}^N) \subset \begin{cases} L^r(\mathbb{R}^N),\ s - \frac{N}{q} \geq -\frac{N}{r},\ q \leq r < \infty & if\ s - \frac{N}{q} < 0 \\ L^r(\mathbb{R}^N),\ 1 \leq r < \infty & if\ s - \frac{N}{q} = 0 \\ C^\eta(\mathbb{R}^N) & if\ s - \frac{N}{q} > \eta \geq 0. \end{cases}$$

Also, for the negative indexes, we have

$$H^{-s,q}(\mathbb{R}^N) = (H^{s,q'}(\mathbb{R}^N))'. \tag{5.2.1}$$

For more details, see [31, pg. 35], [1], [3] [2, I.2] or [50]. In what follows we will denote $E^\alpha := H^{2\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, the Bessel-Lebesgue scale of spaces.

Also it is known, see [31] and [1], that for $1 < q < \infty$ the heat equation

$$\begin{cases} u_t - \Delta u = 0, & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0, & \text{in } \mathbb{R}^N \end{cases} \tag{5.2.2}$$

defines a semigroup $S_{-\Delta}(t)$ in the scale of Bessel spaces $\{E^\alpha\}_{\alpha\in\mathbb{R}} := \{H^{2\alpha,q}(\mathbb{R}^N)\}_{\alpha\in\mathbb{R}}$ that satisfies the smoothing estimates

$$\|S_{-\Delta}(t)u_0\|_{H^{2\alpha,q}(\mathbb{R}^N)} \le \frac{M_{\alpha,\beta}e^{\mu_0 t}}{t^{\alpha-\beta}}\|u_0\|_{H^{2\beta,q}(\mathbb{R}^N)}, \qquad t>0, \quad u_0 \in H^{2\beta,q}(\mathbb{R}^N)$$

for $1<q<\infty$, $\alpha,\beta\in\mathbb{R}$, $\alpha\ge\beta$ and

$$\|S_{-\Delta}(t)u_0\|_{L^r(\mathbb{R}^N)} \le \frac{M_{r,q}e^{\mu_0 t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\mathbb{R}^N)}, \qquad t>0, \quad u_0\in L^q(\mathbb{R}^N)$$

for $1\le q\le r\le\infty$ and some constant $M_{r,q}$. In both estimates above $\mu_0>0$ can be arbitrarily small because type$(-\Delta)=0$.

This as well as some other useful properties of $-\Delta$ and Δ^2 in $L^q(\mathbb{R}^N)$, $1<q<\infty$, are collected in the next Lemma.

Lemma 5.2.1 *Take $1<q<\infty$ and denote $E^0 = L^q(\mathbb{R}^N)$.*
i) The Laplace operator $-\Delta$ in E^0 with domain $E^1 = D(-\Delta) = H^{2,q}(\mathbb{R}^N)$ satisfies the estimate

$$\|(-\Delta-\lambda)^{-1}\|_{\mathcal{L}(E^0)} \le M|\lambda|^{-1} \qquad \textit{for all } \lambda\in S_{0,\phi}$$

for the sector

$$S_{a,\phi} = \{z\in\mathbb{C} : \phi \le |arg(z-a)| \le \pi,\ z\ne a\} \subset \rho(A_0) \tag{5.2.3}$$

with $\phi>0$ arbitrarily small. Furthermore $\sigma(-\Delta)=[0,\infty)$ and therefore

$$type(-\Delta) = \inf\{Re(\sigma(-\Delta))\} = 0.$$

ii) The bi–Laplacian operator Δ^2 in E^0 with domain $E^2 = D(\Delta^2) = H^{4,q}(\mathbb{R}^N)$ satisfies the estimate

$$\|(\Delta^2-\lambda)^{-1}\|_{\mathcal{L}(E^0)} \le M|\lambda|^{-1} \qquad \textit{for all } \lambda\in S_{0,2\phi}$$

with $\phi>0$ arbitrarily small. Furthermore $\sigma(\Delta^2)=[0,\infty)$ and therefore

$$type(\Delta^2) = \inf\{Re(\sigma(\Delta^2))\} = 0.$$

Proof. Part i), that is, the information for the Laplacian, is well known. The resolvent estimate, in particular, can be found in [31, pages 32 and 33].

For proving ii), since in i) $\phi>0$ can be taken arbitrarily small, we can apply [37, Proposition 10.5] (see also Proposition 5.1.1) and we get that Δ^2 is sectorial with sector $S_{0,2\phi}$, where $2\phi>0$ can be arbitrarily small. Then $\sigma(\Delta^2)\subset[0,\infty)$ is an immediate consequence of the fact that $\phi>0$ is arbitrarily small. On the other hand, it can be proved that in the uniform space $\dot{L}^q_U(\mathbb{R}^N)$, we have $\sigma(\Delta^2)=[0,\infty)$, see Proposition 5.3.1 below for more details. Then, $\sigma(\Delta^2)=[0,\infty)$ in $L^q(\mathbb{R}^N)$ as well. From this, we get type$(\Delta^2)=0$. ■

Then we can prove the following.

Lemma 5.2.2 *Consider the problem*

$$\left\{ \begin{array}{ll} u_t + \Delta^2 u = 0, & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0, & in\, \mathbb{R}^N. \end{array} \right. \tag{5.2.4}$$

i) Then for each $1 < q < \infty$, (5.2.4) defines an analytic semigroup, $S_{\Delta^2}(t)$, in the scale $X^\alpha = E^{2\alpha} = H^{4\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, such that for any $\mu_0 > 0$ there exists C such that

$$\|S_{\Delta^2}(t)\|_{\mathcal{L}(H^{4\beta,q}(\mathbb{R}^N),H^{4\alpha,q}(\mathbb{R}^N))} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\mu_0 t} \qquad t > 0,\ \alpha, \beta \in \mathbb{R},\ \alpha \geq \beta.$$

ii) The analytic semigroup $S_{\Delta^2}(t)$, in $L^q(\mathbb{R}^N)$, $1 < q < \infty$, satisfies

$$\|S_{\Delta^2}(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N),L^r(\mathbb{R}^N))} \leq \frac{M_{q,r}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} e^{\mu_0 t} \qquad t > 0$$

for any $\mu_0 > 0$ and $1 < q \leq r \leq \infty$ and some $M_{q,r} > 0$ (which also depends on μ_0).

Proof.
i) This is a consequence of Proposition 5.1.3 for $A_0 = -\Delta$.

Note that from Lemma 5.2.1, type$(\Delta^2) = 0$ and then $\mu_0 > 0$ is arbitrary.

ii) For $1 < q < \infty$, we use i) with $\alpha = 0$ and we have that $-\Delta^2$ defines an analytic semigroup in $L^q(\mathbb{R}^N)$.

Now, if $r \geq q$ we use i) again, now with $\beta = 0$, and choosing α such that $-\frac{N}{r} = 4\alpha - \frac{N}{q}$ and we get

$$\|S_{\Delta^2}(t)u_0\|_{L^r(\mathbb{R}^N)} \leq \|S_{\Delta^2}(t)u_0\|_{H^{4\alpha,q}(\mathbb{R}^N)} \leq \frac{M_\alpha e^{\mu_0 t}}{t^\alpha} \|u_0\|_{L^q(\mathbb{R}^N)},$$

which leads to

$$\|S_{\Delta^2}(t)u_0\|_{L^r(\mathbb{R}^N)} \leq \frac{M_{r,q} e^{\mu_0 t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{L^q(\mathbb{R}^N)}.$$

Again, because of part ii) of Lemma 5.2.1, type$(\Delta^2) = 0$ and then $\mu_0 > 0$ is arbitrary.

■

Remark 5.2.3 *Lemma 5.2.2, ii) above can be considered for $q = 1$, as long as $r > 1$. For $q = 1$, if we take any $r > 1$ and any $\beta > \frac{N}{4r'}$ then we have $H^{4\beta,r'}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ and therefore $L^1(\mathbb{R}^N) \hookrightarrow H^{-4\beta,r}(\mathbb{R}^N)$.*

Now using Lemma 5.2.2, i) with $\alpha = 0$ we get

$$\|S_{\Delta^2}(t)u_0\|_{L^r(\mathbb{R}^N)} \leq \frac{M_{r,1} e^{\mu_0 t}}{t^\beta} \|u_0\|_{H^{-4\beta,r}(\mathbb{R}^N)} \leq \frac{M_{r,1} e^{\mu_0 t}}{t^\beta} \|u_0\|_{L^1(\mathbb{R}^N)}$$

for any $\beta > \frac{N}{4}(1-\frac{1}{r})$. Hence we obtain an estimate similar to the one in Lemma 5.2.2, ii) for $q = 1$ and any $r > 1$, for an exponent as close as we want to $\frac{N}{4}(1-\frac{1}{r})$.

Remark 5.2.4 *Observe that the solution of problem (5.2.4) can be described as the convolution of the initial data with the self-similar kernel for the bi–Laplacian operator, which satisfies suitable Gaussian bounds; see e.g. [21, 22] and [20, 10].*

Remark 5.2.5
i) Observe that the Bessel spaces described above, naturally appear as a result of an abstract procedure, using complex interpolation, to construct spaces associated to sectorial operators; see e.g. [2, 1] and Chapter 2.
ii) Note that using [3, 9.7, pg. 648] we get that $-\Delta$ has bounded imaginary powers in $L^q(\mathbb{R}^N)$ for $1 < q < \infty$. Hence, because of [2, V.1.5.13, pg. 283], see also Remark 2.2.2, the Bessel spaces described above coincide with the usual fractional power spaces of this operator, see [31].
iii) Also, some results on sectorial operators that apply to other higher order differential operators instead of Δ^2 in (5.2.4) can be found in [19, Theorem 5.5]. These operators have always bounded imaginary powers ([19, (2.15), page 25]). Hence again their complex interpolation scale and their fractional power scale coincide, again by [2, V.1.5.13, pg. 283] (see Remark 2.2.2). Note however that [19, Theorem 5.5] does not give the description of these spaces.

Now we can use the results in [47], see Chapter 1, to perturb equation (5.2.4). For this, let D^r denote any partial derivative of order $r \in \mathbb{N}$ and fix $m \in \mathbb{N}$. Then if $m \geq r$, we have $D^r : H^{m,q}(\mathbb{R}^N) \to H^{m-r,q}(\mathbb{R}^N)$. On the other hand, $D^r : H^{-m,q}(\mathbb{R}^N) \to H^{-m-r,q}(\mathbb{R}^N)$, is defined as

$$< D^r u, \varphi >= (-1)^r \int_{\mathbb{R}^N} u D^r \varphi, \quad \text{for all } \varphi \in H^{m+r,q'}(\mathbb{R}^N).$$

Finally, if $m < r$, $D^r : H^{m,q}(\mathbb{R}^N) \to H^{m-r,q}(\mathbb{R}^N)$ is defined as

$$< D^r u, \varphi >= (-1)^{r-m} \int_{\mathbb{R}^N} D^m u D^{r-m} \varphi, \quad \text{for all } \varphi \in H^{r-m,q'}(\mathbb{R}^N)$$

which corresponds to the composition $D^r = D^{r-m} D^m$, where $D^m : H^{m,q}(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ and $D^{r-m} : L^q(\mathbb{R}^N) \to H^{m-r,q}(\mathbb{R}^N)$.

Thus for any $1 < q < \infty$, $r \in \mathbb{N}$ and $m \in \mathbb{Z}$, we have

$$D^r \in \mathcal{L}(H^{m,q}(\mathbb{R}^N), H^{m-r,q}(\mathbb{R}^N)), \quad \|D^r\|_{\mathcal{L}(H^{m,q}(\mathbb{R}^N), H^{m-r,q}(\mathbb{R}^N))} \leq C$$

for some C independent of r, m, q.

Now we extend this definition to non-integer m. For this take $m \in \mathbb{Z}$ and $s \in (m, m+1)$ and take $\theta \in (0,1)$ such that $s = \theta m + (1-\theta)(m+1)$. Then by interpolation

$$D^r : [H^{m+1,q}(\mathbb{R}^N), H^{m,q}(\mathbb{R}^N)]_\theta = H^{s,q}(\mathbb{R}^N) \to [H^{m+1-r,q}(\mathbb{R}^N), H^{m-r,q}(\mathbb{R}^N)]_\theta = H^{s-r,q}(\mathbb{R}^N),$$

and we get that for any $r \in \mathbb{N}$ and $s \in \mathbb{R}$

$$D^r \in \mathcal{L}(H^{s,q}(\mathbb{R}^N), H^{s-r,q}(\mathbb{R}^N)), \quad \|D^r\|_{\mathcal{L}(H^{s,q}(\mathbb{R}^N), H^{s-r,q}(\mathbb{R}^N))} \leq C \tag{5.2.5}$$

for some C independent of r, s, q. Note that above we denoted by $[\cdot,\cdot]_\theta$ the complex interpolation functor, see [2] and [50].

Using this and the results in [47] we get the following result in which we allow perturbations with derivatives of order $k \leq 3$.

Proposition 5.2.6 *Take $J \in \mathbb{N}$ and $a_j \in \mathbb{R}$, $k_j \in \mathbb{N}$ for $j = 1, ..., J$ with $\max_j |a_j| \leq R_0$ and $k = \max_j |k_j| \leq 3$. Then for each $1 < q < \infty$ the problem*

$$\begin{cases} u_t + \Delta^2 u + \sum_{j=1}^J a_j D^{k_j} u = 0, & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0 & in\ \mathbb{R}^N \end{cases} \tag{5.2.6}$$

defines an analytic semigroup, $S(t)$, on the scale $\{X^\alpha\}_{\alpha\in\mathbb{R}}$ with $X^\alpha = E^{2\alpha} = H^{4\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, such that

$$\|S(t)\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N),H^{4\gamma',q}(\mathbb{R}^N))} \leq \frac{C(\gamma'-\gamma)}{t^{\gamma'-\gamma}} e^{\mu t} \qquad t > 0,\ \gamma, \gamma' \in \mathbb{R},\ \gamma' \geq \gamma$$

and also

$$\|S(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N),L^r(\mathbb{R}^N))} \leq \frac{C(q,r)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} e^{\mu t} \qquad t > 0,$$

for $1 < q \leq r \leq \infty$, with μ, $C(\gamma'-\gamma)$, $C(q,r)$ depending on $\{a_j\}$ only through R_0. The constant $C(\gamma'-\gamma)$ is bounded for γ, γ' in bounded sets of $\mathbb{R}$.

Furthermore, if for all $j = 1, ..., J$, we have $a_j^\varepsilon \to a_j$ as $\varepsilon \to 0$ then for any $T > 0$, $\gamma' \geq \gamma$ or $r \geq q$, there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that the corresponding semigroups satisfy

$$\|S_\varepsilon(t) - S(t)\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N),H^{4\gamma',q}(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \leq T$$

and

$$\|S_\varepsilon(t) - S(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N),L^r(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \leq T$$

for $1 < q \leq r \leq \infty$.

Proof. Since $X^\alpha = E^{2\alpha} = H^{4\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, we get from Lemma 5.2.2 i) that

$$\|S_{\Delta^2}(t)\|_{\mathcal{L}(X^\beta,X^\alpha)} \leq \frac{C}{t^{\alpha-\beta}}, \qquad 0 < t \leq 1,\ \alpha, \beta \in \mathbb{R},\ \alpha \geq \beta.$$

From (5.2.5) each of the perturbations $P_j = a_j D^{k_j}$ satisfies $\|P_j\|_{\mathcal{L}(X^\alpha,X^{\alpha-k_j/4})} \leq C$ for all $\alpha \in \mathbb{R}$ with $C = C(R_0)$ independent of j, and we have that

$$P = \sum_{j=1}^J P_j \in \mathcal{L}(X^\alpha, X^{\alpha-k/4}), \qquad \|P\|_{\mathcal{L}(X^\alpha,X^{\alpha-k/4})} \leq C(J, R_0),\ \alpha \in \mathbb{R}.$$

Hence, we can apply [47, Proposition 10] (see also Theorem 1.0.1) with $\alpha \in \mathbb{R}$, $\beta = \alpha - \frac{k}{4}$ and since the scale is nested, we get a semigroup $S(t) = S_P(t)$ in X^γ for any $\gamma \in E(\alpha) := (\alpha - 1, \alpha]$ that satisfies the smoothing estimates

$$\|S(t)\|_{\mathcal{L}(X^\gamma, X^{\gamma'})} \leq \frac{M_{\gamma,\gamma'} e^{\mu t}}{t^{\gamma' - \gamma}} \tag{5.2.7}$$

with μ depending on R_0 and for every γ, γ' such that

$$\gamma \in E(\alpha) := (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) := [\beta, \beta + 1) = [\alpha - k/4, \alpha - k/4 + 1), \qquad \gamma' \geq \gamma.$$

We now want to see the largest range for γ and γ', in which (5.2.7) holds, that can be achieved in 2 "jumps" in the scale. For this we perform a bootstrap argument as follows. Given $\alpha \in \mathbb{R}$, take $\beta = \alpha - \frac{k}{4}$ as above. In this situation the semigroup transforms X^γ into $X^{\gamma'}$ for any $\gamma \in E(\alpha)$ to any $\gamma' \in R(\beta)$. We now choose an $\alpha' > \alpha$ such that $\beta < \alpha' - 1 < \beta + 1 = \alpha - \frac{k}{4} + 1$, so $R(\beta) \cap E(\alpha') \neq \emptyset$. Then we can "jump" again, starting from any space $X^{\gamma'}$, $\gamma' \in R(\beta) \cap E(\alpha')$, into $X^{\gamma''}$ with $\gamma'' \in R(\beta')$, $\beta' = \alpha' - \frac{k}{4}$. Schematically, we write

$$\gamma \in E(\alpha) \to \gamma' \in R(\beta) \cap E(\alpha') \to \gamma'' \in R(\beta').$$

Then, using $S(t) = S(t/2) \circ S(t/2)$ we get

$$\|S(t)u_0\|_{\gamma''} \leq \frac{\tilde{M} e^{\mu(t/2)}}{(t/2)^{\gamma'' - \gamma'}} \|S(t/2)u_0\|_{\gamma'} \leq \frac{\tilde{M} e^{\mu(t/2)}}{(t/2)^{\gamma'' - \gamma'}} \frac{\tilde{M} e^{\mu(t/2)}}{(t/2)^{\gamma' - \gamma}} \|u_0\|_\gamma = \frac{M e^{\mu t}}{t^{\gamma'' - \gamma}} \|u_0\|_\gamma \tag{5.2.8}$$

for $\gamma \in E(\alpha) = (\alpha - 1, \alpha]$ and $\gamma'' \in R(\beta') = [\beta', \beta' + 1)$ and M depending on γ and γ''. Note that the range for $R(\beta')$ moves continuously as we move α', thus

$$\gamma'' \in \bigcup_{\beta < \alpha' - 1 < \beta + 1} R(\beta') = \bigcup_{\beta < \alpha' - 1 < \beta + 1} [\beta', \beta' + 1) = \bigcup_{\beta < \alpha' - 1 < \beta + 1} [\alpha' - \frac{k}{4}, \alpha' - \frac{k}{4} + 1)$$

$$= (\beta + 1 - \frac{k}{4}, \beta + 3 - \frac{k}{4}) = (\alpha - \frac{2k}{4} + 1, \alpha - \frac{2k}{4} + 3).$$

Hence, after one or two "jumps" we get the estimate (5.2.7) for any $\gamma \in E(\alpha) := (\alpha - 1, \alpha]$ and $\gamma' \in [\alpha - \frac{k}{4}, \alpha - \frac{2k}{4} + 3)$ with $\gamma' \geq \gamma$.

Note that this argument can be repeated to obtain that we can have in (5.2.7) $\gamma' \in [\alpha - \frac{k}{4}, \alpha - \frac{nk}{4} + (2n - 1))$ for any $n \in \mathbb{N}$. So, since $\alpha \in \mathbb{R}$ is arbitrary, after a finite number of iterations we get (5.2.7) for any $\gamma, \gamma' \in \mathbb{R}$, $\gamma' > \gamma$.

Now, if $1 < q < \infty$ and $r \geq q$ we take $\gamma = 0$ and γ' such that $H^{4\gamma', q}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$, that is $-\frac{N}{r} = 4\gamma' - \frac{N}{q}$. Then we get

$$\|S(t)u_0\|_{L^r(\mathbb{R}^N)} \leq C \|S(t)u_0\|_{H^{4\gamma', q}(\mathbb{R}^N)} \leq \frac{C(\gamma') e^{\mu t}}{t^{\gamma'}} \|u_0\|_{L^q(\mathbb{R}^N)} = \frac{C_{q,r} e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q} - \frac{1}{r})}} \|u_0\|_{L^q(\mathbb{R}^N)}.$$

The analyticity comes again from Lemma 5.2.2 and [47, Theorem 12] (see Theorem 1.0.3).

The convergence of the semigroups is consequence of [47, Theorem 14] (see Theorem 1.0.2) since if $a_j^\varepsilon \to a_j$ we would have $P_\varepsilon \to P$ in $\mathcal{L}(X^\alpha, X^{\alpha-k/4})$ as $\varepsilon \to 0$ for any $\alpha \in \mathbb{R}$. ■

Remark 5.2.7 *For a similar result with $q = 1$, we can proceed as in Remark 5.2.3.*

Remark 5.2.8 *Note that the estimates in Lemma 5.2.2 and Proposition 5.2.6 give that the solutions of problems (5.2.4) and (5.2.6) satisfy that $u(t) \in H^{4\gamma',r}(\mathbb{R}^N)$, for all $t > 0$, $\gamma' \in \mathbb{R}$ and $q \le r < \infty$. Since the semigroups are analytic have $u_t(t) \in H^{4\gamma',r}(\mathbb{R}^N)$ as well. Therefore, (5.2.4) and (5.2.6) are satisfied in a classical sense.*

Finally, we study more general perturbations in which we allow a space dependence. For this, take $k \in \mathbb{N}$, which is the order of the perturbation, and take $a, b \in \mathbb{N}$ such that $a + b = k$. We define $P_{a,b}$ to be a perturbation of the form

$$P_{a,b}u = D^b(d(x)D^a u) \qquad x \in \mathbb{R}^N$$

for a given function $d(x)$ with $x \in \mathbb{R}^N$, in the sense that for any smooth enough φ

$$< P_{a,b}u, \varphi >= (-1)^b \int_{\mathbb{R}^N} d(x)D^a u D^b \varphi. \tag{5.2.9}$$

We will assume below that the coefficient $d(x)$ belongs to the locally uniform space $L^p_U(\mathbb{R}^N)$ described in Chapter 3.

The following result states for which spaces of the Bessel scale a perturbation $P_{a,b}$ is a "well behaved" linear operator.

Proposition 5.2.9 *Let $P_{a,b}$ be as above, $d \in L^p_U(\mathbb{R}^N)$ and let $s \ge a$, $\sigma \ge b$. Assume also that $1 < q < \infty$ and*

$$(s - a - \frac{N}{q})_- + (\sigma - b - \frac{N}{q'})_- \ge -\frac{N}{p'} \tag{5.2.10}$$

where the inequality is strict when $s - a - \frac{N}{q} = 0$ and $\sigma - b - \frac{N}{q'} = -\frac{N}{p'}$ or $s - a - \frac{N}{q} = -\frac{N}{p'}$ and $\sigma - b - \frac{N}{q'} = 0$ (or both).

Then, we have

$$P_{a,b} \in \mathcal{L}(H^{s,q}(\mathbb{R}^N), H^{-\sigma,q}(\mathbb{R}^N)), \qquad \|P_{a,b}\|_{\mathcal{L}(H^{s,q}(\mathbb{R}^N), H^{-\sigma,q}(\mathbb{R}^N))} \le C\|d\|_{L^p_U(\mathbb{R}^N)}.$$

Proof. Let $\{Q_i\}$, $i \in \mathbb{Z}^N$ be a partition of $\mathbb{R}^N$ in open disjoint cubes centered in $i \in \mathbb{Z}^N$ with sides of length 1, parallel to the axes. Note that $\mathbb{R}^N = \cup_{i \in \mathbb{Z}^N} \overline{Q_i}$ and $Q_i \cap Q_j = \emptyset$ for $i \ne j$. Then

$$|\int_{\mathbb{R}^N} dD^a u D^b \varphi| \le \sum_i |\int_{Q_i} dD^a u D^b \varphi| \le \sum_i (\int_{Q_i} |d|^p)^{\frac{1}{p}} (\int_{Q_i} |D^a u|^n)^{\frac{1}{n}} (\int_{Q_i} |D^b \varphi|^\tau)^{\frac{1}{\tau}}$$

where we have applied Hölder's inequality with $\frac{1}{p}+\frac{1}{n}+\frac{1}{\tau}=1$. If (5.2.10) holds, we can choose n, τ as before such that $s-\frac{N}{q}\geq a-\frac{N}{n}$ and $\sigma-\frac{N}{q'}\geq b-\frac{N}{\tau}$. Now, we can use the embeddings of Bessel spaces and, for some C is independent of the cube Q_i, obtain

$$\begin{aligned}|\int_{\mathbb{R}^N} dD^a u D^b\varphi| &\leq C\|d\|_{L^p_U(\mathbb{R}^N)}\sum_i \|u\|_{H^{s,q}(Q_i)}\|\varphi\|_{H^{\sigma,q'}(Q_i)}\\ &\leq C\|d\|_{L^p_U(\mathbb{R}^N)}\left(\sum_i \|u\|^q_{H^{s,q}(Q_i)}\right)^{1/q}\left(\sum_i \|\varphi\|^{q'}_{H^{\sigma,q'}(Q_i)}\right)^{1/q'}. \end{aligned}\qquad (5.2.11)$$

Then, as in [6, Lemma 2.4], we get for any $0\leq\alpha\leq 2$ and any $1<q<\infty$

$$\sum_i \|\phi\|^q_{H^{2\alpha,q}(Q_i)}\leq C\|\phi\|^q_{H^{2\alpha,q}(\mathbb{R}^N)}\qquad \text{for all}\quad \phi\in H^{2\alpha,q}(\mathbb{R}^N),$$

and we obtain from (5.2.11)

$$|\int_{\mathbb{R}^N} dD^a u^b\varphi|\leq C\|d\|_{L^p_U(\mathbb{R}^N)}\|u\|_{H^{s,q}(\mathbb{R}^N)}\|\varphi\|_{H^{\sigma,q'}(\mathbb{R}^N)}$$

which gives the result. ■

Now we can use again the results in [47] (see Chapter 1) to obtain the following.

Theorem 5.2.10 *Let $P_{a,b}$ be as in (5.2.9) with $k,a,b\in\{0,1,2,3\}$, $k=a+b$. Assume that $\|d\|_{L^p_U(\mathbb{R}^N)}\leq R_0$ with $p>\frac{N}{4-k}$, then for any $1<q<\infty$ and such $P_{a,b}$, there exists an interval $I(q,a,b)\subset(-1+\frac{a}{4},1-\frac{b}{4})$ containing $(-1+\frac{a}{4}+\frac{N}{4p},1-\frac{b}{4}-\frac{N}{4p})$, such that for any $\gamma\in I(q,a,b)$, we have a strongly continuous analytic semigroup, $S_{P_{a,b}}(t)$, in the space $H^{4\gamma,q}(\mathbb{R}^N)$, for the problem*

$$\left\{\begin{array}{ll} u_t+\Delta^2 u+D^b(d(x)D^a u)=0, & x\in\mathbb{R}^N,\ t>0\\ u(0)=u_0 & \mathit{in}\ \mathbb{R}^N.\end{array}\right.\qquad (5.2.12)$$

Moreover the semigroup has the smoothing estimates

$$\|S_{P_{a,b}}(t)u_0\|_{H^{4\gamma',q}(\mathbb{R}^N)}\leq\frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{4\gamma,q}(\mathbb{R}^N)},\quad t>0,\ u_0\in H^{4\gamma,q}(\mathbb{R}^N)$$

for every $\gamma,\gamma'\in I(q,a,b)$ with $\gamma'\geq\gamma$, and

$$\|S_{P_{a,b}}(t)u_0\|_{L^r(\mathbb{R}^N)}\leq\frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\mathbb{R}^N)},\quad t>0,\ u_0\in L^q(\mathbb{R}^N)$$

with $1<q\leq r\leq\infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu\in\mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q,a,b)$ is given by

$$I(q,a,b)=(-1+\frac{a}{4}+\frac{N}{4}(\frac{1}{p}-\frac{1}{q'})_+,1-\frac{b}{4}-\frac{N}{4}(\frac{1}{p}-\frac{1}{q})_+).$$

Finally, if

$$d_\varepsilon \to d \quad in\ L^p_U(\mathbb{R}^N), \quad p > \frac{N}{4-k}$$

then for every $1 < q < \infty$ *and* $T > 0$ *there exists* $C(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*, such that*

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N),H^{4\gamma',q}(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \leq T$$

for all $\gamma, \gamma' \in I(q,a,b)$*,* $\gamma' \geq \gamma$ *and for any* $1 < q \leq r \leq \infty$

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N),L^r(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \leq T.$$

Proof. By Proposition 5.2.9 and using $X^\alpha = E^{2\alpha} = H^{4\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, if we assume for a moment that (5.2.10) is satisfied for some s and σ, then we would have

$$P \in \mathcal{L}(X^{s/4}, X^{-\sigma/4}), \qquad \|P\|_{\mathcal{L}(X_{s/4},X_{-\sigma/4})} \leq C\|d\|_{L^p_U(\mathbb{R}^N)}.$$

Hence we can apply Theorem 1.0.1 with $\alpha = s/4$ and $\beta = -\sigma/4$ provided $0 \leq \alpha - \beta < 1$, that is, $s + \sigma < 4$.

Thus, we check now that (5.2.10) and $s + \sigma < 4$ hold for suitable pairs (s,σ). For this we rewrite the ranges for s, σ in Proposition 5.2.9 in terms of $\tilde{s} = s - a - \frac{N}{q}$ and $\tilde{\sigma} = \sigma - b - \frac{N}{q'}$, so $\tilde{s} \geq -\frac{N}{q}, \tilde{\sigma} \geq -\frac{N}{q'}$ since $s \geq a, \sigma \geq b$. Then (5.2.10) and $s + \sigma < 4$ read

$$\tilde{s} \geq -\frac{N}{q},\ \tilde{\sigma} \geq -\frac{N}{q'}, \qquad -\frac{N}{p'} \leq \tilde{s}_- + \tilde{\sigma}_-, \qquad \tilde{s} + \tilde{\sigma} < 4 - k - N. \tag{5.2.13}$$

Note that since necessarily $-\frac{N}{p'} < 4 - k - N$, we get that $p > \frac{N}{4-k}$.

The set of admissible parameters $(\tilde{s}, \tilde{\sigma})$ given by (5.2.13) depends on the relationship between q, q' and p. Note that (5.2.13) defines a planar trapezium–shaped polygon, $\tilde{\mathcal{P}}$, whose long base is on the line $\tilde{s}+\tilde{\sigma} = 4-k-N$ and the short base is on the line $\tilde{s}+\tilde{\sigma} = -\frac{N}{p'}$ in the third quadrant. As for the lateral sides note that the restriction $-\frac{N}{p'} \leq \tilde{s}_- + \tilde{\sigma}_-$ adds the condition that $\tilde{s} \geq -\frac{N}{p'}$ in the second quadrant and $\tilde{\sigma} \geq -\frac{N}{p'}$ in the fourth. These have to be combined with $\tilde{s} \geq -\frac{N}{q}$ and $\tilde{\sigma} \geq -\frac{N}{q'}$. Therefore the lateral sides are given by the lines $\tilde{s} = \max\{-\frac{N}{p'}, -\frac{N}{q}\}$ and $\tilde{\sigma} = \max\{-\frac{N}{p'}, -\frac{N}{q'}\}$. One of the possible cases is depicted in Figure 5.1.

Note that the polygon $\tilde{\mathcal{P}}$ transforms into a similar shaped polygon $\mathcal{P}$ which determines the region of admissible pairs (s,σ).

In any case, projecting $\tilde{\mathcal{P}}$ onto the axes gives the following ranges for $\tilde{s}$ and $\tilde{\sigma}$

$$\tilde{s} \in [\max\{-\frac{N}{p'}, -\frac{N}{q}\}, 4 - k - N - \max\{-\frac{N}{p'}, -\frac{N}{q'}\})$$

$$\tilde{\sigma} \in [\max\{-\frac{N}{p'}, -\frac{N}{q'}\}, 4 - k - N - \max\{-\frac{N}{p'}, -\frac{N}{q}\}).$$

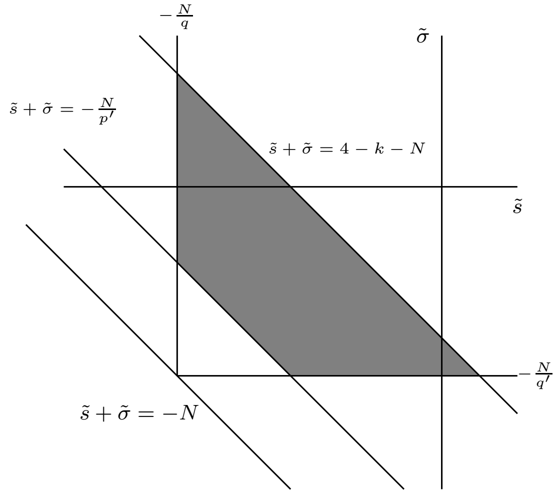

Figure 5.1: Admissible $\tilde{s}$ and $\tilde{\sigma}$ with $p > q, q'$

Thus the projection ranges for s and σ are given by

$$s \in J_1 = [a + (\frac{N}{q} - \frac{N}{p'})_+, 4 - b - (\frac{N}{q'} - \frac{N}{p'})_+) \tag{5.2.14}$$

$$\sigma \in J_2 = [b + (\frac{N}{q'} - \frac{N}{p'})_+, 4 - a - (\frac{N}{q} - \frac{N}{p'})_+). \tag{5.2.15}$$

For each pair of admissible pairs $(s, \sigma) \in \mathcal{P}$, by [47, Proposition 10] (see Theorem 1.0.1) with $\alpha = \frac{s}{4}$ and $\beta = -\frac{\sigma}{4}$, we get a perturbed semigroup and smoothing estimates in the spaces corresponding to

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta + 1), \qquad \gamma' \geq \gamma.$$

Hence, as (s, σ) range in the region $\mathcal{P}$, a repeated bootstrap argument as in (5.2.8) gives that the smoothing estimates hold for $\gamma \in \bigcup_{(s,\sigma)\in\mathcal{P}} E(s/4)$ and $\gamma' \in \bigcup_{(s,\sigma)\in\mathcal{P}} R(\sigma/4)$, $\gamma' \geq \gamma$. This leads to

$$\gamma \in (\frac{\inf J_1}{4} - 1, \frac{\sup J_1}{4}], \qquad \gamma' \in [-\frac{\sup J_2}{4}, 1 - \frac{\inf J_2}{4}), \qquad \gamma' \geq \gamma$$

which, after a simple calculation, reads

$$\gamma, \gamma' \in I(q, a, b) = (-1 + \frac{a}{4} + \frac{N}{4}(\frac{1}{q} - \frac{1}{p'})_+, 1 - \frac{b}{4} - \frac{N}{4}(\frac{1}{q'} - \frac{1}{p'})_+) = (\gamma_{min}, \gamma_{max}).$$

For the estimates in Lebesgue spaces we use the Sobolev inclusions. First note that for any $1 < q < \infty$, $I(q, a, b) \supset (-1 + \frac{a}{4} + \frac{N}{4p}, 1 - \frac{b}{4} - \frac{N}{4p})$ which does not depend on q and is not empty because $p > \frac{N}{4-k}$. Let $\tilde{\gamma} := 1 - \frac{b}{4} - \frac{N}{4p} > 0$ and take $0 \leq \gamma < \tilde{\gamma}$, then $H^{4\gamma,q}(\mathbb{R}^N) \hookrightarrow L^{\tilde{q}}(\mathbb{R}^N)$, for $\tilde{q} \geq q$ such that $-\frac{N}{\tilde{q}} = 4\gamma - \frac{N}{q}$, i.e. $\frac{1}{q} - \frac{1}{\tilde{q}} = \frac{4\gamma}{N}$ and we get

$$\|S_{P_{a,b}}(t)u_0\|_{L^{\tilde{q}}(\mathbb{R}^N)} \leq \|S_{P_{a,b}}(t)u_0\|_{H^{4\gamma,q}(\mathbb{R}^N)} \leq \frac{M_\gamma e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{\tilde{q}})}} \|u_0\|_{L^q(\mathbb{R}^N)}$$

In particular we can take $0 \le \gamma \le \frac{\tilde{\gamma}}{2}$ and we get the estimate above for all $\tilde{q} \ge q$ such that $\frac{1}{q} - \frac{1}{\tilde{q}} \in [0, \frac{2\tilde{\gamma}}{N}]$ and this interval does not depend on q.

We now use a bootstrap argument as in (5.2.8), jumping between different Lebesgue spaces at intermediate times. Starting with $r_0 := q$ and defining the numbers r_i, $i = 1, 2, 3, \ldots$ such that $\frac{1}{r_i} - \frac{1}{r_{i+1}} = \frac{2\gamma_{max}}{N}$, we obtain the estimate above for any $\tilde{q} \ge q$ such that $\tilde{q} \in [q, r_{i+1}]$. Hence in a finite number os steps we can reach any $\tilde{q}$ with $q < \tilde{q} \le \infty$.

The convergence of the semigroups is a direct consequence of [47, Theorem 14] (see Theorem 1.0.2), since Proposition 5.2.9 gives that if $d_\varepsilon \to d$ in $L^p_U(\mathbb{R}^N)$, then $P_\varepsilon \to P$ in $\mathcal{L}(X^{s/4}, X^{-\sigma/4})$ for any pair of admissible $(s, \sigma) \in \mathcal{P}$. The case of Lebesgue spaces follows from this as well.

Finally, the analyticity comes from Lemma 5.2.2 and [47, Theorem 12](see Theorem 1.0.3). ■

Remark 5.2.11 *Now we make precise in what sense equation (5.2.12) is satisfied.*
i) First note that since $p > \frac{N}{4-k}$ we have $4\gamma_{max} > 4-b-\frac{N}{p} > a$, and $4\gamma_{min} < -4+a+\frac{N}{p} < -b$. Hence $[-\frac{b}{4}, \frac{a}{4}] \subset I(q,a,b)$.
ii) Because of the analyticity of the semigroup, and as in [47, Remark 6], the equation $u_t + \Delta^2 u = Pu$ is satisfied in $H^{-b,q}(\mathbb{R}^N)$.

Therefore, we have that $u(t) \in H^{4-b,q}(\mathbb{R}^N)$, for all $t > 0$. In terms of the scale, $u(t) \in X^{\gamma^*}$, $\gamma^* = 1 - \frac{b}{4} \ge \gamma_{max}$. Note that in Theorem 5.2.10 we did not get an estimate of $u(t)$ in the space $H^{4-b,q}(\mathbb{R}^N)$ though.

Also, since the semigroup is analytic in X^γ, $u_t(t) \in X^\gamma$ for all $\gamma \in I(q,a,b)$ and $t > 0$.
iii) In particular, the equation (5.2.12) is always satisfied as

$$\int_{\mathbb{R}^N} u_t\varphi + \int_{\mathbb{R}^N} u\Delta^2\varphi + \int_{\mathbb{R}^N} d(x)D^a u D^b\varphi = 0, \qquad t > 0$$

for any $\varphi \in H^{b,q'}(\mathbb{R}^N)$.

However, for $b = 3$, $a = 0$ we have $\gamma^* \ge \frac{1}{4}$, that is $u(t) \in H^{1,q}(\mathbb{R}^N)$, $t > 0$, and therefore

$$\int_{\mathbb{R}^N} u_t\varphi - \int_{\mathbb{R}^N} \nabla u \nabla(\Delta\varphi) - \int_{\mathbb{R}^N} d(x) u D^3\varphi = 0.$$

For $b = 2$, $a \le 1$ we have $\gamma^* \ge \frac{1}{2}$, that is $u(t) \in H^{2,q}(\mathbb{R}^N)$, $t > 0$, and therefore

$$\int_{\mathbb{R}^N} u_t\varphi + \int_{\mathbb{R}^N} \Delta u \Delta\varphi + \int_{\mathbb{R}^N} d(x)D^a u D^2\varphi = 0.$$

For $b = 1$, $a \le 2$ we have $\gamma^* \ge \frac{3}{4}$, that is $u(t) \in H^{3,q}(\mathbb{R}^N)$, $t > 0$, and therefore

$$\int_{\mathbb{R}^N} u_t\varphi - \int_{\mathbb{R}^N} \nabla(\Delta u)\nabla\varphi - \int_{\mathbb{R}^N} d(x)D^a u D\varphi = 0.$$

Finally, $b = 0$, $a \le 3$ we have $\gamma^* = 1$, that is $u(t) \in H^{4,q}(\mathbb{R}^N)$, $t > 0$, and therefore,

$$\int_{\mathbb{R}^N} u_t\varphi + \int_{\mathbb{R}^N} \Delta^2 u\varphi + \int_{\mathbb{R}^N} d(x)D^a u\varphi = 0.$$

Note the ranges of spaces for which we can solve the equation are determined by the base space in terms of $1 < q < \infty$, the integrability p of the coefficient $d(x)$ and the order of derivatives a, b. Observe that in Theorem 5.2.10 just one perturbation $P_{a,b}$ is considered. Several perturbations can be thus combined together, although not all combinations are allowed. We now discuss a general procedure to determine whether or not some given perturbations can be combined together.

Proposition 5.2.12 *Consider a finite family of perturbations $P_i := P_{a_i,b_i}$ as in (5.2.9) with $\|d_i\|_{L^{p_i}_U(\mathbb{R}^N)} \leq R_0$, with $k_i, a_i, b_i \in \{0,1,2,3\}$, $k_i = a_i + b_i$, $p_i > \frac{N}{4-k_i}$, $i = 1, ..., J$. Denote $P := \sum_i P_i$, then for any $1 < q < \infty$, if*

$$\max_i \{a_i + (\frac{N}{p_i} - \frac{N}{q'})_+\} + \max_i \{b_i + (\frac{N}{p_i} - \frac{N}{q})_+\} < 4 \tag{5.2.16}$$

then there exists an interval $I(q,P) \subset (-1 + \frac{\max_i\{a_i\}}{4}, 1 - \frac{\max_i\{b_i\}}{4})$ containing $(-1 + \max_i\{\frac{a_i}{4} + \frac{N}{4p_i}\}, 1 - \max_i\{\frac{b_i}{4} + \frac{N}{4p_i}\})$, such that for any $\gamma \in I(q,P)$, we have a strongly continuous, analytic semigroup, $S_P(t)$ in the space $H^{4\gamma,q}(\mathbb{R}^N)$, for the problem

$$\begin{cases} u_t + \Delta^2 u + Pu = 0, & x \in \mathbb{R}^N, \ t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimates

$$\|S_P(t)u_0\|_{H^{4\gamma',q}(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{H^{4\gamma,q}(\mathbb{R}^N)}, \quad t > 0, \ u_0 \in H^{4\gamma,q}(\mathbb{R}^N)$$

for every $\gamma, \gamma' \in I(q,P)$ with $\gamma' \geq \gamma$, and

$$\|S_P(t)u_0\|_{L^r(\mathbb{R}^N)} \leq \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{L^q(\mathbb{R}^N)}, \quad t > 0, \ u_0 \in L^q(\mathbb{R}^N)$$

with $1 < q \leq r \leq \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q,P)$ is given by

$$I(q,P) = (-1 + \max_i\{\frac{a_i}{4} + \frac{N}{4}(\frac{1}{p_i} - \frac{1}{q'})_+\}, 1 - \max_i\{\frac{b_i}{4} + \frac{N}{4}(\frac{1}{p_i} - \frac{1}{q})_+\}).$$

Finally, if as $\varepsilon \to 0$

$$d_i^\varepsilon \to d_i \quad \text{in } L^{p_i}_U(\mathbb{R}^N), \quad p_i > \frac{N}{4-k_i}$$

then for every $1 < q < \infty$ and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N), H^{4\gamma',q}(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \leq T$$

for all $\gamma, \gamma' \in I(q,P)$, $\gamma' \geq \gamma$ and for any $1 < q \leq r \leq \infty$

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \leq T.$$

Proof. From Theorem 5.2.10 we know that for each perturbation P_i there exists a non empty trapezoidal polygon $\mathcal{P}_i$ of admissible pairs of spaces (s,σ) described in terms of $\tilde{s} = s - a_i - \frac{N}{q}$ and $\tilde{\sigma} = \sigma - b_i - \frac{N}{q'}$, see (5.2.13).

Therefore the polygon $\mathcal{P}_i$ of the perturbation P_i is given by a planar trapezium whose long base is on the line $s+\sigma = 4$ and the short base is on the line $s+\sigma = k_i + \frac{N}{p_i}$ in the third quadrant, with $k_i = a_i + b_i$. As for the lateral sides they are given by the lines $s = a_i + (\frac{N}{q} - \frac{N}{p_i'})_+$ and $\sigma = b_i + (\frac{N}{q'} - \frac{N}{p_i'})_+$. Thus the projection of $\mathcal{P}_i$ on the axes give the intervals

$$s \in J_1^i = [s^i_{min}, 4 - \sigma^i_{min}) \quad and \quad \sigma \in J_2^i = [\sigma^i_{min}, 4 - s^i_{min})$$

see (5.2.14) and (5.2.15).

According to [47, Lemma 13, iii)], we can consider $P := \sum_i P_i$, that is, all perturbations acting at the same time, if there exists a common region $\mathcal{P}$ of admissible pairs (s,σ), that is if $\mathcal{P} := \cap_i \mathcal{P}_i \neq \emptyset$.

Since the admissible sets $\mathcal{P}_i$ always have the long base on the line $s+\sigma = 4$ and the lateral sides are parallel to the axes, the set $\mathcal{P}$ is non empty if and only if

$$\max_i\{\inf J_1^i\} < \min_i\{\sup J_1^i\} \quad i.e. \quad \max_i\{s^i_{min}\} < \min_i\{4 - \sigma^i_{min}\}$$

and

$$\max_i\{\inf J_2^i\} < \min_i\{\sup J_2^i\} \quad i.e. \quad \max_i\{\sigma^i_{min}\} < \min_i\{4 - s^i_{min}\}$$

which are equivalent to (5.2.16), that is

$$\max_i\{a_i + (\frac{N}{p_i} - \frac{N}{q'})_+\} + \max_i\{b_i + (\frac{N}{p_i} - \frac{N}{q})_+\} < 4.$$

In such a case the projection of $\mathcal{P} = \bigcap_i \mathcal{P}_i$ on the axes gives the intervals

$$s \in J_1 = [\max_i(\inf J_1^i), \min_i(\sup J_1^i)) = [\max_i\{a_i + (\frac{N}{p_i} - \frac{N}{q'})_+\}, 4 - \max_i\{b_i + (\frac{N}{p_i} - \frac{N}{q})_+\})$$

$$\sigma \in J_2 = [\max_i(\inf J_2^i), \min_i(\sup J_2^i)) = [\max_i\{b_i + (\frac{N}{p_i} - \frac{N}{q})_+\}, 4 - \max_i\{a_i + (\frac{N}{p_i} - \frac{N}{q})_+\}).$$

For each pair of admissible pairs $(s,\sigma) \in \mathcal{P}$, by [47, Proposition 10] (see Theorem 1.0.1) with $\alpha = \frac{s}{4}$ and $\beta = -\frac{\sigma}{4}$, we get a perturbed semigroup and smoothing estimates in the spaces corresponding to γ and γ' as in [47], i.e.

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta+1), \qquad \gamma' \geq \gamma.$$

Hence as (s,σ) range in the region $\mathcal{P}$ a repeated bootstrap argument as in (5.2.8) gives that the smoothing estimates hold for $\gamma \in \bigcup_{(s,\sigma)\in\mathcal{P}} E(s/4)$ and $\gamma' \in \bigcup_{(s,\sigma)\in\mathcal{P}} R(-\sigma/4)$, $\gamma' \geq \gamma$, see also the proof of Theorem 5.2.10. This leads to

$$\gamma \in (\frac{\inf J_1}{4} - 1, \frac{\sup J_1}{4}], \qquad \gamma' \in [-\frac{\sup J_2}{4}, 1 - \frac{\inf J_2}{4}), \quad \gamma' \geq \gamma$$

which, after a simple calculation, reads

$$\gamma, \gamma' \in I(q,P) = (-1 + \max_i\{\frac{a_i}{4} + \frac{N}{4}(\frac{1}{p_i} - \frac{1}{q'})_+\}, 1 - \max_i\{\frac{b_i}{4} + \frac{N}{4}(\frac{1}{p_i} - \frac{1}{q})_+\}).$$

Note that this interval is contained in an interval $(-1 + \frac{\max_i\{a_i\}}{4}, 1 - \frac{\max_i\{b_i\}}{4})$ and contains $(-1 + \max_i\{\frac{a_i}{4} + \frac{N}{4p_i}\}, 1 - \max_i\{\frac{b_i}{4} + \frac{N}{4p_i}\})$, which is non empty because $p_i > \frac{N}{4-k_i}$. To see this note that the latter condition gives $\frac{a_i}{4} + \frac{N}{4p_i} < 1 - \frac{b_i}{4} < 1$ and $\frac{b_i}{4} + \frac{N}{4p_i} < 1 - \frac{a_i}{4} < 1$. ■

Remark 5.2.13 *Note that now, since* $p_i > \frac{N}{4-k_i}$, $I(q,a,b) \supset [-\min\{\frac{b_i}{4}\}, \min\{\frac{a_i}{4}\}]$, *thus all the comments on Remark 5.2.11 hold for* $\min\{b_i\}, \min\{a_i\}$ *instead of* b, a.

Remark 5.2.14 *In some cases the condition (5.2.16) can be simplified and simpler description can be given.*
i) If there is only one perturbation, then (5.2.16) is equivalent to $p > \frac{N}{4-k}$ *as in Theorem 5.2.10.*
ii) If $a_i = a$ *and* $b_i = b$ *(thus* $k_i = k$*) for all* i*, then*

$$P = \sum_i D^b(d_i(x)D^a) = D^b(d(x)D^a) \quad \textit{where} \quad d := \sum_i d_i$$

can be considered as a perturbation with $d \in L^p_U(\mathbb{R}^N)$ *for* $p = \min_i\{p_i\}$*. Then (5.2.16) holds if and only if* $p > \frac{N}{4-k}$ *as in Theorem 5.2.10.*
iii) Assume now $p_i = p$ *for all* i*. Then (5.2.16) is equivalent to*

$$\max_i\{a_i\} + \max_i\{b_i\} < 4 - (\frac{N}{p} - \frac{N}{q})_+ - (\frac{N}{p} - \frac{N}{q'})_+. \tag{5.2.17}$$

Hence, if we denote $k := \max\{a_i\} + \max\{b_i\}$*, then (5.2.17) is satisfied provided* $p > \frac{N}{4-k}$*, which resembles the condition in Theorem 5.2.10. Note that* k *can be regarded as the order of the perturbation* $P = \sum_i P_i$.

In particular, if

$$k := \max_i\{a_i\} + \max_i\{b_i\} < 4 \qquad \textit{and} \qquad p > \frac{N}{4-k}$$

are satisfied, then Proposition 5.2.12 applies with an interval for P *given by*

$$I(q,P) = (-1 + \frac{\max_i\{a_i\}}{4} + \frac{N}{4}(\frac{1}{p} - \frac{1}{q'})_+, 1 - \frac{\max_i\{b_i\}}{4} - \frac{N}{4}(\frac{1}{p} - \frac{1}{q})_+).$$

Compare it with $I(q,a,b)$ *in Theorem 5.2.10 to see the resemblance.*
iv) We now describe how to determine if two perturbations as in iii) can be combined.

For example, if we fix a perturbation $P_{a,b}$ *with* $k = 3$*, then any perturbation* $P_{c,d}$ *with* $c \le a$ *and* $d \le b$ *can be combined with it, and the interval is* $I(q,P) = I(q,a,b)$, $P = P_{a,b} + P_{c,d}$.

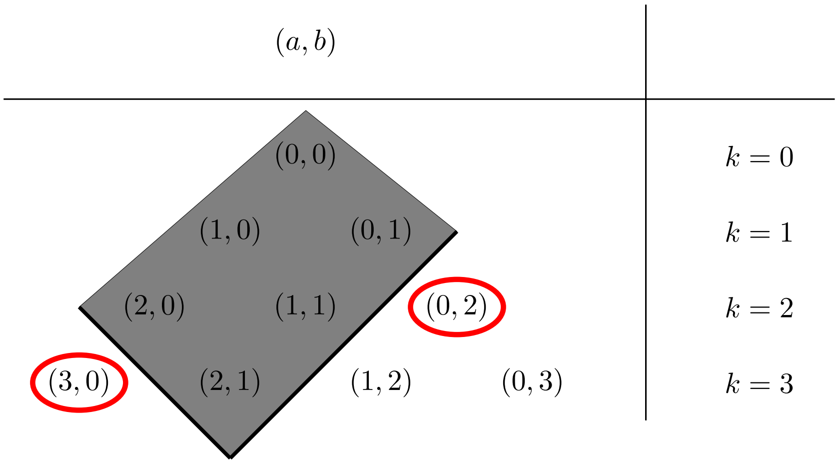

Figure 5.2: Combining perturbations.

Also, a perturbation $P_{2,1}$ can be combined with all the ones included in the shaded area in Figure 5.2 with interval $I(q,2,1)$. However, the encircled perturbations $P_{3,0}$ and $P_{0,2}$ cannot be combined together.

If we fix a perturbation $P_{a,b}$ with $k=2$ then, all perturbations $P_{c,d}$ with $c \leq a$ and $d \leq b$ can be combined with it, and also those with $c-1 \leq a$ or $d-1 \leq b$, but not both at the same time.

The same happens for $P_{a,b}$ with $k=1$, all perturbations $P_{c,d}$ with $k \leq 1$ can be combined with it.

v) There are 127 possible combinations for pairs of perturbations as in iv).

Observe that perturbations in (5.2.9) can be handled as above because we could determine the spaces of the Bessel scale between which a perturbation $P_{a,b}$ is a well behaved linear operator; see Proposition 5.2.9. However the fact that a, b are integer derivatives is not really essential. Therefore, this class of perturbations can be extended to the following one, where derivatives are replaced by fractional powers of the Laplacian as long as this one is well defined in our scale. For example $-\Delta + cI$, with $c>0$ can be used in this way, because the operator $(-\Delta+cI)^{r/2}$, $r>0$ satisfies for any $s \in \mathbb{R}$,

$$(-\Delta+cI)^{r/2} \in \mathcal{L}(H^{s,q}(\mathbb{R}^N), H^{s-r,q}(\mathbb{R}^N)), \quad \|(-\Delta+cI)^{r/2}\|_{\mathcal{L}(H^{s,q}(\mathbb{R}^N),H^{s-r,q}(\mathbb{R}^N))} \leq C$$

for some C independent of s, r, q. Note that this estimate is analogous to (5.2.5) for a non-integer r.

Thus, the perturbations

$$P_{a,b}u = (-\Delta+cI)^{b/2}(d(x)(-\Delta+cI)^{a/2}u), \quad a,b \geq 0$$

for any $0 \leq a,b \in \mathbb{R}$, in the sense that for any smooth enough φ

$$< P_{a,b}u, \varphi >= \int_{\mathbb{R}^N} d(x)(-\Delta+cI)^{a/2}u(-\Delta+cI)^{b/2}\varphi, \tag{5.2.18}$$

with $d \in L^p_U(\mathbb{R}^N)$, satisfy the statement in Proposition 5.2.9.

Then proceeding exactly as in Theorem 5.2.10, we recover the same results for this kind of perturbations, with the only difference that now $k = a+b$ is a real number smaller than 4.

Theorem 5.2.15 *Let $a, b, k \geq 0$ be real numbers such that $k = a + b < 4$ and $P_{a,b}$ be as in (5.2.18). Assume that $\|d\|_{L^p_U(\mathbb{R}^N)} \leq R_0$ with $p > \frac{N}{4-k}$, then for any $1 < q < \infty$ and such $P_{a,b}$ there exists an interval $I(q, a, b) \subset (-1+\frac{a}{4}, 1-\frac{b}{4})$ containing $(-1+\frac{a}{4}+\frac{N}{4p}, 1-\frac{b}{4}-\frac{N}{4p})$, such that for any $\gamma \in I(q, a, b)$, we have a strongly continuous, analytic semigroup, $S_{P_{a,b}}(t)$ in the space $H^{4\gamma,q}(\mathbb{R}^N)$, $1 < q < \infty$, for the problem*

$$\begin{cases} u_t + \Delta^2 u + P_{a,b} u = 0, & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimates

$$\|S_{P_{a,b}}(t)u_0\|_{H^{4\gamma',q}(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{4\gamma,q}(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in H^{4\gamma,q}(\mathbb{R}^N)$$

for every $\gamma, \gamma' \in I(q, a, b)$ with $\gamma' \geq \gamma$, and

$$\|S_{P_{a,b}}(t)u_0\|_{L^r(\mathbb{R}^N)} \leq \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in L^q(\mathbb{R}^N)$$

with $1 < q \leq r \leq \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q, a, b)$ is given by

$$I(q, a, b) = (-1 + \frac{a}{4} + \frac{N}{4}(\frac{1}{p} - \frac{1}{q'})_+, 1 - \frac{b}{4} - \frac{N}{4}(\frac{1}{p} - \frac{1}{q})_+).$$

Finally, if, as $\varepsilon \to 0$,

$$d_\varepsilon \to d \quad \text{in } L^p_U(\mathbb{R}^N), \quad p > \frac{N}{4-k}$$

then for every $1 < q \leq r \leq \infty$ and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N),H^{4\gamma',q}(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \leq T$$

for all $\gamma, \gamma' \in I(q, a, b)$ with $\gamma' > \gamma$ and for any $1 < q \leq r \leq \infty$

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N),L^r(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \leq T.$$

Note that Remark 5.2.3 and Remark 5.2.11, Proposition 5.2.12 and Remark 5.2.14 apply here as well.

5.3 Fourth order equations in uniform spaces in $\mathbb{R}^N$

The heat equation (5.2.2) and therefore the bi-Laplacian equation (5.2.4) can be also considered in much larger spaces than the Bessel spaces above, by taking the initial data in locally uniform spaces.

Using the spaces above and the convolution with the heat kernel, it was proved in Proposition 2.1, Theorem 2.1 and Theorem 5.3 in [5] that the heat equation defines an order preserving analytic semigroup in $L^q_U(\mathbb{R}^N)$ and , for $1 \leq q < \infty$, which is strongly continuous in $\dot{L}^q_U(\mathbb{R}^N)$ and in $E^\alpha := \dot{H}^{2\alpha,q}_U(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$. Moreover, this semigroup satisfies the smoothing estimates

$$\|S_{-\Delta}(t)u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \leq \frac{M_{r,q}e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in \dot{L}^q_U(\mathbb{R}^N)$$

for $1 \leq q \leq r \leq \infty$ for $\mu > 0$ arbitrary, and

$$\|S_{-\Delta}(t)u_0\|_{\dot{H}^{2\alpha,q}_U(\mathbb{R}^N)} \leq \frac{M_{\alpha,\beta}e^{\mu t}}{t^{\alpha-\beta}}\|u_0\|_{\dot{H}^{2\beta,q}_U(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in \dot{H}^{2\beta,q}_U(\mathbb{R}^N)$$

with $\mu > 0$ arbitrary, for any $\alpha, \beta \in \mathbb{R}$, $\alpha \geq \beta$.

It was also proved in [5] using a parabolic argument that type$(-\Delta) = 0$ in the $\dot{L}^q_U(\mathbb{R}^N)$ spaces (and thus in $\dot{H}^{\alpha,q}_U(\mathbb{R}^N)$), which explains why $\mu > 0$ above is arbitrary.

We now show some relevant information on the spectrum and resolvent of $-\Delta$ and Δ^2 in the uniform spaces which is analogous to Lemma 5.2.1.

