

To appear in *Physics Letters A*

On the construction of evolution equations admitting a master symmetry

F. FINKEL*

*Departamento de Física Teórica II
Universidad Complutense
28040 Madrid
Spain*

A.S. FOKAS

*Department of Mathematics
Imperial College
London SW7 2BZ
United Kingdom*

May 18, 2001

Abstract

A method for constructing evolution equations admitting a master symmetry is proposed. Several examples illustrating the method are presented. It is also noted that for certain evolution equations master symmetries can be useful for obtaining new conservation laws from a given one.

Mathematics Subject Classification (2000): 35Q53, 35Q58, 37K10

Keywords: master symmetries, integrable evolution equations, conservation laws

*Corresponding author. E-mail: federico@ciruelo.fis.ucm.es

1 Introduction

The concept of master symmetries was introduced in [1]. This concept, which was further developed in [2–6], is deeply related to the so called bispectral problem [7], as well as to Virasoro algebras [8]. Among the direct applications of master symmetries are the recursive construction of hierarchies of infinitely many symmetries and of infinitely many gradients of conserved densities for nonlinear integrable evolution equations.

In this paper we present an algorithm for constructing evolution equations which are invariant under a given group of scaling transformations and which possess a master symmetry. Moreover, as a further indication of the importance of master symmetries, we note that they can be useful for constructing new conservation laws starting from a given one. We note that the method introduced here can be applied to other types of equations such as the very interesting class of non-linear equations recently introduced in [9].

We first introduce the basic rings used throughout this paper. Then we define symmetries, master symmetries, and scalings.

Definition 1.1 (*The basic rings*). Let $q(t, \mathbf{x})$, $t \in \mathbb{R}$, $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, be a smooth function of the indicated variables. Let $\partial_{x_j}^{-1}$ denote the integral operators $\partial_{x_j}^{-1} = \frac{1}{2}(\int_{-\infty}^{x_j} - \int_{x_j}^{\infty})$, $j = 1, \dots, n$. We shall denote by $\mathcal{R}\{q\}$ the ring of smooth functions of q and its x_j - partial derivatives; $\mathcal{R}\{q, \partial^{-1}q\}$ will denote the ring of smooth functions of q , its x_j - partial derivatives, and its integrals $\partial_{x_j}^{-1}$. If these smooth functions are polynomials, the corresponding rings will be denoted by $\mathcal{P}\{q\}$ and $\mathcal{P}\{q, \partial^{-1}q\}$. Similarly, if these smooth functions depend explicitly on \mathbf{x} and on t we will use the notation $\mathcal{R}\{q, \partial^{-1}q, t, \mathbf{x}\}$.

Definition 1.2 (*Symmetries*). Let $\Delta(q)$ and $\eta(q)$ belong to $\mathcal{R}\{q, \partial^{-1}q, t, \mathbf{x}\}$. We call η a symmetry of the PDE

$$\Delta(q) = 0, \tag{1.1}$$

iff

$$\Delta'\eta = 0 \quad \text{when} \quad \Delta = 0, \tag{1.2}$$

where Δ' denotes the Fréchet derivative of Δ with respect to q , i.e.,

$$\Delta'(q)\eta(q) = \left. \frac{\partial}{\partial \varepsilon} \Delta(q + \varepsilon\eta) \right|_{\varepsilon=0}. \tag{1.3}$$

Constructing master symmetries

Let q satisfy the evolution PDE

$$q_t = K_1(q), \quad K_1(q) \in \mathcal{R}\{q, \partial^{-1}q\}. \quad (1.4)$$

The definition of a symmetry implies that the function $K_2(q) \in \mathcal{R}\{q, \partial^{-1}q\}$ is a symmetry of this evolution PDE iff

$$[K_1(q), K_2(q)]_L = 0, \quad (1.5)$$

where the Lie commutator $[\cdot, \cdot]_L$ is defined by

$$[A(q), B(q)]_L = A'B - B'A.$$

We note that since K_1 and K_2 are x_j -independent they commute with q_{x_j} , i.e.

$$[q_{x_j}, K_1]_L = [q_{x_j}, K_2]_L = 0. \quad (1.6)$$

Thus for evolution equations the question of constructing symmetries which are independent of x_j and of t , reduces to the question of constructing an Abelian Lie algebra with elements $\{q_{x_j}, K_1, K_2\}$. Such a construction can be carried out using a master symmetry.

