EXTREMAL EQUILIBRIA FOR DISSIPATIVE PARABOLIC EQUATIONS IN LOCALLY UNIFORM SPACES

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Abstract. We consider a reaction diffusion equation $u_t = \Delta u + f(x,u)$ in $\mathbb{R}^N$ with initial data in the locally uniform space $\dot{L}^q_U(\mathbb{R}^N)$, $q \in [1,\infty)$, and with dissipative nonlinearities satisfying $sf(x,s) \leq C(x)s^2 + D(x)|s|$, where $C \in L^{r_1}_U(\mathbb{R}^N)$ and $0 \leq D \in L^{r_2}_U(\mathbb{R}^N)$ for certain $r_1, r_2 > \frac{N}{2}$. We construct a global attractor $\mathcal{A}$ and show that $\mathcal{A}$ is actually contained in an ordered interval $[\varphi_m, \varphi_M]$, where $\varphi_m, \varphi_M \in \mathcal{A}$ is a pair of stationary solutions, minimal and maximal respectively, that satisfy $\varphi_m \leq \liminf_{t\to\infty} u(t;u_0) \leq \limsup_{t\to\infty} u(t;u_0) \leq \varphi_M$ uniformly for u_0 in bounded subsets of $\dot{L}^q_U(\mathbb{R}^N)$. A sufficient condition concerning the existence of minimal positive steady state, asymptotically stable from below, is given. Certain sufficient conditions are also discussed ensuring the solutions to be asymptotically small as $|x| \to \infty$. In this case the solutions are shown to enter, asymptotically, Lebesque spaces of integrable functions in $\mathbb{R}^N$, the attractor attracts in the uniform convergence topology in $\mathbb{R}^N$ and is a bounded subset of $W^{2,r}(\mathbb{R}^N)$ for some $r > N/2$. Uniqueness of positive solutions is also discussed.

1. Introduction

In this paper we consider the Cauchy problem

$$\begin{cases} u_t = \Delta u + f(x,u), \quad t>0, \ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

with measurable initial data satisfying for some $q \in [1,\infty)$ the locally uniform conditions

$$\sup_{y\in\mathbb{R}^N} \int_{\{|x-y|<1\}} |u_0(x)|^q \, dx < \infty, \tag{1.2}$$

$$\lim_{|z|\to 0} \sup_{y\in\mathbb{R}^N} \int_{\{|x-y|<1\}} |u_0(x+z) - u_0(x)|^q \, dx = 0, \tag{1.3}$$

and with a nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfying

$$sf(x,s) \leq C(x)s^2 + D(x)|s|, \quad x \in \mathbb{R}^N, \ s \in \mathbb{R}, \tag{1.4}$$

where

$$0 \leq D, \qquad \sup_{y\in\mathbb{R}^N} \int_{\{|x-y|<1\}} (|C(x)|^{r_1} + |D(x)|^{r_2}) \, dx < \infty, \tag{1.5}$$

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for some $r_1, r_2 > \frac{N}{2}$.

Condition (1.2) states that u_0 belongs to the so called *locally uniform space* $L^q_U(\mathbb{R}^N)$

$$L^q_U(\mathbb{R}^N) \stackrel{def}{=} \{\phi \in L^q_{loc}(\mathbb{R}^N) : \|\phi\|_{L^q_U(\mathbb{R}^N)} = \sup_{y\in\mathbb{R}^N} \|u\|_{L^q(B(y,1))} < \infty\}, \quad 1 \le q \le \infty, \tag{1.6}$$

while (1.3) states that u_0 is in the closed subspace $\dot{L}^q_U(\mathbb{R}^N) \subset L^q_U(\mathbb{R}^N)$, consisting of all elements $\phi \in L^q_U(\mathbb{R}^N)$ satisfying the continuity condition

$$\|\tau_z\phi - \phi\|_{L^q_U(\mathbb{R}^N)} \to 0 \ \text{ as } \ |z| \to 0 \tag{1.7}$$

with respect to the group $\{\tau_z, z \in \mathbb{R}^N\}$ of translations in $\mathbb{R}^N$.

Finally, (1.5) states that

$$C \in L^{r_1}_U(\mathbb{R}^N), \quad 0 \le D \in L^{r_2}_U(\mathbb{R}^N) \ \text{ for } r_1, r_2 > \frac{N}{2}. \tag{1.8}$$

Reaction diffusion equations as in (1.1) have been recently investigated in [16, 17] within a setting on Lebesgue spaces of integrable functions in unbounded and bounded domains, Ω, respectively. It has been proved in [16, 17] that a pair of equilibria solutions φ_m and φ_M exists, minimal and maximal respectively, which bound the asymptotic dynamics of the system so that

$$\varphi_m(x) \le \liminf_{t\to\infty} u(t,x;u_0) \le \limsup_{t\to\infty} u(t,x;u_0) \le \varphi_M(x), \quad x \in \Omega, \tag{1.9}$$

hold uniformly for u_0 varying in bounded sets of initial data.

Such a remarkable dynamical behavior of the solutions has been observed in these references assuming (1.4) and suitable integrability properties of $C(x)$ and $D(x)$.

Here, our goal is to extend such results to the case where initial data are considered in much larger spaces $\dot{L}^q_U(\mathbb{R}^N)$ and with weaker assumptions of the type (1.8). Note that a preliminary analysis on the asymptotic behavior of solutions of (1.1) in such a setting can be found in [4], which is, in turn, based on the study of linear equations carried out in [3]. In the former reference only smooth initial data are considered. In this paper, we first construct solutions for nonsmooth initial data in $\dot{L}^q_U(\mathbb{R}^N)$. For this we assume the one side monotonicity condition

$$\frac{\partial f}{\partial s}(x,s) \le L(x), \quad x \in \mathbb{R}^N, s \in \mathbb{R}, \quad L \in L^{r_3}_U(\mathbb{R}^N), \ r_3 > \frac{N}{2}. \tag{1.10}$$

In order to ensure that (1.1) is dissipative we will assume, as in [2, 4] that the solutions of the linear equation

$$V_t - \Delta V = C(x)V, \quad t > 0, \ x \in \mathbb{R}^N. \tag{1.11}$$

decay exponentially to zero, as $t \to \infty$.

Hence, in what follows our goal is to show that under the above assumptions the problem (1.1) is well posed in $\dot{L}^q_U(\mathbb{R}^N)$ for any $q \ge 1$, the associated semiflow of global solutions possesses a global attractor $\mathcal{A}$ (which attracts solutions in a suitable way) and that, the extremal steady states φ_m and φ_M satisfying (1.9) exist. In particular,

$$\mathcal{A} \subset [\varphi_m, \varphi_M]$$

and $\varphi_m, \varphi_M \in \mathcal{A}$ are the "caps" of the attractor.

Note that in [4] structure conditions like (1.4), (1.8) were shown to guarantee global well posedness of (1.1), dissipativeness properties of the solutions and the existence of an attractor in the locally uniform spaces but only for appropriately regular initial data; namely for $u_0 \in \dot{W}^{2\alpha,q}_U(\mathbb{R}^N)$, where $2\alpha - \frac{N}{q} > 0$, $\alpha \in [0,1)$ and $q > N/2$. Also, in [7] a monotonicity condition, stronger than (1.10), was used to construct the semigroup for (1.1) in $\dot{L}^q_U(\mathbb{R}^N)$ with $q > \frac{N}{2}$. However, only orbits of bounded sets consisting of appropriately smooth functions were shown to converge to the attractor and the nonlinear term f was assumed to be independent of the spatial variable. All these results will be considerably extended here.

Now we present our main results. Note that some additional results, not stated here, are proved in the Sections below. Concerning global existence and the existence of the global attractor our result is as follows.

Theorem 1.1. *Suppose that* $q \in [1,\infty)$ *and*

$$f(x,s) = g(x) + m_0(x)s + \sum_{j=1}^{k} m_j(x)h_j(s) + f_0(x,s) =: m_0(x)s + F(x,s), \tag{1.12}$$

where
i) $g \in \dot{L}^{r_0}_U(\mathbb{R}^N)$, $m_j \in \dot{L}^{r_0}_U(\mathbb{R}^N)$, $j = 0,\dots,k$, *with some* $r_0 > \frac{N}{2}$,
ii) $h_j \in C^1(\mathbb{R},\mathbb{R})$ *and* $h_j(0) = 0$, $h_j'(0) = 0$ *for each* $j = 0,\dots,k$,
iii) $f_0(x,s)$ *is Hölder continuous with respect to* $x \in \mathbb{R}^N$ *uniformly for* s *in bounded subsets of* $\mathbb{R}$, *the partial derivative* $\frac{\partial f_0}{\partial s}(x,s)$ *exists and is bounded in* $x \in \mathbb{R}^N$ *for* s *in bounded sets of* $\mathbb{R}$; *in addition,*

$$f_0(x,0) = 0,\ \frac{\partial f_0}{\partial s}(x,0) = 0 \quad \text{for all } x \in \mathbb{R}^N,$$

iv) condition (1.10) *holds. For* $q = 1$ *assume additionally that*

$$L(x) \le \mathcal{K},\ \ x \in \mathbb{R}^N, \tag{1.13}$$

for a certain constant $\mathcal{K} \in \mathbb{R}$.

Then the problem (1.1) *is globally well posed in* $\dot{L}^q_U(\mathbb{R}^N)$.

v) For the asymptotic behavior, assume (1.4) *is satisfied with some* C, D *as in* (1.8) *being such that the solutions of* (1.11) *decay exponentially as* $t \to \infty$.

Then the corresponding semigroup possesses a $(\dot{L}^q_U(\mathbb{R}^N) - W)$ *global attractor* $\mathcal{A}$; *that is*

a) $\mathcal{A}$ *is closed in* $\dot{L}^q_U(\mathbb{R}^N)$, *compact in* W, *invariant for (1.1) and*

b) $\mathcal{A}$ *attracts bounded sets of* $\dot{L}^q_U(\mathbb{R}^N)$ *in the norm of* W, *where* W *is either* $C^{\mu}_{loc}(\mathbb{R}^N)$, *with* $2 - \frac{N}{r_0} > \mu > 0$, *or* $W^{s,r_0}_{\rho}(\mathbb{R}^N)$ *with* $0 \le s < 2$.

In addition $\mathcal{A}$ *is bounded in* $\dot{W}^{2,r_0}_U(\mathbb{R}^N)$.

As for the existence of extremal equilibria for (1.1) we prove

Theorem 1.2. *Under the assumptions of Theorem 1.1 there exist two ordered extremal equilibria for (1.1),* $\varphi_m, \varphi_M \in \dot{W}^{2,r_0}_U(\mathbb{R}^N)$, *minimal and maximal, respectively, in the sense that any equilibrium* ψ *of (1.1) satisfies*

$$\varphi_m \le \psi \le \varphi_M.$$

Furthermore, the ordered set $\{v \in \dot{L}^q_U(\mathbb{R}^N) : \varphi_m \leq v \leq \varphi_M\}$ uniformly attracts the dynamics of the systems, i.e.,

$$\varphi_m(x) \leq \liminf_{t\to\infty} u(t,x;u_0) \leq \limsup_{t\to\infty} u(t,x;u_0) \leq \varphi_M(x) \tag{1.14}$$

uniformly in compact sets in $\mathbb{R}^N$ for bounded sets of initial data. Moreover, the minimal equilibrium is stable from below and the maximal one is stable from above.

Finally, the global attractor $\mathcal{A}$ for problem (1.1) in Theorem 1.1 satisfies

$$\varphi_m \leq \mathcal{A} \leq \varphi_M$$

and $\varphi_m, \varphi_M \in \mathcal{A}$.

With further integrability asumption in the function D we are able to show that solutions of (1.1) become uniformly small at infinity for large times. More precisely, we have

Theorem 1.3. *If the assumptions of Theorem 1.1 hold and*

$$D \in L^r(\mathbb{R}^N), \qquad \text{where } \frac{N}{2} < r \leq r_1, \tag{1.15}$$

then
i) the attractor $\mathcal{A}$ in Theorem 1.1 is a bounded subset of $L^\infty(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$. Moreover there exists a positive function

$$\Phi \in W^{2,r}(\mathbb{R}^N) \subset BUC_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$$

and

$$\Phi(x) \to 0 \quad \text{as} \quad |x| \to \infty,$$

such that for all $u \in \mathcal{A}$ we have $|u| \leq \Phi$ in $\mathbb{R}^N$.
ii) convergence to $\mathcal{A}$ is also in $W = C_{bd}(\mathbb{R}^N)$ with the topology of uniform convergence,
iii) the extremal equilibria of Theorem 1.2 attract solutions uniformly in $\mathbb{R}^N$ from above and from below respectively.That is, (1.14) holds uniformly in $x \in \mathbb{R}^N$.
Furthermore,
iv) the attractor $\mathcal{A}$ is bounded in $W^{2,r}(\mathbb{R}^N)$ provided that, in addition, $r_0 \geq r$.

For the analysis of the dynamics of nonnegative solutions of (1.1) we then give certain sufficient conditions for the existence of a minimal positive solution, asymptotically stable from below.

Theorem 1.4. *Under the assumptions of Theorem 1.1 suppose that $g(x) = f(x,0) = 0$. Furthermore, assume that there exists $\mathcal{M} \in L^p_U(\mathbb{R}^N)$, $p > N/2$, such that for some $s_0 > 0$*

$$f(x,s) \geq \mathcal{M}(x)s, \quad 0 \leq s \leq s_0, \tag{1.16}$$

and 0 is unstable for the linear problem

$$\begin{cases} v_t - \Delta v = \mathcal{M}(x)v & \text{in } \mathbb{R}^N \\ v(0) = v_0; \end{cases}$$

that is, the spectrum of $-\Delta - \mathcal{M}$, $\sigma(-\Delta - \mathcal{M})$, satisfies the condition

$$\sigma(-\Delta - \mathcal{M}) \cap \mathbb{R}^- \neq \emptyset.$$

Then, there exists a minimal positive equilibrium, $0 < \varphi_m \in \dot{W}^{2,r_0}_U(\mathbb{R}^N)$*. Moreover, for all not identically zero initial data* $0 \le u_0 \in \dot{L}^q_U(\mathbb{R}^N)$*, we have*

$$\liminf_{t\to\infty} u(t,x;u_0) \ge \varphi_m(x)$$

uniformly in compact sets of $\mathbb{R}^N$*. In particular,* φ_m *is globally asymptotically stable from below for positive solutions; namely, for all* $u_0 \in \dot{L}^q_U(\mathbb{R}^N)$*,* $0 \le u_0 \le \varphi_m$ *and* $u_0 \not\equiv 0$ *we have*

$$\lim_{t\to\infty} u(t,x;u_0) = \varphi_m(x) \quad \text{in } W, \tag{1.17}$$

where W *is either* $C^{\mu}_{loc}(\mathbb{R}^N)$*, with* $2 - \frac{N}{r_0} > \mu > 0$*, or the weighted space* $W^{s,r_0}_{\rho}(\mathbb{R}^N)$ *with* $0 \le s < 2$*.*

In some cases we can obtain the uniqueness of a positive steady state. In this case, this equilibrium becomes globally asymptotically stable for positive solutions of (1.1).

