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CALCULATING ULTIMATE NON-RUIN PROBABILITIES WHEN CLAIM SIZES FOLLOW A GENERALIZED r-CONVOLUTION DISTRIBUTION FUNCTION

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Calculating ultimate non-ruin probabilities when claim sizes follow a generalized Γ-convolution distribution function.

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Abstract The non-ruin probability, $\Phi(u)$, for initial reserves u, in the classical can be calculated using the so-called Bromwich-Mellin inversion formula, an outstanding result from Residues Theory first introduced for these purposes by Seal(1977) for exponential claim size. We will use this technique when claim sizes follow a generalized $\Gamma-$ *convolution* function distribution. Some of the most frequently used heavy-tailed distributions in actuarial science belongs to this family. Thorin(1977) or Berg(1981) proved that Pareto distributions are members of this family; so Thorin(1977) did with Log-normal distributions.

1. Introduction

The problem of ruin in the collective risk theory has been extensively treated in actuarial literature using integral transforms. Since the paper by Sparre Andersen in 1955 many authors developed approximations for the ruin probability using Laplace- Stieltjes transforms. Cràmer(1955) used the Wiener-Hopf for the classical case and Thorin(1970,71,77) introduced the generalization when epochs of claims form a renewal process. Thorin and Wikstad(1973), Wikstad(1971,77) used the Piessen(1969) inversion method of the Laplace transforms and Bohman(1971,74,75) focussed on inversions of Fourier transforms and Seal(1971,74) dealt with both Laplace and Fourier numerical inversions. Seal(1975) obtained an interesting result for the classical case and exponential claim size distribution using the Bromwich-Mellin inversion formula for Laplace transforms.

We will use here a similar methodology for approximating ultimate non-ruin probabilities in the classical case when the claim sizes follow a generalized $\Gamma-$ *convolution* function. Some of the most frequently used heavy-tailed distributions in actuarial science belongs to this family. Thorin(1977) or Berg(1981) proved that Pareto distributions are members of this family;

so Thorin(1977) did with Log-normal distributions. Other interesting works related with $\Gamma -$ *convolution* functions are Thorin(1978) and Goovaerts et al.(1977).

In section 2 we will obtain the Laplace transform of the ultimate non-ruin probability function and the Bromwich-Mellin inversion formula is introduced. In Theorem 1 of section 3 the abcissa of convergence of the Laplace transform, $\Phi^*(s)$,when claims follow a generalized $\Gamma -$ *convolution* function is obtained. As a theoretical application, in section 4 we introduced a series expansions for $\Phi(u)$. Section 5 is devoted to concluding comments.

2. Laplace transform of the ultimate non-ruin probability function

The non-ruin probability could be expressed using the following integral equation(Voterra integral equation of the second kind):

$$\Phi(U) = \frac{\theta}{1+\theta} + \frac{1}{(1+\theta)\,p_1} \int_0^U \Phi\,(U-x)\,(1-F\,(x))\,dx \tag{2.1}$$

where $F\,(x)$ is the c.d.f. of the claim size and p_1 is the expected value of the claim size and θ the premium loading factor.

Using the Laplace transform of the former expression:

$$\begin{aligned}\Phi^*\,(s) &= \int_0^{+\infty} e^{-sx}\Phi\,(x)\,dx \\ &= \frac{\theta}{1+\theta}\left(\frac{1}{s}\right) + \frac{1}{(1+\theta)\,p_1}\Phi^*\,(s)\left(\frac{1}{s} - F^*\,(s)\right) \\ &\qquad \mathrm{Re}(s) > 0\end{aligned} \tag{2.2}$$

finally:

$$\begin{aligned}\Phi^*\,(s) &= \frac{\frac{\theta}{1+\theta}\left(\frac{1}{s}\right)}{1 - \frac{1}{(1+\theta)p_1}\left(\frac{1}{s} - F^*\,(s)\right)} \\ &= \frac{\frac{\theta}{1+\theta}}{s - \left(\frac{1}{(1+\theta)p_1}\right) + \left(\frac{1}{(1+\theta)p_1}\right) f^*\,(s)}\end{aligned} \tag{2.3}$$

where $f\,(x)$ is the d.f. of the claim size distribution and

$$f^*\,(s) = \int_0^{+\infty} f\,(x)\;e^{-sx}dx = sF^*(s) \tag{2.4}$$

In order to obtain the inverse Laplace transform we can use the Bromwich-Mellin inversion formula,