Definition 1.3 (*Master symmetries*). Suppose that $K_1(q), K_2(q) \in \mathcal{R}\{q, \partial^{-1}q\}$ commute, i.e., suppose that they satisfy Eq. (1.5). We call $\tau(q) \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$ a master symmetry associated with q_{x_j} , K_1 , and K_2 , iff

$$[q_{x_j}, \tau]_L = c_1 K_1, \quad [K_1, \tau]_L = c_2 K_2, \quad (1.7)$$

where c_1 and c_2 are constants.

This definition implies that the Abelian algebra generated by $\{q_{x_j}, K_1, K_2\}$ is constructed solely from a master symmetry.

The existence of a master symmetry implies the existence of a generalized symmetry K_2 .[†] Although in all known examples the recursive use of τ ,

$$[K_j, \tau]_L = c_j K_{j+1},$$

generates infinitely many symmetries, their existence does not follow from Definition 1.3. However, it was conjectured in [10] that the existence of one generalized symmetry implies infinitely many symmetries. This conjecture was shown to be true

[†]It is called generalized in juxtaposition to Lie point because either it depends nonlinearly on first-order derivatives or it involves higher-order derivatives, see [5] for further discussion.

in [11] for a large class of evolution equations in one space dimension. This provides further support for the usefulness of constructing master symmetries, since they guarantee the existence of at least one generalized symmetry. In what follows we present such a construction based on the notion of scaling:

Definition 1.4 (*Scalings*). Suppose that $K(q) \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$ is invariant up to a multiplicative constant under the transformation $q \mapsto \alpha^\lambda q$, $x_j \mapsto \alpha^{\lambda_j} x_j$, i.e.

$$K(\alpha^\lambda q(t, \alpha^{\lambda_1} x_1, \dots, \alpha^{\lambda_n} x_n)) = \mu K(q(t, x_1, \dots, x_n)), \quad (1.8)$$

where α , λ , λ_j , μ are constants. Then we say that $K(q)$ admits the scaling $\tau_0(q)$, where τ_0 is defined by

$$\tau_0(q) = \lambda q - \sum_{j=1}^n \lambda_j x_j q_{x_j}. \quad (1.9)$$

If the constant μ equals one, we say that $K(q)$ is an absolute invariant of the scaling τ_0 .

It is shown in [4] that if $K(q) \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$ admits the scaling $\tau_0(q)$, then

$$[K, \tau_0]_L = cK, \quad (1.10)$$

where c is a constant.

Observation 1.1 (*Candidates for master symmetries*). Suppose that $K_1(q) \in \mathcal{R}\{q, \partial^{-1}q\}$ admits the scaling $\tau_0(q)$ defined by Eq. (1.9). Assume that there exists a function $f(q) \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$ such that τ_0 can be written in the form

$$\tau_0 = [q_{x_j}, f]_L. \quad (1.11)$$

Define $\tau(q) \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$ by

$$\tau = [K_1, f]_L + \tau_1, \quad (1.12)$$

where $\tau_1(q) \in \mathcal{R}\{q, \partial^{-1}q, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n\}$ is an arbitrary x_j -independent function. Then τ satisfies

$$[q_{x_j}, \tau]_L = c_1 K_1. \quad (1.13)$$

Indeed, since τ_1 is x_j -independent, it commutes with q_{x_j} ; thus

$$[q_{x_j}, \tau]_L = [q_{x_j}, [K_1, f]_L]_L = [K_1, [q_{x_j}, f]_L]_L - [f, [q_{x_j}, K_1]_L]_L, \quad (1.14)$$

where we have used the fact that $[\cdot, \cdot]_L$ satisfies the Jacobi identity. Since K_1 is x_k -independent for all k and admits the scaling τ_0 , Eq. (1.14) becomes

$$[q_{x_j}, \tau]_L = [K_1, \tau_0]_L = c_1 K_1.$$

The above observation suggests the following algorithm for constructing equations which admit a given scaling τ_0 and a master symmetry:

Construction 1.1 (*Construction of master symmetries*).