Theorem 1.5. *Assume Theorem 1.4 holds true. Moreover assume* $\frac{\partial f}{\partial s}(x,\cdot)$ *is nonincreasing in* $\mathbb{R}^+$ *for every* $x \in \mathbb{R}^N$*. Assume furthermore that the minimal positive solution* φ_m *from Theorem 1.4 is linearly asymptotically stable, that is, the semigroup* $\{e^{(\Delta + \frac{\partial f}{\partial s}(\cdot,\varphi_m))t}\}$ *generated by the linearized Schrödinger operator* $\Delta + \frac{\partial f}{\partial s}(\cdot,\varphi_m)$ *decays asymptotically.*

Then φ_m *is the unique positive steady state of* (1.1)*. Moreover,* φ_m *is also globally asymptotically stable for nonnegative nontrivial solutions of* (1.1)*. Namely,* (1.17) *holds for all* $u_0 \in \dot{L}^q_U(\mathbb{R}^N)$ *satisfying* $0 \le u_0 \not\equiv 0$*, where* W *denote either* $C^{\mu}_{loc}(\mathbb{R}^N)$*, with* $2 - \frac{N}{r_0} > \mu > 0$*, or the weighted space* $W^{s,r_0}_{\rho}(\mathbb{R}^N)$ *with* $0 \le s < 2$*.*

The assumption on the linear asymptotic stability above will be obtained in many cases with the help of the following lemma of independent interest, see Section 9.

Lemma 1.6. *Assume* $V_0, V \in L^{r_0}_U(\mathbb{R}^N)$ *with* $N/2 < r_0 < \infty$ *are such that*

$$\lambda = 0 \quad \text{is the bottom spectrum of} \quad L_0 = -\Delta + V_0(x)I$$

and $V \ge 0$ *satisfies that there exists* $c > 0$ *and* $\alpha > 0$ *such that for any* $y \in \mathbb{R}^N$

$$|\{x,\ V(x) \le a\} \cap B(y,1)| \le c a^{\alpha}$$

for all sufficiently small $a > 0$*.*

Then the bottom spectrum of $L = L_0 + V(x)I$ *is strictly positive.*

The article is organized as follows. In Section 2 we include certain preliminaries concerning locally uniform spaces and locally uniform Sobolev spaces, their embeddings, properties of Schrödinger operators in these spaces and properties of the solutions of the associated linear problems.

In Section 3 we recall local and global well posedness of (1.1) for smooth initial data u_0 in some locally uniform Sobolev spaces.

In Section 4 we show that with the aid of (1.10) a density argument leads to the construction of the global solution of (1.1) through each $u_0 \in \dot{L}^q_U(\mathbb{R}^N)$, $q \in [1,\infty)$. We also show that these solutions depend continuously on the initial conditions, become smooth for positive times and satisfy the variation of constants formula.

In Section 5 we obtain asymptotic estimates of the solutions and prove the existence of a global attractor, which attracts solutions in the spaces W as in Theorem 1.5, completing the proof of Theorem 1.1.

In Section 6 we prove Theorem 1.2. Note that in [16], where the problem was considered in standard Lebesgue spaces, some technical restrictions on r_0, r_1, r_2 appeared in order to control the tails of the solutions at infinity since the solutions had to remain in suitable Sobolev spaces. Here, as we are dealing with solutions in uniform spaces, those restrictions are greatly simplified. However, in general, we are not able to prove uniform convergence in space to the extremal solutions, but only on compact sets.

In Section 7 we give conditions for the solutions of (1.1) to become suitable small as $|x| \to \infty$, for large times. In particular, we prove Theorem 1.3.

In Section 8 we give sufficient conditions for the existence of a minimal positive steady state which is asymptotically stable from below and prove Theorem 1.4. Following [17] we show that this stationary solution can be approximated by the minimal positive equilibria constructed for initial boundary value problems with Dirichlet boundary conditions, in balls of radius $R \to \infty$. This reflects some resemblance with the technique of handling Cauchy problems developed in [12].

In Section 9 we discuss uniqueness of the minimal positive equilibrium solution and conclude Theorem 1.5 and Lemma 1.6.

In Section 10 we apply the previous results to some model problems. In particular, we discuss uniform and nonuniform convergence of the solutions to the extremal equilibria.

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2. Preliminaries on locally uniform spaces and linear equations

The locally uniform spaces $L^q_U(\mathbb{R}^N)$, that can be traced back to [11], have been extensively applied in a number of references dealing with problems in unbounded domains, see e.g. [1, 2, 3, 4, 7, 8, 13, 14, 18]. They allow to consider initial data and coefficients in the equation, with no prescribed behavior at infinity and local singularities. Also, they enjoy suitable continuous and locally compact embeddings and possess the useful nesting properties

$$L^q_U(\mathbb{R}^N) \hookrightarrow L^p_U(\mathbb{R}^N),\ 1 \le p \le q \le \infty, \tag{2.1}$$

which are also satisfied by the $\dot{L}^q_U(\mathbb{R}^N)$ spaces.

The spaces $L^q_U(\mathbb{R}^N)$, $\dot{L}^q_U(\mathbb{R}^N)$, defined in (1.6), (1.7) can alternatively be characterized in terms of suitable *weighted norms*, see [3]. In fact, fix any weight ρ_0 integrable in $\mathbb{R}^N$ such that

$$\rho_0 \in C^2(\mathbb{R}^N), \quad \rho_0(x) > 0, \quad \rho_0(x) \to 0, \text{ as } |x| \to \infty \tag{2.2}$$

and

$$|\nabla \rho_0(x)| \le c\rho_0(x), \quad |D^2\rho_0(x)| \le c\rho_0^2(x), \qquad x \in \mathbb{R}, \tag{2.3}$$

with some $c > 0$. For example

$$\rho_0(x) = (1 + \epsilon|x|^2)^{-\nu}, \ x \in \mathbb{R}^N, \quad \text{where } \nu > N/2 \text{ and } \epsilon > 0.$$

Denote by ρ any translate of ρ_0, that is

$$\rho(x) = \tau_y \rho_0(x) = \rho_0(x - y), \tag{2.4}$$

and define the weighted spaces

$$L^q_\rho(\mathbb{R}^N) \stackrel{def}{=} \{\phi \in L^q_{loc}(\mathbb{R}^N), \ \|\phi\|_{L^q_\rho(\mathbb{R}^N)} = \big(\int_{\mathbb{R}^N} |\phi(x)|^q \rho(x)\, dx\big)^{\frac{1}{q}} < \infty\}, \quad q \in [1, \infty). \tag{2.5}$$

Then, for $q \in [1, \infty)$ we have

$$L^q_U(\mathbb{R}^N) = L^q_{lu}(\mathbb{R}^N) \stackrel{def}{=} \{\phi \in L^q_{loc}(\mathbb{R}^N), \ \|\phi\|_{L^q_{lu}(\mathbb{R}^N)} = \sup_\rho \|\phi\|_{L^q_\rho(\mathbb{R}^N)} < \infty\}, \tag{2.6}$$

and

$$\dot{L}^q_U(\mathbb{R}^N) = \dot{L}^q_{lu}(\mathbb{R}^N) \stackrel{def}{=} \Big\{\phi \in L^q_{lu}(\mathbb{R}^N) : \quad \|\tau_z\phi - \phi\|_{L^q_{lu}(\mathbb{R}^N)} \to 0 \ \text{ as } \ |z| \to 0\Big\}. \tag{2.7}$$

Weighted and locally uniform Sobolev spaces $W^{k,q}_{\tau_y\rho}(\mathbb{R}^N)$, $W^{k,q}_U(\mathbb{R}^N)$, $W^{k,q}_{lu}(\mathbb{R}^N)$, $\dot{W}^{k,q}_U(\mathbb{R}^N)$, $\dot{W}^{k,q}_{lu}(\mathbb{R}^N)$ can be then defined for $q \geq 1$ and $k \in \mathbb{N}$ in a natural way. Sobolev spaces for non-integer positive indexes can also be defined by interpolation, see [3].

Note that the set $C^\infty_{bd}(\mathbb{R}^N)$ of infinitely many times differentiable functions with bounded derivatives is dense in $\dot{W}^{k,p}_U(\mathbb{R}^N)$, but not in $W^{k,p}_U(\mathbb{R}^N)$, for each $k \in \mathbb{N}$ and every $1 \leq p < \infty$.

Also, some embeddings of these spaces are reflected in the following diagram, see [3],

$$\begin{array}{ccccc} \dot{W}^{k,p}_U(\mathbb{R}^N) & \hookrightarrow & W^{k,p}_U(\mathbb{R}^N) & \hookrightarrow & W^{k,p}_\rho(\mathbb{R}^N) \\ \uparrow & \nwarrow & \uparrow & & \uparrow \\ \dot{W}^{k+1,p}_U(\mathbb{R}^N) & \hookrightarrow & W^{k+1,p}_U(\mathbb{R}^N) & \hookrightarrow & W^{k+1,p}_\rho(\mathbb{R}^N), \quad k \in \mathbb{N}. \end{array} \tag{2.8}$$

We also have

$$\dot{W}^{s,p}_U(\mathbb{R}^N) \hookrightarrow C^{l+\mu}(\mathbb{R}^N), \quad \text{for} \quad l + 1 > s - \frac{N}{p} > l + \mu > l \geq 0, \tag{2.9}$$

and the compact inclusions

$$W^{s_1,p_1}_U(\mathbb{R}^N) \hookrightarrow W^{s_2,p_2}_\rho(\mathbb{R}^N), \tag{2.10}$$

for $s_2 \in \mathbb{N}$, $s_1 > s_2$, $\infty > p_2 \geq p_1 > 1$, $s_1 - \frac{N}{p_1} > s_2 - \frac{N}{p_2}$, and

$$\dot{W}^{s,p}_U(\mathbb{R}^N) \hookrightarrow C^{l+\mu}_{loc}(\mathbb{R}^N), \quad \text{for} \quad l + 1 > s - \frac{N}{p} > l + \mu > l \geq 0. \tag{2.11}$$

The results in [3] on the properties of elliptic operators, in particular Schrödinger operators (see [18]), are summarized in the following results. As shown in [3, Theorem 3.5, Proposition 3.2], we have

Proposition 2.1. *If $C \in L^\sigma_U(\mathbb{R}^N)$ for a certain $\sigma > N/2$, then the Schrödinger operator $\Delta + C$ generates an order preserving analytic semigroup $\{e^{(\Delta+C)t}\}$ in $L^q_U(\mathbb{R}^N)$ for any $1 \leq q \leq \infty$. Furthermore, this semigroup is continuous at $t = 0$ provided that $V_0 \in \dot{L}^q_U(\mathbb{R}^N)$.*

Additionally, the exponential type ν of $\Delta + C$ is the same in all the spaces $L^q_U(\mathbb{R}^N)$ for any $1 \le q \le \infty$ and the $L^q_U(\mathbb{R}^N) - L^p_U(\mathbb{R}^N)$ estimate

$$\|e^{(\Delta+C)t}V_0\|_{L^p_U(\mathbb{R}^N)} \le Me^{at}t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|V_0\|_{L^q_U(\mathbb{R}^N)}, \quad q \le p \le \infty, \tag{2.12}$$

holds with arbitrarily chosen $a > \nu$ and a certain positive constant $M = M(a, N, \sigma, C)$.

Note that under the assumptions of Proposition 2.1 the function $V(t; V_0) = e^{(\Delta+C)t}V_0$ is the solution of the homogeneous equation (1.11) through $V_0 \in L^q_U(\mathbb{R}^N)$.

As for the solutions of the non-homogeneous equation in the locally uniform spaces we have the following lemma.

Lemma 2.2. *Suppose that* (1.8) *holds and let $q \in [1, \infty)$. Then, the solution $U(t; U_0)$ of the equation*

$$U_t - \Delta U = C(x)U + D(x), \quad t > 0,\ x \in \mathbb{R}^N, \tag{2.13}$$

through $U_0 \in L^q_U(\mathbb{R}^N)$ is given by the variation of constants formula

$$U(t; U_0) = e^{(\Delta+C)t}U_0 + \int_0^t e^{(\Delta+C)(t-s)}D\,ds, \quad t \ge 0. \tag{2.14}$$

Furthermore, $U(t; U_0)$ is a classical solution of (2.13) for $t > 0$ and is bounded in $L^\infty(\mathbb{R}^N)$ on compact time intervals away from zero uniformly with respect to U_0 in bounded subsets of $L^q_U(\mathbb{R}^N)$. Furthermore, this solution is continuous at $t = 0$ provided that $U_0 \in \dot{L}^q_U(\mathbb{R}^N)$.

Proof. We merely remark that having defined the analytic semigroup solving the homogeneous equation and using that $r_2 > \frac{N}{2}$ we get the $L^\infty(\mathbb{R}^N)$ bound for U. The smoothness of U comes from the regularizing effect of $e^{(\Delta+C)t}$, see [3]. □

3. Smooth solutions in locally uniform spaces

In this section we prove that (1.1) is locally well posed for smooth initial data in uniform Sobolev spaces.

Proposition 3.1. *Suppose that the assumptions i), ii), iii) of Theorem 1.1 hold. Let $\alpha_0 \in (0, 1)$ be such that $2\alpha_0 - \frac{N}{r_0} > 0$ and consider $u_0 \in \dot{W}^{2\alpha_0,r_0}_U(\mathbb{R}^N)$.*

Then there exists $\tau_{u_0} > 0$ such that the problem (1.1) *has a unique maximally defined mild solution $u = u(\cdot, u_0)$ in the class $C([0, \tau_{u_0}), \dot{W}^{2\alpha_0,r_0}_U(\mathbb{R}^N))$, given by the variation of constants formula*

$$u(t; u_0) = e^{(\Delta+m_0)t}u_0 + \int_0^t e^{(\Delta+m_0)(t-s)}F(\cdot, u(s; u_0))\,ds, \quad t \in [0, \tau_{u_0}). \tag{3.1}$$

In addition, for any $\gamma \in [0, 1)$,

$$u \in C((0, \tau_{u_0}), \dot{W}^{2,r_0}_U(\mathbb{R}^N)) \cap C^1((0, \tau_{u_0}), \dot{W}^{2\gamma,r_0}_U(\mathbb{R}^N)). \tag{3.2}$$

That is, u is a classical solution of (1.1) *defined on the maximal interval of existence $[0, \tau_{u_0})$.*

Furthermore, if $\tau_{u_0} < \infty$ then $\limsup_{t\to\tau_{u_0}^-} \|u(t, u_0)\|_{\dot{W}^{2\alpha_0,r_0}_U(\mathbb{R}^N)} = \infty$.

