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Multiseries Lie groups and asymptotic modules for characterizing and solving integrable models

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A multiseries integrable model (MSIM) is defined as a family of compatible flows on an infinite-dimensional Lie group of N-tuples of formal series around N given poles on the Riemann sphere. Broad classes of solutions to a MSIM are characterized through modules over rings of rational functions, called asymptotic modules. Possible ways for constructing asymptotic modules are Riemann–Hilbert and $\bar{\partial}$ problems. When MSIM's are written in terms of the "group coordinates," some of them can be "contracted" into standard integrable models involving a small number of scalar functions only. Simple contractible MSIM's corresponding to one pole, yield the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy. Two-pole contractible MSIM's are exhibited, which lead to a hierarchy of solvable systems of nonlinear differential equations consisting of $(2+1)$-dimensional evolution equations and of quite strong differential constraints.

I. INTRODUCTION

During the past ten years or so the application of Lie-algebraic methods has clarified and developed essential parts of the theory of integrable systems. These methods lead to simple geometric interpretations of integrable systems and exhibit important algebraic properties.[1–3] Moreover, they provide a link between two basic problems of the theory: namely, to classify integrable models and describe their solutions. The purpose of this paper is to present a Lie-algebraic scheme that allows us to determine new hierarchies of integrable systems and analyze relevant families of their solutions. We use the term integrable because there are natural methods for constructing solutions to these systems. However, we are not concerned here with aspects such a Hamiltonian formalism or constants of motion (see Refs. 1–3). The main points of our approach are as follows.

(1) A general class of integrable models called multiseries integrable models (MSIM's) is introduced. A MSIM is defined as a family of compatible flows on an infinite-dimensional Lie group of N-tuples of formal series with matrix-valued coefficients around N given poles on the Riemann sphere S. These flows arise as a consequence of the presence of a double Lie algebra structure $\mathscr{G} = \mathscr{G}_+ + \mathscr{G}_-$. Here $\mathscr{G}_+$ is isomorphic to a subset of an associative algebra $\mathscr{R}$ of matrix-valued rational functions of a complex variable k with given poles.

(2) When MSIM's are written in terms of the "group coordinates," some of them can be "contracted" into standard integrable models consisting of systems of nonlinear differential equations (NDE's) involving a small number of scalar functions only. In general, reductions of standard integrable models correspond to reductions of MSIM's. Simple contractible MSIM's corresponding to one pole yield the AKNS hierarchy.[4]

Two-pole contractible MSIM's are exhibited that lead to a hierarchy (H) of solvable systems of NDE's consisting of $(2+1)$-dimensional evolution equations and of quite strong differential constraints. Among these systems we mention

$$q_t = \alpha(\tfrac{1}{4} q_{xxx} - \tfrac{3}{2} q^2 q_x) + \alpha'(-\tfrac{1}{4} q'_{xyy} + \tfrac{3}{2}(q')^2 q'_x), \quad (1.1a)$$

$$q'_t = \alpha'(\tfrac{1}{4} q'_{yyy} - \tfrac{3}{2}(q')^2 q'_y) + \alpha(-\tfrac{1}{4} q_{yxx} + \tfrac{3}{2} q^2 q_y), \quad (1.1b)$$

$$q_y = -q'_x, \quad \frac{q_{xy}}{q} = \frac{q'_{xy}}{q'}, \quad \left(\frac{q_{xy}}{4q}\right)^2 = 1 + \frac{(q_y)^2}{4}, \quad (1.1c)$$

where the unknowns are the complex functions $q(x,y,t)$ and $q'(x,y,t)$, α and α' are given complex numbers, and $q_t, q_x, \ldots$ mean $\partial_t q, \partial_x q, \ldots$. Observe that the system (1.1a)–(1.1c) describes a time evolution in the manifold of solutions to the bidimensional NDE's [(1.1b) and (1.1c)]. On the other hand, since (1.1c) implies $q'_y = \int 4q'\sqrt{1+\frac{1}{4}(q'_x)^2}\,dx$, i.e., y derivations correspond to some x integrations, the system (1.1a)–(1.1c) may be interpreted formally as an integrodifferential evolution equation in $(1+1)$ dimensions.

Note also that for $\alpha = -4$ and $\alpha' = 0$, (1.1a) reduces to the modified Korteweg-de Vries (MKdV) equation, $q_t + q_{xxx} - 6q^2 q_x = 0$, relative to the variables (x,t).

(3) Certain objects called asymptotically normalized wave functions (NW functions) and asymptotic modules

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(AM's) play a fundamental role for characterizing broad classes of solutions to MSIM's and as a consequence to their associated standard integrable models [such as (1.1a)–(1.1c)] in the contractible cases. An AM is a module over the previously introduced ring $\mathscr{R}$. Both the NW functions and the elements of AM's are functions of k admitting specific asymptotic expansions (AE's) around the given poles. For a considered MSIM, each NW function determines one solution. On the other hand any AM provides a new NW function from a given NW function, i.e., we thus obtain an iterative procedure for generating solutions and, in particular, for adding solitons. Possible ways for constructing AM's are Riemann–Hilbert problems and $\bar{\partial}$ (DBAR) equations outside of given poles on the Riemann sphere.

Concerning points (1) and (2), we are inspired by the theory of the Kadomtsev–Petviashvili (KP) hierarchy, more specifically, by its formulation as the system of compatible flows,[5,6]

$$\frac{\partial K}{\partial t_r} = (K D^r K^{-1})_+ K - K D^r, \quad r \geqslant 1, \tag{1.2}$$

where K lies on the Volterra group of pseudodifferential operators,

$$K = 1 + \sum_{n=1}^{\infty} a_n(x,t) D^{-n}, \quad t = (t_r), \quad D = \partial_x, \tag{1.3}$$

and the subscript $+$ in (1.2) means the differential operator part. On the other hand, we remark that the notion of loop group used for the AKNS hierarchy[1] is a particular case of that group of formal multiseries considered here.

With respect to point (3), we recall that $\bar{\partial}$ equations were very useful for extending the range of application of the inverse scattering transform method of the solution for (1 + 1) NDE's and (2 + 1) NDE's.[7] Subsequently, $\bar{\partial}$ equations were considered as the starting point for introducing and solving NE's.[8,9(a),10] As a matter of fact, the concept of AM is motivated partly by the analysis of the algebraic structure underlying the use of $\bar{\partial}$ equations in Refs. 9(a) and 10. Further motivations come from other important methods[11] for solving integrable models such as the construction of finite gap solutions in the framework of algebraic geometry and that of rational solutions in the context of the Grassmannian formalism where modules over polynomial rings also occurs.

For these cases the role of the NW functions is played by the Baker functions. In some sense AM's provide the bridge between the Grassmannian formalism and the $\bar{\partial}$ and Zakharov–Shabat dressing methods[12] for integrable systems of the AKNS type. On the other hand, they show the group-theoretical content of these solution methods. However, if one is more interested in the construction of solutions to NDE's than in the group aspects, an economical scheme based on AM's can be restated. On this point we refer to Ref. 9(b), where additional information can be found with regard to the hierarchy (H) containing (1.1a)–(1.1c). Other hierarchies of evolution NDE's, with constraints solvable within the framework of the AM scheme, are investigated in Ref. 9(c) [(2 + 1)-dimensional case] and Ref. 9(d) [$(N+1)$-dimensional case, $N \geqslant 1$]. Genuine (2 + 1)-dimensional equations can also be obtained in the context of AM's [see Ref. 9(e)]. The formulation of the associated Lie-algebraic approach would require the use of pseudodifferential operators in addition to that of formal multiseries [see (1.2) and (1.3) for the KP case]. Finally we notice that the AM scheme can be developed in discrete cases as well: see Ref. 9(f), where an integrable (2 + 1)-dimensional generalization of the Volterra model is derived.

This paper is organized as follows. Section II starts with an abstract introduction to the class of compatible families of flows used in our work. Then we define MSIM's in the one-pole case and we discuss the contractible models corresponding to the AKNS hierarchy. Section III deals with MSIM's in the general N-pole case. Some contractible models are exhibited for $N=2$. They yield the sinh–Gordon equation and the hierarchy (H), including the system (1.1a)–(1.1c). In Sec. IV we analyze solution methods to MSIM's from the point of view of NW functions and AM's. We exhibit Riemann–Hilbert and $\bar{\partial}$ realizations of AM's. Special attention is devoted to soliton solutions and some explicit examples are worked out in detail. In particular, we give a simple derivation of the Blaschke–Potapov factor[3] of soliton dressing. More general procedures for constructing asymptotic modules, based not only on the Riemann sphere but also on higher-genus Riemann surfaces, will be presented elsewhere,[9(g)] allowing us to characterize wide classes of solutions, including the rational and soliton classes as well as those arising in the finite-zone integration method. There are also two appendices. The first one considers some properties of differential polynomials that are used throughout the paper and the other includes the proof of the fundamental property of AM's.

