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Entrelazamiento en teorías de campos con simetrías conformes

Entanglement in conformal field theories

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# Entrelazamiento en teorías conformes de campos

## Resumen:

El entrelazamiento es una de las propiedades características de los sistemas cuánticos. La medición del entrelazamiento se realiza a través de la entropía de Von Neumann. En esta tesis se estudiará el entrelazamiento cuántico en teorías de campos cuánticos, generalizando el estudio de la entropía de entrelazamiento que se hace en mecánica cuántica. El enfoque será el estudio de dicha entropía en teorías de campos conformes. Estas teorías están definidas a partir del grupo de simetrías conformes, que extiende el grupo de Poincaré incluyendo transformaciones de escala y las denominadas transformaciones conformes especiales. Gracias a estas transformaciones será posible encontrar una formulación exacta de la entropía de entrelazamiento en el caso de superficies de entrelazamiento esféricas, o sea, en el caso en el que una superficie esférica divida el espaciotiempo en dos partes y estemos interesados en el entrelazamiento de los campos entre las dos partes.

En este trabajo se darán dos ejemplos de referencia del cálculo de la entropía de entrelazamiento: uno a partir de teorías de campos libres y uno a partir de teorías definidas por medio de la correspondencia AdS/CFT. Esta correspondencia establece una dualidad entre ciertas teorías conformes y teorías de gravedad definidas en un espacio Anti de Sitter. Es útil pensar que la teoría de campos vive en la frontera del espacio Anti de Sitter y por esta razón, este tipo de teorías se denominan teorías holográficas. En todos los casos, daremos una discusión general de la entropía de entrelazamiento, y luego calcularemos dicha entropía en las dos teorías de referencia. Para las teorías holográficas habrá una conexión entre la entropía de agujeros negros y la entropía de entrelazamiento. Esta conexión forma parte del conjunto de observaciones que 'entrelazan' la gravedad y información cuántica.

## Abstract:

Quantum entanglement is one of the distinguishing properties of quantum systems. Its measurement is achieved through the Von Neumann entropy. In this thesis we will study entanglement entropy in quantum field theories (QFTs), generalizing what is usually done in quantum mechanics. Of particular interest to us will be the study of entanglement entropy in conformal field theories (CFTs), which are quantum field theories invariant under an extension of the Poincaré group, the conformal group, that also includes scale transformation and the so called special conformal transformations. Thanks to this extended symmetry it is possible to give an exact formulation of the entanglement entropy in the case in which a sphere divides spacetime into two regions, and one is interested in computing the entanglement of the fields between the two regions.

In this work we will give two examples for computation of the entanglement entropy: one for free theories, and one for theories defined through the AdS/CFT correspondence. This correspondence provides a duality between a class of CFTs and a class of gravitational theories living in a higher dimensional Anti de Sitter spacetime. It is useful to think about the CFT as living at the boundary of the AdS space, and so these theories are called holographic. We will give a general discussion about the entanglement entropy for spherical entangling surfaces, applicable to any theory, and then compute it for our reference theories. For holographic theories there will be a connection between black hole entropy and entanglement entropy. This connection is part of a number of observations that have been recently made to understand gravity from the point of view of quantum information.

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## 1 Introduction

In quantum mechanics, given a collection of product states, entangled states are non-separable states such that the aforementioned product states are found to be in a non trivial superposition. For this reason, entangled states display non classical correlation, therefore, the theory of entanglement provides a fundamental language for describing correlations in many body systems and understanding the emergence of collective phenomena.

A simple question one could ask is the following: divide a quantum system into two parts  $A$  and  $A^c$ , “How entangled are the two parts? is it possible to quantify the entanglement between the two parts?”. In modern physics this question is answered by the entanglement entropy. In particular, in the case of a finite dimensional Hilbert space, say  $\rho_A$  is the reduced density matrix of the sub-system  $A$ , obtained by tracing over the sub-system  $A^c$ , the entanglement entropy between the two parts is defined unambiguously as the Von-Neumann entropy of  $\rho_A$ . This is

$$S_{VN} = -\text{Tr}_A[\rho_A \log \rho_A]. \quad (1.1)$$

In this thesis we will be interested in probing entanglement for quantum fields. To understand the novelties in this problem, compared to the analogous one in quantum mechanics, let us review some basic properties of a Quantum Field Theory (QFT). First of all a QFT is defined by set of fields  $\{\phi(x), \dots\}$ , living on Minkowski spacetime. These fields act as operators on an Hilbert space of physical states  $\mathcal{H}$ , and the action of the fields commutes for space-like separations. The Hilbert space  $\mathcal{H}$  can be given as the tensor product of local Hilbert spaces  $\mathcal{H}_x$  at each point, namely

$$\mathcal{H} = \bigotimes_x \mathcal{H}_x \quad (1.2)$$

What makes a field theory novel compared to quantum mechanics is the existence of space. Therefore, what we will be interested in is the phenomenon of entanglement across spacetime. To start with, say  $A$  is a region of space at fixed time,  $\partial A$  will be called the entangling surface, we will consider

$$\mathcal{H}_A = \bigotimes_{x \in A} \mathcal{H}_x \quad ; \quad \mathcal{H}_{A^c} = \bigotimes_{x \in A^c} \mathcal{H}_x \quad (1.3)$$

While this is not the only possible way to factorize the Hilbert space, it is the simplest one we can use to explore entanglement of quantum fields in a way that involves spacetime.

There are important subtleties when we consider entanglement for quantum fields. First the density matrix is now an “infinite” object that needs to be regularized. Secondly, the log of such an infinite density matrix has to be properly defined. Finally, the fields not only live on space but also in time. It is the purpose of this work to review the computation of the entanglement entropy for spherical entangling surfaces, and show how the above subtleties are resolved in two classes of field theories: free field theories and a special class of interacting field theories known as holographic conformal field theories.

Before motivating the use of conformal symmetry (an extension of the Poincare symmetry), we would like to provide some intuition about the result that we expect to find. To this end it is useful to discretize space and think about the QFT living on the lattice. As in the general setup described above, we consider an entangling surface  $\partial A$  and define a discrete version of  $\mathcal{H}_A$  and  $\mathcal{H}_{A^c}$ . Also on the lattice the axioms of the QFT give constraints on the vacuum: it must respect the spatial structure of the lattice, and preserve some notion of locality, for example, we expect to find only nearest neighbor interactions. This locality manifests itself in the pattern of entanglement, and what we expect is that most of the entanglement is short-ranged, stored in between nearby lattice sites. Thus we expect that the entanglement entropy will be proportional to the number of “bonds” between neighboring sites that are cut by the entangling surface. This means that the entanglement entropy will have a leading contribution that grows like the area of  $\partial A$ . In the continuum limit this will give a short distance divergence whose coefficient is proportional to the area of  $\partial A$ .

Conformal symmetry is an extension of the Poincare group that provides additional structure to the field theory, in particular, the vacuum of a conformal field theory is invariant under conformal transformation. Conformal transformations will allow us to give a concrete (rather than formal) description of the density matrix, and for this reason we will review them as part of this project.

Free field theories are conformal invariant. Now, even though free fields do not interact, by definition, we will see that entanglement across spacetime is non trivial.

Besides free fields, there is another class of CFTs that offer analytic control, under certain approximations, and are very interesting. These are the so called holographic CFTs, that starting from the late 90s have received an incredible amount of attention, since they realize the so called “holographic principle”. The holographic principle refers to the general assertion that the dynamics of a system that lives in a region of space with a boundary is encoded by a lower-dimensional theory that lives on that boundary. While the idea of the holographic principle was known it lacked of a concrete realization until Maldacena provided a precise framework in 1998 [1], within String Theory. Although String Theory remains the only known way of constructing examples of the AdS/CFT correspondence, we will not discuss String Theory in this project, but rather, we will take an axiomatic point of view, by stating what correspondence exists between observables of our interest in the CFTs, and in the theory of gravity, once the parameters are properly identified. It will turn out that the computation of the entanglement entropy in theory is deeply related to the computation of the Bekenstein–Hawking entropy for black holes [2] [1]

## 1.1 Goals and Methodology

This thesis aims at providing two illustrative examples of the calculation of entanglement entropy in conformal field theories (CFTs). Compared to Poincare invariant quantum field theories, those that we study during the regular courses on QFTs, CFTs are defined to be invariant under an extended symmetry group, called the conformal group. They play a crucial role in theoretical physics, since they describe the end points of Wilson RG flow transformations, critical phenomena in condensed matter physics, and the AdS/CFT correspondence [3]

This Bachelor's thesis is divided in three main sections. In the first part we discuss the problem of computing entanglement entropy in QFTs, and CFTs, and then, in the other two parts we discuss concrete computations: in AdS/CFT and free field theory. Along the way we give a short introduction to the AdS/CFT correspondence, for what concerns our computation. The holographic techniques we follow was developed by Casini, Huerta and Myers [4] while the free theory computation is based on the work of Klebanov, Pufu et al. [5] using results from spectral theory and the study of solvable of partial differential equations (PDEs). In sum, this work will use the knowledge acquired in the study of Quantum Field Theory, Differential Geometry, Statistical Physics, General Relativity.