Proof. By the assumptions $\dot{W}_U^{2\alpha_0,r_0}(\mathbb{R}^N) \hookrightarrow C^\mu(\mathbb{R}^N)$ for some $\mu > 0$, see (2.9), and then, from (1.12), the Nemitsky operator defined by $F(x,s)$ is Lipschitz continuous on bounded sets from $\dot{W}_U^{2\alpha_0,r_0}(\mathbb{R}^N)$ into $\dot{L}_U^{r_0}(\mathbb{R}^N)$. Also $\Delta + m_0$ generates an analytic semigroup in $\dot{L}_U^{r_0}(\mathbb{R}^N)$, see Propositon 2.1. Hence the result follows from the results in [10]. □

In order to prove that the solutions in Proposition 3.1 are globally defined, we will assume the structure condition (1.4). We remark that this implies in particular

$$|g(x)| \leq D(x), \quad x \in \mathbb{R}^N.$$

Solutions of (1.1) will be then bounded from above by the solutions $U(\cdot; |u_0|)$ of the linear equation (2.13) with the initial condition $U(0) = |u_0|$.

Lemma 3.2. *Let the assumptions of Propositions 3.1 hold. Assume additionally* (1.4), (1.8), *that is*

$$sf(x,s) \leq C(x)s^2 + D(x)|s|, \quad x \in \mathbb{R}^N,\ s \in \mathbb{R},$$

for some

$$C \in L_U^{r_1}(\mathbb{R}^N), \quad 0 \leq D \in L_U^{r_2}(\mathbb{R}^N) \text{ with } r_1, r_2 > \frac{N}{2}.$$

Then the local solutions of (1.1) *given by (3.1) exist globally in time and satisfy*

$$|u(t;u_0)| \leq U(t;|u_0|), \quad \text{for all} \quad t \geq 0, \tag{3.3}$$

where $U(t;|u_0|)$ is the solution of

$$U_t - \Delta U = C(x)U + D(x), \quad U(0) = u_0, \quad t > 0,\ x \in \mathbb{R}^N.$$

Hence, solutions of (1.1) in Proposition 3.1 remain bounded in $L^\infty(\mathbb{R}^N)$ on compact time intervals away from zero and uniformly with respect to initial conditions u_0 that remain bounded in $\dot{L}_U^q(\mathbb{R}^N)$, $q \in [1,\infty)$.

In particular, if the nonlinear terms satisfies (1.10) then (1.4) *holds for $C(x) = L(x)$ and $D(x) = |g(x)|$.*

Proof. By Proposition 2.1 the semigroup $e^{(\Delta+C)t}$ is order preserving in locally uniform spaces so that we get $U(t,|u_0|) \geq 0$ for all $t \geq 0$.

Now, from (1.4) we have

$$f(x,s) \leq C(x)s + D(x), \quad s \geq 0,\ x \in \mathbb{R}^N,$$
$$f(x,s) \geq C(x)s - D(x), \quad s \leq 0,\ x \in \mathbb{R}^N,$$

and then $\pm U(t,|u_0|)$ are, respectively, super/sub solutions of (1.1). Therefore, by comparison, we get (3.3) and the rest follows from Lemma 2.2.

In particular, the $L^\infty(\mathbb{R}^N)$ bounds on $u(t;u_0)$ and the assumptions in (1.12) give bounds on the $\dot{L}_U^{r_0}(\mathbb{R}^N)$ norm of $F(\cdot, u(t;u_0))$, on finite time intervals. Hence, global existence follows from this and (3.1). □

Remark 3.3. *Note that in Proposition 3.1 and Lemma 3.2 we have used some properties of the Nemitsky operator defined by the nonlinear term, which follow from the assumptions in (1.12).*

In fact the next remarks will be repeatedly used in the rest of the paper.

i) First, note that if $\xi \in \dot{L}^q_U(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for some $q \in [1,\infty)$ then $\xi \in \dot{L}^q_U(\mathbb{R}^N)$ for all $q \in [1,\infty)$.
ii) Now if $\xi \in \dot{L}^q_U(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for some $q \in [1,\infty)$ then $F(\cdot,\xi) \in \dot{L}^\sigma_U(\mathbb{R}^N)$ for each $\sigma \in [1, r_0)$.

Indeed, from (1.12) we immediately get $F(\cdot,\xi) \in L^{r_0}_U(\mathbb{R}^N)$.

On the other hand, to prove the continuity under translations observe that for any $z \in \mathbb{R}^N$, $\|\tau_z(m_j h(\xi)) - m_j h(\xi)\|_{L^\sigma_U(\mathbb{R}^N)}$ is bounded by

$$\|(\tau_z m_j - m_j)\tau_z h(\xi)\|_{L^\sigma_U(\mathbb{R}^N)} + \|(\tau_z h_j(\xi) - h_j(\xi))m_j\|_{L^\sigma_U(\mathbb{R}^N)} =: \mathcal{J}_1 + \mathcal{J}_2.$$

Thus, since $\xi \in L^\infty(\mathbb{R}^N)$ and $m_j \in \dot{L}^{r_0}_U(\mathbb{R}^N)$ we infer that $\mathcal{J}_1 \to 0$ as $|z| \to 0$ for $1 \le \sigma \le r_0$. Now, Hölder's inequality and the Lipschitz continuity of h_j on bounded sets imply that for any $1 \le \sigma < r_0$.

$$\mathcal{J}_2 \le \|m_j\|_{L^{r_0}_U(\mathbb{R}^N)} \|h_j(\tau_z\xi) - h_j(\xi)\|_{L_U^{\frac{\sigma r_0}{r_0-\sigma}}(\mathbb{R}^N)} \le \kappa \|m_j\|_{L^{r_0}_U(\mathbb{R}^N)} \|\tau_z\xi - \xi\|_{L_U^{\frac{\sigma r_0}{r_0-\sigma}}(\mathbb{R}^N)},$$

so that, from i), $\lim_{|z|\to 0} \mathcal{J}_2 = 0$.

Finally, the remaining term $\mathcal{J}_3 := \|\tau_z(f_0(\cdot,\xi)) - f_0(\cdot,\xi)\|_{L^\sigma_U(\mathbb{R}^N)}$, can be estimated by $\|f_0(\cdot + z, \tau_z\xi) - f_0(\cdot,\tau_z\xi)\|_{L^\sigma_U(\mathbb{R}^N)} + \|f_0(\cdot,\tau_z\xi) - f_0(\cdot,\xi)\|_{L^\sigma_U(\mathbb{R}^N)}$. The first term goes to zero with z because ξ is bounded and f_0 is assumed to be Hölder continuous in the first variable. Also the second term is bounded by $C\|\tau_z\xi - \xi\|_{L^\sigma_U(\mathbb{R}^N)}$ and hence $\lim_{|z|\to 0} \mathcal{J}_3 = 0$, for $1 \le \sigma < \infty$.
iii) If moreover ξ is Hölder continuous, then we can take $\sigma = r_0$ in ii). This was actually used in the proof of Proposition 3.1.

4. Global solutions with initial data in $\dot{L}^q_U(\mathbb{R}^N)$, $q \in [1,\infty)$.

In this section we will use (1.10) and a density argument to construct global solutions of (1.1) for nonsmooth initial conditions $u_0 \in \dot{L}^q_U(\mathbb{R}^N)$, $q \in [1,\infty)$. These solutions will satisfy the variation of constants formula (3.1) and the estimate (3.3); in particular they will be smooth for positive times.

We first show that the global smooth solutions of Section 3 have a suitable Cauchy property in $\dot{L}^q_U(\mathbb{R}^N)$ with respect to the initial data, for $1 < q < \infty$. The case $q = 1$ will be considered further below.

Lemma 4.1. *Let the assumptions of Propositions 3.1 hold and moreover assume* (1.10)*.*

Then for each $u_0^1, u_0^2 \in \dot{W}^{2\alpha_0, r_0}_U(\mathbb{R}^N)$, $T > 0$ and $1 < q < \infty$ we have

$$\|u(t;u_0^1) - u(t;u_0^2)\|_{\dot{L}^q_U(\mathbb{R}^N)} \le C(q,T)\|u_0^1 - u_0^2\|_{\dot{L}^q_U(\mathbb{R}^N)}, \ 0 \le t \le T < \infty, \tag{4.1}$$

with some constant $C(q,T) > 0$ independent of u_0^1, u_0^2.

In particular, for any $u_0 \in \dot{L}^q_U(\mathbb{R}^N)$ and any sequence $\{u_0^n\} \subset \dot{W}^{2\alpha_0, r_0}_U(\mathbb{R}^N)$ converging to u_0 in $\dot{L}^q_U(\mathbb{R}^N)$, the sequence of global solutions $\{u(t;u_0^n)\}$ is a Cauchy sequence in $C([0,T], \dot{L}^q_U(\mathbb{R}^N))$ for every $T > 0$.

Furthermore,
i) the limit function satisfies

$$u(\cdot;u_0) \in C([0,\infty), \dot{L}^q_U(\mathbb{R}^N)) \tag{4.2}$$

and is independent of the choice of $\{u_0^n\} \subset \dot{W}_U^{2\alpha_0,r_0}(\mathbb{R}^N)$,
ii) for any $u_0^1, u_0^2 \in \dot{L}_U^q(\mathbb{R}^N)$ *and* $T>0$, *(4.1) is satisfied by the pair of limit functions* $u(\cdot;u_0^1)$ *and* $u(\cdot;u_0^2)$ *resulting from the above construction.*

Proof. By assumption (1.10), Lemma 3.2 holds for $C(x)=L(x)$ and $D(x)=|g(x)|$. In particular, solutions in Proposition 3.1 are globally defined.

For the proof of (4.1) it is enough to get a similar estimate with respect to a weighted spaces as in (2.5), with a constant independent of the weight function and its translates. Therefore below we denote by ρ any traslate of a weight ρ_0 as in (2.2), (2.3), (2.4).

Multiplying the equation for $v(t) := u(t;u_0^1) - u(t;u_0^2)$ by $v|v|^{q-2}\rho$, integrating over $\mathbb{R}^N$ and using (1.10) we have

$$\frac{1}{q}\frac{d}{dt}\int_{\mathbb{R}^N}|v|^q\rho \le \int_{\mathbb{R}^N}\Delta v v|v|^{q-2}\rho + \int_{\mathbb{R}^N}L(x)|v|^q\rho. \tag{4.3}$$

Integrating by parts and using (2.3) we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}\Delta v v|v|^{q-2}\rho &\le \int_{\mathbb{R}^N}|v|^{q-2}\left(-(q-1)|\nabla v|^2 + c|\nabla v||v|\right)\rho\\ &= -\frac{4(q-1)}{q^2}\int_{\mathbb{R}^N}\left|\nabla\left(|v|^{\frac{q}{2}}\right)\right|^2\rho + \frac{2c}{q}\int_{\mathbb{R}^N}|v|^{\frac{q}{2}}\left|\nabla\left(|v|^{\frac{q}{2}}\right)\right|\rho\\ &\le -\frac{4(q-1)}{q^2}\int_{\mathbb{R}^N}\left|\nabla\left(|v|^{\frac{q}{2}}\right)\right|^2\rho + \frac{c}{q}\int_{\mathbb{R}^N}\left(\epsilon\left|\nabla\left(|v|^{\frac{q}{2}}\right)\right|^2 + \epsilon^{-1}\left(|v|^{\frac{q}{2}}\right)^2\right)\rho\\ &= -\frac{2(q-1)}{q^2}\int_{\mathbb{R}^N}\left|\nabla\left(|v|^{\frac{q}{2}}\right)\right|^2\rho + \frac{c^2}{2(q-1)}\int_{\mathbb{R}^N}\left(|v|^{\frac{q}{2}}\right)^2\rho,\end{aligned}$$

where we used the Cauchy inequality with $\epsilon = \frac{2(q-1)}{cq}$. Letting

$$L_q(x) := \frac{c^2q^2}{4(q-1)^2} + \frac{q^2}{2(q-1)}L(x), \qquad L_q \in L_U^{r_3}(\mathbb{R}^N),$$

we get from (4.3)

$$\frac{d}{dt}\int_{\mathbb{R}^N}|v|^q\rho \le \frac{2(q-1)}{q}\int_{\mathbb{R}^N}\left(-\left|\nabla\left(|v|^{\frac{q}{2}}\right)\right|^2 + L_q(x)\left(|v|^{\frac{q}{2}}\right)^2\right)\rho.$$

Using again the Cauchy inequality and (2.3) we also get

$$\begin{aligned}-\left|\nabla\left(|v|^{\frac{q}{2}}\right)\right|^2\rho &= -\left|\nabla\left(|v|^{\frac{q}{2}}\rho^{\frac{1}{2}}\right) - |v|^{\frac{q}{2}}\nabla\left(\rho^{\frac{1}{2}}\right)\right|^2\\ &= -|\nabla\left(|v|^{\frac{q}{2}}\rho^{\frac{1}{2}}\right)|^2 + 2|v|^{\frac{q}{2}}\nabla\left(|v|^{\frac{q}{2}}\rho^{\frac{1}{2}}\right)\nabla\left(\rho^{\frac{1}{2}}\right) - \left(|v|^{\frac{q}{2}}\nabla\left(\rho^{\frac{1}{2}}\right)\right)^2\\ &\le -|\nabla\left(|v|^{\frac{q}{2}}\rho^{\frac{1}{2}}\right)|^2 + c|\nabla\left(|v|^{\frac{q}{2}}\rho^{\frac{1}{2}}\right)||v|^{\frac{q}{2}}\rho^{\frac{1}{2}} + \frac{1}{4}c^2\left(|v|^{\frac{q}{2}}\rho^{\frac{1}{2}}\right)^2\\ &\le -\frac{1}{2}|\nabla\left(|v|^{\frac{q}{2}}\rho^{\frac{1}{2}}\right)|^2 + \frac{3c^2}{4}\left(|v|^{\frac{q}{2}}\rho^{\frac{1}{2}}\right)^2\end{aligned}$$

and hence

$$\frac{d}{dt}\int_{\mathbb{R}^N}|v|^q\rho \le \frac{q-1}{q}\int_{\mathbb{R}^N}\left(-\left|\nabla\left(|v|^{\frac{q}{2}}\rho^{\frac{1}{2}}\right)\right|^2 + \hat{L}_q(x)\left(|v|^{\frac{q}{2}}\rho^{\frac{1}{2}}\right)^2\right),$$

where $\hat{L}_q(x) := 2L_q(x) + \frac{3c^2}{2}$ and $\hat{L}_q \in L^{r_3}_U(\mathbb{R}^N)$.

Now, since $r_3 > N/2$, with $z = |v|^{\frac{q}{2}}\rho^{\frac{1}{2}} \in L^2(\mathbb{R}^N)$, we infer from [18], see also [2], that there is a certain $\mu \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} \big|\nabla z|^2 - \hat{L}_q(x)|z|^2 \geq \mu \int_{\mathbb{R}^N} |z|^2.$$

Since all constants remain independent of ρ, Gronwall's inequality and (2.5), (2.6), leads to (4.1).