II. MULTISERIES INTEGRABLE MODELS (MSIM's): PRELIMINARY MATERIAL AND THE ONE-POLE CASE

A. Double Lie algebras and compatible flows on Lie groups

Much of the geometric content of integrable systems is often related with the presence of a double Lie algebra structure; that is to say, a Lie algebra $\mathscr{G}$ that admits a decomposition into a linear direct sum of two Lie subalgebras,[2,3,13,14]

$$\mathscr{G} = \mathscr{G}_+ + \mathscr{G}_-. \tag{2.1}$$

In this subsection we are going to use this structure for defining compatible flows on Lie groups. Given a double Lie algebra $\mathscr{G}$ and $u \in \mathscr{G}$ we will denote by $u_\pm$ the projections of u on $\mathscr{G}_\pm$ associated with the decomposition (2.1), and by π the projection operator,

$$\pi(u) = u_+, \quad (1-\pi)(u) = u_-.$$

Let $\hat{\mathscr{G}}$ be a Lie group with Lie algebra $\mathscr{G}$ and $\hat{\mathscr{G}}_-$ a subgroup with Lie algebra $\mathscr{G}_-$. Given a commutative family of elements $\{c_i\}_1^s$ in $\mathscr{G}$,

$$[c_i, c_j] = 0, \tag{2.2}$$

we consider the following associated family of flows on $\hat{\mathscr{G}}_-$:

$$\partial_{t_i} g = -(g c_i g^{-1})_- g, \quad g \in \hat{\mathscr{G}}_-, \quad i = 1,\ldots,s. \tag{2.3}$$

Here, $g c_i g^{-1}$ denotes the image of c_i under the adjoint action of $g \in \hat{\mathscr{G}}_-$ on the Lie algebra $\mathscr{G}$ and $(g c_i g^{-1})_- g$ is the image of $(g c_i g^{-1})_- \in \mathscr{G}_-$ under the right-translation action of $g \in \hat{\mathscr{G}}_-$. Thus the right-hand side of (2.3) determines a well-defined vector field on $\hat{\mathscr{G}}_-$. The remarkable property of these flows is that they commute with each other, i.e., that the quantity

$$Y \doteq (\partial_{t_i}\partial_{t_j} g - \partial_{t_j}\partial_{t_i} g)g^{-1}$$

is zero. To prove this, we use the identities

$$Y = [(\partial_{t_j} g)g^{-1}, (\partial_{t_i} g)g^{-1}] + \partial_{t_i}((\partial_{t_j} g)g^{-1}) - \partial_{t_j}((\partial_{t_i} g)g^{-1}),$$

and

$$\partial_{t_i}(g c_j g^{-1}) = [(\partial_{t_i} g)g^{-1}, g c_j g^{-1}],$$

so that (2.3) implies

$$Y = [(q_i)_-, q_j]_- - [(q_j)_-, q_i]_- + [(q_j)_-, (q_i)_-], \tag{2.4}$$

where

$$q_i = g c_i g^{-1}. \tag{2.5}$$

Now from (2.2), $[q_i, q_j] = g[c_i, c_j]\, g^{-1} = 0$, and therefore

$$[(q_j)_-, q_i] = -[(q_j)_+, q_i] = -[(q_j)_+, (q_i)_+] - [(q_j)_+, (q_i)_-].$$

Hence

$$[(q_j)_-, q_i]_- = -[(q_j)_+, (q_i)_-]_-$$

since

$[(q_j)_+, (q_i)_+]_- = 0$ ($\mathscr{G}_+$ is a Lie subalgebra).

By inserting this into (2.4) we get

$$Y = [(q_i)_-, (q_j)_-]_- + [(q_j)_-, (q_i)_-].$$

Hence $Y = 0$ since

$$[(q_i)_-, (q_j)_-]_- = [(q_i)_-, (q_j)_-]$$

($\mathscr{G}_-$ is a Lie subalgebra).

As a consequence of the compatibility, it is reasonable to consider simultaneous solutions $g(t)$ $[t = (t_1, \ldots, t_s)]$ to the family of flows (2.3). These solutions constitute the main elements of our analysis in the specific cases described below where $\mathscr{G}$ is an algebra of formal multiseries.

It is worth noting that $\hat{\mathscr{G}}$, the Lie group with Lie algebra $\mathscr{G}$, plays no role in the above discussion. Indeed, the construction of the flows (2.3) only requires three basic objects: a double Lie algebra $\mathscr{G}$, a Lie group $\hat{\mathscr{G}}_-$ with Lie algebra $\mathscr{G}_-$, such that $\hat{\mathscr{G}}_-$ acts on $\mathscr{G}$ by means of the adjoint action, and some commutative subset $\{c_i\}_1^s$ of $\mathscr{G}$. This is an important fact, since in the applications we have to deal with infinite-dimensional Lie algebras for which $\hat{\mathscr{G}}$ turns out to be a much more complicated object than $\hat{\mathscr{G}}_-$. In the cases relevant to this paper, both structures, $\mathscr{G}$ and $\hat{\mathscr{G}}_-$, are immersed in some associative algebra of formal multiseries that allows us to define the adjoint action of $\hat{\mathscr{G}}_-$ on $\mathscr{G}$ in a natural way.

Finally we notice that it is sometimes useful to consider a family of reduced flows, i.e., a family of flows (2.3) on a subgroup $\hat{\mathscr{G}}'_-$ of $\hat{\mathscr{G}}_-$, which is invariant under certain automorphisms.

B. Definition of MSIM's in the one-pole case

Let $\mathscr{A}$ be the associative algebra of formal series of the form

$$u(k) = \sum_{n=-\infty}^{N} u_n k^n, \quad N \in \mathbb{Z}, \tag{2.6}$$

with $d \times d$ matrix coefficients u_n. The product uv in $\mathscr{A}$ is defined through a term by term series multiplication. The subset $\mathscr{G} = \{u \in \mathscr{A}$ s.t. tr $u_n = 0$ for all $n\}$ is a Lie algebra with the Lie product $[u,v] \doteq uv - vu$. Further, $\mathscr{G}$ admits a double Lie algebra structure determined by the projection operator

$$\pi(u) = \sum_{0 \leqslant n \leqslant N} u_n k^n. \tag{2.7}$$

The corresponding Lie subalgebras $\mathscr{G}_+$ and $\mathscr{G}_-$ of $\mathscr{G}$ are given by

$$\mathscr{G}_+ = \{u \in \mathscr{G} / u_n = 0 \text{ for all } n < 0\},$$
$$\mathscr{G}_- = \{u \in \mathscr{G} / u_n = 0 \text{ for all } n \geqslant 0\}.$$

Products of exponentials of elements in $\mathscr{G}_-$ generate a Lie group $\hat{\mathscr{G}}_- \subset \mathscr{A}$, whose elements are of the form

$$g(k) = \sum_{n=0}^{\infty} a_n k^{-n}, \quad a_0 = 1, \tag{2.8a}$$

and satisfy the constraint

$$\det g(k) = 1. \tag{2.8b}$$

We will refer to the coefficients $\{a_n\}_1^\infty$ as the coordinates of the group element g. Note that the inversion operation in $\hat{\mathscr{G}}_-$ can be performed as follows:

$$g^{-1} = 1 + \sum_{l=1}^{\infty} (-1)^l (g-1)^l = 1 - a_1 k^{-1} + \cdots. \tag{2.9}$$

Thus it is clear that $\hat{\mathscr{G}}_-$ has a well-defined adjoint action on $\mathscr{G}$,

$$\mathscr{G} \to \mathscr{G}, \quad \operatorname{ad} g(u) = g u g^{-1}.$$

Given a commutative family, $\{c_i\}_1^s \subset \mathscr{G}_+$, we define the associated MSIM as the family of compatible flows (2.3) on $\hat{\mathscr{G}}_-$. This definition corresponds to the case of a unique pole $k = \infty$ on the Riemann sphere. A reduction of a MSIM is defined as a family of flows (2.3) on a subgroup $\hat{\mathscr{G}}'_-$ of $\hat{\mathscr{G}}_-$, which is invariant under certain automorphisms.

C. Simple contractible MSIM's: The AKNS hierarchy

Suppose now that $d = 2$ and consider a commutative family in $\mathscr{G}_+$ of the form

$$c_i(k) = \omega_i(k)\sigma_3, \quad i = 1, \ldots, s, \tag{2.10}$$

where $\omega_i(k)$ are arbitrary polynomials in k and σ_3 is the Pauli matrix. The associated MSIM is the family of compatible flows,

$$\partial_{t_i} g = -(\omega_i(k) g \sigma_3 g^{-1})_- g, \quad g \in \hat{\mathscr{G}}_-. \tag{2.11}$$

Each of these equations can be described in terms of the coordinates $\{a_n\}_1^\infty$ of the group element (2.8a), so that (2.11) constitutes a system of equations with an infinite number of dependent scalar variables. However, because of the compatibility of this system, it is possible to deduce standard integrable models, i.e., differential equations involving a small number of dependent scalar variables. We say that the MSIM is "contractible."

Let us make explicit the case $s=2$, $c_1 = -ik\sigma_3$, $c_2 = i\omega(k)\sigma_3$ for an arbitrary polynomial $\omega(k)$. The associated MSIM consists of the two flows

$$\partial_x g = (ikg\sigma_3 g^{-1})_- g, \tag{2.12a}$$

$$\partial_t g = -(i\omega(k) g\sigma_3 g^{-1})_- g, \tag{2.12b}$$

where we have set $x = t_1$ and $t = t_2$. Let us introduce the following element of $\mathscr{G}$:

$$r = ig\sigma_3 g^{-1}, \tag{2.13}$$

which, according to (2.8) and (2.9), is of the form

$$r = \sum_{n=0}^{\infty} r_n k^{-n}, \quad r_0 = i\sigma_3. \tag{2.14}$$

One proves (see Appendix A) that the matrix elements of the coefficients r_n are polynomials in the matrix elements of $[\sigma_3, a_1]$ and their derivatives, with respect to x. Now, Eq. (2.12b) can be rewritten as

$$(\partial_t g) g^{-1} = (\pi - 1)(\omega(k) r(k)). \tag{2.15}$$

Then if

$$\omega(k) = \sum_{l=0}^{N} \alpha_l k^l,$$

by identifying the coefficients of k^{-1} in Eq. (2.15), we find at once

$$\partial_t a_1 = -\sum_{l=0}^{N} \alpha_l r_{l+1},$$

which implies

$$i\partial_t [\sigma_3, a_1] = -i \sum_{l=0}^{N} \alpha_l [\sigma_3, r_{l+1}]. \tag{2.16}$$

These differential equations are the members of the AKNS hierarchy. They involve the matrix elements of $[\sigma_3, a_1]$ only.

