PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 5, May 1997, Pages 1349–1353
S 0002-9939(97)03733-7

POLYNOMIAL CONTINUITY ON ℓ_1

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. A mapping between Banach spaces is said to be polynomially continuous if its restriction to any bounded set is uniformly continuous for the weak polynomial topology. A Banach space X has property (RP) if given two bounded sequences $(u_j), (v_j) \subset X$, we have that $Q(u_j) - Q(v_j) \to 0$ for every polynomial Q on X whenever $P(u_j - v_j) \to 0$ for every polynomial P on X; i.e., the restriction of every polynomial on X to each bounded set is uniformly sequentially continuous for the weak polynomial topology. We show that property (RP) does not imply that every scalar valued polynomial on X must be polynomially continuous.

Throughout, X and Y are Banach spaces, X^* the dual of X, B_X its closed unit ball, S_X its unit sphere, and $\mathbf{N}$ the set of natural numbers. Given $k \in \mathbf{N}$, we denote by $\mathcal{P}(^kX, Y)$ the space of all k-homogeneous (continuous) polynomials from X into Y; $\mathcal{L}_s(^kX, Y)$ is the space of all (continuous) symmetric k-linear mappings from $X^k := X \times \overset{(k)}{\cdots} \times X$ into Y. Whenever Y is omitted, it is understood to be the scalar field $\mathbf{K}$ (real $\mathbf{R}$ or complex $\mathbf{C}$). We identify $\mathcal{P}(^0X) = \mathbf{K}$, and denote $\mathcal{P}(X) := \sum_{k=0}^{\infty} \mathcal{P}(^kX)$. For the general theory of polynomials on Banach spaces, we refer to [6]. As usual, e_n stands for the sequence $(0, \dots, 0, 1, 0, \dots)$ with 1 in the nth position.

To each polynomial $P \in \mathcal{P}(^kX, Y)$ we can associate a unique symmetric k-linear mapping $\hat{P} \in \mathcal{L}_s(^kX, Y)$ so that $P(x) = \hat{P}(x, \dots, x)$ for all $x \in X$, and a (bounded linear) operator $T_P : X \to \mathcal{L}_s(^{k-1}X, Y)$ given by

$$T_P(x)(x_1, \dots, x_{k-1}) = \hat{P}(x, x_1, \dots, x_{k-1}).$$

Following [1], we say that a mapping $f : X \to Y$ is *polynomially continuous* (*P-continuous,* for short) if, for every $\epsilon > 0$ and bounded $B \subset X$, there are a finite set $\{P_1, \dots, P_n\} \subset \mathcal{P}(X)$ and $\delta > 0$ so that $\|f(x) - f(y)\| < \epsilon$ whenever $x, y \in B$ satisfy $|P_j(x - y)| < \delta$ $(1 \le j \le n)$.

Clearly, the definition may be restated assuming that the polynomials $P_1, \dots, P_n$ are homogeneous.

Suppose we require the polynomials $\{P_1, \dots, P_n\} \subset \mathcal{P}(X)$ in the above definition to be of degree one, i.e., to be continuous linear forms on X. Then we obtain that f is weakly uniformly continuous on bounded subsets, a notion that has been studied

Received by the editors October 30, 1995.
1991 *Mathematics Subject Classification.* Primary 46E15; Secondary 46B20.
Key words and phrases. Polynomials on Banach spaces, weak polynomial topology, polynomials on ℓ_1.
The first author was supported in part by DGICYT Project PB 94–1052 (Spain), and the second and third authors by DGICYT Project PB 93–0452 (Spain).

by many authors (see [1]). Since an operator is compact if and only if it is weakly (uniformly) continuous on bounded sets [2, Proposition 2.5], every compact operator is P-continuous. If a polynomial is weakly (uniformly) continuous on bounded sets (such as every scalar valued polynomial on c_0), then it is clearly P-continuous.

We shall need the following result:

Proposition 1. *A polynomial P is P-continuous if and only if so is the associated operator T_P.*

Proof. Suppose $P \in \mathcal{P}(^kX, Y)$ is P-continuous. Given $\epsilon > 0$, we can find $\delta > 0$ and $\{P_1, \dots, P_n\} \subset \mathcal{P}(X)$ so that $\|P(x) - P(y)\| < \epsilon$ whenever $|P_j(x - y)| < \delta$ for all $1 \le j \le n$ and $x, y \in B_X$.