Now the proofs of i) and ii) follow easily. □

To get (4.3) when $q = 1$ we will approximate the sign function by smooth functions; for example we will use

$$h_n(s) = \begin{cases} 1, \ s \geq \frac{1}{n}, \\ -n^2s^2 + 2ns, \ 0 < s < \frac{1}{n}, \end{cases} \qquad h_n(s) = -h_n(-s), \quad h'(s) \geq 0, \quad n \in \mathbb{N}, \tag{4.4}$$

(see e.g. [5]). We will also denote $H_n(v) = \int_0^v h_n(s)\,ds$ which is a smooth approximation of the function $|v|$.

Lemma 4.2. *Under the additional assumption* (1.13), *Lemma 4.1 applies also for* $q = 1$.

Proof. We now multiply the equation for $v(t) := u(t; u_0^1) - u(t; u_0^2)$ by $h_n(v)\rho$ as in (4.4), and integrate the result over $\mathbb{R}^N$ to get

$$\begin{aligned} \int_{\mathbb{R}^N} h_n(v)v_t\rho &= \int_{\mathbb{R}^N} (-|\nabla v|^2 h_n'(v)\rho - h_n(v)\nabla v \nabla \rho) \\ &+ \int_{\mathbb{R}^N} \Big((f(\cdot, u(t; u_0^1)) - f(\cdot, u(t; u_0^2)\Big) h_n(v)\rho \\ &\leq -\int_{\mathbb{R}^N} h_n(v)\nabla v \nabla \rho + \int_{\mathbb{R}^N} \Big(f(\cdot, u(t; u_0^1)) - f(\cdot, u(t; u_0^2)\Big) h_n(v)\rho. \end{aligned}$$

Integrating now over $[0, t]$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} H_n(v)\rho - \int_{\mathbb{R}^N} H_n(v(0))\rho &\leq -\int_0^t \int_{\mathbb{R}^N} h_n(v)\nabla v \nabla \rho \\ &+ \int_0^t \int_{\mathbb{R}^N} \Big(f(\cdot, u(t; u_0^1)) - f(\cdot, u(t; u_0^2))\Big) h_n(v)\rho. \end{aligned}$$

Since we have that (1.10) is satisfied then Lemma 3.2 applies and using (3.3) and the pointwise convergence $h_n(v) \to sgn(v)$ and $H_n(v) \to |v|$ in $[0, t] \times \mathbb{R}^N$, we obtain by the Lebesgue dominated convergence theorem that

$$\begin{aligned} \int_{\mathbb{R}^N} |v|\rho - \int_{\mathbb{R}^N} |v(0)|\rho &\leq -\int_0^t \int_{\mathbb{R}^N} \nabla |v| \nabla \rho \\ &+ \int_0^t \int_{\mathbb{R}^N} \Big(f(\cdot, u(t; u_0^1)) - f(\cdot, u(t; u_0^2))\Big) sgn(v)\rho. \end{aligned}$$

Then we integrate by parts the first term of the right hand side above and use (2.3) and (1.13) to obtain

$$\int_{\mathbb{R}^N} |v|\rho - \int_{\mathbb{R}^N} |v(0)|\rho \le (c+\mathcal{K}) \int_0^t \int_{\mathbb{R}^N} |v|\rho.$$

Hence we obtain again (4.1) by using Gronwall's inequality. The rest follows then as in Lemma 4.1. □

Below we prove that for any $u_0 \in \dot{L}^q(\mathbb{R}^N)$ and $q \in [1,\infty)$, the limit function $u(\cdot;u_0)$ resulting from Lemmas 4.1, 4.2 is in fact a smooth solution of (1.1) for positive times. First we consider the case $1 < q < \infty$.

Theorem 4.3. *Assume, as in Lemma 4.1, that $f(x,s)$ satisfies the assumptions in Proposition 3.1 and* (1.10)*. Moreover, assume also that (1.4) and (1.8) are satisfied. Let $1 < q < \infty$ and $u_0 \in \dot{L}^q_U(\mathbb{R}^N)$.*

Then the limit functions constructed in Lemma 4.1 satisfy

$$|u(t;u_0)| \le U(t;|u_0|) \quad \textit{for} \;\; t \ge 0, \tag{4.5}$$

where $U(t;|u_0|)$ is the solution of the linear equation (2.13) *with initial data $U(0) = |u_0|$; in particular $u(t;u_0)$ remains bounded in $L^\infty(\mathbb{R}^N)$ uniformly on any compact time interval away from zero.*

Furthermore, $u(\cdot;u_0)$ satisfies the variation of constants formula

$$u(t;u_0) = e^{(\Delta+m_0)(t-\tau)}u(\tau;u_0) + \int_\tau^t e^{(\Delta+m_0)(t-s)}F(\cdot,u(s;u_0))ds \;\; \textit{for all}\; t > \tau \ge 0 \tag{4.6}$$

and it is a smooth solution of (1.1) for $t > 0$, that is

$$u(\cdot;u_0) \in C((0,\infty), \dot{W}^{2,r_0}_U(\mathbb{R}^N)) \cap C^1((0,\infty), \dot{W}^{2\gamma,r_0}_U(\mathbb{R}^N)) \;\; \textit{for any}\; \gamma \in [0,1). \tag{4.7}$$

Proof. Note, from Lemma 3.2, that the approximating solutions satisfy

$$|u(t;u_0^n)| \le U(t;|u_0^n|) \quad \text{for all} \;\; t \ge 0, \; n \in \mathbb{N}.$$

Passing to the limit we get (4.5).

To prove (4.6) we will write the variation of constants formula for a sequence $\{u(\cdot;u_0^n)\}$ of approximating solutions and then pass to the limit.

Observe first that from [3], $\{e^{(\Delta+m_0)t}\}$ is a C^0 semigroup on $L^{r_0}_\rho(\mathbb{R}^N)$ and that (1.12) together with the $L^\infty(\mathbb{R}^N)$ bounds for $u(t;u_0)$ (valid on compact time intervals away from zero) imply that $F(\cdot,u(\cdot)) \in L^1([\tau,T], L^{r_0}_\rho(\mathbb{R}^N))$ for any $0 < \tau < T$. Note also that $u(t;u_0) \in \dot{L}^{r_0}_U(\mathbb{R}^N)$, since $u(t;u_0) \in \dot{L}^q_U(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for each $t > 0$. Consequently the right hand side of (4.6) is well defined for the limit solution as well.

Now observe that, as $n \to \infty$,

$$e^{(\Delta+m_0)(t-\tau)}u(\tau;u_0^n) \longrightarrow e^{(\Delta+m_0)(t-\tau)}u(\tau;u_0) \quad \text{in } \dot{L}^{r_0}_U(\mathbb{R}^N), \text{ for } t > \tau$$

and, from the assumptions in (1.12),

$$\|e^{(\Delta+m_0)(t-s)}F(\cdot,u(s;u_0^n)) - e^{(\Delta+m_0)(t-s)}F(\cdot,u(s;u_0))\|_{L^{r_0}_\rho(\mathbb{R}^N)} \le K \;\; \text{for} \;\; s \in [\tau,t],$$

where the constant K is independent of n and s. Also, again the assumptions in (1.12) and Lebesgue's Dominated Convergence Theorem imply

$$\|F(\cdot, u(s; u_0^n)) - F(\cdot, u(s; u_0))\|_{L^{r_0}_\rho(\mathbb{R}^N)} \to 0,$$

since $u(s; u_0^n) \to u(s; u_0)$ a.e. in $\mathbb{R}^N$ and (4.5) holds for the aproximating solutions.

Therefore, passing to the limit in the variation of constants formula for $u(t; u_0^n)$, for $T \geq t \geq \tau > 0$, we get

$$u(t; u_0) = e^{(\Delta+m_0)(t-\tau)} u(\tau; u_0) + \int_\tau^t e^{(\Delta+m_0)(t-s)} F(\cdot, u(s; u_0))\, ds,$$

where we can also pass to the limit as $\tau \to 0$, since $e^{(\Delta+m_0)t}$ is a C^0 analytic semigroup on $\dot{L}^q_U(\mathbb{R}^N)$ (see Proposition 2.1) and $u(\cdot; u_0)$ is continuous at $\tau = 0$ in $\dot{L}^q_U(\mathbb{R}^N)$.

Formula (4.6) and the $L^\infty(\mathbb{R}^N)$ bounds on the solution, away from $t = 0$, give the smoothness of the solution as in Proposition 3.1 (see also Remark 3.3). □

Concerning the case $q = 1$ let us prove that

Proposition 4.4. *Under the additional assumption* (1.13)*, Theorem 4.3 applies also when* $q = 1$.

Proof. Recalling Lemma 3.2 and Lemma 4.2, we again get (4.5). In particular, $u(t; u_0)$ remains bounded in $L^\infty(\mathbb{R}^N)$ uniformly on any compact time intervals away from zero as well as for u_0 in bounded subsets of $L^1_U(\mathbb{R}^N)$. Hence $u(t; u_0) \in \dot{L}^1_U(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for each $t > 0$, which implies that $u(t; u_0) \in \dot{L}^\sigma_U(\mathbb{R}^N)$ for all $t > 0$ and $\sigma \in (1, \infty)$. Conditions (4.6), (4.7) thus follow as in Theorem 4.3. □

The above results allow us to define for every $1 \leq q < \infty$ the nonlinear semigroup associated to (1.1) as

$$S(t) : \dot{L}^q_U(\mathbb{R}^N) \to \dot{L}^q_U(\mathbb{R}^N), \quad S(t)u_0 = u(t; u_0) \text{ for } t \geq 0,\ u_0 \in \dot{L}^q_U(\mathbb{R}^N). \tag{4.8}$$

Recall that the uniform spaces $\dot{L}^q_U(\mathbb{R}^N)$ are nested, so the closer q is to 1, the larger this space is. Whenever (1.13) is satisfied, the nonlinear semigroup in (4.8) is defined in the largest of these spaces $\dot{L}^1_U(\mathbb{R}^N)$.

Finally note that the results above finish the proof of the global existence part in Theorem 1.1.

5. Asymptotic bounds and the global attractor

In this section we prove asymptotic bounds for the solutions of (1.1) and the existence of a global attractor that attracts solutions in a suitable way. In particular we finish the proof of Theorem 1.1.

As in [4] the difficulty is that the semigroup $\{S(t)\}$ in (4.8) associated to (1.1) is not asymptotically compact in the norm of the phase space $\dot{L}^q_U(\mathbb{R}^N)$.

We start from the pointwise and uniform asymptotic bounds in $L^\infty(\mathbb{R}^N)$.

Proposition 5.1. *Suppose that the assumptions of i)–v) in Theorem 1.1 are satisfied.*

Let $0 \leq \Phi \in W^{2,r'}_U(\mathbb{R}^N)$ *with* $r' = \min\{r_1, r_2\}$ *be the unique solution of the elliptic equation*

$$-\Delta\Phi = C(x)\Phi + D(x), \quad x \in \mathbb{R}^N. \tag{5.1}$$

Then the solutions $u(\cdot; u_0)$ *constructed in Theorem 4.3 (resp. Proposition 4.4, if* $q = 1$*) satisfy the asymptotic bounds*

$$\limsup_{t\to\infty} |u(t; u_0)(x)| \leq \Phi(x) \tag{5.2}$$

uniformly for $x \in \mathbb{R}^N$*, and*

$$\limsup_{t\to\infty} \|u(t; u_0)\|_{L^\infty(\mathbb{R}^N)} \leq \|\Phi\|_{L^\infty(\mathbb{R}^N)} \tag{5.3}$$

uniformly for u_0 *in bounded sets in* $\dot{L}^q_U(\mathbb{R}^N)$*.*

Also, if $|u_0(x)| \leq \Phi(x)$ *then* $|u(t; u_0)| \leq \Phi$ *for all* $t \geq 0$*.*

Proof. If $V(\cdot; |u_0| - \Phi)$ is the solution of the homogeneous equation (1.11), then $U(t; |u_0|) = \Phi + V(t; |u_0| - \Phi)$ solves (2.13) and (4.5) reads

$$|u(t; u_0)| \leq \Phi + V(t; |u_0| - \Phi) \quad \text{for } \ t \geq 0. \tag{5.4}$$

By assumptions the estimate (2.12) holds with some $a < 0$ and $p = \infty$ so that $\|V(\cdot; |u_0| - \Phi)\|_{L^\infty_U(\mathbb{R}^N)} \to 0$ as $t \to \infty$ uniformly for u_0 in bounded sets in $\dot{L}^q_U(\mathbb{R}^N)$. The result now follows easily. □

Now we use the variation of constants formula to obtain asymptotic bounds for the solutions in stronger norms of uniform spaces.

Proposition 5.2. *Suppose that the assumptions of Proposition 5.1 hold. Then there exists a constant* $c_\infty > 0$ *such that for any set* B *bounded in* $\dot{L}^q_U(\mathbb{R}^N)$ *there exists* $t_B > 0$ *such that*

$$\|u(t; u_0)\|_{\dot{W}^{2,r_0}_U(\mathbb{R}^N)} \leq c_\infty \quad \text{for all } t \geq t_B \text{ and } u_0 \in B. \tag{5.5}$$

Moreover there exists a bounded, positively invariant, absorbing set in $\dot{W}^{2,r_0}_U(\mathbb{R}^N)$*.*

Proof. Thanks to (5.3) for B bounded in $\dot{L}^q_U(\mathbb{R}^N)$ there exists τ_B such that

$$\|u(t; u_0)\|_{L^\infty(\mathbb{R}^N)} \leq \|\Phi\|_{L^\infty(\mathbb{R}^N)} + 1 = c_0 \quad \text{for all } \ t \geq \tau_B, \ u_0 \in B. \tag{5.6}$$

We also choose β sufficiently large, such that the semigroup $e^{(\Delta+m_0-\beta)t}$ decays exponentially, and from (1.12), we obtain the estimate

$$\|F(\cdot, u(t; u_0)) + \beta u(t; u_0)\|_{\dot{L}^{r_0}_U(\mathbb{R}^N)} \leq c_1 \quad \text{for all } \ t \geq \tau_B, \ u_0 \in B.$$

Using this and applying (4.6) with $\tau = \tau_B$ we get

$$\begin{aligned}\|u(t; u_0)\|_{\dot{W}^{2\alpha,r_0}_U(\mathbb{R}^N)} &\leq Me^{-a(t-\tau_B)}(t-\tau_B)^{-\alpha}\|u(\tau_B; u_0)\|_{\dot{L}^{r_0}_U(\mathbb{R}^N)} \\ &\quad + M\int_{\tau_B}^t e^{-a(t-s)}(t-s)^{-\alpha}\|F(\cdot, u(s; u_0)) + \beta u(s; u_0)\|_{\dot{L}^{r_0}_U(\mathbb{R}^N)} ds \\ &\leq Mc_0\omega_N e^{-a(t-\tau_B)}(t-\tau_B)^{-\alpha} + M\Gamma(1-\alpha)a^{\alpha-1}c_1,\end{aligned}$$

where $\alpha \in [0, 1)$ and M, a are suitable positive constants.