Reductions of AKNS hierarchy equations can be obtained by means of reductions of the MSIM (2.11). For example, the modified KdV hierarchy (MKdV) is characterized as follows. Let $\hat{\mathscr{G}}'_-$ be the set of elements $g \in \hat{\mathscr{G}}_-$, verifying

$$\sigma_1 g(-k)\sigma_1 = g(k). \tag{2.17}$$

Clearly $\hat{\mathscr{G}}'_-$ is a subgroup of $\hat{\mathscr{G}}_-$ and its Lie algebra $\mathscr{G}'_-$ is given by the elements of $\mathscr{G}_-$ satisfying (2.17). Now if we take $c_i(k) = \omega_i(k)\sigma_3$ with $\omega_i(k)$ being odd polynomials in k, then the right-hand side of (2.11) determines a vector field on $\hat{\mathscr{G}}'_-$ and consequently we obtain a reduction on $\hat{\mathscr{G}}'_-$ of the MSIM (2.11). Since $\sigma_1 a_1 \sigma_1 = -a_1$ for the coordinate a_1 of $g \in \hat{\mathscr{G}}'_-$, these reduced MSIM's impose a compatible constraint to the AKNS equations (2.16), which turns out to yield the members of the MKdV hierarchy (see Appendix A).

III. MULTISERIES INTEGRABLE MODELS (MSIM's): THE GENERAL CASE

A. Formal multiseries

In order to apply the construction of compatible flows of Sec. II A to define general MSIM's, we will use algebras of multiseries as the basic algebraic objects, so it is convenient to introduce some appropriate notation conventions.

Given a positive integer d and N different points $\{k_n\}_1^N$ on the Riemann sphere $\mathbb{S}$ with $k_1 = \infty$, let $\mathscr{A}$ be the set of N-tuples of the formal series

$$u = (u_1(k), \ldots, u_N(k)), \tag{3.1a}$$

$$u_1(k) = \sum_{m=-\infty}^{M_1} u_{1m} k^m, \tag{3.1b}$$

$$u_n(k) = \sum_{m=M_n}^{\infty} u_{nm} (k - k_n)^m, \quad n = 2, \ldots, N, \tag{3.1c}$$

where $M_n \in \mathbb{Z}$ and u_{nm} are $d \times d$ matrix coefficients. We point out that all the formal series involved here are of finite order M_n at their corresponding reference points. Therefore, besides the usual notions of sum and multiplication by complex numbers, we can define a product operation in $\mathscr{A}$,

$$uu' = (u_1(k) u'_1(k), \ldots, u_N(k) u'_N(k)),$$

where the products $u_n(k) u'_n(k)$ are understood in the sense of a term by term series multiplication. With these operations $\mathscr{A}$ becomes an associative algebra.

Furthermore, let $\mathscr{R}$ be the associative algebra of $d \times d$ matrix-valued rational functions on $\mathbb{S}$ with possible poles at $\{k_n\}_1^N$ only. Given $U \in \mathscr{R}$, let us denote by $u_n(k)$ its corresponding Laurent series at $k = k_n$. Then the map

$$\mathscr{R} \xrightarrow{\tau} \mathscr{A}, \quad \tau(U) = (u_1(k), \ldots, u_N(k)) \tag{3.2}$$

is an injective homomorphism between the associative algebras $\mathscr{R}$ and $\mathscr{A}$.

The following linear map will be particularly important in our discussion:

$$\mathscr{A} \xrightarrow{p} \mathscr{R}, \quad p(u) = \sum_{0 \leqslant m \leqslant M_1} u_{1m} k^m + \sum_{n=2}^{N} \left(\sum_{M_n \leqslant m < 0} u_{nm} (k - k_n)^m \right). \tag{3.3}$$

Here u_{nm} are the coefficients of the formal series $u_n(k)$ that determine $u \in \mathscr{A}$ [see (3.1)]. Observe that $p(u)$ is obtained by adding the principal parts of the series $u_n(k)$ $(n = 1, \ldots, N)$ and the constant term of $u_1(k)$. Now consider the composition of τ and p,

$$\mathscr{A} \xrightarrow{\tau} \mathscr{A}, \quad \pi = \tau \circ p. \tag{3.4}$$

It follows at once from the partial fractial decomposition theorem that $p \circ \tau = \mathrm{Id}_{\mathscr{R}}$, so that π is a projection operator on the vector space $\mathscr{A}$ that determines a decomposition of $\mathscr{A}$ into a linear direct sum of two subalgebras,

$$\mathscr{A} = \mathscr{A}_+ + \mathscr{A}_-, \tag{3.5}$$

where $\mathscr{A}_+ = \operatorname{Ran} \pi = \tau(\mathscr{R})$ and $\mathscr{A}_- = \operatorname{Ker} \pi$ consists of the elements (3.1), such that $M_1 = -1$ and $M_n = 0$ for $n = 2, \ldots, N$.

B. Definition of MSIM's

We are now ready to generalize the definition of MSIM's given in Sec. II B by considering Lie algebras and groups contained in the associative algebra $\mathscr{A}$ of multiseries (3.1). The subset $\mathscr{G} = \{u \in \mathscr{A} \text{ s.t. tr } u_{nm} = 0 \text{ for all } n,m\}$ is a Lie algebra with the Lie product $[u,v] \doteq uv - vu$. Here $\mathscr{G}$ is an invariant subspace of $\mathscr{A}$ under the projection operator π of (3.4), and the corresponding restriction $\mathscr{G} \xrightarrow{\pi} \mathscr{G}$ determines a double Lie algebra structure on $\mathscr{G}$ with $\mathscr{G}_{\pm} = \mathscr{G} \cap \mathscr{A}_{\pm}$.

By means of the exponentials of elements of $\mathscr{G}_-$ one generates a Lie group $\hat{\mathscr{G}}_-$ that consists of the elements of $\mathscr{A}$,

$$g = (g_1(k),\ldots,g_N(k)),$$

$$g_1(k) = 1 + \sum_{m=1}^{\infty} g_{1m} k^{-m}, \tag{3.6}$$

$$g_n(k) = \sum_{m=0}^{\infty} g_{nm}(k-k_n)^m, \quad n = 2,\ldots,N,$$

such that

$$\det g_n(k) = 1, \quad n = 1,\ldots,N. \tag{3.7}$$

We will refer to the coefficients g_{nm} as the coordinates of $g \in \hat{\mathscr{G}}_-$. It is easy to see that $\hat{\mathscr{G}}_-$ has a well-defined adjoint action on $\mathscr{G}$.

Henceforth formal Lie algebras and groups of N-tuples of $d \times d$ matrix-valued series centered at $\{k_n\}_1^N$ will be called multiseries Lie algebras and groups with reference points $\{k_n\}_1^N$.

At this point we can take any commutative family $\{c_i\}_1^s \subset \mathscr{G}_+$ and define the associated MSIM as the family of compatible flows (2.3) on $\hat{\mathscr{G}}_-$. Equations (2.3) can be described in terms of the coordinates g_{nm} of g so that they constitute a system of equations with an infinite number of dependent scalar variables. However, because of the compatibility of this system, in some cases it is possible to deduce standard integrable models, i.e., differential equations involving a small number of dependent scalar variables. Then we say that the MSIM is contractible. Many integrable models in $1+1$ dimensions can be obtained by the contraction of MSIM's. Among these are the Toda lattice, the sine–Gordon, and the Chiral-field models.

A reduction of a MSIM is defined as a family of flows (2.3) on a subgroup $\hat{\mathscr{G}}'_-$ of $\hat{\mathscr{G}}_-$ that is invariant under certain automorphisms. In general, reductions of standard integrable models correspond to reductions of MSIM's.

As an illustration of the above abstract considerations we now analyze the simple case given by $d=2$, $N=2$, and $\{k_1 = \infty, k_2 = 0\}$. Here the elements of $\mathscr{A}$ can be written as $u = (u_1(k), u_2(k))$ with

$$u_1(k) = \sum_{m=-\infty}^{M_1} u_{1m} k^m,$$
$$u_2(k) = \sum_{m=M_2}^{\infty} u_{2m} k^m, \quad M_1, M_2 \in \mathbb{Z}, \tag{3.8}$$

where u_{nm} are 2×2 matrices. In this particular case the map (3.2) takes a very simple form. Indeed since $\mathscr{R}$ is now the set of rational functions on S with poles at $k_1 = \infty$ and $k_2 = 0$ only, its elements are finite sums of the form $U = \Sigma_{m=N}^{M} u_m k^m$, with $M, N \in \mathbb{Z}$ $(M \geqslant N)$. Hence

$$\tau(U) = (U(K), U(K)), \quad U \in \mathscr{R}. \tag{3.9}$$

The Lie algebra $\mathscr{G}$ and the Lie group $\hat{\mathscr{G}}_-$ are readily characterized as subsets of $\mathscr{A}$. We express the elements $g = (g_1(k), g_2(k))$ of $\hat{\mathscr{G}}_-$ as

$$g_1(k) = 1 + \sum_{m=1}^{\infty} a_m k^{-m}, \quad g_2(k) = \sum_{m=0}^{\infty} b_m k^m. \tag{3.10}$$

In the following subsections we give some examples of contractible MSIM's in the case $d=2$, $N=2$, and $\{k_1 = \infty, k_2 = 0\}$. Some N-pole contractible MSIM's (with $N > 2$) will be investigated elsewhere.