1. Start with a general function $K_1(q) \in \mathcal{R}\{q, \partial^{-1}q\}$ which remains invariant up to a multiplicative constant under the transformation $q \mapsto \alpha^\lambda q$, $x_j \mapsto \alpha^{\lambda_j} x_j$ (see Eq. (1.8)). Let the associated scaling τ_0 be defined by Eq. (1.9). The function $K_1(q)$ involves some absolute invariants $b_j(q) \in \mathcal{R}\{q, \partial^{-1}q\}$ of the scaling τ_0 .
2. Find a function $f(q) \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$ such that Eq. (1.11) is satisfied.
3. Define $\tau(q) \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$ by Eq. (1.12) and $K_2(q)$ by

$$K_2 = [K_1, \tau]_L. \quad (1.15)$$

If there exist absolute invariants $b_j(q)$ and a function $\tau_1(q)$ such that $K_2 \in \mathcal{R}\{q, \partial^{-1}q\}$ and $[K_1, K_2]_L = 0$, then τ is a master symmetry associated with q_{x_j} , K_1 , K_2 .

We note that the requirement $K_2 \in \mathcal{R}\{q, \partial^{-1}q\}$ is automatically satisfied in one space dimension ($n = 1$). Indeed,

$$[q_x, K_2]_L = [q_x, [K_1, \tau]_L]_L = [K_1, [q_x, \tau]_L]_L - [\tau, [q_x, K_1]_L]_L = 0,$$

where we have used Eq. (1.13) and the fact that K_1 commutes with q_x .

Example 1.1 The most general third-order function $K_1(q) \in \mathcal{P}\{q\}$ which remains invariant up to a multiplicative constant under the scaling

$$q \mapsto \alpha^{-1}q, \quad x \mapsto \alpha^2x \quad (1.16)$$

is given by

$$K_1 = q_3 + 3q^2q_2 + \beta_1qq_1^2 + \beta_2q^4q_1 + \beta_3q^7, \quad (1.17)$$

where throughout the paper

$$q_j = \partial_x^j q,$$

and $\beta_1, \beta_2, \beta_3$ are constants.[‡] The scaling (1.16) implies that $\tau_0 = q + 2xq_x$. The class of solutions of the equation $\tau_0 = [q_x, f]_L$ depend on the form of f . If $f \in \mathcal{R}\{q, \partial^{-1}q, x\}$, then

$$f = xq + x^2q_1 + g(q),$$

[‡]Note that any absolute invariant of the scaling (1.16) in $\mathcal{P}\{q\}$ is necessarily a constant.

where g is an arbitrary function in $\mathcal{R}\{q, \partial^{-1}q\}$. We take $g = 0$, define τ by Eq. (1.12) with $\tau_1 = 0$, and determine K_2 from Eq. (1.15). A straightforward computation shows that K_2 commutes with K_1 if and only if

$$\beta_1 = 9, \quad \beta_2 = 3, \quad \beta_3 = 0.$$

We recover in this way the Ibragimov–Shabat equation [12, 13]

$$q_t = q_3 + 3q^2q_2 + 9qq_1^2 + 3q^4q_1; \quad (1.18)$$

this equation admits the master symmetry

$$\tau = 2x(q_3 + 3q^2q_2 + 9qq_1^2 + 3q^4q_1) + 3q_2 + 10q^2q_1 + q^5. \quad (1.19)$$

We note that the master symmetry of Burgers equation, both master symmetries of the Kadomtsev–Petviashvili (KP) equation, and the master symmetries of several other equations (see Eqs. (2.7), (2.14), (2.17), (2.11), (2.20)) can be obtained through Construction 1.1.

Master symmetries and conservation laws

It is well known [14] that certain symmetries can be used to construct a new conservation law from a given one. However, not all symmetries can be used for this purpose since some symmetries yield trivial conservation laws. For example, the Korteweg–de Vries (KdV) equation

$$\partial_t q = \partial_x(q_2 + 3q^2), \quad (1.20)$$

is in a form of a conservation law and also it admits infinitely many symmetries $K_j \in \mathcal{P}\{q\}$. If one starts with the conservation law (1.20), all these symmetries yield a trivial conservation law.

It will be shown in Section 3 that starting from (1.20), the time-dependent generalized symmetry associated with the master symmetry of the KdV equation can be used to generate a hierarchy of nontrivial conservation laws.