This Bachelor thesis originality lies in the use and understanding of extra curricula advanced techniques, such as the path integral formulation of QFTs, the study of basic aspects of CFTs, the geometry of AdS spaces and the study of techniques from spectral theory. This was achieved through the realization of supervised exercises alongside symbolic calculations in *Mathematica* in order to reproduce and gain a deeper understanding of the techniques involved.

## 2 Preliminaries

### 2.1 Entanglement in Quantum Field theory

The partition function of a quantum field theory is computed by a path integral over all possible field configurations:

$$Z = \int \mathcal{D}\phi \exp\left(-\int d^d x \mathcal{L}[\phi(x)]\right) \quad (2.1)$$

Here  $S[\phi] = \int d^d x \mathcal{L}[\phi(x)]$  is the action underlying the QFT.

The contribution of a given field configuration in the partition function is weighted the value of the action. The partition function can be understood to compute the trace of the vacuum density matrix,

$$Z \longrightarrow Tr[\rho] \quad (2.2)$$

It should be said that  $Z$  can be decorated by the addition of sources. The typical source terms that we see in QFT courses are terms of the type  $J(x)\phi(x)$  added to the Lagrangian, where  $J(x)$  is the source. Then, given the knowledge of  $Z[J]$ , one can compute correlation functions  $\langle\phi(x_1)\dots\phi(x_n)\rangle$  by taking functional derivatives of  $Z[J]$ . Another source "hidden" in  $Z$  is the choice of spacetime manifold, in other words  $Z$  depends also on the choice of metric and spacetime where the fields live. To have a concrete example in mind one can think about the standard kinetic term in the Lagrangian, which is  $\partial_\mu\phi g^{\mu\nu} \partial_\nu\phi$  and depends explicitly on the metric  $g^{\mu\nu}$ .

We will devote this first section to explain the problem. First we will give a QFT description of the subsystem  $A$ , defined initially by a taking spacelike slice of Minkowski, an entangling surface

$\partial A$ , and considering the partition

$$\mathcal{H}_A = \bigotimes_{x \in A} \mathcal{H}_x \quad ; \quad \mathcal{H}_{A^c} = \bigotimes_{x \in A^c} \mathcal{H}_x, \quad (2.3)$$

then we will discuss how to actually compute the Von-Neumann entropy  $S(A)$ .

The choice of entangling surface  $\partial A$  at given time defines a subsystem in the QFT. (Even though we will consider  $\partial A$  to be a sphere, this can be defined more generally as a subregion which lies within a Cauchy slice  $\Sigma$  of our spacetime manifold  $M$ ). One may be tempted to think that is enough to restrict the quantum fields to the interior region,  $A$ , at given time, but this is not true since  $A$  is just a snapshot of the time evolution of the quantum fields. For example, we could consider that the fields interact with one another in the causal development of  $\mathcal{D}(A)$ . The latter set can be formally defined as:

**Definition 2.1. Causal development** *Given a subregion contained in a Cauchy slice of our manifold  $A \subset \Sigma$ , the causal development will be the set of points in our manifold where any causal curve through it intersects with  $A$ .*

The reduced density matrix  $\rho_A$  is obtained from  $Z$  by integrating over field configurations outside the causal development  $\mathcal{D}(A)$ . What we are left with is a QFT that lives on  $\mathcal{D}(A)$ . To understand this point we should first understand what space is  $\mathcal{D}(A)$  and what is the metric on  $\mathcal{D}(A)$ .

The entangling surface we work with is a sphere of radius  $r = R$ , within a Cauchy slice  $\Sigma$  at  $t = 0$ , and the causal development  $\mathcal{D}(A)$  is

$$\mathcal{D}(A) : \{x^\pm \leq R \quad ; \quad x^\pm \equiv r \pm t\} \quad (2.4)$$

It is simple to see that  $\mathcal{D}(A)$  is the set of points such that  $t \leq -r + R$  and  $t \geq +r - R$ , in other words  $|t| \leq -r + R$ . The metric on the casual development can be obtained as follows. Starting from the Minkowski metric in polar coordinates

$$r = \sqrt{(x^1)^2 + \dots + (x^n)^2} \quad (2.5)$$

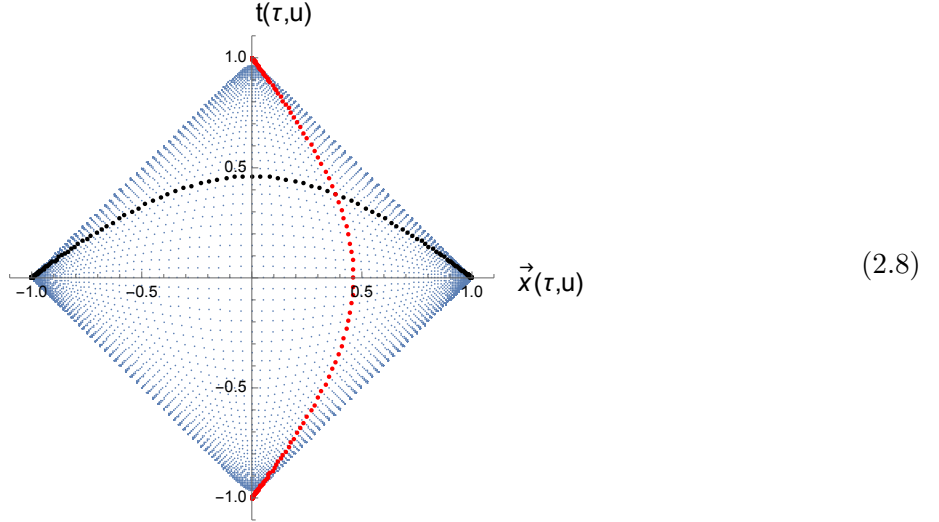
one finds  $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2$ . The following transformation:

$$t = R \frac{\sinh\left(\frac{\tau}{R}\right)}{\cosh u + \cosh \frac{\tau}{R}} \quad (2.6)$$

$$r = R \frac{\sinh(u)}{\cosh u + \cosh \frac{\tau}{R}} \quad (2.7)$$

with  $\tau \in (-\infty, +\infty)$  and  $u \in [0, +\infty)$ , since  $r \geq 0$  in  $d \geq 3$ , restricts the geometry to  $\mathcal{D}(A)$ . (In  $d = 2$  we can take  $u \in (-\infty, +\infty)$ .) In the figure below we give an example, showing what is the patch in  $(t, \vec{x})$  coordinates covered by the  $\tau, u$  coordinates, in a given  $\vec{x}$  direction and for the

reference choice  $R = 1$ ,<sup>1</sup>



The curve in red is computed as function of  $\tau$  for fixed value  $u$ . The curve in black is computed as function of  $u$  for fixed value  $\tau$ . The story of this change of variables goes back to Rindler and the Rindler wedge [CITE]. In (2.8) we can see a full diamond. We will comment more about the Rindler wedge in the Section 2.3

The Minkowski metric transforms as:

$$ds^2 = \Omega^2 [-d\tau^2 + R^2 (du^2 + \sinh^2 u d\Omega_{d-2}^2)] \quad (2.9)$$

where  $\Omega = \frac{1}{\cosh u + \cosh \frac{\tau}{R}}$ .

The partition function of the QFT on  $\mathcal{D}(A)$  with the metric in (2.9) computes  $\rho_A$ . As mentioned above, this partition function depends by construction on the QFT and the data that defines  $\partial A$ : in our particular case it depends on  $R$ . Let us note a feature of  $ds^2$  in (2.9). Up to the prefactor, the topology of the metric is that of  $\mathbb{R} \times \mathbb{H}$  where  $\mathbb{H}$  is hyperbolic.

Now we turn to the computational problem of actually computing  $\log(\rho_A)$ . This is not possible directly, instead we can make use of the Rényi entropy  $S_q$  defined as,

**Definition 2.2. Rényi entropy:**

$$S_q = \frac{1}{1-q} \log \text{Tr}[\rho^q] \quad \text{with } q \geq 0 \quad (2.10)$$

A straightforward calculation shows us that in the limit when  $q \rightarrow 1$  we recover the Von Neumann entropy, thus  $S_1 := S_{VN}$ . The problem of computing  $\text{Tr}[\rho^q]$  for  $q \in \mathbb{N}$  reduces again to a path integral computation, that can be done resorting on to the **replica trick** [7] [8]. As we have said before, the partition function which we attribute to  $\text{Tr}[\rho]$  is computed through the path integral, in order to generalize this to  $\text{Tr}[\rho^q]$  we will consider  $q$  copies of our manifold and do the path integral over fields  $\phi_i$  in each copy, this integral is denoted with  $Z_q$ . Finally, the trace we are interested in is defined as:  $\text{Tr}[\rho^q] = \frac{Z_q}{Z^q}$ .