C. A contractible two-pole MSIM: The sinh–Gordon equation

Let us consider the MSIM defined by the following pair of compatible flows on $\hat{\mathscr{G}}_-$:

$$\partial_x g = -(gcg^{-1})_- g, \quad \partial_y g = -(gc'g^{-1})_- g, \tag{3.11}$$

associated with the following commuting elements in $\hat{\mathscr{G}}_+$:

$$c = \tau(-ik\sigma_3) = (-ik\sigma_3, -ik\sigma_3), \tag{3.12a}$$

$$c' = \tau(-ik^{-1}\sigma_3) = [-i(\sigma_3/k), -i(\sigma_3/k)]. \tag{3.12b}$$

From the definition (3.3) of the map p we have

$$p(gcg^{-1}) = -ik\sigma_3 + i[\sigma_3, a_1],$$

$$p(gc'g^{-1}) = -(i/k) b_0 \sigma_3 b_0^{-1}.$$

Then, taking into account (3.4) and (3.9), Eq. (3.11) implies that

$$\partial_x g_j = ik[g_j, \sigma_3] + i[\sigma_3, a_1] g_j, \tag{3.13a}$$

$$\partial_y g_j = (i/k) b_0 [b_0^{-1} g_j, \sigma_3], \quad j = 1,2. \tag{3.13b}$$

By substituting the expansions (3.10) into (3.13) and identifying the terms in k^0 and $1/k$ in (3.13a) and (3.13b), respectively, we get

$$\partial_x b_0 = i[\sigma_3, a_1] b_0, \quad \partial_y a_1 = i[\sigma_3, b_0] b_0^{-1}, \tag{3.14}$$

which imply a differential equation for b_0,

$$\partial_y[(\partial_x b_0) b_0^{-1}] = [\sigma_3, b_0 \sigma_3 b_0^{-1}]. \tag{3.15}$$

Furthermore, since $\det g_2(k) = 1$, b_0 satisfies the constraint $\det b_0 = 1$.

Now we consider the reduction of the previous MSIM on the group $\hat{\mathscr{G}}'_-$ of elements, $g = (g_1, g_2) \in \hat{\mathscr{G}}_-$, such that g_j $(j = 1,2)$ satisfies (2.17). [This is possible because $-ik\sigma_3$ and $-ik^{-1}\sigma_3$ satisfy (2.17).] Under this assumption we deduce $\sigma_1 b_0 \sigma_1 = b_0$, which together with $\det b_0 = 1$ implies the following form for b_0:

$$b_0 = \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix}, \tag{3.16}$$

where $\varphi \in \mathbb{C}$. Now from (3.15) one finds at once the sinh–Gordon equation for φ,

$$\partial_{xy}\varphi = -2\sinh(2\varphi). \tag{3.17}$$

D. Contractible two-pole MSIM's: A hierarchy of evolution (2+1) NDE's with constraints

Let us consider the MSIM obtained by adding the following flow to the system (3.11):

$$\partial_t g = -(gc''g^{-1})_- g, \tag{3.18}$$

where $c'' \in \mathcal{G}_+$ is defined by

$$c'' = \tau(i\omega(k)\sigma_3) = (i\omega(k)\sigma_3, i\omega(k)\sigma_3), \tag{3.19a}$$

with

$$\omega(k) = \sum_{l=-M}^{N} \alpha_l k^l, \quad N,M>0. \tag{3.19b}$$

Now the solutions to (3.11) and (3.18) will depend on three variables (x,y,t). In order to investigate the differential equations satisfied by the coordinates of g we rewrite (3.18) as

$$(\partial_t g)g^{-1} = (\pi - 1)(g\tau(i\omega\sigma_3)g^{-1}). \tag{3.20}$$

It is easy to see that $g\tau(i\omega\sigma_3)g^{-1}$ can be written in the form

$$g\tau(i\omega\sigma_3)g^{-1} = (\omega r, \omega b_0 r' b_0^{-1}), \tag{3.21a}$$

where r and r' are defined by

$$r = g_1 i\sigma_3 g_1^{-1}, \quad r' = g_1' i\sigma_3 g_1'^{-1}, \tag{3.21b}$$

with

$$g_1' \doteq b_0^{-1} g_2 = 1 + \sum_{m=1}^{\infty} a_m' k^m, \quad a_m' \doteq b_0^{-1} b_m. \tag{3.22}$$

Both r and r' are interesting objects since they depend on a few group coordinates only. Indeed, they are of the form

$$r = \sum_{n=0}^{\infty} r_n k^{-n}, \quad r' = \sum_{n=0}^{\infty} r_n' k^n, \quad r_0 = r_0' = i\sigma_3, \tag{3.23}$$

with the coefficients r_n (resp. r_n') being polynomials in the off-diagonal elements of a_1 (resp. of $a_1' = b_0^{-1} b_1$) and their derivatives with respect to the x (resp. y) variable. To prove this we notice that

$$\operatorname{tr} r = \operatorname{tr} r' = 0, \quad \det r = \det r' = 1. \tag{3.24}$$

On the other hand, we have

$$\partial_x r = -ik[\sigma_3, r] + i[[\sigma_3, a_1], r], \tag{3.25a}$$

$$\partial_y r' = -(i/k)[\sigma_3, r'] + i[[\sigma_3, a_1'], r']. \tag{3.25b}$$

Equation (3.25a) is an immediate consequence of Eq. (3.13a) for g_1, whereas Eq. (3.25b) follows from the equation

$$\partial_y g_1' = (i/k)[g_1', \sigma_3] + i[\sigma_3, a_1']g_1', \tag{3.26}$$

which in turn derives from Eq. (3.13b) for g_2 by taking into account that this latter implies

$$\partial_y b_0 = i b_0 [a_1', \sigma_3]. \tag{3.27}$$

Now, as it is proved in Appendix A, Eqs. (3.23)–(3.25) are all we need to conclude that r_n (resp. r_n') is a polynomial in q, s (resp. q', s') and their x derivatives (resp. y derivatives), where

$$i[\sigma_3, a_1] = -\begin{pmatrix} 0 & q \\ s & 0 \end{pmatrix}, \quad i[\sigma_3, a_1'] = -\begin{pmatrix} 0 & q' \\ s' & 0 \end{pmatrix}. \tag{3.28}$$

Moreover, in view of (3.25) and (3.28), it is clear that r_n' is obtained from r_n by replacing $(\partial_x^m q, \partial_x^m s)$ with $(\partial_y^m q', \partial_y^m s')$ for all $m \geqslant 0$.

We are now ready to write (3.20) in terms of the coordinates of g. However, we are only interested in obtaining differential equations for the first few coordinates. In this sense observe that

$$(\partial_t g)g^{-1} = \left(\frac{\partial_t a_1}{k} + O\left(\frac{1}{k^2}\right), (\partial_t b_0)b_0^{-1} + b_0(\partial_t a_1')b_0^{-1}k + O(k^2)\right).$$

Furthermore, from (3.3) and (3.19)–(3.23) it follows that

$$p(g\tau(i\omega\sigma_3)g^{-1}) = \sum_{n=0}^{N} k^n \left(\sum_{l=n}^{N} \alpha_l r_{l-n}\right) + \sum_{n=1}^{M} \frac{1}{k^n}\left(\sum_{l=n}^{M} \alpha_{-l} b_0 r_{l-n}' b_0^{-1}\right).$$

Then, since $\pi = \tau \circ p$ and taking (3.9) into account by identifying coefficients in (3.20), we get

$$i\partial_t[\sigma_3, a_1] = -i \sum_{l=-1}^{N} \alpha_l [\sigma_3, r_{l+1}] + i \sum_{l=1}^{M} \alpha_{-l}[\sigma_3, b_0 r_{l-1}' b_0^{-1}], \tag{3.29a}$$

$$i\partial_t[\sigma_3, a_1'] = -i \sum_{l=-1}^{M} \alpha_{-l}[\sigma_3, r_{l+1}'] + \sum_{l=1}^{N} \alpha_l [\sigma_3, b_0^{-1} r_{l-1} b_0], \tag{3.29b}$$

$$\partial_t b_0 = \sum_{l=0}^{N} \alpha_l r_l b_0 - \sum_{l=0}^{M} \alpha_{-l} b_0 r_l'. \tag{3.29c}$$

From the properties of r_n and r_n' these differential equations involve the matrix elements of $[\sigma_3, a_1]$, $[\sigma_3, a_1']$, and b_0 only, and contain derivatives with respect to three variables x, y, and t. On the other hand, besides (3.29a)–(3.29c) we have to consider the equations corresponding to the flows (3.11) as well. These later derive from (3.13) and reduce to four additional relations:

$$i[\sigma_3, a_1] = (\partial_x b_0)b_0^{-1}, \quad i[\sigma_3, a_1'] = -b_0^{-1}\partial_y b_0, \tag{3.29d}$$

$$i\partial_y[\sigma_3, a_1] = [\sigma_3, b_0\sigma_3 b_0^{-1}],$$
$$i\partial_x[\sigma_3, a_1'] = [\sigma_3, b_0^{-1}\sigma_3 b_0]. \tag{3.29e}$$