2 Constructing master symmetries

Example 2.1 The most general second-order function $K_1 \in \mathcal{P}\{q\}$ which remains invariant up to a multiplicative constant under the scaling

$$q \mapsto \alpha^{-1}q, \quad x \mapsto \alpha x \quad (2.1)$$

is given by

$$K_1 = q_2 + 2qq_1 + \beta q^3. \quad (2.2)$$

The above scaling implies that

$$\tau_0 = q + xq_1. \quad (2.3)$$

The general solution of Eq. (1.11) in $\mathcal{R}\{q, \partial^{-1}q, x\}$ is given by

$$f = xq + \frac{x^2}{2} q_1 + g(q),$$

where g is an arbitrary function in $\mathcal{R}\{q, \partial^{-1}q\}$. Define τ by Eq. (1.12) with $g = \tau_1 = 0$, i.e.,

$$\tau = [K_1, f]_L = 2xK_1 + 3q_1 + 2q^2. \quad (2.4)$$

A simple computation shows that the function K_2 defined by

$$K_2 = [K_1, \tau]_L = 4\left(q_3 + 3qq_2 + 3q_1^2 + (3 + \beta)q^2q_1 + \frac{3\beta}{2}q^4\right) \quad (2.5)$$

commutes with K_1 iff $\beta = 0$. In summary, the Burgers equation

$$q_t = q_2 + 2qq_1 \quad (2.6)$$

admits the master symmetry

$$\tau = 2x(q_2 + 2qq_1) + 3q_1 + 2q^2. \quad (2.7)$$

Example 2.2 Consider the function $K_1 \in \mathcal{R}\{q\}$ given by

$$K_1 = q_3 + 3b_1(q) \frac{q_2^2}{q_1} + 3b_2(q) q_1 q_2 + b_3(q) q_1^3, \quad (2.8)$$

where $b_j(q)$ are arbitrary smooth functions of q . We assume that $b_1(q)$ is not identically zero. The function K_1 remains invariant up to a multiplicative constant under the scaling $q \mapsto q$, $x \mapsto \alpha x$; this corresponds to the function $\tau_0 = xq_1$. The general solution of Eq. (1.11) in $\mathcal{R}\{q, \partial^{-1}q, x\}$ is of the form

$$f = \frac{x^2}{2} q_1 + g(q),$$

where $g \in \mathcal{R}\{q, \partial^{-1}q\}$. Define τ by Eq. (1.12) with $g = \tau_1 = 0$, i.e.,

$$\tau = [K_1, f]_L = 3(xK_1 + (1 + 2b_1)q_2 + b_2q_1^2). \quad (2.9)$$

A straightforward computation shows that the function K_2 defined by Eq. (1.15) commutes with K_1 iff

$$b_1 = -\frac{1}{4}, \quad b_2 = b(q), \quad b_3 = b(q)^2 + 2b'(q),$$

where $b(q)$ is an arbitrary smooth function of q , and $b'(q) = db/dq$. In summary, the evolution equation

$$q_t = q_3 - \frac{3q_2^2}{4q_1} + 3b q_1 q_2 + (b^2 + 2b')q_1^3 \quad (2.10)$$

admits the master symmetry

$$\tau = x \left(q_3 - \frac{3q_2^2}{4q_1} + 3b q_1 q_2 + (b^2 + 2b')q_1^3 \right) + \frac{1}{2} q_2 + b q_1^2. \quad (2.11)$$

We note that (2.10) can be obtained from the equation [15]

$$u_t = u_3 - \frac{3u_2^2}{4u_1}$$

by the transformation

$$u = \beta_1 \int^q e^{2 \int^s b(r) dr} ds + \beta_2,$$

where β_1 and β_2 are constants.