Proposition 5.3.1 *i) For $1 < q < \infty$, in the space $E^0 := \dot{L}^q_U(\mathbb{R}^N)$ the operator $-\Delta$ with domain $E^1 := D(-\Delta) = \dot{H}^{2,q}_U(\mathbb{R}^N)$, satisfies the estimate*

$$\|(-\Delta - \lambda)^{-1}\|_{\mathcal{L}(E^0)} \leq M|\lambda|^{-1}$$

for all λ in a sector $S_{0,\phi}$ as in (5.2.3) for $\phi > 0$ arbitrarily small.

Furthermore, $\sigma(-\Delta) = [0, \infty)$, and thus, type$(-\Delta) = 0$.

ii) For $1 < q < \infty$, in the space $E^0 := \dot{L}^q_U(\mathbb{R}^N)$ the operator Δ^2 with domain $E^2 := D(\Delta^2) = \dot{H}^{4,q}_U(\mathbb{R}^N)$, satisfies the estimate

$$\|(\Delta^2 - \lambda)^{-1}\|_{\mathcal{L}(E^0)} \leq M|\lambda|^{-1}$$

for all λ in a sector $S_{0,2\phi}$ as in (5.2.3) for $\phi > 0$ arbitrarily small.

Furthermore, $\sigma(\Delta^2) = [0, \infty)$, and thus, type$(\Delta^2) = 0$.

Proof. First recall that from [5, Theorem 2.1] we have that the domain of the Laplacian operator in $\dot{L}^q_U(\mathbb{R}^N)$ is given by $D(-\Delta) = \dot{H}^{2,q}_U(\mathbb{R}^N)$. To prove part i), observe that, as in page 32–33 in [31], we can obtain an expression for the operator $(-\Delta + \mu I)^{-1}$, provided $Re(\sqrt{\mu}) > 0$, as a convolution operator. The expression is

$$u = (-\Delta + \mu)^{-1} f = \Gamma_\mu * f, \qquad Re(\sqrt{\mu}) > 0$$

with

$$\Gamma_\mu(x) = \sqrt{\mu}^{N-2} G_2(\sqrt{\mu}x), \qquad x \in \mathbb{R}^N, \qquad Re(\sqrt{\mu}) > 0$$

where G_2 is as in page 132 in [48] or page 33 in [31], that is

$$G_2(x) = \frac{1}{(4\pi)^{N/2}} \int_0^\infty t^{-N/2} e^{-t-\frac{x\cdot x}{4t}} dt = \frac{\xi^{1-N/2}}{(4\pi)^{N/2}} \int_0^\infty s^{-N/2} e^{-\xi(s+\frac{1}{4s})} ds, \qquad x \in \mathbb{R}^N$$

with $\xi = \sqrt{x \cdot x} > 0$. This definition can be extended to complex variables as

$$G_2(z) = \frac{\xi^{1-N/2}}{(4\pi)^{N/2}} \int_0^\infty s^{-N/2} e^{-\xi(s+\frac{1}{4s})} ds, \quad z \in \mathbb{C}^N, \quad \xi = \sqrt{z \cdot z}, \quad Re(\xi) > 0.$$

According to [31], we have for $z \in \mathbb{C}^N$ with $Re(\xi) > 0$, if $N > 2$

$$|G_2(z)| \leq C|\xi|^{(2-N)/2} (Re\ \xi)^{(2-N)/2} e^{-\frac{1}{2}Re\xi} \qquad \xi = \sqrt{z \cdot z} \tag{5.3.1}$$

and if $N = 2$,

$$|G_2(z)| \leq C \max\{\ln \frac{1}{Re\xi}, 1\} e^{-\frac{1}{2}Re\xi} \qquad \xi = \sqrt{z \cdot z}. \tag{5.3.2}$$

Now observe that if $\lambda \in S_{0,\phi}$ with $\phi > 0$ then for $\mu = -\lambda \in \mathbb{C} \setminus (-\infty, 0]$ we can choose $Re(\sqrt{\mu}) > 0$. For such λ and similarly to Lemma 5.2.1 we are going to check that for $f \in \dot{L}^q_U(\mathbb{R}^N)$ we have the following estimate for $u = \Gamma_\mu * f$,

$$\|u\|_{L^q_U(\mathbb{R}^N)} \leq C \frac{1}{|\lambda|} \|f\|_{L^q_U(\mathbb{R}^N)}, \qquad \lambda \in S_{0,\phi} \qquad \phi > 0.$$

Let $\{Q_i\}$, $i \in \mathbb{Z}^N$, be a partition of $\mathbb{R}^N$ in open disjoint cubes centered in $i \in \mathbb{Z}^N$ with edges of length 1, parallel to the axes. Thus $Q_i \cap Q_j = \emptyset$ for $i \neq j$ and $\mathbb{R}^N = \cup_i \overline{Q_i}$.

Then we fix $i \in \mathbb{Z}^N$ and decompose $f \in \dot{L}^q_U(\mathbb{R}^N)$ in a *far* and a *near* region as in [5, Proposition 2.1]. For this we denote by $N(i)$ the set for indices j such that $\overline{Q_i} \cap \overline{Q_j} \neq 0$. That is, the set for which

$$d_{ij} := \inf\{dist(x,y), x \in Q_i, y \in Q_j\}$$

satisfies that $d_{ij} = 0$. Thus we can define, for each $i \in \mathbb{Z}^N$ fixed

$$Q_i^{near} = \cup_{j \in N(i)} Q_j \qquad and \qquad Q_i^{far} = \mathbb{R}^N \setminus Q_i^{near}.$$

Hence, we decompose $f := f_i^{near} + f_i^{far} := f\chi_{Q_i^{near}} + f\chi_{Q_i^{far}}$, where χ denotes the characteristic function and $u := u_i^{near} + u_i^{far}$ with

$$u_i^{near} := \Gamma_\mu * f_i^{near} \qquad u_i^{far} := \Gamma_\mu * f_i^{far}.$$

The resolvent estimate will follow from the following estimates of the two terms of the decomposition. For λ as above, we have first,

$$\|u_i^{near}\|_{L^q(Q_i)} \leq \frac{C}{|\lambda|} \|f\|_{L^q(Q_i^{near})}, \qquad \lambda \in S_{0,\phi} \tag{5.3.3}$$

and, second,

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leq \frac{C}{|\lambda|}\|f\|_{L^1_U(Q_i^{far})}, \qquad \lambda \in S_{0,\phi} \tag{5.3.4}$$

for some C independent if $i \in \mathbb{Z}^N$.

Using (5.3.3) and (5.3.4), since the constants for the embedding $L^\infty(Q_i) \hookrightarrow L^q(Q_i)$ and the restrictions $L^q_U(\mathbb{R}^N) \hookrightarrow L^q(Q_i^{near})$, $L^q_U(\mathbb{R}^N) \hookrightarrow L^1_U(Q_i^{near})$ depend on N but can be chosen independent of p, q and i, (5.3.3) and (5.3.4) imply

$$\|u\|_{L^q(Q_i)} \leq \frac{C}{|\lambda|}\|f\|_{L^q_U(\mathbb{R}^N)}, \qquad \lambda \in S_{0,\phi}$$

for each $i \in \mathbb{Z}^N$ with C independent of i and $\lambda \in S_{0,\phi}$, which gives the result.

Hence, we first prove (5.3.3). As a consequence of Lemma 5.2.1, we get for all $\lambda \in S_{0,\phi}$

$$\|u_i^{near}\|_{L^q(Q_i)} \leq \|u_i^{near}\|_{L^q(\mathbb{R}^N)} \leq \frac{C}{|\lambda|}\|f_i^{near}\|_{L^q(\mathbb{R}^N)} = \frac{C(N)}{|\lambda|}\|f\|_{L^q(Q_i^{near})}.$$

We show now (5.3.4) for $N > 2$. Observe that $f_i^{far} = f\chi_{Q_i^{far}} = \sum_{j\in\mathbb{Z}^N\setminus N(i)} f\chi_{Q_j}$. Hence, because of (5.3.1) with $z = \sqrt{\mu}x$, $Re(\sqrt{\mu}) > 0$, $x \in \mathbb{R}^N$, $\mu = -\lambda$ and $\lambda \in S_{0,\phi}$, we have for all $x \in Q_i$

$$|u_i^{far}(x)| = \sum_{j\notin N(i)} |(\Gamma_\mu * f\chi_{Q_j})(x)|$$

$$\leq \sum_{j\notin N(i)} C \sup_{y\in Q_j} |\sqrt{\mu}^{N-2} \cdot (\sqrt{\mu}|x-y|)^{1-N/2} Re(\sqrt{\mu}|x-y|)^{1-N/2} e^{-\frac{1}{2}Re\sqrt{\mu}|x-y|}|\|f\|_{L^1(Q_j)}$$

$$\leq C\|f\|_{L^1_U(Q_i^{far})}\sqrt{|\lambda|}^{N/2-1} Re(\sqrt{\mu})^{1-N/2} \sum_{j\notin N(i)} \sup_{y\in Q_i} |x-y|^{2-N} e^{-\frac{1}{2}|x-y|Re\sqrt{\mu}}.$$

Note that for all $x \in Q_i$ and $y \in Q_j$ it holds $|x-y| \geq d_{ij}$, thus

$$|u_i^{far}(x)| \leq C\|f\|_{L^1_U(Q_i^{far})}\Big(\frac{\sqrt{|\lambda|}}{Re(\sqrt{\mu})}\Big)^{N/2-1} \sum_{j\notin N(i)} d_{ij}^{2-N} e^{-\frac{1}{2}d_{ij}Re\sqrt{\mu}}.$$

Hence

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leq C\|f\|_{L^1_U(Q_i^{far})}\Big(\frac{\sqrt{|\lambda|}}{Re(\sqrt{\mu})}\Big)^{N/2-1} \sum_{j\notin N(i)} d_{ij}^{2-N} e^{-\frac{1}{2}d_{ij}Re\sqrt{\mu}}.$$

Now, using that $\sharp\{j \in \mathbb{Z}, d_{ij} = k\} \leq Ck^{N-1}$ we obtain

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leq C\|f\|_{L^1_U(Q_i^{far})}\Big(\frac{\sqrt{|\lambda|}}{Re(\sqrt{\mu})}\Big)^{N/2-1} \sum_{k=1}^{\infty} k e^{-\frac{1}{2}kRe\sqrt{\mu}}$$

$$\leq C\|f\|_{L^1_U(Q_i^{far})}\Big(\frac{\sqrt{|\lambda|}}{Re(\sqrt{\mu})}\Big)^{N/2-1} \int_1^\infty s e^{-\frac{1}{2}sRe\sqrt{\mu}}\, ds.$$

Finally, changing variables in the integral above as $r = Re(\sqrt{\mu})s$, we obtain

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leq C\big(\frac{\sqrt{|\lambda|}}{Re(\sqrt{\mu})}\big)^{N/2-1}\frac{1}{Re(\sqrt{\mu})^2}\|f\|_{L^1_U(Q_i^{far})}$$

which can be arranged as

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leq \big(\frac{\sqrt{|\lambda|}}{Re(\sqrt{\mu})}\big)^{N/2+1}\frac{C}{|\lambda|}\|f\|_{L^1_U(Q_i^{far})}.$$

To conclude, observe that for all $\lambda \in S_{0,\phi}$ we find

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leq \frac{C}{\cos(\phi/2)^{N/2+1}}\frac{1}{|\lambda|}\|f\|_{L^1_U(Q_i^{far})}.$$

Thus, (5.3.4) is proved for $N > 2$.

We show now (5.3.4) for $N = 2$. Proceeding as above and using (5.3.2) we get

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leq C\|f\|_{L^1_U(Q_i^{far})}\sum_{j\notin N(i)}\max\{\ln\frac{1}{d_{ij}Re(\sqrt{\mu})}, 1\}e^{-\frac{1}{2}d_{ij}Re\sqrt{\mu}}.$$

Using again that $\sharp\{j \in \mathbb{Z}, d_{ij} = k\} \leq Ck^{N-1}$ we get

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leq C\|f\|_{L^1_U(Q_i^{far})}\sum_{k=1}^{\infty} k\max\{\ln\frac{1}{kRe(\sqrt{\mu})}, 1\}e^{-\frac{1}{2}kRe\sqrt{\mu}}$$

$$\leq C\|f\|_{L^1_U(Q_i^{far})}\int_0^\infty s\max\{\ln\frac{1}{sRe(\sqrt{\mu})}, 1\}e^{-\frac{1}{2}sRe\sqrt{\mu}}ds$$

and with the change of variables $r = Re(\sqrt{\mu})s$ we obtain,

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leq \|f\|_{L^1_U(Q_i^{far})}\frac{C}{Re(\sqrt{\mu})^2} = \big(\frac{\sqrt{|\lambda|}}{Re(\sqrt{\mu})}\big)^2\frac{C}{|\lambda|}\|f\|_{L^1_U(Q_i^{far})}.$$

Thus for all $\lambda \in S_{0,\phi}$ we find

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leq \frac{C}{\cos(\phi/2)^2}\frac{1}{|\lambda|}\|f\|_{L^1_U(Q_i^{far})}$$

and the result is proved.

In particular since $\phi > 0$ is arbitrary, $\sigma(-\Delta) \subset [0,\infty)$. For the opposite inclusion, note that $u(x) = e^{i\omega x}$, $\omega \in \mathbb{R}^N$ satisfies $u \in \dot{L}^p_U(\mathbb{R}^N)$ and

$$-\Delta u = \lambda u$$

for $\lambda = |\omega|^2 \subset [0,\infty)$, and thus $[0,\infty) \subset \sigma(-\Delta)$.

For part ii), since $-\Delta$ is sectorial with sector $S_{0,\phi}$ with $\phi < \pi/4$ and we have the estimate $\|(-\Delta-\lambda)^{-1}\| \leq \frac{C}{|\lambda|}$ for $\lambda \in S_{0,\phi}$, we apply [37, 10.5] (see Proposition 5.1.1). Therefore, we get that Δ^2 is sectorial with sector $S_{0,2\phi}$. Note that $\sigma(\Delta^2) \subset [0,\infty)$ because $\phi > 0$ is arbitrarily small. Also, note again that $u(x) = e^{i\omega x}$, $\omega \in \mathbb{R}^N$ satisfies $u \in \dot{L}^q_U(\mathbb{R}^N)$ and

$$\Delta^2 u = \lambda u$$

for $\lambda = |\omega|^4 \subset [0,\infty)$. ■

Now, using Proposition 5.1.3 and an argument as in Lemma 5.2.2 we get the next result.

Lemma 5.3.2 *Consider the problem*

$$\begin{cases} u_t + \Delta^2 u = 0 & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{5.3.5}$$

i) Then for each $1 < q < \infty$, (5.3.5) defines an analytic semigroup, $S_{\Delta^2}(t)$, in the scale $X^\alpha := E^{2\alpha} = \dot{H}^{4\alpha,q}_U(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, such that for any $\mu_0 > 0$ there exists C such that

$$\|S_{\Delta^2}(t)u_0\|_{\dot{H}^{4\alpha,q}_U(\mathbb{R}^N)} \leq \frac{M_{\alpha,\beta}e^{\mu t}}{t^{\alpha-\beta}}\|u_0\|_{\dot{H}^{4\beta,q}_U(\mathbb{R}^N)}, \qquad t > 0, u_0 \in \dot{H}^{4\beta,q}_U(\mathbb{R}^N)$$

with $\alpha,\beta \in \mathbb{R},\ \alpha \geq \beta$.
ii) The analytic semigroup $S_{\Delta^2}(t)$, in $\dot{L}^q_U(\mathbb{R}^N)$, $1 < q < \infty$, satisfies

$$\|S_{\Delta^2}(t)u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \leq \frac{M_{q,r}e^{\mu_0 t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \qquad t > 0, u_0 \in \dot{L}^q_U(\mathbb{R}^N)$$

for any $\mu_0 > 0$ and $1 < q \leq r \leq \infty$ and some $M_{q,r} > 0$.

For a similar estimate with $q = 1 < r \leq \infty$, we can proceed as in Remark 5.2.3.

We can now adapt the arguments for Bessel and Lebesgue spaces in Section 5.2 to the uniform Bessel spaces to perturb equation (5.3.5) as follows. First, as in [47, Lemma 26, pg. 43] we have

Lemma 5.3.3 *i) Assume that $m \in L^p_U(\mathbb{R}^N)$, then the multiplication operator*

$$Pu(x) = m(x)u(x)$$

satisfies, for $r \geq p'$ and $\frac{1}{s} = \frac{1}{r} + \frac{1}{p}$, that

$$P \in \mathcal{L}(L^r_U(\mathbb{R}^N), L^s_U(\mathbb{R}^N)), \qquad \|P\|_{\mathcal{L}(L^r_U(\mathbb{R}^N),L^s_U(\mathbb{R}^N))} \leq C\|m\|_{L^p_U(\mathbb{R}^N)}.$$

ii) If moreover $m \in \dot{L}^p_U(\mathbb{R}^N)$ we have for $r \geq p'$ and $\frac{1}{s} = \frac{1}{r} + \frac{1}{p}$, that

$$P \in \mathcal{L}(\dot{L}^r_U(\mathbb{R}^N), \dot{L}^s_U(\mathbb{R}^N)), \qquad \|P\|_{\mathcal{L}(\dot{L}^r_U(\mathbb{R}^N),\dot{L}^s_U(\mathbb{R}^N))} \leq C\|m\|_{L^p_U(\mathbb{R}^N)}.$$

Now, we consider perturbations similar to the perturbations in (5.2.9) with $b = 0$, that is,

$$P_a u = d(x) D^a u \tag{5.3.6}$$

with $d \in \dot{L}^p_U(\mathbb{R}^N)$ and $a \in \mathbb{N}$. Note that since the uniform Bessel spaces are not reflexive (even for $q = 2$), the negative spaces cannot be described as dual spaces, and thus, the approach in Proposition 5.2.9 can not be carried out for $b \neq 0$ in uniform spaces. We will use Proposition 3.0.1 instead.

Proposition 5.3.4 *Let $P_a u = d(x)D^a u$ with $d \in \dot{L}^p_U(\mathbb{R}^N)$, $a \in \{0, 1, 2, 3\}$ and let $s \geq a$, $\sigma \geq 0$. Assume also that $1 < q < \infty$ and*

$$(s - a - \frac{N}{q})_- + (\sigma - \frac{N}{q'})_- \geq -\frac{N}{p'} \tag{5.3.7}$$

where the inequality is strict when $s - a - \frac{N}{q} = 0$ and $\sigma - \frac{N}{q'} = -\frac{N}{p'}$ or $s - a - \frac{N}{q} = -\frac{N}{p'}$ and $\sigma - \frac{N}{q'} = 0$ (or both).

Then, we have

$$P_a \in \mathcal{L}(\dot{H}^{s,q}_U(\mathbb{R}^N), \dot{H}^{-\sigma,q}_U(\mathbb{R}^N)), \qquad \|P_a\|_{\mathcal{L}(\dot{H}^{s,q}_U(\mathbb{R}^N), \dot{H}^{-\sigma,q}_U(\mathbb{R}^N))} \leq C\|d\|_{\dot{L}^p_U(\mathbb{R}^N)}.$$

Proof. First note that $u \in \dot{H}^{s,q}_U(\mathbb{R}^N)$, thus $D^a u \in \dot{H}^{s-a,q}_U(\mathbb{R}^N)$. Because of (5.3.7) we can choose $r, \rho \geq 1$ such that $(s - a - \frac{N}{q})_- \geq -\frac{N}{r}$ and $(\sigma - \frac{N}{q'})_- \geq -\frac{N}{\rho'}$ with $\frac{1}{\rho} = \frac{1}{r} + \frac{1}{p}$ (and so $r \geq p'$).

Therefore we can use the inclusion $\dot{H}^{s-a,q}_U(\mathbb{R}^N) \hookrightarrow \dot{L}^r_U(\mathbb{R}^N)$ and then part ii) in Lemma 5.3.3 gives $P_a u \in \dot{L}^\rho_U(\mathbb{R}^N)$ and finally, because of Proposition 3.0.1, we use the inclusion $\dot{L}^\rho_U(\mathbb{R}^N) \hookrightarrow \dot{H}^{-\sigma,q}_U(\mathbb{R}^N)$ and we get the result. ■

With this, we can obtain the main result for perturbations of (5.3.5).

Theorem 5.3.5 *Let $a \in \{0, 1, 2, 3\}$, $d \in \dot{L}^p_U(\mathbb{R}^N)$ such that $\|d\|_{\dot{L}^p_U(\mathbb{R}^N)} \leq R_0$ with $p > \frac{N}{4-a}$. Then for any $1 < q < \infty$ and any P_a as in (5.3.6) there exists an interval $I(q, a) \subset (-1 + \frac{a}{4}, 1)$ containing $(-1 + \frac{a}{4} + \frac{N}{4p}, 1 - \frac{N}{4p})$, such that for any $\gamma \in I(q, a)$, we have a continuous, analytic semigroup, $S_{P_a}(t)$ in the space $\dot{H}^{4\gamma,q}_U(\mathbb{R}^N)$, for the problem*

$$\begin{cases} u_t + \Delta^2 u + d(x) D^a u = 0, & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimate

$$\|S_{P_a}(t) u_0\|_{\dot{H}^{4\gamma',q}_U(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma} e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{\dot{H}^{4\gamma,q}_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{H}^{4\gamma}_U(\mathbb{R}^N)$$

for every $\gamma, \gamma' \in I(q, a)$ with $\gamma' \geq \gamma$, and

$$\|S_{P_a}(t) u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \leq \frac{M_{q,r} e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q} - \frac{1}{r})}} \|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{L}^q_U(\mathbb{R}^N)$$

for $1 < q \leq r \leq \infty$ *with some* $M_{\gamma',\gamma}$, $M_{q,r}$ *and* $\mu \in \mathbb{R}$ *depending on* d *only through* R_0.

For each P_a, *the interval* $I(q,a)$ *is given by*

$$I(q,a) = (-1 + \frac{a}{4} + \frac{N}{4}(\frac{1}{p} - \frac{1}{q'})_+, 1 - \frac{N}{4}(\frac{1}{p} - \frac{1}{q})_+) \subset (-1 + \frac{a}{4}, 1).$$

Finally, if, as $\varepsilon \to 0$

$$d_\varepsilon \to d \quad \textit{in } \dot{L}^p_U(\mathbb{R}^N), \quad p > \frac{N}{4-k}$$

then for every $T > 0$ *there exists* $C(\varepsilon) \to 0$ *as* $\varepsilon \to 0$, *such that*

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(\dot{H}^{4\gamma,q}_U(\mathbb{R}^N), \dot{H}^{4\gamma',q}_U(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \leq T$$

for all $\gamma, \gamma' \in I(q,a,b)$, $\gamma' \geq \gamma$ *and for all* $1 < q \leq r \leq \infty$,

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(\dot{L}^q_U(\mathbb{R}^N), \dot{L}^r_U(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q} - \frac{1}{r})}}, \qquad \forall\, 0 < t \leq T.$$

Proof. The proof is as in proof of Theorem 5.2.10 but using Proposition 5.3.4 instead of Proposition 5.2.9. The analyticity comes again from [47, Theorem 12] (see Theorem 1.0.3). ■

Note that Remark 5.2.11, Proposition 5.2.12 and Remark 5.2.14 apply here as well. Also, we can replace D^a in (5.3.6) by $(-\Delta + cI)^{a/2}$ with $0 \leq a < 4$ as in Theorem 5.2.15.

Chapter 6

Higher order parabolic equations

In this chapter we show that all the results in Sections 5.2 and 5.3 above also hold true for other natural powers of suitable operators, and in particular, for any power of the Laplacian, $(-\Delta)^m$, with $m \in \mathbb{N}$. The proofs below have barely no changes with respect to the ones above, and we now detail the main points for them.

We start showing how the scale of spaces constructed in Chapter 2 for A_0 can be used for the squared operator $A_0^m := A_0 \circ \cdots \circ A_m$. More precisely, as we did in Section 5.1 for A_0^2, our goal is to relate the scales of the power of an operator, A_0^m, with the scale of the original operator A_0.

As in Chapter 2 we assume

$$A_0 \in \mathcal{H}(E^1, E^0).$$

Observe that by Propositions 2.1.4 and 2.2.1 we can consider the associated interpolation scale $\{E^\alpha\}_{\alpha\in\mathbb{R}}$ or the fractional power scale $\{F^\alpha\}_{\alpha\geq -N}$, $N \in \mathbb{N}$ without assuming $0 \in \rho(A_0)$ or $(-\infty, 0] \in \rho(A_0)$, respectively. Also, note that with the notation from Chapter 2,

$$A_0^m := A_0 \circ \cdots \circ A_m, \qquad A_0^m : E^m \to E^0.$$

Hence, we will assume furthermore that

$$A_0^m \in \mathcal{H}(E^m, E^0).$$

The following result can be found in [37, Proposition 10.5] and gives a criteria for determining when A_0^m is a sectorial operator.

Proposition 6.0.1 *Let $A_0 \in \mathcal{H}(E^1, E^0)$ with $(-\infty, 0] \subset \rho(A_0)$ and satisfying $\|(A_0 - \lambda)^{-1}\| \leq \frac{K}{|\lambda|}$ for $\lambda \in S_{0,\phi}$ with $\phi \in (0, \frac{\pi}{2m})$ where $S_{0,\phi}$ is a sector as (2.0.1) with vertex $a = 0$.*

Then A_0^m satisfies $S_{0,m\phi} \subset \rho(A_0^m)$ and

$$\|(A_0^m - \lambda)^{-1}\|_{E^0} \leq \frac{K}{|\lambda|}$$

for $\lambda \in S_{0,m\phi}$, thus $A_0^m \in \mathcal{H}(E^m, E^0)$.

Remark 6.0.2
i) For the proof we refer to [37, Proposition 10.5]. As an indication f why this holds check Remark 5.1.2
ii) $0 \in \rho(A_0)$ *implies* $0 \in \rho(A_0^m)$.
iii) In general, there is no relationship between type(A_0^m) *and type*(A_0).

With this, we can construct both interpolation and fractional scales for A_0^m following the procedures explained in Chapter 2. As in Section 5.1 the problem is clarify how this scale is related with the one generated by A_0. In the next two results we show that the scales constructed from A_0^m coincide with the ones from A_0 after a suitable labeling.

Proposition 6.0.3 *Let* $A_0 \in \mathcal{H}(E^1, E^0)$ *and assume* $A_0^m := A_0 \circ A_1 \in \mathcal{H}(E^2, E^0)$. *Let* $\{E^\alpha\}_{\alpha\in\mathbb{R}}$ *be the interpolation scale for* A_0 *as in Proposition 2.1.4. Then on the scale* $X^\alpha = E^{m\alpha}$ *with* $\alpha \in \mathbb{R}$ *we have* $A_\alpha^m := A_\alpha \circ \cdots \circ A_{\alpha+m} \in \mathcal{H}(X^{\alpha+m}, X^\alpha)$ *and* A_0^m *defines a semigroup* $S_{A_0^m}(t)$ *in the scale* $\{X^\alpha\}_{\alpha\in\mathbb{R}}$ *that satisfies* $S_{A_0^m}(t)|_{X^\alpha} = e^{-A_\alpha^m t}$ *and*

$$\|S_{A_0^m}(t)\|_{\mathcal{L}(X^\beta,X^\alpha)} \leq \frac{C(\alpha-\beta)}{t^{\beta-\alpha}} e^{\mu t} \qquad t>0, \alpha,\beta\in\mathbb{R}, \alpha\geq\beta$$

for any $\mu >$ *type*(A_0^m). *The constant* $C(\alpha-\beta)$ *is bounded for* α, β *in bounded sets of* $\mathbb{R}$.

If E^0 *is reflexive, the negative side of the scale can be described as*

$$X^{-\alpha} = (X^{\alpha\sharp})' \quad \text{and} \quad A_{-\alpha}^m = (A_\alpha^{m\sharp})', \qquad \alpha > 0$$

and it holds that

$$e^{-A_{-\alpha}^m t} = (e^{-A_\alpha^{m\sharp} t})'.$$

Furthermore, the problem

$$\begin{cases} u_t + A_\alpha^m u = 0, & t > 0 \\ u(0) = u_0 \in X^\alpha \end{cases}$$

for any $\alpha \in \mathbb{R}$ *has a unique solution* $u(t) = S_{A_0^m}(t)u_0 = e^{-A_\alpha^m t}u_0$.

Proof. The proof is analogous to the one in Proposition 5.1.3. Step 1 can be repeated in the same manner just noting that that interpolation works in the general case $X^k = E^{mk}$ in the same way it did for the case $m = 2$. For Step 2, we can again follow the proof in Proposition 5.1.3 with the only difference that now $\tilde{A}_0^m = A_0^m + P$ where $P = \sum_{i=0}^{m-1} \binom{m}{i} c^{m-i} A_0^i$. ■

Now we turn to the fractional power scale to obtain

Proposition 6.0.4 *Let* $A_0 \in \mathcal{H}(E^1, E^0)$ *and assume* $A_0^m := A_0 \circ \cdots \circ A_m \in \mathcal{H}(E^m, E^0)$. *Let* $N \in \mathbb{N}$ *and* $\{F^\alpha\}_{\alpha\geq -mN}$ *be the fractional power scale for* A_0 *as in Proposition 2.2.1. Then on the fractional power scale* $Y^\alpha = F^{m\alpha}$ *with* $\alpha \geq -N$ *we have* $A_\alpha^m := A_\alpha \circ A_{\alpha+m} \in$

$\mathcal{H}(Y^{\alpha+m}, Y^{\alpha})$ and A_0^m defines a semigroup $S_{A_0^m}(t)$ in the scale $\{Y^\alpha\}_{\alpha\geq -N}$ that satisfies $S_{A_0^m}(t)|_{F^\alpha} = e^{-A_\alpha^m t}$ and

$$\|S_{A_0^m}(t)\|_{\mathcal{L}(Y^\beta, Y^\alpha)} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\mu t} \qquad t>0,\ \alpha\geq\beta\geq -N$$

for any $\mu >$ type(A_0^m). The constant $C(\alpha-\beta)$ is bounded for α, β in bounded sets of $\mathbb{R}$.

If E^0 is reflexive, the negative side of the scale can be described as

$$Y^{-\alpha} = (Y^{\alpha\sharp})' \quad \text{and} \quad A_{-\alpha}^m = (A_\alpha^{\sharp m})' \qquad \alpha>0,$$

and it holds that

$$e^{-A_{-\alpha}^m t} = (e^{-A_\alpha^{m\sharp} t})'.$$

Furthermore, the problem

$$\begin{cases} u_t + A_\alpha^m u = 0, & t>0 \\ u(0) = u_0 \in Y^\alpha \end{cases}$$

for any $\alpha \geq -N$ has a unique solution $u(t) = S_{A_0^m}(t)u_0 = e^{-A_\alpha^m t}u_0$.

The proof can be again repeated from the one for Proposition 5.1.4, with the difference the $P = \sum_{i=0}^{m-1}\binom{m}{i}c^{m-i}A_0^i$.

Remark 6.0.5 *According to Remark 2.2.2 if A_0 has bounded imaginary powers, then A_0^m does as well, see (2.2.9). In such case both scales and semigroups in Propositions 6.0.3 and 6.0.4 coincide, that is, $X^\alpha = Y^\alpha$ for $\alpha \geq -N$, see [2, V.1.5.13, pg. 283].*

Lemma 6.0.6 *For $1<q<\infty$, in $E^0 = L^q(\mathbb{R}^N)$ the operator $(-\Delta)^m$ with domain $E^m = D(-\Delta^m) = H^{2m,q}(\mathbb{R}^N)$, satisfies the estimate*

$$\|((-\Delta)^m - \lambda)^{-1}\|_{L^q(\mathbb{R}^N)} \leq M|\lambda|^{-1} \qquad \text{for all } \lambda \in S_{0,m\phi}$$

where $\phi>0$ is arbitrarily small. Furthermore $\sigma((-\Delta)^m) = [0,\infty)$ and therefore

$$\mathit{type}((-\Delta)^m) = 0.$$

The proof is exactly as the one in Lemma 5.2.1, using now Proposition 6.0.1. This together with Proposition 6.0.3 lead to

Lemma 6.0.7 *Consider the problem*

$$\begin{cases} u_t + (-\Delta)^m u = 0 & x\in\mathbb{R}^N, t>0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N \end{cases} \tag{6.0.1}$$

with $m \in \mathbb{N}$.

i) Then for $1 < q < \infty$, *(6.0.1) defines an analytic semigroup,* $S_{(-\Delta)^m}(t)$, *in the scale* $X^\alpha = E^{m\alpha} = H^{2m\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, *such that for any* $\mu_0 > 0$ *there exists* $C(\alpha-\beta)$ *such that*

$$\|S_{(-\Delta)^m}(t)\|_{\mathcal{L}(H^{2m\beta,q}(\mathbb{R}^N),H^{2m\alpha,q}(\mathbb{R}^N))} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\mu_0 t} \qquad t > 0,\ \alpha,\beta \in \mathbb{R},\ \alpha \geq \beta.$$

ii) The analytic semigroup, $S_{(-\Delta)^m}(t)$, *in* $L^q(\mathbb{R}^N)$, $1 < q < \infty$, *satisfies that for any* $\mu_0 > 0$ *there exists* $M_{q,r}$ *such that*

$$\|S_{(-\Delta)^m}(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N),L^r(\mathbb{R}^N))} \leq \frac{M_{q,r}}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}} e^{\mu_0 t} \qquad t > 0$$

for $1 < q \leq r \leq \infty$.

Note that the proof in Lemma 5.2.2 can be carried out now taking $(-\Delta)^m$ instead of Δ^2 in the scale of spaces.

Also note that the solution of problem (6.0.1) can also be described as the convolution of the initial data with the fundamental kernel for the m–Laplacian operator, which satisfies suitable Gaussian bounds; see e.g. [20, 10].

We can now add the perturbations to (6.0.1), as in Theorem 5.2.10.

Theorem 6.0.8 *Let* a, $b \in \mathbb{N}$ *with* $k = a + b \leq 2m-1$ *and* $P_{a,b}$ *be as in (5.2.9). Assume that* $\|d\|_{L^p_U(\mathbb{R}^N)} \leq R_0$ *with* $p > \frac{N}{2m-k}$. *Then for any* $1 < q < \infty$ *and such* $P_{a,b}$ *there exists an interval* $I(q,a,b) \subset (-1+\frac{a}{2m}, 1-\frac{b}{2m})$ *containing* $(-1+\frac{a}{2m}+\frac{N}{2mp}, 1-\frac{b}{2m}-\frac{N}{2mp})$, *such that for any* $\gamma \in I(q,a,b)$, *we have a strongly continuous, analytic semigroup,* $S_{P_{a,b}}(t)$ *in the space* $H^{2m\gamma,q}(\mathbb{R}^N)$, *for the problem*

$$\begin{cases} u_t + (-\Delta)^m u + D^b(d(x)D^a u) = 0, & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimates

$$\|S_{P_{a,b}}(t)u_0\|_{H^{2m\gamma',q}(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma} e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{H^{2m\gamma,q}(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in H^{2m\gamma,q}(\mathbb{R}^N)$$

for every $\gamma, \gamma' \in I(q,a,b)$ *with* $\gamma' \geq \gamma$, *and*

$$\|S_{P_{a,b}}(t)u_0\|_{L^r(\mathbb{R}^N)} \leq \frac{M_{q,r} e^{\mu t}}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{L^q(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in L^q(\mathbb{R}^N)$$

with $1 < q \leq r \leq \infty$ *and some* $M_{\gamma',\gamma}$, $M_{q,r}$ *and* $\mu \in \mathbb{R}$ *depending on* d *only through* R_0.

Furthermore, the interval $I(q,a,b)$ *is given by*

$$I(q,a,b) = (-1 + \frac{a}{2m} + \frac{N}{2m}(\frac{1}{p} - \frac{1}{q'})_+, 1 - \frac{b}{2m} - \frac{N}{2m}(\frac{1}{p} - \frac{1}{q})_+).$$

Finally, if as $\varepsilon \to 0$

$$d_\varepsilon \to d \quad \text{in } L^p_U(\mathbb{R}^N), \quad p > \frac{N}{2m-k}$$

then for every $1 < q < \infty$ *and* $T > 0$ *there exists* $C(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*, such that*

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(H^{2m\gamma,q}(\mathbb{R}^N), H^{2m\gamma',q}(\mathbb{R}^N))} \le \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \le T$$

for all $\gamma, \gamma' \in I(q,a,b)$*, with* $\gamma' \ge \gamma$ *and*

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \le \frac{C(\varepsilon)}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \le T$$

for all $1 < q \le r \le \infty$.

Note that now, the amount of possible combinations of perturbations becomes enormous, however, they can be combined just as explained in Proposition 5.2.12 and Remark 5.2.14. Also, Remark 5.2.11 still holds.

We finally turn into the uniform spaces $\dot{L}^q_U(\mathbb{R}^N)$. First of all, we check the information about the spectrum and resolvent set for $(-\Delta)^m$ in $\dot{L}^q_U(\mathbb{R}^N)$, with the same ideas as in Proposition 5.3.1 and using again Proposition 6.0.1.

Lemma 6.0.9 *For* $1 < q < \infty$*, the operator* $(-\Delta)^m$ *in the space* $E^0 = \dot{L}^q_U(\mathbb{R}^N)$ *with domain* $E^m = D((-\Delta)^m) = \dot{H}^{2m,q}_U(\mathbb{R}^N)$*, satisfies the estimate*

$$\|((-\Delta)^m - \lambda)^{-1}\|_{\dot{L}^q_U(\mathbb{R}^N)} \le M|\lambda|^{-1}$$

for all λ *in a sector* $S_{0,m\phi}$ *as in (5.2.3) for* $\phi > 0$ *arbitrarily small.*

Furthermore, $\sigma((-\Delta)^m) = [0,\infty)$*, and thus, type*$((-\Delta)^m) = 0$.

Again, this leads to

Lemma 6.0.10 *Consider the problem*

$$\begin{cases} u_t + (-\Delta)^m u = 0 & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

i)Then for each $1 < q < \infty$*, (6.0.10) defines an analytic semigroup,* $S_{(-\Delta)^m}(t)$*, in the scale* $X^\alpha := E^{m\alpha} = \dot{H}^{2m\alpha,q}_U(\mathbb{R}^N)$*,* $\alpha \in \mathbb{R}$*, such that for any* $\mu_0 > 0$ *there exists* C *such that*

$$\|S_{(-\Delta)^m}(t)u_0\|_{\dot{H}^{2m\alpha,q}_U(\mathbb{R}^N)} \le \frac{M_{\alpha,\beta}e^{\mu_0 t}}{t^{\alpha-\beta}}\|u_0\|_{\dot{H}^{4\beta,q}_U(\mathbb{R}^N)}, \qquad t > 0, u_0 \in \dot{H}^{4\beta,q}_U(\mathbb{R}^N)$$

with $\alpha, \beta \in \mathbb{R}$*,* $\alpha \ge \beta$.

ii) The analytic semigroup $S_{(-\Delta)^m}(t)$*, in* $\dot{L}^q_U(\mathbb{R}^N)$*,* $1 < q < \infty$*, satisfies*

$$\|S_{(-\Delta)^m}(t)u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \le \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \qquad t > 0, u_0 \in \dot{L}^q_U(\mathbb{R}^N)$$

for any $1 < q \le r \le \infty$ *and* μ_0 *and some* $M_{q,r} > 0$.

Then adding perturbations as above, we have

Theorem 6.0.11 *Let $a \in \mathbb{N}$, $a \leq 2m-1$ and $\|d\|_{\dot{L}^p_U(\mathbb{R}^N)} \leq R_0$ with $p > \frac{N}{2m-a}$, then for any $1 < q < \infty$ and any P_a as in (5.3.6) there exists an interval $I(q,a) \subset (-1+\frac{a}{2m}, 1)$ containing $(-1+\frac{a}{2m}+\frac{N}{2mp}, 1-\frac{N}{2mp})$, such that for any $\gamma \in I(q,a)$, we have a continuous, analytic semigroup, $S_{P_a}(t)$ in the space $\dot{H}^{2m\gamma,q}_U(\mathbb{R}^N)$, for the problem*

$$\begin{cases} u_t + (-\Delta)^m u + d(x) D^a u = 0, & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimate

$$\|S_{P_a}(t)u_0\|_{\dot{H}^{2m\gamma',q}_U(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{\dot{H}^{2m\gamma,q}_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{H}^{2m\gamma,q}_U(\mathbb{R}^N)$$

for every $\gamma, \gamma' \in I(q,a)$ with $\gamma' \geq \gamma$, and

$$\|S_{P_a}(t)u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \leq \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \quad t > 0, \quad u_0 \in \dot{L}^q_U(\mathbb{R}^N)$$

with $1 < q \leq r \leq \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

For each P_a, the interval $I(q,a)$ is given by

$$I(q,a) = (-1+\frac{a}{2m}+\frac{N}{2m}(\frac{1}{p}-\frac{1}{q'})_+, 1-\frac{N}{2m}(\frac{1}{p}-\frac{1}{q})_+) \subset (-1+\frac{a}{2m}, 1).$$

Finally, if as $\varepsilon \to 0$

$$d_\varepsilon \to d \quad \text{in } \dot{L}^p_U(\mathbb{R}^N), \quad p > \frac{N}{2m-k}$$

then for every $1 < q < \infty$ and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(\dot{H}^{2m\gamma,q}_U(\mathbb{R}^N), \dot{H}^{2m\gamma',q}_U(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \leq T$$

for all $\gamma, \gamma' \in I(q,a,b)$, $\gamma' \geq \gamma$ and

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(\dot{L}^q_U(\mathbb{R}^N), \dot{L}^r_U(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \leq T$$

for all $1 < q \leq r \leq \infty$.

The proofs of both Lemma 6.0.10 and Theorem 6.0.11 follow the proofs of Lemma 5.3.2 and Theorem 5.3.5, just replacing Δ^2 by $(-\Delta)^m$ as the order of the operator involved.

Chapter 7

Fourth order problems in bounded domains

In this chapter we study fourth order linear parabolic equations in bounded domains $\Omega \subset \mathbb{R}^N$. More precisely, we consider

$$\begin{cases} u_t + \Delta^2 u + Pu + Qu = 0, & x \in \Omega, t > 0 \\ \textit{boundary conditions} & \\ u(0) = u_0 & \text{in } \mathbb{R}^N \end{cases} \tag{7.0.1}$$

with u_0 a suitable initial data defined in Ω and P and Q linear perturbations. As in previous Chapters, we will consider space dependent perturbations in the interior of the domain of the form $Pu := \sum_{a,b} P_{a,b}u$ with

$$P_{a,b}u := D^b(d(x)D^a u) \qquad x \in \Omega$$

for some $a, b \in \{0, 1, 2, 3\}$ such that $a+b \leq 3$, where D^a, D^b denote any partial derivatives of order a, b, and $d(x)$ is a given function with $x \in \Omega$. Now, we can also consider space dependent perturbations in the boundary of the domain of the form $Qu := \sum_{c,d} Q_{c,d}u$ such that for any smooth φ

$$< Q_{c,d}u, \varphi >= (-1)^d \int_\Gamma \delta(x) D^c u D^d \varphi.$$

for some $c, d \in \{0, 1, 2\}$ such that $c + d \leq 2$ and $\delta(x)$ is a given function with $x \in \partial\Omega$.

We will consider in (7.0.1) some large classes of initial data u_0 in Ω and we will assume that the coefficients $d(x)$ and $\delta(x)$ belong to some locally uniform space $L^p_U(\Omega)$, $1 \leq p < \infty$, and $L^r_U(\partial\Omega)$, $1 < r < \infty$ respectively.

Similar problems have been covered for example in [47] but only for second order operators, so we now consider fourth order ones. As much Chapter 5 is a sort of continuation (into fourth order problems in $\mathbb{R}^N$) of the examples in that paper, this chapter is the natural continuation of the bounded examples in [47] but now, for fourth order operators.

The scale we consider is the Bessel scale of spaces with Neumann conditions, $H^{\alpha,q}_{\mathcal{N}}(\Omega)$, so we first describe it and study the action of the bi-Laplacian on it. For it, we study the elliptic problem and then use it for the parabolic one. An important result here is that of understanding the new class of boundary perturbations. We consider a weak formulation of the elliptic abstract problem (7.1.10) below, and depending on the choice of the perturbation and the space where it lays, we find the problem for which a u satisfying (7.1.10) is a solution.

In Section 7.1 we recall from [1] the construction of the Bessel scale of spaces with Neumann boundary conditions, $H^{\alpha,q}_{\mathcal{N}}(\Omega)$, $\alpha \in [-2,2]$ and extend it to a larger range of indexes, $\alpha \in [-4,4]$. We then use it to study the problem $\Delta^2 u + u = h$ for different ranges of spaces and choices of h, giving in Proposition 7.1.2 an interpretation for the abstract problem (7.1.10).

In Section 7.2 we introduce perturbations. Perturbations in the interior of the domain can be handled as in Chapter 5, so we focus on perturbations on the boundary. We also give a result for combining both perturbations in the interior and in the boundary at the same time.

7.1 Scale of Spaces with Neumann conditions

In this section we study the functional context in which we will work later on. In particular we are going to use the Bessel-Sobolev spaces with Neumann boundary conditions defined as in [1]. However, since we are going to use operators of order higher than 2 we need to extend the scale both in the positive and negative side. Once this is done, we study the action of the bi-Laplacian in this scale.

7.1.1 The scale of spaces for the Laplacian

We start recalling the definition of Bessel spaces with Neumann boundary condition $H^{\alpha,q}_{\mathcal{N}}(\Omega)$ from [1].

For $\alpha \in [0,2] \setminus \{\frac{1}{q}, 1+\frac{1}{q}\}$,

$$H^{\alpha,q}_{\mathcal{N}}(\Omega) = \begin{cases} H^{\alpha,q}(\Omega) & \alpha \in [0,\frac{1}{q}) \cup (\frac{1}{q}, 1+\frac{1}{q}) \\ \{u \in H^{\alpha,q}(\Omega), \frac{\partial u}{\partial \vec{n}} = 0\} & \alpha \in (1+\frac{1}{q}, 2] \end{cases}$$

This is what in [1, p. 34] is called $\mathcal{S}^s_{p,\mathcal{B}}$ with $\delta = 1$ and $c = 0$.

The construction of the scale in [1, p. 34] is abstract, thus we can consider $H^{\alpha,q}_{\mathcal{N}}(\Omega)$ for $\alpha \in [2,4] \setminus \{2+\frac{1}{q}, 3+\frac{1}{q}\}$, studying the elliptic problem for $-\Delta + I : H^{\alpha+2,q}_{\mathcal{N}}(\Omega) \to H^{\alpha,q}_{\mathcal{N}}(\Omega)$, $\alpha \in [0,2] \setminus \{\frac{1}{q}, 1+\frac{1}{q}\}$. A function v is in $H^{\alpha+2,q}_{\mathcal{N}}(\Omega)$ if $v \in H^{\alpha,q}_{\mathcal{N}}(\Omega)$ and satisfies

$$(-\Delta + I)v = f, \quad \frac{\partial v}{\partial \vec{n}} = 0$$

for $f \in H^{\alpha,q}_{\mathcal{N}}(\Omega)$.

Also, for all $\alpha > 0$

$$H^{-\alpha,q}_{\mathcal{N}}(\Omega) = (H^{\alpha,q'}_{\mathcal{N}}(\Omega))'.$$

Throughout the chapter we will say for simplicity that $\alpha \in [-4, 4]$. When we do it, the reader must understand that in fact we are referring to $[-4, 4] \setminus \Sigma_q$, where Σ_q is the set of exceptional points $\Sigma_q = \{-4 + \frac{1}{q}, -3 + \frac{1}{q}, -2 + \frac{1}{q}, -1 + \frac{1}{q}, \frac{1}{q}, 1 + \frac{1}{q}, 2 + \frac{1}{q}, 3 + \frac{1}{q}\}$.

We introduce the following notation for different ranges of α, we call $\mathbf{8}_q = \{4\}$, $\mathbf{7}_q = (3+\frac{1}{q}, 4)$, $\mathbf{6}_q = (2+\frac{1}{q}, 3+\frac{1}{q})$, $\mathbf{5}_q = (2, 2+\frac{1}{q})$, $\mathbf{4}_q = \{2\}$, $\mathbf{3}_q = (1+\frac{1}{q}, 2)$, $\mathbf{2}_q = (\frac{1}{q}, 1+\frac{1}{q})$, $\mathbf{1}_q = (0, \frac{1}{q})$, $\mathbf{0}_q = \{0\}$, $\mathbf{(-1)}_q = (-1 + \frac{1}{q}, 0)$, $\mathbf{(-2)}_q = (-2 + \frac{1}{q}, -1 + \frac{1}{q})$, $\mathbf{(-3)}_q = (-2, -2 + \frac{1}{q})$, $\mathbf{(-4)}_q = \{-2\}$, $\mathbf{(-5)}_q = (-3+\frac{1}{q}, -2)$, $\mathbf{(-6)}_q = (-4+\frac{1}{q}, -3+\frac{1}{q})$, $\mathbf{(-7)}_q = (-4, -4+\frac{1}{q})$, $\mathbf{(-8)}_q = \{-4\}$. Note that this notation satisfies for $\mathbf{J} = [0, 8] \cap \mathbb{Z}$ that

$$\boldsymbol{(J - 4)}_q = \boldsymbol{J}_q - 2 \qquad \textit{and} \qquad \boldsymbol{(-J)}_q = -(\boldsymbol{J}_{q'}). \tag{7.1.1}$$

Thus we can write

$$H^{\alpha,q}_{\mathcal{N}}(\Omega) := \begin{cases} \{u \in H^{\alpha,q}(\Omega), \frac{\partial \Delta u}{\partial \vec{n}} = 0, \frac{\partial u}{\partial \vec{n}} = 0\} & \alpha \in \mathbf{7}_q \cup \mathbf{8}_q \\ \{u \in H^{\alpha,q}(\Omega), \frac{\partial u}{\partial \vec{n}} = 0\} & \alpha \in \mathbf{3}_q \cup \mathbf{4}_q \cup \mathbf{5}_q \cup \mathbf{6}_q \\ H^{\alpha,q}(\Omega) & \alpha \in \mathbf{0}_q \cup \mathbf{1}_q \cup \mathbf{2}_q \\ (H^{-\alpha,q'}(\Omega))' & \alpha \in \mathbf{(-1)}_q \cup \mathbf{(-2)}_q \\ \{u \in H^{-\alpha,q'}(\Omega), \frac{\partial u}{\partial \vec{n}} = 0\}' & \alpha \in \mathbf{(-3)}_q \cup \mathbf{(-4)}_q \cup \mathbf{(-5)}_q \cup \mathbf{(-6)}_q \\ \{u \in H^{-\alpha,q'}(\Omega), \frac{\partial \Delta u}{\partial \vec{n}} = 0, \frac{\partial u}{\partial \vec{n}} = 0\}' & \alpha \in \mathbf{(-7)}_q \cup \mathbf{(-8)}_q \end{cases} \tag{7.1.2}$$

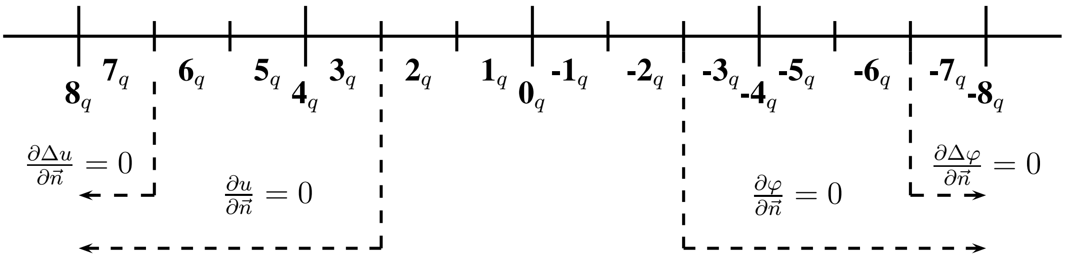

Figure 7.1: Scheme for spaces and boundary conditions

We will abuse of notation and say that

$$u \in \boldsymbol{J}_q \tag{7.1.3}$$

when $u \in H^{\alpha,q}_{\mathcal{N}}(\Omega)$ for $\alpha \in \boldsymbol{J}_q$. Also, we will sometimes consider test functions saying $\varphi \in \boldsymbol{J}_q$. That does not mean that we consider all $\varphi \in H^{\alpha,q}_{\mathcal{N}}(\Omega)$ for any $\alpha \in \boldsymbol{J}_q$. It means that we consider all $\varphi \in H^{\alpha,q}_{\mathcal{N}}(\Omega)$ for a particular $\alpha \in \boldsymbol{J}_q$.

As a direct consequence of the abstract theory developed by [1, pg. 39], the realization of the Laplacian in $H^{\alpha,q}_{\mathcal{N}}(\Omega)$, $\alpha \in [-4, 2]$ denoted by $-\Delta_\alpha$, satisfies

$$-\Delta_\alpha : H^{\alpha+2,q}_{\mathcal{N}}(\Omega) \to H^{\alpha,q}_{\mathcal{N}}(\Omega). \tag{7.1.4}$$

Note that $\sigma(-\Delta_\alpha)$ does not depend on α and $\inf\{\sigma(-\Delta_\alpha)\} = 0$.

In terms of our notation, when in (7.1.4) $\alpha + 2 \in \boldsymbol{J}_q$, then because of (7.1.1), $\alpha \in (\boldsymbol{J-4})_q$, hence we will write schematically (7.1.4) as

$$-\Delta_\alpha : \boldsymbol{J}_q \to (\boldsymbol{J-4})_q \tag{7.1.5}$$

where $\mathbf{J} \in [-4, 8] \cap \mathbb{Z}$.

With this notation, according to [1, p. 34], the realizations of the Laplacian in (7.1.4) are given in the notations (7.1.5) and (7.1.3) by

$$< -\Delta_\alpha u, \varphi >:= \begin{cases} < -\Delta u, \varphi > & \begin{cases} u \in \mathbf{8}_q & \varphi \in (\mathbf{-4})_{q'} \\ u \in \mathbf{7}_q & \varphi \in (\mathbf{-3})_{q'} \\ u \in \mathbf{6}_q & \varphi \in (\mathbf{-2})_{q'} \\ u \in \mathbf{5}_q & \varphi \in (\mathbf{-1})_{q'} \\ u \in \mathbf{4}_q & \varphi \in \mathbf{0}_{q'} \\ u \in \mathbf{3}_q & \varphi \in \mathbf{1}_{q'} \end{cases} \\ \\ < \nabla u, \nabla \varphi > & u \in \mathbf{2}_q \quad \varphi \in \mathbf{2}_{q'} \\ \\ < u, -\Delta \varphi > & \begin{cases} u \in \mathbf{1}_q & \varphi \in \mathbf{3}_{q'} \\ u \in \mathbf{0}_q & \varphi \in \mathbf{4}_{q'} \\ u \in (\mathbf{-1})_q & \varphi \in \mathbf{5}_{q'} \\ u \in (\mathbf{-2})_q & \varphi \in \mathbf{6}_{q'} \\ u \in (\mathbf{-3})_q & \varphi \in \mathbf{7}_{q'} \\ u \in (\mathbf{-4})_q & \varphi \in \mathbf{8}_{q'} \end{cases} \end{cases} \tag{7.1.6}$$

As a direct consequence of the results in [1, Section 10 and 11] we get

Proposition 7.1.1 *i) $-\Delta_\alpha$ in $H^{\alpha,q}_{\mathcal{N}}(\Omega)$ with domain $H^{\alpha+2,q}_{\mathcal{N}}(\Omega)$, $\alpha \in [-4, 2]$, generates an analytic semigroup $S_{-\Delta}(t)$.*

ii) For $1 < q < \infty$, the problem

$$\begin{cases} u_t - \Delta u = 0, \quad x \in \Omega,\ t > 0 \\ \dfrac{\partial u}{\partial \vec{n}} = 0 \\ u(0) = u_0 \end{cases}$$

has a unique solution $u(t) = S_{-\Delta}(t)u_0$ for $u_0 \in H^{\beta,q}_{\mathcal{N}}(\Omega)$, $\beta \in [-2, 2]$, where $S_{-\Delta}(t)$ is an analytic semigroup that satisfies the smoothing estimates

$$\|S_{-\Delta}(t)u_0\|_{H^{\alpha,q}_{\mathcal{N}}(\Omega)} \leq \frac{M_{\alpha,\beta}e^{\mu_0 t}}{t^{\alpha-\beta}}\|u_0\|_{H^{\beta,q}_{\mathcal{N}}(\Omega)}, \qquad t > 0, \quad u_0 \in H^{\beta,q}_{\mathcal{N}}(\Omega)$$

for $1 < q < \infty$, $\alpha, \beta \in [-2, 2]$, $\alpha \geq \beta$, *and*

$$\|S_{-\Delta}(t)u_0\|_{L^r(\Omega)} \leq \frac{M_{r,q}e^{\mu_0 t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\Omega)}, \qquad t > 0, \quad u_0 \in L^q(\mathbb{R}^N)$$

for $1 \leq q \leq r \leq \infty$ *and for any* $\mu_0 > 0$.

7.1.2 The scale of spaces for the bi-Laplacian

We now describe how the bi-Laplacian Δ^2_α acts on the scale $H^{\alpha,q}_{\mathcal{N}}(\Omega)$.

Let $\Delta^2_\alpha := (-\Delta)_\alpha \circ (-\Delta)_{\alpha+2}$ be

$$\Delta^2_\alpha : H^{\alpha+4,q}_{\mathcal{N}}(\Omega) \to H^{\alpha,q}_{\mathcal{N}}(\Omega) \qquad \alpha \in [-4, 0]. \tag{7.1.7}$$

As a direct consequence of Proposition 5.1.3 for the scale $X^\alpha = H^{4\alpha,q}_{\mathcal{N}}(\Omega)$, $-1 \leq \alpha \leq 1$, we get that Δ^2_α generates an analytic semigroup $S_{\Delta^2}(t)$.

Under our notation, as a consequence of (7.1.5) and (7.1.2), we write (7.1.7) as

$$\Delta^2_\alpha : \boldsymbol{J}_q \to \boldsymbol{(J-8)}_q \tag{7.1.8}$$

for $\mathbf{J} \in [0, 8] \cap \mathbb{Z}$. Depending on $\mathbf{J}$, and using (7.1.6), we get

$$< \Delta^2_\alpha u, \varphi > := \begin{cases} < \Delta^2 u, \varphi > & \begin{cases} u \in \mathbf{8}_q & \varphi \in \mathbf{0}_{q'} \\ u \in \mathbf{7}_q & \varphi \in \mathbf{1}_{q'} \end{cases} \\ < \nabla(-\Delta u), \nabla\varphi > & \quad u \in \mathbf{6}_q \quad \varphi \in \mathbf{2}_{q'} \\ < -\Delta u, -\Delta\varphi > & \begin{cases} u \in \mathbf{5}_q & \varphi \in \mathbf{3}_{q'} \\ u \in \mathbf{4}_q & \varphi \in \mathbf{4}_{q'} \\ u \in \mathbf{3}_q & \varphi \in \mathbf{5}_{q'} \end{cases} \\ < \nabla u, \nabla(-\Delta\varphi) > & \quad u \in \mathbf{2}_q \quad \varphi \in \mathbf{6}_{q'} \\ < u, \Delta^2\varphi > & \begin{cases} u \in \mathbf{1}_q & \varphi \in \mathbf{7}_{q'} \\ u \in \mathbf{0}_q & \varphi \in \mathbf{8}_{q'} \end{cases} \end{cases} \tag{7.1.9}$$

Note that since $\inf\{\sigma(-\Delta_\alpha)\} = 0$, then $\inf\{\sigma(\Delta^2_\alpha)\} = 0$, and thus $\Delta^2_\alpha + I$ is invertible. Thus we can consider the problem

$$\Delta^2_\alpha u + u = h$$

for $h \in \boldsymbol{(J-8)}_q$, for different $\mathbf{J} \in [0, 8] \cap \mathbb{Z}$.