Thus for some t_B^α and all $t \geq t_B^\alpha$, $u_0 \in B$, we have

$$\|u(t;u_0)\|_{\dot{W}_U^{2\alpha,r_0}(\mathbb{R}^N)} \leq 2M\Gamma(1-\alpha)a^{\alpha-1}c_1$$

which implies the existence of a bounded absorbing set $\tilde{B}_0^\alpha \in \dot{W}_U^{2\alpha,r_0}(\mathbb{R}^N)$. In particular $\tilde{B}_0^\alpha$ absorbs itself; that is for a certain $t_0^\alpha > 0$ we have $u(t;\tilde{B}_0^\alpha) \subset \tilde{B}_0^\alpha$ for all $t \geq t_0^\alpha$. Then $B_0^\alpha := \bigcup_{t \geq t_0^\alpha} u(t;\tilde{B}_0^\alpha)$ is bounded, absorbing and positively invariant.

Finally, take $\alpha = \alpha_0$ as in Proposition 3.1. Since $\tilde{B}_0^{\alpha_0}$ is bounded in $\dot{W}_U^{2\alpha_0,r_0}(\mathbb{R}^N)$ then (5.5) and the rest of the result is a consequence of [6, Lemma 3.2.1]. $\square$

Remark 5.3. *Observe that if $r_0 = \infty$ or $g = m_j = 0$, $j = 1,\dots,k$, in (1.12), then the asymptotic estimate above holds in $\dot{W}_U^{2,s}(\mathbb{R}^N)$ for any $s \geq 1$, since in this case we get, for every $s \geq 1$,*

$$\|F(\cdot,u(t;u_0)) + \beta u(t;u_0)\|_{\dot{L}_U^s(\mathbb{R}^N)} \leq c_1 \quad \textit{for all } \ t \geq \tau_B, \ u_0 \in B.$$

We are now ready to prove *asymptotic compactness* of the nonlinear semigroup in a certain sense.

Proposition 5.4. *Suppose that the assumptions of Proposition 5.1 hold and let W denote either $C^\mu_{loc}(\mathbb{R}^N)$, with $2 - \frac{N}{p} > \mu > 0$, or the weighted space $W_\rho^{s,r_0}(\mathbb{R}^N)$ with $0 \leq s < 2$.*

Then, for any bounded sequence $\{u_0^n\}$ in $\dot{L}_U^q(\mathbb{R}^N)$ and each sequence $t_n \nearrow \infty$, there exists a subsequence of $\{u(t_n;u_0^n)\}$ (which we denote the same) such that

$$u(t_n;u_0^n) \to v_0 \quad \textit{in } W, \tag{5.7}$$

where $v_0 \in W_U^{2,r_0}(\mathbb{R}^N)$, $|v_0| \leq \Phi$ and

$$\|v_0\|_{W_U^{2,r_0}(\mathbb{R}^N)} = \sup_{y\in\mathbb{R}^N} \|v_0\|_{W^{2,r_0}(B(y,1))} \leq c_\infty, \tag{5.8}$$

with c_∞ as in Proposition 5.2.

Proof. For large enough n and W as above, using (5.5) and the compact embeddings $W_U^{2,r_0}(\mathbb{R}^N) \hookrightarrow W$ (see (2.10)-(2.11)) we get the convergence to v_0.

Also, (5.5) and weak convergence in $W^{2,r_0}(B(y,1))$ lead to (5.8). Finally we get $|v_0| \leq \Phi$ as a result of (5.2). $\square$

Remark 5.5. *Observe that if $r_0 = \infty$ or $g = m_j = 0$, $j = 1,\dots,k$, in (1.12), then , from Remark 5.3, the space W in Proposition 5.4 can be either $C^{1+\mu}_{loc}(\mathbb{R}^N)$, with $0 < \mu < 1$, or $W_\rho^{s,p}(\mathbb{R}^N)$ with $0 \leq s < 2$ and $p \in (1,\infty)$.*

Now we follow the argument in [4, Corollary 2.2], to conclude the existence of a bounded invariant set in $\dot{W}_U^{2,r_0}(\mathbb{R}^N)$ that attracts solutions of (1.1) as in (5.7). For this it is enough to prove the following result.

Proposition 5.6. *Suppose that the assumptions of Proposition 5.1 are satisfied and let W, $\{u_0^n\} \subset \dot{L}_U^q(\mathbb{R}^N)$, $t_n \nearrow \infty$ and $v_0 \in W_U^{2,r_0}(\mathbb{R}^N)$ be as in Proposition 5.4. Moreover, assume that (5.7) holds.*

Then for any $t > 0$

$$u(t+t_n;u_0^n) \longrightarrow u(t;v_0) \quad \textit{in } W. \tag{5.9}$$

Proof. The computations in the proofs of Lemmas 4.1 or 4.2 show that for $0 \le t \le T$

$$\|u(t+t_n;u_0^n) - u(t;v_0)\|_{L^q_\rho(\mathbb{R}^N)} \le C(q,T)\|u(t_n;u_0^n) - v_0\|_{L^q_\rho(\mathbb{R}^N)} \stackrel{n\to\infty}{\longrightarrow} 0, \tag{5.10}$$

whereas Proposition 5.4 implies that for any $t > 0$ each subsequence of $\{u(t+t_n;u_0^n)\}$ has a convergent subsequence in W. Thus, the limit must be $u(t;v_0)$ as stated in (5.9). □

Now we finish this Section with the

Proof of v)–Theorem 1.1.

After Propositions 5.2, 5.4 and 5.6, [4, Corollary 2.2] implies the existence of a global attractor $\mathcal{A}$ in the sense described in Theorem 1.1. In fact (5.5) and the invariance of $\mathcal{A}$ implies this set is bounded in $\dot{W}^{2,r_0}_U(\mathbb{R}^N)$. Note moreover that we have

$$|u| \le \Phi \quad \text{for each } u \in \mathcal{A}. \quad \square \tag{5.11}$$

6. Extremal equilibria

In this section we prove that (1.1) possesses extremal equilibria that bound the asymptotic dynamics. In particular, we prove Theorem 1.2. These sort of results have been already established for problems in bounded domains in [17] and in unbounded ones in [16] and here we follow a similar 'dynamical strategy'. In particular the proof of Theorem 1.2 will follow the abstract result for monotonic flows in [17], which however does not apply here because the nonlinear semigroup associated to (1.1) is neither asymptotically compact, nor decreasing sequences, bounded below, converge in $\dot{L}^q_U(\mathbb{R}^N)$.

Proof of Theorem 1.2. From Proposition 5.1 we have that, for any $\varepsilon > 0$, the ordered interval $I = [\eta_m, \eta_M]$ with $\eta_M = -\eta_m = \Phi + \varepsilon$ is absorbing for (1.1). Hence there exists a time $T \ge 0$ such that

$$\eta_m \le u(t+T;\eta_M) \le \eta_M \tag{6.1}$$

for all $t \ge 0$. Using now the order-preserving property of the semigroup and (6.1) we have

$$u(2T;\eta_M) \le u(T;\eta_M) \le \eta_M.$$

And by iterating the process

$$u(nT;\eta_M) \le u((n-1)T;\eta_M) \le \cdots \le u(T;\eta_M) \le \eta_M$$

for all $n \in \mathbb{N}$. Thus, $\{u(nT;\eta_M)\}$ is a monotonically decreasing sequence.

From the compactness obtained in Proposition 5.4 and the monotonicity we infer that $\{u(nT;\eta_M)\}$ actually converges in the sense of (5.7) to some element $\varphi_M \in W^{2,r_0}_U(\mathbb{R}^N)$.

Now we prove that, in fact, the whole solution $u(t;\eta_M)$ converges as $t \to \infty$, in the same sense as above, to φ_M.

Let $\{t_n\}$ tend to infinity and $k_n \in \mathbb{N}$, $\tau_n \in [0,T)$ be such that $t_n = k_nT + \tau_n$ and $\{k_n\}$ is strictly increasing. Then, on the one hand, taking $t = \tau_n$ in (6.1) we have

$$u(T+\tau_n;\eta_M) \le \eta_M,$$

and, after time $(k_n - 1)T$, we get

$$u(t_n;\eta_M) \le u((k_n-1)T;\eta_M). \tag{6.2}$$

On the other hand, for any $s \in [0,T)$ we take $t = T - s$ in (6.1) and after time s we obtain

$$u(2T;\eta_M) \leq u(s;\eta_M).$$

From this, in turn, after time $k_n T$ and taking $s = \tau_n$ we have

$$u((k_n+2)T;\eta_M) \leq u(t_n;\eta_M). \tag{6.3}$$

Then, passing with n to infinity in (6.2) and (6.3) we get

$$\lim_{n\to\infty} u(t_n;\eta_M) = \varphi_M.$$

Since the previous argument is valid for any time sequence $\{t_n\}$, we actually have

$$\lim_{t\to\infty} u(t;\eta_M) = \varphi_M \tag{6.4}$$

in the sense of (5.7).

Now we show that φ_M is an equilibrium solution of (1.1). For τ large enough and all $t > \tau$ we note that

$$u(t;\eta_M) = e^{(\Delta+m_0-\beta)(t-\tau)}u(\tau;\eta_M) + \int_\tau^t e^{(\Delta+m_0-\beta)(t-s)}\big(F(\cdot,u(s;\eta_M)) + \beta u(s;\eta_M)\big)\,ds, \tag{6.5}$$

where $\beta > 0$ is chosen such that $e^{(\Delta+m_0-\beta)t}$ decays exponentially.

Also, given any $\delta > 0$, if τ is sufficiently large then for $\sigma \in [1,r_0]$ we claim that from (6.4) and (1.12)

$$\|F(\cdot,u(s;\eta_M)) + \beta u(s;\eta_M) - F(\cdot,\varphi_M) - \beta\varphi_M\|_{L^\sigma_\rho(\mathbb{R}^N)} \leq \delta \quad \text{whenever } s \geq \tau. \tag{6.6}$$

Indeed, to prove (6.6), observe that the terms $\|m_j\Big(h_j(u(s;\eta_M)) - h_j(\varphi_M)\Big)\|_{L^\sigma_\rho(\mathbb{R}^N)}$, can be bounded above by a multiple of

$$\|m_j\Big(h_j(u(s;\eta_M)) - h_j(\varphi_M)\Big)\|_{L^\sigma_\rho(B(0,R))} + \|m_j\|_{L^\sigma_\rho(\mathbb{R}^N\setminus B(0,R))} =: \mathcal{I}_1 + \mathcal{I}_2$$

for any $R > 0$. By the integrability of $m_j^\sigma\rho$, we can choose $R > 0$ large enough such that $\mathcal{I}_2 < \frac{\delta}{2}$. On the other hand from (6.4) we have $\mathcal{I}_1 < \frac{\delta}{2}$ for all s sufficiently large. The other terms appearing in (6.6) are handled similarly.

Hence, the first two terms in (6.5) converge, as $t \to \infty$, to φ_M and to zero in the sense of (5.7) and in $L^\infty(\mathbb{R}^N)$ respectively. On the other hand the integral term in (6.5) is as close as we want in $L^\sigma_\rho(\mathbb{R}^N)$ to

$$z(t) = \int_\tau^t e^{(\Delta+m_0-\beta)(t-s)}\big(F(\cdot,\varphi_M) + \beta\varphi_M\big)\,ds$$

uniformly for $t \geq \tau$. But $z(t)$ solves

$$\begin{cases} z_t - \Delta z - m_0(x)z + \beta z = F(\cdot,\varphi_M) + \beta\varphi_M \in \dot{L}^{r_0}_U(\mathbb{R}^N) \\ z(\tau) = 0 \end{cases}$$

and therefore we have that

$$z(t) \to \xi \text{ in } \dot{W}^{2,r_0}_U(\mathbb{R}^N) \text{ as } t \to \infty,$$

where ξ satisfies

$$-\Delta\xi - m_0(x)\xi + \beta\xi = F(\cdot, \varphi_M) + \beta\varphi_M.$$

Putting all these together in (6.5) we get that $\xi = \varphi_M$ and hence φ_M is an equilibrium.

Finally, given any bounded set of initial data in $\dot{L}^q_U(\mathbb{R}^N)$, all the solutions starting at this set enter in finite time below η_M. We also know that the solution starting at η_M converges to φ_M as in (5.7). Then, (1.14) holds uniformly for bounded sets in $\dot{L}^q_U(\mathbb{R}^N)$. In particular for any equilibrium ψ, by (1.14) with $u_0 = \psi$, we get $\psi \leq \varphi_M$; that is φ_M is maximal in the set of equilibria.

The results for φ_m follow analogously. □

Remark 6.1. *Note that Φ satisfies $-\Delta\Phi = C(x)\Phi + D(x)$ and so is formally a supersolution of (1.1). Hence, $u(t;\Phi)$ is decreasing and converges to φ_M in the sense of (5.7). Also note that φ_M does not need to have constant sign unless $g(x) \geq 0$.*

7. Asymptotic small behavior at infinity

In this section we complete the proof of Theorem 1.3. We thus discuss sufficient conditions for the solutions of (1.1) to be asymptotically small as $|x| \to \infty$. In particular, these solutions will enter usual Lebesque spaces and the attractor constructed in Theorem 1.1 will attract in stronger norms.

Proof of Theorem 1.3. Recall that, in addition to the assumptions in Theorem 1.1, we assume now (1.15). Then, elliptic regularity for (5.1), see [2], implies that Φ in (5.1) satisfies

$$\Phi \in W^{2,r}(\mathbb{R}^N) \subset BUC_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$$

and, in particular,

$$\Phi(x) \to 0 \quad \text{as} \quad |x| \to \infty.$$

Consequently we get from (5.2) that for any bounded set of initial data $B \subset \dot{L}^q_U(\mathbb{R}^N)$ and for every $\varepsilon > 0$ there exists $R > 0$ and $T_0 = T_0(B)$ such that

$$|u(t,x;u_0)| \leq \varepsilon \quad \text{for all} \quad |x| \geq R,\ t \geq T_0,\ u_0 \in B. \tag{7.1}$$

From here, (5.11) and the attraction of $\mathcal{A}$ in W as in (5.7), we conclude at once parts i), ii), iii) of Theorem 1.3.

These describe what we can denote the "first stage" of the dynamics of (1.1). In order to describe the "second stage", we assume in what follows that the initial data satisfies

$$|u_0(x)| \leq \Phi(x), \quad x \in \mathbb{R}^N, \tag{7.2}$$

which implies $|u(t;u_0)| \leq \Phi$ for all $t \geq 0$ (see Proposition 5.1).