Example 2.3 Consider the KP equation in the variable $q(t, x, y)$,

$$q_t = K_1, \quad K_1 = q_{xxx} + 6qq_x + 3\partial_x^{-1}q_{yy}. \quad (2.12)$$

It is well-known that the KP equation admits infinitely many symmetries and conservation laws. The function K_1 defining the KP equation admits the scaling

$$\tau_0 = 2q + xq_x + 2yq_y. \quad (2.13)$$

It may be easily verified that the function $f \in \mathcal{R}\{q, \mathbf{x}\}$ given by

$$f = y(2q + xq_x + yq_y)$$

satisfies Eq. (1.11) for q_y , i.e.,

$$\tau_0 = [q_y, f]_L.$$

The function $\tau \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$ defined by Eq. (1.12) with $\tau_1 = 0$,

$$\tau = [K_1, f]_L = 3(yK_1 + 2xq_y + 4\partial_x^{-1}q_y), \quad (2.14)$$

is a master symmetry of the KP equation [3, 16]. Indeed, the function $K_2 \in \mathcal{P}\{q, \partial^{-1}q\}$ defined by

$$K_2 = [K_1, \tau]_L = 36 (q_{xxy} + 4qq_y + 2q_x \partial_x^{-1} q_y + \partial_x^{-2} q_{yyy}) \quad (2.15)$$

commutes with K_1 .

The alternative equation

$$\tau_0 = [q_x, \tilde{f}]_L \quad (2.16)$$

also leads to a master symmetry of the KP equation. Consider the solution of this equation given by

$$\tilde{f} = x \left(2q + \frac{1}{2} x q_x + 2y q_y \right).$$

Define $\tilde{\tau}_1 \in \mathcal{R}\{q, \partial^{-1}q, y\}$ by

$$\tilde{\tau}_1 = 6y (q_{xxy} + 6qq_y + 4q_x \partial_x^{-1} q_y + 3\partial_x^{-2} q_{yyy}) + 3 (q_{xx} + 4q^2 + 2q_x \partial_x^{-1} q + 11\partial_x^{-2} q_{yy}).$$

Then the function $\tilde{\tau} \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$ defined by Eq. (1.12), i.e.,

$$\tilde{\tau} = [K_1, \tilde{f}]_L + \tilde{\tau}_1 = 3xK_1 + \frac{1}{3}yK_2 + 6(2q_{xx} + 4q^2 + q_x \partial_x^{-1} q + 3\partial_x^{-2} q_{yy}) \quad (2.17)$$

is a master symmetry of the KP equation [3].[§] Indeed, it may be verified that the function

$$\begin{aligned} \tilde{K}_2 = [K_1, \tilde{\tau}]_L = & 9(q_{xxxxx} + 10qq_{xxx} + 20q_x q_{xx} + 10q_{xyy} + 30q^2 q_x + 10q_x \partial_x^{-2} q_{yy} \\ & + 20q_y \partial_x^{-1} q_y + 20q \partial_x^{-1} q_{yy} + 10\partial_x^{-1} (qq_{yy}) + 10\partial_x^{-1} q_y^2 + 5\partial_x^{-3} q_{yyyy}) \end{aligned}$$

commutes with K_1 . We note that

$$K_3 = [K_2, \tau]_L = 12\tilde{K}_2.$$

In general, the master symmetry $\tilde{\tau}$ generates the odd time-independent generalized symmetries K_{2j+1} of the KP hierarchy $K_{j+1} = [K_j, \tau]_L$.

Example 2.4 The function $K_1 \in \mathcal{P}\{q\}$ given by

$$K_1 = q_7 + 7qq_5 + 14q_1q_4 + 21q_2q_3 + 14q^2q_3 + 42qq_1q_2 + 7q_1^3 + \frac{28}{3}q^3q_1 \quad (2.18)$$

is the first time-independent higher-order symmetry of the Sawada–Kotera (SK) equation [17]

$$q_t = \tilde{K}_1, \quad \tilde{K}_1 = q_5 + 5qq_3 + 5q_1q_2 + 5q^2q_1. \quad (2.19)$$

[§]Notice that $\tilde{\tau}$ reduces to the master symmetry of the KdV equation if q does not depend on y .