In particular, for $u \in \boldsymbol{J}_q$, we have the problem

$$< \Delta_\alpha^2 u, \varphi > + < u, \varphi >=< h, \varphi > \tag{7.1.10}$$

for any φ in a space in $(\mathbf{8}-\boldsymbol{J})_{q'}$ and look at the different choices of h. We will consider combinations of

$$< h, \varphi >_\Omega = \int_\Omega h\varphi \quad and \quad < h, \varphi >_\Gamma = \int_\Gamma h\varphi$$

where $\Gamma = \partial\Omega$.

Proposition 7.1.2 *Let h satisfy one of the following*

i) $h \in \mathbf{0}_q \cup (\mathbf{-1})_q$

ii) $h \in (\mathbf{-2})_q \cup (\mathbf{-3})_q \cup (\mathbf{-4})_q \cup (\mathbf{-5})_q$ is such that $< h, \varphi >=< f, \varphi >_\Omega + < g_0, \varphi >_\Gamma$.

iii) $h \in (\mathbf{-6})_q \cup (\mathbf{-7})_q \cup (\mathbf{-8})_q$ is such that $< h, \varphi >=< f, \varphi >_\Omega + < g_0, \varphi >_\Gamma + < g_2, \Delta\varphi >_\Gamma$.

Then (7.1.10) is a weak formulation of the respective problems

a)

$$\begin{cases} \Delta^2 u + u = h, & in\ \Omega \\ \frac{\partial u}{\partial \vec{n}} = 0 & on\ \partial\Omega \\ \frac{\partial \Delta u}{\partial \vec{n}} = 0 & on\ \partial\Omega. \end{cases}$$

b)

$$\begin{cases} \Delta^2 u + u = f, & in\ \Omega \\ \frac{\partial u}{\partial \vec{n}} = 0 & on\ \partial\Omega \\ \frac{\partial \Delta u}{\partial \vec{n}} = -g_0 & on\ \partial\Omega. \end{cases}$$

c)

$$\begin{cases} \Delta^2 u + u = f, & in\ \Omega \\ \frac{\partial u}{\partial \vec{n}} = -g_2 & on\ \partial\Omega \\ \frac{\partial \Delta u}{\partial \vec{n}} = -g_0 & on\ \partial\Omega. \end{cases}$$

Proof.

- If $h \in \mathbf{0}_q \cup (\mathbf{-1})_q$ then from (7.1.7) and (7.1.8) $u \in \mathbf{8}_q \cup \mathbf{7}_q$ and $\varphi \in \mathbf{0}_{q'} \cup \mathbf{1}_{q'}$. Because of (7.1.2)

$$\frac{\partial u}{\partial \vec{n}} = 0 \quad and \quad \frac{\partial \Delta u}{\partial \vec{n}} = 0$$

and by (7.1.9) equation (7.1.10) can be read as

$$< \Delta^2 u, \varphi > + < u, \varphi >=< h, \varphi > .$$

In particular, if we choose $\varphi \in C_0^\infty(\Omega)$ then

$$\Delta^2 u + u = h \qquad in\ \mathcal{D}'(\Omega)$$

hence we get u satisfies a).

- If $h \in \mathbf{(-2)}_q$ then from (7.1.7) and (7.1.8) $u \in \mathbf{6}_q$ and $\varphi \in \mathbf{2}_{q'}$. Because of (7.1.2) we have that
$$\frac{\partial u}{\partial \vec{n}} = 0.$$
From (7.1.9) and the choice of h for this range (7.1.10) can be read as
$$< \nabla(-\Delta u), \nabla\varphi > + < u, \varphi >=< f, \varphi > + < g_0, \varphi >_\Gamma \qquad \varphi \in \mathbf{2}_{q'}. \tag{7.1.11}$$
Assume now that $\varphi \in C_0^\infty(\Omega)$ then (7.1.11) implies
$$\Delta^2 u + u = f \qquad in\ \mathcal{D}'(\Omega).$$
Assume $u \in H^4(\Omega)$ (see Lemma 7.1.6 below), then we can integrate by parts in (7.1.11)
$$\int_\Omega \Delta^2 u\varphi + \int_\Omega u\varphi - \int_\Gamma \frac{\partial \Delta u}{\partial \vec{n}}\varphi = \int_\Omega f\varphi + \int_\Gamma g_0\varphi \quad \varphi \in \mathbf{2}_{q'}.$$
Since $\Delta^2 u + u = f$ in Ω, we have that
$$-\int_\Gamma \frac{\partial \Delta u}{\partial \vec{n}}\varphi = \int_\Gamma g_0\varphi \qquad \varphi \in \mathbf{2}_{q'}$$
and by the density of traces of functions $H_{\mathcal{N}}^{-\alpha,q'}(\Omega)$ in $L^{q'}(\Omega)$ we get that $\frac{\partial \Delta u}{\partial \vec{n}} = -g_0$. Thus, u satisfies b).

- If $h \in \mathbf{(-3)}_q \cup \mathbf{(-4)}_q \cup \mathbf{(-5)}_q$ then from (7.1.7) and (7.1.8) $u \in \mathbf{5}_q \cup \mathbf{4}_q \cup \mathbf{3}_q$ and $\varphi \in \mathbf{3}_{q'} \cup \mathbf{4}_{q'} \cup \mathbf{5}_{q'}$ respectively. Because of (7.1.2) we have that
$$\frac{\partial u}{\partial \vec{n}} = 0 \qquad \frac{\partial \varphi}{\partial \vec{n}} = 0.$$
From (7.1.9) and the choice of h for this range (7.1.10) can be read as
$$< -\Delta u, -\Delta\varphi > + < u, \varphi >=< f, \varphi > + < g_0, \varphi >_\Gamma \qquad \varphi \in \mathbf{3}_{q'} \cup \mathbf{4}_{q'} \cup \mathbf{5}_{q'}. \tag{7.1.12}$$
Assume now that $\varphi \in C_0^\infty(\Omega)$ then (7.1.12) implies
$$\Delta^2 u + u = f \qquad in\ \mathcal{D}'(\Omega)$$
Assume $u \in H^4(\Omega)$ (see Lemma 7.1.6 below), then we can integrate by parts (7.1.12)
$$\int_\Omega \Delta^2 u\varphi + \int_\Omega u\varphi - \int_\Gamma \frac{\partial \Delta u}{\partial \vec{n}}\varphi = \int_\Omega f\varphi + \int_\Gamma g_0\varphi \quad \varphi \in \mathbf{3}_{q'} \cup \mathbf{4}_{q'} \cup \mathbf{5}_{q'}.$$
Since $\Delta^2 u + u = f$ in Ω, then we have that
$$-\int_\Gamma \frac{\partial \Delta u}{\partial \vec{n}}\varphi = \int_\Gamma g_0\varphi$$
and as before $\frac{\partial \Delta u}{\partial \vec{n}} = -g_0$. Thus, u satisfies b).

- If $h \in \mathbf{(-6)}_q$ then from (7.1.7) and (7.1.8) $u \in \mathbf{2}_q$ and $\varphi \in \mathbf{6}_{q'}$. Because of (7.1.2) we have that
$$\frac{\partial \varphi}{\partial \vec{n}} = 0.$$
From (7.1.9) and the choice of h for this range (7.1.10) can be read as
$$< \nabla u, \nabla(-\Delta\varphi) > + < u, \varphi > = < f, \varphi > + < g_0, \varphi >_\Gamma \qquad \varphi \in \mathbf{6}_{q'}. \qquad (7.1.13)$$
Assume now that $\varphi \in C_0^\infty(\Omega)$ then (7.1.13) implies
$$\Delta^2 u + u = f \qquad in\ \mathcal{D}'(\Omega)$$
Assume $u \in H^4(\Omega)$ (see Lemma 7.1.6 below), then we can integrate by parts (7.1.13)
$$\int_\Omega \Delta^2 u\varphi + \int_\Omega u\varphi - \int_\Gamma \frac{\partial \Delta u}{\partial \vec{n}}\varphi - \int_\Gamma \frac{\partial u}{\partial \vec{n}}\Delta\varphi = \int_\Omega f\varphi + \int_\Gamma g_1\varphi \quad \varphi \in \mathbf{6}_{q'}.$$
Since $\Delta^2 u + u = f$ in Ω, we have that
$$-\int_\Gamma \frac{\partial \Delta u}{\partial \vec{n}}\varphi - \int_\Gamma \frac{\partial u}{\partial \vec{n}}\Delta\varphi = \int_\Gamma g_0\varphi + \int_\Gamma g_2\Delta\varphi \qquad (7.1.14)$$
Take now $\varphi \in \mathcal{C} := \{\varphi \in C^\infty(\bar{\Omega}) : \frac{\partial \varphi}{\partial \vec{n}} = 0,\ -\Delta\varphi|_\Gamma = 0\}$, so (7.1.14) turns into
$$\int_\Gamma (g_0 + \frac{\partial \Delta u}{\partial \vec{n}})\varphi = 0 \quad \forall \varphi \in \mathcal{C}$$
Because of Lemma 7.1.3 i) below, we have $\frac{\partial \Delta u}{\partial \vec{n}} = -g_0$.
Using this in (7.1.14) we get that
$$-\int_\Gamma \frac{\partial u}{\partial \vec{n}}\Delta\varphi = \int_\Gamma g_2\Delta\varphi$$
Take now $\varphi \in \mathcal{C} := \{\varphi \in C^\infty(\bar{\Omega}) : \frac{\partial \varphi}{\partial \vec{n}} = 0\}$, and because of Lemma 7.1.3 ii) below, we have that $\frac{\partial u}{\partial \vec{n}} = -g_2$.
Hence, u satisfies c).

- If $h \in \mathbf{(-7)}_q \cup \mathbf{(-8)}_q$ then from (7.1.7) and (7.1.8) $u \in \mathbf{1}_q \cup \mathbf{0}_q$ and $\varphi \in \mathbf{7}_{q'} \cup \mathbf{8}_{q'}$ respectively. Because of (7.1.2) we have that
$$\frac{\partial \varphi}{\partial \vec{n}} = 0 \qquad \frac{\partial \Delta\varphi}{\partial \vec{n}} = 0.$$
From (7.1.9) and the choice of h for this range (7.1.10) can be read as
$$< u, \Delta^2\varphi > + < u, \varphi > = < f, \varphi > + < g_0, \varphi >_\Gamma + < g_2, \Delta\varphi >_\Gamma \qquad \varphi \in \mathbf{7}_{q'} \cup \mathbf{8}_{q'}. \qquad (7.1.15)$$

Assume now that $\varphi \in C_0^\infty(\Omega)$ then (7.1.15) implies

$$\Delta^2 u + u = f \qquad in\ \mathcal{D}'(\Omega)$$

Assume $u \in H^4(\Omega)$ (see Lemma 7.1.6 below), then we can integrate by parts (7.1.15)

$$\int_\Omega \Delta^2 u\varphi + \int_\Omega u\varphi - \int_\Gamma \frac{\partial \Delta u}{\partial \vec{n}}\varphi - \int_\Gamma \frac{\partial u}{\partial \vec{n}}\Delta\varphi = \int_\Omega f\varphi + \int_\Gamma g_0\varphi + \int_\Gamma g_2\Delta\varphi \quad \varphi \in \mathbf{7}_{q'} \cup \mathbf{8}_{q'}.$$

Since $\Delta^2 u + u = f$ in Ω, we have that

$$-\int_\Gamma \frac{\partial \Delta u}{\partial \vec{n}}\varphi - \int_\Gamma \frac{\partial u}{\partial \vec{n}}\Delta\varphi = \int_\Gamma g_0\varphi + \int_\Gamma g_2\Delta\varphi$$

As above, $\frac{\partial \Delta u}{\partial \vec{n}} = -g_0$ and $\frac{\partial u}{\partial \vec{n}} = -g_2$, thus u satisfies c).

$\blacksquare$

We now prove the lemma we have used above.

Lemma 7.1.3 *Let f be a function defined in Γ, the boundary of Ω, $f \in L^q(\Gamma)$, $1 < q < \infty$, then*

i) If $\displaystyle\int_\Gamma f\varphi = 0$ for all $\varphi \in \mathcal{C} := \{\varphi \in C^\infty(\bar{\Omega}) : \frac{\partial \varphi}{\partial \vec{n}} = 0,\ -\Delta\varphi|_\Gamma = 0\}$ then $f = 0$.

ii) If $\displaystyle\int_\Gamma f(-\Delta\varphi) = 0$ for all $\varphi \in \mathcal{C}' := \{\varphi \in C^\infty(\bar{\Omega}) : \frac{\partial \varphi}{\partial \vec{n}} = 0\}$ then $f = 0$.

Proof. i) Let $h \in C^\infty(\bar{\Omega})$ be such that $-\Delta h|_\Gamma = 0$ and $\int_\Omega h = 0$. Since $\int_\Omega h = 0$, let φ be the unique solution to the problem

$$\begin{cases} -\Delta\varphi = h \\ \frac{\partial \varphi}{\partial \vec{n}} = 0 \end{cases}$$

with $\int_\Omega \varphi = 0$. Then, φ chosen this way is in the class $\mathcal{C}$.

Now for $f \in L^q(\Gamma)$ note that when $\varphi = 1$, $\int_\Gamma f = 0$. Let ξ be the unique solution of the problem

$$\begin{cases} -\Delta\xi = 0 \\ \frac{\partial \xi}{\partial \vec{n}} = f \end{cases}$$

with $\int_\Gamma \xi = 0$. Thus, for all φ as above, by hypothesis we have that

$$\int_\Omega \nabla\xi\nabla\varphi = \int_\Gamma f\varphi = 0$$

so using the equation satisfied by φ and using ξ as a test function we get $\int_\Omega \nabla\varphi\nabla\xi = \int_\Omega \xi h$ and thus $\int_\Omega h\xi = 0$ for any h as above. We can take in particular $h \in C_0^\infty(\Omega)$ with $\int_\Omega h = 0$

and we get that ξ is a constant. This, together with $\int_\Omega \xi = 0$ leads to $\xi = 0$ in Ω and thus $f = 0$.

ii) We first claim that for all given $g \in C^\infty(\Gamma)$, we can choose φ such that $-\Delta\varphi = g$ on Γ and $\frac{\partial\varphi}{\partial\vec{n}} = 0$ on Γ. Assuming that for a moment, φ is in the class $\mathcal{C}'$, so we have $\int_\Gamma fg = 0$ for all $g \in C^\infty(\Gamma)$, and this implies that $f = 0$.

We now prove the claim. For it, take $g \in C^\infty(\bar\Omega)$ and $\eta \in C_0^\infty(\Omega)$ such that $\int_\Omega (g+\eta) = 0$, then the problem

$$\begin{cases} -\Delta\varphi = g + \eta & in\ \bar\Omega \\ \frac{\partial\varphi}{\partial\vec{n}} = 0 & on\ \Gamma \end{cases}$$

has a unique solution φ with $\int_\Omega \varphi = 0$. And in particular its restriction to the boundary does as well, and therefore the result is proved. ∎

Lemma 7.1.4 *Let $\Omega \subset \mathbb{R}^N$ and Γ smooth be its boundary.*

1. Let g, φ be smooth on Γ and $\tau \in \Xi(\Gamma)$ be a smooth tangent vector field of the tangent bundle. Then

$$\int_\Gamma g\partial_\tau\varphi dS = -\int_\Gamma (\partial_\tau g + div(\tau)g)\varphi dS.$$

2. Let g be smooth on Γ, φ be smooth on $\bar\Omega$. Take any derivative D_i, then there exist $\tau_i \in \Xi(\Gamma)$ a smooth tangent vector field and a smooth function $a_i(x)$ on Γ such that

$$\int_\Gamma gD_i\varphi dS = -\int_\Gamma (\partial_{\tau_i} g + div(\tau_i)g)\varphi dS + \int_\Gamma g a_i \frac{\partial\varphi}{\partial\vec{n}} dS.$$

3. Let g, φ be smooth on $\bar\Omega$. Take any derivative D_i, then there exist $\tau_i \in \Xi(\Gamma)$ a smooth tangent vector field, $a_i(x)$ and $b_i(x)$ smooth functions on Γ such that

$$\int_\Gamma gD_i\varphi dS = -\int_\Gamma (D_i g + div(\tau_i)g)\varphi dS + \int_\Gamma g a_i \frac{\partial\varphi}{\partial\vec{n}} dS + \int_\Gamma \frac{\partial g}{\partial\vec{n}} b_i\varphi dS.$$

Proof. 1. Let L_τ be the Lie derivative in the direction of the flux of τ. Then

$$L_\tau(g\varphi dS) = (L_\tau)g\varphi dS + g(L_\tau\varphi)dS + g\varphi(L_\tau dS).$$

On one hand, when f is a function, $L_\tau f = \partial_\tau f$, and for the volume form, $L_\tau dS = div(\tau)dS$, thus

$$L_\tau(g\varphi dS) = \partial_\tau g\varphi dS + g\partial_\tau\varphi dS + g\varphi div(\tau)dS. \tag{7.1.16}$$

On the other hand, we have Cartan's formula

$$L_\tau\omega = d(i_\tau\omega) + i_\tau d\omega$$

where ω is a $(k+1) - form$ and $(i_\tau\omega)(\cdot,\dots,\cdot) = \omega(\tau,\cdot,\dots,\cdot)$. Using it in our case

$$L_\tau(g\varphi dS) = d(i_\tau(g\varphi dS)) + i_\tau d(g\varphi dS) \tag{7.1.17}$$

But this last term is 0 because it is a N-form in a (N-1)-manifold.

Combining (7.1.16) and (7.1.17) and integrating we get

$$\int_\Gamma g\partial_\tau \varphi dS + \int_\Gamma \partial_\tau g\varphi dS + \int_\Gamma div(\tau)g\varphi dS = \int_\Gamma d(i_\tau(g\varphi dS))$$

But the right hand side is 0 because of the Stokes Theorem, see [51, Theorem 22.8] and the result is proved.

2 and 3. Recall now that there exist unique $a_i(x)$ and $\tau_i(x)$ such that $e_i = a_i(x)\vec{n}(x) + \tau_i(x)$ for all $x \in \Gamma$, thus $D_i\varphi =< \nabla\varphi, e_i >= a_i(x)\frac{\partial\varphi}{\partial\vec{n}} + \partial_{\tau_i}\varphi$ and we get

$$\int_\Gamma (gD_i\varphi)dS = \int_\Gamma (g\partial_{\tau_i}\varphi)dS + \int_\Gamma ga_i(x)\frac{\partial\varphi}{\partial\vec{n}}dS = -\int_\Gamma (\partial_{\tau_i}g + div(\tau_i)g)\varphi dS + \int_\Gamma ga_i(x)\frac{\partial\varphi}{\partial\vec{n}}dS$$

$$= -\int_\Gamma (D_ig + div(\tau_i)g)\varphi dS + \int_\Gamma ga_i(x)\frac{\partial\varphi}{\partial\vec{n}}dS + \int_\Gamma \frac{\partial g}{\partial\vec{n}}b_i(x)\varphi dS$$

■

Remark 7.1.5 *Note that in Proposition 7.1.2 for $\varphi \in \boldsymbol{J}_{q'}$, with $J \geq 3$, we could consider h such that $< h, \varphi >=< f, \varphi >_\Omega + < g_1, D\varphi >_\Gamma$. Since $\frac{\partial\varphi}{\partial\vec{n}} = 0$, $D\varphi = D_\tau\varphi$ and using Lemma 7.1.4, we have*

$$\int_\Gamma g_1 D\varphi = \int_\Gamma g_1 D_\tau\varphi = \int_\Gamma (\partial_\tau + div(\tau))g_1\varphi.$$

For this last result we have used the following lemma

Lemma 7.1.6 *For $f \in L^q(\Omega)$, $g_2 \in H_{\mathcal{N}}^{2+1/q,q}(\Gamma)$ and $g_0 \in H_{\mathcal{N}}^{1/q,q,q}(\Omega)_{\mathcal{N}}(\Gamma)$, if u is a solution to problem a), b) or c) in Proposition 7.1.2, then $u \in H^{4,q}(\Omega)$.*

Proof. Let u satisfy c) for $f \in L^q(\Omega)$, $g_2 \in H_{\mathcal{N}}^{2+1/q,q}(\Gamma)$ and $g_0 \in H_{\mathcal{N}}^{1/q,q,q}(\Omega)_{\mathcal{N}}(\Gamma)$. Then $u \in H^{4,q}_{loc}(\Omega)$.

Let $G \in H^4(\Omega)$ be such that $\frac{\partial G}{\partial\vec{n}} = g_2$ and $\frac{\partial\Delta G}{\partial\vec{n}} = g_0$. Then, taking $v = u - G$ we get

$$\begin{cases} \Delta^2 v + v = \tilde{f} \\ \frac{\partial u}{\partial\vec{n}} = 0 \\ \frac{\partial\Delta u}{\partial\vec{n}} = 0 \end{cases}$$

Thus, since $(\Delta^2 + I) : H_{\mathcal{N}}^{4,q}(\Omega) \to L^q(\Omega)$, $v \in H_{\mathcal{N}}^{4,q}(\Omega)$ and therefore $u \in H^{4,q}(\Omega)$.

If u satisfies a) or b) proceed in the same way. ■

We can now study the parabolic problem. For the homogeneous case, we have

Proposition 7.1.7 *The problem*

$$\begin{cases} u_t + \Delta^2 u = 0, & x \in \Omega \\ \frac{\partial u}{\partial \vec{n}} = 0 & x \in \partial\Omega \\ \frac{\partial \Delta u}{\partial \vec{n}} = 0 & x \in \partial\Omega \\ u(0) = u_0 \end{cases} \tag{7.1.18}$$

has a unique solution and it defines an analytic semigroup, $S_{\Delta^2}(t)$, on the scale $H^{4\alpha,q}_{\mathcal{N}}(\Omega)$, $1 < q < \infty$, for any $\alpha \in [-1,1]$ such that

$$\|S_{\Delta^2}(t)\|_{\mathcal{L}(H^{4\beta,q}_{\mathcal{N}}(\Omega), H^{4\alpha,q}_{\mathcal{N}}(\Omega))} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\mu t} \qquad t > 0,\ \alpha \geq \beta, \alpha, \beta \in [-1,1]$$

and also

$$\|S_{\Delta^2}(t)\|_{\mathcal{L}(L^q(\Omega), L^r(\Omega))} \leq \frac{C(q,r)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} e^{\mu t} \qquad t > 0,$$

for $1 < q \leq r \leq \infty$. The constant $C(\alpha-\beta)$ is bounded for α, β in bounded sets of $\mathbb{R}$.

Proof. Using Proposition 5.1.3 with $X^\alpha = H^{4\alpha,q}_{\mathcal{N}}(\Omega)$ and $A_0 = -\Delta$ proves immediately the result. It only remains to prove the smoothing estimate for the Lebesgue spaces, which follows from the Sobolev embeddings. ■

For the non-homogeneous case, we have

$$\begin{cases} u_t + \Delta^2 u = f, & x \in \Omega \\ \frac{\partial u}{\partial \vec{n}} = g_2 & \partial\Omega \\ \frac{\partial \Delta u}{\partial \vec{n}} = g_1 & \partial\Omega \\ u(t_0) = u_0. \end{cases} \tag{7.1.19}$$

for certain classes of g_1, g_2 and f.

Let $h \in L^1_{loc}((0,T], X^\gamma)$, where X^γ is a space in a scale. Then since Δ^2 generates an analytic semigroup, from [47] the problem

$$u_t + \Delta^2 u = h(t) \quad \textit{in } X^\gamma \quad \gamma \leq 0 \quad u(0) = u_0 \tag{7.1.20}$$

has a unique mild solution given by the variation of constants formula

$$u(t) = S(t)_{\Delta^2} u_0 + \int_0^t S_{\Delta^2}(t-\tau) h(\tau) d\tau$$

where $S_{\Delta^2}(t)$ is the semigroup from (7.1.18). In particular, if $h \in C^1((0,T), X^\gamma)$, then

$$u \in C([0,T], X^\gamma) \cap C^1((0,T), X^{\gamma+1})$$

and $u(t)$ is a strong solution and verifies (7.1.20) as

$$< u_t, \varphi > + < \Delta^2 u, \varphi > = < h(t), \varphi > \tag{7.1.21}$$

for $\varphi \in X^{-\gamma}$.

We use this result for $X^\gamma = H^{4\gamma,q}_{\mathcal{N}}(\Omega)$. Since $u = S_{\Delta^2}(t)u_0$ and $S_{\Delta^2}(t)$ is an analytic semigroup, $u_t \in L^q(\Omega)$. Thus, we can write $\Delta^2 u = h(t) - u_t$, and for each t we have an elliptic problem as above, thus we get the following result

Proposition 7.1.8 *Let h satisfy one of the following*

i) $h \in \mathbf{0}_q \cup \mathbf{(-1)}_q$

ii) $h \in \mathbf{(-2)}_q \cup \mathbf{(-3)}_q \cup \mathbf{(-4)}_q \cup \mathbf{(-5)}_q$ is such that $< h, \varphi > = < f, \varphi >_\Omega + < g_0, \varphi >_\Gamma$.

iii) $h \in \mathbf{(-6)}_q \cup \mathbf{(-7)}_q \cup \mathbf{(-8)}_q$ is such that $< h, \varphi > = < f, \varphi >_\Omega + < g_0, \varphi >_\Gamma + < g_2, \Delta\varphi >_\Gamma$.

Then, respectively, (7.1.21) is a weak formulation of the problems

a)

$$\begin{cases} u_t + \Delta^2 u = h, & in\ \Omega \\ \frac{\partial u}{\partial \vec{n}} = 0 & on\ \partial\Omega \\ \frac{\partial \Delta u}{\partial \vec{n}} = 0 & on\ \partial\Omega. \end{cases}$$

b)

$$\begin{cases} u_t + \Delta^2 u = f, & in\ \Omega \\ \frac{\partial u}{\partial \vec{n}} = 0 & on\ \partial\Omega \\ \frac{\partial \Delta u}{\partial \vec{n}} = -g_0 & on\ \partial\Omega. \end{cases}$$

c)

$$\begin{cases} u_t + \Delta^2 u = f, & in\ \Omega \\ \frac{\partial u}{\partial \vec{n}} = -g_2 & on\ \partial\Omega \\ \frac{\partial \Delta u}{\partial \vec{n}} = -g_0 & on\ \partial\Omega. \end{cases}$$

Now because of [47, Theorem 4 ii)] we have

Proposition 7.1.9 *Let g_1, g_2 be as above, and $f \in L^\sigma((0,T), H^{4\alpha,q}_{\mathcal{N}}(\Omega))$. Then, the problem (7.1.19) has a solution $u \in H^{4\beta,q}_{\mathcal{N}}(\Omega)$ for $\beta \in [\alpha, \alpha + \frac{\sigma-1}{\sigma})$*

Moreover,

$$\|u\|_{C((0,T],H^{4\beta,q}_{\mathcal{N}}(\Omega))} \leq C(\|u_0\|_{H^{\beta,q}_{\mathcal{N}}(\Omega)} + \|h\|_{L^\sigma((0,T),H^{4\alpha,q}_{\mathcal{N}}(\Omega))})$$

7.2 Perturbed parabolic problems

7.2.1 Perturbations in the interior of the domain

We now add perturbations to the problem in the interior of the domain. We can proceed as in Chapter 5 because the fact that the domain is bounded does not affect the proof in any way, so we obtain analogous results.

Let

$$< P_{a,b}u, \varphi > = (-1)^b \int_\Omega d(x) D^a u D^b \varphi, \tag{7.2.1}$$

then we get

Theorem 7.2.1 *Let $P_{a,b}$ be as in (7.2.1) with $k,a,b \in \{0,1,2,3\}$, $k=a+b$. Assume that $\|d\|_{L^p(\Omega)} \leq R_0$ with $p > \frac{N}{4-k}$, then for any $1<q<\infty$ and such $P_{a,b}$ there exists an interval $I(q,a,b) \subset (-1+\frac{a}{4}, 1-\frac{b}{4})$ containing $(-1+\frac{a}{4}+\frac{N}{4p}, 1-\frac{b}{4}-\frac{N}{4p})$, such that for any $\gamma \in I(q,a,b)$, we have a strongly continuous, analytic semigroup, $S_{P_{a,b}}(t)$ in the space $H^{4\gamma,q}(\Omega)$, for the problem*

$$\begin{cases} u_t + \Delta^2 u + D^b(d(x)D^a u) = 0, & x \in \Omega,\ t>0 \\ u(0) = u_0 & \text{in } \Omega. \end{cases}$$

Moreover the semigroup has the smoothing estimates

$$\|S_{P_{a,b}}(t)u_0\|_{H^{4\gamma',q}(\Omega)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{4\gamma,q}(\Omega)}, \quad t>0,\ u_0 \in H^{4\gamma,q}(\Omega)$$

for every $\gamma,\gamma' \in I(q,a,b)$ with $\gamma' \geq \gamma$, and

$$\|S_{P_{a,b}}(t)u_0\|_{L^r(\Omega)} \leq \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\Omega)}, \quad t>0,\ u_0 \in L^q(\Omega)$$

with $1<q\leq r\leq\infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q,a,b)$ is given by

$$I(q,a,b) = (-1+\frac{a}{4}+\frac{N}{4}(\frac{1}{p}-\frac{1}{q'})_+, 1-\frac{b}{4}-\frac{N}{4}(\frac{1}{p}-\frac{1}{q})_+).$$

Finally, if

$$d_\varepsilon \to d \quad \text{in } L^p(\Omega), \quad p > \frac{N}{4-k}$$

then for every $1<q<\infty$ and $T>0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(H^{4\gamma,q}(\Omega),H^{4\gamma',q}(\Omega))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0<t\leq T$$

for all $\gamma,\gamma' \in I(q,a,b)$, $\gamma' \geq \gamma$ and for any $1<q\leq r\leq\infty$

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(L^q(\Omega),L^r(\Omega))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0<t\leq T.$$

Remark 7.2.2 *Now we make precise in what sense the equation from Theorem 7.2.1 is satisfied.*
i) First note that since $p > \frac{N}{4-\kappa}$ we have $4\gamma_{max} > 4-b-\frac{N}{p} > a$, and $4\gamma_{min} < -4+a+\frac{N}{p} < -b$.
ii) Because of the analyticity of the semigroup, and as in [47, Remark 6], the equation $u_t+\Delta^2 u = Pu$ is satisfied in $H^{-b,q}_{\mathcal{N}}(\Omega)$.

Therefore, we have that $u(t) \in H^{4-b,q}_{\mathcal{N}}(\Omega)$*, for all* $t > 0$*, that is,* $\gamma^* = 1 - \frac{b}{4} \geq \gamma_{max}$*. Note that the estimate of* $u(t)$ *is not obtained in the space* $H^{4-b,q}_{\mathcal{N}}(\Omega)$*.*

Also, since the semigroup is analytic in $H^{4\gamma,q}_{\mathcal{N}}(\Omega)$*,* $u_t(t) \in H^{4\gamma,q}_{\mathcal{N}}(\Omega)$ *for all* $\gamma \in I(q,a,b)$ *and* $t > 0$*.*

iii) In particular the equation (7.2.4) is always satisfied as

$$\int_\Omega u_t\varphi + \int_\Omega u\Delta^2\varphi + \int_\Omega d(x)D^a uD^b\varphi = 0, \qquad t > 0$$

for any $\varphi \in H^{4,q'}_{\mathcal{N}}(\Omega)$*.*

- *For* $b = 0$*,* $a \leq 3$*, we have* $\gamma^* = 1$*, that is* $u(t) \in H^{4,q}_{\mathcal{N}}(\Omega)$*, so the problem can be regarded as*

$$\begin{cases} u_t + \Delta^2 u + d(x)D^a u = 0 \quad x \in \Omega \\ \frac{\partial u}{\partial \vec{n}} = 0 \\ \frac{\partial \Delta u}{\partial \vec{n}} = 0 \\ u(t_0) = u_0 \end{cases}$$

- *For* $b = 1$*,* $a \leq 2$*, we have* $\gamma^* \geq \frac{3}{4}$*, that is* $u(t) \in H^{3,q}_{\mathcal{N}}(\Omega)$*. Because of the Divergence Theorem we get that for some* $n_1(x)$

$$\int_\Omega d(x)D^a uD\varphi = -\int_\Omega D(d(x)D^a u)\varphi + \int_\Gamma d(x)n_1(x)D^a u\varphi$$

so, according to Proposition 7.1.2 ii), the problem can be regarded as

$$\begin{cases} u_t + \Delta^2 u - D(d(x)D^a u) = 0 \quad x \in \Omega \\ \frac{\partial u}{\partial \vec{n}} = 0 \\ \frac{\partial \Delta u}{\partial \vec{n}} = -d(x)n_1(x)D^a u \\ u(t_0) = u_0 \end{cases}$$

- *For* $b = 2$*,* $a \leq 1$*, we have* $\gamma^* \geq \frac{1}{2}$*, that is* $u(t) \in H^{2,q}_{\mathcal{N}}(\Omega)$*. We could use twice the Divergence Theorem to get for some* $n_1(x)$ *and* $n_2(x)$

$$\begin{aligned} \int_\Omega d(x)D^a uD^2\varphi &= \int_\Omega D(d(x)D^a u)D\varphi + \int_\Gamma d(x)n_1(x)D^a uD\varphi \\ &= \int_\Omega D^2(d(x)D^a u)\varphi + \int_\Gamma n_2(x)D(d(x)D^a u)\varphi + \int_\Gamma d(x)n_1(x)D^a uD\varphi \end{aligned}$$

but this yields no easy interpretation, so for the shake of simplicity we focus on the particular case where

$$< P_{a,b}u, \varphi >=< d(x)D^a u, \Delta\varphi > .$$

Then, because of the Divergence Theorem we have

$$\int_\Omega d(x)D^a u\Delta\varphi = \int_\Omega \Delta(d(x)D^a u)\varphi - \int_\Gamma \frac{\partial d(x)D^a u}{\partial \vec{n}}\varphi$$

and according to Proposition 7.1.2 ii) the problem can be regarded as

$$\begin{cases} u_t + \Delta^2 u + \Delta(d(x)D^a u) = 0 \quad x \in \Omega \\ \frac{\partial u}{\partial \vec{n}} = 0 \\ \frac{\partial \Delta u}{\partial \vec{n}} = -\frac{\partial d(x)D^a u}{\partial \vec{n}} \\ u(t_0) = u_0 \end{cases}$$

- *For $b = 3$, $a = 0$, we have $\gamma^* \geq \frac{1}{4}$, that is $u(t) \in H^{1,q}_{\mathcal{N}}(\Omega)$. Again, we could use the Divergence Theorem thrice but for simplicity we consider the particular case when*

$$< P_{a,b}u, \varphi >=< d(x)u, D(\Delta\varphi) > .$$

Then, because of the Divergence Theorem we have we get that for some $n_1(x)$

$$\begin{aligned} \int_\Omega d(x)uD(\Delta\varphi) &= -\int_\Omega D(d(x)u)\Delta\varphi + \int_\Gamma d(x)n_1(x)u\Delta\varphi \\ &= -\int_\Omega \Delta(Dd(x)u)\varphi + \int_\Gamma \frac{\partial Dd(x)u}{\partial \vec{n}}\varphi + \int_\Gamma d(x)n_1(x)u\Delta\varphi \end{aligned}$$

and according to Proposition 7.1.2, the problem can be regarded as

$$\begin{cases} u_t + \Delta^2 u - \Delta D(d(x)u) = 0 \quad x \in \Omega \\ \frac{\partial u}{\partial \vec{n}} = -n_1(x)d(x)u \\ \frac{\partial \Delta u}{\partial \vec{n}} = -\frac{\partial Dd(x)u}{\partial \vec{n}} \\ u(t_0) = u_0 \end{cases}$$

But since we are now in a bounded domain, we can consider perturbations in the boundary, as it was done in [47] for the Laplacian.

7.2.2 Perturbations in the boundary

We study now perturbations in the boundary. For this, take $\kappa \in \mathbb{N}$ which is the order of the perturbation and take $c, d \in \mathbb{N}$ such that $\kappa = c + d$. We define $Q_{c,d}u$ to be a perturbation such that for a given $\delta(x)$ and any smooth function φ, satisfies

$$< Q_{c,d}u, \varphi >= (-1)^d \int_\Gamma \delta(x)D^c u D^d \varphi. \tag{7.2.2}$$

Note that R_0 in [47, p. 27] is $Q_{0,0}$ in this notation.

We are going to proceed as above. Similarly to Proposition 5.2.9 we get

Proposition 7.2.3 *Let $Q_{c,d}$ be as above, $\delta \in L^r(\Gamma)$ and let $s \geq c + \frac{1}{q}$, $\sigma \geq d + \frac{1}{q'}$. Then for $1 < q < \infty$ and*

$$(s - c - \frac{N}{q})_- + (\sigma - d - \frac{N}{q'})_- \geq -\frac{N-1}{r'} \tag{7.2.3}$$

we have

$$Q_{c,d} \in \mathcal{L}(H^{s,q}_{\mathcal{N}}(\Omega), H^{-\sigma,q}_{\mathcal{N}}(\Omega)), \qquad \|Q_{c,d}\|_{\mathcal{L}(H^{s,q}_{\mathcal{N}}(\Omega), H^{-\sigma,q}_{\mathcal{N}}(\Omega))} \leq C\|\delta\|_{L^r(\Gamma)}.$$

Proof. Note that for every $u \in H^{s,q}_{\mathcal{N}}(\Omega)$ and $u \in H^{\sigma,q'}_{\mathcal{N}}(\Omega)$, we have:

$$|\int_\Gamma dD^c u D^d \varphi| \leq (\int_\Gamma |\delta|^r)^{\frac{1}{r}} (\int_\Gamma |D^c u|^n)^{\frac{1}{n}} (\int_\Gamma |D^d \varphi|^\tau)^{\frac{1}{\tau}}$$

where we have applied Hölder's inequality with $\frac{1}{p} + \frac{1}{n} + \frac{1}{\tau} = 1$. If (7.2.3) holds, we can choose n, τ as before such that $s - \frac{N}{q} \geq c - \frac{N-1}{n}$ and $\sigma - \frac{N}{q'} \geq d - \frac{N-1}{\tau}$. Now, we can use the trace properties and the embeddings of Bessel spaces to obtain

$$|\int_\Gamma dD^c u D^d \varphi| \leq C\|\delta\|_{L^r(\Gamma)} \|u\|_{H^{s,q}_{\mathcal{N}}(\Omega)} \|\varphi\|_{H^{\sigma,q'}_{\mathcal{N}}(\Omega)}$$

which gives the result. ■

Theorem 7.2.4 *Let $Q_{c,d}$ be as in (7.2.2) with $\kappa, c, d \in \{0, 1, 2\}$, $\kappa = c + d$. Assume that $\|\delta\|_{L^r(\Gamma)} \leq R_0$ with $r > \frac{N-1}{3-\kappa}$, then for any $1 < q < \infty$ and such $Q_{c,d}$ there exists an interval $I(q, c, d) \subset (-1 + \frac{c}{4}, 1 - \frac{c}{4})$ containing $(-\frac{3}{4} + \frac{c}{4} + \frac{N-1}{4r}, \frac{3}{4} - \frac{d}{4} - \frac{N-1}{4r})$, such that for any $\gamma \in I(q, c, d)$, we have a strongly continuous, analytic semigroup, $S_{Q_{a,b}}(t)$ in the space $H^{4\gamma,q}_{\mathcal{N}}(\Omega)$, for the problem*

$$< u_t, \varphi >_\Omega + < \Delta^2_\alpha u, \varphi >_\Omega = < Q_{c,d} u, \varphi >_\Gamma \tag{7.2.4}$$

Moreover the semigroup has the smoothing estimates

$$\|S_{Q_{c,d}}(t) u_0\|_{H^{4\gamma',q}_{\mathcal{N}}(\Omega)} \leq \frac{M_{\gamma',\gamma} e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{H^{4\gamma,q}_{\mathcal{N}}(\Omega)}, \quad t > 0,\ u_0 \in H^{4\gamma,q}_{\mathcal{N}}(\Omega)$$

for every $\gamma, \gamma' \in I(q, c, d)$ with $\gamma' \geq \gamma$, and

$$\|S_{Q_{c,d}}(t) u_0\|_{L^r(\Omega)} \leq \frac{M_{q,r} e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q} - \frac{1}{r})}} \|u_0\|_{L^q(\Omega)}, \quad t > 0,\ u_0 \in L^q(\Omega)$$

with $1 < q \leq r \leq \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on δ only through R_0.

Furthermore, the interval $I(q, c, d)$ is given by

$$I(q, c, d) = (-1 + \frac{c}{4} + \frac{1}{4q} + \frac{N-1}{4}(\frac{1}{r} - \frac{1}{q'})_+, 1 - \frac{d}{4} - \frac{1}{4q'} - \frac{N-1}{4}(\frac{1}{r} - \frac{1}{q})_+).$$

Finally, if

$$\delta_\varepsilon \to \delta \quad in\ L^r(\Gamma), \quad r > \frac{N-1}{3-\kappa}$$

then for every $1 < q < \infty$ *and* $T > 0$ *there exists* $C(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*, such that*

$$\|S_{Q_\varepsilon}(t) - S_Q(t)\|_{\mathcal{L}(H^{4\gamma,q}_{\mathcal{N}}(\Omega), H^{4\gamma',q}_{\mathcal{N}}(\Omega))} \le \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \le T$$

for all $\gamma, \gamma' \in I(q, c, d)$*,* $\gamma \ge \gamma'$ *and for any* $1 < q \le r \le \infty$

$$\|S_{Q_\varepsilon}(t) - S_Q(t)\|_{\mathcal{L}(L^q(\Omega), L^\rho(\Omega))} \le \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{\rho})}}, \qquad \forall\, 0 < t \le T.$$

Proof. By Proposition 7.2.3, if we assume for a moment that (7.2.3) is satisfied for some s and σ, then it would be true that

$$Q \in \mathcal{L}(H^{s,q}_{\mathcal{N}}(\Omega), H^{-\sigma,q}_{\mathcal{N}}(\Omega)), \qquad \|Q\|_{\mathcal{L}(H^{s,q}_{\mathcal{N}}(\Omega), H^{-\sigma,q}_{\mathcal{N}}(\Omega))} \le C\|d\|_{L^r(\Gamma)}.$$

Hence we can apply [47, Theorem 14] (see Theorem 1.0.2) with $\alpha = s/4$ and $\beta = \sigma/4$ provided $0 \le \alpha - \beta < 1$, that is $s + \sigma < 4$.

Thus, if we check that (7.2.3) and $s + \sigma < 4$ hold for suitable pairs (s, σ) the result will be proved. For this we rewrite the ranges for s, σ in Proposition 7.2.3 in terms of $\tilde{s} = s - c - \frac{N}{q}$ and $\tilde{\sigma} = \sigma - d - \frac{N}{q'}$, so $\tilde{s} \ge -\frac{N-1}{q}, \tilde{\sigma} \ge -\frac{N-1}{q'}$ since $s \ge c + \frac{1}{q}, \sigma \ge d + \frac{1}{q'}$. Then (7.2.3) and $s + \sigma < 4$ read

$$\tilde{s} \ge -\frac{N-1}{q},\ \tilde{\sigma} \ge -\frac{N-1}{q'}, \qquad -\frac{N-1}{r'} \le \tilde{s}_- + \tilde{\sigma}_-, \qquad \tilde{s} + \tilde{\sigma} < 4 - \kappa - N. \quad (7.2.5)$$

Note that since necessarily $-\frac{N-1}{r'} < 4 - k - N$, we get that $r > \frac{N-1}{3-k}$.

The set of admissible parameters $(\tilde{s}, \tilde{\sigma})$ given by (7.2.5) depends on the relationship between q, q' and p. Note that (7.2.5) defines a planar trapezium–shaped polygon, $\tilde{\mathcal{P}}$, whose long base is on the line $\tilde{s} + \tilde{\sigma} = 4 - \kappa - N$ and the short base is on the line $\tilde{s} + \tilde{\sigma} = -\frac{N-1}{r'}$ in the third quadrant. As for the lateral sides note that the restriction $-\frac{N-1}{r'} \le \tilde{s}_- + \tilde{\sigma}_-$ adds the condition that $\tilde{s} \ge -\frac{N-1}{r'}$ in the second quadrant and $\tilde{\sigma} \ge -\frac{N-1}{r'}$ in the fourth. These have to be combined with $\tilde{s} \ge -\frac{N-1}{q}$ and $\tilde{\sigma} \ge -\frac{N-1}{q'}$. Therefore the lateral sides are given by the lines $\tilde{s} = \max\{-\frac{N-1}{r'}, -\frac{N-1}{q}\}$ and $\tilde{\sigma} = \max\{-\frac{N-1}{r'}, -\frac{N-1}{q'}\}$. One of the possible cases is depicted in Figure 7.2.

Note that the polygon $\tilde{\mathcal{P}}$ transforms into a similar shaped polygon $\mathcal{P}$ which determines the region of admissible pairs (s, σ).

In any case, projecting $\tilde{\mathcal{P}}$ onto the axes gives the following ranges for $\tilde{s}$ and $\tilde{\sigma}$

$$\tilde{s} \in [\max\{-\frac{N-1}{r'}, -\frac{N-1}{q}\}, 4 - \kappa - N - \max\{-\frac{N-1}{r'}, -\frac{N-1}{q'}\})$$

$$\tilde{\sigma} \in [\max\{-\frac{N-1}{r'}, -\frac{N-1}{q'}\}, 4 - \kappa - N - \max\{-\frac{N-1}{r'}, -\frac{N-1}{q}\}).$$

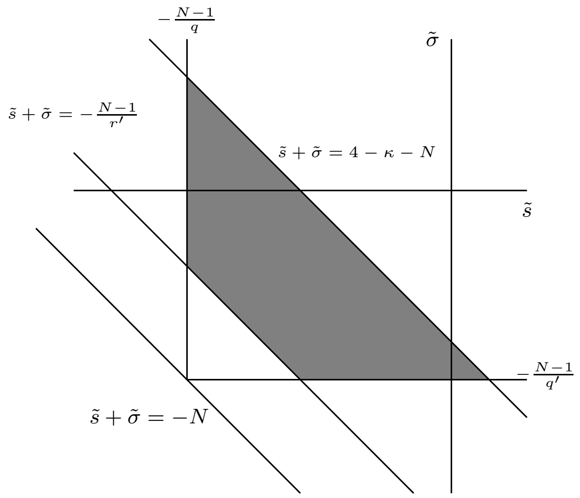

Figure 7.2: Admissible $\tilde{s}$ and $\tilde{\sigma}$ with $r > q, q'$

Thus

$$s \in J_1 = [c + \frac{1}{q} + (\frac{N-1}{q} - \frac{N-1}{r'})_+, 4 - d - \frac{1}{q'} - (\frac{N-1}{q'} - \frac{N-1}{r'})_+) \tag{7.2.6}$$

$$\sigma \in J_2 = [d + \frac{1}{q'} + (\frac{N-1}{q'} - \frac{N-1}{r'})_+, 4 - c - \frac{1}{q} - (\frac{N-1}{q} - \frac{N-1}{r'})_+). \tag{7.2.7}$$

For each pair of admissible pairs $(s, \sigma) \in \mathcal{P}$, by [47, Theorem 14] (see Theorem 1.0.2) with $\alpha = \frac{s}{4}$ and $\beta = \frac{\sigma}{4}$, we get a perturbed semigroup and smoothing estimates

$$\|S_Q(t)u_0\|_{\gamma'} \leq M_0 e^{\omega t} t^{-(\gamma'-\gamma)} \|u_0\|_\gamma, \quad \gamma' \geq \gamma$$

in the spaces corresponding to γ and γ', that is

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta + 1), \qquad \gamma' \geq \gamma.$$

Hence as (s, σ) range in the region $\mathcal{P}$ a repeated bootstrap argument as in (5.2.8) gives that the smoothing estimates hold for $\gamma \in \bigcup_{(s,\sigma)\in\mathcal{P}} E(s/4)$ and $\gamma' \in \bigcup_{(s,\sigma)\in\mathcal{P}} R(\sigma/4)$, $\gamma' \geq \gamma$. This leads to

$$\gamma \in (\frac{\inf J_1}{4} - 1, \frac{\sup J_1}{4}], \qquad \gamma' \in [-\frac{\sup J_2}{4}, 1 - \frac{\inf J_2}{4}), \gamma' \geq \gamma$$

which, after a simple calculation, reads

$$\gamma, \gamma' \in I(q, c, d) = (-1 + \frac{c}{4} + \frac{1}{4q} + \frac{N-1}{4}(\frac{1}{q} - \frac{1}{r'})_+, 1 - \frac{d}{4} - \frac{1}{4q'} - \frac{N-1}{4}(\frac{1}{q'} - \frac{1}{r'})_+).$$

For the estimates in Lebesgue spaces we use the Sobolev inclusions. Taking $1 < q < \infty$, $\gamma = 0$ and $0 < \gamma' \in I(q, c, d)$ we define $\rho > q$ such that $H^{4\gamma',q}_{\mathcal{N}}(\Omega) \hookrightarrow L^\rho(\Omega)$, that is $-\frac{N}{\rho} = 4\gamma' - \frac{N}{q}$. Then we get

$$\|S_Q(t)u_0\|_{L^\rho(\Omega)} \leq \|S_Q(t)u_0\|_{H^{4\gamma',q}_{\mathcal{N}}(\Omega)} \leq \frac{M_{\gamma'} e^{\mu t}}{t^{\gamma'}} \|u_0\|_{L^q(\Omega)}$$

and $\gamma' = \frac{N}{4}(\frac{1}{q} - \frac{1}{\rho})$. Now we follow a bootstrap argument as in (5.2.8) where we take $S_Q(t/2)u_0$ as initial data in $L^\rho(\Omega)$, repeat the argument above to estimate $S_Q(t)u_0$ in $L^{\tilde\rho}(\Omega)$ for $\tilde\rho > \rho > q$. Since $1 - \frac{1}{4q} = \frac{3}{4} + \frac{1}{4q'}$, the intervals $I(\rho, c, d)$ always contain $(-\frac{3}{4} - \frac{1}{q'} + \frac{c}{4} + \frac{N-1}{4r}, \frac{3}{4} + \frac{1}{q} - \frac{d}{4} - \frac{N-1}{4r})$ which do not depend on ρ, repeating the jump process several times we can get the estimate for any $\tilde\rho \geq q$.

The convergence of the semigroups is a direct consequence of [47, Theorem 14], since Proposition 7.2.3 gives that if $\delta_\varepsilon \to \delta$ in $L^r(\Gamma)$, then $Q_\varepsilon \to Q$ in $\mathcal{L}(H^{s,q}_{\mathcal{N}}(\Omega), H^{-\sigma,q}_{\mathcal{N}}(\Omega))$ for any pair of admissible $(s, \sigma) \in \mathcal{P}$. The case of Lebesgue spaces follows from this as well.

Finally, the analyticity comes from [47, Theorem 12], see Theorem 1.0.3. ■

Remark 7.2.5 *Now we make precise in what sense equation (7.2.4) is satisfied.*
i) First note that since $r > \frac{N-1}{3-\kappa}$ we have $4\gamma_{max} > 3 + \frac{1}{q} - d - \frac{N-1}{r} > c + \frac{1}{q}$, and $4\gamma_{min} < -3 - \frac{1}{q'} + c + \frac{N-1}{r} < -d - \frac{1}{q'}$.
ii) Because of the analyticity of the semigroup, and as in [47, Remark 6], the equation $u_t + \Delta^2 u = Qu$ is satisfied in $H^{-d-\frac{1}{q'},q}_{\mathcal{N}}(\Omega)$.

Therefore, we have that $u(t) \in H^{4-d-\frac{1}{q'},q}_{\mathcal{N}}(\Omega)$, for all $t > 0$, that is, $\gamma^* = 1 - \frac{d}{4} - \frac{1}{4q'} \geq \gamma_{max}$. Note that the estimate of $u(t)$ is not obtained in the space $H^{4-d-\frac{1}{q'},q}_{\mathcal{N}}(\Omega)$.

Also, since the semigroup is analytic in $H^{4\gamma,q}_{\mathcal{N}}(\Omega)$, $u_t(t) \in H^{4\gamma,q}_{\mathcal{N}}(\Omega)$ for all $\gamma \in I(q, a, b)$ and $t > 0$.
iii) In particular the equation (7.2.4) is always satisfied as

$$< u_t, \varphi > + < \Delta^2 u, \varphi > + < d(x) D^c u, D^d \varphi >_\Gamma = 0, \qquad t > 0$$

for any $\varphi \in H^{4,q'}_{\mathcal{N}}(\Omega)$.

- *For $d = 0$, $c \leq 2$, we have $\gamma^* = 1 - \frac{1}{4q'}$, that is $u(t) \in \mathbf{6}_q$, so according to Proposition 7.1.2 the problem can be regarded as*

$$\begin{cases} u_t + \Delta^2 u = 0 \quad x \in \Omega \\ \frac{\partial u}{\partial \vec n} = 0 \\ \frac{\partial \Delta u}{\partial \vec n} = d(x) D^c u \\ u(t_0) = u_0 \end{cases}$$

- *For $d = 1$, $c \leq 1$, we have $\gamma^* \geq \frac{3}{4} - \frac{1}{4q'}$, that is $u(t) \in \mathbf{5}_q \cup \mathbf{4}_q \cup \mathbf{3}_q$, and therefore according to Proposition 7.1.2 and Lemma 7.1.4 the problem can be regarded as*

$$\begin{cases} u_t + \Delta^2 u = 0 \quad x \in \Omega \\ \frac{\partial u}{\partial \vec n} = 0 \\ \frac{\partial \Delta u}{\partial \vec n} = (\partial_\tau + div(\tau)) d(x) D^c u \\ u(t_0) = u_0 \end{cases}$$

- *Finally, for* $d=2$, $c=0$, *we have* $\gamma^* \geq \frac{1}{2} - \frac{1}{4q'}$, *that is* $u(t) \in \mathbf{2}_q \cup \mathbf{1}_q \cup \mathbf{0}_q$. *If, in particular we are in the case*

$$< Q_{0,2}u, \varphi > = < d(x)u, \Delta\varphi >,$$

then according to Proposition 7.1.2 the problem can be regarded as

$$\begin{cases} u_t + \Delta^2 u = 0 \quad x \in \Omega \\ \frac{\partial u}{\partial \vec{n}} = d(x)u \\ \frac{\partial \Delta u}{\partial \vec{n}} = 0 \\ u(t_0) = u_0 \end{cases}$$

For any other $Q_{0,2}$, *Lemma 7.1.4 ii) could be used twice and we have that*

$$< u, D^2_{ij}\varphi >_\Gamma = \int_\Gamma a_{ij} u \frac{\partial D_j \varphi}{\partial \vec{n}} + \int_\Gamma (\partial^2_{ij} u + \partial_j(div(\tau_i)u) + div(\tau_j)\partial_j u + div(\tau_i)div(\tau_j)u)\varphi.$$

The second term can be expressed as a condition in the boundary but $\int_\Gamma a_{ij} u \frac{\partial D_j \varphi}{\partial \vec{n}}$ *cannot be handled easily.*

Note that different perturbations Q_i can be combined together, although not all combinations are allowed.

Proposition 7.2.6 *Consider a family of perturbations* $Q_i := Q_{c_i,d_i}$ *as in (7.2.2) with* $\|\delta_i\|_{L^{r_i}(\Gamma)} \leq R_0$, *with* $r_i > \frac{N-1}{3-\kappa_i}$, $\kappa_i = c_i + d_i$, $i = 1, ..., I$. *Denote* $Q := \sum_i Q_i$, *then for any* $1 < q < \infty$ *if*

$$\max_i \{c_i + (\frac{N-1}{r_i} - \frac{N-1}{q'})_+\} + \max_i \{d_i + (\frac{N-1}{r_i} - \frac{N-1}{q})_+\} < 3 \tag{7.2.8}$$

then there exists an interval $I(q,Q) \subset (-1 + \frac{1}{4q} + \frac{\max_i\{c_i\}}{4}, 1 - \frac{1}{4q'} - \frac{\max_i\{d_i\}}{4})$ *containing* $(-\frac{3}{4} + \max_i\{\frac{c_i}{4} + \frac{N-1}{4r_i}\}, \frac{3}{4} - \max_i\{\frac{d_i}{4} + \frac{N-1}{4r_i}\})$, *such that for any* $\gamma \in I(q,Q)$, *we have a strongly continuous, analytic semigroup,* $S_Q(t)$ *in the space* $H^{4\gamma,q}_{\mathcal{N}}(\Omega)$, *for the problem*

$$\begin{cases} u_t + \Delta^2 u + Qu = 0, & x \in \Omega,\ t > 0 \\ u(0) = u_0 & in\ \Omega. \end{cases}$$

Moreover the semigroup has the smoothing estimates

$$\|S_Q(t)u_0\|_{H^{4\gamma',q}_{\mathcal{N}}(\Omega)} \leq \frac{M_{\gamma',\gamma} e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{H^{4\gamma,q}_{\mathcal{N}}(\Omega)}, \quad t > 0,\ u_0 \in H^{4\gamma,q}(\Omega)$$

for every $\gamma, \gamma' \in I(q,Q)$ *with* $\gamma' \geq \gamma$, *and*

$$\|S_Q(t)u_0\|_{L^r(\Omega)} \leq \frac{M_{q,r} e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{L^q(\Omega)}, \quad t > 0,\ u_0 \in L^q(\Omega)$$

with $1 < q \leq r \leq \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q,Q)$ is given by

$$I(q,Q) = (-1 + \frac{1}{4q} + \max_i\{\frac{c_i}{4} + \frac{N-1}{4}(\frac{1}{r_i} - \frac{1}{q'})_+\}, 1 - \frac{1}{4q'} - \max\{\frac{d_i}{4} + \frac{N-1}{4}(\frac{1}{r_i} - \frac{1}{q})_+\}).$$

Finally, if

$$\delta_i^\varepsilon \to \delta_i \quad in\ L^{r_i}(\Gamma), \quad r_i > \frac{N-1}{3-\kappa_i}$$

then for every $1 < q < \infty$ and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{Q_\varepsilon}(t) - S_Q(t)\|_{\mathcal{L}(H_{\mathcal{N}}^{4\gamma,q}(\Omega), H_{\mathcal{N}}^{4\gamma',q}(\Omega))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \leq T$$

for all $\gamma, \gamma' \in I(q,Q)$, $\gamma \geq \gamma'$ and for any $1 < q \leq r \leq \infty$

$$\|S_{Q_\varepsilon}(t) - S_Q(t)\|_{\mathcal{L}(L^q(\Omega), L^r(\Omega))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q} - \frac{1}{r})}}, \qquad \forall\, 0 < t \leq T.$$

Proof. From Theorem 7.2.4 we know that for each perturbation Q_i there exists a non empty trapezoidal polygon $\mathcal{P}_i$ of admissible pairs of spaces (s,σ) described in terms of $\tilde{s} = s - c_i - \frac{N}{q}$ and $\tilde{\sigma} = \sigma - d_i - \frac{N}{q'}$, see (7.2.5).

Therefore the polygon $\mathcal{P}_i$ of the perturbation Q_i is given by a planar trapezium whose long base is on the line $s + \sigma = 4$ and the short base is on the line $s + \sigma = \kappa_i + N - \frac{N-1}{r_i'}$, with $\kappa_i = c_i + d_i$, in the third quadrant. As for the lateral sides they are given by the lines $s = a_i + \frac{1}{q} + (\frac{N-1}{r_i} - \frac{N-1}{q'})_+$ and $\sigma = b_i + \frac{1}{q'} + (\frac{N-1}{r_i} - \frac{N-1}{q})_+$. Thus the projection of $\mathcal{P}_i$ on the axes give the intervals

$$s \in J_1^i = [s_{min}^i, 4 - \sigma_{min}^i) \quad and \quad \sigma \in J_2^i = [\sigma_{min}^i, 4 - s_{min}^i)$$

see (7.2.6) and (7.2.7).

