VŨ NGỌC'S CONJECTURE ON FOCUS-FOCUS SINGULAR FIBERS WITH MULTIPLE PINCHED POINTS

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ABSTRACT. We classify, up to fiberwise symplectomorphisms, a saturated neighborhood of a singular fiber of an integrable system (which is proper onto its image and has connected fibers) containing $k \geqslant 1$ focus-focus critical points. Our result shows that there is a one-to-one correspondence between such neighborhoods and k formal power series, up to a $(\mathbb{Z}_2 \times D_k)$-action, where D_k is the k-th dihedral group. The k formal power series determine the dynamical behavior of the Hamiltonian vector fields associated to the components of the momentum map on the symplectic manifold (M, ω) near the singular fiber containing the k focus-focus critical points. This proves a conjecture of San Vũ Ngọc from 2002.

1. INTRODUCTION

Our goal in this paper is to provide a classification of saturated neighborhoods of a compact connected fiber $\mathcal{F}$ containing only non-degenerate focus-focus singular points of integrable systems $F = M \to \mathbb{R}^2$ on symplectic 4-manifolds up to isomorphism. Here a point $m \in \mathcal{F}$ being of *focus-focus type* means that there is an $E \in \mathrm{GL}(2, \mathbb{R})$ such that if $(J, H) = E \circ (F - F(m))$ then the Hessians $\mathcal{H}_J(m)$ of J and $\mathcal{H}_H(m)$ of H are simultaneously symplectically conjugate to the following matrices

$$\mathcal{H}_J(m) \sim \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{H}_H(m) \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Eliasson's normal form theorem (see Eliasson [9] and Vũ Ngọc–Wacheux [21]) states that the aforementioned normal form is not only achieved linearly but also nonlinearly, in the sense that any focus-focus singular point of F has a neighborhood such that (M, ω, F) restricted to it is isomorphic to $(\mathbb{R}^4, \omega_0, q)$ restricted to a neighborhood of the origin, where

$$q(x_1, y_1, x_2, y_2) = (x_1 y_2 - x_2 y_1, x_1 y_1 + x_2 y_2).$$

In particular, focus-focus singular points are isolated (and $\mathcal{F}$ is allowed to have any finite number of such singular points). Throughout this paper, with very few exceptions that we point out explicitly, we assume that F *is proper onto its image with connected fibers*. If all singular points in a compact connected fiber $\mathcal{F}$ have focus-focus type then there are finitely many such points, say $k \in \mathbb{N}$ is the number, which is the only topological invariant of such fibers, and then $\mathcal{F}$ is homeomorphic to a torus pinched k times (Zung [23, 24]).

In 2003, Vũ Ngọc proved [20] that germs of integrable systems at a compact connected fiber with exactly one non-degenerate singular point of focus-focus type are classified (up to a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action if one does not specify directions, as clarified in [18]) by a formal power series $\mathsf{s} \in \mathsf{R}_{2\pi X}$, where R is the space of formal power series in two variables X, Y, without the constant term, and $\mathsf{R}_{2\pi X} \stackrel{\text{def}}{=} \mathsf{R}/(2\pi X)\mathbb{Z}$. In his paper, he also stated a claim for the case $k > 1$, which already appeared in the arXiv version of the paper in 2002, and in 2003 [20, Section 7] with a sketch of the argument. Our proof for the general case is different than Vũ Ngọc' s proof for $k = 1$ and in particular gives another proof for this case.

He claimed that a neighborhood of a compact connected fiber with precisely $k \in \mathbb{N}$ non-degenerate critical points of focus-focus type is classified up to isomorphisms by k formal power series in R, with $(k-1)$ of which measure the obstruction to construct a semiglobal momentum map in the Eliasson normal form simultaneously at two different singular points, and the other of which is the Taylor series of the action integral in a neighborhood of the critical fiber, vanishing at the origin, desingularized at each singular point. Essentially, this turns out to be the case, as we verify in the current paper. Indeed, let R_+ be the group of formal power series in R with the coefficients of the Y term positive, and let $\mathbb{Z}_k = \{\overline{0}, \overline{1}, \ldots, \overline{k-1}\}$ and let D_k stand for the k-th Dihedral group of order $2k$. Our main theorem, which is more technical so we state it later once we have introduced the necessary ingredients for its precise formulation, will essentially say that in a small symplectic neighborhood of a focus-focus fiber with k singular points of focus-focus type, an integrable system on a symplectic 4-manifold is classified up to isomorphisms, by k formal power series

$$\mathsf{s}_{\overline{0}} \in \mathsf{R}_{2\pi X}, \mathsf{g}_{\overline{0},\overline{1}}, \mathsf{g}_{\overline{1},\overline{2}}, \ldots, \mathsf{g}_{\overline{k-2},\overline{k-1}} \in \mathsf{R}_+$$

modulo an effective $(\mathbb{Z}_2 \times D_k)$-action. Here the D_k-action controls the way in which the focus-focus points are ordered and $\mathbb{Z}_2$-action reflects the direction of the natural $\mathbb{S}^1$-action.

Remark 1.1. Focus-focus singular points appear naturally as singularities of semitoric [15, 16] and almost-toric systems [11, 19]. In fact, any semitoric or almost-toric integrable system satisfies the assumptions above. An explicit example of a semitoric system which includes a twice pinched torus for certain values of the parameters is given in [10].

This formulation however hides the fact that we give a step-by-step explicit construction of the invariants $\mathsf{s}_{\overline{0}} \in \mathsf{R}_{2\pi X}$ and $\mathsf{g}_{\overline{1},\overline{2}}, \ldots, \mathsf{g}_{\overline{k-2},\overline{k-1}} \in \mathsf{R}_+$; for the case of $k = 1$, the only invariant is $\mathsf{s}_{\overline{0}}$, and this has been computed (at least some of its terms have been computed) by several authors for important cases such as the coupled spin-oscillator system [1, 17], the spherical pendulum [8], and the coupled angular-momenta [2, 12]. Roughly speaking, $\mathsf{s}_{\overline{0}}$ measures the global singular behavior of the Hamiltonian vector fields $\mathcal{X}_{f_1}$ and $\mathcal{X}_{f_2}$, where $F = (f_1, f_2)$, near the fiber containing the focus-focus singular point. The travel times of the flows of these vector fields exhibit a singular behavior, of logarithmic type, as they approach the singular points. The remaining $(k-1)$ Taylor series $\mathsf{g}_{j,j+\overline{1}}$ account for the difference between the Eliasson normal forms at the singular points m_j and $m_{j+\overline{1}}$.