Assume x, y satisfy the above conditions, and $z_1, \dots, z_{k-1} \in B_X$. The polarization formula [6, Theorem 1.10] yields:

$$\begin{aligned}
&(T_P(x) - T_P(y))\,(z_1, \dots, z_{k-1}) \\
&\quad = \hat{P}(x, z_1, \dots, z_{k-1}) - \hat{P}(y, z_1, \dots, z_{k-1}) \\
&\quad = \frac{k^k}{k!2^k} \sum_{\epsilon_j = \pm 1} \epsilon_1 \cdots \epsilon_k \left[P\left(\frac{\epsilon_1 x + \epsilon_2 z_1 + \cdots + \epsilon_k z_{k-1}}{k} \right) \right. \\
&\qquad\qquad\qquad \left. - P\left(\frac{\epsilon_1 y + \epsilon_2 z_1 + \cdots + \epsilon_k z_{k-1}}{k} \right) \right].
\end{aligned}$$

Assuming that every P_j is homogeneous, we have

$$\begin{aligned}
&\left| P_j \left(\frac{\epsilon_1 x + \epsilon_2 z_1 + \cdots + \epsilon_k z_{k-1}}{k} - \frac{\epsilon_1 y + \epsilon_2 z_1 + \cdots + \epsilon_k z_{k-1}}{k} \right) \right| \\
&\quad < \; |P_j(\epsilon_1 x - \epsilon_1 y)| \\
&\quad = \; |P_j(x - y)| \\
&\quad < \; \delta
\end{aligned}$$

for $1 \le j \le n$, and so

$$\|T_P(x) - T_P(y)\| \le \frac{\epsilon k^k}{k!} \, .$$

Conversely, let T_P be P-continuous. For $0 < \epsilon < 1$, there is $\delta > 0$ and $\{P_1, \dots, P_n\} \subset \mathcal{P}(X)$ so that $\|T_P(x) - T_P(y)\| < \epsilon$, whenever $|P_j(x - y)| < \delta$ for any $1 \le j \le n$ and $x, y \in B_X$. For such x, y we have

$$\begin{aligned}
&\|P(x) - P(y)\| \\
&\quad \le \; \|\hat{P}(x, \dots, x) - \hat{P}(x, y, x, \dots, x)\| + \|\hat{P}(x, y, x, \dots, x) - \hat{P}(x, y, y, x, \dots, x)\| \\
&\qquad + \cdots + \|\hat{P}(x, y, \dots, y) - \hat{P}(y, \dots, y)\| \\
&\quad = \; \|(T_P(x) - T_P(y))(x, \dots, x)\| + \|(T_P(x) - T_P(y))(x, y, x, \dots, x)\| \\
&\qquad + \cdots + \|(T_P(x) - T_P(y))(y, \dots, y)\| \\
&\quad < \; k\epsilon \, ,
\end{aligned}$$

and the proof is complete. □

We say that a net $(x_\alpha) \subset X$ converges to x in the *weak polynomial topology* (*pw-topology,* for short) [3, §6] if for every $P \in \mathcal{P}(X)$ we have $P(x_\alpha) \to P(x)$.

It is clear that a mapping $f : X \to Y$ is *pw*-continuous on bounded sets if and only if for every $x \in X$, $\epsilon > 0$ and bounded $B \subset X$ with $x \in B$, there are $\delta > 0$ and

$\{P_1, \dots, P_n\} \subset \mathcal{P}(X)$ so that we have $\|f(x) - f(y)\| < \epsilon$ whenever $|P_j(x - y)| < \delta$ for $1 \le j \le n$ and $y \in B$. Obviously, an operator is P-continuous if and only if it is *pw*-continuous on bounded sets.

We now relate the P-continuity with property (RP) of Aron, Choi and Llavona [1]. We say that X has *property* (RP) if given two bounded sequences (u_j) and (v_j) in X, we have that $Q(u_j) - Q(v_j) \to 0$ for every $Q \in \mathcal{P}(X)$ whenever $P(u_j - v_j) \to 0$ for every $P \in \mathcal{P}(X)$.

Every superreflexive space and every space with the DPP not containing ℓ_1 have property (RP) [1]. Clearly, if every scalar valued (continuous) polynomial on X is P-continuous, then X has property (RP). It is proved in [1] that $C[0,1]$, $L_1[0,1]$ and $L_\infty[0,1]$ do not satisfy property (RP), and that there are 3-homogeneous polynomials on the spaces $C[0,1]$ and $L_\infty[0,1]$ which are not P-continuous. Similarly, there is a non-P-continuous 2-homogeneous polynomial on $L_1[0,1]$.