The function K_1 in Eq. (2.18) admits the scaling

$$\tau_0 = q + \frac{1}{2} x q_1.$$

A straightforward computation shows that the function $f \in \mathcal{R}\{q, x\}$ given by

$$f = x q + \frac{1}{4} x^2 q_1$$

solves Eq. (1.11). Define $\tau_1 \in \mathcal{R}\{q, \partial^{-1}q\}$ by

$$\frac{2}{7} \tau_1 = 3q_6 + \tilde{K}_1 \partial_x^{-1} q + 22qq_4 + 32q_1q_3 + 17q_2^2 + 38q^2q_2 + 43qq_1^2 - 5q_1 \partial_x^{-1} q_1^2 + \frac{5}{3} q_1 \partial_x^{-1} q^3 + \frac{16}{3} q^4.$$

Then the function $\tau \in \mathcal{R}\{q, \partial^{-1}q, x\}$ determined by Eq. (1.12), i.e.,

$$\begin{aligned} \tau = [K_1, f]_L + \tau_1 = \frac{7}{2} & \left(xK_1 + \tilde{K}_1 \partial_x^{-1} q + 8q_6 + 46qq_4 + 75q_1q_3 + 44q_2^2 + 68q^2q_2 \right. \\ & \left. + 79qq_1^2 - 5q_1 \partial_x^{-1} q_1^2 + \frac{5}{3} q_1 \partial_x^{-1} q^3 + 8q^4 \right), \end{aligned} \quad (2.20)$$

is a master symmetry associated with q_1 , K_1 , and $K_2 = [K_1, \tau]_L$. In fact, the master symmetry τ generates two hierarchies of infinitely many symmetries of the SK equation [18], namely

$$K_{j+1} = [K_j, \tau]_L, \quad \tilde{K}_{j+1} = [\tilde{K}_j, \tau]_L, \quad j \geq 1.$$

3 Master symmetries and conservation laws

In this section we shall show that for certain equations the master symmetries can be useful for constructing new conservation laws from a given one. We first formally define conservation laws and trivial conservation laws. Then we recall a well-known result [14] on the action of a symmetry on a conservation law.

Definition 3.1 (*Conservation laws*). *We say that Eq. (1.1) admits the conservation law characterized by $\{T_j(q)\}_0^n$, where $T_j \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$, $j = 0, \dots, n$, iff*

$$D_t T_0 + \sum_{j=1}^n D_j T_j = 0 \quad \text{when} \quad \Delta = 0, \quad (3.1)$$

where D_t and D_j denote total differentiation with respect to t and x_j , i.e.,

$$D_t = \frac{\partial}{\partial t} + q_t \frac{\partial}{\partial q} + \dots, \quad D_j = \frac{\partial}{\partial x_j} + q_{x_j} \frac{\partial}{\partial q} + \dots. \quad (3.2)$$

The component $T_0(q)$ of a conservation law is called the conserved density. The vector formed by the remaining components $(T_1(q), \dots, T_n(q))$ is called the flux. A conservation law is local if $T_j(q) \in \mathcal{R}\{q, \mathbf{x}\}$, $j = 0, \dots, n$.

Definition 3.2 (Trivial conservation laws). A conservation law is trivial if there exist smooth functions $Q_{jk} \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$, $j, k = 0, \dots, n$, such that $Q_{jk} = -Q_{kj}$ and

$$T_j = D_t Q_{j0} + \sum_{k=1}^n D_k Q_{jk}, \quad j = 0, \dots, n, \quad \text{when} \quad \Delta = 0. \quad (3.3)$$

A trivial conservation law will be denoted as $\{T_j\}_0^n \sim 0$. Two conservation laws are equivalent if they differ by a trivial conservation law.

The Definitions 1.2 and 3.1 of symmetries and conservation laws imply:

Observation 3.1 (Symmetries and conservation laws). Assume that the PDE (1.1) possesses a symmetry $\eta(q) \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$ and a conservation law characterized by $\{T_j(q)\}_0^n$, $T_j \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$, see Definitions 1.2 and 3.1. Then this equation also possesses a conservation law characterized by $\{T'_j(q)\eta(q)\}_0^n$.

Indeed, it follows from Definitions 1.2 and 3.1 that

$$\frac{\partial}{\partial \varepsilon} D_t T_0(q + \varepsilon \eta) \Big|_{\varepsilon=0} + \frac{\partial}{\partial \varepsilon} \sum_{j=1}^n D_j T_j(q + \varepsilon \eta) \Big|_{\varepsilon=0} = 0, \quad (3.4)$$

which implies the desired result.

This observation provides an effective way to compute conserved densities for self-adjoint linear systems [14].