We will attempt to make the proof as self-contained as possible, and to facilitate this we include a section on preliminaries with a quick review of the ingredients we need for the proof of the aforementioned classification result. We would like to point out a related recent work by Bolsinov–Izosimov [4] where the authors give a *smooth* classification of semiglobal germs at compact focus-focus leaves. The smooth invariants they define can be computed from the symplectic invariants of the present paper, see Remark 3.15. We refer to the recent article [14] for an introduction and review of recent progress on the symplectic geometry of integrable systems.

2. Preliminaries

The goal of this section is twofold. First, we briefly review the basic terminology and results which we need to state the main theorem of the paper in the following sections; this is done with the goal of making the paper as self-contained as possible. Second, once we have set up the basic notions, we derive some consequences and some variations of them which we will need to prove the main theorem in the following section; most of the new content of this section is concentrated on Section 2.2 about automorphisms of local normal forms, where we prove technical results about local symplectomorphisms and their possible extensions.

2.1. **Basic review of integrable systems.** In this subsection, we recall the definition of integrable systems and semitoric integrable systems and discuss some of the concepts and their properties that are related to finding the semiglobal symplectic invariants.

Let (M, ω) be a $2n$-dimensional symplectic manifold. For a smooth map $f\colon M \to \mathbb{R}$ we denote by $\mathcal{X}_f = -\omega^{-1}(\mathrm{d}f)$ the *Hamiltonian vector field* of f. For any smooth maps $f, g\colon M \to \mathbb{R}$ we define their *Poisson bracket* $\{f, g\} = \omega(\mathcal{X}_f, \mathcal{X}_g)$ and say they are *Poisson commutative* if their Poisson bracket vanishes. Let B be a smooth manifold with $b \in B$. Let $\mathcal{N}(B, b)$ denote the collection of open neighborhoods of b in B. If $F\colon M \to B$ is a surjection and $\mathcal{F}$ is a fiber of F, then let $\mathcal{N}_F(M, \mathcal{F})$ denote the collection of open saturated neighborhoods of $\mathcal{F}$ in M with respect to F, where a subset $W \subseteq M$ is *saturated* with respect to F if $F^{-1}(F(W)) = W$.

Definition 2.1. Let (M, ω) be a $2n$-dimensional symplectic manifold. Let

$$F = (f_1, \ldots, f_n)\colon M \to \mathbb{R}^n$$

be a smooth map such that $f_1, \ldots, f_n$ are functionally independent (i.e. $\mathrm{d}f_1, \ldots, \mathrm{d}f_n$ are linearly independent) almost everywhere and pairwise Poisson commutative. In this case we call F a *momentum map* on M and (M, ω, F) an *integrable system*. Let $\mathcal{IS}$ be the collection of all integrable systems.

It is worth noting that the fiber of a momentum map F is Lagrangian near any regular point of F it goes through. For any regular point $x \in M$ of F, let $b = F(x)$. Since $\mathcal{X}_1(x), \ldots, \mathcal{X}_n(x)$ are linearly independent and $\dim T_xF^{-1}(b) = n$, they span $T_xF^{-1}(b)$. But then $\{f_i, f_j\}(x) = \omega(\mathcal{X}_i, \mathcal{X}_j)(x) = 0$ implies that $T_xF^{-1}(b) \subset T_xM$ is Lagrangian. Therefore, integrable systems are sometimes considered as *singular Lagrangian fibrations*.

Let $B = F(M)$ and let $U \subseteq B$ be an open subset. For each $\beta \in \Omega^1(U)$, its *Hamiltonian vector field* $\mathcal{X}_\beta = -\omega^{-1}(F^*\beta)$ is a vector field on $F^{-1}(U)$. When exists, let $\Psi_\beta\colon F^{-1}(U) \to F^{-1}(U)$ be the time-1 map of the flow of $\mathcal{X}_\beta$. On each fiber $F^{-1}(b)$ for $b \in B$ and $\beta_b \in T_x^*B$ let $\Psi_{\beta_b}\colon F^{-1}(b) \to F^{-1}(b)$ be the time-1 map of the flow of $-\omega^{-1}(F^*\beta_b)$ when it exists. Throughout this paper we use the following definition for smooth functions or forms on subsets $B \subseteq \mathbb{R}^n$: let X be a smooth manifold. A map $f\colon B \to X$ is *smooth* if for any $b \in B$ there are $U \in \mathcal{N}(\mathbb{R}^n, b)$ and a smooth $\widetilde{f}\colon U \to X$ which coincides with f in $B \cap U$.

Definition 2.2. An integrable system (M, ω, F) is called *complete* if Ψ_β exists for any open subset $U \subseteq B$ and each $\beta \in \Omega^1(U)$. Equivalently, the flow of $\mathcal{X}_\beta$ exists for all time.

A complete integrable system (M, ω, F) is called *vertically transitive* if for smooth sections $P, Q\colon B \to M$ of F and $b \in B = F(M)$, there are $U \in \mathcal{N}(B, b)$ and $\beta \in \Omega^1(U)$ such that $\Psi_\beta \circ P|_U = Q|_U$ and moreover, if $P(b) = Q(b)$ then one could ask $\beta(b) = 0$.

Remark 2.3. The completeness in Definition 2.2 is automatic when the moment map has compact fibers, or more specifically, is proper onto its image.

Let (M,ω,F) be a complete integrable system. For any open subset $U \subseteq B = F(M)$ let $\Lambda^{(M,\omega,F)}(U) = \{\beta \in \Omega^1(U) \mid \Psi_{2\pi\beta} = \mathrm{id}\}$. We will use Λ omitting the superscripts if there is no ambiguity.

Definition 2.4. We call Λ the *period sheaf* of (M,ω,F), which is a sheaf of abelian groups over B. The local sections of Λ are *period forms*.

Consider the presheaf $U \mapsto \Omega^1(U)/2\pi\Lambda(U)$, denoted by $\Omega^1/2\pi\Lambda$ of abelian groups on B. On one hand, any global section $\tau \in (\Omega^1/2\pi\Lambda)(B)$ yields a diffeomorphism Ψ_τ of (M,ω) as a fiberwise translation by the representatives of stalks of τ. On the other hand, let $P, Q, R\colon B \to M$ be smooth sections of F. If every $b \in B$ has a $U \in \mathcal{N}(B,b)$ such that

$$\tau_U^{PQ} = \left\{ \widetilde{\tau}_U^{PQ} \in \Omega^1(U) \,\middle|\, \Psi_{\widetilde{\tau}_U^{PQ}} \circ P|_U = Q|_U \right\}$$

is nonempty, then τ_U^{PQ} is a coset of $2\pi\Lambda(U)$ in $\Omega^1(U)$, whose germ at b is an element of the stalk of $\Omega^1/2\pi\Lambda$ at b. Then those τ_U^{PQ}, for b ranging in B and U being neighborhoods as above, glue to a global section $\tau^{PQ} \in (\Omega^1/2\pi\Lambda)(B)$. In this case we call τ^{PQ} the *translation form* from P to Q. The translation forms satisfy the additivity property $\tau^{PQ} + \tau^{QR} = \tau^{PR}$, whenever the two forms on the left-hand side are defined.