It is natural to ask whether property (RP) implies that every scalar valued polynomial is P-continuous. We show that the answer is no by giving examples of polynomials on ℓ_1 which are not P-continuous. We first need to construct a *pw*-null net in the sphere of ℓ_1. We need a previous lemma.

Lemma 2. *Let U be a weak zero neighbourhood in ℓ_1. Then, for each $m \in \mathbf{N}$ we can find $x = (x_n) \in S_{\ell_1} \cap U$ and $r > m$ so that $x_n = 0$ whenever $n < m$ and $n > r$.*

Proof. We can find $\xi_1, \dots, \xi_k \in B_{\ell_\infty}$ and $\epsilon > 0$ such that

$$U \supseteq \{x \in \ell_1 : |\xi_j(x)| < \epsilon \text{ for } 1 \le j \le k\} .$$

Let $\xi_j = \left(\xi_j^n\right)_{n=1}^\infty$. There is an infinite set $A \subset \mathbf{N}$ so that $\left|\xi_j^p - \xi_j^q\right| < 2\epsilon$ whenever $1 \le j \le k$ and $p, q \in A$. Fix $p, q \in A$ $(m \le p < q)$, and set $x := (e_p - e_q)/2$ and $r = q$. Then $|\xi_j(x)| = \left|\xi_j^p - \xi_j^q\right|/2 < \epsilon$ for $1 \le j \le k$, and the proof is complete. □

The following two results use the idea of [4].

Lemma 3. *Let $\mathcal{F}$ be a finite family of continuous symmetric multilinear forms on ℓ_1, $\epsilon > 0$ and $N \ge 1$. Then there exist $x_1, \dots, x_N \in S_{\ell_1}$, with disjoint supports, such that $|F(x_{i_1}, \dots, x_{i_m})| < \epsilon$ whenever $F \in \mathcal{F}$ is an m-form and $i_1, \dots, i_m$ are distinct indices between 1 and N.*

Proof. Since each $F \in \mathcal{F}$ is symmetric, it is enough to obtain the estimate when $i_1 < \dots < i_m$.

By Lemma 2, we can find $n_1 \in \mathbf{N}$ and $x_1 \in S_{\ell_1}$, having all but the first n_1 coordinates equal to zero, so that $|F(x_1)| < \epsilon$ for all $F \in \mathcal{F} \cap \ell_1^*$. Again by Lemma 2, we can choose $n_2 \in \mathbf{N}$ and $x_2 \in S_{\ell_1}$ having disjoint support with x_1 and all but the first n_2 coordinates equal to zero, so that $|F(x_2)| < \epsilon$ for all $F \in \mathcal{F} \cap \ell_1^*$, and $|F(x_1, x_2)| < \epsilon$ for all $F \in \mathcal{F} \cap \mathcal{L}_s(^2\ell_1)$. In this way, we obtain x_j's with disjoint supports, so that $|F(x_{i_1}, \dots, x_{i_m})| < \epsilon$ for all $F \in \mathcal{F} \cap \mathcal{L}_s(^m\ell_1)$ and all $i_1 < \dots < i_m$. □

Theorem 4. *There is a pw-null net in S_{ℓ_1}.*

Proof. It is enough to show that for every finite family $\mathcal{F} \subset \mathcal{P}(\ell_1)$ and $\epsilon > 0$, there is an $x \in S_{\ell_1}$ so that $|P(x)| < \epsilon$ for all $P \in \mathcal{F}$.

Fix N large, choose $x_1, \dots, x_N \in S_{\ell_1}$ with disjoint supports satisfying the conditions of Lemma 3 for the family $\{\hat{P} : P \in \mathcal{F}\}$ of symmetric multilinear forms, and set

$$x := \frac{1}{N}(x_1 + \dots + x_N) \in S_{\ell_1} .$$

If $P \in \mathcal{F} \cap \mathcal{P}(^m\ell_1)$, we write

$$P(x) = \frac{1}{N^m} \sum_{i_1,\dots,i_m=1}^{N} \hat{P}\left(x_{i_1},\dots,x_{i_m}\right) = \Sigma_1 + \Sigma_2 ,$$

where Σ_1 is the sum over m-tuples of distinct indices, and Σ_2 is the sum over the remaining indices.

By Lemma 3, $|\Sigma_1| < \epsilon/2$. Since there are $N^m - N(N-1)\cdots(N-m+1)$ summands in Σ_2, we obtain

$$|\Sigma_2| \le \left[1 - \left(1 - \frac{1}{N}\right)\cdots\left(1 - \frac{m-1}{N}\right)\right] \cdot \|\hat{P}\| < \frac{\epsilon}{2} ,$$

for N large enough. □

As a consequence, if X contains a copy of ℓ_1, then the unit sphere of X contains a *pw*-null net as well. We now give the main result.