According to [47, Lemma 13, iii)], we can consider $Q := \sum_i Q_i$, that is, all perturbations acting at the same time, if there exists a common region $\mathcal{P}$ of admissible pairs (s,σ), that is if $\mathcal{P} := \cap_i \mathcal{P}_i \neq \emptyset$.

Since the admissible sets always have the long base on the line $s + \sigma = 4$ and the lateral sides are parallel to the axes, the set $\mathcal{P}$ is non empty if and only if

$$\max_i(\inf J_1^i) < \min_i(\sup J_1^i) \quad i.e. \quad \max_i(s_{min}^i) < \min_i(4 - \sigma_{min}^i)$$

and

$$\max_i(\inf J_2^i) < \min_i(\sup J_2^i) \quad i.e. \quad \max_i(\sigma_{min}^i) < \min_i(4 - s_{min}^i)$$

which are equivalent to (5.2.16), that is

$$\max_i\{c_i + (\frac{N-1}{r_i} - \frac{N-1}{q'})_+\} + \max_i\{d_i + (\frac{N-1}{r_i} - \frac{N-1}{q})_+\} < 3$$

In such a case the projection of $\mathcal{P}$ on the axes gives the intervals

$$s \in J_1 = [\max_i(\inf J_1^i), \min_i(\sup J_1^i))$$

$$= [\frac{1}{q} + \max_i\{c_i + (N-1)(\frac{1}{r_i} - \frac{1}{q'})_+\}, 4 - \frac{1}{q'} - \max_i\{d_i + (N-1)(\frac{1}{r_i} - \frac{1}{q})_+\})$$

$$\sigma \in J_2 = [\max_i(\inf J_2^i), \min_i(\sup J_2^i))$$

$$= [\frac{1}{q'} + \max_i\{c_i + (N-1)(\frac{1}{r_i} - \frac{1}{q})_+\}, 4 - \frac{1}{q} - \max_i\{d_i + (N-1)(\frac{1}{r_i} - \frac{1}{q})_+\}).$$

For each pair of admissible pairs $(s,\sigma) \in \mathcal{P}$, by [47, Proposition 10] (see Theorem 1.0.1) with $\alpha = \frac{s}{4}$ and $\beta = \frac{\sigma}{4}$, we get a perturbed semigroup and smoothing estimates

$$\|S_P(t)u_0\|_{\gamma'} \leq \frac{M_0 e^{\omega t}}{t^{\gamma'-\gamma}}\|u_0\|_\gamma, \quad \gamma' \geq \gamma$$

with

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta+1), \qquad \gamma' \geq \gamma.$$

Hence as (s,σ) range in the region $\mathcal{P}$ a repeated bootstrap argument as (5.2.8) gives that the smoothing estimates hold for $\gamma \in \bigcup_{(s,\sigma)\in\mathcal{P}} E(s/4)$ and $\gamma' \in \bigcup_{(s,\sigma)\in\mathcal{P}} R(\sigma/4)$, $\gamma' \geq \gamma$. This leads to

$$\gamma \in (\frac{\inf J_1}{4} - 1, \frac{\sup J_1}{4}], \qquad \gamma' \in [-\frac{\sup J_2}{4}, 1 - \frac{\inf J_2}{4}), \gamma' \geq \gamma$$

which, after a simple calculation, reads

$$\gamma, \gamma' \in I(q,Q) = (-1 + \frac{1}{4q} + \max_i\{\frac{c_i}{4} + \frac{N-1}{4}(\frac{1}{r_i} - \frac{1}{q'})_+\}, 1 - \frac{1}{4q'} - \max\{\frac{d_i}{4} + \frac{N-1}{4}(\frac{1}{r_i} - \frac{1}{q})_+\}).$$

Note that this interval is contained in an interval $(-1 + \frac{1}{4q} + \frac{\max_i\{c_i\}}{4}, 1 - \frac{1}{4q'} - \frac{\max_i\{d_i\}}{4})$ and contains $(-\frac{3}{4} + \max_i\{\frac{c_i}{4} + \frac{N-1}{4r_i}\}, \frac{3}{4} - \max_i\{\frac{d_i}{4} + \frac{N-1}{4r_i}\})$.

■

For some cases the condition (7.2.8) can be simplified and a more visual description can be given.

Remark 7.2.7 *i) If there is only one perturbation, then (7.2.8) is equivalent to $r > \frac{N-1}{3-\kappa}$ as in Theorem 7.2.4.*
ii) If $c_i = c$ and $d_i = d$ (thus $\kappa_i = \kappa$) for all i, then

$$< Qu, \varphi >= (-1)^d \int_{\partial\Omega} \delta D^d u D^c \varphi \quad \textit{where} \quad \delta := \sum_i \delta_i$$

can be considered as a perturbation with $\delta \in L^r_U(\Gamma)$ *for* $r = \min_i\{r_i\}$. *Then (7.2.8) holds if and only if* $r > \frac{N-1}{3-\kappa}$ *as in Theorem 7.2.4.*
iii) Assume now $r = r_i$ *for all* i. *Then (7.2.8) is equivalent to*

$$\max_i\{c_i\} + \max_i\{d_i\} < 3 - (\frac{N-1}{r} - \frac{N-1}{q})_+ - (\frac{N-1}{r} - \frac{N-1}{q'})_+. \tag{7.2.9}$$

Hence, if we denote $k := \max\{c_i\} + \max\{d_i\}$, *then (7.2.9) is satisfied provided* $r > \frac{N-1}{3-\kappa}$, *which resembles the condition in Theorem 7.2.4. Note that* κ *can be regarded as the order of the perturbation* $Q = \sum_i Q_i$.

In particular, if

$$r > \frac{N-1}{3-\kappa} \quad \textit{and} \quad \max_i\{c_i\} + \max_i\{d_i\} < 3$$

are satisfied, then Proposition 7.2.6 applies with an interval for Q *given by*

$$I(q,Q) = (-1 + \frac{1}{4q} + \frac{\max_i\{c_i\}}{4} + \frac{N-1}{4}(\frac{1}{r} - \frac{1}{q'})_+, 1 - \frac{1}{4q'} - \frac{\max_i\{d_i\}}{4} - \frac{N-1}{4}(\frac{1}{r} - \frac{1}{q})_+).$$

Compare it with $I(q,c,d)$ *in Theorem 7.2.4 to see the resemblance.*
iv) We now describe how to determine if two perturbations as in iii) can be combined.

For example, if we fix a perturbation Q_{c_0,d_0} *with* $\kappa_0 = 2$, *then, all perturbations* $Q_{c,d}$ *with* $c \leq c_0$ *and* $d \leq d_0$ *can be combined with it, and the interval is* $I(q,Q) = I(q,c_0,d_0)$.

For example a perturbation $Q_{1,1}$ *can be combined with all the ones included in the shaded area in Figure 7.3 with interval* $I(q,1,1)$. *However, the encircled perturbations* $Q_{2,0}$ *and* $Q_{0,2}$ *cannot be combined together.*

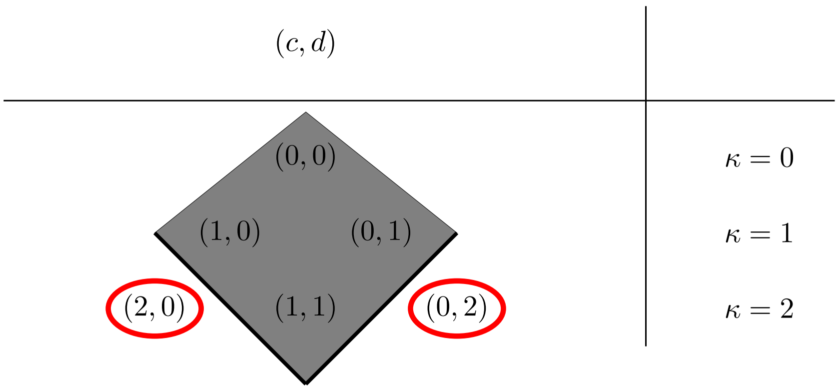

Figure 7.3: Combining perturbations.

For Q_{c_0,d_0} *with* $\kappa_0 = 1$, *all perturbations* $Q_{c,d}$ *with* $\kappa \leq 1$ *can be combined with it.*
v) There are 23 possible combinations for pairs of perturbations as in iv).

7.2.3 Perturbations in the interior and in the boundary

We now want to combine perturbations in the boundary and in the interior at the same time. For that, we proceed as above.

Proposition 7.2.8 *Consider a family of perturbations $P_i := P_{a_i,b_i}$ as in (7.2.1) with $\|d_i\|_{L^{p_i}(\Omega)} \leq R_0$, with $p_i > \frac{N}{4-k_i}$, $k_i = a_i + b_i \leq 3$, $i = 1, ..., I$ and $Q_j := Q_{c_j,d_j}$ as in (7.2.2) with $\|\delta_j\|_{L^{r_j}(\Gamma)} \leq R_0$, with $r_j > \frac{N-1}{3-\kappa_j}$, $\kappa_j = c_j + d_j \leq 2$, $j = 1, ..., J$. Denote $P := \sum_i P_i$ and $Q := \sum_j Q_j$, then for any $1 < q < \infty$ if*

$$\max\left\{\max_i\{a_i + (\frac{N}{p_i} - \frac{N}{q'})_+\}+, \frac{1}{q} + \max_j\{c_j + (\frac{N-1}{r_j} - \frac{N-1}{q'})_+\}\right\} + \tag{7.2.10}$$

$$\max\left\{\max_i\{b_i + (\frac{N}{p_i} - \frac{N}{q})_+\}, \frac{1}{q'} + \max_j\{d_j + (\frac{N-1}{r_j} - \frac{N-1}{q})_+\}\right\} < 4$$

then there exists an interval $I(q, P, Q) \subset (-1 + \frac{\max_i\{a_i\}}{4}, 1 - \frac{\max_i\{b_i\}}{4})$ containing $(-\frac{3}{4} + \max_i\{\frac{a_i}{4} + \frac{N-1}{4r_i}\}, \frac{3}{4} - \max_i\{\frac{b_i}{4} + \frac{N-1}{4r_i}\})$, such that for any $\gamma \in I(q, P, Q)$, we have a strongly continuous, analytic semigroup, $S_{PQ}(t)$ in the space $H^{4\gamma,q}_{\mathcal{N}}(\Omega)$, for the problem

$$< u_t, \varphi >_\Omega + < \Delta^2 u, \varphi >_\Omega + < Pu, \varphi >_\Omega + < Qu, \varphi >_\Gamma = 0 \tag{7.2.11}$$

Moreover the semigroup has the smoothing estimates

$$\|S_{PQ}(t)u_0\|_{H^{4\gamma',q}_{\mathcal{N}}(\Omega)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{4\gamma,q}_{\mathcal{N}}(\Omega)}, \quad t > 0,\ u_0 \in H^{4\gamma,q}_{\mathcal{N}}(\Omega)(\Omega)$$

for every $\gamma, \gamma' \in I(q, P, Q)$ with $\gamma' \geq \gamma$, and

$$\|S_Q(t)u_0\|_{L^r(\Omega)} \leq \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\Omega)}, \quad t > 0,\ u_0 \in L^q(\Omega)$$

with $1 < q \leq r \leq \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q, P, Q)$ is given by

$$\gamma, \gamma' \in I(q, P, Q) =$$

$$(-1 + \frac{\max}{4}\left\{\max_i\{a_i + (\frac{N}{p_i} - \frac{N}{q'})_+\}+, \frac{1}{q} + \max_j\{c_j + (\frac{N-1}{r_j} - \frac{N-1}{q'})_+\}\right\},$$

$$1 - \frac{\max}{4}\left\{\max_i\{b_i + (\frac{N}{p_i} - \frac{N}{q})_+\}, \frac{1}{q'} + \max_j\{d_j + (\frac{N-1}{r_j} - \frac{N-1}{q})_+\}\right\}).$$

Finally, if

$$d_i^\varepsilon \to d_i \quad in\ L^{p_i}(\Omega), \quad p_i > \frac{N}{4-k_i}$$

and

$$\delta_j^\varepsilon \to \delta_j \quad in\ L^{r_j}(\Gamma), \quad r_j > \frac{N-1}{3-\kappa_j}$$

then for every $1 < q < \infty$ *and* $T > 0$ *there exists* $C(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*, such that*

$$\|S_{PQ_\varepsilon}(t) - S_{PQ}(t)\|_{\mathcal{L}(H_{\mathcal{N}}^{4\gamma,q}(\Omega), H_{\mathcal{N}}^{4\gamma',q}(\Omega))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \leq T$$

for all $\gamma, \gamma' \in I(q, P, Q)$*,* $\gamma \geq \gamma'$ *and for any* $1 < q \leq r \leq \infty$

$$\|S_{PQ_\varepsilon}(t) - S_{PQ}(t)\|_{\mathcal{L}(L^q(\Omega), L^r(\Omega))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \leq T.$$

Proof. From Theorem 5.2.10 and Theorem 7.2.4 we know that for each perturbation P_i and Q_j there exists a non empty trapezoidal polygon $\mathcal{P}_i$ and $\mathcal{Q}_j$ of admissible pairs of spaces (s, σ) described in terms of $\tilde{s} = s - a_i - \frac{N}{q}$ and $\tilde{\sigma} = \sigma - b_i - \frac{N}{q'}$.

The polygon $\mathcal{P}_i$ of the perturbation P_i is given by a planar trapezium whose long base is on the line $s + \sigma = 4$ and the short base is on the line $s + \sigma = k_i + N - \frac{N}{p_i'}$, with $k_i = a_i + b_i$, in the third quadrant. As for the lateral sides they are given by the lines $s = a_i + (\frac{N}{p_i} - \frac{N}{q'})_+$ and $\sigma = b_i + (\frac{N}{p_i} - \frac{N-1}{q})_+$. Thus the projection of $\mathcal{P}_i$ on the axes give the intervals

$$s \in J_1^i = [s_{min}^i, 4 - \sigma_{min}^i) \quad and \quad \sigma \in J_2^i = [\sigma_{min}^i, 4 - s_{min}^i)$$

The polygon $\mathcal{Q}_j$ of the perturbation Q_j is given by a planar trapezium whose long base is on the line $s+\sigma = 4$ and the short base is on the line $s+\sigma = \kappa_j + N - \frac{N-1}{r_j'}$, with $\kappa_j = c_j + d_j$, in the third quadrant. As for the lateral sides they are given by the lines $s = d_j + \frac{1}{q}(\frac{N-1}{r_j} - \frac{N-1}{q'})_+$ and $\sigma = d_j + \frac{1}{q'} + (\frac{N-1}{r_j} - \frac{N-1}{q})_+$. Thus the projection of $\mathcal{Q}_j$ on the axes give the intervals

$$s \in J_1^j = [s_{min}^j, 4 - \sigma_{min}^j) \quad and \quad \sigma \in J_2^j = [\sigma_{min}^j, 4 - s_{min}^j)$$

see (7.2.6) and (7.2.7).

According to [47, Lemma 13, iii)], we can consider $P + Q := \sum_i P_i + \sum_j Q_j$, that is, all perturbations acting at the same time, if there exists a common region $\mathcal{P}$ of admissible pairs (s, σ), that is if $\mathcal{P} := (\cap_i \mathcal{P}_i) \cap (\cap_j \mathcal{Q}_j) \neq \emptyset$.

Since the admissible sets always have the long base on the line $s + \sigma = 4$ and the lateral sides are parallel to the axes, the set $\mathcal{P}$ is non empty if and only if

$$\max(\inf J_1^i, \inf J_1^j) < \min(\sup J_1^i, \sup J_1^j) \quad i.e. \quad \max(s_{min}^i, s_{min}^j) < \min(4-\sigma_{min}^i, 4-\sigma_{min}^j)$$

and

$$\max(\inf J_2^i, \inf J_2^j) < \min(\sup J_2^i, \sup J_2^j) \quad i.e. \quad \max(\sigma_{min}^i, \sigma_{min}^j) < \min(4-s_{min}^i, 4-s_{min}^j)$$

which are equivalent to (7.2.10).

In such a case the projection of $\mathcal{P}$ on the axes gives the intervals

$$s \in J_1 = [\max_{i,j}(\inf J_1^{i,j}), \min_{i,j}(\sup J_1^{i,j}))$$

$$\sigma \in J_2 = [\max_{i,j}(\inf J_2^{i,j}), \min_{i,j}(\sup J_2^{i,j})).$$

For each pair of admissible pairs $(s,\sigma) \in \mathcal{P}$, by [47, Proposition 10] (see Theorem 1.0.1) with $\alpha = \frac{s}{4}$ and $\beta = \frac{\sigma}{4}$, we get a perturbed semigroup and smoothing estimates

$$\|S_P(t)u_0\|_{\gamma'} \le \frac{M_0 e^{\omega t}}{t^{\gamma'-\gamma}} \|u_0\|_{\gamma}, \quad \gamma' \ge \gamma$$

with

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta + 1), \qquad \gamma' \ge \gamma.$$

Hence as (s,σ) range in the region $\mathcal{P}$ a repeated bootstrap argument as in (5.2.8) gives that the smoothing estimates hold for $\gamma \in \bigcup_{(s,\sigma)\in\mathcal{P}} E(s/4)$ and $\gamma' \in \bigcup_{(s,\sigma)\in\mathcal{P}} R(\sigma/4)$, $\gamma' \ge \gamma$. This leads to

$$\gamma \in (\frac{\inf J_1}{4} - 1, \frac{\sup J_1}{4}], \qquad \gamma' \in [-\frac{\sup J_2}{4}, 1 - \frac{\inf J_2}{4}), \gamma' \ge \gamma$$

which, after a simple calculation, reads

$$\gamma, \gamma' \in I(q,P) =$$

$$(-1 + \frac{\max}{4}\left\{\max_i\{a_i + (\frac{N}{p_i} - \frac{N}{q'})_+\}+, \frac{1}{q} + \max_j\{c_j + (\frac{N-1}{r_j} - \frac{N-1}{q'})_+\}\right\},$$
$$1 - \frac{\max}{4}\left\{\max_i\{b_i + (\frac{N}{p_i} - \frac{N}{q})_+\}, \frac{1}{q'} + \max_j\{d_j + (\frac{N-1}{r_j} - \frac{N-1}{q})_+\}\right\}).$$

Note that this interval is contained in the interval $(-1 + \frac{\max_i\{a_i\}}{4}, 1 - \frac{\max_i\{b_i\}}{4})$ and contains $(-\frac{3}{4} + \max_i\{\frac{a_i}{4} + \frac{N-1}{4r_i}\}, \frac{3}{4} - \max_i\{\frac{b_i}{4} + \frac{N-1}{4r_i}\})$.

■

Again, in some cases the condition (7.2.10) can be simplified.

Remark 7.2.9 *i) If $a_i = a$ and $b_i = b$ (thus $k_i = k$) for all i and $c_j = c$ and $d_j = d$ (thus $\kappa_j = \kappa$) for all j, then*

$$P = \sum_i D^b(d_i D^a) = D^b(dD^a) \quad where \quad d := \sum_i d_i$$

can be considered as a perturbation with $d \in L^p_U(\mathbb{R}^N)$ for $p = \min_i\{p_i\}$, and

$$Q = \sum_j D^d(\delta_j D^c) = D^d(\delta D^c) \quad where \quad \delta := \sum_j \delta_j$$

can be considered as a perturbation with $\delta \in L^r_U(\Gamma)$ *for* $r = \min_j\{r_j\}$ *and (7.2.10) is equivalent to*

$$\max\left\{a + (\frac{N}{p} - \frac{N}{q'})_+, \frac{1}{q} + c + (\frac{N-1}{r} - \frac{N-1}{q'})_+\right\}$$
$$+\max\left\{b + (\frac{N}{p} - \frac{N}{q})_+, \frac{1}{q'} + d + (\frac{N-1}{r} - \frac{N-1}{q})_+\right\} < 4$$

In this situation, (7.2.10) implies $r > \frac{N-1}{3-\kappa}$ *and* $p > \frac{N}{4-k}$ *as in Theorem 7.2.4 and Theorem 7.2.1 respectively.*

ii) A perturbation $P_{a,b}$ *in the interior can be combined with another one* $Q_{c,d}$ *in the boundary using the same arguments. However, note that the condition has now two maximums, entire parts and the relation between* r *and* p*, therefore we do not include the full list of cases. Still we explain a case where we assume enough integrability, that is, when* $p > q, q'$ *and* $r > q, q'$.

Under that circumstances (7.2.10) turns into

$$\max\{a, \frac{1}{q} + c\} + \max\{b, \frac{1}{q'} + d\} < 4.$$

Now fix $P_{a,b}$*, with* $a + b < 4$*, and take* $Q_{c,b}$*. Then for all* $c \leq a$ *and* $d < b$ *or* $c < a$ *and* $d \leq b$*, the condition above holds, thus all perturbations of such form can be combined.*

Remark 7.2.10 *In order to make precise in what sense equation (7.2.11) is satisfied we need, as in Remarks 7.2.2, 7.2.5, to consider many different possibilities of perturbations. Since the amount of possible combinations is now enormous, we just focus on a single case. Any other problem that the reader might consider can be treated in the same way.*

We focus on the case where $b = 2$, $a \leq 1$, $c = a + 1$ *and* $d = 0$*, with* $< P_{a,b}u, \varphi >=< d(x)D^a u, \Delta\varphi >_\Omega$ *and* $< Q_{c,d}u, \varphi >=< \frac{\partial d(x)D^a u}{\partial \vec{n}}, \varphi >_\Gamma$*. Then, because of the Divergence Theorem we have*

$$\int_\Omega d(x)D^a u\Delta\varphi = \int_\Omega \Delta(d(x)D^a u)\varphi - \int_\Gamma \frac{\partial d(x)D^a u}{\partial \vec{n}}\varphi$$

so (7.2.11) turns into

$$< u_t, \varphi >_\Omega + < \Delta u, \varphi >_\Omega + < \Delta(d(x)D^a u), \varphi >_\Omega = 0$$

and according to Proposition 7.1.2 ii) the problem can be regarded as

$$\begin{cases} u_t + \Delta^2 u + \Delta(d(x)D^a u) = 0 & x \in \Omega \\ \frac{\partial u}{\partial \vec{n}} = 0 \\ \frac{\partial \Delta u}{\partial \vec{n}} = 0 \\ u(t_0) = u_0 \end{cases}$$

Part II

Nonlinear parabolic problems

We are now going to consider parabolic problems of the form

$$\begin{cases} u_t + Au = f(u), & t > 0 \\ u(0,x) = u_0(x) \end{cases}$$

where the main operator A is a sectorial operator of either second or higher order, f is a linear nonlinear function and u_0 is the initial data in some space to be considered.

In order to study the problem, consider a family of Banach spaces $\{X^\gamma\}_{\gamma\in\mathcal{J}}$ where $\mathcal{J}$ is an interval of real indexes. The norm of the space X^γ is denoted by $\|\cdot\|_\gamma$. We assume that we have given a linear semigroup defined in each of these spaces, that is,

$$\begin{aligned} &\{S(t) : t \geq 0\} \text{ is for any } \gamma \in \mathcal{J} \text{ a semigroup of bounded linear operators in} \\ &X^\gamma \text{ such that the map } (0,\infty) \times X^\gamma \ni (t, u_0) \to S(t)u_0 \in X^\gamma \text{ is continuous} \\ &\text{and, moreover, } \|S(t)\|_{\mathcal{L}(X^\gamma, X^{\gamma'})} \leq \frac{M_0}{t^{\gamma'-\gamma}} \text{ for all } 0 < t \leq T, \gamma, \gamma' \in \mathcal{J}, \gamma' \geq \gamma, \end{aligned} \tag{ii.0.12}$$

where $M_0 := M_0(\gamma, \gamma', T)$ is a positive constant which can be chosen uniformly for T in bounded time intervals. Hence we say that we have a semigroup $\{S(t) : t \geq 0\}$ in the scale $\{X^\gamma\}_{\gamma\in\mathcal{J}}$.

Observe that we do not assume continuity for the semigroup at $t = 0$, that is for $u_0 \in X^\gamma$, $\gamma \in \mathcal{J}$,

$$S(t)u_0 \to u_0 \quad \text{in } X^\gamma \text{ as } t \to 0^+ \tag{ii.0.13}$$

unless explicitly stated. Some of the results we obtain, however, will require initial data $u_0 \in X^\gamma$ such that (ii.0.13) holds. In case we assume (ii.0.13), we say $\{S(t) : t \geq 0\}$ in (ii.0.12) is a C^0 semigroup in the scale $\{X^\gamma\}_{\gamma\in\mathcal{J}}$.

As stated in the introduction, given a semigroup in a scale as above, we will consider a nonlinear mapping satisfying

$$f : X^\alpha \to X^\beta \text{ for some } \alpha, \beta \in \mathcal{J} \text{ with } 0 \leq \alpha - \beta < 1 \tag{ii.0.14}$$

and that there exist $\rho \geq 1$, $L > 0$ such that

$$\|f(u) - f(v)\|_\beta \leq L\big(1 + \|u\|_\alpha^{\rho-1} + \|v\|_\alpha^{\rho-1}\big)\|u - v\|_\alpha, \quad u, v \in X^\alpha. \tag{ii.0.15}$$

Thus f is continuous and

$$\|f(u)\|_\beta \leq L\big(1 + \|u\|_\alpha^\rho\big), \quad u \in X^\alpha \tag{ii.0.16}$$

where the constants in (ii.0.15) and (ii.0.16) can be chosen the same. Note that this setting includes a particular case where $f : X^\alpha \to X^\beta$ is globally Lipschitz, that is $\rho = 1$.

Hence, our main goal is the analysis of the abstract nonlinear integral equation

$$u(t; u_0) = S(t)u_0 + \int_0^t S(t-\tau) f(u(\tau; u_0))\, d\tau, \qquad 0 < t \leq T, \tag{ii.0.17}$$

where u_0 is taken from some space X^γ in the scale. Notice that (ii.0.17) is the corresponding variation of constants formula for mild solutions of the nonlinear problem

$$u_t + Au = f(u), \quad u(0) = u_0.$$

Of course the first step before attempting to solve (ii.0.17) is to define a suitable notion of solution and many options could be available. In any case, to make sense of (ii.0.17), any definition of solution has to include the minimal requirements that $u : (0, T] \to X^\alpha$ and, that for any $0 < \tau < T$ and for all $\tau \leq t \leq T$, $u(t)$ satisfies

$$u(t) = S(t-\tau)u(\tau) + \int_\tau^t S(t-s)f(u(s))\, ds.$$

Additionally, it is also natural to require that for any $\tau > 0$, $u \in L^\infty([\tau, T], X^\alpha)$. Also, any suitable notion of solution must incorporate information on the initial data and the behavior of the solution near $t = 0$. In particular, we define

Definition ii.0.11 *If $u_0 \in X^\gamma$, then $u \in L^\infty_{loc}((0, T], X^\alpha)$ that satisfes $t^{\alpha-\gamma}\|u(t)\|_\alpha \leq M$, $t \in (0, T]$ for some $M > 0$, $u(0) = u_0$ and (ii.0.17) for $0 < t \leq T$ is called a γ–solution of (ii.0.17) in $[0, T]$.*

Notice that, from (1.0.1), the behavior of the γ–solution at $t = 0$ is the same as that of the linear semigroup $S(t)u_0$.

For this class of solutions we can show existence, uniqueness and continuous dependence with respect to the initial data, for the following ranges of γ:

$$\gamma \in E(\alpha, \beta, \rho) = \begin{cases} (\alpha - \frac{1}{\rho}, \alpha], & \text{if } 0 \leq \alpha - \beta \leq \frac{1}{\rho} \\ [\frac{\alpha\rho-\beta-1}{\rho-1}, \alpha], & \text{if } \frac{1}{\rho} < \alpha - \beta < 1. \end{cases}$$

The case $\gamma = \frac{\alpha\rho-\beta-1}{\rho-1}$ is called critical and subcritical otherwise. In particular, we will prove (see Theorems 9.1.7 and 9.1.8 for more complete statements)

Theorem ii.0.12 *Assume $\gamma \in E(\alpha, \beta, \rho)$ as above. Then there exists $r > 0$ such for any $v_0 \in X^\gamma$ there exists $T > 0$ such that for any u_0 such that $\|u_0 - v_0\|_\gamma < r$ there exits a γ–solution of (ii.0.17) with initial data u_0 defined in $[0, T]$. In the subcritical case r can be taken arbitrarily large.*

These solutions will be shown to regularize, that is, to enter continuously in other spaces of the scale. We will also give estimates on the existence time and the possible blow–up rate for different norms. We will also show that the conditions of the theorem are essentially optimal. Finally, since Theorem ii.0.12 is for γ–solutions we will also improve uniqueness for functions that, satisfying the minimum requirements described above and (ii.0.17), are bounded at $t = 0$ in X^γ in the subcritical case, or continuous at $t = 0$ in X^γ in the critical one. See Chapter 9 for full details and precise statements.

When applying these techniques to concrete problems one finds that there typically exist many admissible pairs (α, β) such that the nonlinear term f satisfies (ii.0.15). Such admissible pairs make up the admissible region $\mathcal{S}$ for the problem considered. In this situation we develop in Chapter 10 a general bootstrapping argument that leads to the largest range of γ for which the solution can be constructed as well as to the largest range of spaces into which the solution regularizes. Next, Chapters 11 and 12 are devoted to apply the previous abstract arguments to concrete PDE problems in concrete scales of spaces. For example in Chapter 11 we consider parabolic problems of the type

$$u_t + (-\Delta)^m u = f(x,u) := D^b(h(x, D^a u)),\ t > 0,\ x \in \mathbb{R}^N, \tag{ii.0.18}$$

with $m \in \mathbb{N}$, where D^c represents any partial derivative of order $c \in \mathbb{N}$, $h(\cdot, 0) = 0$ and for some $\rho > 1$, $L > 0$ we have

$$|h(x,u) - h(x,v)| \le L|u - v|(|u|^{\rho-1} + |v|^{\rho-1}),\ u, v \in \mathbb{R},\ x \in \mathbb{R}^N.$$

Then problem (ii.0.18) is solved in several different scales of spaces.

We will also study uniqueness in several classes, continuous dependence with respect to the initial data, blow up estimates when blow up occurs and smoothing estimates.

In Section 11.1 the problem is considered assuming $a = b = 0$ and in the Lebesgue scale. The results there recover and slightly improve the results in [52], [53], [11], [4] and [46, Chapter 2], and most of them are essentially optimal, see the discussion in Section 11.1.

As for Section 11.2 the problem is considered in the Bessel scale of spaces with $a \neq 0 \neq b$.

Later, in Section 11.3, (ii.0.18) is considered, with $b = 0$, in the uniform Bessel scale, see Chapter 3. In this setting we obtain similar results for (ii.0.18) to those in Section 11.2 with $b = 0$.

Finally, in Chapter 12 we consider the following strongly damped wave equation

$$\begin{cases} w_{tt} - \Delta w_t + w_t - \Delta w = h(x,w),\ t > 0,\ x \in \mathbb{R}^N, \\ w(0,x) = w_0(x), \quad w_t(0,x) = z_0(x),\ x \in \mathbb{R}^N, \end{cases}$$

which, after being written as a system with $z = w_t$, is considered in the scale

$$X^\alpha = \begin{cases} H^1(\mathbb{R}^N) \times H^{2\alpha}(\mathbb{R}^N), & \alpha \in [-\frac{1}{2}, \frac{1}{2}], \\ \{\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : w + z \in H^{2\alpha}(\mathbb{R}^N)\}, & \alpha \in [\frac{1}{2}, 1]. \end{cases}$$

Chapter 8

Smoothing effect of the variation of constants formula

In this chapter we collect some preparatory material based on the smoothing effect of the variations of constants formula. These results are the nonlinear version of the results for linear perturbations from [47] which we gathered in Chapter 1. Note that all spaces and norms below are for indexes in the interval $\mathcal{J}$, although we will not write this all the time for the sake of brevity. Hereafter we extensively use the following spaces.

Definition 8.0.1 *Let $T > 0$, $\sigma \in \mathcal{J}$ and $\varepsilon \geq 0$. For u in $L^\infty_{loc}((0, T], X^\sigma)$ we define*

$$|||u|||_{\sigma,\varepsilon,T} = \sup_{t\in(0,T]} t^\varepsilon \|u(t)\|_\sigma$$

which becomes a norm on the set $\mathcal{L}^\infty_{\sigma,\varepsilon}((0, T]) \subset L^\infty_{loc}((0, T], X^\sigma)$ where it is finite.

Remark 8.0.2 *i) The space $\mathcal{L}^\infty_{\sigma,\varepsilon}((0, T])$ was denoted by $\mathcal{L}^\infty_\varepsilon((0, T], X^\sigma)$ in [47] and it is a Banach space with the norm $|||u|||_{\sigma,\varepsilon,T}$.*
ii) From (ii.0.12), the semigroup $\{S(t) : t \geq 0\}$ satisfies

$$|||S(\cdot)u_0|||_{\gamma',\gamma'-\gamma,T} = \sup_{t\in(0,T]} t^{\gamma'-\gamma}\|S(t)u_0\|_{\gamma'} \leq \sup_{t\in(0,T]} t^{\gamma'-\gamma}\frac{M_0}{t^{\gamma'-\gamma}}\|u_0\|_\gamma = M_0\|u_0\|_\gamma. \quad (8.0.1)$$

Thus for any $T > 0$ and $\gamma', \gamma \in \mathcal{J}$ with $\gamma' \geq \gamma$ the map

$$S(\cdot) : X^\gamma \longrightarrow \mathcal{L}^\infty_{\gamma',\gamma'-\gamma}((0, T]), \qquad u_0 \mapsto S(\cdot)u_0$$

is linear and continuous.
iii) The constant $M_0 := M_0(\gamma, \gamma', T)$ in (ii.0.12), (8.0.1) can be assumed to be $M_0(\gamma, \gamma', T) \geq 1$ and increasing in T.

For f as in (ii.0.14)-(ii.0.16), $u_0 \in X^\gamma$, $\gamma \in \mathcal{J}$ and for $u : (0, T] \to X^\alpha$ we denote

$$\mathcal{F}(u, u_0)(t) = S(t)u_0 + \int_0^t S(t-\tau)f(u(\tau))\, d\tau, \qquad 0 < t \leq T \qquad (8.0.2)$$

provided that the right hand side is well defined.

The following lemma gives estimates of $\mathcal{F}(u, u_0)(t)$ in various norms. Note that below $B(\cdot,\cdot)$ denotes Euler's beta function.

Lemma 8.0.3 *Assume (ii.0.12) and (ii.0.14), (ii.0.16). Assume also that $\varepsilon \geq 0$, $\delta \geq 0$ and γ' satisfy*

$$\beta \leq \gamma' < \beta + 1, \quad 0 \leq \rho\varepsilon < 1. \tag{8.0.3}$$

and $u \in \mathcal{L}^{\infty}_{\alpha,\varepsilon}((0,T])$. Then all of the following hold.
i) For $0 < t \leq T$

$$t^{\delta} \| \int_0^t S(t-\tau) f(u(\tau))\, d\tau \|_{\gamma'} \leq M(T) t^{\beta+\delta+1-\gamma'-\rho\varepsilon} \big(t^{\rho\varepsilon} + |||u|||^{\rho}_{\alpha,\varepsilon,T} \big) \tag{8.0.4}$$

where $M(T) = M(T,\gamma') = LM_0(\beta,\gamma',T)B(1-\rho\varepsilon, 1+\beta-\gamma')$.
ii) If $\gamma \leq \gamma'$ and $u_0 \in X^{\gamma}$ then for $0 < t \leq T$

$$t^{\delta} \|\mathcal{F}(u,u_0)(t)\|_{\gamma'} \leq t^{\delta} \|S(t)u_0\|_{\gamma'} + M(T) t^{\beta+\delta+1-\gamma'-\rho\varepsilon} \big(t^{\rho\varepsilon} + |||u|||^{\rho}_{\alpha,\varepsilon,T} \big) \tag{8.0.5}$$

with $M(T)$ as in i) above.
iii) If, in addition,

$$\delta = \gamma' - \gamma, \quad \gamma \leq \beta + 1 - \rho\varepsilon, \quad \gamma \neq \beta + 1 \tag{8.0.6}$$

then

$$|||\mathcal{F}(u,u_0)|||_{\gamma',\delta,T} \leq |||S(\cdot)u_0|||_{\gamma',\delta,T} + C(T) \big(T^{\rho\varepsilon} + |||u|||^{\rho}_{\alpha,\varepsilon,T} \big) \tag{8.0.7}$$

with $C(T) = M(T) T^{\beta+1-\gamma-\rho\varepsilon}$ and $M(T)$ as in i) above.
iv) Assuming moreover (ii.0.15), if $u_0, v_0 \in X^{\gamma}$ and $u, v \in \mathcal{L}^{\infty}_{\alpha,\varepsilon}((0,T])$ then

$$\begin{aligned} |||\mathcal{F}(u,u_0) - \mathcal{F}(v,v_0)|||_{\gamma',\delta,T} &\leq |||S(\cdot)(u_0 - v_0)|||_{\gamma',\delta,T} + \\ & C(T) \big(T^{(\rho-1)\varepsilon} + |||u|||^{\rho-1}_{\alpha,\varepsilon,T} + |||v|||^{\rho-1}_{\alpha,\varepsilon,T} \big) |||u - v|||_{\alpha,\varepsilon,T} \end{aligned} \tag{8.0.8}$$

with $C(T) = M(T) T^{\beta+1-\gamma-\rho\varepsilon}$ and $M(T)$ as in i) above.

If the scale is nested, see (1.0.4), then the results above also hold for $\gamma' < \beta$, with different constants and with $\beta = \gamma'$ in the exponents of t or T in the right hand side of (8.0.4), (8.0.5), (8.0.7) and (8.0.8).

Proof. Using (ii.0.12) and (ii.0.16), we have for $\gamma' \geq \beta$

$$t^{\delta} \| \int_0^t S(t-\tau) f(u(\tau))\, d\tau \|_{\gamma'} \leq M_0 t^{\delta} \int_0^t \frac{1}{(t-\tau)^{\gamma'-\beta}} L \big(1 + \|u(\tau)\|^{\rho}_{\alpha} \big)\, d\tau$$

$$\leq M_0 L t^{\delta} \Big(\int_0^t \frac{1}{(t-\tau)^{\gamma'-\beta}}\, d\tau + |||u|||^{\rho}_{\alpha,\varepsilon,T} \int_0^t \frac{1}{(t-\tau)^{\gamma'-\beta} \tau^{\rho\varepsilon}}\, d\tau \Big),$$

with M_0 as in (ii.0.12). Now, since $\gamma' - \beta < 1$ and $\rho\varepsilon < 1$ as in the assumptions, the change of variables $\tau = tr$ gives the result of part i) for $M(T) = LM_0 B(1-\rho\varepsilon, 1+\beta-\gamma')$.

Parts ii) and iii) are now immediate from i) and (8.0.2).

For part iv) we observe that (ii.0.12), (ii.0.15) lead to

$$t^{\delta}\| \int_0^t S(t-\tau)\big(f(u(\tau)) - f(v(\tau))\big)\, d\tau\|_{\gamma'}$$

$$\leq M_0 L|||u-v|||_{\alpha,\varepsilon,T} t^{\delta}\Big(\int_0^t \frac{1}{(t-\tau)^{\gamma'-\beta}\tau^{\varepsilon}}\, d\tau + \big(|||u|||^{\rho-1}_{\alpha,\varepsilon,T} + |||v|||^{\rho-1}_{\alpha,\varepsilon,T}\big) \int_0^t \frac{1}{(t-\tau)^{\gamma'-\beta}\tau^{\rho\varepsilon}}\, d\tau\Big).$$

Hence, after change of variables $\tau = tr$ we get the result taking supremum for $0 < t \leq T$.

Finally, if the scale is nested and $\gamma' < \beta$, we first use the embedding $X^{\beta} \hookrightarrow X^{\gamma'}$ for the integral involving nonlinearity and then use for the integral the estimates above for $\gamma' = \beta$. ■

We now prove continuity of $\mathcal{F}(u, u_0)(t)$ for positive times.

Lemma 8.0.4 *Assume (ii.0.12) and (ii.0.14), (ii.0.16).*

Then, given $u_0 \in X^{\gamma}$, $\varepsilon \geq 0$, $0 \leq \rho\varepsilon < 1$ and $u \in \mathcal{L}^{\infty}_{\alpha,\varepsilon}((0,T])$, we have

$$\mathcal{F}(u, u_0) \in C((0,T], X^{\gamma'}) \quad \textit{for} \quad \gamma' \geq \gamma,\ \beta \leq \gamma' < \beta + 1. \tag{8.0.9}$$

If the scale is nested (8.0.9) holds for $\gamma' < \beta + 1$.

Proof. Given $t \in (0,T]$ and any $h \in \mathbb{R}$ satisfying $\frac{t}{2} \leq t + h \leq T$ we use (8.0.2) to get

$$\begin{aligned}\|\mathcal{F}(u,u_0)(t+h) - \mathcal{F}(u,u_0)(t)\|_{\gamma'} &\leq \|\Big(S\big(\frac{t}{2}+h\big) - S\big(\frac{t}{2}\big)\Big)S\big(\frac{t}{2}\big)u_0\|_{\gamma'} \\ &+ \|\int_0^{t+h} S(t+h-s)f(u(s))ds - \int_0^t S(t-s)f(u(s))ds\|_{\gamma'} =: I_1(h) + I_2(h).\end{aligned}$$

From (ii.0.12) we have $\lim_{h\to 0} I_1(h) = 0$ because $S\big(\frac{t}{2}\big)u_0 \in X^{\gamma'}$ and the semigroup is continuous in $X^{\gamma'}$ for $t > 0$. Hence it suffices to prove that $\lim_{h\to 0} I_2(h) = 0$.

First, for $h > 0$ we obtain

$$\begin{aligned}I_2(h) &\leq \|\int_0^t (S(t+h-s) - S(t-s))f(u(s))ds\|_{\gamma'} + \int_t^{t+h} \|S(t+h-s)f(u(s))ds\|_{\gamma'} \\ &=: I^+_{21}(h) + I^+_{22}(h).\end{aligned}$$

Due to (ii.0.12) and (ii.0.16), we have $I^+_{22}(h) \leq \frac{LM_0}{1+\beta-\gamma'}(1 + |||u|||^{\rho}_{\alpha,\varepsilon,T} t^{-\varepsilon\rho})h^{1+\beta-\gamma'}$, that is, $\lim_{h\to 0^+} I^+_{22}(h) = 0$. Furthermore, $\|(S(t+h-s) - S(t-s))f(u(s))ds\|_{\gamma'}$ is bounded by $G(s) = \frac{2LM_0}{(t-s)^{\gamma'-\beta}} + \frac{2LM_0|||u|||^{\rho}_{\alpha,\varepsilon,T}}{(t-s)^{\gamma'-\beta}s^{\varepsilon\rho}}$ which is integrable for $s \in (0,t)$ whereas, given $s \in (0,t)$ and $r \in (0, t-s)$, we also have $S(r)f(u(s)) \in X^{\gamma'}$ and obtain as $h \to 0^+$

$$\begin{aligned}&\|(S(t+h-s) - S(t-s))f(u(s))\|_{\gamma'} \\ &\qquad = \|(S(t+h-s-r) - S(t-s-r))S(r)f(u(s))\|_{\gamma'} \to 0.\end{aligned} \tag{8.0.10}$$

Hence, using Lebesgue's dominated convergence theorem we get $\lim_{h\to 0^+} I_{21}^+(h) = 0$ which proves that $\lim_{h\to 0^+} I_2(h) = 0$.

Then for $h < 0$ we get

$$I_2(h) \le \int_0^{t+h} \|(S(t+h-s) - S(t-s))f(u(s))\|_{\gamma'} ds + \int_{t+h}^{t} \|S(t-s)f(u(s))\|_{\gamma'} ds$$
$$=: I_{21}^-(h) + I_{22}^-(h).$$

Since $\frac{t}{2} \le t+h$, from (ii.0.12), (ii.0.16) we have $I_{22}^-(h) \le \frac{LM_0}{1+\beta-\gamma'}(1+|||u|||^{\rho}_{\alpha,\varepsilon,T}\big(\frac{t}{2}\big)^{-\varepsilon\rho})|h|^{1+\beta-\gamma'}$. On the other hand, given any ξ such that $\frac{t}{4} \le t-\xi \le t+h$, $I_{21}^-(h)$ is bounded by

$$\int_0^{t-\xi} \|(S(t+h-s) - S(t-s))f(u(s))\|_{\gamma'} ds$$
$$+ \int_{t-\xi}^{t+h} (\|S(t+h-s)f(u(s))\|_{\gamma'} + \|S(t-s)f(u(s))\|_{\gamma'}) ds =: J(h,\xi) + K(h,\xi).$$

Applying (ii.0.12), (ii.0.16) observe that $K(h,\xi) \le \frac{2LM_0}{1+\beta-\gamma'}\big(1 + |||u|||^{\rho}_{\alpha,\varepsilon,T}\big(\frac{t}{4}\big)^{-\varepsilon\rho}\big)\xi^{1+\beta-\gamma'}$. Hence, given $\eta > 0$, there exists $\xi > 0$ such that $K(h,\xi) < \eta$ for all $h \in (-\xi, 0)$. Having fixed such ξ, note that $(0, t-\xi) \subset (0, t+h)$ and that due to (ii.0.12), (ii.0.16) $\|(S(t+h-s) - S(t-s))f(u(s))\|_{\gamma'}$ is bounded by a function $H(s) = LM_0\big(\frac{1}{(t-\xi-s)^{\gamma'-\beta}} + \frac{1}{(t-s)^{\gamma'-\beta}}\big) + LM_0|||u|||^{\rho}_{\alpha,\varepsilon,T}\big(\frac{1}{(t-\xi-s)^{\gamma'-\beta}s^{\varepsilon\rho}} + \frac{1}{(t-s)^{\gamma'-\beta}s^{\varepsilon\rho}}\big)$ which is integrable for $s \in (0, t-\xi)$. Furthermore, for each $s \in (0, t-\xi)$ and $r \in (0, t-\xi-s)$ we also have $S(r)f(u(s)) \in X^{\gamma'}$ and by (ii.0.12) we observe that (8.0.10) holds as $h \to 0^-$. Using Lebesgue's dominated convergence theorem we get $\lim_{h\to 0^-} J(h,\xi) = 0$. Therefore $\lim_{h\to 0^-} I_2(h) = 0$, which leads to (8.0.9).

If the scale is nested (8.0.9) also holds by embedding for $\gamma' < \max\{\beta, \gamma\}$. ■

We finally analyze continuity of $\mathcal{F}(u, u_0)(t)$ at $t = 0$.

Lemma 8.0.5 *Assume (ii.0.12), (ii.0.14), (ii.0.16) and let $u \in \mathcal{L}^{\infty}_{\alpha,\varepsilon}((0,T])$, $\varepsilon \ge 0$. Also, assume*

$$\beta \le \gamma' < \beta + 1 - \rho\varepsilon, \quad 0 \le \rho\varepsilon < 1 \tag{8.0.11}$$

or if $\gamma' = \beta + 1 - \rho\varepsilon \ne \beta + 1$ assume moreover that $|||u|||_{\alpha,\varepsilon,t} \to 0$ as $t \to 0^+$.

Then for $u_0 \in X^{\gamma'}$ such that $\lim_{t\to 0^+} \|S(t)u_0 - u_0\|_{\gamma'} = 0$ we have

$$\mathcal{F}(u, u_0)(t) \to u_0 \quad in \quad X^{\gamma'} \quad as \quad t \to 0^+.$$

Assuming (ii.0.13), the above holds for each $u_0 \in X^{\gamma'}$. If, in addition, the scale is nested, the above holds also for $\gamma' < \beta$.

Proof. Applying Lemma 8.0.3 i) with $\delta = 0$ we have

$$\|\int_0^t S(t-\tau)f(u(\tau))\,d\tau\|_{\gamma'} \le M(T,\gamma')t^{\beta+1-\gamma'-\rho\varepsilon}(t^{\rho\varepsilon} + |||u|||^{\rho}_{\alpha,\varepsilon,t}). \tag{8.0.12}$$

Since $\lim_{t\to 0^+} \|S(t)u_0 - u_0\|_{\gamma'} = 0$, if either (8.0.11) or $\gamma' = \beta + 1 - \rho\varepsilon \neq \beta + 1$ and $|||u|||_{\alpha,\varepsilon,t} \to 0$ as $t \to 0^+$, we get the statement.

If the scale is nested and $\gamma' < \beta$, we use the embedding $X^\beta \hookrightarrow X^{\gamma'}$ for the integral involving nonlinearity and (8.0.12) for $\gamma' = \beta$. $\blacksquare$

Chapter 9

Nonlinear perturbation of the semigroup

In this chapter we assume (ii.0.12) and proceed with the analysis of (ii.0.17) with f as in (ii.0.14)-(ii.0.16). Recall again that all spaces and norms below are for indexes in the interval $\mathcal{J}$, although we will not write this all the time for the sake of brevity.

9.1 Existence and uniqueness of solutions

We must first define the type of solutions of (ii.0.17) that we are considering. In this direction and in view of (ii.0.14) and (ii.0.17), any suitable notion of solution in $[0,T]$ must at least satisfy that
$(S1)$. $u : (0,T] \to X^\alpha$.
$(S2)$. For any $0 < \tau < T$ and for all $\tau \le t \le T$, $u(t)$ satisfies

$$u(t) = S(t-\tau)u(\tau) + \int_\tau^t S(t-s)f(u(s))\,ds$$

Additionally, it is also natural to require that
$(S3)$. For any $\tau > 0$, $u \in L^\infty([\tau,T], X^\alpha)$.

The following result shows that to prove uniqueness it is enough to have local uniqueness.

Lemma 9.1.1 Local uniqueness implies global uniqueness.
Assume (ii.0.12), (ii.0.14)-(ii.0.16) and let u, v be functions satisfying (S1), (S2), (S3) for some $T > 0$. Then, if there exists $T_0 < T$ such that $u = v$ for all $t \in (0,T_0]$ then $u = v$ in $[0,T]$.

Proof. From Lemma 8.0.4 with $\varepsilon = 0$ and $\gamma' = \alpha$, we have that $\tilde{u}_1(t) = u_1(t+T_0)$ and $\tilde{u}_2(t) = u_2(t+T_0)$ are continuous and bounded in X^α in $[0, T-T_0]$ with $\tilde{u}_1(0) = \tilde{u}_2(0)$. Hence $z(t) = \tilde{u}_1(t) - \tilde{u}_2(t)$ satisfies $z(0) = 0$ and

$$z(t) = \int_0^t S(t-s)\Big(f(\tilde{u}_1(s)) - f(\tilde{u}_2(s))\Big)\,ds, \quad 0 \le t \le T - T_0$$

and taking the X^α norm we get using (ii.0.12), $\|z(t)\|_\alpha \le \int_0^t \frac{M_0}{(t-s)^{\alpha-\beta}} \|f(\tilde u_1(s)) - f(\tilde u_2(s))\|_\beta \, ds$ for $0 \le t \le T - T_0$. Using (ii.0.15) and the bound in X^α of $\tilde u_1, \tilde u_2$ we get

$$\|z(t)\|_\alpha \le \int_0^t \frac{M_0 C}{(t-s)^{\alpha-\beta}} \|z(s)\|_\alpha \, ds, \quad 0 \le t \le T - T_0.$$

Now the singular Gronwall lemma in [31, 1.2.1], gives $z(t) = 0$ in $[0, T - T_0]$, that is $u_1 = u_2$ in $[T_0, T]$. ■

We also have the following regularity result for functions satisfying $(S1), (S2)$ and $(S3)$.

Lemma 9.1.2 *Assume (ii.0.12), (ii.0.14), (ii.0.16),*

$$\alpha \ge \gamma \ge \frac{\alpha\rho - \beta - 1}{\rho - 1} \tag{9.1.1}$$

and let

$$\kappa(\alpha, \beta, \gamma) := 1 + \beta - (\alpha - \gamma)\rho - \gamma \ge 0. \tag{9.1.2}$$

Assume for some $T > 0$ we have that u satisfies $(S1), (S2)$ and $(S3)$.

Then, for $\gamma' \in [\alpha, \beta + 1)$, all the following hold.

i) $u \in C((0, T], X^{\gamma'})$.

ii) If

$$|||u|||_{\alpha, \alpha-\gamma, T} \le K_0 \tag{9.1.3}$$

then

$$|||u|||_{\gamma', \gamma'-\gamma, T} \le C(K_0 + K_0^\rho + T^{1+\beta-\gamma}) \tag{9.1.4}$$

for some constant $C > 0$ which depends only on parameters $\alpha, \beta, \rho, \gamma, \gamma'$, the constants $M_0(\gamma', \alpha, T)$, $M_0(\gamma', \beta, T)$ as in (ii.0.12) and $T^{\kappa(\alpha,\beta,\gamma)}$. Hence, if

$$|||u|||_{\alpha, \alpha-\gamma, \tau} \to 0^+ \ \text{as } \tau \to 0^+ \quad \text{then} \ \ |||u|||_{\gamma', \gamma'-\gamma, \tau} \to 0^+ \ \text{as } \tau \to 0^+.$$

iii) Assume (ii.0.15) and let $u, \tilde u$ satisfy $(S1), (S2)$ and $(S3)$. If

$$\max\{|||u|||_{\alpha,\alpha-\gamma,T}, |||\tilde u|||_{\alpha,\alpha-\gamma,T}\} \le K_0 \quad |||u - \tilde u|||_{\alpha,\alpha-\gamma,T} \le L_0 \tag{9.1.5}$$

then

$$|||u - \tilde u|||_{\gamma', \gamma'-\gamma, T} \le \bar C L_0, \tag{9.1.6}$$

for some constant $\bar C > 0$ which depends only on the parameters $\alpha, \beta, \rho, \gamma, \gamma'$, the constants $M_0(\gamma', \alpha, T)$, $M_0(\gamma', \beta, T)$ in (ii.0.12), K_0 in (9.1.5), and $T^{1+\beta-\alpha}$, $T^{\kappa(\alpha,\beta,\gamma)}$.

Proof. For i), note that for any $\tau > 0$ small, writing $v(t) = u(t + \tau)$ with $t \in [0, T - \tau]$, $(S2)$ becomes

$$v(t) = S(t)v(0) + \int_0^t S(t-s) f(v(s)) \, ds.$$

Using Lemma 8.0.4 with $\varepsilon = 0$, $\gamma = \alpha$, $\gamma' \geq \gamma$ gives that $u \in C((\tau, T], X^{\gamma'})$, $\gamma' \in [\alpha, \beta+1)$.

For part ii), given $t \in (0, T]$ we write $u(t) = S\big(\frac{t}{2}\big)u\big(\frac{t}{2}\big) + \int_{\frac{t}{2}}^{t} S(t-s)f(u(s))ds$ and using (ii.0.12), (ii.0.14), (ii.0.16) we get

$$t^{\gamma'-\gamma}\|u(t)\|_{\gamma'} \leq \frac{M_0(\gamma', \alpha, T)}{(\frac{t}{2})^{\gamma'-\alpha}} t^{\gamma'-\gamma}\|u(\frac{t}{2})\|_{\alpha} + t^{\gamma'-\gamma}\int_{\frac{t}{2}}^{t} \frac{M_0(\gamma', \beta, T)}{(t-s)^{\gamma'-\beta}} L(1 + \|u(s)\|_{\alpha}^{\rho})ds.$$

For $N := \max\{2^{\gamma'-\gamma}M_0(\gamma', \alpha, T), LM_0(\gamma', \beta, T)\}$ and K_0 as in (9.1.3) we then have

$$t^{\gamma'-\gamma}\|u(t)\|_{\gamma'} \leq K_0 N + t^{\gamma'-\gamma}N\int_{\frac{t}{2}}^{t} \frac{ds}{(t-s)^{\gamma'-\beta}} + t^{\gamma'-\gamma}K_0^{\rho}N\int_{\frac{t}{2}}^{t} \frac{ds}{(t-s)^{\gamma'-\beta}s^{(\alpha-\gamma)\rho}}$$

and using the change of variables $s = tr$

$$t^{\gamma'-\gamma}\|u(t)\|_{\gamma'} \leq N\big(K_0 + \frac{t^{1+\beta-\gamma}}{2^{1+\beta-\gamma'}(1+\beta-\gamma')} + K_0^{\rho}t^{\kappa(\alpha,\beta,\gamma)}\int_{\frac{1}{2}}^{1} \frac{dr}{(1-r)^{\gamma'-\beta}r^{(\alpha-\gamma)\rho}}\big).$$

Because of (9.1.1) both $1+\beta-\gamma$ and $\kappa(\alpha, \beta, \gamma)$ are nonnegative so we get (9.1.4).

Writing $u(t) - \tilde{u}(t) = S\big(\frac{t}{2}\big)\big(u\big(\frac{t}{2}\big) - \tilde{u}\big(\frac{t}{2}\big)\big) + \int_{\frac{t}{2}}^{t} S(t-s)(f(u(s)) - f(\tilde{u}(s)))ds$, and using (ii.0.15) we get (9.1.6) in a similar manner. ■

Observe that none of conditions $(S1), (S2)$ or $(S3)$ have any information about the initial data. Therefore we define a suitable notion of solution of (ii.0.17) by incorporating some information on the initial data and the behaviour of the solution near $t = 0$.

Definition 9.1.3 *Given $f : X^{\alpha} \to X^{\beta}$ satisfying (ii.0.15) and $u_0 \in X^{\gamma}$, then a function $u \in \mathcal{L}^{\infty}_{\alpha,\alpha-\gamma}((0, T])$, $u(0) = u_0$ and satisfying (ii.0.17) for $0 < t \leq T$ is called a γ–solution of (ii.0.17) in $[0, T]$.*

Note that in particular, a γ–solution in $[0, T]$ satisfies $(S1)$, $(S2)$, $(S3)$ and $t^{\alpha-\gamma}\|u(t)\|_{\alpha} \leq M$, $t \in (0, T]$ for some $M > 0$.

Remark 9.1.4 *i) Observe that since γ–solutions satisfy (ii.0.17) they must be fixed points of (8.0.2), i.e. $u(t) = \mathcal{F}(t, u(t))$, and we are lead to use Lemma 8.0.3 with $\varepsilon = \alpha - \gamma \geq 0$. Then conditions (8.0.3), (8.0.6) read $0 \leq \rho(\alpha-\gamma) < 1$ and $\gamma \leq \beta + 1 - \rho(\alpha - \gamma)$, respectively, which lead to*

$$\alpha \geq \gamma, \quad \gamma > \alpha - \frac{1}{\rho} \quad and \quad \gamma \geq \frac{\alpha\rho - \beta - 1}{\rho - 1} \tag{9.1.7}$$

respectively.

ii) Now note that $\alpha - \frac{1}{\rho} \geq \frac{\alpha\rho-\beta-1}{\rho-1}$ if and only if $\alpha - \beta \leq \frac{1}{\rho}$. Thus the range (9.1.7) can be written as

$$\gamma \in E(\alpha, \beta, \rho) = \begin{cases} (\alpha - \frac{1}{\rho}, \alpha], & \text{if } 0 \leq \alpha - \beta \leq \frac{1}{\rho} \\ [\frac{\alpha\rho-\beta-1}{\rho-1}, \alpha], & \text{if } \frac{1}{\rho} < \alpha - \beta < 1. \end{cases} \tag{9.1.8}$$

For convenience, we introduce the following function

$$G(\alpha,\beta) := \begin{cases} G_1(\alpha,\beta) = \alpha - \frac{1}{\rho}, & \text{if } 0 \le \alpha-\beta \le \frac{1}{\rho}, \\ G_2(\alpha,\beta) = \frac{\alpha\rho-\beta-1}{\rho-1}, & \text{if } \frac{1}{\rho} < \alpha-\beta < 1. \end{cases} \tag{9.1.9}$$

Note that $G(\alpha,\beta) > \beta$ *iff* $\frac{1}{\rho} < \alpha-\beta < 1$ *and equality is only for* $\alpha-\beta = \frac{1}{\rho}$*. Thus the interval* $E(\alpha,\beta,\rho)$ *contains* β *iff* $0 \le \alpha-\beta < \frac{1}{\rho}$*.*
iii) When $\gamma \in (G(\alpha,\beta),\alpha]$ *we will say we are in the subcritical case, while the case* $\frac{1}{\rho} < \alpha-\beta < 1$ *and* $\gamma = \frac{\alpha\rho-\beta-1}{\rho-1}$ *is denoted the critical case.*

Then we have the following a priori result on the smoothness of γ–solutions.