$$\Phi(u) = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} e^{su} \Phi^*(s)\, ds \tag{2.5}$$

where c is a positive real constant that exceeds the real part of all singularities of $\Phi^*(s)$ or abcissa of convergence. As long as $\Phi^*(s)$ is a fraction, we will be interested in the zeros of the denominator of 2.3

$$\begin{aligned} s - \left(\frac{1}{(1+\theta)\, p_1}\right) + \left(\frac{1}{(1+\theta)\, p_1}\right) f^*(s) &= 0 + i0 \\ \mathrm{Re}(s) &> 0 \end{aligned} \tag{2.6}$$

that can be expressed

$$\begin{aligned} \mathrm{Re}\left(s + \partial f^*(s)\right) &= \partial \\ \mathrm{Im}\left(s + \partial f^*(s)\right) &= 0 \end{aligned} \tag{2.7}$$

where, hereafter for simplicity

$$\partial = \left(\frac{1}{(1+\theta)\, p_1}\right)$$

If we write

$$\Phi^*(s) = \Phi^*(c+iy) = A(y) + iB(y)$$

where

$$\begin{aligned} A(y) &\equiv \mathrm{Re}(\Phi^*(c+iy)) \\ B(y) &\equiv \mathrm{Im}(\Phi^*(c+iy)) \end{aligned}$$

Seal(1977) obtained the following expression

$$\Phi(u) = \frac{2e^{cu}}{\pi} \int_0^{\infty} A(y) \cos(uy) dy \tag{2.8}$$

very suitable for numerical integration purposes.

Let us focus our attention in finding the constant c when the distribution function of the claim size is a generalized Γ−convolution.

3. Singularities of the Laplace transform of $\Phi(u)$ when claim sizes follow a generalized Γ−convolution function

A distribution function defined on the non-negative Real axis is a generalized $\Gamma -$ *convolution* if its Laplace transform can be written,

$$f^*(s) = \int_0^{+\infty} f(x)\ e^{-sx} dx = e^{-as}\ e^{\int_0^\infty \ln\left(\frac{1}{1+\left(\frac{s}{y}\right)}\right) dU(y)} \qquad \mathrm{Re}(s) > 0$$

where $a \geq 0$ and U(y) is *nondecreasing* and such that

$$\begin{aligned} U(0) &= 0 \\ \int_0^1 |\ln(y)|\, dU(y) &< \infty \\ \int_0^\infty \frac{dU(y)}{y} &< \infty \end{aligned}$$

Some of the most frequently used heavy-tailed distributions in actuarial science belongs to this family. Thorin(1977) or Berg(1981) proved that Pareto distributions are members of this family; so Thorin(1977) did with Log-normal distributions.

Theorem 1. *The abcissa of convergence, c, of the Laplace transform of the non-ruin probability has a lower bound at* 2∂

$$c \geq 2\partial \qquad c > 0 \tag{3.1}$$

Proof. *Substituting in 2.7 we get,*

$$\begin{aligned} \mathrm{Re}\left(s + \partial e^{-as}\ e^{\int_0^\infty \ln\left(\frac{1}{1+\left(\frac{s}{y}\right)}\right) dU(y)}\right) &= \partial \\ \mathrm{Im}\left(s + \partial e^{-as}\ e^{\int_0^\infty \ln\left(\frac{1}{1+\left(\frac{s}{y}\right)}\right) dU(y)}\right) &= 0 \end{aligned} \tag{3.2}$$

defining

$$s = \alpha + \beta i \qquad \alpha > 0$$

we can write,

$$\begin{aligned} e^{-as}\ e^{\int_0^\infty \ln\left(\frac{1}{1+\left(\frac{s}{y}\right)}\right) dU(y)} &= \left(e^{-a\alpha} r\right)\ e^{i(-a\beta+\theta)} = R\ e^{i\Theta} \\ &= R\cos(\Theta) + iR\sin(\Theta) \end{aligned}$$

bearing in mind that,

$$\ln\left(\frac{y}{y+\alpha+\beta i}\right) = \ln(y) - \ln\left((y+\alpha)+\beta i)\right)$$
$$= \ln(y) - \ln(\sqrt{(y+\alpha)^2+\beta^2}) - \arctan\left(\frac{\beta}{y+\alpha}\right) i$$