By using (3.29d) we can express r_l and r_l' in terms of b_0 and its derivatives with respect to x and y. In this way (3.29c) becomes an evolution equation for the matrix b_0. The remaining constraints on b_0 follow from (3.29d) and (3.29e) and are resumed by the equation

$$\partial_y((\partial_x b_0)b_0^{-1}) = [\sigma_3, b_0\sigma_3 b_0^{-1}]. \tag{3.30}$$

Observe that Eqs. (3.29a) and (3.29b) are a consequence of (3.29c) and (3.29d). To see this point, it is enough to differentiate (3.29c) with respect to x and y, taking into account (3.29d) and the following equations:

$$b_0^{-1}(\partial_y r)b_0 = -(i/k)[\sigma_3, b_0^{-1} r b_0],$$
$$b_0(\partial_x r')b_0^{-1} = -ik[\sigma_3, b_0 r' b_0^{-1}],$$

which derive from (3.13) and imply

$$b_0^{-1}(\partial_y r_{l+1})b_0 = -i[\sigma_3, b_0^{-1} r_l b_0],$$
$$b_0(\partial_x r'_{l+1})b_0^{-1} = -i[\sigma_3, b_0 r'_l b_0^{-1}].$$

We also remark [see Ref. 9(b)] that by eliminating b_0 in Eqs. (3.29a) and (3.29b) it is possible to obtain a system of four evolution scalar NDE's with some differential constraints. These NDE's involve the four scalar functions q, s, q', and s' defined by (3.28).

As an illustration of the rich structure [Ref. 9(b)] that underlies Eqs. (3.29), we will analyze one of the reductions of the MSIM [(3.11)–(3.18)]. Suppose that $\omega(k)$ is an odd polynomial,

$$\omega(K) = \sum_{l=-M}^{N} \alpha_{2l+1} k^{2l+1}. \quad (3.31)$$

Then $i\omega(k)\sigma_3$ satisfies (2.17) and the three flows (3.11), (3.18) can be defined on the subgroup $\hat{\mathscr{G}}'_-$ of elements $g=(g_1,g_2)\in\hat{\mathscr{G}}_-$, such that g_j verifies (2.17) for $j=1,2$. Let g be a solution on the reduced group; then we have

$$\sigma_1 g_1(-k)\sigma_1 = g_1(k), \quad \sigma_1 g'_1(-k)\sigma_1 = g'_1(k), \quad (3.32)$$
$$\sigma_1 b_0 \sigma_1 = b_0. \quad (3.33)$$

As it is shown in Appendix A, (3.32) implies $q=s$ and $q'=s'$, that is to say

$$i[\sigma_3, a_1] = -q\sigma_1, \quad i[\sigma_3, a'_1] = -q'\sigma_1. \quad (3.34)$$

Furthermore, if we set

$$r_n = \begin{pmatrix} \xi_n & \eta_n \\ \gamma_n & -\xi_n \end{pmatrix}, \quad r'_n = \begin{pmatrix} \xi'_n & \eta'_n \\ \gamma'_n & -\xi'_n \end{pmatrix},$$

then (3.32) means that relations (A12) of Appendix A hold for both r_n and r'_n. Hence

$$r_{2n+1} = \eta_{2n+1}\sigma_1, \quad r'_{2n+1} = \eta'_{2n+1}\sigma_1. \quad (3.35)$$

On the other hand, as we saw in Sec. III C, (3.33) enables us to write b_0 in the form (3.16), which implies

$$(\partial_t b_0)b_0^{-1} = \varphi_t\sigma_1, \quad (\partial_x b_0)b_0^{-1} = \varphi_x\sigma_1,$$
$$b_0^{-1}(\partial_y b_0) = \varphi_y\sigma_1. \quad (3.36)$$

In this way, one easily finds that Eqs. (3.29c) and (3.29d) become

$$\varphi_t = \sum_{l=0}^{N} \alpha_{2l+1}\eta_{2l+1} - \sum_{l=1}^{M} \alpha_{-2l+1}\eta'_{2l-1}, \quad (3.37)$$
$$q = -\varphi_x, \quad q' = \varphi_y, \quad (3.38)$$
$$q_y = -q'_x = 2\sinh(2\varphi). \quad (3.39)$$

By using (3.38) we can express η_{2l+1} and η'_{2l-1} in terms of φ so that (3.37) represents a hierarchy of evolution NDE's in (2 + 1) dimensions for the single function φ. However, the function φ is subject to a differential constraint; it derives from (3.38) and (3.39) and takes the form

$$\varphi_{xy} = -2\sinh(2\varphi), \quad (3.40)$$

as it should be expected in view of (3.30) and of the results of Sec. III C. In addition, it is easy to see that differentiation of (3.37) with respect to x and y yields evolution NDE's for q and q' in (2 + 1) dimensions. Note also that (3.38) and (3.39) determine the differential constraints,

$$q_y = -q'_x, \quad \frac{q_{xy}}{q} = \frac{q'_{xy}}{q'}, \quad \left(\frac{q_{xy}}{4q}\right)^2 = 1 + \frac{(q_y)^2}{4}. \quad (3.41)$$

The first nontrivial example of (3.37) corresponds to $\omega(k) = \alpha k^3 + \alpha' k^{-3}$ and leads to

$$\varphi_t = \alpha(\tfrac{1}{4}\varphi_{xxx} - \tfrac{1}{2}(\varphi_x)^3) + \alpha'(\tfrac{1}{4}\varphi_{yyy} - \tfrac{1}{2}(\varphi_y)^3). \quad (3.42)$$

Differentiation of (3.42), with respect to x and y, gives

$$q_t = \alpha(\tfrac{1}{4}q_{xxx} - \tfrac{3}{2}q^2 q_x) + \alpha'(-\tfrac{1}{4}q'_{xyy} + \tfrac{3}{2}(q')^2 q'_x), \quad (3.43a)$$

$$q'_t = \alpha'(\tfrac{1}{4}q'_{yyy} - \tfrac{3}{2}(q')^2 q'_y) + \alpha(-\tfrac{1}{4}q_{yxx} + \tfrac{3}{2}q^2 q_y). \quad (3.43b)$$

In addition to these evolution equations, the functions φ and (q,q') satisfy the constraints (3.40) and (3.41), respectively.

IV. ASYMPTOTIC MODULES (AM's) AND SOLUTION METHODS

The MSIM's described in the preceding sections admit natural solution methods based on the construction of particular objects called normalized wave functions and asymptotic modules. In particular, these methods provide solutions to the standard integrable models associated with contractible MSIM's.

A. Normalized wave (NW) functions

Again we will use the multipole structures $\mathscr{A}$, $\mathscr{R}$, $\mathscr{G}$, $\mathscr{G}_+$, $\mathscr{G}_-$, and $\hat{\mathscr{G}}_-$ with reference points $\{k_n\}_1^N$ and the maps τ, p, $\pi = \tau \circ p$ defined in Sec. III. Let us take a commutative family $\{c_i\}_1^s \subset \mathscr{G}_+$. Since $\mathscr{G}_+ = \mathscr{G} \cap \tau(\mathscr{R})$ there is a commutative family $\{C_i\}_1^s \subset \mathscr{R}$ with tr $C_i = 0$, such that

$$c_i = \tau(C_i). \quad (4.1)$$

We look for solutions to the MSIM (2.3) associated with (4.1). This system can be rewritten as

$$\partial_{t_i} g = \pi(g c_i g^{-1})g - g c_i, \quad g \in \hat{\mathscr{G}}_-. \quad (4.2)$$

We now introduce the following associative functional algebras $\mathscr{A}(D)$. Let D be a subset of the Riemann sphere S such that $\{k_n\}_1^N$ are limit points of D. By $\mathscr{A}(D)$ we will denote the set of $d \times d$ matrix-valued functions $H = H(k)$, defined on D, for which there is an element $h = (h_1(k),\ldots,h_N(k))$ in $\mathscr{A}$, such that $H(k)$ admits $h_n(k)$ as its asymptotic expansion (AE) as $k \to k_n$ $(n = 1,\ldots,N)$. Obviously, $\mathscr{R} \subset \mathscr{A}(D)$ and the map (3.2) admits an extension,

$$\mathscr{A}(D) \xrightarrow{\tau} \mathscr{A}, \quad \tau(H) = (h_1(k),\ldots,h_N(k)), \quad (4.3)$$

which is a homomorphism between the associative algebras $\mathscr{A}(D)$ and $\mathscr{A}$. Next we define the projection operator,

$$\mathscr{A}(D) \xrightarrow{\Pi} \mathscr{A}(D), \quad \Pi = p \circ \tau. \quad (4.4)$$

From (3.3)–(3.5) it follows at once that $\Pi_{\mathscr{R}} = Id_{\mathscr{R}}$, Ran $\Pi = \mathscr{R}$, while Ker $\Pi = \tau^{-1}(\mathscr{A}_-)$. The maps τ and Π enable us to formulate a version of (4.2) on $\mathscr{A}(D)$. Indeed, consider the system

$$\partial_{t_i} G = \Pi(G C_i G^{-1})G - G C_i, \quad (4.5a)$$

with the conditions valid for any $t = (t_1,\ldots,t_s)$,

$$G(t),\quad G^{-1}(t)\in\mathscr{A}(D),\quad \tau(G(t))\in\widehat{\mathscr{G}}_{-}. \tag{4.5b}$$