Example 3.1 Let q satisfy,

$$q_{tt} = q_{xx} + q_{yy}. \quad (3.5)$$

This equation admits the conservation law characterized by

$$T_0(q) = q_t q_x, \quad T_1(q) = -q q_{yy} - \frac{1}{2} (q_t^2 + q_x^2 + q_y^2), \quad T_2(q) = q q_{xy}.$$

Consider the Lorentz-boost symmetry $\eta(q) = x q_t + t q_x$. Then

$$\{T'_j \eta\}_0^2 \sim \left\{ x q_x q_{yy} + \frac{1}{2} (q_t^2 + q_x^2), -q_t (q_x + x q_{yy}), x (q_t q_{xy} - q_x q_{ty}) \right\} \quad (3.6)$$

is also a conservation law of equation (3.5). We have subtracted here the trivial conservation law determined by (see Eq. (3.3))

$$Q_{01} = tq_t q_x + \frac{x}{2} (q_t^2 + q_x^2), \quad Q_{02} = 0, \quad Q_{12} = -q(tq_{xy} + xq_{ty}).$$

We note that if $T_j(q)$, $j = 0, \dots, n$, are invariant with respect to the symmetry $\eta(q)$ under consideration then $\{T'_j \eta\}_0^n \sim 0$, thus the Observation 3.1 does not yield a new conservation law.

Example 3.2 Consider the KdV equation (1.20). The KdV equation admits infinitely many symmetries $K_j \in \mathcal{P}\{q\}$ and local conservation laws. The KdV equation is in a form of a conservation law, where

$$T_0(q) = q, \quad T_1(q) = -q_2 - 3q^2. \quad (3.7)$$

However, since $\{T'_0 K_j, T'_1 K_j\} \sim 0$ none of the symmetries K_j yield new (nontrivial) conservation laws.

The KdV equation also possesses x, t -dependent symmetries $\sigma_j(q) \in \mathcal{R}\{q, \partial^{-1}q, t, x\}$. These symmetries are intimately related to the master symmetry, see the Appendix. In particular, a time-dependent symmetry of the KdV equation is given by

$$\sigma = tK_2 + \tau, \quad (3.8)$$

where

$$\tau = \frac{1}{3} (xK_1 + 4q_2 + 8q^2 + 2q_1 \partial_x^{-1} q)$$

is the master symmetry of the KdV equation and

$$K_2 = [K_1, \tau]_L = q_5 + 10qq_3 + 20q_1q_2 + 30q^2q_1$$

is its first generalized time-independent symmetry. The symmetry σ gives rise to the conservation law

$$T_0^{(1)}(q) = q^2, \quad T_1^{(1)}(q) = -2qq_2 + q_1^2 - 4q^3. \quad (3.9)$$

Indeed, on solutions of the KdV equation we have

$$\{T'_0 \sigma, T'_1 \sigma\} = \{T_0^{(1)}, T_1^{(1)}\} + \{D_x Q, -D_t Q\},$$

where

$$Q = t(q_4 + 10qq_2 + 5q_1^2 + 10q^3) + x\left(\frac{1}{3}q_2 + q^2\right) + q_1 + \frac{2}{3}q\partial_x^{-1}q.$$

It should be emphasized that the conserved density $T_0^{(1)}$ of the conservation law (3.9) can be generated solely by τ , up to a total x -derivative. The recursive application of the Fréchet derivative with respect to σ can be used to obtain the well-known sequence of conservation laws associated with the KdV equation:

$$\begin{aligned}\{T_0^{(2)}, T_1^{(2)}\} &= \{-q_1^2 + 2q^3, 2q_1q_3 - q_2^2 - 6q^2q_2 + 12qq_1^2 - 9q^4\}, \\ \{T_0^{(3)}, T_1^{(3)}\} &= \{q_2^2 - 10qq_1^2 + 5q^4, -2q_2q_4 + q_3^2 + 20qq_1q_3 - 16qq_2^2 - 10q_1^2q_2 - 20q^3q_2 \\ &\quad + 90q^2q_1^2 - 24q^5\}, \\ \{T_0^{(4)}, T_1^{(4)}\} &= \{-q_3^2 + 14qq_2^2 - 70q^2q_1^2 + 14q^5, 2q_3q_5 - q_4^2 - 28qq_2q_4 + 20qq_3^2 + 28q_1q_2q_3 \\ &\quad + 140q^2q_1q_3 - 2q_2^3 - 154q^2q_2^2 - 140qq_1^2q_2 - 70q^4q_2 + 35q_1^4 + 560q^3q_1^2 - 70q^6\},\end{aligned}$$