Theorem 5. *For every $k \in \mathbf{N}$ $(k \ge 2)$, there is a k-homogeneous scalar valued polynomial on ℓ_1 which is not P-continuous.*

Proof. Suppose first that $k = 2$ and ℓ_1 is constructed over the real numbers. We need a sequence $(x_j) \subset \ell_\infty$, equivalent to the ℓ_1-basis, such that $x_i^j = x_j^i$ for all $i, j \in \mathbf{N}$, where $x_j^i := x_j(e_i)$ (then we say that the sequence is *symmetric*).

We select a Rademacher-like sequence $(y_j) \subset \ell_\infty$, taking $y_1 := (1, -1, 1, -1, \dots)$, and letting y_j be the sequence consisting of infinitely many times the following block of 2^j integers:

$$1, \overset{(2^{j-1})}{\dots}, 1, -1, \overset{(2^{j-1})}{\dots}, -1 .$$

Clearly, (y_j) is 1-equivalent to the unit vector basis of ℓ_1; i.e., for every finite set of real numbers $\alpha_1, \dots, \alpha_n$, we have

$$\sum_{j=1}^{n} |\alpha_j| = \left\| \sum_{j=1}^{n} \alpha_j y_j \right\|_\infty . \tag{1}$$

If we take $x_1 := y_1$ and, for $j > 1$, modify the first $j-1$ coordinates of y_j in the obvious way, then we get a symmetric sequence (x_j) which is still 1-equivalent to the unit vector basis of ℓ_1.

Now, define an operator $T : \ell_1 \to \ell_\infty$ by $T(e_j) := x_j$. Since T is an embedding, we conclude from Theorem 4 that it is not P-continuous. Therefore, by Proposition 1, the 2-homogeneous polynomial $P : \ell_1 \to \mathbf{R}$ given by $P(y) = (T(y))(y)$ for $y \in \ell_1$ is not P-continuous.

The same sequence can be used in the complex case, since, for every finite set of complex numbers $\alpha_1, \dots, \alpha_n$, we have (see [5, XI, Proposition 4])

$$\sum_{j=1}^{n} |\alpha_j| \le 4 \left\| \sum_{j=1}^{n} \alpha_j y_j \right\|_\infty . \tag{2}$$

The sequence (y_j) will be used in the case $k > 2$ as well. Letting

$$A_j\left(e_{i_2},\dots,e_{i_k}\right) := \begin{cases} y_j^{i_2+\cdots+i_k} & \text{if } j \le \min\{i_2,\dots,i_k\} , \\ y_{i_r}^{j+i_2+\cdots+i_{r-1}+i_{r+1}+\cdots+i_k} & \text{if } i_r = \min\{i_2,\dots,i_k\} < j , \end{cases}$$

we obtain $A_j \in \mathcal{L}_s({}^{k-1}\ell_1)$. Moreover, the sequence (A_j) is equivalent to the unit vector basis of ℓ_1. Indeed, given real numbers $\alpha_1, \ldots, \alpha_n$, using (1), choose $i_2, \ldots, i_k \in \mathbf{N}$ such that $n \leq \min\{i_2, \ldots, i_k\}$ and

$$\sum_{j=1}^{n} |\alpha_j| = \left| \sum_{j=1}^{n} \alpha_j y_j^{i_2+\cdots+i_k} \right| = \left\| \sum_{j=1}^{n} \alpha_j A_j \right\| .$$

Note that the sequence (A_j) is symmetric in the sense that $A_{i_1}(e_{i_2}, \ldots, e_{i_k})$ is invariant under permutation of the indices $i_1, \ldots, i_k$. In the complex case we proceed similarly, using (2) in place of (1). In both cases we define $T : \ell_1 \to \mathcal{L}_s({}^{k-1}\ell_1)$ by $T(e_j) = A_j$. Then the polynomial $P \in \mathcal{P}({}^k\ell_1)$ given by

$$P(\alpha) := \sum_{i_1,\ldots,i_k=1}^{\infty} \alpha_{i_1} \cdots \alpha_{i_k} A_{i_1}(e_{i_2}, \ldots, e_{i_k}) , \quad \text{for } \alpha = (\alpha_j)_{j=1}^{\infty} \in \ell_1 ,$$

is not P-continuous, since the associated operator T is an isomorphism. $\square$

The authors wish to thank J. A. Jaramillo, M. Lacruz and S. Troyanski for helpful discussions.

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