Given a solution $G(t)$ of (4.5), since $\tau\circ\Pi=\pi\circ\tau$ and τ is an algebra homomorphism, it follows at once that $g(t)=\tau(G(t))$ is a solution of (4.2). We may still perform a further reformulation of our problem by means of the function

$$F=G\exp\left(\sum_{i=1}^{s}t_iC_i\right). \tag{4.6}$$

It is clear that (4.5a) is equivalent to

$$\partial_{t_i}F=\Pi(FC_iF^{-1})F, \tag{4.7}$$

while (4.5b) is verified if F satisfies

$$\det F=1 \tag{4.8}$$

and admits AE's of the form

$$F(k,t)\sim\left(1+\sum_{m=1}^{\infty}g_{1m}(t)k^{-m}\right)\exp\left(\sum_{i=1}^{s}t_iC_i(k)\right),\quad k\to\infty, \tag{4.9a}$$

$$F(k,t)\sim\left(\sum_{m=0}^{\infty}g_{nm}(t)(k-k_n)^m\right)\exp\left(\sum_{i=1}^{s}t_iC_i(k)\right),\quad k\to k_n,\quad n=2,\ldots,N. \tag{4.9b}$$

A function $F(k,t)$ (such that $k\in D\subset S$ and $\{k_n\}_1^N$ are limit points of D), which satisfies (4.7)–(4.9), will be called an asymptotically normalized wave function (NW function) on D for the MSIM (4.2). To summarize we can state that each NW function $F(k,t)$ determines a solution of (4.2) in the form

$$g(t)=\tau\left(F(k,t)\exp\left(-\sum_{i=1}^{s}t_iC_i\right)\right),\quad t=(t_1,\ldots,t_s). \tag{4.10}$$

We notice that there is an elementary NW function on $S-\{k_n\}_1^N$ given by

$$E(k,t)=\exp\left(\sum_{i=1}^{s}t_iC_i\right),$$

which corresponds to the trivial solution $g=1$ to (4.2).

B. Asymptotic modules

A set $\mathscr{W}$ of $d\times d$ matrix-valued functions defined on a subset $D\subset S$ is said to be a (left) $\mathscr{R}$ module if it satisfies

$$H_1+H_2\in\mathscr{W},\quad\text{for all } H_1,H_2\in\mathscr{W}, \tag{4.11a}$$

$$UH\in\mathscr{W},\quad\text{for all } U\in\mathscr{R},\quad H\in\mathscr{W}. \tag{4.11b}$$

We are going to see how some $\mathscr{R}$ modules, called asymptotic modules (AM's), allow us to reproduce NW functions. Let D_0 and D_1 be subsets of S such that $\{k_n\}_1^N$ are limit points for $D_0\cap D_1$. We are given a NW function, $F_0=F_0(k,t)$, on D_0 for the MSIM (4.2) and we want to produce another NW function, $F_1=F_1(k,t)$, on D_1 for (4.2). The simplest case corresponds to $F_0(k,t)=E(k,t)$. The strategy is the following. First it is convenient to look for a function F differing from F_1 by a normalization factor, i.e., condition (4.8) is not required for F_1. Here F is generally defined on a subset D_1' bigger than D_1. On one hand we characterize the behavior of F in D_1' by looking for $F(t)$ in a fixed $\mathscr{R}$ module for all t (isospectrality condition). Furthermore, in order to guarantee the uniqueness of F, we impose that FF_0^{-1} satisfies some regularity condition in some appropriate set D_{01} containing $D_0\cap D_1'$ and admits some asymptotic structure at $\{k_n\}_1^N$.

More precisely, we will say that a set $\mathscr{W}$ of $d\times d$ matrix-valued functions $H(k)$, defined on D_1', is an AM around the NW function $F_0(k,t)$ on D_0 for the MSIM (4.2) if the following conditions hold:

(1) $\mathscr{W}$ is an $\mathscr{R}$ module; (2) for each value of $t=(t_1,\ldots,t_s)$, there is a *unique* function $F(t)\in\mathscr{W}$ such that (i) the function $\widehat{F}(t)\doteq F(t)\,F_0^{-1}(t)$ defined for $k\in D_0\cap D_1'$ has a "smooth" extension in D_{01} and (ii) $\widehat{F}(t)$ belongs to $\mathscr{A}(D_{01})$ with $\Pi(\widehat{F}(t))=1$, i.e., $\widehat{F}(k,t)$ admits AE's of the form

$$\widehat{F}(k,t)\sim 1+\sum_{m=1}^{\infty}\varphi_{1m}(t)k^{-m},\quad k\to\infty, \tag{4.12a}$$

$$\widehat{F}(k,t)\sim\sum_{m=0}^{\infty}\varphi_{nm}(t)(k-k_n)^m,\quad k\to k_n,\quad n=2,\ldots,N; \tag{4.12b}$$

and (3) $\det F(k,t)\neq 0$ for $k\in D_1\subset D_1'$.

Then we have the following important property whose proof is given in Appendix B. If $\mathscr{W}$ is an AM around the NW function F_0 on D_0 for the MSIM (4.2), then it turns out that

$$F_1(k,t)=F(k,t)\,[\det F(k,t)]^{-1/d} \tag{4.13}$$

is a NW function on D_1 for (4.2).

By using (4.10) we conclude that starting from the solution

$$g_0(t)=\tau\left(F_0(k,t)\exp\left(-\sum_{i=1}^{s}t_iC_i\right)\right)$$

of the MSIM (4.2) we can construct the new solution

$$g_1(t)=\tau\left(F_1(k,t)\exp\left(-\sum_{i=1}^{s}t_iC_i\right)\right).$$

Thus we obtain an iterative procedure for generating solutions to MSIM's and as a consequence to their associated standard integrable models [such as the AKNS hierarchy and the system (1.1a)–(1.1c) identical with (3.43) and (3.41)] in the contractible cases.

Note that the terminology in this paper is slightly different from Ref. 9(b) where we use the ring $\widetilde{\mathscr{R}}$ of $\mathscr{R}$-valued functions of t and we name AM the $\widetilde{\mathscr{R}}$ module $\widetilde{\mathscr{W}}$ of dimension 1 and of basis $F(k,t)$.

C. Construction of asymptotic modules

First we remark that $\mathscr{R}$ modules can be constructed in a natural way by means of Riemann–Hilbert and $\bar{\partial}$ problems.

Let γ be an oriented curve in $S-\{k_n\}_1^N$ and let $G(k)$ be a $d\times d$ matrix-valued function defined on γ. Let us denote by $\mathscr{W}$ the set of $d\times d$ matrix-valued functions, defined on $D_1'=S-(\gamma\cup\{k_n\}_1^N)$, whose left and right boundary values $H_\pm$ on γ exist and satisfy

$$H_-(k)=H_+(k)G(k). \tag{4.14}$$

Then $\mathscr{W}$ defines an obvious $\mathscr{R}$ module. We recall that such Riemann–Hilbert problems appear in the Zakharov–Shabat dressing method.[12]

Now let us consider a $d\times d$ matrix-valued distribution $R(k)$ with support in $\mathbb{S}-\{k_n\}_1^N$. The set $\mathscr{W}$ of $d\times d$ matrix-valued functions $H(k)$, defined on $D_1' = \mathbb{S}-\{k_n\}_1^N$, which satisfy the $\bar\partial$ equation

$$\frac{\partial H}{\partial \bar k}(k) = H(k)R(k), \quad k\in\mathbb{S}-\{k_n\}_1^N, \tag{4.15a}$$

is also an obvious $\mathscr{R}$ module.

It is known that a Riemann–Hilbert problem can be considered formally as a particular $\bar\partial$ problem. For this reason we will only investigate the construction of AM's associated with $\bar\partial$ problems.

Let us prove that for an appropriate choice of the input function $R(k)$, the $\mathscr{R}$ module $\mathscr{W}$ associated with (4.15a) is an AM around each of the NW functions $F_0(k,t)$ considered in (a) and (b).

(a) Here $F_0(k,t)=E(k,t)$ (the elementary NW function). Then $D_0 = D_1' = D_1 = D_{01} = S-\{k_n\}_1^N$. The function $\hat F \doteq F F_0^{-1}$ must satisfy the $\bar\partial$ equation

$$\frac{\partial \hat F}{\partial \bar k}(k) = \hat F(k)\hat R(k), \quad k\in\mathbb{S}-\{k_n\}_1^N, \tag{4.15b}$$

with $\hat R(k)\doteq F_0(k)R(k)F_0^{-1}(k)$. Here $\hat F$ must also admit AE's of the form (4.12). By applying the generalized Cauchy formula in a way similar to Refs. 9(a), 10(a), (c) one can see that $\hat F$ is a solution of the integral equation

$$(1-J)\hat F = 1, \tag{4.15c}$$

where J is the integral operator

$$J\hat F(k) = \frac{1}{2i\pi}\iint_{\mathbb{R}^2} \frac{dq\wedge d\bar q}{q-k}\hat F(q)\hat R(q).$$

With reasonable assumptions on $R(k)$, (4.15c) has a unique solution and $\det \hat F(k)$ does not vanish. As a consequence $\mathscr{W}$ is an AM around F_0. Note that if $\operatorname{tr} R(k)=0$, then we also have $\operatorname{tr}\tilde R(k)=0$ so that formula (4.15b) implies $(\partial/\partial\bar k)\det\hat F = 0$. By using (4.12) we find $\det F = \det \hat F = 1$ and the formula (4.13) becomes merely $F_1 = F$. Note also that the new NW function (4.13) can be computed easily when $R(k)$ is a linear combination of delta functions. The corresponding new solution of the MSIM (4.2) will be called, in a wider sense, a multisoliton solution since it yields in some cases a multisoliton solution to a standard integrable model.