Proposition 9.1.5 *Assume (ii.0.12), (ii.0.14), (ii.0.16) and let* $\gamma \in E(\alpha,\beta,\rho)$ *as in (9.1.8).*

Assume that for some $u_0 \in X^\gamma$ *there exists a* γ*-solution of (ii.0.17) in* $[0,T]$ *as in Definition 9.1.3, denoted* $u(\cdot,u_0)$*, then the* γ*–solution satisfies the following properties.*
i) **Time continuity***: for* $\beta \le \gamma' < \beta+1$ *and* $\gamma' \ge \gamma$*,*

$$u(\cdot,u_0) \in C((0,T],X^{\gamma'})$$

ii) **Continuity at** $t=0$*: Assume* $\beta \le \gamma' < \beta+1-\rho(\alpha-\gamma)$*,* $\gamma' \ge \gamma$ *or, if* $\gamma' = \beta+1-\rho(\alpha-\gamma)$ *assume moreover that* $|||u(\cdot,u_0)|||_{\alpha,\alpha-\gamma,t} \to 0$ *as* $t \to 0^+$*. If* $u_0 \in X^{\gamma'}$ *is such that* $\lim_{t\to 0^+} \|S(t)u_0 - u_0\|_{\gamma'} = 0$*, then*

$$u(\cdot,u_0) \in C([0,T],X^{\gamma'}).$$

If the scale is nested, i) and ii) hold also for $\gamma' < \beta$ *and* $\gamma' < \gamma$*.*

In particular, if $\gamma \le \beta$ *then we can take* $\gamma' = \beta$*, if* $\gamma \ge \beta$ *then we can take* $\gamma' = \gamma$*, and if* $\gamma \ge \frac{(\rho+1)\alpha-\beta-1}{\rho}$ *we can take* $\gamma' = \alpha$*. If the scale is nested, we can always take* $\gamma' = \gamma$*.*
iii) **A priori bounds***: for* $\beta \le \gamma' < \beta+1$ *and* $\gamma' \ge \gamma$*, we have the estimate*

$$t^{\gamma'-\gamma}\|u(t,u_0)\|_{\gamma'} \le K, \quad t \in (0,T] \tag{9.1.10}$$

for some $K > 0$ *which depends on the norm* $|||u(\cdot,u_0)|||_{\alpha,\alpha-\gamma,T}$ *of the* γ*–solution,* T *and* $\|u_0\|_\gamma$*.*
iv) **Smallness at** $t=0$*: Assume* $u_0 \in X^\gamma$ *is such that*

$$|||S(\cdot)u_0|||_{\gamma',\gamma'-\gamma,t} \to 0 \quad as \quad t \to 0^+, \tag{9.1.11}$$

for some $\beta \le \gamma' < \beta+1$ *and* $\gamma' \ge \gamma$*, and either* $\gamma \in (G(\alpha,\beta),\alpha]$ *(subcritical case) or* $\frac{1}{\rho} < \alpha-\beta < 1$ *and* $\gamma = \frac{\alpha\rho-\beta-1}{\rho-1}$ *(critical case) and* $|||u(\cdot,u_0)|||_{\alpha,\alpha-\gamma,t} \overset{t\to 0^+}{\to} 0$*.*

Then

$$|||u(\cdot,u_0)|||_{\gamma',\gamma'-\gamma,t} \overset{t\to 0^+}{\to} 0. \tag{9.1.12}$$

When the scale is nested we also obtain (9.1.10), (9.1.12) for any $\gamma' \in [\gamma,\beta+1)$*.*

Proof. Part i) comes from (ii.0.17) and Lemma 8.0.4 with $\varepsilon = \alpha - \gamma$.

For part ii) we use Lemma 8.0.5 with $\varepsilon = \alpha - \gamma$. Now it is clear that if $\gamma \leq \beta$ then we can take $\gamma' = \beta$. Also, since $\gamma \in E(\alpha, \beta, \rho)$ then $\gamma \leq \beta + 1 - \rho(\alpha - \gamma)$ and then, if $\gamma \geq \beta$ then we can take $\gamma' = \gamma$. Finally, if $\gamma \geq \frac{(\rho+1)\alpha - \beta - 1}{\rho}$ we have $\alpha \leq \beta + 1 - \rho(\alpha - \gamma)$ and we can take $\gamma' = \alpha$. If the scale is nested, all the arguments above hold also for $\gamma' < \beta$ or $\gamma' < \gamma$. In particular, we can always take $\gamma' = \gamma$.

For part iii) we use (8.0.7) with $\varepsilon = \alpha - \gamma$, so that (8.0.6) is satisfied because $\gamma \in E(\alpha, \beta, \rho)$ as in (9.1.8). Then we have

$$|||u(\cdot, u_0)|||_{\gamma', \gamma' - \gamma, T} \leq |||S(\cdot)u_0|||_{\gamma', \gamma' - \gamma, T} + M(T, \gamma') T^{\kappa(\alpha, \beta, \gamma)} \big(T^{\rho(\alpha - \gamma)} + |||u(\cdot, u_0)|||^{\rho}_{\alpha, \alpha - \gamma, T} \big).$$

By definition the γ–solution satisfies $|||u(\cdot, u_0)|||_{\alpha, \alpha - \gamma, T} \leq M$, for some $M > 0$ and by (8.0.1)

$$|||S(\cdot)u_0|||_{\gamma', \gamma' - \gamma, T} \leq M_0(\gamma, \gamma', T) \|u_0\|_{\gamma}.$$

Hence, we obtain (9.1.10).

Finally, to prove iv), note that as in Lemma 8.0.3 iii) we have for $t \leq T$,

$$|||u(\cdot, u_0)|||_{\gamma', \gamma' - \gamma, t} \leq |||S(\cdot)u_0|||_{\gamma', \gamma' - \gamma, t} + M(T, \gamma') t^{\kappa(\alpha, \beta, \gamma)} \big(t^{\rho(\alpha - \gamma)} + |||u(\cdot, u_0)|||^{\rho}_{\alpha, \alpha - \gamma, t} \big).$$

and the right hand side above goes to 0 as $t \to 0^+$ because of assumption (9.1.11) and because in the subcritical case we have $\kappa(\alpha, \beta, \gamma) > 0$ while in the critical case $\kappa(\alpha, \beta, \gamma) = 0$ and $|||u(\cdot, u_0)|||_{\alpha, \alpha - \gamma, t} \overset{t \to 0^+}{\to} 0$ by assumption.

When the scale is nested, due to Lemma 8.0.3, with minor changes in the proof above, one also obtains (9.1.10), (9.1.12) for $\gamma' \in [\gamma, \beta + 1)$ such that $\gamma' < \beta$. ∎

Now we give some natural condition guaranteeing the additional assumption (9.1.11).

Lemma 9.1.6 *Assume that $\gamma' > \gamma$ and there exists $\delta \in (\gamma, \gamma']$ such that*

$$X^{\delta} \cap X^{\gamma} \quad \textit{dense in } X^{\gamma}. \tag{9.1.13}$$

Then for any $u_0 \in X^{\gamma}$ we have $t^{\gamma' - \gamma} \|S(t)u_0\|_{\gamma'} \to 0$ as $t \to 0^+$ or equivalently

$$|||S(\cdot)u_0|||_{\gamma', \gamma' - \gamma, t} \to 0 \quad \textit{as} \quad t \to 0^+. \tag{9.1.14}$$

Convergence in (9.1.14) is actually uniform for u_0 in compact subsets of X^{γ}.

Proof. In what follows, assume $0 < t \leq 1$. Then from (ii.0.12), for any $v_0 \in X^{\delta} \cap X^{\gamma}$ we have

$$t^{\gamma' - \gamma} \|S(t)v_0\|_{\gamma'} \leq t^{\gamma' - \gamma} \frac{M_0(\delta, \gamma')}{t^{\gamma' - \delta}} \|v_0\|_{\delta} = t^{\delta - \gamma} M_0(\delta, \gamma') \|v_0\|_{\delta} \overset{t \to 0^+}{\longrightarrow} 0.$$

Now for any $u_0 \in X^{\gamma}$ and $\varepsilon > 0$, take $v_0 \in X^{\delta} \cap X^{\gamma}$ such that $\|u_0 - v_0\|_{\gamma} \leq r < \frac{\varepsilon}{2M_0(\gamma, \gamma')}$ and $T \leq 1$ such that $t^{\gamma' - \gamma} \|S(t)v_0\|_{\gamma'} \leq \frac{\varepsilon}{2}$, $\quad t \in (0, T]$. Then, for $t \in (0, T]$,

$$t^{\gamma' - \gamma} \|S(t)u_0\|_{\gamma'} = t^{\gamma' - \gamma} \|S(t)(u_0 - v_0)\|_{\gamma'} + t^{\gamma' - \gamma} \|S(t)v_0\|_{\gamma'} \leq M_0(\gamma, \gamma') r + \frac{\varepsilon}{2} \leq \varepsilon,$$

where we have used again (ii.0.12) in the first term. Thus we get the result.

Finally, assume that the convergence is not uniform for u_0 in compact subsets of X^γ. Then, there exist a compact set K in X^γ and sequences $K \ni u_n \to u_0 \in K$ in X^γ and $t_n \to 0$ and $\varepsilon > 0$ such that $t_n^{\gamma'-\gamma}\|S(t_n)u_n\|_{\gamma'} > \varepsilon$. But then $t_n^{\gamma'-\gamma}\|S(t_n)(u_n - u_0)\|_{\gamma'} \leq M_0(\gamma,\gamma')\|u_n - u_0\|_\gamma \to 0$ and hence $t_n^{\gamma'-\gamma}\|S(t_n)u_0\|_{\gamma'} \geq t_n^{\gamma'-\gamma}\|S(t_n)u_n\|_{\gamma'} - t_n^{\gamma'-\gamma}\|S(t_n)(u_n - u_0)\|_{\gamma'} \geq \frac{\epsilon}{2}$ for almost all $n \in \mathbb{N}$, which contradicts (9.1.14). ■

We now prove the existence result which we divide into two cases, the subcritical and the critical case respectively. The main reason for this is that in the critical case, $\frac{1}{\rho} < \alpha - \beta < 1$ and $\gamma = \frac{\alpha\rho-\beta-1}{\rho-1}$, the constant $C(T)$ in (8.0.7) and (8.0.8) is not small for small T, since $\kappa(\alpha,\beta,\gamma) = 0$, and we must proceed in a different way in the proofs.

Theorem 9.1.7 (Existence and uniqueness of γ-solutions. Subcritical case) *Assume (ii.0.12), (ii.0.14)-(ii.0.16) and let $G(\alpha,\beta)$ be as in (9.1.9). Then for*

$$\gamma \in (G(\alpha,\beta), \alpha]$$

all of the following hold.
i) For any $u_0 \in X^\gamma$ and any $T > 0$ there exists at most a γ–solution of (ii.0.17) in $[0,T]$ in the sense of Definition 9.1.3.

ii) For any $R_0 > 0$, there exists a $T > 0$ such that for any $u_0 \in X^\gamma$ with $\|u_0\|_\gamma \leq R_0$, there exits a (unique) γ–solution of (ii.0.17) in $[0,T]$ in the sense of Definition 9.1.3.

In particular $T \geq T(u_0)$ where

$$T(u_0) = \frac{C}{(1+\|u_0\|_\gamma)^{\frac{1}{\gamma - \frac{\rho\alpha-\beta-1}{\rho-1}}}}, \tag{9.1.15}$$

where C is a positive constant which depends on α,β,γ,ρ but does not depend on $u_0 \in X^\gamma$.

iii) For these solutions in ii), Proposition 9.1.5 applies. In particular, if $|||S(\cdot)u_0|||_{\alpha,\alpha-\gamma,t} \to 0$ as $t \to 0^+$, then

$$|||u|||_{\alpha,\alpha-\gamma,t} \to 0 \quad \text{as } t \to 0^+.$$

Also when $\gamma \geq \beta$ or the scale is nested, we have $u \in C([0,T], X^\gamma)$ provided $\lim_{t\to 0^+} \|S(t)u_0 - u_0\|_\gamma = 0$.

Proof. We first prove the existence and then the uniqueness.
Existence. We first show that $\mathcal{F}(\cdot, u_0)$ in (8.0.2) is a contraction in a closed subset of $\mathcal{L}^\infty_{\alpha,\alpha-\gamma}((0,T])$ for some $T \leq 1$. Fix $\gamma \in (G(\alpha,\beta),\alpha]$ and $R_0 > 0$. Consider u_0 such that $\|u_0\|_\gamma \leq R_0$ and define for $K_0 > R_0 M_0$, with $M_0 = M_0(\gamma,\alpha,1)$ as in (ii.0.12), the set

$$\mathcal{K}_{T,K_0} = \{\varphi \in \mathcal{L}^\infty_{\alpha,\alpha-\gamma}((0,T]),\ |||\varphi|||_{\alpha,\alpha-\gamma,T} \leq K_0\} \tag{9.1.16}$$

and $T \leq 1$ is chosen below. Observe that $\gamma < \beta + 1 - \rho(\alpha - \gamma)$ because $\gamma \in (G(\alpha,\beta),\alpha]$ and then we can use Lemma 8.0.3 iii) with $\gamma' = \alpha$, $\varepsilon = \delta = \alpha - \gamma$ for $u \in \mathcal{L}^\infty_{\alpha,\alpha-\gamma}((0,T])$ since (8.0.6) is satisfied. Then in (8.0.7) we get

$$|||\mathcal{F}(u,u_0)|||_{\alpha,\alpha-\gamma,T} \leq |||S(\cdot)u_0|||_{\alpha,\alpha-\gamma,T} + C(T,\alpha)\big(T^{\rho(\alpha-\gamma)} + |||u|||^\rho_{\alpha,\alpha-\gamma,T}\big)$$

where $C(T,\alpha) = M(T,\alpha)T^{\kappa(\alpha,\beta,\gamma)}$ as in (9.1.2). Since $\kappa(\alpha,\beta,\gamma) > 0$, using (8.0.1) and $u \in \mathcal{K}_{T,K_0}$,

$$|||\mathcal{F}(u,u_0)|||_{\alpha,\alpha-\gamma,T} \leq R_0 M_0(\gamma,\alpha,T) + C(T,\alpha)\big(T^{\rho(\alpha-\gamma)} + K_0^{\rho}\big) \leq K_0$$

for some $T = T(K_0)$ small enough. That is, $\mathcal{F}$ maps $\mathcal{K}_{T,K_0}$ into itself.

Now for u_1, u_2 in $\mathcal{K}_{T,K_0}$, we can use Lemma 8.0.3 iv), with $\gamma' = \alpha$, $\varepsilon = \delta = \alpha - \gamma$ to get in (8.0.8)

$$\begin{aligned} |||\mathcal{F}(u_1,u_0)-&\mathcal{F}(u_2,u_0)|||_{\alpha,\alpha-\gamma,T} \\ &\leq C(T,\alpha)\big(T^{(\rho-1)(\alpha-\gamma)} + |||u_1|||^{\rho-1}_{\alpha,\alpha-\gamma,T} + |||u_2|||^{\rho-1}_{\alpha,\alpha-\gamma,T}\big)|||u_1-u_2|||_{\alpha,\alpha-\gamma,T} \\ &\leq C(T,\alpha)\big(T^{(\rho-1)(\alpha-\gamma)} + 2K_0^{\rho-1}\big)|||u_1-u_2|||_{\alpha,\alpha-\gamma,T} \end{aligned} \tag{9.1.17}$$

with $C(T,\alpha) = M(T,\alpha)T^{\kappa(\alpha,\beta,\gamma)}$. Thus, again $\kappa(\alpha,\beta,\gamma) = 1+\beta-(\alpha-\gamma)\rho-\gamma > 0$ and for small enough $T = T(K_0)$, $\mathcal{F}(\cdot,u_0)$ is a contraction.

Time of existence. In particular, taking $\|u_0\|_\gamma := R_0$ and $K_0 := \|u_0\|_\gamma(M_0(\gamma,\alpha,1)+1)+1$ in the proof above, if we require that $T \leq 1$ satisfies

$$C(T,\alpha)\big(T^{\rho(\alpha-\gamma)} + K_0^{\rho}\big) \leq \|u_0\|_\gamma + 1 \qquad \textit{and} \qquad C(T,\alpha)\big(T^{(\rho-1)(\alpha-\gamma)} + 2K_0^{\rho-1}\big) \leq \frac{1}{2}$$

then $\mathcal{F}(\cdot,u_0)$ is a contraction in $\mathcal{K}_{T,K_0}$. In particular, this holds if $T \leq 1$ satisfies

$$T^{\kappa(\alpha,\beta,\gamma)}N(1+\|u_0\|_\gamma)^{\rho-1} = \frac{1}{2}$$

with $N := M\max\{(1+M_0)^\rho + 1, 2(M_0+1)^{\rho-1}+1\}$. Hence (9.1.15) holds with $C := (1/2N)^{\frac{1}{\kappa(\alpha,\beta,\gamma)}} < 1$.

Uniqueness. Assume u_1, u_2 are two γ–solutions in $[0,T]$ as in Definition 9.1.3 with $u_1(0) = u_2(0) = u_0$. Then we can take K_0 in (9.1.16) large enough such that $u_1, u_2 \in \mathcal{K}_{T,K_0}$ and use the contraction (9.1.17) for a small enough $T_0 = T(K_0)$. Hence $u_1 = u_2$ on $[0,T_0]$. Then Lemma 9.1.1 concludes.

Finally, part iii) comes directly from Proposition 9.1.5. ■

We now consider the critical case, that is, $\frac{1}{\rho} < \alpha - \beta < 1$ and $\gamma = \frac{\alpha\rho-\beta-1}{\rho-1}$.

Theorem 9.1.8 (Existence of γ-solutions. Critical case)
Assume (ii.0.12), (ii.0.14)-(ii.0.16), $\frac{1}{\rho} < \alpha-\beta < 1$ and

$$\gamma = \frac{\alpha\rho-\beta-1}{\rho-1}.$$

Assume also that for any $v_0 \in X^\gamma$ we have

$$t^{\alpha-\gamma}\|S(t)v_0\|_\alpha \to 0 \quad \textit{as} \quad t \to 0^+, \tag{9.1.18}$$

see Lemma 9.1.6 for sufficient conditions. Then all of the following hold.
i) For any $u_0 \in X^\gamma$ *and any* $T > 0$ *there exists at most a* γ*–solution of (ii.0.17) in* $[0,T]$ *in the sense of Definition 9.1.3 with* $u(0) = u_0$ *and such that*

$$|||u|||_{\alpha,\alpha-\gamma,t} \to 0 \quad \text{as } t \to 0^+.$$

ii) There exists $r > 0$ *and* $K_0 > 0$ *such that for all* $v_0 \in X^\gamma$*, there exists certain* $T = T(v_0)$ *such that for each* $u_0 \in B_{X^\gamma}(v_0, r)$*, (8.0.2) has a unique* γ*–solution* u *in the sense of Definition 9.1.3, defined in* $[0,T]$ *with* $u(0) = u_0$ *and such that*

$$|||u|||_{\alpha,\alpha-\gamma,T} \le K_0.$$

iii) The existence time in ii) above is uniform for initial data in bounded sets in X^γ *with Hausdorff measure of non-compactness smaller than* r*.*
iv) The solutions in ii) satisfy

$$|||u|||_{\alpha,\alpha-\gamma,t} \to 0 \quad \text{as } t \to 0^+. \tag{9.1.19}$$

In particular, by Proposition 9.1.5 ii), we get that $u \in C([0,T], X^\gamma)$ *provided* $\lim_{t\to 0^+} \|S(t)u_0 - u_0\|_\gamma = 0$*.*

Proof. We first prove the existence and then the uniqueness.
Existence. Observe that here we take in Lemma 8.0.3 $\gamma' = \alpha$, $\varepsilon = \delta = \alpha - \gamma$. Since now $\kappa(\alpha, \beta, \gamma) = 0$, in parts ii) and iii) of Lemma 8.0.3 we have $C(T, \alpha) = M(T, \alpha)$ and recall that the constant $M(T, \alpha)$ is uniform in bounded time intervals. Thus, $M(1, \alpha) \ge M(T, \alpha)$ for $T \le 1$. Then for any K_0 such that

$$0 < K_0^{\rho-1} \le \frac{1}{4M(1,\alpha)}, \tag{9.1.20}$$

using (9.1.18), we choose $T = T(K_0, v_0) \le 1$ such that

$$|||S(\cdot)v_0|||_{\alpha,\alpha-\gamma,T} \le \frac{K_0}{4}, \qquad M(T,\alpha)T^{\rho(\alpha-\gamma)} < \frac{K_0}{4}, \qquad M(T,\alpha)T^{(\rho-1)(\alpha-\gamma)} < \frac{1}{4} \tag{9.1.21}$$

and define $\mathcal{K}_{T,K_0}$ as in (9.1.16). Now define $r = \frac{K_0}{4M_0}$ with $M_0 = M_0(\gamma, \alpha, 1)$ as in (ii.0.12) and take $u_0 \in X^\gamma$ such that $\|u_0 - v_0\|_\gamma < r$. Using Lemma 8.0.3 iii) with $\gamma' = \alpha$, $\varepsilon = \delta = \alpha - \gamma$ for u_0 as above and $u \in \mathcal{K}_{T,K_0}$ we have in (8.0.7)

$$\begin{aligned}
|||\mathcal{F}(u,u_0)|||_{\alpha,\alpha-\gamma,T} &\le |||S(\cdot)u_0|||_{\alpha,\alpha-\gamma,T} + M(T,\alpha)\big(T^{\rho(\alpha-\gamma)} + |||u|||^\rho_{\alpha,\alpha-\gamma,T}\big) \\
&\le |||S(\cdot)(u_0 - v_0)|||_{\alpha,\alpha-\gamma,T} + |||S(\cdot)v_0|||_{\alpha,\alpha-\gamma,T} \\
&\qquad\qquad + M(T,\alpha)\big(T^{\rho(\alpha-\gamma)} + |||u|||^\rho_{\alpha,\alpha-\gamma,T}\big) \\
&\le M_0 r + |||S(\cdot)v_0|||_{\alpha,\alpha-\gamma,T} + M(T,\alpha)\big(T^{\rho(\alpha-\gamma)} + K_0^\rho\big) \\
&< \frac{K_0}{4} + \frac{K_0}{4} + \frac{K_0}{4} + \frac{K_0}{4} = K_0.
\end{aligned} \tag{9.1.22}$$

Hence, $\mathcal{F}$ maps $\mathcal{K}_{T,K_0}$ into itself.

For u_0 as above and u, v in $\mathcal{K}_{T,K_0}$ we can also use Lemma 8.0.3 iv) with $\gamma' = \alpha$, $\varepsilon = \delta = \alpha - \gamma$ to get in (8.0.8)

$$\begin{aligned} |||\mathcal{F}(u,u_0) - \mathcal{F}(v,u_0)|||_{\alpha,\alpha-\gamma,T} & \\ &\leq M(T,\alpha)\big(T^{(\rho-1)(\alpha-\gamma)} + |||u|||^{\rho-1}_{\alpha,\alpha-\gamma,T} + |||v|||^{\rho-1}_{\alpha,\alpha-\gamma,T}\big)|||u-v|||_{\alpha,\alpha-\gamma,T} \\ &\leq M(T,\alpha)\big(T^{(\rho-1)(\alpha-\gamma)} + 2K_0^{\rho-1}\big)|||u-v|||_{\alpha,\alpha-\gamma,T} \leq \frac{3}{4}|||u-v|||_{\alpha,\alpha-\gamma,T}, \end{aligned}$$

because of (9.1.21). Then $\mathcal{F}$ is a strict contraction in $\mathcal{K}_{T,K_0}$ and part ii) is proved.

Part iii) is immediate from ii) since r is independent of $v_0 \in X^\gamma$ and a set of measure of non-compactness less than r can be covered by a finite number of balls of radius r.

In order to prove (9.1.19) observe that as in (9.1.22) we have for the fixed point of $\mathcal{K}_{T,K_0}$ and for $0 < t \leq T \leq 1$,

$$|||u|||_{\alpha,\alpha-\gamma,t} \leq |||S(\cdot)u_0|||_{\alpha,\alpha-\gamma,t} + M(T,\alpha)\big(t^{\rho(\alpha-\gamma)} + |||u|||^{\rho}_{\alpha,\alpha-\gamma,t}\big)$$

and $|||u|||^{\rho}_{\alpha,\alpha-\gamma,t} \leq K_0^{\rho-1}|||u|||_{\alpha,\alpha-\gamma,t} \leq \frac{1}{4M(1,\alpha)}|||u|||_{\alpha,\alpha-\gamma,t}$. Hence, by (9.1.18),

$$\frac{3}{4}|||u|||_{\alpha,\alpha-\gamma,t} \leq |||S(\cdot)u_0|||_{\alpha,\alpha-\gamma,t} + M(T,\alpha)t^{\rho(\alpha-\gamma)} \to 0 \text{ as } t \to 0.$$

The rest comes from Proposition 9.1.5 using (9.1.19).

Uniqueness. Assume u_1, u_2 are two γ–solutions in $[0,T]$ as in Definition 9.1.3 with $u_1(0) = u_2(0) = u_0$ and

$$|||u_i|||_{\alpha,\alpha-\gamma,t} \to 0 \text{ as } t \to 0^+, \qquad i = 1,2.$$

Then taking $v_0 = u_0$ and with the notations in part ii), we can take $T_0 \leq T(u_0)$ small enough such that $|||u_i|||_{\alpha,\alpha-\gamma,T_0} \leq K_0$ and both are fixed points of $\mathcal{F}$. Hence $u_1 = u_2$ on $[0,T_0]$. Then Lemma 9.1.1 concludes. ■

Remark 9.1.9 *Observe that the existence parts in Theorems 9.1.7 and 9.1.8 prove Theorem ii.0.12.*

Note that Theorems 9.1.7 and 9.1.8 state the uniqueness of γ–solution (in a certain class) for initial data $u_0 \in X^\gamma$. Since the initial data could belong to several spaces of the scale (for example if the scale is nested) the following results gives the consistency of such solutions.

Proposition 9.1.10 (Consistency of solutions)

With the notations in Theorems 9.1.7 or 9.1.8 and with $u_0 \in X^\gamma$, let u be a γ–solution in the subcritical case or a γ–solution such that

$$|||u|||_{\alpha,\alpha-\gamma,t} \to 0 \quad \text{as } t \to 0^+$$

in the critical case, defined in $[0,T]$.

If $u_0 \in X^{\tilde{\gamma}}$ *for some* $\tilde{\gamma} \in (\gamma, \alpha]$, *then* u *is in fact a* $\tilde{\gamma}$*–solution defined in* $[0,T]$. *In particular, for any* $0 < s < T$, *and* $\tilde{\gamma} \in [\gamma, \alpha]$, *assume* $u(s) \in X^{\tilde{\gamma}}$ *(this is true for example for* $\tilde{\gamma} \in [\beta, \alpha]$, $\tilde{\gamma} \geq \gamma$ *by Proposition 9.1.5 i)). Then* $u(\cdot + s)$ *defined in* $[0, T-s]$ *is the* $\tilde{\gamma}$*-solution with initial data* $u(s)$. *In other words,* $u(t, u(s)) = u(t+s)$ *for all* $s \in [0,T]$ *and all* $t \in [0, T-s]$.

Proof. Since $\tilde{\gamma} \in (\gamma, \alpha] \subset E(\alpha, \beta, \rho)$, independently of whether γ is critical or subcritical, we can use Theorem 9.1.7 in $X^{\tilde{\gamma}}$ so there exists a unique $\tilde{\gamma}$–solution with initial data u_0, $\tilde{u}$ and $t^{\alpha - \tilde{\gamma}} \|\tilde{u}(t)\|_\alpha \leq M$ for small enough $0 < t < 1$ and some constant M.

However, since $\gamma < \tilde{\gamma}$, we also have $t^{\alpha-\gamma}\|\tilde{u}(t)\|_\alpha = t^{\tilde{\gamma}-\gamma} t^{\alpha-\tilde{\gamma}} \|\tilde{u}(t)\|_\alpha \leq t^{\tilde{\gamma}-\gamma} M$ and therefore, in the subcritical case $\tilde{u}$ is a γ–solution, and moreover $|||\tilde{u}|||_{\alpha, \alpha-\gamma, t} \to 0$ as $t \to 0^+$, in the critical case. By the uniqueness in Theorems 9.1.7 or 9.1.8, $u = \tilde{u}$, and then, in turn, u is a $\tilde{\gamma}$–solution.

For the second part, just observe that form Lemma 9.1.2 i) we have that $u(\cdot + s) \in C([0, T-s], X^\alpha)$. Hence we get the result. ∎

We now prove continuous dependence of γ–solutions.

Proposition 9.1.11 (Continuous dependence)

Assume (ii.0.12), (ii.0.14)-(ii.0.16) and let $G(\alpha, \beta)$ *be as in (9.1.9). Also assume either*

i) **Subcritical case.** $\gamma \in (G(\alpha, \beta), \alpha]$ *and* $u_0, u_1 \in X^\gamma$ *and let* $u(\cdot, u_0), u(\cdot, u_1)$ *be corresponding* γ*–solutions, defined in* $[0,T]$.

ii) **Critical case.** *(9.1.18) holds true,* $\frac{1}{\rho} < \alpha - \beta < 1$ *and* $\gamma = \frac{\alpha\rho - \beta - 1}{\rho - 1}$, *and let* $r > 0$ *be as in Theorem 9.1.8. For any* $v_0 \in X^\gamma$ *and for any* $u_0, u_1 \in B_{X^\gamma}(v_0, r)$ *let* $u(\cdot, u_0), u(\cdot, u_1)$ *be corresponding* γ*–solutions, defined in* $[0,T]$ *such that*

$$|||u(\cdot, u_i)|||_{\alpha, \alpha-\gamma, t} \to 0 \quad \text{as } t \to 0^+, \qquad i = 1, 2.$$

Then there exists a constant $K > 0$, *which depends on the norm* $|||u(\cdot, u_i)|||_{\alpha, \alpha-\gamma, T}$, $i = 0, 1$, *of the* γ*–solutions and* T, *such that for any* $\beta \leq \gamma' < \beta + 1$ *and* $\gamma' \geq \gamma$

$$t^{\gamma' - \gamma} \|u(t, u_0) - u(t, u_1)\|_{\gamma'} \leq K \|u_0 - u_1\|_\gamma, \ t \in (0, T]. \tag{9.1.23}$$

When the scale is nested we also obtain (9.1.23) for any $\gamma' \in [\gamma, \beta + 1)$ *such that* $\gamma' < \beta$.

Proof. Subcritical case. Observe that since $\gamma < \beta + 1 - \rho(\alpha - \gamma)$ because $\gamma \in (G(\alpha, \beta), \alpha]$, then (8.0.6) is satisfied with $\varepsilon = \delta = \alpha - \gamma$ and then from (ii.0.17) and (8.0.8) in Lemma 8.0.3 iv), we have

$$\begin{aligned} |||u(\cdot, u_0) - u(\cdot, u_1)|||_{\gamma', \gamma'-\gamma, t} \leq |||S(\cdot)(u_0 - u_1)|||_{\gamma', \gamma'-\gamma, t} + C(t, \gamma')\Big(t^{(\rho-1)(\alpha-\gamma)} \\ + |||u(\cdot, u_0)|||^{\rho-1}_{\alpha, \alpha-\gamma, t} + |||u(\cdot, u_1)|||^{\rho-1}_{\alpha, \alpha-\gamma, t}\Big) |||u(\cdot, u_0) - u(\cdot, u_1)|||_{\alpha, \alpha-\gamma, t}, \end{aligned} \tag{9.1.24}$$

where $C(t,\gamma') = M(t,\gamma')t^{\kappa(\alpha,\beta,\gamma)}$ and $t \in (0,T]$. Using that $|||u(\cdot,u_i)|||_{\alpha,\alpha-\gamma,T} \le K_0$, for $i = 0,1$, and some K_0, we have

$$C(t,\gamma')\Big(t^{(\rho-1)(\alpha-\gamma)} + |||u(\cdot,u_0)|||^{\rho-1}_{\alpha,\alpha-\gamma,t} + |||u(\cdot,u_1)|||^{\rho-1}_{\alpha,\alpha-\gamma,t}\Big)$$
$$\le M(T,\gamma')t^{\kappa(\alpha,\beta,\rho)}\Big(t^{(\rho-1)(\alpha-\gamma)} + 2K_0^{\rho-1}\Big).$$

Since $\kappa(\alpha,\beta,\gamma) > 0$ we can find $t_0 = t_0(K_0) \le T$ such that

$$M(T,\gamma')t_0^{\kappa(\alpha,\beta,\rho)}\Big(t_0^{(\rho-1)(\alpha-\gamma)} + 2K_0^{\rho-1}\Big) \le \frac{3}{4}. \tag{9.1.25}$$

Choosing first $\gamma' = \alpha$ we get from (9.1.24), (9.1.25) and from (8.0.1) that

$$\frac{1}{4}|||u(\cdot,u_0) - u(\cdot,u_1)|||_{\alpha,\alpha-\gamma,t_0} \le |||S(\cdot)(u_0 - u_1)|||_{\alpha,\alpha-\gamma,t_0} \le M_0\|u_0 - u_1\|_\gamma. \tag{9.1.26}$$

Then, using (9.1.25), (9.1.26) in (9.1.24) we have

$$|||u(\cdot,u_0) - u(\cdot,u_1)|||_{\gamma',\gamma'-\gamma,t_0} \le |||S(\cdot)(u_0 - u_1)|||_{\gamma',\gamma'-\gamma,t_0} + 3M_0\|u_0 - u_1\|_\gamma$$

which together with (8.0.1) proves (9.1.23) for $t \in (0,t_0]$.

Now $\tilde{u}_0(t) = u(t+t_0,u_0)$ and $\tilde{u}_1(t) = u(t+t_0,u_1)$ are continuous and bounded in X^α in $[0,T-t_0]$ and

$$\tilde{u}_0(t) - \tilde{u}_1(t) = S(t)(\tilde{u}_0(0) - \tilde{u}_1(0)) + \int_0^t S(t-s)\Big(f(\tilde{u}_0(s)) - f(\tilde{u}_1(s))\Big)\,ds, \quad 0 \le t \le T - t_0$$

and taking the X^α norm we get using (ii.0.12) that for $0 \le t \le T - t_0$,

$$\|\tilde{u}_0(t) - \tilde{u}_1(t)\|_\alpha \le \|S(t)(\tilde{u}_0(0) - \tilde{u}_1(0))\|_\alpha + \int_0^t \frac{M_0}{(t-s)^{\alpha-\beta}}\|f(\tilde{u}_0(s)) - f(\tilde{u}_1(s)\|_\beta ds.$$

Using (ii.0.15) and the bound in X^α of $\tilde{u}_0, \tilde{u}_1$ we get, for some C,

$$\|\tilde{u}_0(t) - \tilde{u}_1(t)\|_\alpha \le M_0\|\tilde{u}_0(0) - \tilde{u}_1(0)\|_\alpha + \int_0^t \frac{M_0 C}{(t-s)^{\alpha-\beta}}\|\tilde{u}_0(s) - \tilde{u}_1(s)\|_\alpha\,ds, \quad 0 \le t \le T - t_0. \tag{9.1.27}$$

Now the singular Gronwall lemma in [31, 1.2.1], gives

$$\|\tilde{u}_0(t) - \tilde{u}_1(t)\|_\alpha \le C\|\tilde{u}_0(0) - \tilde{u}_1(0)\|_\alpha \quad 0 \le t \le T - t_0$$

which together with (9.1.26) gives $|||u(\cdot,u_0) - u(\cdot,u_1)|||_{\alpha,\alpha-\gamma,T} \le C\|u_0 - u_1\|_\gamma$. Plugging this into (9.1.24) with $t = T$ and using (8.0.1), we get (9.1.23), which proves i).

Critical case. Now as in (9.1.24), we have for $t \in (0,T]$ and $\beta \le \gamma' < \beta + 1$, $\gamma' \ge \gamma$

$$\begin{aligned}|||u(\cdot,u_0) - u(\cdot,u_1)|||_{\gamma',\gamma'-\gamma,t} &\le |||S(\cdot)(u_0 - u_1)|||_{\gamma',\gamma'-\gamma,t} + C(t,\gamma')\Big(t^{(\rho-1)(\alpha-\gamma)} \\ &+ |||u(\cdot,u_0)|||^{\rho-1}_{\alpha,\alpha-\gamma,t} + |||u(\cdot,u_1)|||^{\rho-1}_{\alpha,\alpha-\gamma,t}\Big)|||u(\cdot,u_0) - u(\cdot,u_1)|||_{\alpha,\alpha-\gamma,t},\end{aligned} \tag{9.1.28}$$

where now $C(t,\gamma') = M(t,\gamma')t^{\kappa(\alpha,\beta,\gamma)} = M(t,\gamma')$, because in the critical case $\kappa(\alpha,\beta,\gamma) = 0$. Now we can chose t such that $|||u(\cdot,u_i)|||_{\alpha,\alpha-\gamma,t} \leq K_0$ for $i = 0,1$, with K_0 as in (9.1.20) and then

$$C(t,\gamma')\Big(t^{(\rho-1)(\alpha-\gamma)}+|||u(\cdot,u_0)|||^{\rho-1}_{\alpha,\alpha-\gamma,t}+|||u(\cdot,u_1)|||^{\rho-1}_{\alpha,\alpha-\gamma,t}\Big) \leq M(T,\gamma')\Big(t^{(\rho-1)(\alpha-\gamma)}+2K_0^{\rho-1}\Big).$$

Thus we can choose $t_0 = t_0(K_0) \leq T$ such that

$$M(T,\gamma')\Big(t_0^{(\rho-1)(\alpha-\gamma)} + 2K_0^{\rho-1}\Big) \leq M(T,\gamma')t_0^{\beta+1-\alpha} + \frac{M(T,\gamma')}{2M(1,\alpha)} \leq \frac{1}{4} + \frac{M(T,\gamma')}{2M(1,\alpha)},$$

(compare with (9.1.25) in the subcritical case).

Choosing first $\gamma' = \alpha$ and recalling that $M(1,\alpha) \geq M(T,\alpha)$ as in Theorem 9.1.8, we get from the above and (8.0.1) that

$$\frac{1}{4}|||u(\cdot,u_0) - u(\cdot,u_1)|||_{\alpha,\alpha-\gamma,t_0} \leq |||S(\cdot)(u_0-u_1)|||_{\alpha,\alpha-\gamma,t_0} \leq M_0\|u_0-u_1\|_\gamma. \qquad (9.1.29)$$

Then, using (9.1.29) in (9.1.28), we have

$$|||u(\cdot,u_0) - u(\cdot,u_1)|||_{\gamma',\gamma'-\gamma,t_0} \leq |||S(\cdot)(u_0-u_1)|||_{\gamma',\gamma'-\gamma,t_0} + (1+\frac{2M(T,\gamma')}{M(1,\alpha)})M_0\|u_0-u_1\|_\gamma,$$

which together with (8.0.1) proves (9.1.23) for each $t \in (0,t_0] \subset (0,T]$.

Now we argue as in (9.1.27) to get $|||u(\cdot,u_0) - u(\cdot,u_1)|||_{\alpha,\alpha-\gamma,T} \leq C\|u_0-u_1\|_\gamma$ and plugging this into (9.1.28) with $t = T$ and using (8.0.1), we get (9.1.23) for $t \in (0,T]$.

When the scale is nested we proceed as above for $\gamma' < \beta$ using Lemma 8.0.3. ■

We now study the continuation of γ–solutions.

Proposition 9.1.12 *Let $u_0 \in X^\gamma$.*
i) Then the γ–solution in either Theorem 9.1.7 or 9.1.8 can be continued to the maximal interval of existence $[0,\tau_{u_0})$ where $\tau_{u_0} \leq \infty$.
ii) If $\tau_{u_0} < \infty$, then for any $\gamma' \in [\beta,\alpha]$ such that either $\gamma \leq \gamma'$, in the **subcritical case** *or $\gamma < \gamma'$, in the* **critical case***, we have*

$$\limsup_{t\to\tau_{u_0}^-} \|u(t,u_0)\|_{\gamma'} = \infty.$$

Furthermore, there is a constant $c > 0$ such that the estimate

$$\|u(t,u_0)\|_{\gamma'} \geqslant \frac{c}{(\tau_{u_0}-t)^{\gamma'-\frac{\alpha\rho-\beta-1}{\rho-1}}} \qquad (9.1.30)$$

holds for $t < \tau_{u_0}$ close enough to τ_{u_0}.
iii) If the scale is nested, part ii) holds also for $\gamma \leq \gamma' < \beta$ ($\gamma < \gamma'$ in the critical case).

Proof. Observe that γ–solutions in Theorems 9.1.7 or 9.1.8 enter in X^α for $t > 0$. Then, restarting the solution from X^α, Theorems 9.1.7 with $\gamma = \alpha$ extends the interval of existence for the solution. Repeating this process we obtain the continuation of the solutions.

For $\gamma' \in [\beta, \alpha]$, $\gamma' \geq \gamma$ (with strict inequality for the critical case), using Proposition 9.1.5 i), $u(t, u_0) \in X^{\gamma'}$ for $t \in [0, \tau_{u_0})$. When the scale is nested, this is also true for $\gamma \leq \gamma' < \beta$. Then, using Proposition 9.1.10, for any $\tau \in [0, \tau_{u_0})$, $u(\cdot + \tau, u_0)$ is a γ'-solution in $[0, \tau_{u_0} - \tau)$, and we can use (9.1.15) in Theorem 9.1.7,

$$\tau_{u_0} - t > \frac{C}{(1 + \|u(t, u_0)\|_{\gamma'})^{\frac{1}{\gamma' - \frac{\rho\alpha - \beta - 1}{\rho - 1}}}}$$

for all $t < \tau_{u_0}$ close enough to τ_{u_0}. ∎

We can also prove the following.

Corollary 9.1.13 Uniform estimates. Subcritical case
Under the assumptions of Theorem 9.1.7, assume that for a γ–solution, or a family of γ–solutions, we have an estimate of either type:

$$\|u(t)\|_\gamma \leq C, \quad \textit{for } t \in [0, T], \quad \textit{or} \quad \|u(t)\|_\gamma \leq C, \quad \textit{for all } t \geq T.$$

Then for any $\beta \leq \gamma' < \beta + 1$ and $\gamma' \geq \gamma$ and for any $\tau > 0$ we have,

$$\|u(t)\|_{\gamma'} \leq C(\tau), \quad \textit{for } t \in [\tau, T], \quad \textit{or} \quad \|u(t)\|_{\gamma'} \leq C(\tau), \quad \textit{for all } t \geq T + \tau,$$

respectively, where $C(\tau)$ depends on C and τ.

Proof. Consider $u_0 = u(s)$, $s \in [0, T]$ in the first case and $u_0 = u(s)$, $s \geq T$, in the second, which are bounded in X^γ. Then by Theorem 9.1.7 there exists $\tau_0 > 0$ such that the γ–solution starting at these u_0, $u(\cdot, u(s))$ is defined in $[0, \tau_0]$ and the corresponding γ–solution are bounded independent of such u_0. Since $u(\cdot + s)$ is continuous in X^α then by Proposition 9.1.10 we have $u(\cdot, u(s)) = u(\cdot + s)$.

Hence there exists K such that for any such u_0 and any $0 < \tau < \tau_0$ we have, from (9.1.10), with $t = \tau$,

$$\tau^{\gamma' - \gamma} \|u(s + \tau)\|_{\gamma'} \leq K,$$

for all $s \in [0, T - \tau]$ and $s \geq T$ respectively. ∎

Although Theorems 9.1.7 and 9.1.8 are for nonlinear terms satisfying (ii.0.14) and (ii.0.15) with $\rho > 1$ we can also consider Lipschitz nonlinear terms, that is $\rho = 1$.

Proposition 9.1.14 *Let $f : X^\alpha \to X^\beta$ be Lipschitz, then for any $\gamma \in E(\alpha) = (\alpha - 1, \alpha]$ the results from Theorem 9.1.7 hold. Furthermore, in this case, the solutions are defined globally.*

Proof. Note that for any $\gamma \in E(\alpha) = (\alpha-1, \alpha]$, we can take $\rho > 1$ such that $0 \le \alpha-\beta < \frac{1}{\rho}$ and $\gamma \in (\alpha - \frac{1}{\rho}, \alpha]$. Then we can use Theorem 9.1.7.

To show the global existence, we follow the proof of Theorem 9.1.7 but using $\mathcal{K}_T := \mathcal{L}^{\infty}_{\alpha,\alpha-\gamma}((0,T])$ instead of $\mathcal{K}_{T,K_0}$ in (9.1.16). Then, $\rho = 1$ in (9.1.17) gives a contraction for a time which is independent of u_0. Thus the solution can be prolonged globally. ■

Remark 9.1.15 *i) Global existence can also be proved using a Gronwall type argument. ii) In particular we can consider unbounded linear perturbations as in [47] and Part I.*

$$f \in \mathcal{L}(X^{\alpha}, X^{\beta}) \quad \text{with} \quad 0 \le \alpha - \beta < 1.$$

Remark 9.1.16 ε-regular maps and ε-regular solutions

In [4] the authors consider a densely defined sectorial operator A in a Banach space X and consider the scale of fractional power spaces associated to A, $\{Y^{\alpha}\}_{\alpha\ge 0}$, which is a nested scale in which the semigroup $S(t) = e^{-At}$ satisfies (ii.0.12) and (ii.0.13); see [31], [2]. In order to construct solutions for $u_0 \in Y^1$ they consider ε-regular maps which satisfy $f : Y^{1+\varepsilon} \to Y^{\gamma(\varepsilon)}$ with

$$\|f(u) - f(v)\|_{\gamma(\varepsilon)} \le c\|u - v\|_{1+\varepsilon}(1 + \|u\|^{\rho-1}_{1+\varepsilon} + \|v\|^{\rho-1}_{1+\varepsilon}), \ u, v \in Y^{1+\varepsilon},$$

for some constants $c > 0$, $\rho > 1$, $\varepsilon > 0$ and $\rho\varepsilon \le \gamma(\varepsilon) < 1$. Then they prove the existence of ε-regular solutions for the problem (ii.0.17) for initial data $u_0 \in Y^1$. Their critical case corresponds to the case $\gamma(\varepsilon) = \rho\varepsilon$.

In our setting, we can take $\alpha = 1 + \varepsilon$, $\beta = \gamma(\varepsilon)$ and $X^t = Y^t$ for all $t \ge 0$. Then observe that $G_1(\alpha, \beta) = \alpha - \frac{1}{\rho} = 1 + \varepsilon - \frac{1}{\rho} < 1$, since $\rho\varepsilon < 1$. On the other hand, $G_2(\alpha, \beta) = \frac{\alpha\rho-\beta-1}{\rho-1} = 1 + \frac{\rho\varepsilon-\gamma(\varepsilon)}{\rho-1} \le 1$ since $\gamma(\varepsilon) \ge \rho\varepsilon$, with equality if $\gamma(\varepsilon) = \rho\varepsilon$. This implies that in any case the interval $E(\alpha, \beta, \rho)$ in (9.1.8) always contains $\gamma = 1$. Also, notice that in this setting $\alpha - \beta = 1 + \varepsilon - \gamma(\varepsilon) < 1$, since $\gamma(\varepsilon) \ge \rho\varepsilon > \varepsilon$, and $\frac{1}{\rho} < \alpha - \beta$ iff $\rho\varepsilon \le \gamma(\varepsilon) < 1 + \varepsilon - \frac{1}{\rho}$ (< 1).

Thus our setting includes that of [4] and in their case, we can construct solutions for more spaces of initial data and not only for X^1.

Conversely, given α, β, γ, in their setting, we can choose $\varepsilon = \alpha-\gamma$, $\gamma(\varepsilon) = \beta+1-\gamma$ and $Y^{t+(1-\gamma)} = X^t$. Then, condition $\rho\varepsilon \le \gamma(\varepsilon)$ gives $\gamma \ge \frac{\rho\alpha-\beta-1}{\rho-1}$ while $\rho\varepsilon < 1$ gives $\gamma > \alpha - \frac{1}{\rho}$ as in our setting, however, $\gamma(\varepsilon) < 1$ yields $\gamma > \beta$ which adds an extra restriction compared to our case.

Therefore, our setting extends the one in [4] not only by using scales which are not necessarily nested, but even when using the fractional power scale, since we can solve the case when $\gamma \le \beta$.

9.2 Improved uniqueness

Note that Theorem 9.1.7 gives existence and uniqueness of γ–solutions and that Theorem 9.1.8 gives existence of a unique γ–solution such that $|||u|||_{\alpha,\alpha-\gamma,T} \le K_0$ with K_0 as in (9.1.20), which is independent of $v_0 \in X^\gamma$ but relatively small, and $T = T(K_0, v_0) \leq 1$, defined in (9.1.21), is small. Therefore in the latter case it is not clear whether or not there exist some other γ–solutions of larger norm. Furthermore, one could consider different (may be larger) classes where to find solutions, for example $u \in C([0,T], X^\gamma)$.

Therefore, our goal in this section is to improve the uniqueness result. For this recall that by Lemma 9.1.1 it is enough to improve local uniqueness. Before continuing, we prove the following auxiliary lemma. Note that this states that "solutions" satisfying $(S1)$, $(S2)$ and $(S3)$ in Section 9.1 above, become γ–solutions in positive time.

Lemma 9.2.1 *Let u be a function satisfying $(S1)$, $(S2)$, $(S3)$ for some $T > 0$. Assume (ii.0.12), (ii.0.14)-(ii.0.16) and let $G(\alpha, \beta)$ be as in (9.1.9). Then*
i) **Subcritical case.** *Assume $\gamma \in (G(\alpha, \beta), \alpha]$ and u is bounded in X^γ in a neighborhood of $t = 0$. Then there exist $M < \infty$, $t^* > 0$ and $h^* > 0$ such that*

$$t^{\alpha-\gamma}\|u(t+h)\|_\alpha \leq M \quad \textit{for all } 0 < t < t^* \textit{ and } 0 < h < h^*.$$

ii) **Critical case.** *Assume (9.1.18), $\frac{1}{\rho} < \alpha - \beta < 1$ and $\gamma = \frac{\alpha\rho-\beta-1}{\rho-1}$ and u is right continuous in X^γ when $t \to 0^+$. Then for any K_0 as in (9.1.20) there exist $t^* > 0$ and $h^* > 0$ such that*

$$t^{\alpha-\gamma}\|u(t+h)\|_\alpha \leq K_0 \quad \textit{for all } 0 < t < t^* \textit{ and } 0 < h < h^*.$$

Proof. Observe that there exists h^* such that for all $h \in (0, h^*]$ we have that $u(h)$ is bounded in X^γ. Also, from Lemma 9.1.2 i), $u \in C((0,T], X^\alpha)$ and therefore $u(\cdot + h)$ is a γ-solution in $[0, T - h^*]$ with initial data $u(h)$ because of Proposition 9.1.10.

In order to prove part i), from Theorem 9.1.7, there exist t_0 and M such that there exists a unique γ-solution with initial data $u(h)$, $\mathcal{U}_h$ which satisfies $|||\mathcal{U}_h|||_{\alpha,\alpha-\gamma,t_0} \leq M$.

Therefore, $u(\cdot + h) = \mathcal{U}_h$ in the common interval of existence $(0, t^*]$ with $t^* = \min\{t_0, T - h^*\}$, and so $|||u(\cdot + h)|||_{\alpha,\alpha-\gamma,t^*} \leq M$ which concludes the proof of part i).

For part ii), take any K_0 as in (9.1.20). Because of the right continuity in X^γ of u, when $t \to 0^+$, there exists h^* such that for all $h \in (0, h^*]$, $u(h) \in B_{X^\gamma}(u_0, r)$ for $r = \frac{K_0}{4M_0}$ as in Theorem 9.1.8. Now, from part ii) in Theorem 9.1.8, there exists $T(K_0) = T(K_0, u_0)$ for which there exists a unique γ–solution with initial data $u(h)$, $\mathcal{U}_h \in \mathcal{K}_{T(K_0),K_0}$ as in (9.1.16). From (9.1.19) we also have that $|||\mathcal{U}_h|||_{\alpha,\alpha-\gamma,t} \to 0$ as $t \to 0^+$.

On the other hand, because of (S3) and $\alpha > \gamma$, for any $0 < h < h^*$,

$$s^{\alpha-\gamma}\|u(s+h)\|_\alpha \to 0 \ \text{ for } s \to 0,$$

that is, $|||u(\cdot+h)|||_{\alpha,\alpha-\gamma,t} \to 0$ as $t \to 0^+$, so $u(\cdot+h)$ is a γ-solution because of Proposition 9.1.10. Due to Theorem 9.1.8 i) we thus have $u(\cdot + h) = \mathcal{U}_h$ on $[0, \min\{T - h^*, T(K_0)\}]$ and since $\mathcal{U}_h \in \mathcal{K}_{T(K_0),K_0}$ part ii) now follows easily. ■

With this preparatory result we can prove the following uniqueness result. Observe that this result states that suitable classes of solutions are in fact γ–solutions. Also note that below we use γ–solutions as constructed in Theorems 9.1.7 or 9.1.8, which have been prolongued to a maximal interval of time as in Proposition 9.1.12.

Theorem 9.2.2 (Improved uniqueness)

Assume (ii.0.12), (ii.0.14)-(ii.0.16) and let $G(\alpha, \beta)$ be as in (9.1.9). Let $u_0 \in X^\gamma$ and u satisfies (ii.0.17) and $(S1)$, $(S2)$, $(S3)$ for some $T > 0$ and $u(0) = u_0$. Then
i) **Subcritical case.** *Assume $\gamma \in (G(\alpha, \beta), \alpha]$, u is bounded in X^γ in a neighborhood of $t = 0$.*

Then u is a γ-solution and coincides in $[0, T]$ with the one in Theorem 9.1.7 prolonged as in Proposition 9.1.12.
ii) **Critical case.** *Assume (9.1.18), $\frac{1}{\rho} < \alpha - \beta < 1$, $\gamma = \frac{\alpha\rho - \beta - 1}{\rho - 1}$ and u is right continuous in X^γ when $t \to 0^+$,*

Then u is a γ-solution satisfying $|||u|||_{\alpha, \alpha-\gamma, t} \to 0$ when $t \to 0^+$, $\lim_{t\to 0^+} \|S(t)u_0 - u_0\|_\gamma = 0$ and coincides in $[0, T]$ with the one in Theorem 9.1.8 prolonged as in Proposition 9.1.12.

Proof. Note that for any $h > 0$ and $0 < t < t + h \leq T$, we can write

$$t^{\alpha-\gamma}\|u(t)\|_\alpha \leq t^{\alpha-\gamma}\|u(t) - u(t+h)\|_\alpha + t^{\alpha-\gamma}\|u(t+h)\|_\alpha. \tag{9.2.1}$$

Then for part i), using Lemma 9.2.1 part i) there exist M, t^*, h^* such that $t^{\alpha-\gamma}\|u(t+h)\|_\alpha \leq M$ for all $0 < t < t^*$ and $0 < h < h^*$. Hence using (9.2.1) for $0 < t < t^*$ and $0 < h < h^*$, we have

$$t^{\alpha-\gamma}\|u(t)\|_\alpha \leq t^{\alpha-\gamma}\|u(t) - u(t+h)\|_\alpha + M.$$

Now, by continuity of u in X^α, see Lemma 9.1.2 i), for any $0 < t < t^*$ there exists $0 < h < h^*$ such that $t^{\alpha-\gamma}\|u(t) - u(t+h)\|_\alpha < M$, so $t^{\alpha-\gamma}\|u(t)\|_\alpha \leq 2M$ for $t \in (0, t^*]$.

Thus, by Theorem 9.1.7 i), u coincides with the solution in that Theorem in $(0, t^*]$ and Lemma 9.1.1 concludes the proof.

For part ii), note that for any K_0 as in (9.1.20), $K_0/2$ is also as in (9.1.20) and using Lemma 9.2.1 part ii) for $K_0/2$ and (9.2.1), there exist t^*, h^* such that

$$t^{\alpha-\gamma}\|u(t)\|_\alpha \leq t^{\alpha-\gamma}\|u(t) - u(t+h)\|_\alpha + \frac{K_0}{2}, \quad 0 < t < t^* \quad 0 < h < h^*.$$

Again, by continuity of u in X^α, see Lemma 9.1.2 i), for any $0 < t < t^*$ there exists $0 < h < h^*$ such that $t^{\alpha-\gamma}\|u(t) - u(t+h)\|_\alpha < \frac{K_0}{2}$, and so $t^{\alpha-\gamma}\|u(t)\|_\alpha \leq K_0$ for $t \in (0, t^*]$.

Thus, by Theorem 9.1.8, u coincides with the solution in that Theorem in $(0, t^*]$. In particular $|||u|||_{\alpha,\alpha-\gamma,t} \overset{t\to 0^+}{\to} 0$. Since we are in the critical case, $\gamma \geq \beta$, and then as in the proof of part ii) of Proposition 9.1.5 with $\gamma' = \gamma$, we have $\| \int_0^t S(t-\tau) f(u(\tau)) d\tau\|_\gamma \to 0$ as $t \to 0^+$ and thus $\lim_{t\to 0^+} \|S(t)u_0 - u_0\|_\gamma = 0$. Then, Lemma 9.1.1 concludes the proof.
■

9.3 Optimality of the well-posedness results

In this section we give some arguments to show that the well posedness results in Sections 9.1 and 9.2 are essentially optimal. For this, we show that in general if $\gamma < G(\alpha, \beta)$

then one can not expect uniqueness nor continuous dependence of solutions. Hence, the problem is in general not well posed in the sense of Hadamard.

We now show an example of non-uniqueness for γ less than $G(\alpha,\beta)$. The proof is based on the example of M. Miklavčič (see p. 204, [42]).

Proposition 9.3.1 *There exist a nested scale of spaces and a semigroup as in (ii.0.12)-(ii.0.13) and a nonlinear map f satisfying (ii.0.14) and (ii.0.15) such that all of the following hold.*
i) There exists a certain threshold value $G_f < G(\alpha,\beta)$ such that for $\gamma < G_f$, there exist $T > 0$ and an uncountable family $\mathcal{U}_f$ in $C([0,T], X^\gamma)$ satisfying $(S1)$, $(S2)$, $(S3)$ in Chapter 9.1 and (ii.0.17). Moreover, each function $u \in \mathcal{U}_f$ satisfies $u(0) = 0$ and

$$u \in \mathcal{L}^\infty_{\alpha,\alpha-\gamma}((0,T]) \text{ and } u \text{ is bounded in } X^\gamma \text{ in a neighborhood of } t = 0.$$

ii) Given $\epsilon > 0$ one can also choose f in such a way that $G(\alpha,\beta) - \epsilon < G_f$.

Proof. Let A be a sectorial operator in a Banach space X and consider the fractional power scale associated to A, $\{X^\alpha\}_{\alpha\geq 0}$, which is a nested scale in which the semigroup $S(t) = e^{-At}$ satisfies (ii.0.12); see [31], [2]. Fix $q > 1$ and $\alpha > \frac{1}{q}$, and consider the following equation

$$u_t + Au = f(u) = \|u\|_\alpha^q u \tag{9.3.1}$$

Then $f : X^\alpha \to X^\alpha$ satisfies (ii.0.14) and (ii.0.15) with $\beta = \alpha$ and $\rho = q+1$. Therefore we are in a subcritical case and $G(\alpha,\beta) = \alpha - \frac{1}{q+1}$.