Here we specify the morphisms of integrable systems we consider, and then give some properties of such morphisms.

Definition 2.5. Let (M,ω,F), (M',ω',F') be integrable systems and set $B = F(M)$, $B' = F'(M')$. If $G\colon B \to B'$ and $\varphi\colon M \to M'$ satisfy that $F' \circ \varphi = G \circ F$, then we say that φ *lifts* G. A *morphism* from (M,ω,F) to (M',ω',F') is a pair $(\varphi\colon M \to M', G\colon B \to B')$ of smooth maps with φ lifting G such that $\varphi^*\omega' = \omega$. A morphism (φ, G) is said to be an *isomorphism* if both φ and G are diffeomorphisms.

Remark 2.6. The concept of an isomorphism of semitoric systems [15] is more restrictive than Definition 2.5 and is similar to an isomorphism of integrable systems preserving the direction as in Definitions 3.1 and 3.8.

Lemma 2.7. *Let (M,ω,F) be a complete integrable system, let $U \subseteq B = F(M)$ be an open subset and let $\tau \in \Omega^1(U)$. Then $\Psi_\tau\colon F^{-1}(U) \to F^{-1}(U)$ is a symplectomorphism if and only if τ is closed.*

Proof. By Cartan's formula, $\mathcal{L}_{\mathcal{X}_\tau}\omega = \mathrm{d}(\mathcal{X}_\tau \lrcorner\, \omega) = -\,\mathrm{d}(F^*\tau)$. So for any $t \in \mathbb{R}$ we have $\frac{\mathrm{d}}{\mathrm{d}t}\Psi_{t\tau}^*\omega = \Psi_{t\tau}^*\mathcal{L}_{\mathcal{X}_\tau}\omega = -\,\mathrm{d}(\Psi_{t\tau}^*F^*\tau) = -\,\mathrm{d}(F^*\tau)$. Integrating for $t \in [0,1]$, we obtain $\Psi_\tau^*\omega = \omega - \mathrm{d}(F^*\tau)$. Since F^* is injective on almost every cotangent space, $\mathrm{d}(F^*\tau) = F^*(\mathrm{d}\tau) = 0$ if and only if $\mathrm{d}\tau = 0$. ◻

As a result, $2\pi\Lambda(U) \subset Z^1(U)$, the space of closed 1-forms on U for any open $U \subseteq B$, and we can naturally define d in $(\Omega^1/2\pi\Lambda)(B)$ with kernel $(Z^1/2\pi\Lambda)(B)$.

Corollary 2.8. *Let (M,ω,F) be a complete integrable system and $B = F(M)$. Let $\tau \in (\Omega^1/2\pi\Lambda)(B)$. Then $\Psi_\tau\colon M \to M$ is a symplectomorphism if and only if τ is closed.*

Lemma 2.9. *Let (M,ω,F) and (M',ω',F') be complete integrable systems and set $B=F(M)$, $B'=F'(M')$. Suppose that (M,ω,F) admits a Lagrangian section $P\colon B\to M$. Let $\varphi\colon M\to M'$ and $G\colon B\to B'$ be diffeomorphisms with φ lifting G. Then φ is a symplectomorphism if and only if for any $\tau'\in\Omega^1(B)$, we have $\varphi\circ\Psi_{G^*\tau'}=\Psi_{\tau'}\circ\varphi$ and $P'=\varphi\circ P\circ G^{-1}$ is a Lagrangian section of F'.*

Proof. For $\tau'\in\Omega^1(B')$ let $\tau=G^*\tau'\in\Omega^1(B)$. Suppose that $\Psi_{t\tau}=\varphi^{-1}\circ\Psi_{t\tau'}\circ\varphi$ for any $t\in\mathbb{R}$ and we obtain $\mathcal{X}_\tau=\varphi_*^{-1}\mathcal{X}_{\tau'}$ by taking the t-derivative, which is to say that

$$\omega^{-1}(F^*\tau)=\varphi_*^{-1}(\omega')^{-1}((F')^*\tau')=(\varphi^*\omega')^{-1}(\varphi^*(F')^*\tau')=(\varphi^*\omega')^{-1}(F^*\tau). \tag{2.1}$$

Let $b\in B$ and $x=P(b)\in M$. Note that $P(B)$ only contains regular points of F. Let $\mathcal{Y}_1,\mathcal{Y}_2\in T_xM$. If $\mathcal{Y}_1$ is vertical, namely $F_*\mathcal{Y}_1=0$, we can define $\tau_b\in T_b^*B$ by $\tau_b(F_*Z)=\omega(\mathcal{Y}_1,Z)$ for any $Z\in T_xM$. The definition of τ_b is consistent since $\omega(\mathcal{Y}_1,Z)=0$ whenever Z is vertical, as a result of $F^{-1}(b)$ being Lagrangian near x. Now we ask τ to extend τ_b and then (2.1) implies that

$$\varphi^*\omega'(\mathcal{Y}_1,\mathcal{Y}_2)=\omega(\mathcal{Y}_1,\mathcal{Y}_2). \tag{2.2}$$

Suppose P' is a Lagrangian section of F'. Then if both of $\mathcal{Y}_1,\mathcal{Y}_2$ are tangent to $P(B)$, both sides of (2.2) vanish. Hence (2.2) always holds and $\varphi^*\omega'=\omega$.

If φ is a symplectomorphism, then φ preserves Lagrangian sections and (2.1) holds for any $\tau\in\Omega^1(B)$. Multiplying (2.1) by t and integrating for $t\in[0,1]$, we obtain $\Psi_{G^*\tau}=\varphi^{-1}\circ\Psi_\tau\circ\varphi$. □

The next lemma follows from Duistermaat [7].