Then, from (1.12), the nonlinear term satisfies on such solution

$$|F(x,u)| \leq |g(x)| + \sum_{j=1}^{k} |m_j(x)||h_j(u)| + |f_0(x,u)| \leq |g(x)| + K\Big(\sum_{j=1}^{k} |m_j(x)| + 1\Big)|u|,$$

and hence

$$|F(x,u)| \leq |g(x)| + K\Big(\sum_{j=1}^{k} |m_j(x)| + 1\Big)\Phi(x) =: |g(x)| + \mathcal{V}(x)\Phi(x)$$

with $\mathcal{V} \in \dot{L}^{r_0}_U(\mathbb{R}^N)$ and $|g| \leq D \in L^r(\mathbb{R}^N)$.

As we assume $r_0 \geq r$, then from [2, Lemma 2.3],

$$\|\mathcal{V}\Phi\|_{L^r(\mathbb{R}^N)} \leq K_0 \|\mathcal{V}\|_{L^{r_0}_U(\mathbb{R}^N)} \|\Phi\|_{W^{2,r}(\mathbb{R}^N)},$$

thus

$$\|F(\cdot, u)\|_{L^r(\mathbb{R}^N)} \leq K_1, \quad \text{for all} \quad t \geq 0. \tag{7.3}$$

Using again that $r_0 \geq r$ and applying (4.6) (note that (4.6) now makes sense also in $L^r(\mathbb{R}^N)$; see [2]) for any $\tau > 0$ we get

$$\|u(t; u_0)\|_{W^{2,r}(\mathbb{R}^N)} \leq K_2 \quad \text{for all} \quad t \geq \tau.$$

In particular the attractor is bounded in $W^{2,r}(\mathbb{R}^N)$. □

Remark 7.1. *Observe that if $m_j \in L^\infty(\mathbb{R}^N)$, $j = 1, \dots, k$, and if $g \in L^\sigma(\mathbb{R}^N)$ for some $\sigma \geq r$, arguing as above instead of* (7.3) *we get*

$$\|F(\cdot, u)\|_{L^\sigma(\mathbb{R}^N)} \leq K_1 \tag{7.4}$$

for all $t \geq 0$. In particular, we get the bound on the attractor in $W^{2,\sigma}(\mathbb{R}^N)$, provided $r_0 \geq \sigma$.

Remark 7.2. *Note from (7.2) that for every $r \leq \sigma < \infty$ and for every $\varepsilon > 0$ there exists $R > 0$ such that*

$$\int_{|x| \geq R} |u(t, x; u_0)|^\sigma \, dx \leq \varepsilon \tag{7.5}$$

whenever $t \geq 0$. In particular, the attractor attracts these solutions in $L^\sigma(\mathbb{R}^N)$.

With this we can prove the following extra regularity of the attractor.

Corollary 7.3. *Assume $m_j \in L^\infty(\mathbb{R}^N)$, $j = 1, \dots, k$, and $g \in L^\sigma(\mathbb{R}^N)$ for some $r \leq \sigma \leq r_0$. Then the attactor attracts solutions for initial data that satisfy (7.2) in $W^{2\beta,\sigma}(\mathbb{R}^N)$ and is also compact in this space, for $0 < \beta < 1$.*

Proof: Now, let $\sigma \geq r$ be as in the estimates above, (7.4), (7.5), for the nonlinear term (so that we can always take at least $\sigma = r$). Then the uniform bounds in $W^{2,\sigma}(\mathbb{R}^N)$ on the solutions, imply that the attractor attracts these solutions and is compact in $W^{2\beta,\sigma}(\mathbb{R}^N)$ for $0 < \beta < 1$. Indeed, we do as in the proof of Theorem 5.5 in [2] to show first asymptotic compactness in $L^\sigma(\mathbb{R}^N)$ and then in $W^{2\beta,\sigma}(\mathbb{R}^N)$. □

Remark 7.4. *If the coeffcients m_j, $j = 1, \dots, k$, are genuinely $\dot{L}^{r_0}_U(\mathbb{R}^N)$ functions, the above results seem optimal. If however they are bounded functions, the arguments above work for bounded sets of initial data in $\dot{L}^q_U(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ and not only for initial data as in (7.2) but indeed in $L^\sigma(\mathbb{R}^N)$ for any $r \leq \sigma \leq r_0$ such that $g \in L^\sigma(\mathbb{R}^N)$. Note that from (4.5) we then get (7.5) for large $t \geq T_0(B)$ and we get also the bound* (7.4) *for all $t \geq T_0(B)$.*

Remark 7.5. *When $m_j = 0$, $j = 1, \dots, k$, we fall into the setting of* [2] *for initial data in $L^q(\mathbb{R}^N) \subset \dot{L}^q_U(\mathbb{R}^N)$, where the assumptions imply, in addition, that $\Phi \in L^q(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Hence the attractor constructed in* [2] *and the one constructed here coincide. Clearly one inclusion is obvious by invariance and the other one follows in the similar manner since the attractor in the larger space is, due to (7.2), bounded in $L^q(\mathbb{R}^N)$. Note that the attractor* *attracts in different norms solutions starting at $L^q(\mathbb{R}^N)$ or $\dot{L}^q_U(\mathbb{R}^N)$, but attracts in the same norms solutions satisfying (7.2).*

8. Minimal positive solutions

In this section we analyze the dynamics of nonnegative solutions of (1.1); in particular we complete the proof of Theorem 1.4.

In the simplest case, when $g(x) = f(x, 0) \geq 0$ is not identically zero, the existence of a minimal equilibrium follows easily. In fact we have that 0 is a subsolution of (1.1). Then, $u(t, x; 0)$ is increasing and, thanks to Proposition 5.4, the limit $\varphi_m = \lim_{t\to\infty} u(t; 0)$ exists in W as in (5.7) and $\varphi_m \in W^{2,r_0}_U(\mathbb{R}^N)$. Then we conclude that φ_m is the minimal equilibrium. If, in addition, the assumptions in Section 7 are met, then the convergence is uniform in $\mathbb{R}^N$.

A more interesting case is when 0 is equilibrium. In such a case we proceed with the proof of Theorem 1.4.

Proof of Theorem 1.4. For $R > 0$ we denote by λ_1^R the first eigenvalue of $-\Delta - \mathcal{M}$ in the ball B_R of radius R with Dirichlet boundary conditions. Then for large enough R we have $\lambda_1^R < 0$.

Let φ_M be the maximal solution in Theorem 1.2, which is now positive. Then φ_M restricted to B_R is a supersolution for the Dirichlet problem in B_R:

$$\begin{cases} u^R_t - \Delta u^R &= f(x, u^R) \quad \text{in} \quad B_R \\ u^R &= 0 \quad\quad\quad\;\; \text{on} \quad \partial B_R \\ u^R(0) &= v_0. \end{cases} \tag{8.1}$$

The solution of (8.1) starting at $u^R(0) = \varphi_{M|B_R}$ is globally bounded since

$$0 \leq u^R(t, x; \varphi_{M|B_R}) \leq \varphi_{M|B_R}.$$

Thus, from Theorem 4.2 in [17] we have that the minimal positive equilibrium φ^R_m for (8.1) exists and

$$0 \leq \varphi^R_m \leq \varphi_M \text{ in } B_R.$$

Note that φ^R_m is asymptotically stable from below for (8.1); i.e., for all nonzero $v_0 \in C_0(\overline{B_R})$, $0 \leq v_0 \leq \varphi^R_m$ in B_R, we have

$$u^R(t, x; v_0) \to \varphi^R_m(x) \quad \text{uniformly in } x \in B_R \text{ as } t \to \infty.$$

Even more, for all $0 \leq v_0 \in C(\overline{B_R})$, not identically zero,

$$\liminf_{t\to\infty} u^R(t, x; v_0) \geq \varphi^R_m(x) \quad \text{uniformly for} \quad x \in B_R. \tag{8.2}$$

Also, the extension by zero to $\mathbb{R}^N$ of φ^R_m, that we denote the same, belongs to $\dot{L}^q_U(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$ and is a subsolution for the elliptic problem associated to (1.1). Indeed, for any $0 \leq \eta \in \mathcal{D}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \nabla\varphi^R_m \nabla\eta = \int_{B_R} \nabla\varphi^R_m \nabla\eta = \int_{B_R} -\eta\Delta\varphi^R_m + \int_{\partial B_R} \eta \frac{\partial \varphi^R_m}{\partial n} \leq \int_{B_R} f(x, \varphi^R_m)\eta = \int_{\mathbb{R}^N} f(x, \varphi^R_m)\eta,$$

where we have used that $\frac{\partial \varphi^R_m}{\partial n} \leq 0$ on ∂B_R, $\eta \geq 0$ on ∂B_R, $f(x, 0) = 0$ and $\varphi^R_m = 0$ out of B_R.

From here, $u(t,x;\varphi_m^R)$ is monotonically increasing and we also have

$$0 \le u(t,x;\varphi_m^R(x)) \le u(t,x;\varphi_M(x)) = \varphi_M(x), \quad x \in \mathbb{R}^N \quad \text{for all} \quad t \ge 0.$$

Thus the monotonic limit

$$\varphi_m(x) := \lim_{t\to\infty} u(t,x;\varphi_m^R) \le \varphi_M(x), \quad x \in \mathbb{R}^N \tag{8.3}$$

exists for $x \in \mathbb{R}^N$ and in W as in Proposition 5.4. In particular, the limit in (8.3) is uniform in compact sets of $\mathbb{R}^N$.

We now show that φ_m is the minimal positive equilibrium. For this, given a nontrivial $u_0 \in BUC(\mathbb{R}^N)$, $0 \le u_0$, we set $v_0 = u_{0|B_R}$, which is nonzero for sufficiently large R. Then, we have

$$0 \le u^R(t,x;v_0) \le u(t,x;u_0), \quad x \in B_R \tag{8.4}$$

and extending by zero u^R to $\mathbb{R}^N$, (8.4) holds in $\mathbb{R}^N$. By (8.2), taking limits as t goes to infinity, we have

$$\varphi_m^R(x) \le \liminf_{t\to\infty} u^R(t,x;u_{0|B_R}) \le \liminf_{t\to\infty} u(t,x;u_0), \quad x \in B_R. \tag{8.5}$$

Let ψ be any equilibrium for (1.1). Then $\psi \in BUC(\mathbb{R}^N)$ and from (8.5) with $u_0 = \psi$ we have

$$\varphi_m^R \le \psi \quad \text{in} \quad B_R$$

and extending φ_m^R by zero to $\mathbb{R}^N$ the inequality holds in $\mathbb{R}^N$. Letting act the nonlinear semigroup on both sides and taking limits as $t \to \infty$, by (8.3), we have

$$\varphi_m \le \psi \quad \text{in} \quad \mathbb{R}^N.$$

Thus, φ_m is a minimal equilibria for (1.1).

For the asymptotic stability, take first $u_0 \in BUC(\mathbb{R}^N)$ non identically zero and $0 \le u_0 \le \varphi_m$. In fact we can assume that $u_{0|B_R}$ is positive, since otherwise we take $S(t)u_0$, for some small time, as initial data. We consider the restriction to $u_{0|B_R}$ extended by zero to $\mathbb{R}^N$. From (8.4) we have $u^R(t;u_{0|B_R}) \le \varphi_m$ for all t and $u^R(t;u_{0|B_R}) \to \varphi_m^R$ as $t \to \infty$.

Then, for $s > 0$, from the continuity condition (5.10) we infer that

$$\lim_{t\to\infty} S(s)u^R(t;u_{0|B_R}) = S(s)\lim_{t\to\infty} u^R(t;u_{0|B_R}) = S(s)\varphi_m^R \quad \text{in } \mathbb{R}^N.$$

Additionally, (8.4) implies

$$u(t+s,x;u_0) = S(s)u(t,x;u_0) \ge S(s)u^R(t,x;u_{0|B_R}) \quad \text{in} \quad \mathbb{R}^N.$$

Now, taking limit as $t \to \infty$ we have

$$\liminf_{t\to\infty} u(t,x;u_0) \ge (S(s)\varphi_m^R)(x) = u(s,x;\varphi_m^R) \quad x \in \mathbb{R}^N$$

and taking limit as $s \to \infty$ we have, by (8.3),

$$\liminf_{t\to\infty} u(t,x;u_0) \ge \varphi_m(x) \quad x \in \mathbb{R}^N. \tag{8.6}$$

Note that given any $u_0 \in \dot{L}_U^q(\mathbb{R}^N)$, $0 \le u_0 \le \varphi_m$, we have $0 \le u(t,x;u_0) \le \varphi_m(x)$ and by Proposition 5.4 the ω-limit set $\omega(u_0)$ exists in W and satisfies $\omega(u_0) \le \varphi_m$. But from (8.6),

$\omega(u_0) \geq \varphi_m$ and therefore $\omega(u_0) = \{\varphi_m\}$; that is $u(t,x;u_0) \to \varphi_m(x)$ in W as $t \to \infty$. In particular, the convergence is uniform on compact sets of $\mathbb{R}^N$. $\square$

Concerning the behaviour of the minimal solutions constructed above we prove

Proposition 8.1. *If the hypotheses of Theorem 1.4 are satisfied and φ_m^R denote the minimal positive equilibria of (8.1), then*

$$\lim_{R\to\infty} \varphi_m^R(x) = \varphi_m(x) \quad \text{in} \quad W^{2-\epsilon,\sigma}_{\text{loc}}(\mathbb{R}^N), \tag{8.7}$$

for all $q \leq \sigma < \infty$ and for every $\epsilon \in (0, 2]$.

Proof: As shown before, for all $R > 0$ large enough,

$$0 \leq \varphi_m^R \leq \varphi_m. \tag{8.8}$$

Furthermore, given $R_1 < R_2$, we have that $\varphi_m^{R_1}$ and $\varphi_m^{R_2}$ satisfy the same equation in B_{R_1}. Moreover, $\varphi_m^{R_2} > 0 = \varphi_m^{R_1}$ in ∂B_{R_1}. Thus, $\varphi_m^{R_1} \leq \varphi_m^{R_2}$. So φ_m^R is increasing as $R \to \infty$ and there exists the pointwise limit

$$\lim_{R\to\infty} \varphi_m^R(x) = \xi(x) \leq \varphi_m(x), \quad x \in \mathbb{R}^N. \tag{8.9}$$

Now, since $\varphi_m \in L^\infty(\mathbb{R}^N)$, we also have $\xi \in L^q_U(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and as a consequence of the Dominated Convergence Theorem,

$$\varphi_m^R \to \xi \text{ in } L^\sigma_{loc}(\mathbb{R}^N) \text{ as } R \to \infty \text{ for all } q \leq \sigma < \infty.$$

In what follows we prove that ξ is actually an element of $\dot{L}^q_U(\mathbb{R}^N)$. Recalling that $\xi \in L^\infty(\mathbb{R}^N)$, if we show that $\xi \in \dot{L}^\sigma_U(\mathbb{R}^N)$ for a certain $\sigma \in [1,\infty)$, we will immediately have this property for every $\sigma \in [1,\infty)$.