Now we fix $\chi \in X^{\gamma_0}$, for some $\gamma_0 < \alpha$ to be chosen below, and look for a solution of (9.3.1) of the form $u(t) = c(t)e^{-At}\chi$. For this, a simple computations shows that $c(t)$ must satisfy $c'(t) = |c(t)|^q c(t)\|e^{-At}\chi\|_\alpha^q$. Setting $c(1) = 1$ and $t \in (0,1)$ we get that $c(t) = (1 + q\int_t^1 \|e^{-As}\chi\|_\alpha^q ds)^{-\frac{1}{q}}$ and $u(t)$ is a strong solution of (9.3.1) with $t \in (0,1)$. In particular u satisfies $(S1)$, $(S2)$, $(S3)$ in Section 9.1.

Now we figure out the behavior of u near $t = 0$ and check if it satisfies (ii.0.17). For this, assume now that for $t \in (0,1)$ and any given $\varepsilon > 0$, χ is chosen such that

$$\frac{C_1}{t^{\alpha-\gamma_0-\varepsilon}} \leq \|e^{-At}\chi\|_\alpha \leq \frac{C_2}{t^{\alpha-\gamma_0}}. \tag{9.3.2}$$

Notice that the lower bound in (9.3.2) implies that $0 \leq c(t) \leq Ct^{\alpha-\gamma_0-\varepsilon-\frac{1}{q}}$, for $t \approx 0$, and then for any $\gamma_0 \leq \gamma < \alpha$ we have $\|u(t)\|_\gamma \leq Ct^{\alpha-\gamma_0-\varepsilon-\frac{1}{q}}\frac{1}{t^{\gamma-\gamma_0}} = Ct^{\alpha-\gamma-\varepsilon-\frac{1}{q}}$. Hence for $\alpha - \frac{1}{q} > \gamma$, by choosing $\varepsilon > 0$ small and χ as above, we get $u \in C([0,1], X^\gamma)$ with $u(0) = 0$. Since any multiple of χ also satisfies (9.3.2) we have already uncountable many solutions of (9.3.1) with $u(0) = 0$ and satisfying $(S1)$, $(S2)$, $(S3)$ in Section 9.1.

On the other hand, we also have, $\|f(u(t))\|_\gamma = \|u(t)\|_\alpha^q\|u(t)\|_\gamma \leq Ct^{-\varepsilon q-1}t^{\alpha-\gamma-\varepsilon-\frac{1}{q}}$ for $t \approx 0$, and again for $\alpha - \frac{1}{q} > \gamma$, by choosing $\varepsilon > 0$ small and χ as above, we have that $\|f(u(t))\|_\gamma$ is integrable at $t = 0$. With this and the continuity of u above we can pass to the limit as $\tau \to 0$ in $(S2)$ in Section 9.1 to get that u satisfies (ii.0.17).

Finally observe that $t^{\alpha-\gamma}\|u(t)\|_\alpha \le Ct^{\alpha-\gamma}t^{-\varepsilon-\frac{1}{q}}$ is bounded near $t=0$ when $\gamma < \alpha - \frac{1}{q}$ and $\varepsilon > 0$ is small. Hence $u \in \mathcal{L}^\infty_{\alpha,\alpha-\gamma}((0,1])$.

The results above hold for $\gamma < G_f := \alpha - \frac{1}{q} < G(\alpha,\beta)$ and by choosing q as large as needed we get that G_f and $G(\alpha,\beta)$ are as close as we want.

It just remains to show that (9.3.2) can indeed be satisfied. In particular, it is enough to choose $X := L^1(\mathbb{R}^+)$ and $A\phi = h\phi$ for $\phi \in dom(A) := \{\phi \in L^1(\mathbb{R}^+) : h\phi \in L^1(\mathbb{R}^+)\}$, where $h : \mathbb{R}^+ \to [1,\infty)$, $h(x) = \begin{cases} 1, & x \in (0,1], \\ x, & x \in (1,\infty). \end{cases}$ Such A is densely defined, the resolvent set $\rho(A)$ contains a half-plane $\{Re(\lambda) \le 0\}$ and $\|\lambda(\lambda Id - A)^{-1}\| \le 1$ whenever $Re(\lambda) \le 0$. Hence $-A$ generates in X an analytic C^0-semigroup of contractions $\{S(t) : t \ge 0\}$ and, solving $\dot{u} = -Au$, we get $S(t)u_0 = e^{-ht}u_0$ for each $u_0 \in X$, $t \ge 0$. We also have $A^{-\alpha}\phi = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-th}\phi dt = h^{-\alpha}\phi$, $\phi \in X$, $\alpha > 0$, which shows that the fractional powers A^α are $A^\alpha\phi = h^\alpha\phi$ for $\phi \in R(A^{-\alpha}) = h^{-\alpha}X = dom(A^\alpha) =: X^\alpha$, $\alpha > 0$ with the norm $\|u\|_\alpha = \|h^\alpha u\|_{L^1(\mathbb{R}^+)}$. Let $\chi \in X$ be given by

$$\chi(x) = \begin{cases} 0, & x \in (0,1] \\ x^{-1-\kappa}, & x \in (1,\infty) \end{cases}$$

for some fixed small number $0 < \kappa < \alpha - \frac{1}{q}$, so in particular, $\chi \in X^{\gamma_0}$ for any $\gamma_0 < \kappa$.

In this setting, since $\kappa < \alpha$, and using the change $y = tx$ we have that

$$\|e^{-At}\chi\|_\alpha = \int_0^\infty h^\alpha(x)e^{-h(x)t}|\chi(x)|dx = \int_1^\infty x^\alpha e^{tx}x^{-1-\kappa}dx = \frac{1}{t^{\alpha-\kappa}}\int_t^\infty y^{\alpha-1-\kappa}e^{-y}dy.$$

Let $F(t) = \int_t^\infty y^{\alpha-1-\kappa}e^{-y}dy$ and observe that for $t \in (0,1)$, $F(1) \le F(t) \le F(0)$. Thus the right hand inequality in (9.3.2) holds for all $\gamma_0 < \kappa$. Finally, given $\varepsilon > 0$, the left inequality in (9.3.2) holds choosing $\gamma_0 = \kappa - \varepsilon$. ■

Remark 9.3.2 *A similar non-uniqueness result was stated for $A = -\Delta$ in $\mathbb{R}^N$ in [29]. Later on, [9] showed non-uniqueness without assuming $u(0) = 0$, for positive, radial, decreasing solutions in bounded domains. Similar results can also be found in [43], Theorems 3 and 4.*

We now exhibit optimality of the well posedness result in the critical case, that is, when $G(\alpha,\beta) = G_2(\alpha,\beta) = \frac{\alpha\rho-\beta-1}{\rho-1}$ and $\frac{1}{\rho} < \alpha - \beta < 1$. We also show the optimality of part iii) in Theorem 9.1.8.

Proposition 9.3.3 *There exist a densely embedded nested scale $\{X^\alpha\}_{\alpha\in\mathcal{J}}$ and semigroup satisfying (ii.0.12) and (ii.0.13) and there is a nonlinear map f as in (ii.0.14), (ii.0.15), with $\frac{1}{\rho} < \alpha - \beta < 1$, $f(0) = 0$ (so for $u_0 = 0$ (ii.0.17) has a global solution $u(\cdot;u_0) = 0$) such that the following holds.*

There exists a sequence of initial conditions $u_0^n \in \bigcap_{\sigma\in\mathcal{J}} X^\sigma$ satisfying,

i) u_0^n is bounded in X^{γ_*}, $\gamma_* := G(\alpha,\beta) = \frac{\rho\alpha-\beta-1}{\rho-1}$ and $u_0^n \to 0$, as $n\to\infty$ in X^γ for any $\gamma < \gamma_*$.
ii) There exists a solution $u^n = u^n(\cdot; u_0^n)$ of (ii.0.17) in the class $C([0,\tau_n), X^\alpha)$. Also, for any $\gamma \le \gamma_*$ $u^n \in \mathcal{L}^\infty_{\alpha,\alpha-\gamma}((0,T]) \cap C([0,T], X^\gamma)$, for any $T<\tau_n$, u^n is a γ-solution in $[0,T]$, and the maximal existence time τ_n satisfies

$$\tau_n \to 0 \quad as \quad n\to\infty.$$

In particular, for $\gamma<\gamma_*$, there is no continuous dependence of γ-solutions and for $\gamma=\gamma_*$, the existence time of γ-solutions is not uniform in bounded sets.

Proof. The proof is based on the example given in [18, Section 5]. Consider the bi-Laplacian Δ^2 in $L^2(\mathbb{R}^N)$ and the scale of Bessel spaces $X^\gamma := H^{4\gamma}(\mathbb{R}^N)$, $\gamma\in\mathbb{R}$, in which $S(t)=e^{-\Delta^2 t}$ satisfies (ii.0.12) and (9.1.18), see Chapter 5 above or Section 11.2 below. Consider the following fourth order problem

$$\begin{cases} u_t + \Delta^2 u = f(u), \quad t>0,\ x\in\mathbb{R}^N, \\ u(0,x) = u_0(x),\ x\in\mathbb{R}^N, \end{cases} \tag{9.3.3}$$

where $f(u) = u|u|^{\rho-1}$ with $\rho = 1+\frac{8}{N}$. Fix $N=4$, thus $\rho=3$ and then, because of the Sobolev embeddings and Lemma 11.2.1 below, we have $f: H^1(\mathbb{R}^N)\to H^{-1}(\mathbb{R}^N)$, that is, $f: X^\alpha \to X^\beta$, $\alpha=\frac14$, $\beta=-\frac14$. Note that $\frac{1}{\rho}<\alpha-\beta<1$ and $\gamma_* := G(\alpha,\beta)=0$, that is $X^{\gamma_*} = L^2(\mathbb{R}^N)$. We recall from [18, Remarks 5.1, 5.3 and Theorem 5.2] that there exists $u_0\in C_0^\infty(\mathbb{R}^N)$ and a time $\tau>0$ for which there exists a regular solution $u=u(\cdot;u_0)$ of (9.3.3), $u\in C([0,\tau), H^4(\mathbb{R}^N))\cap C^1([0,\tau), L^2(\mathbb{R}^N))$. Also, u ceases to exist at time τ. Since u is regular this solution satisfies (S1), (S2), (S3), and is continuous in $L^2(\mathbb{R}^N) = X^{\gamma_*}$ as $t\to 0$.

Given $n\in\mathbb{N}$ consider the following scaling

$$u^n(t,x) = n^{\frac{4}{\rho-1}} u(n^4 t, nx),\ t\in[0,\tau_n),\ \tau_n = \frac{\tau}{n^4},\ x\in\mathbb{R}^N,$$

which preserves (9.3.3) and since u ceases to exist at time τ then u^n ceases to exist at time $\tau_n\to 0$ as $n\to\infty$. Now, $u^n(t,x)$ is a solution of (9.3.3), $u^n\in C([0,\tau_n), H^4(\mathbb{R}^N))\cap C^1([0,\tau_n), L^2(\mathbb{R}^N))$, which satisfies (S1), (S2), (S3) and $u^n\in\mathcal{L}^\infty_{\alpha,\alpha-\gamma}((0,T])\cap C([0,T], X^\gamma)$, for any $T<\tau_n$ and any $\gamma\le 1$.

If $\gamma<\gamma_*=0$, since $X^\gamma = H^{4\gamma}(\mathbb{R}^N)$, then for $s<2$ such that $-4\gamma+\frac{N}{2}=\frac{N}{s}$ we have that $L^s(\mathbb{R}^N)\hookrightarrow H^{4\gamma}(\mathbb{R}^N)$, and thus

$$\|u_0^n\|_{H^{4\gamma}(\mathbb{R}^N)} \le C\|u_0^n\|_{L^s(\mathbb{R}^N)} = Cn^{-\frac{N}{s}+\frac{N}{2}}\|u_0\|_{L^s(\mathbb{R}^N)} \to 0, \quad as\ n\to\infty,$$

which gives the result. In the critical case, since $\frac{N}{2}=\frac{4}{\rho-1}$ we have that

$$\|u_0^n\|_{L^2(\mathbb{R}^N)} = n^{\frac{4}{\rho-1}-\frac{N}{2}}\|u_0\|_{L^2(\mathbb{R}^N)} = \|u_0\|_{L^2(\mathbb{R}^N)}.$$

This and $\tau_n\to 0$ prove that the existence time is not uniform in bounded sets. ■

9.4 Optimality of the blow up rate

We now study optimality of Proposition 9.1.12. Note that the exponent on the right hand side in (9.1.30), is increasing with γ'. Since $\gamma' \in [\max\{\gamma, \beta\}, \alpha]$ the maximal exponent is $\frac{1+\beta-\alpha}{\rho-1} =: b(\alpha)$. We now show that the estimate (9.1.30) near a blow up point is optimal for the maximal exponent.

Proposition 9.4.1 *There exist a scale and a semigroup as in (ii.0.12) and (ii.0.13) and there is a nonlinear map f satisfying (ii.0.14) and (ii.0.15) such that, given any growth $\rho \in (1, \infty)$, any positive time τ and any $\gamma \in (G(\alpha, \beta), \alpha]$, a certain initial condition $u_0 \in X^\gamma$ can be chosen for which the τ_{u_0} of the corresponding γ-solution $u = u(\cdot, u_0)$ is finite and equal τ and, in addition,*

$$(\tau_{u_0} - t)^{b(\alpha)} \|u(t, u_0)\|_\alpha \to c \quad as \quad t \to \tau_{u_0}^- \tag{9.4.1}$$

for some constant $c > 0$.

Proof. We consider the equation (9.3.1), with A, f, X, $\chi(x)$ as in the proof of Proposition 9.3.1. Consequently, $b(\alpha) = \frac{1+\beta-\alpha}{\rho-1} = \frac{1}{q}$ and $G(\alpha, \beta) = \alpha - \frac{1}{q+1}$.

Given τ we now look for a function $u(t) = c(t)e^{-At}\chi$ satisfying (9.3.1) such that it ceases to exist on $t = \tau$. Thus $c(t) = (\eta - q\int_0^t \|e^{-As}\chi\|_\alpha^q ds)^{-\frac{1}{q}}$ with $\eta := q\int_0^\tau \|e^{-As}\chi\|_\alpha^q ds$ so $u(0) = \eta^{-\frac{1}{q}}\chi = u_0$.

Note that, given $\gamma \in (G(\alpha, \beta), \alpha]$, we have $u_0 = \eta^{-\frac{1}{q}}\chi \in X^\gamma$, $u \in C([0, T], X^\alpha)$ and that (ii.0.17) holds for any $T < \tau$. In particular, u is a γ-solution on $[0, T]$ for any positive time $T < \tau$. Also, $\lim_{t\to\tau^-} \|u(t)\|_\alpha = (\eta - q\int_0^t \|e^{-As}\chi\|_\alpha^q ds)^{-\frac{1}{q}} \|e^{-At}\chi\|_\alpha = \infty$ so $[0, \tau)$ is the maximal interval of existence of this solution. Since $b(\alpha) = \frac{1}{q}$, using L'Hôpital's rule

$$\lim_{t\to\tau^-} (\tau - t)^{b(\alpha)} \|u(t)\|_\alpha = \left(\lim_{t\to\tau^-} \frac{\tau - t}{\eta - q\int_0^t \|e^{-As}\chi\|_\alpha^q ds} \right)^{\frac{1}{q}} \lim_{t\to\tau^-} \|e^{-At}\chi\|_\alpha = \frac{1}{q^{\frac{1}{q}}},$$

which proves (9.4.1). ■

Chapter 10

General bootstrap argument

Before turning into the particular details of particular PDE problems, we start with some general bootstrap argument. First of all recall that we are dealing with a scale of spaces $\{X^\alpha\}_{\alpha\in\mathcal{J}}$ where $\mathcal{J}$ is an interval of real indexes. Therefore in all the arguments below the ranges for the indexes have to be intersected with $\mathcal{J}$. For the sake of simplicity we will not write this all the time.

Step 1. Admissible Region.

We will find below that once we fix a suitable scale of spaces in which to set a PDE problem, there will typically exist many admissible pairs (α, β) satisfying $0 \le \alpha - \beta < 1$ and that the nonlinear term f is defined from X^α to X^β and satisfies (ii.0.15). Such admissible pairs make up the admissible region $\mathcal{S}$ for the problem considered.

According to (9.1.9) recall that

$$G(\alpha,\beta) := \begin{cases} G_1(\alpha,\beta) = \alpha - \frac{1}{\rho}, & 0 \le \alpha - \beta \le \frac{1}{\rho}, \\ G_2(\alpha,\beta) = \frac{\alpha\rho-\beta-1}{\rho-1}, & \frac{1}{\rho} < \alpha - \beta < 1. \end{cases} \tag{10.0.1}$$

Hence $\mathcal{S}$ is split in a natural way in the two disjoint subregions $\mathcal{S}_1 := \{(\alpha,\beta) \in \mathcal{S} : 0 \le \alpha - \beta \le \frac{1}{\rho}\}$, where Theorem 9.1.7 can be applied, and $\mathcal{S}_2 := \{(\alpha,\beta) \in \mathcal{S} : \frac{1}{\rho} < \alpha - \beta < 1\}$ where Theorems 9.1.7 and 9.1.8 apply. Note that some of these regions could be empty.

Remark 10.0.1 *Note that if the scale is nested and $(\alpha_0, \beta_0) \in \mathcal{S}$ then any (α, β) such that $\alpha \ge \alpha_0$, $\beta \le \beta_0$ and $\alpha - \beta < 1$ also belongs to $\mathcal{S}$, since $f : X^\alpha \subset X^{\alpha_0} \to X^{\beta_0} \subset X^\beta$. This set is a triangle with left upper vertex (α_0, β_0), sides parallel to the axes and opposite side on the line $\alpha - \beta = 1$.*

Step 2. Local Uniqueness.

Using Theorems 9.1.7 and 9.1.8 we will obtain a well posedness result of γ–solutions for $u_0 \in X^\gamma$, for any

$$\gamma \in \mathcal{E} := \bigcup_{(\alpha,\beta)\in\mathcal{S}} E(\alpha,\beta,\rho). \tag{10.0.2}$$

In particular, $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, where

$$\mathcal{E}_1 := \bigcup_{(\alpha,\beta)\in\mathcal{S}_1} (G_1(\alpha,\beta),\alpha] \qquad \mathcal{E}_2 := \bigcup_{(\alpha,\beta)\in\mathcal{S}_2} [G_2(\alpha,\beta),\alpha]. \tag{10.0.3}$$

Also, because of Theorems 9.1.7 and 9.1.8, and assuming that $\{S(t) : t \geq 0\}$ is a C^0 semigroup in the scale, i.e. (ii.0.13), the γ–solution is continuous in X^γ at $t = 0$ whenever $\gamma \in \mathcal{E}$ and the scale is nested or, otherwise, for

$$\gamma \in \mathcal{E}_1^c \cup \mathcal{E}_2 \quad \text{where} \quad \mathcal{E}_1^c := \bigcup_{\substack{(\alpha,\beta)\in\mathcal{S}_1 \\ \alpha-\beta<\frac{1}{\rho}}} [\beta,\alpha] \cup \bigcup_{\substack{(\alpha,\beta)\in\mathcal{S}_1 \\ \alpha-\beta=\frac{1}{\rho}}} (\beta,\alpha]. \tag{10.0.4}$$

To be more precise, let $\gamma \in \mathcal{E}$ and let $(\alpha_0,\beta_0) \in \mathcal{S}$ be a point for which $\gamma \in E(\alpha_0,\beta_0,\rho)$. Then, from Theorems 9.1.7 and 9.1.8, there exists $r > 0$ such that for any $v_0 \in X^\gamma$ there exists $T > 0$ such that for any u_0 such that $\|u_0 - v_0\|_\gamma < r$ there exists a γ–solution with initial data u_0. From Proposition 9.1.5 we get that for all $\gamma' \in [\beta_0, \beta_0 + 1)$, $\gamma' \geq \gamma$, $u(\cdot, u_0) \in C((0,T], X^{\gamma'})$ and

$$t^{\gamma'-\gamma}\|u(t,u_0)\|_{\gamma'} \leq K, \quad t \in (0,T] \tag{10.0.5}$$

and assuming (9.1.11)

$$|||u(\cdot,u_0)|||_{\gamma',\gamma'-\gamma,t} \to 0 \ \text{ as } t \to 0^+. \tag{10.0.6}$$

Also, from Proposition 9.1.11, if $\|u_0^i - v_0\|_\gamma < r$, $i = 1,2$, we get for all $\gamma' \in [\beta_0, \beta_0+1)$, $\gamma' \geq \gamma$

$$\|u(t,u_0^1) - u(t,u_0^2)\|_{\gamma'} \leq \frac{M}{t^{\gamma'-\gamma}}\|u_0^1 - u_0^2\|_\gamma \quad t \in (0,T]. \tag{10.0.7}$$

Step 3. A general bootstrap argument.

In many cases, we can use a bootstrap argument to prove that, in fact, the solution enters $X^{\gamma'}$, for a larger set of γ' than the one in (10.0.5), (10.0.6) and (10.0.7) preserving these estimates. Assume that $\mathcal{S}$ has the property that:

$$\begin{gathered}\text{for } (\alpha_0,\beta_0) \in \mathcal{S}, \text{ there exists } \alpha_1 \in (\alpha_0, \beta_0 + 1) \text{ such that} \\ \text{we can find a } \beta_1 > \beta_0 \text{ such that } (\alpha_1,\beta_1) \in \mathcal{S}.\end{gathered} \tag{10.0.8}$$

In particular, take $\gamma' = \alpha_1 > \alpha_0$ in (10.0.5), as close as possible to $\beta_0 + 1$. Choose then β_1 as large as possible among those such that $(\alpha_1,\beta_1) \in \mathcal{S}$ and assume

$$\gamma \geq \frac{\rho\alpha_1 - \beta_1 - 1}{\rho - 1}. \tag{10.0.9}$$

Then we can use Lemma 9.1.2 with (α_1,β_1) to obtain that for any $\gamma'' \in [\beta_0, \beta_1 + 1)$, $\gamma'' \geq \gamma$, $u(\cdot,u_0) \in C((0,T], X^{\gamma''})$ and

$$t^{\gamma''-\gamma}\|u(t,u_0)\|_{\gamma''} \leq C(K), \quad t \in (0,T] \tag{10.0.10}$$

$$|||u(\cdot,u_0)|||_{\gamma'',\gamma''-\gamma,\tau} \to 0^+ \quad \text{as } \tau \to 0^+.$$

Similarly, using (9.1.6) for any $\gamma'' \in [\beta_0, \beta_1 + 1)$, $\gamma'' \geq \gamma$, we get

$$\|u(t,u_0^1) - u(t,u_0^2)\|_{X^{\gamma''}} \leq \frac{C(M)}{t^{\gamma''-\gamma}} \|u_0^1 - u_0^2\|_{X^\gamma}. \tag{10.0.11}$$

Repeating this process we expect to construct a finite family of points $(\alpha_j, \beta_j) \in \mathcal{S}$, $j = 0, 1, 2, \dots$ belonging to $\mathcal{S}$, such that

$$\alpha_j < \alpha_{j+1} < \beta_j + 1 \quad \beta_j < \beta_{j+1}, \quad \text{and} \quad \gamma \geq \frac{\rho\alpha_j - \beta_j - 1}{\rho - 1} \tag{10.0.12}$$

so that $u \in C((0,T], X^{\gamma''})$ and (10.0.10) and (10.0.11) hold for all $\gamma'' \in [\beta_0, \beta_j + 1)$, $\gamma'' \geq \gamma$. In particular, in (10.0.12), it suffices to choose points such that $\rho\alpha_j - \beta_j \leq \rho\alpha_0 - \beta_0$.

Summarizing, if $\mathcal{S}$ is a suitable region, we expect to be able to perform steps (10.0.5)-(10.0.12) to obtain that for $\gamma \in \mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ there exists $r > 0$ such that for any $v_0 \in X^\gamma$ there exists $T > 0$ such that for any u_0 such that $\|u_0 - v_0\|_\gamma < r$ there exists a solution of (ii.0.17) which satisfies $u(\cdot, u_0) \in C((0,T], X^{\gamma'})$

$$t^{\gamma'-\gamma}\|u(t,u_0)\|_{\gamma'} \leq C, \tag{10.0.13}$$

$$|||u(\cdot,u_0)|||_{\gamma',\gamma'-\gamma,\tau} \to 0^+ \quad \text{as } \tau \to 0^+ \tag{10.0.14}$$

and if $\|u_0^i - v_0\|_\gamma < r$, $i = 1, 2$,

$$\|u(t,u_0^1) - u(t,u_0^2)\|_{\gamma'} \leq \frac{C}{t^{\gamma'-\gamma}} \|u_0^1 - u_0^2\|_\gamma \tag{10.0.15}$$

for any $\gamma' \geq \gamma$, $\gamma' \geq \beta_0$ and

$$\gamma' \in \mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2 = \bigcup_{(\alpha,\beta)\in\mathcal{S}_1} (G_1(\alpha,\beta), \beta+1) \cup \bigcup_{(\alpha,\beta)\in\mathcal{S}_2} [G_2(\alpha,\beta), \beta+1). \tag{10.0.16}$$

Step 4. Determining the ranges.

As it is often the case, if the regions $\mathcal{S}_1$ and $\mathcal{S}_2$ are such that the intervals in the definition of the sets $\mathcal{E}$ and $\mathcal{R}$ as in (10.0.2) and (10.0.16) are overlapping, so $\mathcal{E}$ and $\mathcal{R}$ are also intervals. Let $I_i = \inf_{\mathcal{S}_i} G_i$ and denote by α^i_{min}, β^i_{min} and α^i_{max}, β^i_{max} the extremal values of the projections of $\mathcal{S}_i$ onto the axis. Then $I_1 = \alpha^1_{min} - \frac{1}{\rho}$ and

$$\mathcal{E}_1 \subset (I_1, \alpha^1_{max}], \quad \text{and} \quad \mathcal{E}_2 \subset [I_2, \alpha^2_{max}], \quad \text{and} \quad \mathcal{E}_1^c \subset [\beta^1_{min}, \alpha^1_{max}] \tag{10.0.17}$$

with the same endpoints. Note that since $\mathcal{S}_1 := \{(\alpha,\beta) \in \mathcal{S} : 0 \leq \alpha - \beta \leq \frac{1}{\rho}\}$, we have that $\beta^1_{min} \geq \alpha^1_{min} - \frac{1}{\rho}$. In particular if the "most to the left and below" point in $\mathcal{S}_1$ is in the line $\alpha - \beta = \frac{1}{\rho}$, then $\beta^1_{min} = \alpha^1_{min} - \frac{1}{\rho}$ and thus, in such a case,

$$\mathcal{E}_1^c = \mathcal{E}_1.$$

Finally, denoting $I = \inf_{\mathcal{S}} G$, we have

$$\mathcal{E} \subset [I, \alpha_{max}], \quad \text{and} \quad \mathcal{R} \subset [I, \beta_{max} + 1] \tag{10.0.18}$$

and with the same endpoints.

Recall that when computing the sets $\mathcal{E}$ and $\mathcal{R}$ as in (10.0.2), (10.0.16) or as in (10.0.17) and (10.0.18) above, we must take the intersection with the interval $\mathcal{J}$ of admissible values determined by the scale, see (ii.0.12). Also, whether the endpoints above belong to the sets (10.0.2) and (10.0.16) must be studied in each particular case.

Remark 10.0.2 *In view of (10.0.18), if the scale is nested, as it is many times the case, $I = \inf_{\mathcal{S}} G$ gives the largest space X^{γ} for which the particular problem is well posed.*

From (10.0.2) and (10.0.3) notice that for each $\gamma \in \mathcal{E}$ there could be several admissible couples $(\alpha, \beta) \in \mathcal{S}$ such that $\gamma \in E(\alpha, \beta, \rho)$. For some of these couples γ could be critical, but subcritical for others. This motivates the following definition.

Definition 10.0.3 *A value $\gamma \in \mathcal{E}$, as in (10.0.2), is "critical" for the problem (ii.0.17) in the scale (ii.0.12) if and only if for every couple $(\alpha, \beta) \in \mathcal{S}$ such that $\gamma \in E(\alpha, \beta, \rho)$ we have $(\alpha, \beta) \in \mathcal{S}_2$ and $\gamma = \frac{\rho\alpha - \beta - 1}{\rho - 1}$.*

Step 5. Minimizing G.

To minimize G in (10.0.1) we note that

i) If we denote $I = \inf_{\mathcal{S}} G$ and $I_i = \inf_{\mathcal{S}_i} G_i$ then $I = \min\{I_1, I_2\}$. To compute I_1 we just need to find the smallest projection of $\mathcal{S}_1$ into the first axis.

On the other hand to compute I_2, following the level sets of G_2, we need to find the line of the form $\beta = \rho\alpha - D$ with the infimum value of D that cuts $\mathcal{S}_2$; this is the "left–most" line of slope ρ that cuts $\mathcal{S}_2$. In such a case $I_2 = \frac{D-1}{\rho-1}$.

ii) On $\mathcal{S}_1$, we have $G_2(\alpha, \beta) \leq G_1(\alpha, \beta) \leq \beta$, and we can have either one equality only on the common boundary of $\mathcal{S}_1$ and $\mathcal{S}_2$, that is when $\alpha - \beta = \frac{1}{\rho}$. In this case $G_1(\alpha, \beta) = G_2(\alpha, \beta) = \beta = \alpha - \frac{1}{\rho}$.

On the other hand, on $\mathcal{S}_2$ we have $\beta < G_1(\alpha, \beta) < G_2(\alpha, \beta)$.

Remark 10.0.4 *i) Notice that the arguments above use the local existence in Theorem 9.1.7 or 9.1.8 only once, while the improved regularity is obtained using Lemma 9.1.2 repeatedly. In this way, (10.0.13), (10.0.14) and (10.0.15) hold up to $t = 0$. Also, the relationship between T and r is determined at the only time local existence is used. Hence, when Theorem 9.1.7 is used then r can be taken arbitrarily large.*
ii) Alternatively the whole bootstrap argument in Step 3 above can be performed without assuming (10.0.9) nor the last part of (10.0.12) but then, (10.0.13) and (10.0.15) do not hold up to $t = 0$. In particular, instead of using Lemma 9.1.2 we can use either Theorem 9.1.7 or 9.1.8 repeatedly in the following way; solve for u_0 to obtain a local solution up to at least $u(\tau)$ for some $\tau > 0$. Consider $u(\tau)$ as the new initial data, solve again and

repeat the process. In this way, instead of (10.0.13) and (10.0.15) we now get, for any $\varepsilon > 0$ *and* $t \in [\varepsilon, T]$ *and* γ' *as in (10.0.16)*

$$t^{\gamma'-\gamma}\|u(t,u_0)\|_{\gamma'} \le K(\varepsilon), \quad \|u(t,u_0^1) - u(t,u_0^2)\|_{\gamma'} \le \frac{M(\varepsilon)}{t^{\gamma'-\gamma}}\|u_0^1 - u_0^2\|_{\gamma}.$$

Chapter 11

Applications to $2m$-th order parabolic problems

In this chapter we apply the general results in Theorems 9.1.7, 9.1.8 to some particular parabolic problems, using also the bootstrap arguments in Chapter 10. In particular, we consider problems which can be written as

$$u_t + (-\Delta)^m u = f(x, u),\ t > 0,\ x \in \mathbb{R}^N, \tag{11.0.1}$$

with $m \in \mathbb{N}$ and a nonlinear term of the form

$$f(x, u) = D^b(h(x, D^a u)),\ x \in \mathbb{R}^N, \tag{11.0.2}$$

where D^c represents any partial derivative of order $c \in \mathbb{N}$, in the sense that for any smooth function φ we have

$$< f(x, u), \varphi >= (-1)^b \int_{\mathbb{R}^N} h(x, D^a u) D^b \varphi(x)$$

for some h, $a, b \in \mathbb{N}$ to be specified below.

11.1 The problem in the scale of Lebesgue spaces

We consider first the case $a = b = 0$ in the Lebesgue scale, then for $1 \leq q \leq \infty$ we denote

$$L^q(\mathbb{R}^N) := X^{\gamma(q)}, \quad \gamma(q) = \frac{-N}{2mq} \in \mathcal{J} = [\frac{-N}{2m}, 0]. \tag{11.1.1}$$

Note that this scale is not nested and the semigroup generated by $-(-\Delta)^m$ satisfies (ii.0.12), (ii.0.13) (except when $\gamma = 0$, that is $q = \infty$), and

$$\|S(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^p(\mathbb{R}^N))} \leq \frac{M_0}{t^{\frac{N}{2m}(\frac{1}{q} - \frac{1}{p})}} \text{ for all } 0 < t \leq T, \quad 1 \leq q \leq p \leq \infty. \tag{11.1.2}$$

Observe that the result for $1 < q \leq p \leq \infty$ and $q < \infty$ can be obtained from Part I. For $q = \infty$, the results in [30] imply that the semigroup is well defined, although not strongly continuous. Also for $u_0 \in BUC(\mathbb{R}^N)$, (ii.0.13) holds true, see Remark 11.3.1. On the other hand, for $q = 1$ the upper Gaussian bounds on the heat kernel for $(-\Delta)^m$, see [20], combined with the results in [32], imply that the semigroup is strongly continuous and analytic in $L^1(\mathbb{R}^N)$. Also, this Gaussian upper bounds imply that (11.1.2) holds with $q = 1$ and $p = \infty$. This and interpolation gives again (11.1.2) for $q = 1$ and $1 \leq p \leq \infty$.

Also, assumption (9.1.18) holds because (9.1.13) holds for this scale of spaces.

Now assume that $h(\cdot, 0) = 0$ and for some $\rho > 1$, $L > 0$ we have

$$|h(x,u) - h(x,v)| \leq L|u-v|(|u|^{\rho-1} + |v|^{\rho-1}),\ u,v \in \mathbb{R},\ x \in \mathbb{R}^N. \tag{11.1.3}$$

Then from (11.1.3) and using Hölder's inequality we get that for any $1 \leq q < \infty$ and for $u, v \in L^{\rho q}(\mathbb{R}^N)$

$$\|h(\cdot,u) - h(\cdot,v)\|_{L^q(\mathbb{R}^N)} \leq L\|u-v\|_{L^{\rho q}(\mathbb{R}^N)}(\|u\|^{\rho-1}_{L^{\rho q}(\mathbb{R}^N)} + \|v\|^{\rho-1}_{L^{\rho q}(\mathbb{R}^N)}), \tag{11.1.4}$$

while for $q = \infty$

$$\|h(\cdot,u) - h(\cdot,v)\|_{L^\infty(\mathbb{R}^N)} \leq L\|u-v\|_{L^\infty(\mathbb{R}^N)}(\|u\|^{\rho-1}_{L^\infty(\mathbb{R}^N)} + \|v\|^{\rho-1}_{L^\infty(\mathbb{R}^N)}). \tag{11.1.5}$$

In terms of the scale, (11.1.1), (11.1.4) and (11.1.5) read $f : X^\alpha \to X^\beta$ and satisfies (ii.0.15), with $\alpha = \frac{-N}{2m\rho q}$ and $\beta = \frac{-N}{2mq}$, that is $\beta = \rho\alpha$ and $\alpha_* := \frac{-N}{2m\rho} \leq \alpha \leq 0$.

Therefore the admissible region, $\mathcal{S}$, for problem (11.0.1), (11.0.2) in the Lebesgue scale (11.1.1) is a segment of the line $\beta = \rho\alpha$ determined by the conditions

$$\alpha_* := \frac{-N}{2m\rho} \leq \alpha \leq 0, \quad \beta = \rho\alpha, \quad \alpha - \beta = \alpha(1-\rho) < 1. \tag{11.1.6}$$

Also denote $\alpha_0 := \frac{-N}{2m}$. Then we have the following result.

Lemma 11.1.1 *The region $\mathcal{S}$ defined in (11.1.6) is nonempty and if we define*

$$I = \max\{-\frac{N}{2m}, -\frac{1}{\rho-1}\}$$

then the ranges $\mathcal{E}$ in (10.0.2) and $\mathcal{R}$ in (10.0.16) are both the same and equal to $[I, 0]$ except when $\frac{-1}{\rho-1} = -\frac{N}{2m}$, where they are equal to $(I, 0]$. In any case,

$$\mathcal{E}_1^c = \mathcal{E}_1 \quad \textit{and} \quad \mathcal{E} = \mathcal{R}.$$

Only when $-\frac{N}{2m} < -\frac{1}{\rho-1}$ the value I is critical for problem (11.0.1), (11.0.2) (with $a = b = 0$) in the Lebesgue scale (11.1.1), in the sense of Definition 10.0.3, and it is the only critical value.

Proof. Observe that the region $\mathcal{S}_1$ is determined by $\alpha \geq \alpha_I := \frac{-1}{\rho(\rho-1)}$ while the region $\mathcal{S}_2$ is determined by $\alpha_I > \alpha > \alpha_{II} := \frac{-1}{\rho-1}$, restrictions that have to be combined with $\alpha_* \leq \alpha \leq 0$. Thus the α coordinates of $\mathcal{S}_1$ are given by $[\max\{\alpha_*, \alpha_I\}, 0]$ and by $[\alpha_*, \alpha_I) \cap (\alpha_{II}, \alpha_I)$ for $\mathcal{S}_2$. Note that we always have

$$\alpha_0 := \rho\alpha_*, \qquad \alpha_{II} = \rho\alpha_I = \alpha_I - \frac{1}{\rho}.$$

Note that the intervals for $\mathcal{E}$ in (10.0.2) and $\mathcal{R}$ in (10.0.16) are overlapping and then, using the interval $\mathcal{J}$ from (11.1.1) we have that $\alpha^1_{max} = \alpha_{max} = 0$, $\beta_{max} = 0$ in (10.0.17) and (10.0.18). Also, the region $\mathcal{S}_1$ is such that its "most to the left and below" point is in $\alpha - \beta < 1$, so as stated in Chapter 10, Step 4 we have $\mathcal{E}^c_1 = \mathcal{E}_1$; see (10.0.17). Then we have the following cases.

Case A. Assume $\alpha_* > \alpha_I$, or equivalently $\alpha_0 > \alpha_{II}$. Then $\mathcal{S}_2$ is empty and $\mathcal{S}_1$ is the segment $\rho\alpha = \beta$, $\alpha \in [\alpha_*, 0]$.

Then, $\alpha^1_{min} = \alpha_*$, $I_1 = \inf_{\mathcal{S}_1} G_1 = \alpha_* - \frac{1}{\rho} = -\frac{N}{2m\rho} - \frac{1}{\rho} < \alpha_0$ and therefore $I = \alpha_0 = -\frac{N}{2m}$ and is attained in $\mathcal{S}_1$ so, by (10.0.18), $\mathcal{E} = \mathcal{E}_1 = \mathcal{E}^c_1 = \mathcal{R} = [-\frac{N}{2m}, 0]$.

Case B. Assume $\alpha_* = \alpha_I$ or equivalently $\alpha_0 = \alpha_{II}$. Then $\mathcal{S}_2$ is still empty and $\mathcal{S}_1$ is the segment $\rho\alpha = \beta$, $\alpha \in [\alpha_*, 0]$.

Then, $\alpha^1_{min} = \alpha_*$, $I_1 = \inf_{\mathcal{S}_1} G_1 = \alpha_* - \frac{1}{\rho} = \alpha_I - \frac{1}{\rho} = \alpha_{II} = \alpha_0$ and therefore $I = \alpha_0 = -\frac{N}{2m}$ but is not attained in $\mathcal{S}_1$ so, by (10.0.18), $\mathcal{E} = \mathcal{E}_1 = \mathcal{E}^c_1 = \mathcal{R} = (-\frac{N}{2m}, 0]$.

Case C. Assume $\alpha_* < \alpha_I$ or equivalently $\alpha_0 < \alpha_{II}$. Then $\mathcal{S}_1$ is the segment $\rho\alpha = \beta$, $\alpha \in [\alpha_I, 0]$ and $\mathcal{S}_2$ is the segment $\rho\alpha = \beta$, $\alpha \in [\alpha_*, \alpha_I) \cap (\alpha_{II}, \alpha_I)$.

Then, $\alpha^1_{min} = \alpha_I$, $I_1 = \inf_{\mathcal{S}_1} G_1 = \alpha_I - \frac{1}{\rho} = \alpha_{II} = -\frac{1}{\rho-1} > \alpha_0$ but is not attained in $\mathcal{S}_1$. Hence in (10.0.17) we get $\mathcal{E}_1 = \mathcal{E}^c_1 = (-\frac{1}{\rho-1}, 0]$.

On the other hand, G_2 is constant in $\mathcal{S}_2$, so $I_2 = \inf_{\mathcal{S}_2} G_2 = -\frac{1}{\rho-1} = I_1 = I$ and is attained in $\mathcal{S}_2$. Thus by (10.0.18), $\mathcal{E} = \mathcal{E}_2 = \mathcal{R} = [-\frac{1}{\rho-1}, 0]$.

Clearly in cases A and B there are no critical values in the sense of Definition 10.0.3, while I is the only one in case C. ■

Therefore we get the following

Theorem 11.1.2 *(Existence and regularity) Assume h satisfies (11.1.3) for some $\rho > 1$, $L > 0$ and $h(\cdot, 0) = 0$. Assume also that $a = b = 0$ and define $p_0 = \frac{N}{2m}(\rho - 1)$. Then*
i) if $p_0 < 1$, or equivalently, $\rho < \rho^* = 1 + \frac{2m}{N}$, then take $1 \leq p \leq \infty$, or
ii) if $p_0 = 1$, or equivalently, $\rho = \rho^* = 1 + \frac{2m}{N}$, then take $1 < p \leq \infty$, or
iii) if $p_0 > 1$, or equivalently, $\rho > \rho^* = 1 + \frac{2m}{N}$, then take $p_0 \leq p \leq \infty$.

Then for any $p < \infty$ as above there exist $r > 0$ and $T = T(r, p) > 0$ such that for $v_0 \in L^p(\mathbb{R}^N)$, and any u_0 satisfying $\|u_0 - v_0\|_{L^p(\mathbb{R}^N)} < r$, there exists a function $u(\cdot, u_0)$ such that for all $p \leq q \leq \infty$, $u(\cdot, u_0) \in C([0, T], L^p(\mathbb{R}^N)) \cap C((0, T], L^q(\mathbb{R}^N))$,

$$t^{\frac{N}{2m}(\frac{1}{p}-\frac{1}{q})} \|u(t, u_0)\|_{L^q(\mathbb{R}^N)} \leq M(u_0, q) \qquad \text{for } 0 < t \leq T \tag{11.1.7}$$

$$t^{\frac{N}{2m}(\frac{1}{p}-\frac{1}{q})} \|u(t, u_0)\|_{L^q(\mathbb{R}^N)} \to 0, \qquad \text{as } t \to 0, \quad p \neq q \tag{11.1.8}$$

and satisfies

$$u(t,u_0) = S(t)u_0 + \int_0^t S(t-s)h(u(s,u_0))\,ds \quad t \in [0,T] \tag{11.1.9}$$

with $S(t)$ as in (11.1.2). For $p=\infty$, the only difference is that $u(\cdot,u_0) \in L^\infty((0,T),L^\infty(\mathbb{R}^N)) \cap C((0,T],L^\infty(\mathbb{R}^N))$. Also, $u(\cdot,u_0) \in C([0,T],L^\infty(\mathbb{R}^N))$ provided $\lim_{t\to 0^+} \|S(t)u_0 - u_0\|_{L^\infty(\mathbb{R}^N)} = 0$.

If $p_0 < p$, then r can be taken arbitrarily large, that is, the existence time is uniform in bounded sets in $L^p(\mathbb{R}^N)$.

Furthermore, there exists $M > 0$ such that for all $u_0^i \in L^p(\mathbb{R}^N)$, $i = 1,2$ such that $\|u_0^i - v_0\|_{L^p(\mathbb{R}^N)} < r$, we have the following continuous dependence result

$$\|u(t,u_0^1) - u(t,u_0^2)\|_{L^q(\mathbb{R}^N)} \le \frac{M}{t^{\frac{N}{2m}(\frac{1}{p}-\frac{1}{q})}} \|u_0^1 - u_0^2\|_{L^p(\mathbb{R}^N)}, \quad t \in (0,T], \tag{11.1.10}$$

for any $p \le q \le \infty$.

Proof. Recall that in this case the admissible region is given by (11.1.6) and by Lemma 11.1.1 we have that $I = \max\{-\frac{N}{2m}, \frac{-1}{\rho-1}\}$ and $\mathcal{E} = \mathcal{R} \subset [I,0]$.

If $p_0 < 1$ then we are in Case A in the proof of Lemma 11.1.1 and then $\gamma \in \mathcal{E} = \mathcal{R} = [-\frac{N}{2m}, 0]$ and since $\gamma = \gamma(p) := -\frac{N}{2mp}$ is as in (11.1.1), we have $1 \le p \le \infty$. If $p_0 = 1$ then we are in Case B in the proof of Lemma 11.1.1 and then $\gamma \in \mathcal{E} = \mathcal{R} = (-\frac{N}{2m}, 0]$ which by (11.1.1) leads to $1 < p \le \infty$. Finally, if $p_0 > 1$ we are in Case C in the proof of Lemma 11.1.1 and then $\gamma \in \mathcal{E} = \mathcal{R} = [-\frac{1}{\rho-1}, 0]$ which leads to $p_0 \le p \le \infty$.

Now, in any case, for such p and $\gamma = \gamma(p) := -\frac{N}{2mp} \in \mathcal{E}$, take $(\alpha_0,\beta_0) \in \mathcal{S}$, $\beta_0 = \rho\alpha_0$, such that $\gamma(p) \in E(\alpha_0,\beta_0,\rho)$. Thus, following Chapter 10, Step 2 above, for any $u_0 \in X^\gamma = L^p(\mathbb{R}^N)$ we obtain a solution for (11.0.1), (11.0.2), as in Theorems 9.1.7 or 9.1.8 which satisfies (11.1.9) and is continuous in X^γ at $t=0$ because of (10.0.4) and Lemma 11.1.1. The solution satisfies (10.0.5), (10.0.6), (10.0.7) for any $\gamma \le \gamma' < \beta_0 + 1$. Note that the critical case in Theorem 9.1.8 corresponds to $\gamma = -\frac{1}{\rho-1}$, i.e. $p = p_0 > 1$.

If $\beta_0 + 1 > 0$, then (10.0.13), (10.0.14), (10.0.15) hold for any $\gamma \le \gamma' \le 0$, i.e. $\gamma' \in \mathcal{E}$, $\gamma' \ge \gamma$, which gives (11.1.7), (11.1.8), (11.1.10) for any $p \le q \le \infty$.

If $\beta_0 + 1 \le 0$, then we follow Chapter 10, Step 3 and because of (11.1.6) we can take $\alpha_1 = \gamma'$ very close to $\beta_0 + 1$, and $\beta_1 = \rho\alpha_1 > \beta_0$. For this choice (10.0.8) and (10.0.9) are satisfied, so (10.0.13), (10.0.14), (10.0.15) hold for any $\gamma \le \gamma' < \beta_1 + 1$.

If $\beta_1 + 1 > 0$, then (10.0.13), (10.0.14), (10.0.15) hold for any $\gamma \le \gamma' \le 0$, i.e. $\gamma' \in \mathcal{E}$, $\gamma' \ge \gamma$, which gives (11.1.7), (11.1.8), (11.1.10) for any $q \ge p$ and we have finished as above. If $\beta_1 + 1 \le 0$, then we can iterate the process taking $\alpha_{j+1} = \gamma'$ very close to $\beta_j + 1$ and $\beta_{j+1} = \rho\alpha_{j+1} > \beta_j$ to construct (α_j,β_j), $j = 0,1,2,...$ satisfying (10.0.12). Note that $\beta_{j+1} \sim \rho\beta_j + \rho$ is increasing in j since $\beta_0 = \rho\alpha_0 \ge -\frac{\rho}{\rho-1}$ so in a finite number J of steps, we have $\beta_J + 1 > 0$ and then we have finished as above.

From the bootstrap in Step 3 in Chapter 10 we also get that $u \in C((0,T],L^q(\mathbb{R}^N))$, $p \le q \le \infty$. ∎

We now get the following blow–up estimate, which recovers and extends some previous result. In particular, the estimates in page 39 in [53], Remark 16.2.(iii) page 89 in [46] and Theorem 3, page 199 in [28], are the same as (11.1.11) below for $m = 1$, $q = p$ and $p > p_0$.

Corollary 11.1.3 *(Blow–up estimate). Let p be as in Theorem 11.1.2 with the same notations, $u(\cdot, u_0)$ be the solution in the theorem for some $u_0 \in L^p(\mathbb{R}^N)$, and assume $\tau_{u_0} < \infty$. Then,*

i) if $p > p_0$, for any $p \le r < \frac{Np}{N\rho-2mp}$ when $p < \frac{N\rho}{2m}$, or $p \le r \le \infty$ otherwise, or

ii) if $p = p_0$, for any $p_0 < r < \rho p_0$,

we have, for $t < \tau_{u_0}$ close enough to τ_{u_0},

$$\|u(t; u_0)\|_{L^r(\mathbb{R}^N)} \ge \frac{c}{(\tau_{u_0} - t)^{-\frac{N}{2mr}+\frac{1}{\rho-1}}} = \frac{c}{(\tau_{u_0} - t)^{\frac{N}{2m}(\frac{1}{p_0}-\frac{1}{r})}}. \tag{11.1.11}$$

Proof. We will get (11.1.11) from Proposition 9.1.12. For this, given $\gamma = -\frac{N}{2mp} \in \mathcal{E}$ we find admissible pairs $(\alpha, \beta) \in \mathcal{S}$ (recall $\beta = \rho\alpha$ and $\alpha = -\frac{N}{2mq\rho}$) such that $\gamma \in E(\alpha, \beta, \rho)$. In order to do it we follow the notations in the proof of Lemma 11.1.1.

i) If $\gamma \ge \alpha_I$ i.e. $p \ge \rho p_0$, then the pair $(\alpha, \beta) \in \mathcal{S}_1$ and thus is such that $\gamma \in E(\alpha, \beta, \rho)$ if and only if $\alpha \in [\alpha_*, 0]$ i.e. $q \in [1, \infty]$, $\alpha \ge \gamma$ i.e. $q \ge \frac{p}{\rho}$ and $\gamma > \alpha - \frac{1}{\rho}$ i.e. $q < \frac{Np}{N\rho-2mp}$ when $p < \frac{N\rho}{2m}$, $p \le q \le \infty$ otherwise. Observe that $\frac{Np}{N\rho-2mp} > p$ when $p < \frac{N\rho}{2m}$ since $p > p_0$, and thus these conditions give a nonempty set for q.

On the other hand, if $\alpha_{II} < \gamma < \alpha_I$ i.e. $p_0 < p < \rho p_0$, then a pair (α, β) such that $\gamma \in E(\alpha, \beta, \rho)$ can be taken in two ways; either $(\alpha, \beta) \in \mathcal{S}_1$ or $(\alpha, \beta) \in \mathcal{S}_2$. To take $(\alpha, \beta) \in \mathcal{S}_1$ we need $\alpha \ge \alpha_I$, i.e. $q \ge p_0$, $\alpha \in [\alpha_*, 0]$ i.e. $q \in (1, \infty]$, $\alpha \ge \gamma$ i.e. $q \ge \frac{p}{\rho}$ and $\gamma > \alpha - \frac{1}{\rho}$ i.e. $q < \frac{Np}{N\rho-2mp}$ when $p < \frac{N\rho}{2m}$, $p \le q \le \infty$ otherwise. Again, since $p > p_0$ we have $\frac{Np}{N\rho-2mp)} > p$ when $p < \frac{N\rho}{2m}$ and we get a nonempty set for q. To take $(\alpha, \beta) \in \mathcal{S}_2$, we need $\alpha_{II} < \alpha < \alpha_I$, i.e. $\frac{p_0}{\rho} < q < p_0$, $\alpha \ge \alpha_*$ i.e. $q > 1$, $\alpha \ge \gamma$ i.e. $q \ge \frac{p}{\rho}$ and $\gamma \ge -\frac{1}{\rho-1} = \alpha_{II}$ which holds true. Again the range of q is nonempty since in this case $p_0 > 1$.

In summary, if $p > p_0$ we can take any q such that $q \ge \frac{p}{\rho}$ and $q < \frac{Np}{N\rho-2mp}$ when $p < \frac{N\rho}{2m}$, $p \le q \le \infty$ otherwise.

ii) If $\gamma = \alpha_{II} = -\frac{1}{\rho-1}$, i.e. if $p = p_0$, we need to take $(\alpha, \beta) \in \mathcal{S}_2$, so we need $\alpha \in [\alpha_*, \alpha_I) \cap (\alpha_{II}, \alpha_I)$. Hence, $\alpha \ge \alpha_*$ i.e. $q > 1$, $\alpha > \alpha_{II} = \gamma$ i.e. $q > \frac{p}{\rho} = \frac{p_0}{\rho}$ and $\alpha < \alpha_I$, i.e. $q < p = p_0$.

Therefore, the pairs (α, β) such that $\gamma \in E(\alpha, \beta, \rho)$ are determined by $\frac{p}{\rho} \le q < \frac{Np}{(N\rho-2mp)_+}$ when $p > p_0$ and $\frac{p_0}{\rho} < q < p_0$ when $p = p_0$. We now use Proposition 9.1.12 for $\gamma' \in [\beta, \alpha]$ and $\gamma' \ge \gamma$ (with strict inequality if $\gamma = \gamma_c$). Letting $\gamma' = -\frac{N}{2mr}$, we get that $\gamma' \in [\beta, \alpha]$, $\gamma' \ge \gamma$ corresponds to $r \in [q, \rho q]$, $r \ge p$ (with strict inequality when $p = p_0$), which leads to the conditions of the theorem. In such case, (9.1.30) gives (11.1.11) observing that for any $(\alpha, \beta) \in \mathcal{S}$, $\frac{\alpha\rho-\beta-1}{\rho-1} = \frac{1}{\rho-1} = \frac{N}{2mp_0}$. ■

Remark 11.1.4 *i) When $p = p_0$, $\frac{Np_0\rho}{N\rho - 2mp_0} = \rho p_0$, so the upper bounds of r from cases i) and ii) in Corollary 11.1.3 match.*
ii) Notice that when $p > \frac{N\rho}{2m}$ then (11.1.11) holds in particular for $r = \infty$. In the particular case $f(u) = |u|^\rho$, $\rho \leq 1 + \frac{2m}{N}$ it was shown in [25, Theorem 1.2, Proposition 3.3] that (11.1.11), with $r = \infty$, actually holds for any nonnegative nontrivial $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ (thus in $L^p(\mathbb{R}^N)$ for $p > \frac{N\rho}{2m}$). Notice that (11.1.11) for $r = \infty$ is the blow–up rate of the associated ode $\dot{u} = u^p$.

Now we show that the solution obtained in Theorem 11.1.2 is unique in a large class of functions. For this we first make the following remark which will be very useful in this and other examples below.

Remark 11.1.5 *Assume the spaces of the scale $\{X^\alpha\}_{\alpha \in \mathcal{J}}$ satisfy the interpolation property: for any $\alpha, \alpha' \in \mathcal{J}$ and $\theta \in [0, 1]$ and for every $u \in X^\alpha \cap X^{\alpha'}$*

$$\|u\|_{\theta\alpha + (1-\theta)\alpha'} \leq C\|u\|_\alpha^\theta \|u\|_{\alpha'}^{1-\theta}.$$

This happens for example when these spaces are the fractional power spaces of some sectorial operator, see Section 1.3 in [31]. It also holds when the spaces are constructed by interpolation as in [2]. Then if $u \in L^\infty([0,T], X^\alpha) \cap L^\infty([0,T], X^{\alpha'})$ we have for $\theta \in [0, 1]$

$$\|u\|_{L^\infty([0,T],\ X^{\theta\alpha+(1-\theta)\alpha'})} \leq C\|u\|^\theta_{L^\infty([0,T],\ X^\alpha)} \|u\|^{1-\theta}_{L^\infty([0,T],\ X^{\alpha'})}. \tag{11.1.12}$$

Also, if $u \in \mathcal{L}^\infty_{\alpha,\alpha-\gamma}((0,T]) \cap \mathcal{L}^\infty_{\alpha',\alpha'-\gamma}((0,T])$ then for $\theta \in [0, 1]$

$$t^{\theta\alpha+(1-\theta)\alpha'-\gamma} \|u(t)\|_{\theta\alpha+(1-\theta)\alpha'} \leq C\big(t^{\alpha-\gamma}\|u(t)\|_\alpha\big)^\theta \big(t^{\alpha'-\gamma}\|u\|_{\alpha'}\big)^{1-\theta}$$

and therefore

$$|||u|||_{\theta\alpha+(1-\theta)\alpha',\theta\alpha+(1-\theta)\alpha'-\gamma,T} \leq C|||u|||^\theta_{\alpha,\alpha-\gamma,T} |||u|||^{1-\theta}_{\alpha',\alpha'-\gamma,T}. \tag{11.1.13}$$

With this we have the following uniqueness result.

Theorem 11.1.6 *(Uniqueness) The solution obtained in Theorem 11.1.2 is unique in the following classes:*
i) If $p \geq p_0 = \frac{N}{2m}(\rho - 1)$ we take the functions $u : (0; T) \to L^p(\mathbb{R}^N)$ such that $u(0) = u_0$, $u(t)$ is bounded in $L^p(\mathbb{R}^N)$ as $t \to 0$ and

$$u \in L^\infty((\tau, T), L^q(\mathbb{R}^N)) \quad 0 < \tau < T$$

for some $q \geq p$ and $q \geq \rho$. If $p = p_0$ we furthermore require $q > p$ and $u(t) \to u_0$ in $L^p(\mathbb{R}^N)$ as $t \to 0$.
ii) If $p = p_0 = \frac{N}{2m}(\rho - 1) > 1$ we take the functions $u : (0; T) \to L^p(\mathbb{R}^N)$ such that $u(0) = u_0$ and

$$\|u(t)\|_{L^p(\mathbb{R}^N)} \leq M, \quad t^{\frac{N}{2m}(\frac{1}{p}-\frac{1}{q})}\|u(t)\|_{L^q(\mathbb{R}^N)} \leq M, \quad \text{and} \quad t^{\frac{N}{2m}(\frac{1}{p}-\frac{1}{q})}\|u(t)\|_{L^q(\mathbb{R}^N)} \to 0, \quad \text{as } t \to 0$$

for some $q > p$ and $q \geq \rho$.

Proof. First observe that the solutions constructed in Theorem 11.1.2 for $u_0 \in L^p(\mathbb{R}^N)$ satisfy the conditions in cases i) and ii), of the statement.

Then consider any function as in cases i) or ii) that satisfies (11.1.9) and observe that $u_0 \in L^p(\mathbb{R}^N) = X^{\gamma(p)}$ for $\gamma = \gamma(p) = -\frac{N}{2mp} \in \mathcal{E}$. In case i) when $q = p \geq \rho$, let $\alpha = \gamma = -\frac{N}{2mp}$, $\beta = \rho\alpha$, then since $p \geq \rho$ and $p > p_0$, the pair (α, β) actually belongs to $\mathcal{S}$ so $u \in L^\infty((0,T], L^p(\mathbb{R}^N)) = \mathcal{L}^\infty_{\alpha,\alpha-\gamma}((0,T])$. Then, Theorem 9.1.7 i) concludes the proof for this case.

In case i) with $q > p$ observe that the estimates in the statement can be read as bounds in $u \in L^\infty([0,T], X^\gamma)$ and $u \in L^\infty([\tau,T], X^{\alpha_0})$, for sufficiently small T, for some pair (α_0, β_0) in the admissible region $\mathcal{S}$ with $\alpha_0 = -\frac{N}{2mq} > \gamma$. Thus by interpolation as in (11.1.12), we get bounds in $u \in L^\infty([\tau,T], X^\alpha)$ for $\gamma \leq \alpha \leq \alpha_0$.