<sup>1</sup>Using Plot functions in *Mathematica* [6]

**Definition 2.3.** *Replica trick:*

$$Z_q = \int_{B.C.} \mathcal{D}\phi_1 \dots \mathcal{D}\phi_q \exp \left( -\frac{1}{\hbar} \sum_{i=1}^q S_E[\phi_i] \right) \quad (2.11)$$

$$\text{Tr}[\rho^q] = \frac{Z_q}{Z^q} \quad (2.12)$$

In order to properly define the theory on the replica manifold, boundary conditions *B.C.* must be applied to the field configurations over which we are integrating. We must relate the fields in the  $M_j$  copy with the ones in the following one  $M_{j+1}$ . Summarizing, the boundary conditions for the path integral on the  $q$ -sheeted manifold are:

$$\phi_j(x, 0^+) = \phi_{j+1}(x, 0^-) \quad ; \quad x \in A \quad (2.13)$$

where  $j \in \{1, 2, \dots, q\}$  and  $q+i \equiv i$ . Setting these particular boundary conditions is usually known as sewing the  $q$ -sheeted manifold.

Finally, the problem of computing  $S_1$  is equivalent to the problem of finding an analytic continuation of  $Z_q$  so that one can take the limit  $q \rightarrow 1$ .

We will be interested in calculating the entanglement entropy not in generic QFT where the symmetry group is the Poincaré group, but rather in a class of theories with a richer symmetry known as conformal symmetry. The reason behind this is because the additional symmetry transformations that come with this symmetry group will allow us to relate the entanglement entropy in the causal diamond (2.9) to a thermal entropy in  $S^1 \times \mathbb{H}$ , essentially by dropping the  $\Omega$  factor and performing a Wick rotation from  $\tau \in \mathbb{R}$  to  $\tau \in S^1$ .

## 2.2 Basics elements of conformal field theories

We will now give a short introduction to the axioms of CFTs and conformal transformations in general, following [9]

**Definition 2.4. Conformal transformation:** *Let  $(M, g)$  and  $(M', g')$  be two semi-Riemannian manifolds. Given  $U$  and  $V$  two open sets of each manifold, a conformal transformation between them will be an orientation preserving diffeomorphism  $f : U \rightarrow V$  with  $x' = f(x)$  such that the metrics are related by:*

$$g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x) = \frac{\partial x'_\alpha}{\partial x^\mu} \frac{\partial x'_\beta}{\partial x^\nu} g_{\alpha\beta}(x) \quad (2.14)$$

where  $\Omega(x)^2 : U \rightarrow \mathbb{R}^+$  is called the conformal factor.

The transformations that obey the constraint in (2.14) can be classified, see Table 1, and define a group structure known as the conformal group. The generators of the conformal group satisfy the conformal algebra commutation relations given in Table 2.

We can think of the Conformal group as an extended Poincaré group by adding dilations and the special conformal transformations (SCT), the latter can also be written as:

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - b^\mu \quad (2.15)$$

Transformation	Group expression	Infinitesimal generator
Translations	$x'_\mu = x_\mu + a_\mu$	$P_\mu = -i\partial_\mu$
Dilations	$x'_\mu = \alpha x_\mu$	$D = -ix^\mu\partial_\mu$
Rotations	$x'_\mu = M^{\mu\nu}x_\nu$	$L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$
Special Conformal Transformation	$x'_\mu = \frac{x_\mu - b_\mu x^2}{1 - 2b \cdot x + b^2 x^2}$	$K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$

Table 1: Expressions for the conformal group and algebra structure.

$$\begin{aligned}
[D, P_\mu] &= iP_\mu & [K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\
[D, K_\mu] &= -iK_\mu & [K_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\
[P_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) & [L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho})
\end{aligned}$$

Table 2: Conformal algebra commutation relations [9]

The rewriting in (2.15) makes more explicit their interpretation as the conjugation of an inversion, a translation, and a final inversion. From above one finds  $x'^2 = x^2/(1 - 2b \cdot x + b^2 x^2)$ , thus

$$x'^\mu = \left( \frac{x^\mu}{x^2} - b^\mu \right) \frac{x^2}{(1 - 2b \cdot x + b^2 x^2)} \quad (2.16)$$

which is the result quoted in table 2. SCTs are very useful since they allow to bring the point at infinity to a finite location. As an example, say  $b = (-1, \vec{0})$ , then the direction perpendicular to  $b$  are such that  $x'_\perp = x_\perp / ((1 + x^1)^2 + (x_\perp)^2)$ , but

$$(x')^1 \Big|_{x_\perp=0} = \frac{x^1 + (x^1)^2 + (x_\perp)^2}{(1 + x^1)^2 + (x_\perp)^2} \Big|_{x_\perp=0} = \frac{x^1}{1 + x^1} \quad (2.17)$$

Therefore the point  $x^1 = 0$  goes to  $(x')^1 = 0$ , while, the point  $x^1 = \infty$  goes to  $(x')^1 = 1$ , and viceversa, the point  $x^1 = 1$  goes to  $(x')^1 = \infty$ .

In a conformal field theory, the fields are representation of the conformal group, thus have properties under conformal transformations. A classical conformal field theory, defined by an action, is field theory whose action is invariant under the conformal group.

Invariance under conformal transformations implies that the quantities that are preserved are no longer distances but rather angles. This fact implies a rigid structure in the structure of correlation functions.

### 2.3 Entanglement entropy in a CFT as generalized thermal entropy

In this section we review the argument that shows that in a CFT the vacuum correlators in  $\mathcal{D}(A)$  are conformally mapped to thermal correlators in the space

$$ds^2 = -d\tau^2 + R^2(du^2 + \sinh^2(u)d\Omega_{d-2}^2) \quad (2.18)$$

This argument then implies that the density matrix on  $\mathcal{D}(A)$  is a Gibbs state (to be defined below) corresponding to the local operator that represents  $\tau$  translations in (2.18), namely

$$\frac{\partial}{\partial \tau} \quad (2.19)$$

To go through the argument we will use what is called as the Rindler wedge. This Rindler wedge is defined by the following construction: Consider a two dimensional subspace with coordinates  $X^0, X^1$  and metric  $ds^2 = -dX_0^2 + dX_1^2$ , the Rindler wedge is the region of space

$$W : \{X^\pm \geq 0 \quad ; \quad X^\pm \equiv X^1 \pm X^0\} \quad (2.20)$$

The inequalities are solved by

$$X^\pm = e^{u \pm \tau} \quad (2.21)$$

where  $\tau \in (-\infty, +\infty)$  and  $u \in (-\infty, \infty)$ . Going back to  $X^0, X^1$ ,

$$X^1 = e^u \cosh(\tau) \quad ; \quad X^0 = e^u \sinh(\tau) \quad (2.22)$$

One can also divide the range of  $u$  in two parts  $(-\infty, 0] \cup [0, +\infty)$ .

Coming back to  $\mathcal{D}(A)$ , the crucial observation at this point is that the transformation:

$$t = R \frac{\sinh\left(\frac{\tau}{R}\right)}{\cosh u + \cosh \frac{\tau}{R}} \quad (2.23)$$

$$r = R \frac{\sinh(u)}{\cosh u + \cosh \frac{\tau}{R}} \quad (2.24)$$

can be written as a conformal transformation, namely it is the same as

$$\begin{aligned} \frac{t}{2R} &= \frac{X^0}{1 + 2X^1 + (X^1)^2 - (X^0)^2} & \text{where} & & X^1 &= e^u \cosh\left(\frac{\tau}{R}\right) \\ \frac{1}{2} \left(\frac{r}{R} + 1\right) &= \frac{X^1 + (X \cdot X)}{1 + 2X^1 + (X^1)^2 - (X^0)^2} & & & X^0 &= e^u \sinh\left(\frac{\tau}{R}\right) \end{aligned} \quad (2.25)$$

This shows that  $\mathcal{D}(A)$  can be mapped to the Rindler wedge, and that  $\Omega = \frac{1}{\cosh u + \cosh \frac{\tau}{R}}$  is a conformal factor that can be dropped for CFTs.