Lemma 2.10. *Let (M,ω,F) and (M',ω',F') be vertically transitive integrable systems. Let $B=F(M)$, $B'=F'(M')$. Suppose φ_P is a diffeomorphism from a Lagrangian section P of F to a Lagrangian section P' of F' lifting a diffeomorphism $G\colon B\to B'$. If $(G^{-1})^*\Lambda^{(M,\omega,F)}\subseteq\Lambda^{(M',\omega',F')}$, then φ_P has a unique extension as a surjective local diffeomorphism $\varphi\colon M\to M'$ such that (φ,G) is a morphism of integrable systems. If $(G^{-1})^*\Lambda^{(M,\omega,F)}=\Lambda^{(M',\omega',F')}$, the pair (φ,G) is an isomorphism.*

Proof. Suppose $(G^{-1})^*\Lambda^{(M,\omega,F)}\subseteq\Lambda^{(M',\omega',F')}$. Let $Q\colon B\to M$ be a smooth section of F. Since (M,ω,F) is vertically transitive, we have a translation form $\tau^{PQ}\in(\Omega^1/2\pi\Lambda)(B)$, and then $(\tau')^{PQ}=(G^{-1})^*\tau^{PQ}$ belongs to $(\Omega^1/2\pi\Lambda)(B')$. Since (M',ω',F') is also vertically transitive, $\Psi_{(\tau')^{PQ}}\colon M'\to M'$ is a well-defined diffeomorphism. Let $\varphi_Q=\Psi_{(\tau')^{PQ}}\circ\varphi_P\circ\Psi^{-1}_{\tau^{PQ}}|_{Q(B)}\colon Q(B)\to M'$.

To show that, patching φ_Q for all smooth sections Q of F yields a well-defined map $\varphi\colon M\to M'$, let R be another smooth section of F, with $\varphi_R=\Psi_{(\tau')^{PR}}\circ\varphi_P\circ\Psi^{-1}_{\tau^{PR}}|_{R(B)}\colon R(B)\to M'$. We need to check that for any $b\in B$ with $Q(b)=R(b)$, we have $\varphi_Q\circ Q(b)=\varphi_R\circ R(b)$. By the vertical transitivity of (M,ω,F), we have translation forms $\tau^{PR},\tau^{QR}\in(\Omega^1/2\pi\Lambda)(B)$, and then $(\tau')^{PR}=(G^{-1})^*\tau^{PR}$ and $(\tau')^{QR}=(G^{-1})^*\tau^{QR}$ belong to $(\Omega^1/2\pi\Lambda)(B')$. Moreover, for any $U\in\mathcal{N}(B,b)$, τ_U^{QR} has a representative $\widetilde{\tau}_U^{QR}$ with $\widetilde{\tau}_U^{QR}(b)=0$, and then $(\widetilde{\tau}')_U^{QR}=(G^{-1})^*\widetilde{\tau}_U^{QR}$ is a representative of $(\tau')_U^{QR}$ with $(\widetilde{\tau}')_U^{QR}\circ G(b)=0$. In particular, $\varphi_R\circ R(b)=\Psi_{(\tau')^{QR}}\circ\varphi_Q\circ Q(b)=\varphi_Q\circ Q(b)$ as well as $R(b)=\Psi_{\tau^{QR}}\circ Q(b)=Q(b)$. Then $\varphi\circ Q(b)=\varphi_Q\circ Q(b)$ defines a local diffeomorphism $\varphi\colon M\to M'$ lifting G.

The surjectivity of φ is due to the vertical transitivity of (M', ω', F'). Applying Lemma 2.9 locally, $\varphi^*\omega' = \omega$ (the completeness condition of Lemma 2.9 is not used here) and then (φ, G) is a morphism of integrable systems.

When $(G^{-1})^*\Lambda^{(M,\omega,F)} = \Lambda^{(M',\omega',F')}$, $(\tau')^{PQ} = (\tau')^{PR} \in (\Omega^1/2\pi\Lambda)(B')$ would imply $\tau^{PQ} = \tau^{PR} \in (\Omega^1/2\pi\Lambda)(B)$, then $\varphi_Q \circ Q(b) = \varphi_R \circ R(b)$ implies $Q(b) = R(b)$. In this case, φ is injective. □

2.2. **Local and semiglobal normal forms.** The goal of this subsection is to recall the existence of local action-angle coordinates *à la Duistermaat* [7], explicit calculations on the Eliasson's local normal form of focus-focus invariants, and classification of automorphisms of the local normal form from the literature. They are collected here in a systematic and self-contained way.

Throughout this paper, we use $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{T}^n = (\mathbb{S}^1)^n$ for $n \in \mathbb{N}$. Let $(M, \omega, F) \in \mathcal{IS}$, let $B = F(M)$, and let B_{r} be the set of regular values of F in B. The next lemma follows from the ideas in Duistermaat [7], of which we include a proof for completeness.

Lemma 2.11. *If $U \subseteq B_{\mathrm{r}}$ is a simply connected open set over which F has compact and connected fibers and admits a Lagrangian section, then there are $\alpha_1, \ldots, \alpha_n \in Z^1(U)$ such that $\Lambda(U) = \oplus_{i=1}^n \alpha_i\mathbb{Z}$. Moreover, for any $b \in U$, $2\pi\alpha_1(b), \ldots, 2\pi\alpha_n(b)$ form a $\mathbb{Z}$-basis of the isotropy subgroup of T_b^*B under the action of Ψ.*

Proof. Let $b \in B_{\mathrm{r}}$. Consider the action of T_b^*B on $F^{-1}(b)$ by Ψ. Since $F^{-1}(b)$ consists of regular points, the action is locally free, and the orbits are open. Since $F^{-1}(b)$ is connected, the action is transitive. Since $F^{-1}(b)$ is compact, the isotropy subgroup of this action has to be an n-lattice, with $2\pi\alpha_{1,b}, \ldots, 2\pi\alpha_{n,b}$ a $\mathbb{Z}$-basis, in which case $F^{-1}(b)$ is diffeomorphic to an n-torus.

Choose a Lagrangian section $P\colon U \to F^{-1}(U)$ and the map

$$\begin{aligned} \lambda\colon T^*U &\to F^{-1}(U) \\ \beta_b \in T_b^*U &\mapsto \Psi_{2\pi\beta_b}P(b) \end{aligned}$$

is smooth by the smooth dependence of ordinary differential equations on parameters, and we have $F \circ \lambda = \pi$, where $\pi\colon T^*U \to U$ is the projection. We conclude that λ is a local diffeomorphism as it is a local diffeomorphism on each fiber of F. In particular, we obtain a $U_b \in \mathcal{N}(B, b)$ such that λ restricted to the connected component $V_{i,b}$ of $\lambda^{-1}(P(U_b))$ containing $\alpha_{i,b}$ is a diffeomorphism onto U_b. Then there are $\alpha_1, \ldots, \alpha_n \in Z^1(U_b)$ such that $\alpha_i(b) = \alpha_{i,b}$ and the images of $\alpha_i|_{U_b}$, $i = 1, \ldots, n$, as sections, coincide with $V_{i,b}$.