we get

$$e^{\int_0^\infty \ln\left(\frac{1}{1+\left(\frac{s}{y}\right)}\right) dU(y)} = e^{\int_0^\infty \ln\left(\frac{y}{y+s}\right) dU(y)}$$
$$= \left(e^{-\frac{1}{2}\int_0^\infty \ln\left(1+2\left(\frac{\alpha}{y}\right)+\left(\frac{\alpha}{y}\right)^2+\left(\frac{\beta}{y}\right)^2\right) dU(y)}\right)$$
$$e^{i\left(-\int_0^\infty \arctan\left(\frac{\beta}{y+\alpha}\right) dU(y)\right)}$$
$$= r\, e^{i\theta}$$

as long as,

$$1+2\left(\frac{\alpha}{y}\right)+\left(\frac{\alpha}{y}\right)^2+\left(\frac{\beta}{y}\right)^2 > 1 \Longrightarrow$$
$$\ln\left(1+2\left(\frac{\alpha}{y}\right)+\left(\frac{\alpha}{y}\right)^2+\left(\frac{\beta}{y}\right)^2\right) > 0$$

and

$$\int_0^\infty \ln\left(1+2\left(\frac{\alpha}{y}\right)+\left(\frac{\alpha}{y}\right)^2+\left(\frac{\beta}{y}\right)^2\right) dU(y) > 0$$

because U(y) is nondecreasing($U'(y) \geq 0$),

$$\begin{aligned} r &< 1 \Longrightarrow R<1 \\ a &\geq \quad \alpha > 0 \end{aligned} \tag{3.3}$$

finally in 3.2

$$\begin{aligned} \alpha + \partial R\ \cos(\Theta) &= \partial \\ \beta + \partial R\ \sin(\Theta) &= 0 \end{aligned} \tag{3.4}$$

it is clear from 3.3 that

$$R\ \cos(\Theta) \in (-1, 1) \tag{3.5}$$

Let us suppose now that

$$\begin{aligned} \alpha &= 2\partial + x \quad x \geq 0 \\ \alpha &\geq 2\partial \end{aligned}$$

we get from the first equation of 3.4 the following inequality

$$R \cos(\Theta) = -1 - \frac{x}{\partial} \leq -1$$

what it is a contradiction with 3.5. ∎

4. Series expansions

As an application we will now obtain a series expansion for the non-ruin probability. Using 2.8 and expanding cos(uy)

$$\begin{aligned} \Phi(u) &= \frac{2e^{cu}}{\pi} \int_0^\infty A(y) \cos(uy) dy \\ &= \frac{2e^{cu}}{\pi} \sum_{n=0}^{\infty} a_n \frac{u^{2n}}{(2n)!} \end{aligned}$$

where,

$$a_n = (-1)^n \int_0^\infty A(y)\, y^{2n}\, dy$$

and the series expansion for the cosinus

$$\cos(uy) = \sum_{n=0}^{\infty} (-1)^n \frac{(uy)^{2n}}{(2n)!}$$

5. Concluding comments

The non-ruin probability, $\Phi(u)$, for initial reserves u, in the classical can be calculated using the so-called Bromwich-Mellin inversion formula, an outstanding result from Residues Theory first introduced for these purposes by Seal(1977) for exponential claim size. We will use this technique when claim sizes follow a generalized $\Gamma - \mathit{convolution}$ function distribution. Some of the most frequently used heavy-tailed distributions in actuarial science belongs to this family. Thorin(1977) or Berg(1981) proved that Pareto distributions are members of this family; so Thorin(1977) did with Log-normal distributions.

We proved in theorem 1 that the abcissa of convergence, c (in other words, the positive real constant that exceeds the real part of all singularities of $\Phi^*(s)$) has lower bound at $\frac{2}{(1+\theta)p_1}(2\partial)$. Subsequently, the ultimate non-ruin probability can be obtained using 2.8,

$$\Phi(u) = \frac{2e^{2\partial u}}{\pi} \int_0^\infty A(y) \cos(uy) dy \tag{5.1}$$

and

$$A(y) \equiv \mathrm{Re}(\Phi^* (2\partial + iy))$$

where $\Phi^* (2\partial + iy)$is calculated using 2.3.

Numerical approximations of integral 5.1 could be performed using Gaussian integration, Newton-Cotes based techniques or Clenshaw-Curtis method until the necessary precision is required. Let us state that the number of significant digits required will be,at least, of the magnitud of $\log_{10}(e^{2\partial u})$,increasing linearly with u.

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