Finally, the Bisognano-Whichmann theorem for the Rindler wedge gives the statement that we were looking for,

**Theorem 2.5. Bisognano-Whichmann:** *The vacuum state of Minkowski space  $|\Omega\rangle$  is mapped to a thermal state with respect to time translations  $\partial_\tau$  in the Rindler wedge. [10]*

**Definition 2.6. Generalized thermal state:** *A state is said to be a (generalized) thermal state – Gibbs state – with respect to an operator  $K$  if it can be written as:*

$$\rho = \frac{e^{-\beta K}}{\text{Tr } e^{-\beta K}} \quad (2.26)$$

While the definition of a generalized thermal state does not say anything about the operator  $K$  (not even that it has to be local) in the argument reviewed above for  $\mathcal{D}(A)$ , it turns out that  $K = H_\tau$  is the local operator associated to  $\partial_\tau$ , and  $\beta = 2\pi R$ .

A straightforward calculation tells us that for a Gibbs state

$$\log \rho = -\beta \sum_{i=1}^n (E_i - \log Z) |E_i\rangle \langle E_i| \quad (2.27)$$

$$\rho \log \rho = -\beta \sum_{i=1}^n \frac{e^{-\beta E_i}}{\text{Tr } e^{-\beta H}} (E_i - \log Z) |E_i\rangle \langle E_i| \quad (2.28)$$

Then the Von Neumann entropy is:

$$S = -\text{Tr}[\rho \log \rho] = \beta \langle H_\tau \rangle_\rho - \beta \log Z \quad (2.29)$$

The last equality is a restatement of the second law of thermodynamical,  $F = E - TS$ . We have then proven that we can interpret the entanglement entropy as the thermal entropy on (2.18), after a Wick rotation is performed to obtain  $S^1 \times \mathbb{H}$ .

For the Renyi entropies,

$$\text{Tr}[\rho_{\mathcal{H}}^q] = \frac{\text{Tr}[e^{-2\pi q R H_\tau}]}{\text{Tr}[e^{-2\pi R H_\tau}]^q} = \frac{Z(T_0/q)}{Z(T_0)^q} \quad (2.30)$$

where  $T_0 = \frac{1}{2\pi R}$ . The thermal state relates nicely with the replica trick.

Finally, applying the definitions of the free energy  $F(T) = -T \log Z(T)$  and thermodynamical entropy  $S_{\text{therm}} = -\frac{\partial F}{\partial T}$  we get two useful expressions for the Rényi entropy:

$$S_q = \frac{q\mathcal{F}_1 - \mathcal{F}_q}{1-q} = \frac{q}{1-q} \frac{1}{T_0} \int_{T_0/q}^{T_0} dT S_{\text{therm}}(T) \quad (2.31)$$

Where  $F_q = -T \log Z_q$  and  $\mathcal{F}_q = 2\pi R q F_q$ .

From this equation we identify that the Von Neumann entropy will be:

$$S_1 = -S_{\text{therm}} \left( \frac{1}{2\pi R} \right) \quad (2.32)$$

For conformally flat spaces the vacuum energy which we set at  $T = \frac{1}{2\pi R}$  is null:  $E \left( \frac{1}{2\pi R} \right) = 0$ . Then, from the second law of thermodynamics we deduce that  $S_1 = -2\pi R F \left( \frac{1}{2\pi R} \right) = -\mathcal{F}_1$ .

Let's briefly summarize the previous observations: Starting from the Minkowski vacuum and a spherical entangling region  $\partial A$  at (reference) time  $t = 0$ , we have been able to construct a conformal map to the hyperbolic cylinder  $\mathbb{R} \times \mathbb{H}^{d-1}$ , and interpret this map as a Rindler wedge construction. This allow us to justify the fact that the entangling entropy for  $\mathcal{D}(A)$  in the CFT can be interpreted as a thermal entropy in  $S^1 \times \mathbb{H}^{d-1}$  by making use of the special properties of the Rindler wedge.

## 2.4 AdS geometry and holographic CFTs

In the section we will give a very brief introduction to the Anti de Sitter space and review some axiomatic notions about a special class of QFTs known as Holographic CFTs [11].

### AdS geometry

Consider  $\mathbb{R}^{d,2}$  then the  $d + 1$  dimensional anti de Sitter space is defined as the hyperboloid given by:

$$-X_0^2 - X_1^2 + \dots + X_{d+1}^2 = -R^2, \quad X_0 > 0 \quad (2.33)$$

From this expression it can be easily seen that the symmetry group is  $SO(d, 2)$  which is actually the conformal group.

AdS is a maximally symmetric manifold, as such it also arises naturally as the solution of Einstein's equation with a negative cosmological constant and constant Ricci curvature.

A natural set of coordinates for AdS are the Poincaré coordinates  $(z, x_1, \dots, x_d)$ , such that

$$\begin{aligned} X_0 &= L \frac{1 + x^2 + z^2}{2z} \\ X^\mu &= L \frac{x^\mu}{z} \\ X_{d+1} &= L \frac{1 - x^2 - z^2}{2z} \end{aligned}$$

where  $x_\mu \in \mathbb{R}^d$  and  $z > 0$ . The metric of AdS in the Poincaré patch is

$$ds^2 = L^2 \frac{dz^2 + \eta^{\mu\nu} dx_\mu dx_\nu}{z^2} \quad (2.34)$$

The foliation of AdS given by this metric is through spacetime Minkowski slices at constant values of  $z$ . The boundary of spacetime is located at  $z \rightarrow 0$ .

Another useful set of coordinates is the following one:

$$\begin{pmatrix} X_{d+1} \\ X_0 \end{pmatrix} = \begin{pmatrix} \tilde{\rho} \cosh(\tau) \\ \rho \cosh(u) \end{pmatrix} \quad (2.35)$$

$$X^1 = \tilde{\rho} \sinh(\tau) \quad (2.36)$$

$$X^\mu = \rho \sinh(u) \Omega_\mu \quad (2.37)$$

where  $\tilde{\rho}^2 = \rho^2 - L^2$  is enforced by (2.33). The AdS metric in this patch is

$$ds^2 = - \left( \frac{\rho^2}{L^2} - 1 \right) d\tau^2 + \frac{d\rho^2}{\frac{\rho^2}{L^2} - 1} + \rho^2 (du^2 + \sinh^2(u) d\Omega_{d-2}^2) \quad (2.38)$$

The foliation of AdS given by this metric is through hyperbolic surfaces  $d\Sigma_{d-1}^2 = du^2 + \sinh^2(u) d\Omega_{d-2}^2$ . These leaves intersect the Poincaré boundary. This intersection is maximal for  $\rho = L$ . In fact, the AdS metric in this coordinates can be interpreted as a (topological) black hole with horizon at  $\rho = L$ , since the term  $g_{\tau\tau} = 0$  as  $\rho = L$  (linearly). To see the intersection, consider the two equations,

$$L \cosh(u) = X_0 + X_{d+1} = \frac{L}{z} \quad ; \quad L \sinh^2(u) = X^\mu \cdot X^\mu = \frac{L^2}{z^2} x^\mu \cdot x^\mu \quad (2.39)$$

We can solve for both  $z = \cosh^{-1}(u)$  and  $r^2$ . For the latter we find

$$r^2 \equiv x_\mu \cdot x_\mu = z^2 \sinh^2(u) = L^2 \tanh^2(u) \rightarrow L^2 \quad \text{in the limit} \quad u \rightarrow \infty \quad (2.40)$$

This shows that the horizon ends on on a sphere of radius  $r = L$  at the Poincaré boundary.

## AdS/CFT

The *AdS/CFT* correspondence was discovered by Juan Maldacena [1] in the context of string theory. The holographic dictionary, as we know it today, was put forward by Edward Witten in [12], and many other scientists contributed to its development.

Besides details, some features of the AdS/CFT correspondence do not need string theory to be explained. Let us begin by mentioning general features of the two sides of the AdS/CFT correspondence:

- A theory of gravity and matter in asymptotically  $AdS \times M$  spacetime with  $M$  some (possibly trivial) compact manifold.
- An CFT that is also an interacting gauge theory with many degrees of freedom and a low energy sparse spectrum.

The correspondence is explicitly achieved by a dictionary between both theories. To explain the dictionary, let us start from the AdS side.

Any gravitational theory in AdS is first of all determined by the strength of the Newton's constant,  $16\pi G_N$ , and the value of the radius  $L$  of AdS (asymptotically). If  $G_{\mu\nu}$  is the bulk metric and  $\Phi$  denotes collectively the matter fields, formally, the partition function of the gravitational theory is

$$W_{gravity}[J] = \int \mathcal{D}\Phi \mathcal{D}G \exp \left[ -\frac{1}{16\pi G_N} \int_{AdS} dz dx^d \sqrt{g} (R - \Lambda) - S[\Phi] \right] \quad (2.41)$$

with  $R$  the curvature and  $\Lambda = -\frac{d(d-1)}{L^2}$  the cosmological constant.