The smooth section P is a proper map and then $P(U_b)$ is closed in $F^{-1}(U_b)$ as well as $L_{U_b} = \lambda^{-1}(P(U_b))$ in T^*U_b. By shrinking U_b if necessary, we could assume that in the fundamental domain $W_b = \{\sum_{i=1}^n s_i\alpha_i(b_1) \in T_*U_b \mid b_1 \in U_b, s_i \in [-\frac{1}{2}, \frac{1}{2}]\}$ there is no other elements of L_{U_b} outside of 0_{U_b}, by the closedness of L_{U_b}; then $\lambda^{-1}(P(U_b))$ coincides with the union of images of $\mathbb{Z}$-spans of $\alpha_i|_{U_b}$, $i = 1, \ldots, n$. Then U_b is evenly covered.

Hence $\lambda|_{L_U}\colon L_U \to P(U)$ for $L_U = \lambda^{-1}(P(U))$ is a smooth covering map, which is a trivial covering by the simple connectedness of U. We arrive at the conclusions in the statement. □

Theorem 2.12 (Action-angle coordinates [3, 13])**.** *Let $U \subseteq B_{\mathrm{r}}$ be a simply connected open subset over which F has compact and connected fibers and admits a Lagrangian section. Let $(\alpha_1, \ldots, \alpha_n)$ be a $\mathbb{Z}$-basis of $\Lambda(U)$. There are coordinate systems $(A_1, \ldots, A_n)\colon U \to \mathbb{R}^n$ and $(\theta_1, \ldots, \theta_n, a_1, \ldots, a_n)\colon F^{-1}(U) \to \mathbb{T}^n \times \mathbb{R}^n$ such that*

- $\mathrm{d}A_i = \alpha_i$;
- $a_i = F^* A_i$;
- $\omega = \sum_{i=1}^n \mathrm{d}\theta_i \wedge \mathrm{d}a_i$.

We call A_i *the* action integrals*,* a_i *the* action coordinates *and* θ_i *the* angle coordinates*.*

Definition 2.13. Let (x_1, ξ_1, x_2, ξ_2) be the coordinates of $\mathbb{R}^4$. Let $\omega_0 = \mathrm{d}x_1 \wedge \mathrm{d}\xi_1 + \mathrm{d}x_2 \wedge \mathrm{d}\xi_2$ be the standard symplectic form on $\mathbb{R}^4$. Let $q = (q_1, q_2)\colon \mathbb{R}^4 \to \mathbb{R}^2$ be

$$q_1 = x_1\xi_2 - x_2\xi_1, \qquad q_2 = x_1\xi_1 + x_2\xi_2.$$

We call $(\mathbb{R}^4, \omega_0, q)$ the *local normal form of focus-focus singular points.*

Now we compute the action Ψ associated with $(\mathbb{R}^4, \omega_0, q)$. Let $z = x_1 + \mathrm{i}x_2$, $\zeta = \xi_2 + \mathrm{i}\xi_1$, then $q_1 + \mathrm{i}q_2 = z\zeta$, and

$$\mathcal{X}_{q_1} = -\omega_0^{-1}\,\mathrm{d}q_1 = x_2\partial_{x_1} - x_1\partial_{x_2} + \xi_2\partial_{\xi_1} - \xi_1\partial_{\xi_2},$$
$$\mathcal{X}_{q_2} = -\omega_0^{-1}\,\mathrm{d}q_2 = -x_1\partial_{x_1} - x_2\partial_{x_2} + \xi_1\partial_{\xi_1} + \xi_2\partial_{\xi_2}.$$

Let $c = (c_1, c_2)$ be the coordinates of $\mathbb{R}^2$, and let $(t_1, t_2) \in \mathbb{R}^2$. Then the action of $\Omega^1(\mathbb{R}^2)$ is

$$\Psi_{t_1\,\mathrm{d}c_1 + t_2\,\mathrm{d}c_2}(z, \zeta) = \big(e^{-t_2 - \mathrm{i}t_1} z, e^{t_2 + \mathrm{i}t_1}\zeta\big). \tag{2.3}$$

Then $(\mathbb{R}^4, \omega_0, q)$ is a complete integrable system whose period sheaf $\Lambda^{(\mathbb{R}^4, \omega_0, q)}$, later simply denoted by Λ, is the constant sheaf with stalks $(\mathrm{d}c_1)\mathbb{Z}$. We will use the identifications

$$\mathbb{R}^4 \to \mathbb{C}^2, \qquad \mathbb{R}^2 \to \mathbb{C},$$
$$(x_1, \xi_1, x_2, \xi_2) \mapsto (z, \zeta), \qquad (c_1, c_2) \mapsto c = c_1 + \mathrm{i}c_2$$

throughout this paper.

Let $\mathbb{R}^2_{\mathrm{r}} \simeq \mathbb{C}_{\mathrm{r}} = \{c \in \mathbb{C} \mid c \neq 0\}$. Let $P, Q\colon \mathbb{R}^2 \to \mathbb{R}^4$ be the two Lagrangian sections of q defined by $P(c) = (1, c)$, $Q(c) = (c, 1)$. Then let $\kappa \in (Z^1/2\pi\Lambda)(\mathbb{R}^2_{\mathrm{r}})$ denote the translation form

$$\kappa = \tau^{PQ} = -\Im \ln c\,\mathrm{d}c_1 - \Re \ln c\,\mathrm{d}c_2. \tag{2.4}$$

Define subsets of $\mathbb{R}^4 \simeq \mathbb{C}^2$ as follows:

$$D^0_{\mathrm{u}} = \big\{(z, \zeta) \in \mathbb{C}^2 \bigm| z = 0\big\}, \qquad D^0_{\mathrm{s}} = \big\{(z, \zeta) \in \mathbb{C}^2 \bigm| \zeta = 0\big\},$$
$$\mathbb{R}^4_{\mathrm{r}} = \mathbb{C}^2_{\mathrm{r}} = \big\{(z, \zeta) \in \mathbb{C}^2 \bigm| q(z, \zeta) \neq 0\big\}, \qquad \mathcal{F}_0 = \big\{(z, \zeta) \in \mathbb{C}^2 \bigm| q(z, \zeta) = 0\big\}.$$

Here D^0_{u} and D^0_{s} are respectively, the unstable and the stable manifolds of $(0, 0)$ under the flow of $\mathcal{X}_{\mathrm{d}c_2}$. For any $(t_1, t_2) \in \mathbb{R}^2$ with $t_2 > 0$, the origin is the only α-limit point for the flow lines of $\mathcal{X}_{t_1\,\mathrm{d}c_1 + t_2\,\mathrm{d}c_2}$ in D^0_{u}, and the ω-limit point for the flow lines in D^0_{s}. Let $\mathrm{pr}_1, \mathrm{pr}_2\colon \mathbb{R}^2 \to \mathbb{R}$ be respectively the projection onto the first and the second component.