We fix $\epsilon > 0$, $y \in \mathbb{R}$ and choose any radius $R_{\epsilon,y} > 0$ for which $B(y,2) \Subset B_{R_{\epsilon,y}}$. Then by the monotonic convergence we have, for all $\sigma \geq 1$,

$$\|\xi - \varphi_m^{R_{\epsilon,y}}\|_{L^\sigma(B(y,1))} + \|\tau_z(\xi - \varphi_m^{R_{\epsilon,y}})\|_{L^\sigma(B(y,1))} < \frac{\epsilon}{2} \text{ whenever } |z| < 1. \tag{8.10}$$

From the characterization of Sobolev spaces as in [3, Remark 4.5] we have

$$\|\tau_z \varphi_m^{R_{\epsilon,y}} - \varphi_m^{R_{\epsilon,y}}\|_{L^\sigma(B(y,1))} \leq C|z| \|\nabla \varphi_m^{R_{\epsilon,y}}\|_{L^\sigma(B(y,2))} \text{ for } |z| < 1, \tag{8.11}$$

and we also have

$$\|\nabla \varphi_m^{R_{\epsilon,y}}\|_{L^\sigma(B(y,2))} \leq C\|\varphi_m^{R_{\epsilon,y}}\|_{L^\sigma(B(y,2))} + C\|\Delta \varphi_m^{R_{\epsilon,y}}\|_{L^\sigma(B(y,2))}.$$

Since also

$$\|\Delta \varphi_m^{R_{\epsilon,y}}\|_{L^\sigma(B(y,2))} = \|f(\cdot, \varphi_m^{R_{\epsilon,y}})\|_{L^\sigma(B(y,2))},$$

from (1.12) and (8.8), we get

$$\|\Delta \varphi_m^{R_{\epsilon,y}}\|_{L^\sigma(B(y,2))} \leq \|m_0 \varphi_m\|_{L^\sigma(B(y,2))} + \sum_{j=1}^{k} \|m_j h_j(\varphi_m)\|_{L^\sigma(B(y,2))} + \|f_0(\cdot, \varphi_m)\|_{L^\sigma(B(y,2))}.$$

As a consequence of the assumptions on m_j and f_0 in (1.12) and recalling that $\varphi_m \in L^\infty(\mathbb{R}^N)$ we take $\sigma = r_0$ and we now have that

$$\|\tau_z \varphi_m^{R_{\epsilon,y}} - \varphi_m^{R_{\epsilon,y}}\|_{L^{r_0}(B(y,1))} \leq C|z|, \quad |z| < 1, \tag{8.12}$$

where the constant C depends on the $L^{r_0}_U(\mathbb{R}^N)$-norms of m_j as well as on the $L^\infty(\mathbb{R}^N)$-bounds for $h_j(\varphi_m)$, $f_0(\cdot, \varphi_m)$, but is independent of $\epsilon > 0$, $y \in \mathbb{R}^N$, $|z| < 1$.

From (8.10) and (8.12) we now infer that

$$\|\tau_z \xi - \xi\|_{L^{r_0}(B(y,1))} \leq \epsilon \quad \text{whenever} \quad |z| \leq (2C)^{-1}\epsilon,$$

and since y is arbitrary we get $\xi \in \dot{L}^{r_0}_U(\mathbb{R}^N)$.

Now we prove that ξ is actually an equilibrium. Let $\eta \in C^\infty_c(\mathbb{R}^N)$ denote a function with compact support in $\mathbb{R}^N$ and $\mathcal{R} > 0$ be such that $\mathrm{supp}(\eta) \subset B_{\mathcal{R}}$. Then

$$\int_{B_{\mathcal{R}}} -\eta \Delta \varphi^R_m = \int_{B_{\mathcal{R}}} f(x, \varphi^R_m)\eta.$$

Integrating by parts, we get

$$\int_{\mathrm{supp}(\eta)} -\varphi^R_m \Delta \eta = \int_{\mathrm{supp}(\eta)} f(x, \varphi^R_m)\eta.$$

For the left hand side in the equation we can apply the Dominated Convergence Theorem and we have

$$\lim_{R\to\infty} \int_{\mathrm{supp}(\eta)} -\varphi^R_m \Delta \eta = \int_{\mathrm{supp}(\eta)} -\xi \Delta \eta.$$

For the right hand side, notice that on the one hand, using that $0 \leq \varphi^R_m \leq \varphi_m \in L^\infty(\mathbb{R}^N)$ and f_0 is locally Lipschitz, we have

$$|f_0(x, \varphi^R_m)| \leq \kappa |\varphi_m(x)| \tag{8.13}$$

for some constant $\kappa > 0$. On the other hand

$$|m_0(x)\varphi^R_m(x)| + |F(x, \varphi^R_m(x))| \leq K\big(1 + \sum_{j=0}^{k} |m_j(x)|\big)|\varphi_m(x)| \in \dot{L}^{r_0}_U(\mathbb{R}^N). \tag{8.14}$$

So, we can pass to the limit by the Dominated Convergence Theorem to get

$$\lim_{R\to\infty} \int_{\mathrm{supp}(\eta)} F(x, \varphi^R_m)\eta = \int_{\mathrm{supp}(\eta)} F(x, \xi)\eta.$$

Thus,

$$-\Delta \xi - m_0(x)\xi = F(x, \xi) \quad x \in \mathbb{R}^N$$

in the sense of distributions. Furthermore, from (8.14) we have $F(\cdot, \xi) \in L^{r_0}_U(\mathbb{R}^N)$.

Using now that $\xi \in \dot{L}^q_U(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for any $q \in [1, \infty)$ we conclude, as in Remark 3.3, that $F(\cdot, \xi) \in \dot{L}^\sigma_U(\mathbb{R}^N)$ for each $\sigma \in [1, r_0)$.

Thus, by elliptic regularity (see [3]), $\xi \in \dot{W}^{2,\sigma}_U(\mathbb{R}^N)$ for all $1 < \sigma < r_0$. In particular, ξ is an equilibrium for (1.1). Since $0 \leq \xi \leq \varphi_m$ we have $\xi = \varphi_m$.

We now show that φ^R_m converges to φ_m in $W^{2-\epsilon,\sigma}_{\mathrm{loc}}(\mathbb{R}^N)$. Given $L > 0$ let $0 \leq \chi \in C^\infty_c(B_{2L})$ be such that $\chi \equiv 1$ in B_L. Let $\eta = \varphi^R_m \chi$. Then, η solves

$$\begin{cases} -\Delta \eta - m_0(x)\eta &= H_{R,L}(x) \quad \text{in} \quad B_{2L} \\ \eta &= 0 \quad \text{on} \quad \partial B_{2L} \end{cases}$$

with $H_{R,L}(x) = -2\nabla\varphi_m^R\nabla\chi + F(x,\varphi_m^R)\chi - \varphi_m^R\Delta\chi$. Since $0 \le \varphi_m^R \le \varphi_m \in \dot{L}_U^q(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we have $||\varphi_m^R||_{L^\sigma(B_{2L})} \le C(L)$ for all $q \le \sigma < \infty$, not depending on R. Thus, by (8.13), (8.14),

$$||H_{R,L}||_{W^{-1,\sigma}(B_{2L})} \le C(L)$$

and by elliptic regularity theory

$$\eta \in W_0^{1,\sigma}(B_{2L}), \quad ||\eta||_{W_0^{1,\sigma}(B_{2L})} \le C(L)$$

for certain constant not depending on R. As a consequence $\{\varphi_m^R\}_R$ is a bounded set of $W^{1,\sigma}_{\mathrm{loc}}(\mathbb{R}^N)$.

But we can repeat the argument above taking now into account that, for all $q \le \sigma < \infty$, $||\nabla\varphi_m^R\nabla\chi||_{L^\sigma(B_{2L})} \le C(L)$ not depending on R. Thus, $||H_{R,L}||_{L^\sigma(B_{2L})} \le C(L)$. Therefore, $\{\varphi_m^R\}_R$ is a bounded set of $W^{2,\sigma}_{\mathrm{loc}}(\mathbb{R}^N)$. So, for every $\epsilon > 0$

$$\lim_{R\to\infty} \varphi_m^R = \varphi_m \quad \text{in} \quad W^{2-\epsilon,\sigma}(B_{2L}) \quad \text{for all} \quad L > 0;$$

that is,

$$\lim_{R\to\infty} \varphi_m^R = \varphi_m \quad \text{in} \quad W^{2-\epsilon,\sigma}_{\mathrm{loc}}(\mathbb{R}^N).$$

In particular, taking $\sigma > N/2$, we have $W^{2-\epsilon,\sigma}(B_{2L}) \subset C^\theta(B_{2L})$ and the convergence holds in $C^\theta(B_{2L})$ for some $\theta > 0$.

9. Uniqueness of positive equilibrium

In this section we prove Theorem 1.5. We remark that in [16] the maximal equilibrium of (1.1) was shown to be a unique positive steady state whenever $\frac{f(x,s)}{s}$ was decreasing with respect to $s > 0$ on a set of positive measure. This was obtained with the aid of integration by parts over large balls B_R with $R \to \infty$. In the locally uniform spaces such argument cannot be directly applied due to the fact that the solutions may not be in general close to zero at infinity.

A certain uniqueness criterion for the existence of positive equilibrium can be provided if concavity of $f(x,s)$ with respect to $s > 0$ is assumed for each $x \in \mathbb{R}^N$ (see [17, Remark 4.13]).

Lemma 9.1. *Suppose that the assumptions i), ii), iii) of Theorem 1.1 hold, $f(x,0) = 0$ and $\frac{\partial f}{\partial s}(x,\cdot)$ is nonincreasing in $\mathbb{R}^+$ for every $x \in \mathbb{R}^N$.*

If $\phi, \psi \in \dot{W}_U^{2,r_0}(\mathbb{R}^N)$ are two stationary solutions of (1.1) satisfying $0 \le \phi \le \psi$ and ϕ is linearly asymptotically stable, i.e., the solutions of the linearized equation

$$z_t - \Delta z = \frac{\partial f}{\partial s}(x,\phi)z, \ t > 0, \ x \in \mathbb{R}^N, \tag{9.1}$$

are asymptotically decaying, then ϕ and ψ coincide.

Proof. Letting $v = \psi - \phi \ge 0$ and using the assumptions for f we obtain

$$-\Delta v = f(x,\psi) - f(x,\phi) \le \frac{\partial f}{\partial s}(x,\phi)v.$$

Thus, v is a subsolution of (9.1) and we get

$$0 \le v \le z(t;\psi - \phi), \ t \ge 0.$$

Since $z(t; \psi - \phi) \to 0$ as $t \to \infty$, we have $\psi \equiv \phi$. $\square$

Lemma 9.1 and the existence of the maximal solution in Theorem 1.2 lead to the following conclusion.

Corollary 9.2. *Suppose that the hypothesis of Theorem 1.1 hold, $f(x, 0) = 0$ and $\frac{\partial f}{\partial s}(x, \cdot)$ is nonincreasing in $\mathbb{R}^+$ for every $x \in \mathbb{R}^N$.*

If $\varphi \in \dot{W}_U^{2,r_0}(\mathbb{R}^N)$ is a nonnegative stationary solution of the problem (1.1) *and is linearly asymptotically stable, then φ is the maximal equilibrium solution of* (1.1) *and thus φ is globally asymptotically stable from above; that is for all $u_0 \in \dot{L}_U^q(\mathbb{R}^N)$ satisfying $\varphi \leq u_0$ we have*

$$\lim_{t \to \infty} u(t, x; u_0) = \varphi \quad \textit{in } W,$$

where W denote either $C_{loc}^{\mu}(\mathbb{R}^N)$, with $2 - \frac{N}{r_0} > \mu > 0$, or the weighted space $W_\rho^{s,r_0}(\mathbb{R}^N)$ with $0 \leq s < 2$.

Recalling the existence of the minimal positive solution in Theorem 1.4, from Corollary 9.2, the proof of Theorem 1.5 is now straightforward.

Note that the assumption that $f(x, 0) = 0$ and $\frac{\partial f}{\partial s}(x, \cdot)$ is nonincreasing in $\mathbb{R}^+$ for every $x \in \mathbb{R}^N$, implies $f(x, s) \geq s \frac{\partial f}{\partial s}(x, s)$ for $(x, s) \in \mathbb{R}^N \times \mathbb{R}^+$ and hence $\frac{f(x,s)}{s}$ is then nonincreasing with respect to $s > 0$ for every $x \in \mathbb{R}^N$. Also, by assumptions in (1.12), the potential $\frac{\partial f}{\partial s}(x, \varphi)$ is in $L_U^{r_0}(\mathbb{R}^N)$ for any equilibrium of (1.1).

Observe that uiqueness of positive solutions has been obtained in [17], for the case of bounded domains, and in [16] for the case of unbouned ones, under the assumption that $\frac{f(x,s)}{s}$ is nonincreasing with respect to $s > 0$, strictly in a set of $x \in \mathbb{R}^N$ of positive measure. In both cases uniqueness is obtained by integration by parts using, in the latter case, that solutions of (1.1), decay in a suitable way as $|x| \to \infty$. Therefore, under the assumption in Theorem 1.3 we have

Corollary 9.3. *Assume Theorem 1.3 applies and*

$$\frac{f(x, s)}{s} \quad \textit{is nonincreasing in } s > 0,$$

strictly in a set of $x \in \mathbb{R}^N$ of positive measure,

Then there exists a unique positive equilibrium of (1.1), which is globally asymptotically stable for positive solutions, as in Theorem 1.5.

Proof: Note that r in Theorem 1.3 satisfies

$$r > \frac{N}{2} \geq \frac{2N}{N+3}.$$

and we can follow the proof of [16, Theorem 5.1]. $\square$

In general, however, compared with the result in [16], we have to ask here for some extra assumption on the linearized Schrödinger operator around the minimal equilibrium to obtain uniqueness.

Consier then a nonnegative stationary solution of the problem (1.1), $\varphi \in \dot{W}^{2,r_0}_U(\mathbb{R}^N)$, and assume as above that $f(x,0)=0$ and $\frac{\partial f}{\partial s}(x,\cdot)$ is nonincreasing in $\mathbb{R}^+$ for every $x \in \mathbb{R}^N$. Then we can write the linearized Schrödinger operator as

$$L(z) = -\Delta z - \frac{\partial f}{\partial s}(x,\varphi)z = -\Delta z - \frac{f(x,\varphi)}{\varphi}z + V(x)z = L_0(z) + V(x)z$$

where

$$V(x) = \frac{f(x,\varphi)}{\varphi} - \frac{\partial f}{\partial s}(x,\varphi) \geq 0, \quad V \in L^{r_0}_U(\mathbb{R}^N). \tag{9.2}$$

Now note that $z = \varphi$ is a bounded nonnegative solution of $L_0(z) = 0$. This implies that $\lambda = 0$ is the bottom spectrum of L_0, see e.g. [15]. Therefore we give below conditions to ensure that $V \geq 0$ is able of "moving the spectrum to the right". In such a case we obtain the linear asymptotic stablity of φ as desired. Of course this would hold trivially if $V(x) \geq a > 0$ for all $x \in \mathbb{R}^N$. As $V \geq 0$ may vanish somewhere, we have the following result of independent interest.