Then, since AdS has a boundary, a theory of gravity in AdS is specified by boundary conditions for each one of the fields in the theory. In the Poincare patch these are

$$G_{\mu\nu}(z \rightarrow 0, \vec{x}) \rightarrow \frac{g_{\mu\nu}(x)}{z^2} \quad ; \quad \Phi(z \rightarrow 0, \vec{x}) \rightarrow z^{d-\Delta} J(x) \quad (2.42)$$

Frobenius analysis close to the boundary  $z \rightarrow 0$  shows what is the most general near boundary expansion: for any bulk field there are two independent modes, a normalizable and a non normalizable mode. The coefficient of the non normalizable modes is associated to a source. This analysis gives the result in (2.42) where  $\Delta$  is related to the mass of the bulk field through the relation  $m^2 = \Delta(\Delta - d)$  and  $\Delta \geq 0$ .

The AdS/CFT postulates that  $g_{\mu\nu}(x)$  and  $J(x)$  are the boundary sources for the partition function *in the dual CFT*, namely

$$W_{CFT}[J] = \int \mathcal{D}\psi \exp \left[ -S_{CFT}[\psi] + \int d^d x J(x) \mathcal{O}_\Delta(x) \right] \quad (2.43)$$

and that

$$W_{CFT}[g, J] = W_{gravity}[g, J] \quad (2.44)$$

The AdS/CFT duality is a weak/strong duality. In other words, the gravity side is classical (and semiclassical) when the gauge theory is strongly coupled in the sense of 't Hooft. In this regime we can consider the radius of AdS to be very large, and  $G_N \rightarrow 0$ . In this approximation, the path integral is given at leading order by a saddle point approximation for  $W_{gravity}$ . On the other hand, this saddle computes a very complicated path integral on the CFT side at strong 't Hooft coupling.

### 3 Holographic calculation of the Rényi entropy

From the equality of partition function in the AdS/CFT framework (2.44) we can compute the EE as the Hawking entropy of the AdS metric given in (2.38) which we identify as a topological black

hole due to the presence of a singularity when  $\rho \rightarrow L$ :

$$ds^2 = - \left( \frac{\rho^2}{L^2} - 1 \right) d\tau^2 + \frac{d\rho^2}{\frac{\rho^2}{L^2} - 1} + \rho^2 (du^2 + \sinh^2(u) d\Omega_{d-2}^2) \quad (3.1)$$

The conformal boundary of the previous metric is the hyperbolic cylinder  $\mathbb{R} \times \mathbb{H}$ . In particular, the Hawking temperature will coincide with the temperature  $T_0$  obtained from Bisognano-Whichmann theorem:

$$T_0 = \frac{1}{2\pi L} \quad (3.2)$$

We will detail the calculation for a generic  $d + 1$  solution and then particularize for our case of interest  $d = 3$ . Therefore, consider an AdS topological black hole as our bulk theory with the Einstein-Hilbert action:

$$S = \frac{1}{2l_p^{d-1}} \int d^{d+1}x \sqrt{-g} \left( R + \frac{d(d-1)}{L^2} \right) \quad (3.3)$$

Where the metric is given by the following expression:

$$ds^2 = - \left( \frac{\rho^2}{L^2} f(\rho) - 1 \right) N^2 dt^2 + \frac{dr^2}{\frac{\rho^2}{L^2} f(\rho) - 1} + \rho^2 d\Sigma_{d-1}^2 \quad (3.4)$$

With  $f$  a  $C^\infty(\mathbb{R})$  radial bounded function such that  $f(r \rightarrow \infty) = f_\infty$

The metric in (3.4) generalizes the metric in (3.1). In particular, the conformal boundary is again given by  $\mathbb{R} \times \mathbb{H}^{d-1}$  with the choice of  $N^2 = \frac{L^2}{f_\infty R^2}$ .

The event horizon of the black hole is given by  $\frac{\rho_h^2}{L^2} f(\rho_h) = 1$ . After performing a Wick rotation to Euclidean space, the temperature of the black hole can be calculated by performing a Taylor expansion around the horizon. If we introduce the following coordinate change:  $\rho = \rho_h + \frac{\rho^2}{4\rho_h^2}$  then the metric will be:

$$ds^2 \simeq N^2 g'(\rho_h)^2 \frac{\rho^2}{4\rho_h^2} dt^2 + d\rho^2 + g'(\rho_h) r_h^2 d\Sigma_{d-1}^2 \quad (3.5)$$

where  $g(\rho) = 1 - f(\rho) \frac{\rho^2}{L^2}$ .

With the periodic identification on  $\tau \sim \tau + \beta$  where  $\beta = \frac{4\pi}{g'(\rho_h)N} \rho_h^2$  we avoid the conical singularity when  $\rho = 0$ . Said procedure will change our conformal boundary to  $\mathbb{S}^1 \times \mathbb{H}^{d-1}$  and we can finally identify the temperature of the black hole as:

$$T = \frac{N}{4\pi} \frac{\partial}{\partial \rho} \left( 1 - f(\rho) \frac{\rho^2}{L^2} \right) \Big|_{\rho=\rho_h} = \frac{-N}{4\pi} \left( \frac{2\rho_h}{L^2} f(\rho_h) + f'(\rho_h) \frac{\rho_h^2}{L^2} \right) \quad (3.6)$$

To find  $f(\rho)$  we need to impose Einstein's equation. In particular for  $d + 1$  dimensions, this gives the following one-parameter solution [13]:

$$\left( -1 + \frac{\rho^2}{L^2} f(\rho) \right) = -1 + \frac{\rho^2}{L^2} - \frac{w_d}{L^2 \rho^{d-2}} \quad (3.7)$$

$$f(\rho) = 1 - \frac{w_d}{\rho^d} \quad (3.8)$$

with  $w_d$  a parameter.

With the above expression for  $f(\rho)$  we can rewrite  $w_d$  in terms of the position of the event horizon as  $w_d = \rho_h^d - L^2 \rho_h^{d-2}$ . Moreover, now  $f(\rho \rightarrow \infty) = 1$ . Taking all this into account and the choice for  $N$  we mentioned earlier, then expression (3.6) leads to:

$$T = \frac{1}{4\pi R} \left( d \frac{\rho_h}{L} - (d-2) \frac{L}{\rho_h} \right) \quad (3.9)$$

For the calculation of Rényi entropy we need the entropy of the black hole computed by the Bekenstein-Hawking formula:  $S = \frac{A_{hor}}{4G_N^{d-1}}$  and in terms of the Planck length  $S = \frac{2\pi}{l_p^{d-1}} A_{hor}$ .

The horizon metric is conformal to that of the hyperbolic cylinder, for a fixed Cauchy slice the area will be proportional to the volume of the hyperbolic plane with negative unit curvature  $V_{\Sigma_{d-1}}$ . In particular, the expression for the entropy will be:

$$S = 2\pi \rho_h^{d-1} V_{\Sigma_{d-1}} \quad (3.10)$$

As is often the case in AdS/CFT calculations we will need to introduce a regularization, in the bulk this is done by setting a maximum radius since the space is non-compact. The hyperbolic plane metric in global coordinates is:  $ds^2 = du^2 + \sinh^2 u d\Omega_{d-1}^2$ , by a suitable change of coordinates:  $y = \cosh u$  then:  $ds^2 = \frac{dy^2}{y^2-1} + (y^2-1)d\Omega_{d-1}^2$ . We will parametrize the maximum radius as:  $y_{max} = \frac{R}{\delta}$ , as we will see the  $\delta$  will relate to an UV scale in the CFT. The analytical expression of the volume is given in terms of hyper-geometric functions, nevertheless, if we expand in powers of  $\frac{R}{\delta}$  for small  $\delta$  then for  $d \geq 3$ :

$$V_{\Sigma_{d-1}} = \int_{\Omega_{d-1}} \int_1^{y_{max}} \sqrt{\frac{(y^2-1)^{d-2}}{y^2-1}} d\Omega_{d-1} dy = \quad (3.11)$$

$$\simeq \Omega_{d-1} \left\{ \frac{1}{(d-2)} \left( \frac{R}{\delta} \right)^{d-1} - \frac{1}{(d-4)} \left( \frac{R}{\delta} \right)^{d-3} + \mathcal{O} \left( \frac{R}{\delta} \right)^{d-6} \right\} \quad (3.12)$$

As we have seen, we can write the Rényi entropy in terms of an integral of the entropy over the temperature (2.31). Instead of integrating over  $T$  we will integrate over  $x \equiv \frac{\rho_h}{L}$ .

$$S_q = \frac{q}{1-q} \frac{1}{T_0} \int_{x_q}^1 S(x) \frac{dT}{dx} dx \quad (3.13)$$