In case ii) we can read the estimates as bounds on

$$\|u(t)\|_\gamma \quad \text{and} \quad t^{\alpha_0-\gamma}\|u(t)\|_{\alpha_0} \qquad \text{for } 0 < t \leq T$$

for some pair (α_0, β_0) in the admissible region $\mathcal{S}$ with $\alpha_0 = -\frac{N}{2mq} > \gamma$. Now observe that the bounds as above are bounds on $|||u|||_{\gamma,0,T}$ and $|||u|||_{\alpha_0,\alpha_0-\gamma,T}$, respectively, so, by (11.1.13) in Remark 11.1.5, these bounds imply bounds on

$$t^{\alpha-\gamma}\|u(t)\|_\alpha \qquad \text{for } 0 < t \leq T$$

for any $\gamma \leq \alpha \leq \alpha_0$.

In both cases, we are going to show that, if additionally to $\alpha_0 > \gamma$, we have $\alpha_0 \geq \alpha_{min} := \inf_{(\alpha,\beta)\in\mathcal{S}} \alpha$, then we can find a pair (α, β) in the admissible region $\mathcal{S}$ such that $\gamma \leq \alpha \leq \alpha_0$ and $\gamma \in E(\alpha, \beta, \rho)$. That is, we check that γ is in the set

$$\mathfrak{E}(\gamma, \alpha_0) := \bigcup_{\substack{(\alpha,\beta)\in\mathcal{S} \\ \gamma\leq\alpha\leq\alpha_0}} E(\alpha, \beta, \rho). \tag{11.1.14}$$

Once this is done, if a function as in cases i) or ii) satisfies (11.1.9) then it satisfies $(S1)$, $(S2)$ and $(S3)$ in Chapter 9 and then, in case i), Theorem 9.2.2 will conclude that u coincides with the solution in Theorem 11.1.2 and in case ii) the uniqueness part of Theorem 9.1.8 will conclude that u coincides with the solution in Theorem 11.1.2.

So to prove that $\gamma \in \mathfrak{E}(\gamma, \alpha_0)$ when $\alpha_0 > \gamma$ and $\alpha_0 \geq \alpha_{min} := \inf_{(\alpha,\beta)\in\mathcal{S}} \alpha$ we follow the cases in the proof of Lemma 11.1.1.

Case A: $p_0 < 1$. Here $\mathcal{S}_2$ is empty, $\alpha_{min} = -\frac{N}{2m\rho}$, $I = -\frac{N}{2m}$ and $\mathcal{E} = [-\frac{N}{2m}, 0]$. Then from (11.1.14) $\mathfrak{E}(\gamma, \alpha_0) \subset [\max\{\gamma - \frac{1}{\rho}, I\}, \alpha_0]$ and with the same endpoints. Then, since $\gamma < \alpha_0$ and $I \leq \gamma$, we have $\gamma \in \mathfrak{E}(\gamma, \alpha_0)$.

Case B: $p_0 = 1$. Again $\mathcal{S}_2$ is empty but now $\alpha_{min} = -\frac{1}{\rho(\rho-1)} = -\frac{N}{2m\rho}$, $I = -\frac{N}{2m}$ and $\mathcal{E} = (-\frac{N}{2m}, 0]$. From (11.1.14), $\mathfrak{E}(\gamma, \alpha_0) \subset [\max\{\gamma - \frac{1}{\rho}, I\}, \alpha_0]$ and with the same endpoints. Then, since $\gamma < \alpha_0$ and $I < \gamma$, we have $\gamma \in \mathfrak{E}(\gamma, \alpha_0)$.

Case C: $p_0 > 1$. Now $\mathcal{S}_1$ is the segment $\rho\alpha = \beta$, $\alpha \in [\frac{-1}{\rho(\rho-1)}, 0]$ and $\mathcal{S}_2$ is the segment $\rho\alpha = \beta$, $\alpha \in [-\frac{N}{2m\rho}, \frac{-1}{\rho(\rho-1)}) \cap (\frac{-1}{\rho-1}, \frac{-1}{\rho(\rho-1)})$ so $\alpha_{min} = \max\{-\frac{N}{2m\rho}, -\frac{1}{\rho-1}\}$, $I = -\frac{1}{\rho-1}$ and

$\mathcal{E} = [-\frac{1}{\rho-1}, 0]$. Also, from (11.1.14)

$$\mathfrak{E}(\gamma,\alpha_0) = \bigcup_{\substack{(\alpha,\beta)\in\mathcal{S}_1\\ \gamma\leq\alpha\leq\alpha_0}} E(\alpha,\beta,\rho) \cup \bigcup_{\substack{(\alpha,\beta)\in\mathcal{S}_2\\ \gamma\leq\alpha\leq\alpha_0}} E(\alpha,\beta,\rho) \subset [\max\{\gamma - \frac{1}{\rho}, I_1\}, \alpha_0] \cup [I, \min\{\alpha_0, \alpha^1_{min}\}]$$

with the same endpoints and with $\alpha^1_{min} = -\frac{1}{\rho(\rho-1)}$, $I_1 = \alpha^1_{min} - \frac{1}{\rho} = -\frac{1}{\rho-1}$. Then, either $\gamma < \alpha^1_{min}$, which together with $I \leq \gamma$ and $\gamma < \alpha_0$ gives $\gamma \in [I, \min\{\alpha_0, \alpha^1_{min}\}]$, or $\gamma \geq \alpha^1_{min} > \alpha^1_{min} - \frac{1}{\rho} = I_1$, which together with $\gamma < \alpha_0$ yields $\gamma \in [\max\{\gamma - \frac{1}{\rho}, I_1\}, \alpha_0]$, so in any case $\gamma \in \mathfrak{E}(\gamma,\alpha_0)$.

Note that in all cases we require $\alpha_0 > \gamma$ and $\alpha_0 \geq \alpha_{min}$, that is $q > p (\geq p_0)$ and $q \geq \rho$ as stated in the result. ∎

Remark 11.1.7 *Observe that, when $p = p_0 > 1$ one cannot take $q = p$ in Theorem 11.1.6 i) because in some cases nonuniqueness occurs (see [41]).*

Remark 11.1.8 *In [11], [4] and for the heat equation (i.e. $m = 1$) in a bounded domain with Dirichlet boundary conditions (which makes the Lebesgue scale to be nested), the uniqueness in Theorem 11.1.6 was stated in the class of classical solutions of (11.0.1) (11.0.2) (with $a = b = 0$) such that $u \in C([0,T], L^p(\mathbb{R}^N))$ which is a particular case of Theorem 11.1.6.*

In [28], and again for the heat equation (i.e. $m = 1$) in a bounded domain with Dirichlet boundary conditions, the uniqueness was stated for $p = p_0$ in the class $L^r((0,T], L^q(\mathbb{R}^N))$ with $\frac{1}{r} = \frac{N}{2}(\frac{1}{p} - \frac{1}{q})$, $q, r > \rho$, $q > p$. The class of uniqueness in Theorem 11.1.6 ii) is a subclass of $u \in L^r((0,T], L^q(\mathbb{R}^N))$ with $\frac{1}{r} > \frac{N}{2}(\frac{1}{p} - \frac{1}{q})$.

The uniqueness result in Theorem 3 in [52] was stated only for $q = p$ and in a smaller class than the one in Theorem 11.1.6, see also Theorem 2.a).i) in [52].

11.2 The problem in the scale of Bessel potentials spaces

In this section we consider the scale of Bessel potential spaces $\{X^\alpha\}_{\alpha\in\mathbb{R}}$ with

$$X^\alpha = H^{2m\alpha,p}(\mathbb{R}^N), \quad \alpha \in \mathcal{J} := \mathbb{R} \tag{11.2.1}$$

for some $1 < p < \infty$, which is a nested scale; see [50] for details.

Note that by the results in Part I, the semigroup generated by $-(-\Delta)^m$ satisfies (ii.0.12) and (ii.0.13) and for any $\gamma' \geq \gamma$

$$\|S(t)\|_{\mathcal{L}(H^{\gamma,p}(\mathbb{R}^N), H^{\gamma',p}(\mathbb{R}^N))} \leq \frac{M_0}{t^{\gamma'-\gamma}} \quad \text{for all } \ 0 < t \leq T.$$

Also, assumption (9.1.18) holds because (9.1.13) holds for this scale of spaces.

We now analyze how the Nemytckii operator associated to f as in (11.0.2), which we denote by f as well, acts between some spaces of the scale.

Lemma 11.2.1 *Suppose that $h(\cdot,0)=0$ and for some $\rho>1$, $L>0$ we have*

$$|h(x,u)-h(x,v)|\leq L|u-v|(|u|^{\rho-1}+|v|^{\rho-1}),\ u,v\in\mathbb{R},\ x\in\mathbb{R}^N. \tag{11.2.2}$$

Assume also that $a,b\in\mathbb{N}$, $s\geq a$, $\sigma\geq b$ and

$$\rho\left(s-a-\frac{N}{p}\right)+\left(\sigma-b-\frac{N}{p'}\right)\geq -N \tag{11.2.3}$$

with $\rho\left(s-a-\frac{N}{p}\right)\geq -N$ and if $\rho(s-a-\frac{N}{p})=-N$ then $\sigma-b-\frac{N}{p'}>0$.

Then, the nonlinear term $D^b(h(\cdot,D^a u))$ takes $H^{s,p}(\mathbb{R}^N)$ into $H^{-\sigma,p}(\mathbb{R}^N)$ and satisfies

$$\|D^b h(\cdot,D^a u)-D^b h(\cdot,D^a v)\|_{H^{-\sigma,p}(\mathbb{R}^N)}\leq cL\|u-v\|_{H^{s,p}(\mathbb{R}^N)}\Big(\|u\|^{\rho-1}_{H^{s,p}(\mathbb{R}^N)}+\|v\|^{\rho-1}_{H^{s,p}(\mathbb{R}^N)}\Big).$$

Proof. Note that from (11.2.2) and (11.1.4) we have for any $1\leq q<\infty$ and for $u,v\in L^{\rho q}(\mathbb{R}^N)$

$$\|h(\cdot,u)-h(\cdot,v)\|_{L^q(\mathbb{R}^N)}\leq L\|u-v\|_{L^{\rho q}(\mathbb{R}^N)}(\|u\|^{\rho-1}_{L^{\rho q}(\mathbb{R}^N)}+\|v\|^{\rho-1}_{L^{\rho q}(\mathbb{R}^N)}) \tag{11.2.4}$$

Now, for any $u\in H^{s,p}(\mathbb{R}^N)$ and $\psi\in H^{\sigma,p'}(\mathbb{R}^N)$ we have

$$|\int_{\mathbb{R}^N}(h(\cdot,D^a u)-h(\cdot,D^a v))D^b\psi|\leq\|D^b\psi\|_{L^{q'}(\mathbb{R}^N)}\|h(\cdot,D^a u)-h(\cdot,D^a v)\|_{L^q(\mathbb{R}^N)}$$

for a certain $1\leq q<\infty$ and $\frac{1}{q}+\frac{1}{q'}=1$ to be chosen below. Then using (11.2.4) we have

$$|\int_{\mathbb{R}^N}(h(\cdot,D^a u)-h(\cdot,D^a v))D^b\psi|\leq L\|D^b\psi\|_{L^{q'}(\mathbb{R}^N)}\|D^a u-D^a v\|_{L^{\rho q}(\mathbb{R}^N)}(\|D^a u\|^{\rho-1}_{L^{\rho q}(\mathbb{R}^N)}+\|D^a v\|^{\rho-1}_{L^{\rho q}(\mathbb{R}^N)}).$$

Then using the embeddings $H^{\sigma-b,p'}(\mathbb{R}^N)\hookrightarrow L^{q'}(\mathbb{R}^N)$ and $H^{s-a,p}(\mathbb{R}^N)\hookrightarrow L^{\rho q}(\mathbb{R}^N)$ we have

$$|\int_{\mathbb{R}^N}(h(\cdot,D^a u)-h(\cdot,D^a v))D^b\psi|\leq cL\|\psi\|_{H^{\sigma,p'}(\mathbb{R}^N)}\|u-v\|_{H^{s,p}(\mathbb{R}^N)}\Big(\|u\|^{\rho-1}_{H^{s,p}(\mathbb{R}^N)}+\|v\|^{\rho-1}_{H^{s,p}(\mathbb{R}^N)}\Big).$$

The conditions for these embeddings read $q'\geq p'$, $\sigma-b-\frac{N}{p'}\geq-\frac{N}{q'}$ and $\rho q\geq p$, $s-a-\frac{N}{p}\geq-\frac{N}{q\rho}$, with the only exceptional case that if $q=1$ we must take $\sigma-b-\frac{N}{p'}>0$. These conditions can be rewritten as to find $1\leq q<\infty$ such that for $z=-\frac{N}{q}$

$$-\frac{N\rho}{p}\leq z\leq-\frac{N}{p},\qquad -\sigma+b-\frac{N}{p}\leq z\leq\rho(s-a-\frac{N}{p})$$

and if $q=1$ then $-\sigma+b-\frac{N}{p}<-N$.

First note that because of (11.2.3), $s\geq a$, $\sigma\geq b$ and $\rho\geq 1$, we have $\max\{-\frac{N\rho}{p},-\sigma+b-\frac{N}{p}\}\leq\min\{-\frac{N}{p},\rho(s-a-\frac{N}{p})\}$. Therefore the set of such z is always a nonempty interval.

Now we check that we can take $z = -\frac{N}{q} \in [-N, 0)$. For this observe that $\max\{-\frac{N\rho}{p}, -\sigma + b - \frac{N}{p}\} < 0$ and that if $\rho(s - a - \frac{N}{p}) > -N$ then $-N < \min\{-\frac{N}{p}, \rho(s - a - \frac{N}{p})\}$ and we can take $z = -\frac{N}{q} \in (-N, 0)$ for some $1 < q < \infty$.

Finally, if $\rho(s - a - \frac{N}{p}) = -N$ then we can only take $z = -N$ and then, since $q = 1$ we must have $-\sigma + b - \frac{N}{p} < -N$, which we can write as $\sigma - b - \frac{N}{p'} > 0$. ■

Remark 11.2.2 *Note that following [16, Lemma 3.1] one can replace (11.2.2) by the condition*

$$|h(x, u) - h(x, v)| \leq L|u - v|(1 + |u|^{\rho-1} + |v|^{\rho-1}), \;\; u, v \in \mathbb{R},$$

for which the results will hold true without essential changes in the proof.

Restating Lemma 11.2.1 in terms of the indexes of the scale (11.2.1), we have that $f : X^{\alpha} \to X^{\beta}$ and satisfies (ii.0.15) for $\alpha = \frac{s}{2m}$ and $\beta = \frac{-\sigma}{2m}$, with s, σ as in Lemma 11.2.1.

Therefore the admissible region, $\mathcal{S}$, for problem (11.0.1), (11.0.2) in the Bessel scale (11.2.1) is determined by conditions (11.2.3) and $\alpha - \beta < 1$. Note that the latter requires that $a + b < 2m$, since $s \geq a$ and $\sigma \geq b$.

For the sake of clarity we consider first the case $a = 0 = b$. Thus, (11.2.3) together with $\alpha \geq 0$, $\beta \leq 0$, $\alpha - \beta < 1$ implies that $\mathcal{S}$ is determined by

$$\beta \leq \rho\alpha - \rho B, \qquad \alpha \geq A, \qquad \beta \leq 0, \qquad \alpha - \beta < 1 \tag{11.2.5}$$

where $A := \frac{N}{2m}\left(\frac{1}{p} - \frac{1}{\rho}\right)_{+}$ and $B := \frac{N(\rho-1)}{2mp\rho}$. Note that $A < B$ since $B = \frac{N}{2m}\left(\frac{1}{p} - \frac{1}{p\rho}\right)$.

Recall that if $G(\alpha, \beta)$ is as in (10.0.1) then on $\mathcal{S}$ we have then $G_1(\alpha, \beta) \geq A - \frac{1}{\rho}$, while $G_2(\alpha, \beta) \geq \frac{\rho B - 1}{\rho - 1}$. Also on $\mathcal{S}_1$, we have $G_1 \geq G_2$ and on $\mathcal{S}_2$ we have $G_2 > G_1$. Thus

$$I = \inf_{(\alpha,\beta)\in\mathcal{S}} G(\alpha, \beta) \geq \max\{A - \frac{1}{\rho}, \; \frac{B\rho - 1}{\rho - 1}\}.$$

Also note that the lines $\beta = \rho\alpha - \rho B$ and $\alpha - \beta = 1$ cut the α-axis at $\alpha = B$ and $\alpha = 1$ respectively. Thus if $B \geq 1$ then the region $\mathcal{S}$ in (11.2.5) is empty.

We now prove the following lemma

Lemma 11.2.3 *The region $\mathcal{S}$ defined in (11.2.5) is nonempty if and only if*

$$B < 1. \tag{11.2.6}$$

In such a case, if $G(\alpha, \beta)$ is as in (10.0.1) with $(\alpha, \beta) \in \mathcal{S}$ as in (11.2.5), then,

$$I = \inf_{(\alpha,\beta)\in\mathcal{S}} G(\alpha, \beta) = \max\{A - \frac{1}{\rho}, \; \frac{B\rho - 1}{\rho - 1}\} = I_2 = \inf_{\mathcal{S}_2} G_2 < 1$$

and I is attained in $\mathcal{S}$. In particular, I is attained in $\mathcal{S}_2$ if and only if $\frac{B\rho-1}{\rho-1} > A - \frac{1}{\rho}$.

Only in the latter case the value I is critical for problem (11.0.1), (11.0.2) (with $a = b = 0$) in the Bessel scale (11.2.1), in the sense of Definition 10.0.3, and it is the only critical value.

The ranges $\mathcal{E}$ in (10.0.2) and $\mathcal{R}$ in (10.0.16) are both the same and equal to $[I, 1)$ when $\frac{B\rho-1}{\rho-1} > A - \frac{1}{\rho}$ or $(I, 1)$ otherwise.

In any case $\mathcal{E} = \mathcal{E}_2$ and when $\mathcal{S}_1$ is non-empty, $\mathcal{E}_1 = \mathcal{E}_1^c = (I, 1)$.

Proof. Recall that we follow the steps in Chapter 10. Observe that the lines $\alpha = A$ and $\alpha - \beta = 1$ meet at $P = (A, A-1)$. The line of slope ρ though P is $\beta = \rho\alpha - \rho B_c$ with $B_c = \frac{\rho-1}{\rho}A + \frac{1}{\rho}$, which satisfies $\frac{1}{\rho} < B_c < 1$ because $A < B < 1$. In particular $\mathcal{S}$ is nonempty if $B < 1$ as claimed. Because of this, $\max\{A - \frac{1}{\rho}, \frac{B\rho-1}{\rho-1}\} < 1$.

Note that the intervals in the definition of both $\mathcal{E}$ in (10.0.2) and $\mathcal{R}$ in (10.0.16) are overlapping and then in (10.0.18) we have $\alpha_{max} = 1$, $\beta_{max} = 0$. In particular $\mathcal{E} = \mathcal{R}$ and are equal to $(I, 1)$ or $[I, 1)$.

Case A: $B_c \le B < 1$

In this case the lines $\beta = \rho\alpha - \rho B$ and $\alpha - \beta = 1$ meet at $Q = (\frac{B\rho-1}{\rho-1}, \frac{\rho(B-1)}{\rho-1})$ and $\mathcal{S}$ is a triangle with vertices Q, $R = (B, 0)$ and $S = (1, 0)$. Note in particular that the "upper–left" segment QR is in a line of slope ρ.

Since the line $\alpha - \beta = \frac{1}{\rho}$ cuts the α–axis at $\alpha = \frac{1}{\rho}$ and $B_c \le B < 1$, we have necessarily $\frac{1}{\rho} < B$ and therefore $\mathcal{S}_1 = \emptyset$, $\mathcal{S}_2 = \mathcal{S}$.

The "left–most" line of slope ρ that cuts $\mathcal{S}_2$ passes through $R = (B, 0)$. Hence $I = I_2 = G_2(B, 0) = \frac{B\rho-1}{\rho-1} > A - \frac{1}{\rho}$ and is attained in $\mathcal{S}_2$. Therefore, by (10.0.18), $\mathcal{E} = \mathcal{R} = [I, 1)$. In this case, $\mathcal{E}_2 = [I, 1)$ and $\mathcal{E}_1 = \mathcal{E}_1^c = \emptyset$.

Case B: $\frac{1}{\rho} < B < B_c$

Now the lines $\beta = \rho\alpha - \rho B$ and $\alpha = A$ meet at $Q = (A, (A - B)\rho)$. In this case $\mathcal{S}$ is a quadrilateral of vertex P, Q, R and S. Note in particular that the "upper–left" segment QR is in a line of slope ρ.

However, as in Case A, since $B > \frac{1}{\rho}$ we still have $\mathcal{S}_1 = \emptyset$ and $I = I_2 = G_2(B, 0) = \frac{B\rho-1}{\rho-1} > A - \frac{1}{\rho}$ and is attained in $\mathcal{S}_2$. Therefore, by (10.0.18), $\mathcal{E} = \mathcal{R} = [I, 1)$. In this case, $\mathcal{E}_2 = [I, 1)$ and $\mathcal{E}_1 = \mathcal{E}_1^c = \emptyset$.

In order to discuss the remaining cases, observe that the line of slope 1 through $Q = (A, (A - B)\rho)$ cuts the α–axis at $\alpha = (1 - \rho)A + \rho B$.

Case C: $B \le \frac{1}{\rho}$ **and** $A - \frac{1}{\rho} < \frac{B\rho-1}{\rho-1}$

Since now $B < B_c$, $\mathcal{S}$ is still a quadrilateral of vertex P, Q, R and S. Note that $A - \frac{1}{\rho} < \frac{B\rho-1}{\rho-1}$ is equivalent to $\frac{1}{\rho} < (1 - \rho)A + \rho B$. Therefore, $\mathcal{S}_1$ is a triangle, with one of its vertex being R and $\alpha - \beta = \frac{1}{\rho}$ defines its opposite side. Also a piece of the segment QR belongs to $\mathcal{S}_2$.

In this case, the "left–most" line of slope ρ that cuts $\mathcal{S}_2$ passes through the segment QR, hence again $I_2 = G_2(B, 0) = \frac{B\rho-1}{\rho-1}$ and is attained in $\mathcal{S}_2$. On the other hand, the smallest projection of $\mathcal{S}_1$ into the first axis is given by the first coordinate of the intersection of the lines $\beta = \rho\alpha - \rho B$ and $\alpha - \beta = \frac{1}{\rho}$. A simple computation gives that $I_1 = \frac{B\rho-1}{\rho-1}$ as well. Therefore, by (10.0.18), $\mathcal{E} = \mathcal{R} = [I, 1)$. Also, $\mathcal{E}_2 = [I, 1)$ and $\mathcal{E}_1 = \mathcal{E}_1^c = (I, 1)$.

Case D: $B \le \frac{1}{\rho}$ **and** $A - \frac{1}{\rho} \ge \frac{B\rho-1}{\rho-1}$
Since yet $B < B_c$, $\mathcal{S}$ is still a quadrilateral of vertex P, Q, R and S. Now, since $\frac{1}{\rho} \ge (1-\rho)A + \rho B$ then $\mathcal{S}_1$ might be a triangle or a quadrilateral, but in any case no point in the segment QR belongs to $\mathcal{S}_2$.

In this case, the "left–most" line of slope ρ that cuts $\mathcal{S}_2$ passes through a point in $\mathcal{S}_1$ (that is, in the common boundary of $\mathcal{S}_1$ and $\mathcal{S}_2$) and that also has the smallest projection of $\mathcal{S}_1$ into the first axis, namely A. At this point G_1 and G_2 coincide and therefore $I = I_1 = I_2 = A - \frac{1}{\rho}$ and it is not attained in $\mathcal{S}_2$. Therefore, by (10.0.18), $\mathcal{E} = \mathcal{R} = (I, 1)$. In this case, $\mathcal{E}_2 = \mathcal{E}_1 = \mathcal{E}_1^c = (I, 1)$.

Here, there are no critical values in the sense of Definition 10.0.3, whereas I is the only one in Cases A, B, C. ■

Due to Lemmas 11.2.1 and 11.2.3 the results of Chapter 9 lead to the following theorem concerning well posedness of (11.0.1) when $a = 0 = b$ for $p_0 < \rho p$ with $p_0 := \frac{N}{2m}(\rho - 1)$. Observe that these conditions can be read as $\frac{N}{2m} \le p$ and any $\rho > 1$, or $p < \frac{N}{2m}$ and $\rho < \rho^* = 1 + \frac{2mp}{N-2mp}$.

Theorem 11.2.4 *Assume h satisfies (11.2.2) for some $\rho > 1$, $L > 0$ and $h(\cdot, 0) = 0$. Assume also that $a = b = 0$ and $p_0 < \rho p$ with $p_0 := \frac{N}{2m}(\rho - 1)$. Then for any*

$$\gamma_c := \max\left\{\frac{N}{2m}\left(\frac{1}{p} - \frac{1}{\rho}\right)_+ - \frac{1}{\rho},\ \frac{N}{2mp} - \frac{1}{\rho - 1}\right\} < \gamma < 1 \qquad (11.2.7)$$

there exist $r > 0$ and $T > 0$, such that for any $v_0 \in H^{2m\gamma,p}(\mathbb{R}^N)$ and any u_0 satisfying $\|u_0 - v_0\|_{H^{2m\gamma,p}(\mathbb{R}^N)} < r$, there exists a function $u(\cdot, u_0)$ in $[0, T]$ such that for all $1 > \gamma' \ge \gamma$, $u(\cdot, u_0) \in C((0, T], H^{2m\gamma',p}(\mathbb{R}^N)) \cap C([0, T], H^{2m\gamma,p}(\mathbb{R}^N))$ and

$$t^{\gamma'-\gamma}\|u(t, u_0)\|_{H^{2m\gamma',p}(\mathbb{R}^N)} \le M(u_0, \gamma') \qquad \text{for } 0 < t < T, \qquad (11.2.8)$$

$$t^{\gamma'-\gamma}\|u(t, u_0)\|_{H^{2m\gamma',p}(\mathbb{R}^N)} \to 0 \qquad \text{as } t \to 0, \quad \gamma \ne \gamma' \qquad (11.2.9)$$

and satisfies

$$u(t, u_0) = S(t)u_0 + \int_0^t S(t-s)h(u(s, u_0))ds \quad t \in [0, T]. \qquad (11.2.10)$$

Also, there exists $M > 0$ such that for all $u_0^i \in H^{2m\gamma,p}(\mathbb{R}^N)$, $i = 1, 2$ such that $\|u_0^i - v_0\|_{H_p^{2m\gamma}(\mathbb{R}^N)} < r$, we have for any $\gamma \le \gamma' < 1$

$$\|u(t, u_0^1) - u(t, u_0^2)\|_{H^{2m\gamma',p}(\mathbb{R}^N)} \le \frac{M}{t^{\gamma'-\gamma}}\|u_0^1 - u_0^2\|_{H^{2m\gamma,p}(\mathbb{R}^N)}, \quad t \in (0, T]. \qquad (11.2.11)$$

When $\frac{N}{2mp} - \frac{1}{\rho-1} > \frac{N}{2m}(\frac{1}{p} - \frac{1}{\rho})_+ - \frac{1}{\rho}$, all the above hold also for $\gamma = \gamma_c = \frac{N}{2mp} - \frac{1}{\rho-1}$.
If $\gamma_c < \gamma < 1$ then r can be taken arbitrarily large, that is, the existence time is uniform in bounded sets in $H^{2m\gamma,p}(\mathbb{R}^N)$.

Proof. Recall that in this case the admissible region is given by (11.2.5) with $A := \frac{N}{2m}\left(\frac{1}{p}-\frac{1}{\rho}\right)_+$ and $B := \frac{N(\rho-1)}{2mp\rho}$. In particular, (11.2.6), reads $p_0 < p\rho$ as in the statement.

Then using Lemma 11.2.3 we have that $I = \inf_{\mathcal{S}} G = \gamma_c$ is as in (11.2.7) and is attained in $\mathcal{S}$ if $\frac{N}{2mp} - \frac{1}{\rho-1} > \frac{N}{2m}(\frac{1}{p} - \frac{1}{\rho})_+ - \frac{1}{\rho}$.

For $\gamma \in \mathcal{E} = \mathcal{R} = [I, 1)$, take $(\alpha_0, \beta_0) \in \mathcal{S}$ such that $\gamma \in E(\alpha_0, \beta_0, \rho)$. Thus, following Chapter 10, Step 2, above we obtain a solution u for (11.0.1)–(11.0.2) which is continuous in X^γ at $t = 0$ because the scale is nested, see Proposition 9.1.5. The solution satisfies (10.0.5), (10.0.6) and (10.0.7) for any $\gamma' \in [\beta_0, \beta_0 + 1)$, $\gamma' \geq \gamma$.

If $\beta_0 = 0$, then in fact (10.0.13), (10.0.14) and (10.0.15) hold for any $\gamma' \in \mathcal{R} = \mathcal{E}$, $\gamma' \geq \gamma$.

If $\beta_0 < 0$ then define the segment

$$\ell_{(\alpha_0,\beta_0)} := \{(\alpha, \beta) \in \mathcal{S} : \beta = \rho\alpha + (\beta_0 - \rho\alpha_0) \text{ and } \beta \in [\beta_0, 0]\}$$

and note that $\ell_{(\alpha_0,\beta_0)} \subset \mathcal{S}$ for any $(\alpha_0, \beta_0) \in \mathcal{S}$. Then, following Step 3, we can take $\alpha_1 = \gamma'$ very close to $\beta_0 + 1$ and $\beta_1 = \rho\alpha_1 + (\beta_0 - \rho\alpha_0)$, so that $(\alpha_1, \beta_1) \in \ell_{(\alpha_0,\beta_0)}$. For this choice (10.0.9) is satisfied, so (10.0.13), (10.0.14) and (10.0.15) hold for any $\gamma' \in [\beta_0, \beta_1 + 1)$, $\gamma' \geq \gamma$.

If $\beta_1 = 0$, then in fact (10.0.13), (10.0.14) and (10.0.15) hold for any $\gamma' \in \mathcal{R} = \mathcal{E}$, $\gamma' \geq \gamma$.

If $\beta_1 < 0$, then we can iterate the process taking $\alpha_{j+1} = \gamma'$ very close to $\beta_j + 1$ and $\beta_{j+1} = \rho\alpha_{j+1} + (\beta_0 - \rho\alpha_0)$ so that $(\alpha_{j+1}, \beta_{j+1}) \in \ell_{(\alpha_0,\beta_0)}$ for $j = 0, 1, 2, ...$ satisfy (10.0.12) with $\beta_j < 0$.

Note that β_{j+1} is increasing in j so in a finite number J of steps, we have $\beta_J = 0$ and then (10.0.13), (10.0.14) and (10.0.15) hold for any $\gamma' \in \mathcal{R} = \mathcal{E}$, $\gamma' \geq \gamma$. This proves (11.2.8), (11.2.9) and (11.2.11). ∎

Remark 11.2.5 *We now analyze γ_c in (11.2.7) in terms of (p, ρ). For a given $1 < p < \infty$, condition $\frac{N}{2mp} - \frac{1}{\rho-1} > \frac{N}{2m}(\frac{1}{p} - \frac{1}{\rho})_+ - \frac{1}{\rho}$ can be read as $\min\{\frac{N}{2mp}, \frac{N}{2m\rho}\} > \frac{1}{\rho(\rho-1)}$ which can be read as $p_0 > 1$ and $p < p_0\rho$ where $p_0 := \frac{N}{2m}(\rho - 1)$. This also gives $\rho_*(p) < \rho$ where $\rho_*(p)$ is defined as follows. Let $p_* := 1 + \frac{2m}{N}$, then,*

$$\rho_*(p) = \begin{cases} 1 + \frac{2m}{N} & \text{if } 1 < p \leq p_* \\ \frac{1+\sqrt{1+\frac{8mp}{N}}}{2} & \text{if } p_* < p. \end{cases}$$

Notice that $\rho_*(p) \geq p$ for $1 < p \leq p_*$ and $\rho_*(p) < p$ for $p > p_*$. Therefore, the solution of (11.0.1) in Theorem 11.2.4 exists for $\gamma > \gamma_c(p, \rho)$

$$\gamma_c(p, \rho) = \begin{cases} -\frac{1}{\rho} & \text{if } \rho \leq \rho_*(p) < p \\ \frac{N}{2m}(\frac{1}{p} - \frac{N+2m}{N\rho}) & \text{if } p \leq \rho \leq \rho_*(p) \\ \frac{N}{2mp} - \frac{1}{\rho-1} & \text{if } \rho > \rho_*(p) \end{cases}$$

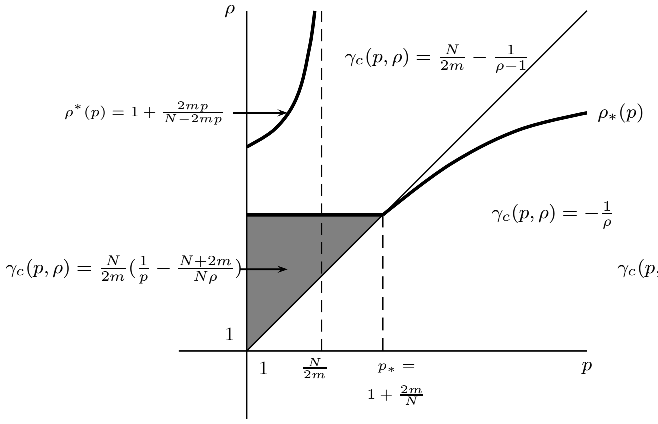

Figure 11.1: Critical curve for ρ when $N \leq 2m$

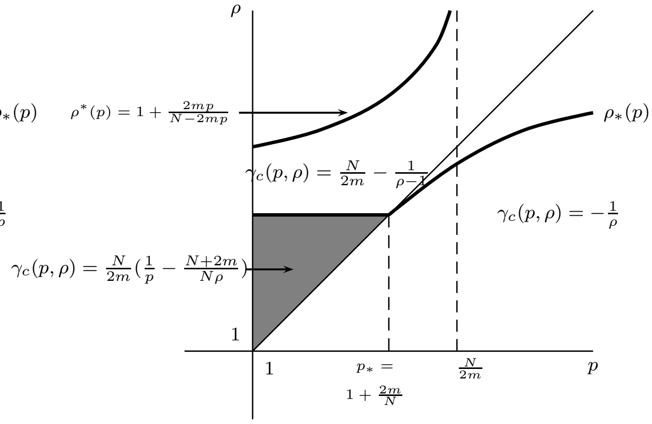

Figure 11.2: Critical curve for ρ when $N > 2m$

Note that when $\rho_*(p) < \rho$, the solution exists also for $\gamma = \gamma_c(p,\rho)$. Finally note that above the curve $\rho = \rho^*(p) = 1 + \frac{2mp}{N-2mp}$, the admissible set $\mathcal{S}$ is empty. See Figures 11.1 and 11.2.

We now study the blow up, using Proposition 9.1.12. Given $\gamma \in \mathcal{E}$, depending on the choice of the pair $(\alpha,\beta) \in \mathcal{S}$, (9.1.30) leads to different rates of blow-up for different ranges of γ'. We focus below in maximizing the blow-up rate for $\gamma' = \gamma$. Since the scale is nested, for any $\gamma' > \gamma$, the same rate holds as well. Thus to maximize the rate for a given γ, we have to minimize $G_2(\alpha,\beta)$ among (α,β) such that $\gamma \in E(\alpha,\beta,\rho)$. Therefore, we search for the smallest α, that is, the closest to γ. For α fixed, $G_2(\alpha,\beta)$ is minimized when β is maximum.

Corollary 11.2.6 *(Blow–up estimate) With the notations in Theorem 11.2.4, let $1 > \gamma \geq \gamma_c$, $u(\cdot,u_0)$ be the solution in the theorem for some $u_0 \in H^{2m\gamma,p}(\mathbb{R}^N)$, and assume $\tau_{u_0} < \infty$.*

Then, for any $\gamma' \geq \gamma$ (with strict inequality when $\gamma = \gamma_c = \frac{N}{2mp} - \frac{1}{\rho-1}$) we have,

$$\limsup_{t\to\tau_{u_0}^-} \|u(t,u_0)\|_{H^{2m\gamma',p}(\mathbb{R}^N)} = \infty. \tag{11.2.12}$$

In particular, for $\gamma \in (\gamma_c, \frac{N(\rho-1)}{2mp\rho}]$, for any $t < \tau_{u_0}$ close enough to τ_{u_0},

$$\|u(t;u_0)\|_{H^{2m\gamma,p}(\mathbb{R}^N)} \geq \frac{c}{(\tau_{u_0}-t)^{\gamma-\frac{N}{2mp}+\frac{1}{\rho-1}}} = \frac{c}{(\tau_{u_0}-t)^{\gamma+\frac{N}{2m}(\frac{1}{p_0}-\frac{1}{p})}} \tag{11.2.13}$$

and for $\gamma \in (\frac{N(\rho-1)}{2mp\rho}, 1)$, for any $t < \tau_{u_0}$ close enough to τ_{u_0},

$$\|u(t;u_0)\|_{H^{2m\gamma,p}(\mathbb{R}^N)} \geq \frac{c}{(\tau_{u_0}-t)^{\frac{1-\gamma}{\rho-1}}}. \tag{11.2.14}$$

Proof. To get (11.2.12), given $\gamma \in \mathcal{E}$ we take admissible pairs $(\alpha, \beta) \in \mathcal{S}$ such that $\gamma \in E(\alpha, \beta, \rho)$. Thus from Proposition 9.1.12 iii) and nestedness, we get (11.2.12) for any $\gamma' \geq \gamma$ (with $\gamma' > \gamma$ if $\gamma = \gamma_c$).

We now show the rate of the blow-up, following the cases in Lemma 11.2.3.

Observe that in all cases from Lemma 11.2.3, the segment $\overline{RS}$ is in $\mathcal{S}$. Therefore, if $B < \gamma < 1$, $B = \frac{N(\rho-1)}{2mp\rho}$, then $G_2(\alpha, \beta)$ is minimum for the smallest α, and biggest β, i.e. $(\alpha, \beta) = (\gamma, 0) \in \overline{RS}$. For such pair, Proposition 9.1.12 iii) and (9.1.30) gives (11.2.14).

Also, note that in all cases from Lemma 11.2.3, the segment $\overline{QR}$ is in $\mathcal{S}$. Therefore, if $\inf_{\alpha \in \mathcal{S}} \alpha < \gamma \leq B$, then $G_2(\alpha, \beta)$ is minimum for the smallest α, and biggest β, i.e. $(\alpha, \beta) = (\gamma, \rho(\gamma - B)) \in \overline{QR}$. For such pair, Proposition 9.1.12 iii) and (9.1.30) gives (11.2.13).

Note that in Case A from Lemma 11.2.3 $\inf_{\alpha \in \mathcal{S}} \alpha = I$ so this case is finished. For Cases B, C, D from Lemma 11.2.3, it remains to deal with the case $I < \gamma < A$. In that case, $G_2(\alpha, \beta)$ is minimum for the smallest α, and biggest β, $(\alpha, \beta) = (A, \rho(A - B))$. For such pair, Proposition 9.1.12 iii) and (9.1.30) gives (11.2.13). ■

Remark 11.2.7 *i) Following [26, Remark 2, p. 986], note that in the critical case it is generally unclear how the blow up actually occurs and note that for the critical case, $\gamma = \gamma_c$, Proposition 9.1.12 does not give the rate of blow-up for the γ_c-norm. However, in this case, $\gamma_c \in E(\alpha, \beta, \rho)$ for any $\alpha \in [\inf_{\alpha \in \mathcal{S}} \alpha, B]$ and $\beta = \rho(\alpha - B)$. Therefore, we can choose $(\alpha, \beta) = (B, 0)$ and (9.1.30) gives*

$$\|u(t; u_0)\|_{H^{2m\gamma', p}(\mathbb{R}^N)} \geq \frac{c}{(\tau_{u_0} - t)^{\gamma' - \gamma_c}}$$

for any $\gamma' \in (\gamma_c, \frac{N(\rho-1)}{2mp\rho}]$.

ii) As stated above, since the scale is nested, the rate of blow-up for $\|u(t; u_0)\|_{H^{2m\gamma, p}(\mathbb{R}^N)}$ holds also for $\|u(t; u_0)\|_{H^{2m\gamma', p}(\mathbb{R}^N)}$, $\gamma' > \gamma$. However, other rates can be obtained in some cases for γ'.

Theorem 11.2.8 *(Uniqueness) The solution obtained in Theorem 11.2.4 is unique in the following class: For $1 > \gamma \geq \gamma_c = \max\{\frac{N}{2mp} - \frac{1}{\rho-1}, \frac{N}{2mp}(\frac{1}{p} - \frac{1}{\rho})_+ - \frac{1}{\rho}\}$ we take functions $u : (0, T) \to H^{2m\gamma, p}(\mathbb{R}^N)$ such that $u(0) = u_0$, $u(t)$ is bounded in $H^{2m\gamma, p}(\mathbb{R}^N)$ as $t \to 0$ and*

$$u \in L^\infty((\tau, T), H^{2m\gamma', p}(\mathbb{R}^N)), \quad 0 < \tau < T$$

for some $\gamma' > \gamma$ and $\gamma' \geq A := \frac{N}{2m}\left(\frac{1}{p} - \frac{1}{\rho}\right)_+$. Also, if $\frac{N}{2mp} - \frac{1}{\rho-1} > \frac{N}{2m}(\frac{1}{p} - \frac{1}{\rho})_+ - \frac{1}{\rho}$ and $\gamma = \gamma_c = \frac{N}{2mp} - \frac{1}{\rho-1}$ we also require $u(t) \to u_0$ in $H^{2m\gamma, p}(\mathbb{R}^N)$ as $t \to 0$.

Proof. First observe that the solutions constructed in Theorem 11.2.4 for $u_0 \in H^{2m\gamma, p}(\mathbb{R}^N)$ satisfy conditions in the statement.

Then consider any function as in the statement that satisfies (11.2.10) and observe that $u_0 \in H^{2m\gamma, p}(\mathbb{R}^N) = X^\gamma$ for $\gamma \in \mathcal{E}$. Observe that the estimates in the statement can be read as bounds in $u \in L^\infty([0,T], X^\gamma)$ and $u \in L^\infty([\tau,T], X^{\gamma'})$, for sufficiently small T. Thus, by interpolation as in (11.1.12) we get bounds in $u \in L^\infty([\tau,T], X^\alpha)$ for $\alpha \in [\gamma,\gamma']$.

To conclude, we are going to show that if additionally to $\gamma' > \gamma$ we have $\gamma' \geq \alpha_{min} := \inf_{(\alpha,\beta)\in\mathcal{S}} \alpha$, we can find a pair (α,β) in the admissible region $\mathcal{S}$ such that $\gamma \leq \alpha \leq \gamma'$ and $\gamma \in E(\alpha,\beta,\rho)$. That is, we check that γ is in the set

$$\mathfrak{E}(\gamma,\gamma') := \bigcup_{\substack{(\alpha,\beta)\in\mathcal{S}\\ \gamma\leq\alpha\leq\gamma'}} E(\alpha,\beta,\rho).$$

Once this is done, if moreover u verifies (11.2.10) then u satisfies $(S1)$, $(S2)$ and $(S3)$ in Chapter 9 and then either Theorem 9.2.2 part i) or ii) will conclude that u coincides with the solution in Theorem 11.2.4.

So to prove that $\gamma \in \mathfrak{E}(\gamma,\gamma')$ when $\alpha_0 > \gamma$ and $\alpha_0 \geq \alpha_{min} := \inf_{(\alpha,\beta)\in\mathcal{S}} \alpha$, observe that for fixed α, then if the maximum β such that $(\alpha,\beta) \in \mathcal{S}_2$ is $\beta = 0$ then $\inf_\beta G(\alpha,\beta) = \frac{\alpha\rho-1}{\rho-1}$, while $\inf_\beta G(\alpha,\beta) = I$ in any other case, and we follow the cases from Lemma 11.2.3.

Case A: $B_c \leq B < 1$**.** In this case $\mathcal{S}_1 = \emptyset$, $\mathcal{S} = \mathcal{S}_2$ is a triangle, $\alpha_{min} = \frac{B\rho-1}{\rho-1} = I > A$ and $\mathcal{E} = [I,1)$.

Case B: $\frac{1}{\rho} < B < B_c$**.** In this case $\mathcal{S}$ is a trapezoid, $\mathcal{S}_1 = \emptyset$, $\mathcal{S}_2 = \mathcal{S}$, $\alpha_{min} = A$ and $\mathcal{E} = (I,1)$.

For Cases A and B, observe that for any $\alpha \leq B$ then $\inf_\beta G(\alpha,\beta) = I$ while if $\alpha > B$, then $\inf_\beta G(\alpha,\beta) = \frac{\alpha\rho-1}{\rho-1}$. So if $\gamma \leq B$ then $\mathfrak{E}(\gamma,\gamma') \subset [I,\gamma']$ with the same endpoints whereas if $\gamma > B$ then $\mathfrak{E}(\gamma,\gamma') \subset [\frac{\gamma\rho-1}{\rho-1},\gamma']$ with the same endpoints. In both cases, $\gamma \in \mathfrak{E}(\gamma,\gamma')$.

Case C: $B \leq \frac{1}{\rho}$ **and** $A - \frac{1}{\rho} < \frac{B\rho-1}{\rho-1}$**.** In this case $\mathcal{S}$ is a trapezoid, $\alpha_{min} = A$ and $\mathcal{E} = [I,1)$.

Case D: $B \leq \frac{1}{\rho}$ **and** $A - \frac{1}{\rho} \geq \frac{B\rho-1}{\rho-1}$**.** In this case $\mathcal{S}$ is a trapezoid, $\alpha_{min} = A$ and $\mathcal{E} = (I,1)$.

For Cases C and D, observe that for any $\alpha \leq \frac{1}{\rho}$ then $\inf_\beta G(\alpha,\beta) = I$ while if $\alpha > \frac{1}{\rho}$, then $\inf_\beta G(\alpha,\beta) = \frac{\alpha\rho-1}{\rho-1}$. So if $\gamma \leq \frac{1}{\rho}$ then $\mathfrak{E}(\gamma,\gamma') \subset [I,\gamma']$ with the same endpoints whereas if $\gamma > \frac{1}{\rho}$ then $\mathfrak{E}(\gamma,\gamma') \subset [\frac{\gamma\rho-1}{\rho-1},\gamma']$ with the same endpoints. In both cases, $\gamma \in \mathfrak{E}(\gamma,\gamma')$.

Note that in all cases we require $\gamma' > \gamma$ and $\gamma' \geq A$ (which is satisfied implicitly in Case A) as stated in the result. ■

Remark 11.2.9 *For fixed γ, (11.2.7) gives the admissible growth ρ for the nonlinearity to have well-possedness, and we can recover some known results.*

In particular if $\gamma = 0$, that is for initial data in $L^p(\mathbb{R}^N)$, we have $\rho \leq 1 + \frac{2mp}{N}$. When $m = 1$ this leads to the results in [11]. When $m = 2$ the results in [16] are recovered.

Similarly, for $\gamma = \frac{1}{2}$, that is for initial data in $H^{m,p}(\mathbb{R}^N)$, we need $\rho \leq 1 + \frac{2mp}{N-mp}$ when $N > mp$ and any ρ otherwise. When $m = 1$ this means taking the initial data in $H^{1,p}(\mathbb{R}^N)$, and we recover the results in [4]. When $m = 2$ and $p = 2$ this means taking the initial data in $H^{2,2}(\mathbb{R}^N)$, and we obtain the results in [16].

Finally, for $\gamma = -\frac{1}{2}$, that is for initial data in $H^{-m,p}(\mathbb{R}^N)$, we have $\rho \leq 1 + \frac{2mp}{N+mp}$.

Now we turn to the case $a, b \neq 0$. In this case (11.2.3) together with $\alpha \geq \frac{a}{2m}$, $\beta \leq \frac{-b}{2m}$, $\alpha - \beta < 1$ implies that the admissible region $\mathcal{S}$ is now determined by

$$\beta \leq \rho\alpha - \rho B, \qquad \alpha \geq A, \qquad \beta \leq -C, \qquad \alpha - \beta < 1 \tag{11.2.15}$$

with $A := \frac{N}{2m}\left(\frac{1}{p} - \frac{1}{\rho}\right)_+ + \frac{a}{2m}$, $B := \frac{a\rho+b}{2m\rho} + \frac{N(\rho-1)}{2mp\rho}$ and $C := \frac{b}{2m} < 1$. Note that in this case $A + \frac{C}{\rho} < B$. Also, if $b = 0$, then we are in the situation of region (11.2.5), for the values of A and B above. Recall that, as in the case without derivatives, if $G(\alpha, \beta)$ as in (10.0.1) then

$$I = \inf_{(\alpha,\beta)\in\mathcal{S}} G(\alpha, \beta) \geq \max\{A - \frac{1}{\rho}, \frac{B\rho - 1}{\rho - 1}\}.$$

Also note that the lines $\beta = \rho\alpha - \rho B$ and $\alpha - \beta = 1$ cut $\beta = -C$ at $\alpha = B - \frac{C}{\rho}$ and $\alpha = 1 - C$. Thus if $B + \frac{\rho-1}{\rho}C \geq 1$ the region $\mathcal{S}$ is empty. In this case we have the following.

Lemma 11.2.10 *The region $\mathcal{S}$ defined in (11.2.15) is nonempty if and only if*

$$B + \frac{\rho - 1}{\rho}C < 1. \tag{11.2.16}$$

In such a case, if $G(\alpha, \beta)$ is as in (10.0.1) with $(\alpha, \beta) \in \mathcal{S}$ as in (11.2.15), then

$$I = \inf_{(\alpha,\beta)\in\mathcal{S}} G(\alpha, \beta) = \max\{A - \frac{1}{\rho},\ \frac{B\rho - 1}{\rho - 1}\} = I_2 = \inf_{\mathcal{S}_2} G_2 < 1 - C$$

and is attained in $\mathcal{S}$. In particular, I is attained in $\mathcal{S}_2$ if and only if $\frac{B\rho-1}{\rho-1} > A - \frac{1}{\rho}$.

Only in the latter case the value I is critical for problem (11.0.1), (11.0.2) in the Bessel scale (11.2.1), in the sense of Definition 10.0.3, and it is the only critical value.

The ranges $\mathcal{E}$ in (10.0.2) and $\mathcal{R}$ in (10.0.16) are both the same and equal to $[I, 1 - C)$ when $\frac{B\rho-1}{\rho-1} > A - \frac{1}{\rho}$ or $(I, 1 - C)$ otherwise.

In any case $\mathcal{E} = \mathcal{E}_2$ and when $\mathcal{S}_1$ is non-empty, $\mathcal{E}_1 = \mathcal{E}_1^c = (I, 1 - C)$.

Proof. Observe that the lines $\alpha = A$ and $\alpha - \beta = 1$ meet at $P = (A, A - 1)$. The line of slope ρ through P is $\beta = \rho\alpha - \rho B_c$ with $B_c = \frac{\rho-1}{\rho}A + \frac{1}{\rho}$, which satisfies $\frac{1}{\rho} < B_c < 1$ because $A < B - \frac{C}{\rho} < B + \frac{\rho-1}{\rho}C < 1$. In particular $\mathcal{S}$ is nonempty if $B + \frac{\rho-1}{\rho}C < 1$ as claimed. Because of this, $\max\{A - \frac{1}{\rho}, \frac{B\rho-1}{\rho-1}\} < 1 - C$.

Note that the intervals in the definition of both $\mathcal{E}$ in (10.0.2) and $\mathcal{R}$ in (10.0.16) are overlapping and then in (10.0.18) we have $\alpha_{max} = 1 - C$, $\beta_{max} = -C$. In particular $\mathcal{E} = \mathcal{R}$ and are equal to $(I, 1 - C)$ or $[I, 1 - C)$.

Case A: $B_c \leq B < 1$

In this case the lines $\beta = \rho\alpha - \rho B$ and $\alpha - \beta = 1$ meet at $Q = (\frac{B\rho-1}{\rho-1}, \frac{\rho(B-1)}{\rho-1})$ and $\mathcal{S}$ is a triangle with vertices Q, $\tilde{R} = (B - \frac{C}{\rho}, -C)$ and $\tilde{S} = (1 - C, -C)$. Note in particular that the "upper–left" segment $Q\tilde{R}$ is in a line of slope ρ.

Since the line $\alpha-\beta=\frac{1}{\rho}$ cuts the α-axis at $\alpha=\frac{1}{\rho}$ and $B_c \le B < 1$, we have necessarily $\frac{1}{\rho} < B$ and therefore $\mathcal{S}_1=\emptyset$, $\mathcal{S}_2=\mathcal{S}$. The "left–most" line of slope ρ that cuts $\mathcal{S}_2$ passes through $\tilde{R}=(B-\frac{C}{\rho},-C)$. Hence $I=I_2=G_2(B-\frac{C}{\rho},-C)=\frac{B\rho-1}{\rho-1} > A-\frac{1}{\rho}$ and is attained in $\mathcal{S}_2$. Therefore, by (10.0.18), $\mathcal{E}=\mathcal{R}=[I,1)$. In this case, $\mathcal{E}_2=[I,1)$ and $\mathcal{E}_1=\mathcal{E}_1^c=\emptyset$.

Case B: $\frac{1}{\rho} < B < B_c$

Now the lines $\beta=\rho\alpha-\rho B$ and $\alpha=A$ meet at $Q=(A,(A-B)\rho)$. In this case, since $(A-B)\rho < -C$, $\mathcal{S}$ is a quadrilateral of vertex P, Q, $\tilde{R}$ and $\tilde{S}$. Note in particular that the "upper–left" segment $Q\tilde{R}$ is in a line of slope ρ.

However, as in Case A, since $B > \frac{1}{\rho}$ we still have $\mathcal{S}_1=\emptyset$ and $I=I_2=G_2(B-\frac{C}{\rho},-C)=\frac{B\rho-1}{\rho-1} > A-\frac{1}{\rho}$ and is attained in $\mathcal{S}_2$. Therefore, by (10.0.18), $\mathcal{E}=\mathcal{R}=[I,1)$. In this case, $\mathcal{E}_2=[I,1)$ and $\mathcal{E}_1=\mathcal{E}_1^c=\emptyset$.

In order to discuss the remaining cases, observe that the line of slope 1 through $Q=(A,(A-B)\rho)$ cuts the α–axis at $\alpha=(1-\rho)A+\rho B$.

Case C: $B \le \frac{1}{\rho}$ **and** $A-\frac{1}{\rho} < \frac{B\rho-1}{\rho-1}$

Since now $B < B_c$, $\mathcal{S}$ is still a quadrilateral of vertex P, Q, $\tilde{R}$ and $\tilde{S}$. Note that $A-\frac{1}{\rho} < \frac{B\rho-1}{\rho-1}$ is equivalent to $\frac{1}{\rho} < (1-\rho)A+\rho B$. Therefore, $\mathcal{S}_1$ is a triangle, with one of its vertex being $\tilde{R}$ and $\alpha-\beta=\frac{1}{\rho}$ defines its opposite side. Also a piece of the segment $Q\tilde{R}$ belongs to $\mathcal{S}_2$. In this case, the "left–most" line of slope ρ that cuts $\mathcal{S}_2$ passes through the segment $Q\tilde{R}$, hence again $I_2=G_2(B-\frac{C}{\rho},-C)=\frac{B\rho-1}{\rho-1}$ and is attained in $\mathcal{S}_2$. On the other hand, the smallest projection of $\mathcal{S}_1$ into the first axis is given by the first coordinate of the intersection of the lines $\beta=\rho\alpha-\rho B$ and $\alpha-\beta=\frac{1}{\rho}$. A simple computation gives that $I_1=\frac{B\rho-1}{\rho-1}$ as well. Therefore, by (10.0.18), $\mathcal{E}=\mathcal{R}=[I,1)$. Also, $\mathcal{E}_2=[I,1)$ and $\mathcal{E}_1=\mathcal{E}_1^c=(I,1)$.

Case D: $B \le \frac{1}{\rho}$ **and** $A-\frac{1}{\rho} \ge \frac{B\rho-1}{\rho-1}$

Since now $B < B_c$, $\mathcal{S}$ is still a quadrilateral of vertex P, Q, $\tilde{R}$ and $\tilde{S}$. Now, since $\frac{1}{\rho} \ge (1-\rho)A+\rho B$ then $\mathcal{S}_1$ might be a triangle or a quadrilateral, but in any case no point in the segment $Q\tilde{R}$ belongs to $\mathcal{S}_2$. In this case, the "left–most" line of slope ρ that cuts $\mathcal{S}_2$ passes through a point in $\mathcal{S}_1$ (that is, in the common boundary of $\mathcal{S}_1$ and $\mathcal{S}_2$) and that also has the smallest projection of $\mathcal{S}_1$ into the first axis, namely A. At this point G_1 and G_2 coincide and therefore $I=I_1=I_2=A-\frac{1}{\rho}$ and it is not attained in $\mathcal{S}_2$. Therefore, by (10.0.18), $\mathcal{E}=\mathcal{R}=(I,1)$. In this case, $\mathcal{E}_2=\mathcal{E}_1=\mathcal{E}_1^c=(I,1)$.

Here, there are no critical values in the sense of Definition 10.0.3, whereas I is the only one in Cases A, B, C. ■

Therefore we get the following Theorem for $p_0 < \rho p(1-\frac{k}{2m})$ with $p_0 := \frac{N}{2m}(\rho-1)$, that is either $\frac{N}{2m-k} \le p$ and any $\rho > 1$, or $p < \frac{N}{2m-k}$ and $\rho < \rho^* = 1+\frac{p(2m-k)}{N-p(2m-k)}$.

Theorem 11.2.11 *Assume h satisfies (11.2.2) for some $\rho > 1$, $L > 0$. Denote $k =$*

$a+b<2m$ and assume $p_0 < \rho p(1-\frac{k}{2m})$ with $p_0 := \frac{N}{2m}(\rho-1)$. Then for

$$\gamma_c = \max\left\{\frac{N}{2m}\left(\frac{1}{p}-\frac{1}{\rho}\right)_+ + \frac{a}{2m}-\frac{1}{\rho},\ \frac{N}{2mp}+\frac{a\rho+b}{2m(\rho-1)}-\frac{1}{\rho-1}\right\} < \gamma < 1-\frac{b}{2m} \tag{11.2.17}$$

there exist $r>0$ and $T>0$, such that for any $v_0 \in H^{2m\gamma,p}(\mathbb{R}^N)$ and any u_0 satisfying $\|u_0-v_0\|_{H^{2m\gamma,p}(\mathbb{R}^N)} < r$, there exists a function $u(\cdot,u_0)$ such that for all $\gamma \le \gamma' < 1-\frac{b}{2m}$, $u(\cdot,u_0) \in C((0,T], H^{2m\gamma',p}(\mathbb{R}^N)) \cap C([0,T], H^{2m\gamma,p}(\mathbb{R}^N))$ and

$$t^{\gamma'-\gamma}\|u(t,u_0)\|_{H^{2m\gamma',p}(\mathbb{R}^N)} \le M(u_0,\gamma') \qquad \textit{for } 0<t<T$$

$$t^{\gamma'-\gamma}\|u(t,u_0)\|_{H^{2m\gamma',p}(\mathbb{R}^N)} \to 0 \qquad \textit{as } t\to 0, \gamma'\neq\gamma$$

and satisfies

$$u(t,u_0) = S(t)u_0 + \int_0^t S(t-s)h(u(s,u_0))ds \quad t\in[0,T].$$

Also, there exists $M>0$ such that for all $u_0^i \in H^{2m\gamma,p}(\mathbb{R}^N)$, $i=1,2$ such that $\|u_0^i - v_0\|_{H^{2m\gamma,p}(\mathbb{R}^N)} < r$, we have for $\gamma' \in [\gamma, 1-\frac{b}{2m})$

$$\|u(t,u_0^1)-u(t,u_0^2)\|_{H^{2m\gamma',p}(\mathbb{R}^N)} \le \frac{M}{t^{\gamma'-\gamma}}\|u_0^1-u_0^2\|_{H^{2m\gamma,p}(\mathbb{R}^N)}, \quad t\in(0,T].$$

When $\frac{N}{2mp}+\frac{a\rho+b}{2m(\rho-1)}-\frac{1}{\rho-1} > \frac{N}{2m}\left(\frac{1}{p}-\frac{1}{\rho}\right)_+ + \frac{a}{2m}-\frac{1}{\rho}$ then the above hold also for $\gamma=\gamma_c=\frac{N}{2mp}+\frac{a\rho+b}{2m(\rho-1)}-\frac{1}{\rho-1}$.