where we have omitted an irrelevant global constant and a trivial conservation law. The time-dependent symmetry σ thus plays for conservation laws the same role as the adjoint of the recursion operator for their *conserved gradients*; see Ref. [5]. The formalism based on the time-dependent symmetry σ associated with the master symmetry is more convenient, because it provides explicit expressions for the conserved densities $T_0^{(k)}$, and what is more important, for the corresponding fluxes $T_1^{(k)}$.

Remark 3.1 In some cases, the time-dependent symmetry associated with the master symmetry yields trivial conservation laws. For example, the Burgers equation (2.6) is also in the form of a conservation law, with

$$T_0(q) = q, \quad T_1(q) = -q_1 - q^2. \quad (3.10)$$

Unlike the KdV equation, the Burgers equation is diffusive thus does not possess infinitely many conservation laws. In this case $\sigma = tK_2 + \tau$, where τ and K_2 are respectively given by Eqs. (2.7) and (2.5) with $\beta = 0$, yields a trivial conservation law, namely

$$\{T_0'\sigma, T_1'\sigma\} = \{D_x Q, -D_t Q\},$$

where

$$Q = 4t(q_2 + 3qq_1 + q^3) + 2x(q_1 + q^2) + q.$$

The situation with Eq. (1.18) is similar to the Burgers equation.

Acknowledgements

F.F. would like to express his gratitude to the Department of Mathematics of Imperial College for its hospitality. A.S.F. is grateful to J. Sanders for useful suggestions. This work was partially supported by EPSRC grant No. GR/M61450 and by DGES grant PB98-0821.

Appendix

For the convenience of the reader we summarize some known results of the theory of master symmetries. We first review two additional methods for constructing master symmetries which also rely on the existence of a scaling. We then recall the relation between master symmetries and time-dependent symmetries.

Construction 1 [5] *Let Φ be a hereditary (Nijenhuis) operator [19]. Assume that Φ is invariant under the scaling τ_0 defined by Eq. (1.9). Then $\tau = \Phi\tau_0$ is a master symmetry associated with*

$$\{q_x, K_1 = \Phi q_x, K_2 = \Phi^2 q_x\}.$$

For example, the hereditary operator associated with the KdV equation (1.20) is

$$\Phi = D_x^2 + 4q + 2q_1 \partial_x^{-1}.$$

This operator is invariant under the scaling $q \mapsto \alpha^{-2}q$, $x \mapsto \alpha x$, which gives rise to $\tau_0 = 2q + xq_1$. Thus

$$\tau = \Phi\tau_0 = x(q_3 + 6qq_1) + 4q_2 + 8q^2 + 2q_1 \partial_x^{-1}q$$

is a master symmetry of the KdV equation.

Construction 2 [4] *Suppose that $K_1(q), K_2(q) \in \mathcal{R}\{q, \partial^{-1}q\}$ commute, and that K_2 admits a scaling τ_0 . Assume that there exists a function $F \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$ such that*

$$\tau_0 = [K_1, F]_L. \quad (3.11)$$

Define $\tau(q) \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$ by

$$\tau = [K_2, F]_L. \quad (3.12)$$

Then τ satisfies $[K_1, \tau]_L = c_2 K_2$.

Indeed,

$$[K_1, \tau]_L = [K_1, [K_2, F]_L]_L = [K_2, [K_1, F]_L]_L - [F, [K_1, K_2]_L]_L = [K_2, \tau_0]_L = c_2 K_2,$$

where the fact that τ_0 is a scaling of K_2 has been used in the last equality.

Time-dependent symmetries. Let $\tau \in \mathcal{R}\{q, \partial^{-1}q, \mathbf{x}\}$ be a master symmetry associated with $\{q_x, K_1, K_2\}$, $K_1, K_2 \in \mathcal{R}\{q, \partial^{-1}q\}$, normalized so that $[K_1, \tau]_L = K_2$. Then

$$\sigma = tK_2 + \tau$$

is a time-dependent generalized symmetry of the evolution equation $q_t = K_1$.

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