Note that the upper limit of integration is  $x = 1$ , since when  $\rho_h = L$  we get  $T = \frac{1}{2\pi R}$ . In order to compute the integral all we are left to do is rewrite our previous expressions in terms of  $x$ :

$$S(x) = \frac{2\pi}{l_p^{d-1}} x^{d-1} L^{d-1} V_{\Sigma_{d-1}} \quad (3.14)$$

$$T = \frac{1}{4\pi R} \left( dx - \frac{d-2}{x} \right) \quad (3.15)$$

To find  $x_q$  we must solve the following equation:

$$T = \frac{T_0}{q} = \frac{1}{2\pi R q}$$

which becomes  $-2x_q + qdx_q^2 - q(d-2) = 0$ . The positive solution is  $x_q = \frac{1+\sqrt{1+(d-2)dq^2}}{qd}$  and represent the temperature. Taking all this into account, we arrive at the following expressions for the Rényi entropy, for  $q \geq 2$ :

$$S_q = V_{\Sigma_{d-1}} \pi \left( \frac{L}{l_p} \right)^{d-1} \frac{q}{q-1} (-2 + x_q^d + x_q^{d-2}) \quad (3.16)$$

The entanglement entropy can be computed from the expression (3.15) or from the limit of the Rényi entropy when  $q \rightarrow 1$ :

$$S_{EE} = \frac{-2\pi}{l_p^{d-1}} L^{d-1} V_{\Sigma_{d-1}} \quad (3.17)$$

From here we can compute a parameter free expression by computing the ratio  $S_q/S_1$ :

$$\frac{S_q}{S_{EE}} = \frac{q}{q-1} \left( -2 + x_q^d + x_q^{d-2} \right) \quad (3.18)$$

## 4 CFT calculation of the Rényi entropy

In this section we will perform the calculation of the Rényi entropies in a free field theory, evaluating the path integral explicitly. The procedure will be straightforward, beginning with the path integral we will diagonalize the Laplacian operator that we read from the equations of motion.

The metric for the hyperbolic cylinder is given by:

$$ds_H = d\tau^2 + du^2 + \sinh^2 u d\phi^2 \quad (4.1)$$

The range of the time coordinate is:  $0 \leq \tau < 2\pi q$ . The integer  $q = 1, \dots$  labels the replica manifold. The rest of the coordinates are defined in the range:  $0 \leq \phi < 2\pi$  and  $0 \leq u < \infty$ .

The action for a conformally coupled (complex) scalar field is

$$S = \int \sqrt{g} d^D x (|\partial_\mu \phi|^2 + \xi \mathcal{R} |\phi|^2) \quad (4.2)$$

where  $\sqrt{g}$  es el determinante de la metrica y  $\xi = \frac{D-2}{4(D-1)}$ .<sup>2</sup> Since the Ricci scalar for the hyperbolic cylinder with unit radius is  $\mathcal{R} = -2$  then our action simplifies to:

$$S[\phi, g] = \int d^3 x \sqrt{g} \left( |\partial_\mu \phi|^2 - \frac{1}{4} |\phi|^2 \right) \quad (4.3)$$

We can therefore consider a term  $M^2 = \frac{-1}{4}$  and parametrize the massive deformations:  $m^2 = M^2 + \frac{1}{4}$ . In order to proceed we introduce the Laplace-Beltrami operator (LBO) whose expression is:

$$\Delta \phi = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \phi) \quad (4.4)$$

From it we are able to write up to a boundary terms that vanishes, the action as:

$$S[\phi, g] = \int d^3 x \sqrt{g} \phi \left( -\Delta - \frac{1}{4} + m^2 \right) \phi \quad (4.5)$$

---

<sup>2</sup>We will have to address the fact that the hyperbolic space is non-compact, thus we will later introduce a cut off that will regularize the volume.

Manipulations with the path integral are simple to understand if we first look at a case with a discrete spectrum. Then by expanding the fields in terms of the eigen-functions  $\phi = \sum_n c_n \phi_n$ , we can use this expansion to rewrite the integration measure of the path integral:

$$\langle \phi_\lambda, \phi_{\lambda'} \rangle = \int d^3x \sqrt{g} \phi_i^* \phi_j = \delta_{ij} \quad (4.6)$$

$$\|\delta\phi\|^2 = \sum_n |\delta c_n|^2 \quad (4.7)$$

$$\int \mathcal{D}\phi = \prod_n \int dc_n dc_n^\dagger \quad (4.8)$$

From here the action can be expressed as:

$$S[\phi, g] = - \sum_n \left( \lambda_n + m^2 - \frac{1}{4} \right) |c_n|^2 \quad (4.9)$$

Substituting this in the path integral alongside we recognize a Gaussian integral whose solution is given by (??):

$$Z = \prod_n \int dc_n dc_n^\dagger e^{-(\lambda_n + m^2 - \frac{1}{4})|c_n|^2} = \prod_n \sqrt{\frac{2\pi}{\lambda_n + m^2 - \frac{1}{4}}} \quad (4.10)$$

Therefore the expression of the generating functional would be:

$$\mathcal{F}_q = \text{Tr} \log \left( -\Delta + m^2 - \frac{1}{4} \right) = \sum_n \log \sqrt{\frac{2\pi}{\lambda_n - m^2 + \frac{1}{4}}} \quad (4.11)$$

The generalization to a differential operator in a non-compact space involves dealing with a continuous spectrum. In particular, the sum becomes an integral over the density of eigenvalues.

Since  $\partial_\tau$  is an isometry, we can write the eigenfunctions as the product of those which arise from the Laplacian of the hyperbolic plane and those for  $\partial_\tau^2$ . Due to the periodicity in  $\tau$ , the frequency in the  $\tau$  direction is quantized, so that we can write the eigenvalue as

$$\lambda + \frac{n_\tau^2}{q^2} + m^2 \quad (4.12)$$

where  $\lambda$  correspond to the contribution to the eigenvalue from the Laplacian:  $-\Delta_{\mathbb{H}^2} - \frac{1}{4}$ , where  $\Delta_{\mathbb{H}^2}$  is read off from (4.4),

$$\Delta_{\mathbb{H}^2} = \frac{\partial^2}{\partial u^2} + \coth u \frac{\partial}{\partial u} + \frac{1}{\sinh^2 u} \frac{\partial^2}{\partial \phi^2} \quad (4.13)$$

The computation of  $\mathcal{F}_q$  gives now

$$\mathcal{F}_q = \sum_{n=-\infty}^{+\infty} \int d\lambda \Phi(\lambda) \log \left( \lambda + \frac{n_\tau^2}{q^2} + m^2 \right) \quad (4.14)$$

where  $\Phi(\lambda)$  is the density of eigenvalues.

The study of the spectrum of the Laplacian can be done through the resolvent.

**Definition 4.1. Resolvent:** Let  $A$  be an invertible operator then the resolvent set  $r(A)$  is defined as:

$$r(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is invertible}\} \quad (4.15)$$

Given an element  $\lambda \in r(A)$  we define the resolvent of  $A$  at  $\lambda$  as  $(\lambda I - A)^{-1}$

Studying the resolvent operator  $G_\lambda = (\lambda I - A)^{-1}$  is equivalent to studying the Green functions defined as:

$$(\lambda I - A)G_\lambda(x, y) = \delta(x - y) \quad (4.16)$$

In particular, if we let our differential operator be the Laplacian then through the resolvent we can study the solutions to the eigenvalue problem and the spectrum. In fact the density of states is uniquely defined from the resolvent through the Stieltjes inversion formula [14].

The measure can be recovered from the resolvent as:

$$\Phi(\lambda)d\lambda = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi} \text{Im} \langle G_{\lambda+i\epsilon} \rangle \quad (4.17)$$

Where  $\langle G_{\lambda+i\epsilon} \rangle$  denotes the integral of the resolvent over the manifold, in this case  $\mathbb{H}^2$ .

Since the space is homogeneous we can perform the following trick. We can choose a particular point and calculate the density of states there, call it  $\mu(\lambda)$ . Taking the point  $u = 0$ .

$$\mu(\lambda)d\lambda = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi} \text{Im} (G_{\lambda+i\epsilon}(u = 0)) \quad (4.18)$$

Then the total density is  $\Phi(\lambda) = \text{Vol}(\mathbb{H}^2)\mu(\lambda)$ .