Definition 2.14. A singular point $m \in M$ of F is of *focus-focus type* if the Hessians of the components of F under some symplectic coordinates of M near m and some smooth coordinates of $F(M)$ near $F(m)$ equal those of q near $0 \in \mathbb{R}^4$ which is given in Definition 2.13.

Definition 2.15. An *Eliasson local chart* at a singular point $m \in M$ of F of focus-focus type is an isomorphism (ψ, E) from the integrable system $(V, \omega, F|_V)$ to $(V_0, \omega_0, q|_{V_0})$ where

$V \in \mathcal{N}_F(M,m)$, $V_0 \in \mathcal{N}_q(\mathbb{R}^4,0)$, $U \in \mathcal{N}(B,0)$, $U_0 \in \mathcal{N}(\mathbb{R}^2,0)$, and q is given in Definition 2.13. In other words, the following diagram commutes:

$$\begin{array}{ccc} (V,\omega) & \xrightarrow{\psi} & (V_0,\omega_0) \\ \downarrow F & & \downarrow q \\ U & \xrightarrow{E} & U_0 \end{array}.$$

Theorem 2.16 (Eliasson's theorem [6, 9, 21])**.** *There is an Eliasson local chart at any singular point of focus-focus type.*

Let $\varphi_X, \varphi_Y \colon (\mathbb{R}^4,\omega_0) \to (\mathbb{R}^4,\omega_0)$ and $G_X, G_Y \colon \mathbb{R}^2 \to \mathbb{R}^2$ be

$$\begin{aligned} \varphi_X(z,\zeta) &= (\mathrm{i}\bar{z}, \mathrm{i}\bar{\zeta}), \qquad G_X(c) = -\bar{c}, \\ \varphi_Y(z,\zeta) &= (\mathrm{i}\bar{\zeta}, -\mathrm{i}\bar{z}), \quad G_Y(c) = \bar{c}. \end{aligned} \tag{2.5}$$

Then φ_X, φ_Y are symplectomorphisms lifting G_X, G_Y respectively (with respect to q) and hence they form automorphisms of the standard local model $(\mathbb{R}^4,\omega_0,q)$.

Define a function

$$\begin{aligned} r \colon \mathbb{R}^4 \setminus D^0_{\mathrm{u}} \simeq \mathbb{C}^2 \setminus D^0_{\mathrm{u}} &\to \mathbb{R}, \\ r(z,\zeta) &= \ln|z| \end{aligned} \tag{2.6}$$

as a measurement of the fiberwise translation. In fact, if $\tau = t_1\,\mathrm{d}c_1 + t_2\,\mathrm{d}c_2$, we have

$$\begin{aligned} r \circ \Psi_\tau &= r - t_2, \\ r \circ \Psi_\kappa &= r + \ln|q|. \end{aligned} \tag{2.7}$$

Note that the integrable system $(\mathbb{R}^4,\omega_0,q)$ restricted onto $\mathbb{R}^4 \setminus D^0_{\mathrm{u}}$ and $\mathbb{R}^4 \setminus D^0_{\mathrm{s}}$ respectively remains complete but becomes vertically transitive.

Lemma 2.17. *The map $\Psi_\kappa \colon (\mathbb{R}^4_{\mathrm{r}},\omega_0) \to (\mathbb{R}^4_{\mathrm{r}},\omega_0)$ as in Corollary 2.8, where κ is defined in (2.4), can be extended to a symplectomorphism $\widetilde{\Psi}_\kappa \colon (\mathbb{R}^4 \setminus D^0_{\mathrm{u}},\omega_0) \to (\mathbb{R}^4 \setminus D^0_{\mathrm{s}},\omega_0)$.*

Proof. Since the map

$$\begin{aligned} \widetilde{\Psi}_\kappa \colon \mathbb{R}^4 \setminus D^0_{\mathrm{u}} &\to \mathbb{R}^4 \setminus D^0_{\mathrm{s}}, \\ (z,\zeta) &\mapsto (z^2\zeta, z^{-1}), \end{aligned} \tag{2.8}$$

coincides with Ψ_κ on $\mathbb{R}^4_{\mathrm{r}}$, $\widetilde{\Psi}_\kappa$ is an extension of Ψ_κ as a diffeomorphism. Since $\mathrm{d}\kappa = 0$ in $\mathbb{R}^2_{\mathrm{r}}$, by Corollary 2.8, Ψ_κ is symplectomorphism of $(\mathbb{R}^4_{\mathrm{r}},\omega_0)$. By continuity, $\widetilde{\Psi}_\kappa$ is a symplectomorphism. Alternatively, one can verify $\widetilde{\Psi}^*_\kappa\omega_0 = \omega_0$ by explicit computations. □

Lemma 2.18. *Let $G \colon U \to U'$ be a diffeomorphism where $U, U' \in \mathcal{N}(\mathbb{R}^2,0)$ such that $G(c_1,c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$. Then $G^*\kappa = \kappa + \mathcal{O}(c^\infty)\,\mathrm{d}c_1 + \mathcal{O}(c^\infty)\,\mathrm{d}c_2$ as an element of $(\Omega^1/2\pi\Lambda)(U \cap \mathbb{R}^2_{\mathrm{r}})$.*

Proof. For $c \neq 0$,

$$\begin{aligned} G^*\kappa(c) - \kappa(c) = &- \ln|G(c)| \frac{\partial G_2}{\partial c_1}(c)\,\mathrm{d}c_1 - \arg\frac{G(c)}{c}\,\mathrm{d}c_1 \\ &- \ln\left|\frac{G(c)}{c}\right| \frac{\partial G_2}{\partial c_2}(c)\,\mathrm{d}c_2 - \ln|c|\left(\frac{\partial G_2}{\partial c_2}(c) - 1\right)\mathrm{d}c_2. \end{aligned} \tag{2.9}$$

By the fact of $x \mapsto \ln(1+x)$ being analytic and $c \mapsto \frac{G(c)}{c} - 1$ being flat, both components of $\ln \frac{G}{c}$ are flat, which is to say that $-\ln\left|\frac{G}{c}\right|, \arg \frac{G}{c} \in \mathcal{O}(c^\infty)$. Since $\frac{\partial G_2}{\partial c_2} - 1, \frac{\partial G_2}{\partial c_1} \in \mathcal{O}(c^\infty)$, by Lemma A.3, we have $\ln|\cdot|(\frac{\partial G_2}{\partial c_2} - 1), \ln|G| \frac{\partial G_2}{\partial c_1} \in \mathcal{O}(c^\infty)$. Hence the form in (2.9) has the shape of $\mathcal{O}(c^\infty)\,\mathrm{d}c_1 + \mathcal{O}(c^\infty)\,\mathrm{d}c_2$. □