(b) Here $F_0(k,t)$ satisfies a $\bar\partial$ equation,

$$\frac{\partial F_0}{\partial\bar k}(k) = F_0(k)R_0(k), \quad k\in\mathbb{S}-\{k_n\}_1^N.$$

Then $D_0 = D_1 = D_1' = D_{01} = \mathbb{S}-\{k_n\}_1^N$. The function $\hat F \doteq FF_0^{-1}$ must satisfy the $\bar\partial$ equation (4.15b) with $\hat R = F_0(R-R_0)F_0^{-1}$ and admit AE's of the form (4.12). Similar to (a), $\hat F$ is a solution of the integral equation (4.15c) and, with reasonable assumptions on $R(k)$ and $R_0(k)$, we conclude that $\mathscr{W}$ is an AM around F_0. If $\operatorname{tr} R_0(k) = \operatorname{tr} R(k) = 0$, the formula (4.13) becomes $F_1 = F$.

D. Solitons

There are other ways than using Riemann–Hilbert and $\bar\partial$ problems for constructing AM's. For example, we will now construct some AM's that are particularly appropriate for analyzing soliton solutions.

Let F_0 be a NW function analytic on a dense open set D_0 of $\mathbb{S}$, where $\mathbb{S}-D_0$ is made up of isolated points including the $\{k_n\}_1^N$ and of some curves. Let k_0 and k_0' be two different complex numbers in D_0 and let M and N be two subspaces of $\mathbb{C}^d$, such that $\mathbb{C}^d = M+N$. Denote by $\mathscr{W}$ the set of $d\times d$ matrix-valued functions $H(k)$ analytic on $D_1' = D_0-\{k_0\}$ with, at most, a single pole at $k=k_0$ and verifying (a) the coefficient R of $(k-k_0)^{-1}$ in the Laurent expansion of $H(k)$ at $k=k_0$ satisfies

$$R(M)=\{0\}; \tag{4.16}$$

and (b) the value S of $H(k)$ at k_0' satisfies

$$S(N)=\{0\}. \tag{4.17}$$

It is clear that $\mathscr{W}$ is an $\mathscr{R}$ module. To show that it is an AM around F_0, let us look for functions $F(t)\in\mathscr{W}$ such that $\hat F = FF_0^{-1}$ has a continuous extension in $D_{01}\doteq \mathbb{S}-\{k_0\}$ and admits AE's of the form (4.12). It is easy to see that $\hat F$ must be, in fact, analytic in $\mathbb{S}-\{k_0\}$ (note that $\det F_0 = 1$). It follows that F can be written as

$$F(k,t) = [1 + A(t)/(k-k_0)]F_0(k,t). \tag{4.18}$$

Thus conditions (a) and (b) become

$$A(t)F_0(k_0,t)(M) = \{0\}, \tag{4.19a}$$

$$(k_0' - k_0 + A(t))F_0(k_0',t)(N) = \{0\}. \tag{4.19b}$$

If we assume that $\mathbb{C}^d$ can be decomposed into a direct sum of the subspaces $F_0(k_0,t)(M)$ and $F_0(k_0',t)(N)$ for all t, then we can determine projection operators $P(t)$ on $\mathbb{C}^d$ from the conditions

$$\operatorname{Ker} P(t) = F_0(k_0,t)(M),$$

$$\operatorname{Ran} P(t) = F_0(k_0',t)(N).$$

In this way (4.19) can be rewritten in the simpler form,

$$A(1-P)=0, \quad (k_0'-k_0+A)P=0, \tag{4.20}$$

which immediately gives $A=(k_0-k_0')P$. Therefore there is a unique function $F(t)$ satisfying the required conditions and $\mathscr{W}$ is an AM around F_0. By using (4.13) we get the new NW function on $D_1 = D_0-\{k_0,k_0'\}$,

$$F_1(k,t) = [(k-k_0)/(k-k_0')]^{d'/d} \times(1-[(k_0'-k_0)/(k-k_0)]P(t))F_0(k,t), \tag{4.21}$$

where $d'=\dim N$. The new NW function differs from the old one just by the presence of a Blaschke–Potapov factor.[3] The interpretation of this result is that the new solution of the MSIM (4.2) has an additional soliton. This is in agreement with Refs. 3 and 12 for the standard integrable models.

As an example we consider the MSIM defined by the system of compatible flows (3.11) and (3.18) and choose for F_0 the elementary NW function on $D_0=\mathbb{C}-\{0\}$:

$$E(k,x,y,t) = \exp(-i(kx+y/k-\omega(k)t)\sigma_3), \tag{4.22}$$

while M and N are defined as

$$M=\text{lin}\{(\begin{smallmatrix}1\\-1\end{smallmatrix})\},\quad N=\text{lin}\{(\begin{smallmatrix}1\\1\end{smallmatrix})\}.$$

Then we get

$$P(x,y,t)=\frac{1}{2\cosh\rho_+}\begin{pmatrix}e^{\rho_+} & e^{\rho_-}\\ e^{-\rho_-} & e^{-\rho_+}\end{pmatrix},$$

where

$$\rho_\pm=\pm i[(k_0\mp k'_0)x+[1/k_0\mp 1/k'_0]y - (\omega(k_0)\mp\omega(k'_0))t].$$

The corresponding solution of the hierarchy (3.29) is characterized by

$$a_1=(k_0-k'_0)(P-\tfrac{1}{2}),\quad a'_1=(1/k_0k'_0)a_1,$$
$$b_0=(k_0/k'_0)^{1/2}(1+[(k'_0-k_0)/k_0]P).$$

Clearly this solution represents a plane soliton.

E. Reductions

AM's are also suitable for constructing solutions to the MSIM (4.2), subject to lie on a reduced subgroup $\hat{\mathscr{G}}'_-$ of $\hat{\mathscr{G}}_-$. To illustrate this fact we consider the system (3.11)–(3.18) with the reduction (2.17). Let F_0 be a NW function, verifying

$$\sigma_1F_0(-k,t)\sigma_1=F_0(k,t) \tag{4.23}$$

and let $\mathscr{W}$ be an AM around F_0 such that $\sigma_1H(-k)\sigma_1$ belongs to $\mathscr{W}$ for all $H\in\mathscr{W}$ (the subsets D, D'_1, D_{01} are supposed to be symmetric with respect to 0). Under these assumptions it is not difficult to see that condition (2) for AM's implies that (4.23) holds for the new NW function F_1. From (4.10) we conclude that the solution $g_1(t)$, associated with F_1, lies on the reduced subgroup.

As an example, again let us take the elementary NW function (4.22) with $\omega(k)$ being an odd polynomial in order to satisfy condition (4.23). Let k_0 and k'_0 be two different complex numbers in $D_0=\mathbb{C}-\{0\}$ and let M and N be two subspaces such that $\mathbb{C}^2=M\oplus N$. We define $\mathscr{W}$ as the set of 2×2 matrix functions $H(k)$, analytic on $D'_1=\mathbb{C}-\{0,\pm k_0\}$, with at most single poles at $k=\pm k_0$, and verifying that (a) the coefficients $R_\pm$ of $(k\mp k_0)^{-1}$ in the Laurent expansions of $H(k)$ at $k=\pm k_0$ satisfy

$$R_+(M)=\{0\},\quad R_-\sigma_1(M)=\{0\},$$

and (b) the values $S_\pm$ of $H(k)$ at $k=\pm k'_0$ satisfy

$$S_+(N)=\{0\},\quad S_-\sigma_1(N)=\{0\}.$$

It follows that $\mathscr{W}$ is an AM around $F_0=E$ such that $\sigma_1H(-k)\sigma_1$ belongs to $\mathscr{W}$ for all $H\in\mathscr{W}$. Thus if we define

$$M=\text{lin}\{(\begin{smallmatrix}1\\0\end{smallmatrix})\},\quad N=\text{lin}\{(\begin{smallmatrix}1\\1\end{smallmatrix})\},$$

the new NW function F_1 on $D'_1=\mathbb{C}-\{0,\pm k_0,\pm k'_0\}$ turns out to be

$$F_1(k,x,y,t)=\left(\frac{k^2-k_0^2}{k^2-k_0'^2}\right)^{1/2}\left(1+\frac{k'_0-k_0}{k+k_0}P_2\right)\times\left(1+\frac{k_0-k'_0}{k-k_0}P_1\right)E,$$

where

$$P_1=\begin{pmatrix}0 & e^{2\rho}\\ 0 & 1\end{pmatrix},$$
$$P_2=\frac{1}{\gamma}\begin{pmatrix}(1+\alpha e^{4\rho})(1+\beta) & -(1+\alpha e^{4\rho})\beta e^{2\rho}\\ (1+\alpha)(1+\beta)e^{2\rho} & -(1+\alpha)\beta e^{4\rho}\end{pmatrix},$$

with

$$\alpha=(k'_0-k_0)/(k_0+k'_0),\quad \beta=(k'_0-k_0)/2k_0,$$
$$\rho=-ik'_0x-i(y/k'_0)+i\omega(k'_0)t,$$
$$\gamma=1+\beta+(\alpha-\beta)\,e^{4\rho}.$$

It is not difficult to compute the corresponding solution to the hierarchy (3.37). It is given by

$$\sinh(2\varphi)=2i\sinh(2\rho+\theta)/\cosh^2(2\rho+\theta),$$

where $\theta=\log(i\beta/i+\beta)$.