The main result we will reach is known as the Plancherel measure, in which the density of states per hyperbolic volume is given by:

$$\mu_2(\lambda)d\lambda = \frac{1}{4\pi} \tanh \pi \sqrt{\lambda} d\lambda \quad (4.19)$$

Thus

$$\Phi(\lambda)d\lambda = \frac{\text{Vol}(\mathbb{H}^2)}{4\pi} \tanh \pi \sqrt{\lambda} d\lambda \quad (4.20)$$

#### 4.1 The Laplacian and the density of states on $\mathbb{H}^2$

In this section we collect a number of result that are useful to understand how  $\Phi(\lambda)$  can be derived. We will follow [15] [16]

The Laplacian on  $\mathbb{H}^2$  decomposes into a spherical angular part and a radial part. For convenience we repeat below the expression in (4.13),

$$\Delta_{\mathbb{H}^2} = \frac{\partial^2}{\partial u^2} + \coth u \frac{\partial}{\partial u} + \frac{1}{\sinh^2 u} \frac{\partial^2}{\partial \phi^2} \quad (4.21)$$

The eigenfunctions can be written on the basis of the spherical harmonics, corresponding to  $\partial_\phi^2$ . These spherical harmonics are  $Y_l = e^{il\phi}$  where  $l \in \mathbb{Z}$ . Taking this into account, the problem for the radial eigenfunction  $f(\eta)$  is:

$$f_\lambda''(u) + \coth u f_\lambda'(u) - \frac{l^2}{\sinh^2 u} f_\lambda(u) = \lambda f_\lambda(u) \quad (4.22)$$

By multiplying by  $\sinh u$  we can rewrite this equation as

$$\frac{d}{d\eta}(\sinh u f'_\lambda(u)) - \frac{l^2}{\sinh u} = -\lambda \sinh u f_\lambda(u) \quad (4.23)$$

Further introducing  $x = \cosh \eta$  we can rewrite the radial equation as

$$\frac{d}{dx} \left( (1-x^2) \frac{df_\lambda}{dx} \right) + \left( \nu(\nu+1) - \frac{l^2}{1-x^2} \right) f_\lambda(x) = 0 \quad (4.24)$$

where

$$\nu = -\frac{1}{2} + ir \quad ; \quad r = \sqrt{\lambda - \frac{1}{4}} \quad (4.25)$$

For normalization we are going to introduce the boundary condition  $f_\lambda(0) = 1$ . The general solutions are given by the Legendre functions [15]

$$f_\lambda(x) = e^{il\phi} P_{-\frac{1}{2}+ir}^l(x) \quad (4.26)$$

The case  $l = 0$  is the simplest. However, it is useful to point out that for fixed  $\lambda$ , there are infinitely many values of  $l$  that are allowed, since  $l \in \mathbb{Z}$ , thus infinitely many eigenfunctions with the same value of  $\lambda$ .

We now turn to the resolvent. It turns out that the resolvent (4.16) can be written in terms of the Legendre functions of the second type:

$$G_r(x) = \frac{1}{2\pi} Q_{-\frac{1}{2}+ir}(x) \quad (4.27)$$

From the properties of the Legendre functions [15] we can rewrite it as an integral over  $P_{-\frac{1}{2}+r}^{l=0}(x)$ :

$$\frac{1}{2\pi} Q_{-\frac{1}{2}+r}(x) = \frac{1}{4\pi} \int_0^\infty \frac{2\nu \tanh \pi\nu}{\nu^2 + r^2} P_{-\frac{1}{2}+i\nu}(x) d\nu \quad (4.28)$$

Where the previous expression is well defined for  $\text{Re}(r) > 0$ , this fact allows us analytically continue the previous expression with  $r \rightarrow \epsilon \pm ir$ . Finally, with this result we can write the resolvent (4.27) as:

$$G_r(u) = \frac{1}{4\pi} \int_0^\infty \frac{2\nu \tanh \pi\nu}{\nu^2 - r^2 - 2i\epsilon r} P_{-\frac{1}{2}+i\nu}(\cosh u) d\nu \quad (4.29)$$

As explained already, in order to calculate the density of states it suffices to consider the case  $u = 0$  which leads us to:

$$G_r(0) = \frac{1}{4\pi} \int_0^\infty \frac{2\nu \tanh \pi\nu}{\nu^2 - r^2 - 2i\epsilon r} d\nu \quad (4.30)$$

In order to recover the measure we compute the limit when  $\epsilon \rightarrow 0^+$  and use the Cauchy Principal Value identity:

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{4\pi} \int_0^\infty \frac{2\nu \tanh \pi\nu}{\nu^2 - r^2 - 2i\epsilon r} d\nu = \mathcal{P} \left( \frac{1}{\pi} \int_0^\infty \frac{\nu \tanh \pi\nu}{\nu^2 - r^2} d\nu \right) + i \int_0^\infty r\nu \tanh \pi\nu \delta(\nu^2 - r^2) d\nu \quad (4.31)$$

Since we are only interested in the imaginary part then the resulting integral is:

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{4\pi} \text{Im} \left( \int_0^\infty \frac{2r \tanh \pi r}{\nu^2 - r^2 - 2i\epsilon r} dr \right) = \frac{r}{2} \tanh \pi r \quad (4.32)$$

Comparing this expression to (4.18) we can extract the Plancherel measure (4.19). Since we assumed that  $\lambda \geq \frac{1}{4}$  in order to get a bounded expression, we will then redefine  $r = \sqrt{\lambda}$  with  $\lambda \geq 0$

$$\mu_2(r) d\lambda = \frac{1}{4\pi} \tanh \pi\sqrt{\lambda} \quad (4.33)$$

Taking into account the volume of the space over which we are considering our system, we finally find,

$$\Phi(\lambda)d\lambda = \frac{\text{Vol}(\mathbb{H}^2)}{4\pi} \tanh \pi\sqrt{\lambda}d\lambda \quad (4.34)$$

The hyperbolic plane is not compact and we are forced to consider a radius  $\eta_0$  to regularize its volume. The regularized volume can be set to be equal to  $-2\pi$  by taking the finite part of the integral, as done in [5]. However, we shall leave it as  $\text{Vol}(\mathbb{H}^2)$  in the next section.

## 4.2 An explicit formula for the Rényi entropy

We can now compute  $\mathcal{F}_q$ , by using that the eigenvalues of  $-\Delta + m^2 - \frac{1}{4}$  are given by  $\lambda + \frac{n_\tau^2}{q^2} + m^2$ , with  $\lambda \geq 0$  and thus we are left with the following expression:

$$\mathcal{F}_q(m) = \sum_{n_\tau=-\infty}^{\infty} \int_0^{\infty} \frac{\text{Vol}(\mathbb{H}^2)}{4\pi} \tanh(\pi\sqrt{\lambda}) \log\left(\lambda + m^2 + \frac{n_\tau^2}{q^2}\right) d\lambda \quad (4.35)$$

We still need to further manipulate the sum in order to get a numerical result for the Rényi entropy. Firstly, the sum can be computed using some identities, and in our case we need [5]:

$$\sum_{n_\tau=-\infty}^{\infty} \log\left(\frac{n_\tau^2}{q^2} + a^2\right) = \log(2 \cosh(2\pi q|a|) - 2) = 2 \log\left(1 - e^{-2\pi q|a|}\right) + 2\pi q|a| \quad (4.36)$$

Comparing to our above expression, we have to set  $|a| = \sqrt{\lambda + m^2}$ . From here we will consider  $m^2 = 0$  since we are interested in the conformal case. Thus, applying (4.36) to our previous expression we arrive at:

$$\mathcal{F}_q(m=0) = + \int_0^{\infty} \frac{\text{Vol}(\mathbb{H}^2)}{2\pi} d\lambda \tanh(\pi\sqrt{\lambda}) \left(\log\left(1 - e^{-2\pi q\sqrt{\lambda}}\right) + \pi q\sqrt{\lambda}\right) \quad (4.37)$$

There is an additional (but expected) problem: The integral is divergent in the region where  $\lambda \gg 1$ . This is expected because we know that the entanglement entropy is divergent due to short distances or high energy modes. These are precisely the modes at  $\lambda \gg 1$ . Then, we can regularize the result by subtracting the term that causes the divergence, namely

$$\lim_{\lambda \rightarrow \infty} \frac{\text{Vol}(\mathbb{H}^2)}{2\pi} \tanh(\pi\lambda)(\pi q\sqrt{\lambda}) = \frac{\text{Vol}(\mathbb{H}^2)}{2} \sqrt{\lambda} \quad (4.38)$$

Subtracting this we find

$$\frac{\text{Vol}(\mathbb{H}^2)}{2} \int_0^{\infty} d\lambda \left(\tanh(\pi\sqrt{\lambda}) - 1\right) \sqrt{\lambda} = -\frac{\text{Vol}(\mathbb{H}^2)}{2\pi} \frac{3\zeta(3)}{4\pi^2} \quad (4.39)$$

We prove this equation in Appendix A.