Lemma 2.19. *Let $G\colon U \to U'$ be a diffeomorphism where $U, U' \in \mathcal{N}(\mathbb{R}^2, 0)$ such that $G(c_1, c_2) = (c_1, g(c_1, c_2))$ with $\frac{\partial g}{\partial c_2} > 0$. Note that $G^*(\kappa|_{U' \cap \mathbb{R}^2_{\mathrm{r}}}) \in (\Omega^1/2\pi\Lambda)(U \cap \mathbb{R}^2_{\mathrm{r}})$. Then the symplectomorphism $\Psi_{G^*\kappa}\colon (q^{-1}(U \cap \mathbb{R}^2_{\mathrm{r}}), \omega_0) \to (q^{-1}(U' \cap \mathbb{R}^2_{\mathrm{r}}), \omega_0)$ can be extended to a symplectomorphism*

$$\widetilde{\Psi}_{G^*\kappa}\colon (q^{-1}(U) \setminus D^0_{\mathrm{u}}, \omega_0) \to (q^{-1}(U') \setminus D^0_{\mathrm{s}}, \omega_0) \tag{2.10}$$

if and only if $G(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$.

Proof. Notice that $q^{-1}(U \cap \mathbb{R}^2_{\mathrm{r}}) = q^{-1}(U) \cap \mathbb{R}^4_{\mathrm{r}}$ is a punctured neighborhood of $\mathcal{F}_0$ in $\mathbb{R}^4$ and

$$q^{-1}(U) \setminus D^0_{\mathrm{u}} \in \mathcal{N}_q(\mathbb{R}^4 \setminus D^0_{\mathrm{u}}, D^0_{\mathrm{s}}), \qquad q^{-1}(U') \setminus D^0_{\mathrm{s}} \in \mathcal{N}_q(\mathbb{R}^4 \setminus D^0_{\mathrm{s}}, D^0_{\mathrm{u}}).$$

Recall that

$$G^*\kappa(c) = -\ln|G(c)| \frac{\partial G_2}{\partial c_1}(c)\,\mathrm{d}c_1 - \arg G(c)\,\mathrm{d}c_1 - \ln|G(c)| \frac{\partial G_2}{\partial c_2}(c)\,\mathrm{d}c_2.$$

Let $(z, \zeta) = P(c) = (1, c)$, so $c = q(z, \zeta)$. Let $h_1\colon U \cap \mathbb{C}_{\mathrm{r}} \to \mathbb{C}$ and $h_2\colon U \to \mathbb{C}$ be

$$h_1(c) = \frac{G(c)}{c}, \qquad h_2(c) = \frac{\partial G_2}{\partial c_2}(c) - 1 + \mathrm{i}\frac{\partial G_2}{\partial c_1}(c).$$

Then for $c \in U \cap \mathbb{C}_{\mathrm{r}}$ we have

$$\Psi_{G^*\kappa} \circ P(c) = \Big(|G(c)|^{h_2(c)} G(c), |G(c)|^{-h_2(c)} h_1(c)^{-1}\Big).$$

Suppose $\Psi_{G^*\kappa}$ can be extended to $\widetilde{\Psi}_{G^*\kappa}$ as (2.10). By continuity, we conclude that $\lim_{c\to 0}|G(c)|^{-h_2(c)} h_1(c)^{-1}$ as the second complex component of $\widetilde{\Psi}_{G^*\kappa}(1, 0)$ is nonzero. For any fixed $c \in U \cap \mathbb{C}_{\mathrm{r}}$, $t \mapsto h_1(tc)$ is smooth at 0 and $\lim_{t\to 0} h_1(tc) \neq 0$. The map $t \mapsto |t|^{-h_2 \circ G^{-1}(tc)}$ is smooth and has nonzero limit at 0. So $t \mapsto h_2 \circ G^{-1}(tc) \ln|tc| = h_2 \circ G^{-1}(tc) \ln|t| + \mathrm{C}^\infty$ is smooth at 0. Hence by an analogous 1-dimensional version of Lemma A.3, $t \mapsto h_2 \circ G^{-1}(tc)$ is flat at 0. By arbitrarity of c, we have $h_2 \circ G^{-1} \in \mathcal{O}(c^\infty)$, so $h_2 \in \mathcal{O}(c^\infty)$. Therefore $G(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$.

On the other hand, if it is known that $G(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$, then h_1 can be extended onto U with $h_1(0) \neq 0$ and $h_2 \in \mathcal{O}(c^\infty)$. Moreover, $h_2 \ln|G| \in \mathcal{O}(c^\infty)$ by Lemma A.3, so $|G|^{h_2}$ is smooth taking value 1 at 0. Then $\Psi_{G^*\kappa}$ can be extended to a diffeomorphism $\widetilde{\Psi}_{G^*\kappa}$ as (2.10) sending D^0_{s} to D^0_{u}. The pair $(\widetilde{\Psi}_{G^*\kappa}, G)$ is an isomorphism since $\widetilde{\Psi}_{G^*\kappa}$ is a symplectomorphism on $q^{-1}(U \cap \mathbb{R}^2_{\mathrm{r}})$. □

Lemma 2.20. *Let $G\colon U \to U'$ be a diffeomorphism where $U, U' \in \mathcal{N}(\mathbb{R}^2, 0)$ such that $G(c_1, c_2) = (c_1, g(c_1, c_2))$ with $\frac{\partial g}{\partial c_2} > 0$. Then there is a unique symplectomorphism*

$$\varphi_G\colon (q^{-1}(U) \setminus D^0_{\mathrm{u}}, \omega_0) \to (q^{-1}(U') \setminus D^0_{\mathrm{u}}, \omega_0)$$

lifting G characterized by $\varphi_G(1,c) = (1, G(c))$ for $c \in U$. Or equivalently, φ_G is characterized by

$$r = \left(\frac{\partial g}{\partial c_2} \circ q\right) \cdot (r \circ \varphi_G) \tag{2.11}$$

in $q^{-1}(U)$ with r as in (2.6). If $G(c_1,c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$, then φ_G can be uniquely extended to a symplectomorphism

$$\widetilde{\varphi}_G \colon (q^{-1}(U), \omega_0) \to (q^{-1}(U'), \omega_0).$$