Lemma 9.4. *Assume $V_0, V \in L^{r_0}_U(\mathbb{R}^N)$ with $N/2 < r_0 < \infty$ are such that*

$$\lambda = 0 \quad \textit{is the bottom spectrum of} \quad L_0 = -\Delta + V_0(x)I$$

and $V \geq 0$ satisfies that there exists $c > 0$ and $\alpha > 0$ such that for any $y \in \mathbb{R}^N$

$$|\{x,\ V(x) \leq a\} \cap B(y,1)| \leq ca^{\alpha}$$

for all sufficiently small $a > 0$.

Then the bottom spectrum of $L = L_0 + V(x)I$ is strictly positive.

Proof: We use the following perturbation argument. For $a > 0$ small enough denote

$$V_1(x) = \max\{V(x), a\} \geq a > 0, \quad x \in \mathbb{R}^N$$

and

$$V_2(x) = V(x) - V_1(x) = \min\{0, V(x) - a\} \leq 0, \quad x \in \mathbb{R}^N.$$

Then $L = L_0 + V_1(x)I + V_2(x)I$ and the bottom spectrum of $L_1 = L_0 + V_1(x)I$ is not smaller than $a > 0$.

Now according to Lemma 2.2. in [2] we get that if $\|V_2\|_{L^{r_0}_U(\mathbb{R}^N)} = o(a)$ the result follows. But note that there exists $c_0 > 0$ such that for any $y \in \mathbb{R}^N$

$$\|V_2\|_{L^{r_0}(B(y,1))} \leq c_0 a^{1+\frac{\alpha}{r_0}} = o(a). \quad \square$$

Observe that the assumption of the Lemma does not allow $V(x)$ to have "flat vanishing" regions nor $V(x) \to 0$ as $|x| \to \infty$. In particular, for the problem (1.1), were V is given by (9.2) the Lemma above does not apply if $\varphi \to 0$ as $|x| \to \infty$. Note that this is just the case in [16] and in Corollary 9.3.

10. Examples and applications

In this section we discuss the applications of our pevious results to some model problems. In particular we discuss uniform and nonuniform attracting properties of the extremal equilibrium solutions.

Example 10.1. *Consider the nonlinear term of the form*

$$f(x,s) = m(x)s - s^3. \tag{10.1}$$

Then, with the notations in (1.12), we have $f_0(x,s) \equiv 0$, $m_0(x) = m(x)$, $m_1 \equiv -1$, $h_1(s) = s^3$ *and in (1.10) we have* $\frac{\partial f}{\partial s}(x,s) = m(x) - 3s^2 \leq L(x) := m(x)$*. From the Young's inequality we infer that for each* $s \in \mathbb{R}$, $x \in \mathbb{R}^N$ *and for any* $A > 0$

$$sf(x,s) \leq -As^2 + (m(x)+A)s^2 - s^4 \leq -As^2 + c|m(x)+A|^{\frac{3}{2}}|s|.$$

for some $c > 0$.

Hence (1.4) *is satisfied with* $C(x) = -A$ *and* $D(x) = |m(x)+A|^{\frac{3}{2}}$*. Hence, the solutions of* (1.11) *are asymptotically decaying.*

Consequently, Theorems 1.1 and 1.2 hold provided that

$$m \in \dot{L}^{r_0}_U(\mathbb{R}^N) \quad \textit{with} \quad r_0 > \frac{3N}{4}$$

and, in addition, $m(x) \leq \mathcal{K}$ *if* $q = 1$*. Thus, the global attractor and the extremal solutions exist for (10.1).*

Note that whenever the semigroup generated by $\Delta + m(x)$ *is asymptotically decaying then*

$$sf(x,s) \leq m(x)s^2 - s^4 \leq m(x)s^2$$

and then (1.4) holds with $C(x) = m(x)$ *and* $D = 0$*. Thus, zero is globally asymptotically stable equilibrium and the attractor reduces to* $\mathcal{A} = \{0\}$.

So, we will assume that the semigroup generated by $\Delta + m(x)$ *is not asymptotically decaying.*

Now, assume $m(x) = M_1(x) + M_2(x)$*, with* $M_1, M_2 \in \dot{L}^{r_0}_U(\mathbb{R}^N)$ *and such that the semigroup generated by* $\Delta + M_1(x)$ *is asymptotically decaying and* $M_2 \geq 0$*. Then, for some* $c > 0$,

$$sf(x,s) \leq M_1(x)s^2 + M_2(x)s^2 - s^4 \leq M_1(x)s^2 + cM_2(x)^{\frac{3}{2}}|s|.$$

Hence, we can take in (1.4) $C(x) = M_1(x)$ *and* $D(x) = cM_2(x)^{\frac{3}{2}}$*. Therefore, if*

$$M_2 \in L^{\sigma}(\mathbb{R}^N) \quad \textit{with} \quad \sigma > \frac{3N}{4} \tag{10.2}$$

then i)–iii) in Theorem 1.3 apply. If, additionally, $\sigma \leq \frac{3r_0}{2}$ *then iv) in Theorem 1.3 also applies.*

Also, if $\sigma(-\Delta - m(x)) \cap \mathbb{R}^- \neq \emptyset$ *then* $\sigma(-\Delta - m(x) + \delta) \cap \mathbb{R}^- \neq \emptyset$ *for a certain* $\delta > 0$*. Now*

$$f(x,s) = (m(x) - s^2)s \geq (m(x) - s_0^2)s =: \mathcal{M}(x)s \quad \textit{for} \quad 0 < s < s_0, \ s_0^2 \leq \delta$$

and $\mathcal{M} \in L^{r_0}_U(\mathbb{R}^N)$*. Consequently, via Theorem 1.4, there exists a minimal positive equilibrium* φ_m*, which is globally asymptotically stable from below for positive solutions.*

Concerning uniqueness of positive solutions, note that as $\frac{f(x,s)}{s} = m(x) - s^2$ *is strictly decreasing in* $s > 0$*, for* $x \in \mathbb{R}^N$*, then Corollary 9.3 applies if (10.2) holds.*

In the general case, when the semigroup generated by the linearized Schrödinger operator $\Delta + m(x) - 3\varphi_m^2(x)$ *decays asymptotically Theorem 1.5 ensures that* φ_m *is the unique positive steady state, which is globally asymptotically stable for nonnegative nontrivial solutions. According to Lemma 9.4 and (9.2), we have in this case*

$$V(x) = \frac{f(x, \varphi_m)}{\varphi_m} - \frac{\partial f}{\partial s}(x, \varphi_m) = 2\varphi_m^2(x).$$

But if

$$0 < \mathcal{K} \le m(x), \quad x \in \mathbb{R}^N \tag{10.3}$$

then, from [2, Corollary 6.2] *for any nonnegative nontrivial initial condition* $u_0 \in C_0^\infty(\mathbb{R}^N)$ *we have* $\sqrt{\mathcal{K}} \le \liminf_{t\to\infty} u(t, x; u_0)$ *uniformly in compact sets of* $\mathbb{R}^N$*. Consequently,* $V(x)$ *is strictly positive everywhere and Theorem 1.5 applies.*

In the particular case when

$$m(x) \equiv \mathcal{K} > 0 \tag{10.4}$$

all the above results apply and we also have

$$sf(x, s) = s(\mathcal{K}s - s^3) < 0, \ |s| \ge s_0 = \sqrt{\mathcal{K}}, \ x \in \mathbb{R}^N.$$

Then, solutions of (1.1) (with bounded initial data, say) are below the solutions of the ODE

$$\dot{z} = f(z).$$

Thus, solution for (1.1) with initial data above $\varphi_M = \sqrt{\mathcal{K}}$ *converge uniformly in* $\mathbb{R}^N$ *to* φ_M*.*

On the other hand for dimension $N = 1$*, from* [10, §5.4., Exercise 6]*, there are positive monotonic traveling waves of (1.1) connecting the trivial equlibrium* $u = 0$ *at* $t = -\infty$*, with* φ_M *at* $t = \infty$*. This shows that the convergence to* φ_M *from below is not uniform in space.*

A little more involved example is as follows.

Example 10.2. *For the nonlinearities of the form*

$$f(x, s) = m(x)s - n(x)s^3 + g(x) \tag{10.5}$$

assume that $g, m, n \in \dot{L}_U^{r_0}(\mathbb{R}^N)$*, with* $r_0 > N/2$*, and* n *is a nonnegative function. Then* (1.10) *holds with* $L(x) = m(x)$ *and* f *satisfies (1.12).*

Observe that if $n(x) \ge \delta > 0$ *for* $x \in \mathbb{R}^N$ *then the analysis can be carried out as in the former example. Hence, let us consider the case* $n(x) \ge 0$*. For simplicity we will also assume* $g(x) = 0$*.*

Assume then that $\sigma(-\Delta - m(x)) \cap \mathbb{R}^- \ne \emptyset$ *but* $m(x) = M_1(x) + M_2(x)$*, with* $M_1, M_2 \in \dot{L}_U^{r_0}(\mathbb{R}^N)$*, such that the semigroup generated by* $\Delta + M_1(x)$ *is asymptotically decaying and* $M_2 \ge 0$*. Then*

$$sf(x, s) \le M_1(x)s^2 + M_2(x)s^2 - n(x)s^4 \le M_1(x)s^2 + c\frac{M_2^{\frac{3}{2}}(x)}{n^{1/2}(x)}|s|.$$

Hence, we can take in (1.4) $C(x) = M_1(x)$ *and* $D(x) = c\frac{M_2^{\frac{3}{2}}(x)}{n^{1/2}(x)}$*. Consequently, if*

$$\frac{M_2^3(x)}{n(x)} \in L^\sigma_U(\mathbb{R}^N) \quad \textit{with} \quad \sigma > N/4.$$

Theorems 1.1 and 1.2 apply and we have the existence of the global attractor and of the extremal solutions.

Additionally, if

$$\frac{M_2^3(x)}{n(x)} \in L^\sigma(\mathbb{R}^N) \quad \textit{with} \quad N/4 < \sigma \le r_0/2$$

then Theorem 1.3 and the uniqueness result, Corollary 9.3, apply.

For the minimal positive equilibria, observe that for $0 < s < s_0$*, inequality* (1.16) *is satisfied with* $\mathcal{M} = m - s_0^2 n \in L^{r_0}_U(\mathbb{R}^N)$*. Then for sufficiently small* s_0*, from Lemma 2.2 in* [2]*, Theorem 1.4 applies.*

As for the uniqueness we have now

$$V(x) = \frac{f(x,\varphi_m)}{\varphi_m} - \frac{\partial f}{\partial s}(x,\varphi_m) = 2n(x)\varphi_m^2(x)$$

and uniqueness of positive solutions holds provided one can show the assumptions in Lemma 9.4.

Consider now the particular case $n(x) = m(x) \ge 0$ *in* (10.5)*; that is*

$$f(x,s) = m(x)(s - s^3). \tag{10.6}$$

Also assume the semigroup generated by $\Delta - m(x)$ *is asymptotically decaying, while* $\sigma(-\Delta - m(x)) \cap \mathbb{R}^- \neq \emptyset$*. Then we can take* $M_1 = -m$*,* $M_2 = 2m$ *and all the above applies with the only assumption that* $m \in \dot{L}^{r_0}_U(\mathbb{R}^N)$*.*

Note that in this case $\varphi = 1$ *is an equilibrium which is linearly stable, since the linearization reads* $\Delta - 2m(x)$*. Hence, Corollary 9.2 implies that* $\varphi = 1$ *is the maximal solution of (1.1).*

From the analysis above it is clear that we can apply the results in this paper to nonlinearities of the form

$$f(x,s) = m(x)s - n(x)s^{2k+1} + g(x)$$

with any $k \in \mathbb{N}$ and $n(x) \ge 0$, or even

$$f(x,s) = g(x) + m_0(x)s + \sum_{j=1}^{2k+1} m_j(x)s^j$$

with $m_{2k+1}(x) \ge 0$.

Now we turn into the question of the uniform in space attraction towards the extremal solutions. As can be easily seen from Example 10.1, such uniform convergence to the maximal solution, from above, is easily obtained when the nonlinearity f is independent of x and

$$sf(s) < 0, \quad |s| \ge s_0 > 0.$$

In fact in this case the maximal solution is the largest root of f. Also this example shows that, in general, one can not expect uniform convergence from below, even if it is the unique positive solution. This contrast with the results in [16].

In the next example we show that such uniform attraction does not hold in general. In fact, as we will see uniform attraction fails because attraction is "lost at infinity", i.e. as $|x| \to \infty$.

Example 10.3. *Assume in dimension $N = 1$ that $f(x, s)$ is such that $f(x, s) = 0$ only for $s = 0$, is odd and decreasing in s for each $x \in \mathbb{R}$ and, as a function of s, on $[-1, 1]$ it has a positive maximum $a(x)$ (at $s = -1$) such that $a(x) \to 0$ as $x \to \infty$.*

Then $f(x, s)s < 0$ for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^N$, and in fact $f(x, s) \leq C(x)s + D(x)$, for $s > 0$, with e.g. $C(x) = -\alpha$, $\alpha > 0$ and $D(x) = 1$.

Then for, say $u_0 = 1$, we have $u(t; u_0) \to 0$ as $t \to \infty$, monotonically and in compact sets of $\mathbb{R}$. In fact all solutions will converge to the unique equilibrium $\varphi = 0$.

Note however that for large enough x, we have $f(x, s) \geq g(s) = -\delta s$, with $0 \leq s \leq 1$, with $\delta > 0$ small.

Comparing with the Dirichlet problem in (R, ∞) with "nonlinearity" $g(s)$, we get that given any $T > 0$ there exists R large enough such that $u(t, x; u_0) \geq 1/2$ on $0 \leq t \leq T$ and $x \geq R$.

It is clear from the example above that a similar construction is possible for a nonlinear term $f(x, s)$ for which the asymptotic behavior is nontrivial. In fact, assume that for all $x \in \mathbb{R}$, $f(x, s)$ is odd in s and on $[-1, 1]$ it is equal to $s - s^3$. Above $s = 1$ assume f is strictly decreasing in s and tends to $-\infty$, while in $[1, 2]$ it has a negative minimum $j(x) \to 0$ as $|x| \to \infty$. Then it is not difficult to prove that $\varphi_M = 1$ but it does not attract uniformly from above (nor from below).

On the other hand there is another general situation in which uniform attraction from above to φ_M holds true. This happens when the function Φ in (5.1) and (5.11) captures the right asymptotic behavior at $|x| \to \infty$ of φ_M, that is when

$$0 \leq \Phi(x) - \varphi_M(x) \to 0 \quad \text{as} \quad |x| \to \infty.$$

Note that this is precisely the case in [16] and Theorem 1.3.

Indeed in such a case, from (5.2) and the asymptotic compactness in Proposition 5.4, it is easy to get the result.

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