If $\gamma_c<\gamma<1-\frac{b}{2m}$ then r can be taken arbitrarily large, that is, the existence time is uniform in bounded sets in $H^{2m\gamma,p}(\mathbb{R}^N)$.

Proof. First note that (11.2.16) gives $\frac{\rho a+b}{2m\rho}+\frac{N(\rho-1)}{2mp\rho}+\frac{(\rho-1)b}{2m\rho}<1$ which we can write as $\frac{a+b}{2m}+\frac{N(\rho-1)}{2mp\rho}<1$ which in turn, using that $k=a+b<2m$, is satisfied if and only if p and ρ are as in the statement.

According to Lemma 11.2.10 with $A := \frac{N}{2m}\left(\frac{1}{p}-\frac{1}{\rho}\right)_+ + \frac{a}{2m}$, $B := \frac{a\rho+b}{2m\rho}+\frac{N(\rho-1)}{2mp\rho}$ and $C := \frac{b}{2m} < 1$ we have that $I = \inf_{\mathcal{S}} G = \gamma_c$ as in (11.2.17) and is attained in $\mathcal{S}$ if $\frac{N}{2mp}+\frac{a\rho+b}{2m(\rho-1)}-\frac{1}{\rho-1} > \frac{N}{2m}\left(\frac{1}{p}-\frac{1}{\rho}\right)_+ + \frac{a}{2m}-\frac{1}{\rho}$

For $\gamma \in \mathcal{E} = \mathcal{R} = [I, 1-C)$ take $(\alpha_0,\beta_0)\in\mathcal{S}$ and we can construct a solution and perform the bootstrap argument following the proof in Theorem 11.2.4. The only difference with that proof is that we now perform the bootstrap until $\beta_j = -C$ for some j. ■

In the same manner as Corollary 11.2.6 we obtain the following and Theorem 11.2.8 we obtain the following two results

Corollary 11.2.12 *(Blow–up estimate) With the notations in Theorem 11.2.11, let* $1 - \frac{b}{2m} > \gamma \geq \gamma_c$, $u(\cdot, u_0)$ *be the solution in the theorem for some* $u_0 \in H^{2m\gamma,p}(\mathbb{R}^N)$, *and assume* $\tau_{u_0} < \infty$.

Then, for any $\gamma' \geq \gamma$ *(with strict inequality when* $\gamma = \gamma_c = \frac{N}{2mp} + \frac{a\rho+b}{2m(\rho-1)} - \frac{1}{\rho-1}$*) we have,*

$$\limsup_{t \to \tau_{u_0}^-} \|u(t, u_0)\|_{H^{2m\gamma',p}(\mathbb{R}^N)} = \infty.$$

In particular, for $\gamma \in (\gamma_c, \frac{a\rho+b}{2mp} + \frac{N(\rho-1)}{2mp\rho}]$ *, for any* $t < \tau_{u_0}$ *close enough to* τ_{u_0}*,*

$$\|u(t;u_0)\|_{H^{2m\gamma,p}(\mathbb{R}^N)} \geq \frac{c}{(\tau_{u_0} - t)^{\gamma - \frac{N}{2mp} - \frac{a\rho+b}{2m(\rho-1)} + \frac{1}{\rho-1}}} = \frac{c}{(\tau_{u_0} - t)^{\gamma + \frac{N}{2m}(\frac{1}{p_0} - \frac{1}{p}) - \frac{a\rho+b}{2m(\rho-1)}}}$$

and for $\gamma \in (\frac{a\rho+b}{2mp} + \frac{N(\rho-1)}{2mp\rho}, 1 - \frac{b}{2m})$*, for any* $t < \tau_{u_0}$ *close enough to* τ_{u_0}*,*

$$\|u(t;u_0)\|_{H^{2m\gamma,p}(\mathbb{R}^N)} \geq \frac{c}{(\tau_{u_0} - t)^{\frac{1 - \frac{b}{2m} - \gamma}{\rho-1}}}.$$

For the critical case, $\gamma = \gamma_c$*, we have* $\|u(t;u_0)\|_{H^{2m\gamma',p}(\mathbb{R}^N)} \geq \frac{c}{(\tau_{u_0}-t)^{\gamma' + \frac{N}{2m}(\frac{1}{p_0} - \frac{1}{p}) - \frac{a\rho+b}{2m(\rho-1)}}}$ *for any* $\gamma' \in (\gamma_c, \frac{a\rho+b}{2mp} + \frac{N(\rho-1)}{2mp\rho})$.

Theorem 11.2.13 *(Uniqueness) The solution obtained in Theorem 11.2.11 is unique in the following class: For* $1 - \frac{b}{2m} > \gamma \geq \gamma_c = \max\{\frac{N}{2mp} + \frac{a\rho+b}{2m(\rho-1)} - \frac{1}{\rho-1}, \frac{N}{2mp}(\frac{1}{p} - \frac{1}{\rho})_+ + \frac{a}{2m} - \frac{1}{\rho}\}$ *we take functions* $u : (0,T) \to H^{2m\gamma,p}(\mathbb{R}^N)$ *such that* $u(0) = u_0$, $u(t)$ *is bounded in* $H^{2m\gamma,p}(\mathbb{R}^N)$ *as* $t \to 0$ *and*

$$u \in L^\infty((\tau, T), H^{2m\gamma',p}(\mathbb{R}^N)), \quad 0 < \tau < T$$

for some $\gamma' > \gamma$ *and* $\gamma' \geq A := \frac{N}{2m}\left(\frac{1}{p} - \frac{1}{\rho}\right)_+ + \frac{a}{2m}$. *Also, if* $\frac{N}{2mp} + \frac{a\rho+b}{2m(\rho-1)} - \frac{1}{\rho-1} > \frac{N}{2m}(\frac{1}{p} - \frac{1}{\rho})_+ + \frac{a}{2m} - \frac{1}{\rho}$ *and* $\gamma = \gamma_c = \frac{N}{2mp} + \frac{a\rho+b}{2m(\rho-1)} - \frac{1}{\rho-1}$ *we also require* $u(t) \to u_0$ *in* $H^{2m\gamma,p}(\mathbb{R}^N)$ *as* $t \to 0$.

As a particular case we have the following analysis for the Cahn-Hilliard equation in Bessel potentials spaces as in [17] (see also [49, 56] and references therein). This problem reads

$$\begin{cases} u_t + \Delta^2 u + \Delta h(x,u) = 0, \ t > 0, \ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^N, \end{cases} \tag{11.2.18}$$

which has a formal energy functional, $E(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} - \int_{\mathbb{R}^N} \int_0^u h(x,s)dsdx$, which is naturally defined on $H^1(\mathbb{R}^N)$. Hence, $H^1(\mathbb{R}^N)$ is the natural space in which one may want to have this problem well posed, which corresponds, in the scale (11.2.1) to $m = 2$, $p = 2$ and $\gamma = \frac{1}{4}$.

Corollary 11.2.14 *Consider the Cahn–Hilliard equation (11.2.18) in the scale*

$$X^{\gamma} = H^{4\gamma}(\mathbb{R}^N), \qquad \gamma \in \mathbb{R}.$$

Then there exist some $1 < \rho_* < 2$ such for $1 < \rho \le \rho_*$ (11.2.18) is well posed in $H^{4\gamma}(\mathbb{R}^N)$ for $\gamma > \gamma_c := -\frac{1}{\rho}$ while for $\gamma \ge \gamma_c := \frac{N}{8} - \frac{1}{2(\rho-1)}$ if $\rho > \rho_*$.

In particular (11.2.18) is well posed in the energy space $H^1(\mathbb{R}^N)$, provided $N \le 2$ and no restriction on $\rho > 1$, or $1 < \rho \le \rho_c := 1 + \frac{4}{N-2}$ when $N \ge 3$.

Proof. We apply Theorem 11.2.11 with $m = 2$, $p = 2$, $a = 0$ and $b = 2$. Thus if we assume that for (11.2.18), either $N \le 4$ and $\rho > 1$, or $5 \le N$ and $\rho < \rho^* = 1 + \frac{4}{N-4}$, then we have $\gamma_c = \max\left\{\frac{N}{4}\left(\frac{1}{2} - \frac{1}{\rho}\right)_+ - \frac{1}{\rho},\ \frac{N}{8} - \frac{1}{2(\rho-1)}\right\}$. A simple computation shows that there exist some $1 < \rho_* < 2$ such that for any $\rho > \rho_*$ we have $\frac{N}{8} - \frac{1}{2(\rho-1)} > \frac{N}{4}\left(\frac{1}{2} - \frac{1}{\rho}\right)_+ - \frac{1}{\rho}$. Thus for $\rho > \rho_*$ we can solve (11.2.18) in $H^{4\gamma}(\mathbb{R}^N)$ for $\gamma \ge \gamma_c = \frac{N}{8} - \frac{1}{2(\rho-1)}$ while for $1 < \rho \le \rho_*$ we must take $\gamma > \gamma_c = \frac{N}{4}\left(\frac{1}{2} - \frac{1}{\rho}\right)_+ - \frac{1}{\rho} = -\frac{1}{\rho}$, since $\rho_* < 2$.

In particular, comparing γ_c with $\frac{1}{4}$, it is clear that in the latter case, we can always take $\gamma = \frac{1}{4}$, while in the former we need either $N \le 2$ and no restriction on $\rho > 1$, or $\rho \le 1 + \frac{4}{N-2}$ when $N > 2$. ■

11.3 The problem in the uniform Lebesgue-Bessel scale

We now consider problem (11.0.1) with the nonlinear term (11.0.2) with $a = b = 0$ in the uniform Lebesgue scale $\dot{L}^q_U(\mathbb{R}^N)$ described in Chapter 3. Then for $1 < q \le \infty$ we denote

$$\dot{L}^q_U(\mathbb{R}^N) := X^{\gamma(q)}, \quad \gamma(q) = \frac{-N}{2mq} \in \mathcal{J} := (\frac{-N}{2m}, 0]. \tag{11.3.1}$$

Note that this scale is nested and by the results in Part I, (ii.0.12) and (ii.0.13) hold and

$$\|S(t)\|_{\mathcal{L}(\dot{L}^q_U(\mathbb{R}^N), \dot{L}^p_U(\mathbb{R}^N))} \le \frac{M_0}{t^{\frac{N}{2m}(\frac{1}{q} - \frac{1}{p})}} \ \text{ for all } \ 0 < t \le T, 1 < q \le p \le \infty.$$

Also, assumption (9.1.18) holds because (9.1.13) holds for this scale of spaces.

Remark 11.3.1 *Notice that for $q = \infty$, due to the results in [40] (see Th. 3.2.4, (3.2.13), p. 115 and the proof of Theorem 3.1.7, p. 80) we have a strongly continuous analytic semigroup $S(t)$ in $BUC(\mathbb{R}^N) = \dot{L}^{\infty}_U(\mathbb{R}^N)$.*

Now, as for the non-uniform case, assume that $h(\cdot, 0) = 0$ and for some $\rho > 1$, $L > 0$ we have

$$|h(x,u) - h(x,v)| \leq L|u-v|(|u|^{\rho-1} + |v|^{\rho-1}),\ u,v \in \mathbb{R},\ x \in \mathbb{R}^N. \tag{11.3.2}$$

Then from (11.3.2) and using Hölder's inequality we get that for any ball with center c and radius 1, $B(c)$, and any $1 \leq q < \infty$ and for $u, v \in L^{\rho q}(B(c))$

$$\|h(\cdot,u) - h(\cdot,v)\|_{L^q(B(c))} \leq L\|u-v\|_{L^{\rho q}(B(c))}(\|u\|^{\rho-1}_{L^{\rho q}(B(c))} + \|v\|^{\rho-1}_{L^{\rho q}(B(c))}).$$

Since L is independent of c, we have

$$\|h(\cdot,u) - h(\cdot,v)\|_{\dot{L}^q_U(\mathbb{R}^N)} \leq L\|u-v\|_{\dot{L}^{\rho q}_U(\mathbb{R}^N)}(\|u\|^{\rho-1}_{\dot{L}^{\rho q}_U(\mathbb{R}^N)} + \|v\|^{\rho-1}_{\dot{L}^{\rho q}_U(\mathbb{R}^N)}), \tag{11.3.3}$$

When $q = \infty$, $\dot{L}^\infty_U(\mathbb{R}^N) = BUC(\mathbb{R}^N)$ and in the same manner

$$\|h(\cdot,u) - h(\cdot,v)\|_{BUC(\mathbb{R}^N)} \leq L\|u-v\|_{BUC(\mathbb{R}^N)}(\|u\|^{\rho-1}_{BUC(\mathbb{R}^N)} + \|v\|^{\rho-1}_{BUC(\mathbb{R}^N)}). \tag{11.3.4}$$

Note that now, in terms of the scale (11.3.1), (11.3.3) and (11.3.4) read $f : X^\alpha \to X^\beta$ and satisfies (ii.0.15), with $\alpha \geq \frac{-N}{2m\rho q}$ and $\beta \leq \frac{-N}{2mq}$, $\beta \leq \rho\alpha$ and $\alpha - \beta < 1$.

Therefore the admissible region, $\mathcal{S}$, for problem (11.0.1), (11.0.2) in the uniform Lebesgue scale (11.3.1) is a triangle delimited by $\beta \leq \rho\alpha$, $\frac{-N}{2m} \leq \alpha \leq 0$ and $0 \leq \alpha - \beta < 1$. Note that this set is related to the one in the standard Lebesgue scale as described in Remark 10.0.1. Also, I, $\mathcal{E}$ and $\mathcal{R}$ are the same as in Lemma 11.1.1 and we get the following theorem.

Theorem 11.3.2 *Assume h satisfies (11.3.2) for some $\rho > 1$, $L > 0$ and $h(\cdot, 0) = 0$. Assume also that $a = b = 0$ and define $p_0 = \frac{N}{2m}(\rho - 1)$*
i) if $p_0 \leq 1$, or equivalently, $\rho \leq \rho^* = 1 + \frac{2m}{N}$, then take $1 < p \leq \infty$, or
ii) if $p_0 > 1$, or equivalently, $\rho > \rho^* = 1 + \frac{2m}{N}$, then take $p_0 \leq p \leq \infty$.

Then for any p as above there exist $r > 0$ and $T = T(r,p) > 0$ such that for $v_0 \in \dot{L}^p_U(\mathbb{R}^N)$, and any u_0 satisfying $\|u_0 - v_0\|_{\dot{L}^p_U(\mathbb{R}^N)} < r$, there exists a function $u(\cdot, u_0)$ on $[0,T]$, such that for all $p \leq q \leq \infty$, $u(\cdot,u_0) \in C([0,T], \dot{L}^p_U(\mathbb{R}^N)) \cap C((0,T], \dot{L}^q_U(\mathbb{R}^N))$ and

$$t^{\frac{N}{2m}(\frac{1}{p}-\frac{1}{q})}\|u(t,u_0)\|_{\dot{L}^q_U(\mathbb{R}^N)} \leq M(u_0,q) \qquad \text{for } 0 < t \leq T$$

$$t^{\frac{N}{2m}(\frac{1}{p}-\frac{1}{q})}\|u(t,u_0)\|_{\dot{L}^q_U(\mathbb{R}^N)} \to 0, \qquad \text{as } t \to 0, \quad p \neq q$$

and satisfies

$$u(t,u_0) = S(t)u_0 + \int_0^t S(t-s)h(u(s,u_0))\,ds \quad t \in [0,T]$$

with $S(t)$ as in (11.1.2). If $p_0 < p$, then r can be taken arbitrarily large, that is, the existence time is uniform in bounded sets in $\dot{L}^p_U(\mathbb{R}^N)$.

Furthermore, there exists $M > 0$ such that for all $u^i_0 \in \dot{L}^p_U(\mathbb{R}^N)$, $i = 1,2$ such that $\|u^i_0 - v_0\|_{\dot{L}^p_U(\mathbb{R}^N)} < r$, we have for any $p \leq q \leq \infty$

$$\|u(t,u^1_0) - u(t,u^2_0)\|_{\dot{L}^q_U(\mathbb{R}^N)} \leq \frac{M}{t^{\frac{N}{2m}(\frac{1}{p}-\frac{1}{q})}}\|u^1_0 - u^2_0\|_{\dot{L}^p_U(\mathbb{R}^N)}, \quad t \in (0,T].$$

In the same manner as described before we obtain the following two results.

Corollary 11.3.3 *(Blow–up estimate) Let p, $u_0 \in \dot{L}^p_U(\mathbb{R}^N)$ and $u(\cdot, u_0)$ be as in Theorem 11.3.2 and assume $\tau_{u_0} < \infty$. Then,*
i) if $p > p_0$, for any $p \le r < \frac{Np}{N\rho - 2mp}$ when $p < \frac{N\rho}{2m}$, or $p \le r \le \infty$ otherwise, or
ii) if $p = p_0$, for any $p_0 < r < \rho p_0$,
we have, for $t < \tau_{u_0}$ close enough to τ_{u_0},

$$\|u(t; u_0)\|_{\dot{L}^r_U(\mathbb{R}^N)} \ge \frac{c}{(\tau_{u_0} - t)^{-\frac{N}{2mr} + \frac{1}{\rho - 1}}} = \frac{c}{(\tau_{u_0} - t)^{\frac{N}{2m}(\frac{1}{p_0} - \frac{1}{r})}}.$$

Theorem 11.3.4 *(Uniqueness) The solution obtained in Theorem 11.3.2 is unique in the following classes:*
i) If $p \ge p_0 = \frac{N}{2m}(\rho - 1)$ we take the functions $u : (0; T) \to \dot{L}^p_U(\mathbb{R}^N)$ such that $u(0) = u_0$, $u(t)$ is bounded in $\dot{L}^p_U(\mathbb{R}^N)$ as $t \to 0$ and

$$u \in L^\infty((\tau, T), \dot{L}^q_U(\mathbb{R}^N)) \quad 0 < \tau < T$$

for some $q \ge p$ and $q \ge \rho$. If $p = p_0$ we furthermore require $q > p$ and $u(t) \to u_0$ in $\dot{L}^p_U(\mathbb{R}^N)$ as $t \to 0$.
ii) If $p = p_0 = \frac{N}{2m}(\rho - 1) > 1$ we take the functions $u : (0; T) \to \dot{L}^p_U(\mathbb{R}^N)$ such that $u(0) = u_0$ and

$$\|u(t)\|_{\dot{L}^p_U(\mathbb{R}^N)} \le M, \quad t^{\frac{N}{2m}(\frac{1}{p} - \frac{1}{q})} \|u(t)\|_{\dot{L}^q_U(\mathbb{R}^N)} \le M, \quad \text{and} \quad t^{\frac{N}{2m}(\frac{1}{p} - \frac{1}{q})} \|u(t)\|_{\dot{L}^q_U(\mathbb{R}^N)} \to 0, \quad \text{as } t \to 0$$

for some $q > p$ and $q \ge \rho$.

In the same way, we obtain now results for the problem (11.0.1) with the nonlinear term (11.0.2) with $a \ge 0$, $b = 0$ in the uniform Bessel-Sobolev scale $H^{k,q}_U(\mathbb{R}^N)$, with $k \in \mathbb{N} \cup \{0\}$, defined as in Chapter 3.

Note that in the standard Bessel scale we were able to consider $b \ne 0$, however, now, since the uniform Bessel spaces are not reflexive (even for $q = 2$), the negative spaces cannot be described as dual spaces, and thus, $b \ne 0$ cannot be considered with this approach. For some $1 < p < \infty$, we denote

$$X^\alpha := \dot{H}^{2m\alpha, p}_U(\mathbb{R}^N)$$

Note that this scale is nested, see [5] for details, and by the results in Part I, (ii.0.12) holds, that is for $\gamma, \gamma' \in \mathbb{R}$, $\gamma' \ge \gamma$

$$\|S(t)\|_{\mathcal{L}(\dot{H}^{\gamma,p}_U(\mathbb{R}^N), \dot{H}^{\gamma',p}_U(\mathbb{R}^N))} \le \frac{M_0}{t^{\gamma' - \gamma}} \quad \text{for all} \quad 0 < t \le T.$$

Also, assumption (9.1.18) holds because (9.1.13) holds for this scale of spaces.

As above, we analyze how the Nemytckii operator associated to f as in (11.0.2), which we denote by f as well, acts between some spaces of the scale. Now, using (11.3.3) instead of (11.1.4), we get the following Lemma, analogous to Lemma 11.2.1.

Lemma 11.3.5 *Suppose that $h(\cdot,0)=0$ and for some $\rho>1$, $L>0$ we have*

$$|h(x,u)-h(x,v)|\leq L|u-v|(|u|^{\rho-1}+|v|^{\rho-1}),\ u,v\in\mathbb{R},\ x\in\mathbb{R}^N.$$

Assume also that $a\in\mathbb{N}$, $s\geq a$, $\sigma\geq 0$ and

$$\rho\left(s-a-\frac{N}{p}\right)+\left(\sigma-\frac{N}{p'}\right)\geq -N$$

with $\rho\left(s-a-\frac{N}{p}\right)\geq -N$ and if $\rho(s-a-\frac{N}{p})=-N$ then $\sigma-\frac{N}{p'}>0$.

Then, the nonlinear term $h(\cdot,D^a u)$ takes $\dot H^{s,p}_U(\mathbb{R}^N)$ into $\dot H^{-\sigma,p}_U(\mathbb{R}^N)$ and satisfies

$$\|h(\cdot,D^a u)-h(\cdot,D^a v)\|_{\dot H^{-\sigma,p}_U(\mathbb{R}^N)}\leq cL\|u-v\|_{\dot H^{s,p}_U(\mathbb{R}^N)}\Big(\|u\|^{\rho-1}_{\dot H^{s,p}_U(\mathbb{R}^N)}+\|v\|^{\rho-1}_{\dot H^{s,p}_U(\mathbb{R}^N)}\Big).$$

Therefore Lemma 11.2.10 remains the same with $B=\frac{a}{2m}+\frac{N(\rho-1)}{2mp\rho}$ and $C=0$ (since $b=0$) so we get the following theorem for $p_0<\rho p(1-\frac{a}{2m})$ with $p_0:=\frac{N}{2m}(\rho-1)$, that is either $\frac{N}{2m-a}\leq p$ and any $\rho>1$, or $p<\frac{N}{2m-a}$ and $\rho<\rho^*=1+\frac{p(2m-a)}{N-p(2m-a)}$.

Theorem 11.3.6 *Assume h satisfies (11.2.2) for some $\rho>1$, $L>0$. Assume $a<2m$ and $p_0<\rho p(1-\frac{a}{2m})$ with $p_0:=\frac{N}{2m}(\rho-1)$. Then for any*

$$\gamma_c:=\max\left\{\frac{N}{2m}\left(\frac{1}{p}-\frac{1}{\rho}\right)_+ +\frac{a}{2m}-\frac{1}{\rho},\ \frac{N}{2mp}+\frac{a\rho}{2m(\rho-1)}-\frac{1}{\rho-1}\right\}<\gamma<1$$

there exist $r>0$ and $T>0$, such that for any $v_0\in\dot H^{2m\gamma,p}_U(\mathbb{R}^N)$ and any u_0 satisfying $\|u_0-v_0\|_{\dot H^{2m\gamma,p}_U(\mathbb{R}^N)}<r$, there exists a function $u(\cdot,u_0)$ such that for all $\gamma\leq\gamma'<1$, $u(\cdot,u_0)\in C((0,T],\dot H^{2m\gamma',p}_U(\mathbb{R}^N))\cap C([0,T],\dot H^{2m\gamma,p}_U(\mathbb{R}^N))$ and

$$t^{\gamma'-\gamma}\|u(t,u_0)\|_{\dot H^{2m\gamma',p}_U(\mathbb{R}^N)}\leq M(u_0,\gamma')\qquad \textit{for } 0<t<T$$

$$t^{\gamma'-\gamma}\|u(t,u_0)\|_{\dot H^{2m\gamma',p}_U(\mathbb{R}^N)}\to 0\qquad \textit{as } t\to 0,\quad \gamma'\neq\gamma$$

and satisfies

$$u(t,u_0)=S(t)u_0+\int_0^t S(t-s)h(u(s,u_0))ds\quad t\in[0,T].$$

Also, there exists $M>0$ such that for all $u_0^i\in\dot H^{2m\gamma,p}_U(\mathbb{R}^N)$, $i=1,2$ such that $\|u_0^i-v_0\|_{\dot H^{2m\gamma,p}_U(\mathbb{R}^N)}<r$, we have for $\gamma'\in[\gamma,1)$

$$\|u(t,u_0^1)-u(t,u_0^2)\|_{\dot H^{2m\gamma',p}_U(\mathbb{R}^N)}\leq\frac{M}{t^{\gamma'-\gamma}}\|u_0^1-u_0^2\|_{\dot H^{2m\gamma}_U(\mathbb{R}^N)},\quad t\in(0,T].$$

When $\frac{N}{2mp}+\frac{a\rho}{2m(\rho-1)}-\frac{1}{\rho-1}>\frac{N}{2m}\left(\frac{1}{p}-\frac{1}{\rho}\right)_+ +\frac{a}{2m}-\frac{1}{\rho}$ then the above hold also for $\gamma=\gamma_c=\frac{N}{2mp}+\frac{a\rho}{2m(\rho-1)}-\frac{1}{\rho-1}$.

If $\gamma_c<\gamma<1$ then r can be taken arbitrarily large, that is, the existence time is uniform in bounded sets in $\dot H^{2m\gamma,p}_U(\mathbb{R}^N)$.

Again, results on blow-up and uniqueness can be obtained.

Corollary 11.3.7 *(Blow–up estimate) With the notations in Theorem 11.3.6, let* $1 > \gamma \geq \gamma_c$*,* $u(\cdot, u_0)$ *be the solution in the theorem for some* $u_0 \in \dot{H}_U^{2m\gamma,p}(\mathbb{R}^N)$*, and assume* $\tau_{u_0} < \infty$*.*

Then, for any $\gamma' \geq \gamma$ *(with strict inequality when* $\gamma = \gamma_c = \frac{N}{2mp} + \frac{a\rho}{2m(\rho-1)} - \frac{1}{\rho-1}$*) we have,*

$$\limsup_{t \to \tau_{u_0}^-} \|u(t, u_0)\|_{\dot{H}_U^{2m\gamma',p}(\mathbb{R}^N)} = \infty.$$

In particular, for $\gamma \in (\gamma_c, \frac{a\rho}{2mp} + \frac{N(\rho-1)}{2mp\rho}]$ *, for any* $t < \tau_{u_0}$ *close enough to* τ_{u_0}*,*

$$\|u(t; u_0)\|_{\dot{H}_U^{2m\gamma,p}(\mathbb{R}^N)} \geq \frac{c}{(\tau_{u_0} - t)^{\gamma - \frac{N}{2m} - \frac{a\rho}{2m(\rho-1)} + \frac{1}{\rho-1}}} = \frac{c}{(\tau_{u_0} - t)^{\gamma + \frac{N}{2m}(\frac{1}{p_0} - \frac{1}{p}) - \frac{a\rho}{2m(\rho-1)}}}$$

and for $\gamma \in (\frac{a\rho}{2mp} + \frac{N(\rho-1)}{2mp\rho}, 1)$*, for any* $t < \tau_{u_0}$ *close enough to* τ_{u_0}*,*

$$\|u(t; u_0)\|_{\dot{H}_U^{2m\gamma,p}(\mathbb{R}^N)} \geq \frac{c}{(\tau_{u_0} - t)^{\frac{1-\gamma}{\rho-1}}}.$$

For the critical case, $\gamma = \gamma_c$*, we have* $\|u(t; u_0)\|_{\dot{H}_U^{2m\gamma',p}(\mathbb{R}^N)} \geq \frac{c}{(\tau_{u_0} - t)^{\gamma' + \frac{N}{2m}(\frac{1}{p_0} - \frac{1}{p}) - \frac{a\rho}{2m(\rho-1)}}}$ *for any* $\gamma' \in (\gamma_c, \frac{a\rho}{2mp} + \frac{N(\rho-1)}{2mp\rho})$*.*

Theorem 11.3.8 *(Uniqueness) The solution obtained in Theorem 11.2.11 is unique in the following class: If* $1 > \gamma \geq \gamma_c = \max\{\frac{N}{2mp} + \frac{a\rho}{2m(\rho-1)} - \frac{1}{\rho-1}, \frac{N}{2mp}(\frac{1}{p} - \frac{1}{\rho})_+ + \frac{a}{2m} - \frac{1}{\rho}\}$ *we take functions* $u : (0, T) \to \dot{H}_U^{2m\gamma,p}(\mathbb{R}^N)$ *such that* $u(0) = u_0$*,* $u(t)$ *is bounded in* $\dot{H}_U^{2m\gamma,p}(\mathbb{R}^N)$ *as* $t \to 0$ *and*

$$u \in L^\infty((\tau, T), \dot{H}_U^{2m\gamma',p}(\mathbb{R}^N)), \quad 0 < \tau < T$$

for some $\gamma' > \gamma$ *and* $\gamma' \geq A := \frac{N}{2m}\left(\frac{1}{p} - \frac{1}{\rho}\right)_+ + \frac{a}{2m}$*. Also, if* $\frac{N}{2mp} + \frac{a\rho}{2m(\rho-1)} - \frac{1}{\rho-1} > \frac{N}{2m}(\frac{1}{p} - \frac{1}{\rho})_+ + \frac{a}{2m} - \frac{1}{\rho}$ *and* $\gamma = \gamma_c = \frac{N}{2mp} + \frac{a\rho}{2m(\rho-1)} - \frac{1}{\rho-1}$ *we also require* $u(t) \to u_0$ *in* $\dot{H}_U^{2m\gamma,p}(\mathbb{R}^N)$ *as* $t \to 0$*.*

Chapter 12

Application to a strongly damped wave equation

Consider the following strongly damped wave equation

$$\begin{cases} w_{tt} - \Delta w_t + w_t - \Delta w = h(x,w), \ t > 0, \ x \in \mathbb{R}^N, \\ w(0,x) = w_0(x), \quad w_t(0,x) = z_0(x), \ x \in \mathbb{R}^N, \end{cases} \tag{12.0.1}$$

with a nonlinear right hand side satisfying for some $\rho > 1$ and any $x \in \mathbb{R}^N$, $s_1, s_2 \in \mathbb{R}$,

$$h(x,0) = 0 \quad \text{and} \quad |h(x,s_1) - h(x,s_1)| \le L|s_1 - s_2|(|s_1|^{\rho-1} + |s_2|^{\rho-1}). \tag{12.0.2}$$

Note that letting $z = w_t$ and

$$Au = \left[\begin{smallmatrix} -z \\ (-\Delta+I)(w+z) \end{smallmatrix} \right] \quad \text{for} \quad u = \left[\begin{smallmatrix} w \\ z \end{smallmatrix} \right],$$

(12.0.1) can be written as

$$\dot{u} + Au = f(u) := \left[\begin{smallmatrix} 0 \\ h(w)+w \end{smallmatrix} \right], \quad u(0) = u_0 := \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix} \right]. \tag{12.0.3}$$

Also note that the problem has a formal energy in $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$

$$E(w,w_t) = \frac{1}{2}\|w_t\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\|\nabla w\|^2_{L^2(\mathbb{R}^N)} - \int_{\mathbb{R}^N} \int_0^w h(x,s)\, ds dx$$

and in $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ the operator A with domain $\mathrm{dom}(A) = \{\left[\begin{smallmatrix} w \\ z \end{smallmatrix} \right] \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : w + z \in H^2(\mathbb{R}^N)\}$ is a negative generator of a C^0 analytic semigroup $\{S(t) : t \ge 0\}$ (see [15, 13] and references therein). Hence, we let $X^0 = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ and define $\{X^\alpha\}_\alpha$ as the extrapolated fractional power scale of order m generated by (X^0, A) (see [2, Chapter V, p 266] or [31]), for which

$$\|S(t)\|_{\mathcal{L}(X^\beta, X^\alpha)} \le \frac{M_0}{t^{\alpha-\beta}}, \quad t > 0, \ \alpha \ge \beta \ge -m. \tag{12.0.4}$$

The next results gives the description of these spaces when $m = 1/2$.

Proposition 12.0.1 *With the previous notations,*

$$X^\alpha = \begin{cases} H^1(\mathbb{R}^N) \times H^{2\alpha}(\mathbb{R}^N), & \alpha \in [-\frac{1}{2}, \frac{1}{2}], \\ \{[\begin{smallmatrix} w \\ z \end{smallmatrix}] \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : w + z \in H^{2\alpha}(\mathbb{R}^N)\}, & \alpha \in [\frac{1}{2}, 1]. \end{cases} \tag{12.0.5}$$

Proof. The description of X^α in (12.0.5) for $\alpha \in [0,1]$ comes from [15, Theorem 1.1] and [12, Lemma 1]. The description of X^α for $\alpha \in [-\frac{1}{2}, 0]$ can be done using the same techniques. ■

Hence, from (12.0.4) the semigroup satisfies (ii.0.12) for $\mathcal{J} := [-\frac{1}{2}, 1]$. Also, assumption (9.1.18) holds because (9.1.13) holds for this scale of spaces.

The admissible region for the nonlinear term (12.0.2) is given by the following result.

Lemma 12.0.2 *The admissible region $\mathcal{S}$ for problem (12.0.3) is nonempty if and only if*

$$\text{either } N = 1, 2 \text{ and } \rho > 1 \qquad \text{or} \qquad N \geq 3 \text{ and } 1 < \rho \leq \frac{N+2}{N-2} \tag{12.0.6}$$

and it is determined by

$$0 \leq \alpha - \beta < 1, \quad \alpha \geq -\frac{1}{2}, \quad \beta \leq -\beta(\rho), \tag{12.0.7}$$

where $\beta(\rho) = 0$ for $N = 1, 2$ and for $N \geq 3$

$$\beta(\rho) = \left\{ \begin{array}{ll} 0, & \rho \in (1, \frac{N}{N-2}] \\ \frac{1}{2}(\frac{N}{2} - 1)\rho - \frac{N}{4}, & \rho \in (\frac{N}{N-2}, \frac{N+2}{N-2}] \end{array} \right\} \leq \frac{1}{2}. \tag{12.0.8}$$

Moreover

$$I = \inf_{(\alpha,\beta) \in \mathcal{S}} G(\alpha, \beta) = -\frac{1}{2} - \frac{1}{\rho}$$

which is not attained in $\mathcal{S}$ and thus, there is no critical value for the problem (12.0.1), (12.0.2) in the scale (12.0.5), in the sense of Definition 10.0.3.

The ranges $\mathcal{E}$ in (10.0.2) and $\mathcal{R}$ in (10.0.16) are both the same and equal to $(I, 1 - \beta(\rho))$. Furthermore, $\mathcal{E} = \mathcal{E}_2$.

Proof. We follow the steps in Chapter 10.

Step 1. Description of $\mathcal{S}$.

First note that if (12.0.6) holds, then Lemma 11.2.1 can be used with $a = b = 0$, $s = 1$, $p = p' = 2$, $\sigma = 2\beta(\rho)$ where $2\beta(\rho) \geq \frac{N}{2}(\rho - 1) - \rho$ and $\beta(\rho) \geq 0$, which gives (12.0.8). Hence we get

$$\|h(\cdot, w_1) - h(\cdot, w_2)\|_{H^{-2\beta(\rho)}(\mathbb{R}^N)} \leq cL\Big(\|w_1\|^{\rho-1}_{H^1(\mathbb{R}^N)} + \|w_2\|^{\rho-1}_{H^1(\mathbb{R}^N)}\Big)\|w_1 - w_2\|_{H^1(\mathbb{R}^N)}.$$

Because of (12.0.5) we choose $\beta(\rho) \leq \frac{1}{2}$, so $\rho \leq \frac{N+2}{N-2}$ whenever $N \geq 3$ which gives (12.0.6). Hence

$$\|f(u) - f(v)\|_{X^{-\beta(\rho)}} \leq C(1 + \|u\|^{\rho-1}_{X^{-1/2}} + \|v\|^{\rho-1}_{X^{-1/2}})\|u - v\|_{X^{-1/2}}$$

for some $C > 0$. Since the scale is nested, see Remark 10.0.1, we have that

$$\|f(u) - f(v)\|_{X^\beta} \leq C(1 + \|u\|_{X^\alpha}^{\rho-1} + \|v\|_{X^\alpha}^{\rho-1})\|u - v\|_{X^\alpha}$$

for all $\beta \leq -\beta(\rho)$ and $\alpha \geq -\frac{1}{2}$. When the point $(-\frac{1}{2}, -\beta(\rho)) \in \mathcal{S}$, that is when $\beta(\rho) = \frac{1}{2}$, i.e. $\rho = \frac{N+2}{N-2}$, because of Remark 10.0.1, $\mathcal{S}$ is a triangle delimited by $\beta \leq -\beta(\rho)$, $\alpha \geq -\frac{1}{2}$ and $\alpha - \beta < 1$ (note that $\alpha - \beta \geq 0$ holds as well). This is a triangle with upper-left vertex $(-\frac{1}{2}, -\beta(\rho))$, two sides parallel to the axes, and the opposite side given by $\alpha - \beta = 1$. When $(-\frac{1}{2}, -\beta(\rho)) \notin \mathcal{S}$, that is $\rho < \frac{N+2}{N-2}$, the point $(-\frac{1}{2}, -\beta(\rho))$ is above the line $\alpha - \beta = 0$ so the condition $\alpha - \beta \geq 0$ adds a restriction and $\mathcal{S}$ is a trapezoid with short and long base on the lines $\alpha - \beta = 0$, $\alpha - \beta = 1$ respectively and sides $\alpha = -\frac{1}{2}$ and $\beta = -\beta(\rho)$ parallel to the axes.

Step 2. Description of $\mathcal{S}_1$ and $\mathcal{S}_2$.

When $\rho = \frac{N+2}{N-2}$, $\mathcal{S}_1$ is a triangle and $\mathcal{S}_2$ is a trapezoid and both are nonempty. When $\rho < \frac{N+2}{N-2}$ both $\mathcal{S}_1$ and $\mathcal{S}_2$ are non-empty trapezoids.

Note that the intervals in the definition of both $\mathcal{E}$ in (10.0.2) and $\mathcal{R}$ in (10.0.16) are overlapping and hence in (10.0.18) we have $\alpha_{max} = 1 - \beta(\rho)$ and $\beta_{max} = -\beta(\rho)$. In particular $\mathcal{E} = \mathcal{R}$ and are equal to $(I, 1 - \beta(\rho))$ or $[I, 1 - \beta(\rho))$ depending on whether I is attained or not.

Step 3. Minimization of G.

Following Step 5 in Chapter 10 we minimize $G(\alpha, \beta)$ in $\mathcal{S}_1$ and $\mathcal{S}_2$. For $\mathcal{S}_1$, $I_1 = \inf_\alpha \alpha - \frac{1}{\rho} = -\frac{1}{2} - \frac{1}{\rho}$. For $\mathcal{S}_2$, we find that the 'left-most' line $\beta = \alpha\rho - D$ cutting $\mathcal{S}_2$ passes through $(-\frac{1}{2}, -\frac{1}{2} - \frac{1}{\rho})$. That is, $D = -\frac{\rho}{2} + \frac{1}{2} + \frac{1}{\rho}$ and thus $I_2 = \frac{D-1}{\rho-1} = -\frac{1}{2} - \frac{1}{\rho}$. Since $I_1 = I_2$ and $(-\frac{1}{2}, -\frac{1}{2} - \frac{1}{\rho}) \in \mathcal{S}_1$ the infimum is not attained and we have $\mathcal{E} = \mathcal{R} = (-\frac{1}{2} - \frac{1}{\rho}, 1 - \beta(\rho))$, see (10.0.2), (10.0.16) and (10.0.18). Finally note that $\mathcal{E}_2 = (I_2, 1 - \beta(\rho))$ and thus $\mathcal{E}_2 = \mathcal{E}$.
■

Due to Lemma 12.0.2 the results in Chapter 9 now lead to the following theorem.

Theorem 12.0.3 *Assume the conditions in Proposition 12.0.1 and that h satisfies (12.0.2) for some ρ as in (12.0.6) and $L > 0$. Then for any*

$$\gamma_c := -\frac{1}{2} - \frac{1}{\rho} < \gamma < 1 - \beta(\rho) \tag{12.0.9}$$

there exist $r > 0$ and $T > 0$, such that for any $v_0 \in X^\gamma$ and any u_0 satisfying $\|u_0 - v_0\|_{X^\gamma(\mathbb{R}^N)} < r$, there exists a function $u(\cdot, u_0)$ in $[0, T]$ such that for all $\gamma \leq \gamma' < \min\{1 - \beta(\rho), 1 + \gamma + \frac{1}{\rho-1}\}$, $u(\cdot, u_0) \in C([0, T], X^\gamma) \cap C((0, T], X^{\gamma'})$ and

$$t^{\gamma'-\gamma}\|u(t, u_0)\|_{X^{\gamma'}} \leq M(u_0, \gamma') \qquad for\ 0 < t < T, \tag{12.0.10}$$

$$t^{\gamma'-\gamma}\|u(t, u_0)\|_{X^{\gamma'}} \to 0 \qquad as\ t \to 0, \quad \gamma' \neq \gamma \tag{12.0.11}$$

and satisfies

$$u(t, u_0) = S(t)u_0 + \int_0^t S(t-s)h(u(s, u_0))ds \quad t \in [0, T] \tag{12.0.12}$$

with $S(t)$ *as in (12.0.4).*

Also, there exists $M > 0$ *such that for all* $u_0^i \in X^\gamma$, $i = 1, 2$ *satisfying* $\|u_0^i - v_0\|_{X^\gamma} < r$, *we have for all* $\gamma \le \gamma' < \min\{1 - \beta(\rho), 1 + \gamma + \frac{1}{\rho-1}\}$

$$\|u(t, u_0^1) - u(t, u_0^2)\|_{X^{\gamma'}} \le \frac{M}{t^{\gamma'-\gamma}} \|u_0^1 - u_0^2\|_{X^\gamma}, \quad t \in (0, T]. \tag{12.0.13}$$

For each γ *as above radius* r *as above can be taken arbitrarily large. In particular, the existence time is uniform in bounded sets of* X^γ.

Proof. Recall that in this case the admissible region is given by (12.0.7). Then, using Lemma 12.0.2 we have that $I = \inf_{\mathcal{S}} G = \gamma_c$ is as in (12.0.9) and is not attained in $\mathcal{S}$.

Consider first the case when $N \ge 3$ and $\rho = \frac{N+2}{N-2}$ and thus $\mathcal{S}$ is a triangle. Note that $\mathcal{E} = (-\frac{1}{2} - \frac{1}{\rho}, \frac{1}{2}) = \bigcup_{\substack{(\alpha,\beta)\in\mathcal{S} \\ \beta=-\beta(\rho)}} E(\alpha, \beta, \rho)$. Thus, for any $\gamma \in \mathcal{E}$ we can take $(\alpha_0, \beta_0) \in \mathcal{S}$ such that $\beta_0 = -\beta(\rho)$, so that $\gamma \in E(\alpha_0, \beta_0, \rho)$. Thus, following Chapter 10, Step 2, above we obtain a solution u for (12.0.3), and thus for (12.0.1), which is continuous in X^γ at $t = 0$ because the scale is nested, see Proposition 9.1.5. The solution satisfies (10.0.5), (10.0.6) and (10.0.7) for any $\gamma' \in [\beta_0, \beta_0 + 1)$, $\gamma' \ge \gamma$.

Since $\beta_0 = -\beta(\rho)$, then in fact (10.0.13), (10.0.14) and (10.0.15) hold for any $\gamma' \in \mathcal{R} = \mathcal{E}$, $\gamma' \ge \gamma$. This proves (12.0.10), (12.0.11) and (12.0.13).

Consider now the case when either $N \ge 3$ and $\rho < \frac{N+2}{N-2}$ or $N = 1, 2$ and $\rho > 1$, that is, when $\mathcal{S}$ is a trapezoid. For any $\gamma \in \mathcal{E} = \mathcal{E}_2 = (-\frac{1}{2} - \frac{1}{\rho}, 1 - \beta(\rho))$, we can take $(\alpha_0, \beta_0) \in \mathcal{S}_2$, such that $\gamma \in E(\alpha_0, \beta_0, \rho)$. In particular, we can always take $\gamma = G_2(\alpha_0, \beta_0) + \varepsilon$ for some $\varepsilon > 0$. To perform the bootstrap satisfying (10.0.12) we are going to use the segment

$$\ell_{(\alpha_0,\beta_0)} := \{(\alpha, \beta) \in \mathcal{S} : \beta = \rho\alpha + (\beta_0 - \rho\alpha_0) \text{ and } \beta \in [\beta_0, -\beta(\rho)]\}.$$

When the intersection of $\ell_{(\alpha_0,\beta_0)}$ and $\beta = -\beta(\rho)$ is a point in $\mathcal{S}$, the bootstrap can be performed up to $\beta_j = -\beta(\rho)$ for some j and will give (12.0.10), (12.0.11) and (12.0.13) for $\gamma \le \gamma' < 1 - \beta(\rho)$. When the intersection is not in $\mathcal{S}$, the bootstrap can be performed up to the line $\beta = \alpha$, that is $\beta = \gamma - \varepsilon + \frac{1}{\rho-1}$, and thus will give (12.0.10), (12.0.11) and (12.0.13) for $\gamma \le \gamma' < 1 + \gamma + \frac{1}{\rho-1}$. This leads to considering the critical line $\ell_* := \{\beta = \rho\alpha + \beta(\rho)(\rho - 1)\}$, which is the line of slope ρ passing through the point $(-\beta(\rho), -\beta(\rho))$ which is the upper vertex of $\mathcal{S}$. Therefore, when $\ell_{(\alpha_0,\beta_0)}$ is below ℓ_* the bootstrap can be performed up to $-\beta(\rho)$ and when $\ell_{(\alpha_0,\beta_0)}$ is above ℓ_* it can be performed up to $\gamma + \frac{1}{\rho-1}$.

Observe that on ℓ_*, G_2 is constant and equal to $-\beta(\rho) - \frac{1}{\rho-1}$. Thus, when $\gamma > -\beta(\rho) - \frac{1}{\rho-1}$ we have that $\ell_{(\alpha_0,\beta_0)}$ is below ℓ_*, while if $-\frac{1}{2} - \frac{1}{\rho} < \gamma \le -\beta(\rho) - \frac{1}{\rho-1}$ we have that $\ell_{(\alpha_0,\beta_0)}$ is above (or coincides with) ℓ_*.

Therefore, for $\gamma \in (-\beta(\rho) - \frac{1}{\rho-1}, 1 - \beta(\rho))$, take $(\alpha_0, \beta_0) \in \mathcal{S}_2$ satisfying that $\ell_{(\alpha_0,\beta_0)}$ is below ℓ_* such that $\gamma = G_2(\alpha_0, \beta_0) + \varepsilon$, for some $\varepsilon > 0$. Thus, following Chapter 10, Step 2, above we obtain a solution u for (12.0.3), and thus for (12.0.1), which is continuous in X^γ at $t = 0$ because the scale is nested, see Proposition 9.1.5. The solution satisfies (10.0.5), (10.0.6) and (10.0.7) for any $\gamma' \in [\beta_0, \beta_0 + 1)$, $\gamma' \ge \gamma$.

If $\beta_0 = -\beta(\rho)$, then in fact (10.0.13), (10.0.14) and (10.0.15) hold for any $\gamma' \in \mathcal{R} = \mathcal{E}$, $\gamma' \geq \gamma$.

If $\beta_0 < -\beta(\rho)$ then, following Step 3 in Chapter 10,, we can take $\alpha_1 = \gamma'$ very close to $\beta_0 + 1$ and $\beta_1 = \rho\alpha_1 + (\beta_0 - \rho\alpha_0)$, so that $(\alpha_1, \beta_1) \in \ell_{(\alpha_0,\beta_0)}$. For this choice (10.0.9) is satisfied, so (10.0.13), (10.0.14) and (10.0.15) hold for any $\gamma' \in [\beta_0, \beta_1 + 1)$, $\gamma' \geq \gamma$.

If $\beta_1 = -\beta(\rho)$, then in fact (10.0.13), (10.0.14) and (10.0.15) hold for any $\gamma' \in \mathcal{R} = \mathcal{E}$, $\gamma' \geq \gamma$.

If $\beta_1 < -\beta(\rho)$, then we can take $\alpha_2 = \gamma'$ very close to $\beta_1 + 1$ and $\beta_2 = \rho\alpha_2 + (\beta_0 - \rho\alpha_0)$ so that $(\alpha_2, \beta_2) \in \ell_{(\alpha_0,\beta_0)}$ satisfies (10.0.12).

Note that $\max_{\mathcal{S}} \beta - \min_{\mathcal{S}} \beta = -\beta(\rho) - (-\frac{3}{2}) \leq \frac{3}{2}$, so in fact in at most two jumps $-\beta(\rho)$ can be reached and then (10.0.13), (10.0.14) and (10.0.15) hold for any $\gamma' \in \mathcal{R} = \mathcal{E}$, $\gamma' \geq \gamma$. This proves (12.0.10), (12.0.11) and (12.0.13).

On the other hand, for $\gamma \in (-\frac{1}{2} - \frac{1}{\rho}, -\beta(\rho) - \frac{1}{\rho-1}]$ take $(\alpha_0, \beta_0) \in \mathcal{S}_2$ satisfying that $\ell_{(\alpha_0,\beta_0)}$ is above ℓ_* such that $\gamma = G_2(\alpha_0, \beta_0) + \varepsilon$ for some $\varepsilon > 0$. Thus, following Chapter 10, Step 2, above we obtain a solution u for (12.0.3), and thus for (12.0.1), which is continuous in X^γ at $t = 0$ because the scale is nested, see Proposition 9.1.5. The solution satisfies (10.0.5), (10.0.6) and (10.0.7) for any $\gamma' \in [\beta_0, \beta_0 + 1)$, $\gamma' \geq \gamma$.

If $\beta_0 = \gamma - \varepsilon + \frac{1}{\rho-1}$, then in fact (10.0.13), (10.0.14) and (10.0.15) hold for any $1 + \gamma + \frac{1}{\rho-1} > \gamma' \geq \gamma$.

If $\beta_0 < \gamma - \varepsilon + \frac{1}{\rho-1}$ then define

$$\ell_{(\alpha_0,\beta_0)} := \{(\alpha, \beta) \in \mathcal{S} : \beta = \rho\alpha + (\beta_0 - \rho\alpha_0) \text{ and } \beta \in [\beta_0, \gamma + \frac{1}{\rho - 1}]\}.$$

Note that $\ell_{(\alpha_0,\beta_0)} \subset \mathcal{S}$ for any $(\alpha_0, \beta_0) \in \mathcal{S}$. Then, following Step 3 in Chapter 10, we can take $\alpha_1 = \gamma'$ very close to $\beta_0 + 1$ and $\beta_1 = \rho\alpha_1 + (\beta_0 - \rho\alpha_0)$, so that $(\alpha_1, \beta_1) \in \ell_{(\alpha_0,\beta_0)}$. For this choice (10.0.9) is satisfied, so (10.0.13), (10.0.14) and (10.0.15) hold for any $\gamma' \in [\beta_0, \beta_1 + 1)$, $\gamma' \geq \gamma$.

If $\beta_1 = \gamma - \varepsilon + \frac{1}{\rho-1}$, then in fact (10.0.13), (10.0.14) and (10.0.15) hold for any $1 + \gamma + \frac{1}{\rho-1} > \gamma' \geq \gamma$.

If $\beta_1 < \gamma - \varepsilon + \frac{1}{\rho-1}$, then we can take $\alpha_2 = \gamma'$ very close to $\beta_1 + 1$ and $\beta_2 = \rho\alpha_2 + (\beta_0 - \rho\alpha_0)$ so that $(\alpha_2, \beta_2) \in \ell_{(\alpha_0,\beta_0)}$ satisfies (10.0.12).

Note that $\max_{\mathcal{S}} \beta - \min_{\mathcal{S}} \beta \leq -\beta(\rho) - (-\frac{3}{2}) \leq \frac{3}{2}$, so in fact in at most two jumps $\gamma - \varepsilon + \frac{1}{\rho-1}$ can be reached and then (10.0.13), (10.0.14) and (10.0.15) hold for any $1 + \gamma + \frac{1}{\rho-1} > \gamma' \geq \gamma$. This proves (12.0.10), (12.0.11) and (12.0.13). ■

We now study blow up of solutions of the strongly damped wave problem (12.0.3) using Proposition 9.1.12.

Corollary 12.0.4 *With the notations of Theorem 12.0.3, let* $\gamma \in (-\frac{1}{2} - \frac{1}{\rho}, 1 - \beta(\rho))$, $u = \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]$ *be the solution in the theorem for some* $u(0) = \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] =: u_0 \in X^\gamma$, *and assume* $\tau_{u_0} < \infty$. *Then, for any* $\gamma' \geq \gamma$ *we have,*

$$\limsup_{t \to \tau_{u_0}^-} \|u\|_{X^{\gamma'}} = \infty. \tag{12.0.14}$$

In particular, for $\gamma \in (-\frac{1}{2} - \frac{1}{\rho}, -\frac{1}{2})$ *and for any* $t < \tau_{u_0}$ *close enough to* τ_{u_0}*,*

$$\|u(t)\|_{X^\gamma} \geq \frac{c}{(\tau_{u_0} - t)^{\gamma + \frac{1}{2} + \frac{1}{\rho - 1}}} \tag{12.0.15}$$

for $\gamma \in [-\frac{1}{2}, -\beta(\rho)]$ *and for any* $t < \tau_{u_0}$ *close enough to* τ_{u_0}*,*

$$\|u(t)\|_{X^\gamma} \geq \frac{c}{(\tau_{u_0} - t)^{\frac{1}{\rho - 1}}} \tag{12.0.16}$$

and for $\gamma \in (-\beta(\rho), 1 - \beta(\rho))$ *and for any* $t < \tau_{u_0}$ *close enough to* τ_{u_0}*,*

$$\|u(t)\|_{X^\gamma} \geq \frac{c}{(\tau_{u_0} - t)^{\frac{1 - \beta(\rho) - \gamma}{\rho - 1}}}. \tag{12.0.17}$$

Proof. To get (12.0.14), given $\gamma \in \mathcal{E}$ we take admissible pairs $(\alpha, \beta) \in \mathcal{S}$ such that $\gamma \in E(\alpha, \beta, \rho)$. Thus from Proposition 9.1.12 iii) we get (12.0.14) for any $\gamma' \geq \gamma$.

We now show the rate of the blow-up. If $-\beta(\rho) < \gamma < 1 - \beta(\rho)$, we take $(\alpha, \beta) = (\gamma, -\beta(\rho)) \in \mathcal{S}$. For such pair, Proposition 9.1.12 iii) and (9.1.30) gives (12.0.17).

If $-\frac{1}{2} \leq \gamma \leq -\beta(\rho)$, we take $(\alpha, \beta) = (\gamma, \gamma) \in \mathcal{S}$. For such pair, Proposition 9.1.12 iii) and (9.1.30) gives (12.0.16).

If $-\frac{1}{2} - \frac{1}{\rho} < \gamma < -\frac{1}{2}$, we take $(\alpha, \beta) = (-\frac{1}{2}, -\frac{1}{2})$. For such pair, Proposition 9.1.12 iii) and (9.1.30) gives (12.0.15). ■

Theorem 12.0.5 *(Uniqueness) The solution obtained in Proposition 12.0.3 is unique in the following class: For* $1 - \beta(\rho) > \gamma > \gamma_c = -\frac{1}{2} - \frac{1}{\rho}$ *we take functions* $u : (0, T) \to X^\gamma$ *such that* $u(0) = u_0$*,* $u(t)$ *is bounded in* X^γ *as* $t \to 0$ *and*

$$u \in L^\infty((\tau, T), X^{\gamma'}), \quad 0 < \tau < T$$

for some $\gamma' > \gamma$ *and* $\gamma' \geq -\frac{1}{2}$*.*

Proof. First observe that the solutions constructed in Proposition 12.0.3 for $u_0 \in X^\gamma$ satisfy conditions in the statement.

Then consider any function as in the statement that satisfies (12.0.12) and observe that $u_0 \in X^\gamma$ for $\gamma \in \mathcal{E}$. Observe that the estimates in the statement can be read as bounds in $u \in L^\infty([0, T], X^\gamma)$ and $u \in L^\infty([\tau, T], X^{\gamma'})$, for sufficiently small T. Thus, by interpolation as in (11.1.12) we get bounds in $u \in L^\infty([\tau, T], X^\alpha)$ for $\alpha \in [\gamma, \gamma']$.

To conclude, we are going to show that if additionally to $\gamma' > \gamma$ we have $\gamma' \geq \alpha_{min} := \inf_{(\alpha,\beta) \in \mathcal{S}} \alpha$, we can find a pair (α, β) in the admissible region $\mathcal{S}$ such that $\gamma \leq \alpha \leq \gamma'$ and $\gamma \in E(\alpha, \beta, \rho)$. That is, we check that γ is in the set

$$\mathfrak{E}(\gamma, \gamma') := \bigcup_{\substack{(\alpha,\beta) \in \mathcal{S} \\ \gamma \leq \alpha \leq \gamma'}} E(\alpha, \beta, \rho).$$

Once this is done, if moreover u verifies (12.0.12) then u satisfies $(S1)$, $(S2)$ and $(S3)$ in Chapter 9 and then Theorem 9.2.2 part i) will conclude that u coincides with the solution in Proposition 12.0.3.

So to prove that $\gamma \in \mathfrak{E}(\gamma, \gamma')$ when $\alpha_0 > \gamma$ and $\alpha_0 \geq \alpha_{min} := \inf_{(\alpha,\beta)\in\mathcal{S}} \alpha$, observe that for fixed α, then if the maximum β such that $(\alpha, \beta) \in \mathcal{S}_2$ is $\beta = -\beta(\rho)$ then $\inf_\beta G(\alpha, \beta) = \frac{\alpha\rho+\beta(\rho)-1}{\rho-1}$, while $\inf_\beta G(\alpha, \beta) = \alpha - \frac{1}{\rho}$ in any other case.

Observe that $\alpha_{min} = -\frac{1}{2}$ and $\mathcal{E} = (I, 1 - \beta(\rho))$. For any $\alpha \leq -\beta(\rho) + \frac{1}{\rho}$ then $\inf_\beta G(\alpha, \beta) = \alpha - \frac{1}{\rho}$ while if $\alpha > -\beta(\rho) + \frac{1}{\rho}$, then $\inf_\beta G(\alpha, \beta) = \frac{\alpha\rho+\beta(\rho)-1}{\rho-1}$. So if $\gamma \leq -\beta(\rho)+\frac{1}{\rho}$ then $\mathfrak{E}(\gamma, \gamma') \subset (\gamma-\frac{1}{\rho}, \gamma']$ with the same endpoints whereas if $\gamma > -\beta(\rho)+\frac{1}{\rho}$ then $\mathfrak{E}(\gamma, \gamma') \subset [\frac{\gamma\rho+\beta(\rho)-1}{\rho-1}, \gamma']$ with the same endpoints. In both cases, $\gamma \in \mathfrak{E}(\gamma, \gamma')$.

Note that in all cases we require $\gamma' > \gamma$ and $\gamma' \geq -\frac{1}{2}$ as stated in the result. ■

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