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APPENDIX A: STRUCTURE OF THE COEFFICIENTS r_n

In this appendix we give a simple argument that proves that the coefficients r_n of the formal series r in (2.14) are differential polynomials in the off-diagonal elements of a_1. Moreover, we study some properties of the reduction (2.17).

We begin by obtaining a differential equation for r. From (2.12a) and taking into account that r commutes with kr, we have

$$\partial_x r=[(kr)_-,r]=-[(kr)_+,r].$$

Now

$$(kr)_+=(ikg\sigma_3g^{-1})_+=ik\sigma_3-i[\sigma_3,a_1].$$

Therefore

$$\partial_x r=-ik[\sigma_3,r]+i[[\sigma_3,a_1],r]. \tag{A1}$$

Next we introduce some notation,

$$i[\sigma_3,a_1]=-\begin{pmatrix}0 & q\\ s & 0\end{pmatrix}, \tag{A2}$$

$$r=\begin{pmatrix}\xi & \eta\\ \gamma & -\xi\end{pmatrix}. \tag{A3}$$

Observe that according to (2.13), tr $r=0$. Note also that the matrix elements q and s in (A2) are proportional to the off-diagonal elements of a_1 $[q=-2i(a_1)_{12}, s=2i(a_1)_{21}]$. From (2.14) we deduce that the matrix elements of r are of the form

$$\xi=\sum_{n=0}^\infty \xi_nk^{-n},\quad \eta=\sum_{n=0}^\infty \eta_nk^{-n},$$
$$\gamma=\sum_{n=0}^\infty \gamma_nk^{-n},\quad \xi_0=i,\quad \eta_0=\gamma_0=0, \tag{A4}$$

so that (A1)–(A4) imply

$$\partial_x\xi_n=s\eta_n-q\gamma_n, \tag{A5a}$$
$$2i\eta_{n+1}=-\partial_x\eta_n+2q\xi_n, \tag{A5b}$$
$$2i\gamma_{n+1}=\partial_x\gamma_n+2s\xi_n. \tag{A5c}$$

With the only information given by (A5) we cannot guarantee that the coefficients of the formal series ξ, η, and γ are differential polynomials in q and s. Indeed, solving (A5a) would require boundary conditions on the x dependence of ξ_n, which are absent from our analysis. Nevertheless from (2.13) it is clear that r satisfies the constraint $\det r = 1$, so that

$$\xi^2 + \gamma\eta = -1. \tag{A6}$$

This relation and (A4) imply

$$2\xi_{n+1} = -\sum_{l=1}^{n} (\xi_l \xi_{n+1-l} + \gamma_l \eta_{n+1-l}). \tag{A7}$$

Now it is obvious that (A5b), (A5c), and (A7) form a system of recursion relations that enable us to express the unknowns ξ_n, η_n, and γ_n as polynomials in q, s, and their derivatives, with respect to x. For the sake of completeness, we list some of these polynomials:

$$\begin{aligned} &\eta_1 = q, \quad \gamma_1 = s,\\ &\eta_2 = (i/2) q_x, \quad \gamma_2 = -(i/2) s_x,\\ &\eta_3 = -\tfrac{1}{4} q_{xx} + \tfrac{1}{2} q^2 s, \quad \gamma_3 = -\tfrac{1}{4} s_{xx} + \tfrac{1}{2} q s^2,\\ &\eta_4 = -(i/8) q_{xxx} + (3i/4) q s q_x,\\ &\gamma_4 = (i/8) s_{xxx} - (3i/4) q s s_x. \end{aligned} \tag{A8}$$

The AKNS equations (2.16) can thus be written in the form

$$\partial_t q = 2i \sum_{l=0}^{N} \alpha_l \eta_{l+1}, \quad \partial_t s = -2i \sum_{l=0}^{N} \alpha_l \gamma_{l+1}. \tag{A9}$$

Let us assume now that the constraint (2.17) is satisfied. Then $\sigma_1 a_1 \sigma_1 = -a_1$ and therefore

$$q = s. \tag{A10}$$

On the other hand, (2.13) implies $\sigma_1 r(-k)\sigma_1 = -r(k)$ and consequently

$$\sigma_1 r_{2n} \sigma_1 = -r_{2n}, \quad \sigma_1 r_{2n+1} \sigma_1 = r_{2n+1}, \tag{A11}$$

or equivalently,

$$\gamma_{2n} = -\eta_{2n}, \quad \xi_{2n+1} = 0, \quad \gamma_{2n+1} = \eta_{2n+1}. \tag{A12}$$

In this way, Eqs. (A9) reduce to a single equation, provided $\alpha_l = 0$ for even l. For example, if $\omega(k) = \alpha_3 k^3$ we get the MKDV equation for q.

APPENDIX B: CONSTRUCTION OF NW FUNCTIONS FROM ASYMPTOTIC MODULES

Here we prove that the function $F_1(k,t)$ of (4.13) defines a NW function for the MSIM (4.2). To this end let us consider the functions $\partial_{t_i} F$. Since $\mathscr{W}$ is an $\mathscr{R}$ module and $\mathbb{C} \subset \mathscr{R}$, it is clear that $\mathscr{W}$ is also a complex vector space, so that under mild assumptions $\partial_{t_i} F \in W$. The function F_0 satisfies Eq. (4.7), so that

$$(\partial_{t_i} F) F_0^{-1} = \partial_{t_i} \hat{F} + \hat{F} U_{0i}, \tag{B1}$$

where

$$U_{0i} = \Pi(F_0 C_i F_0^{-1}) \in \mathscr{R}.$$

On the other hand, by virtue of conditions (2) and (3) for AM's, we have

$$\hat{F}, \hat{F}^{-1} \quad \text{and} \quad \hat{F} U_{0i} - \Pi(\hat{F} U_{0i} \hat{F}^{-1}) \hat{F} \in \mathscr{A}(D_{01}),$$

$$\begin{aligned} &\hat{F} U_{0i} - \Pi(\hat{F} U_{0i} \hat{F}^{-1}) \hat{F}\\ &\quad = [(1-\Pi)(\hat{F} U_{0i} \hat{F}^{-1})] \hat{F} \in \operatorname{Ker} \Pi. \end{aligned} \tag{B2}$$

Further, we assume that $\partial_{t_i} \hat{F} \in \mathscr{A}(D_{01})$ and that $\partial_{t_i} \hat{F}$ can be asymptotically expanded through a term by term differentiation of the series (4.12). Hence

$$\partial_{t_i} \hat{F} \in \operatorname{Ker} \Pi. \tag{B3}$$

Next, we consider the function

$$F' \doteq \partial_{t_i} F - \Pi(\hat{F} U_{0i} \hat{F}^{-1}) F.$$

We already know that $\partial_{t_i} F \in \mathscr{W}$. Furthermore, since $\mathscr{W}$ is an $\mathscr{R}$ module and $\operatorname{Ran} \Pi = \mathscr{R}$ then $\Pi(\hat{F} U_{0i} \hat{F}^{-1}) F \in \mathscr{W}$. Therefore $F' \in \mathscr{W}$. Now from (B1)–(B3) it follows that

$$F' F_0^{-1} = \partial_{t_i} \hat{F} + \hat{F} U_{0i} - \Pi(\hat{F} U_{0i} \hat{F}^{-1}) \hat{F}$$

belongs to $\mathscr{A}(D_{01})$ and, more precisely, is an element of $\operatorname{Ker} \Pi$. As a consequence $F + F' \in \mathscr{W}$, $(F + F') F_0^{-1} \in \mathscr{A}(D_{01})$ and $\Pi((F + F') F_0^{-1}) = 1$. Therefore from condition (2) for AM's we deduce that $F + F' = F$. Hence $F' = 0$ and

$$\partial_{t_i} F = \Pi(\hat{F} U_{0i} \hat{F}^{-1}) F.$$

In addition, we note that (4.12) implies

$$\hat{F} (\operatorname{Ker} \Pi) \hat{F}^{-1} \subset \operatorname{Ker} \Pi,$$

so that

$$\begin{aligned} \Pi(\hat{F} U_{0i} \hat{F}^{-1}) &= \Pi(\hat{F} F_0 C_i F_0^{-1} \hat{F}^{-1})\\ &\quad - \Pi(\hat{F} [(1-\Pi)(F_0 C_i F_0^{-1})] \hat{F}^{-1})\\ &= \Pi(F C_i F^{-1}), \end{aligned}$$

and the differential equations for F become

$$\partial_{t_i} F = \Pi(F C_i F^{-1}) F. \tag{B4}$$

Finally, as $\operatorname{tr} \Pi(F C_i F^{-1}) = 0$, Eq. (B4) implies that $\partial_{t_i} \det F = 0$, while conditions (2) and (3) for AM's lead to AE's of the form

$$\begin{aligned} &\det F(k,t) \sim 1 + \sum_{m=1}^{\infty} d_{1m} k^{-m}, \qquad k \to \infty,\\ &\det F(k,t) \sim \sum_{m=0}^{\infty} d_{nm} (k - k_n)^m, \qquad k \to k_n,\\ &\qquad n = 2,\ldots,N, \end{aligned} \tag{B5}$$

with $d_{n0} \neq 0$. In this way, from (B4), (B5), (4.12), and (4.9) for F_0, it readily follows that (4.13) is a NW function on D_1 for (4.2).

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