Finally, after the above regularization procedure we can give a finite expression for the free energy which is given by:

$$\mathcal{F}_q(m=0) = \frac{\text{Vol}(\mathbb{H}^2)}{2} \left[ -q \frac{3\zeta(3)}{4\pi^2} + \int_0^\infty d\lambda \tanh(\pi\sqrt{\lambda}) \log(1 - e^{-2\pi q\sqrt{\lambda}}) \right] \quad (4.40)$$

This integral can now be evaluated numerically for different values of  $q$ . Through expression (2.31) we can compute different values for the Rényi entropy, and in particular we computed it for  $q \in [1, 20]$ .

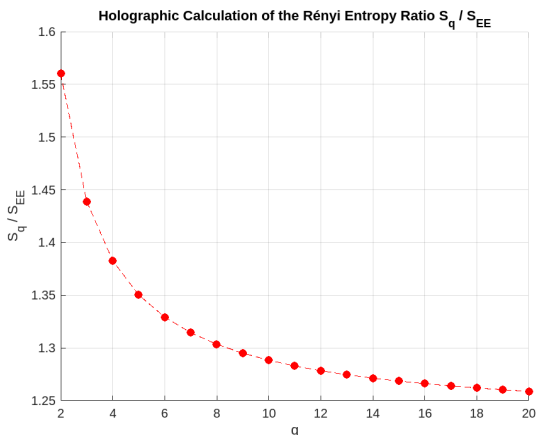
An important observation about (4.40) is that it is analytic in the parameter  $q$ .

Even though we will use the numerical expressions when comparing both results, there are analytical expressions for the integrals mentioned before [5]. For the first three values of the Rényi entropy, applying (2.31) we are left with the following expressions:

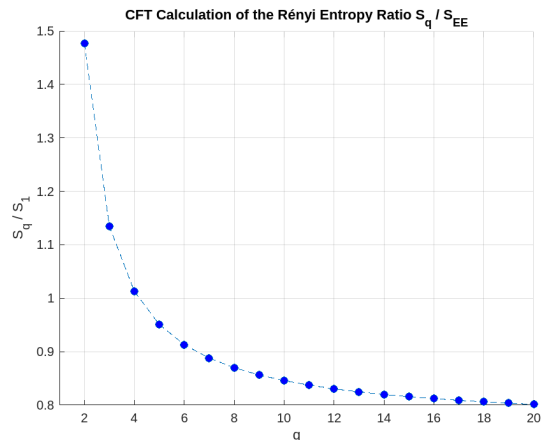
$$S_{\text{EE}} = -\mathcal{F}_1 = -\frac{\log 2}{4} + \frac{3\zeta(3)}{8\pi^2} \quad (4.41)$$

$$S_2 = \mathcal{F}_2 - 2\mathcal{F}_1 = -\frac{\log 2}{4} + \frac{7\zeta(3)}{8\pi^2} \quad (4.42)$$

$$S_3 = \frac{\mathcal{F}_3 - 3\mathcal{F}_1}{3} = -\frac{\log 2}{4} + \frac{13\zeta(3)}{24\pi^2} + \frac{\psi_1(\frac{1}{6}) + \psi_1(\frac{1}{3}) - \psi_1(\frac{2}{3}) - \psi_1(\frac{5}{6})}{144\sqrt{3}\pi} \quad (4.43)$$



(a) Holographic calculation for  $S_q/S_{\text{EE}}$



(b) CFT calculation for  $S_q/S_{\text{EE}}$

Figure 1: Comparison between holographic and CFT calculations of Rényi entropy.

## 5 Conclusions

We can now compare the results obtained in the previous two sections, namely the result for the Rényi entropy computed in the free theory and holographic CFTs in three dimensions. We consider the (normalization free) ratios  $S_q/S_1$ . As we see in Figure 1, even though the theories are very different, both results exhibit a very similar behavior.

Our main goal at the beginning of this work was to study entanglement entropies in the context of Quantum Field Theory. To achieve our goal we described first the theoretical setting, and then we focus on two explicit computations. The first one used holographic techniques from the

AdS/CFT correspondence. In particular we learned methods from General Relativity and Differential Geometry, that we used to study AdS and analyze a (topological) black hole geometry. For the other calculation we learned the path integral method and we applied results from PDEs and spectral theory in order to make explicit calculations.

With this thesis we have reviewed an important piece of knowledge in theoretical physics, and the most recent techniques in the field.

At the same time, we learned about many more areas of lively research and interest, in particular the possibility of applying quantum information theory to gravity. We also touched upon some other developments that have been left uncovered in thesis, such as the Ryu-Takayanagi formula [17], and other properties of the CFTs, and the path integral formalism that are useful to compute correlation functions.

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## A Trace anomaly and zeta function regularization

The regularization procedure we resorted to in the CFT calculation relied on subtracting the vacuum free energy for a box in  $\mathbb{R}^3$ . This is closely related to a characteristic behavior in conformal field theories known as the **Trace anomaly**. For classical field theories with conformal symmetry the trace of the energy momentum tensor vanishes. Nevertheless when quantifying the theory this may no longer hold. In particular for the discussion we will restrict ourselves to two dimensional theories. In this case the expectation value of  $T_\mu^\mu$  is given by:

$$\langle T_\mu^\mu \rangle = -\frac{c}{24\pi} \mathcal{R} \tag{A.1}$$

This indicates a failure in preserving conformal invariance when quantifying the theory. Considering locally conformally flat metrics, an infinitesimal transformation of the form  $g_{\mu\nu} \rightarrow e^{-2\lambda} \eta_{\mu\nu}$  where  $\eta_{\mu\nu}$  denotes the flat metric. The variation of the free energy is given by:

$$\delta\mathcal{F} = -\frac{1}{2} \int_M dx^2 \lambda \langle T_\mu^\mu(x) \rangle = -\mathcal{R} \frac{c\lambda}{12\pi} \text{Vol}(M) \tag{A.2}$$

As we can see for non-compact manifolds with non zero curvature the trace anomaly makes us expect divergences in the free energy which was indeed the case for  $\mathbb{H}^2$ .

Another procedure to regularize divergent behavior is through **zeta function regularization**. We begin first with our main definition

**Definition A.1. Generalized zeta function:** Given the discrete spectrum of the differential operator  $A$ :  $\{\lambda_n\}_{i=n}^{\infty}$ , then the generalized zeta function is defined as:

$$\zeta(s) = \sum_n \lambda_n^{-s} \quad (\text{A.3})$$

From the properties of Gaussian integrals (??) the calculation of the partition function relies on the determinant of the differential operator  $\det A$ . This can be calculated using the following identity of the zeta function:

$$\frac{d}{ds} \zeta(s)|_{s=0} = -\log \left( \prod_n \lambda_n \right) = \log(\det A) \quad (\text{A.4})$$

Following the calculations in (4.11) we see that the free energy can be calculated from the derivative of the regularized zeta function:

$$\mathcal{F}_q \propto -\frac{1}{2} \zeta'(0) \quad (\text{A.5})$$

Thus, many of the properties we have dealt with along this work can be rewritten in terms of zeta-functions. Moreover they play a crucial role in analyzing asymptotic behavior from the expansion around different values of  $s$ .

For continuous spectrum the definition of the zeta function is more involved. For invertible elliptic differential operators<sup>3</sup> the generalized zeta function is defined through the Mellin transform as:

$$\zeta(s|A) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \text{Tr}[e^{-tA}] dt \quad (\text{A.6})$$

We are going to apply this definition to the divergent integral (4.37). Following our results on the Plancherel measure, the zeta function is:

$$\zeta(s|A) = \frac{\pi q}{\Gamma(s)} \int_0^{\infty} \lambda^{s-\frac{1}{2}} \tanh(\pi\sqrt{\lambda}) d\lambda = \frac{\pi q}{\Gamma(s)} \int_0^{\infty} \frac{-2\lambda^{s-\frac{1}{2}}}{e^{2\pi\sqrt{\lambda}} + 1} + \frac{1}{\Gamma(s)} \int_0^{\infty} \lambda^{s-1/2} \quad (\text{A.7})$$

From here we see that we recover the original expression for the value  $s = 1$ . The first integral is convergent around  $s = 1$ , but the second is not. We will remove from the divergent component of the zeta function. The expression we are left with is:

$$\zeta_{\text{reg}}(s|A) = \frac{\pi q}{\Gamma(s)} \int_0^{\infty} \frac{-2\sqrt{\lambda}}{e^{2\pi\sqrt{\lambda}} + 1} \quad (\text{A.8})$$

Evaluating this integral at  $s = 1$  gives us the previous result:

$$\zeta_{\text{reg}}(s|A) = \pi q \frac{3\zeta(3)}{4\pi^3} = q \frac{3\zeta(3)}{4\pi^2} \quad (\text{A.9})$$

---

<sup>3</sup>Although this definition is general, we are only interested in the case of the Laplacian which verifies said properties.