Proof. The first part is a result of Lemma 2.10, as $c \mapsto (1,c)$ and $c \mapsto (1, G(c))$ are Lagrangian sections of q, and $G^* \,\mathrm{d}c_1 = \mathrm{d}c_1$. If $\varphi_G \colon (q^{-1}(U) \setminus D^0_{\mathrm{u}}, \omega_0) \to (q^{-1}(U') \setminus D^0_{\mathrm{u}}, \omega_0)$ is a symplectomorphism lifting G and (2.11), then $\varphi_G(1,c) = (1, G(c))$ holds automatically. We then show that the φ_G given by $\varphi_G(1,c) = (1, G(c))$ has the property (2.11). In fact, if $(z,\zeta) \in q^{-1}(U \cap \mathbb{R}^2_{\mathrm{r}})$, $c = q(z,\zeta) \in U \cap \mathbb{R}^2_{\mathrm{r}}$ and let $\tau' = t'_1 \,\mathrm{d}c_1 + t'_2 \,\mathrm{d}c_2 \in \Omega^1(U' \cap \mathbb{R}^2_{\mathrm{r}})$ be so that $\Psi_{\tau'}(1, G(c)) = \varphi_G(z,\zeta)$, then since $\varphi_G \circ \Psi_{G^*\tau'} = \Psi_{\tau'} \circ \varphi_G$, we have $\Psi_{G^*\tau'}(1,c) = (z,\zeta)$, and then by (2.7), we obtain

$$r(z,\zeta) = r \circ \Psi_{G^*\tau'}(1,c) = -\frac{\partial g}{\partial c_2}(c) t'_2(c) = \frac{\partial g}{\partial c_2}(c) \cdot (r \circ \Psi_{\tau'}(1, G(c))) = \frac{\partial g}{\partial c_2}(c) \cdot (r \circ \varphi_G(z,\zeta)).$$

For the second part, consider

$$\begin{aligned} &\widetilde{\varphi}_G \colon q^{-1}(U) \to q^{-1}(U'), \\ &(z,\zeta) \mapsto \left(z e^{(\frac{\partial g}{\partial c_2} \circ q(z,\zeta) - 1 + \mathrm{i} \frac{\partial g}{\partial c_1} \circ q(z,\zeta)) \ln|z|}, \frac{G \circ q(z,\zeta)}{q(z,\zeta)} \zeta e^{-(\frac{\partial g}{\partial c_2} \circ q(z,\zeta) - 1 + \mathrm{i} \frac{\partial g}{\partial c_1} \circ q(z,\zeta)) \ln|z|} \right). \end{aligned} \tag{2.12}$$

Since $\frac{\partial g}{\partial c_2} \circ q - 1 + \mathrm{i} \frac{\partial g}{\partial c_1} \circ q$ is flat at the origin, the exponents in (2.12) are flat at the origin by Lemma A.3 and therefore (2.12) is smooth. By explicit calculations using Lemma 2.9, $\widetilde{\varphi}_G$ extends φ_G. By continuity, the extension is a symplectomorphism and such an extension is unique. □

Lemma 2.21 ([20, Lemma 4.1, Lemma 5.1])**.** *Let $G \colon U \to U'$ be a diffeomorphism where $U, U' \in \mathcal{N}(\mathbb{R}^2, 0)$. Then there is a symplectomorphism φ in a neighborhood of the origin in $\mathbb{R}^4$ lifting G if and only if $G(c_1,c_2) = (e_1 c_1, e_2 c_2 + \mathcal{O}(c^\infty))$, with $e_i = \pm 1$, $i = 1, 2$.*

Proof. If such a $\varphi \colon (V, \omega_0) \to (V', \omega_0)$ exists, by possibly shrinking U and V, both $\Lambda^{(V,\omega_0,q|_V)}(U)$ and $\Lambda^{(V',\omega_0,q|_{V'})}(U')$ would have rank one, generated by $\mathrm{d}c_1$ on U and U' respectively. But since G is a diffeomorphism and $G^*\Lambda^{(V',\omega_0,q|_{V'})}(U') = \Lambda^{(V,\omega_0,q|_V)}(U)$, we must have $G(c_1,c_2) = (e_1 c_1, g(c_1,c_2))$, where $e_1 = \pm 1$, for some smooth function $g \colon U \to \mathbb{R}$. Let $W = q^{-1}(U)$ and $W' = q^{-1}(U')$. Now we *de facto* have $G^*\Lambda^{(W',\omega_0,q|_{W'})} = \Lambda^{(W,\omega_0,q|_W)}$ and by applying Lemma 2.10 to the vertically transitive integrable systems $(W \setminus D^0_{\mathrm{u}}, \omega_0, q|_{W \setminus D^0_{\mathrm{u}}})$ and $(W \setminus D^0_{\mathrm{s}}, \omega_0, q|_{W \setminus D^0_{\mathrm{s}}})$ respectively we can extend φ to a symplectomorphism $\widetilde{\varphi} \colon (W, \omega_0) \to (W', \omega_0)$ lifting G. Since φ preserves the singular fiber $\mathcal{F}_0$, and the punctured fiber $\mathcal{F}_0 \setminus \{0\}$ has two components $D^0_{\mathrm{u}} \setminus \{0\}$ and $D^0_{\mathrm{s}} \setminus \{0\}$, either φ preserves the two components or it exchanges them. In the first case, $\widetilde{\varphi}(D^0_{\mathrm{s}}) = D^0_{\mathrm{s}}$; in the latter case, $\widetilde{\varphi}(D^0_{\mathrm{s}}) = D^0_{\mathrm{u}}$. In any of the four cases above ($e_1 = \pm 1$, $\widetilde{\varphi}(D^0_{\mathrm{s}}) = D^0_{\mathrm{s}}$ or D^0_{u})), there is exactly one choice of (φ_0, G_0) from the set (with maps defined in (2.5))

$$\{(\mathrm{id}, \mathrm{id}), (\varphi_X, G_X), (\varphi_Y, G_Y), (\varphi_Y \circ \varphi_X, G_Y \circ G_X)\} \tag{2.13}$$

such that $\varphi_0 \circ \widetilde{\varphi}\colon (W \setminus D^0_{\mathrm{u}}, \omega_0) \to (W \setminus D^0_{\mathrm{u}}, \omega_0)$, and $\frac{\partial(\mathrm{pr}_2 \circ G_0 \circ G)}{\partial c_2} > 0$. Now we have that

$$(\varphi_0 \circ \widetilde{\varphi})^{-1} \circ \Psi_\kappa \circ (\varphi_0 \circ \widetilde{\varphi}) = \Psi_{(G_0 \circ G)^* \kappa}\colon (W \setminus D^0_{\mathrm{u}}, \omega_0) \to (\mathbb{R}^4 \setminus D^0_{\mathrm{s}}, \omega_0)$$

is a symplectomorphism and $q \circ (\varphi_0 \circ \widetilde{\varphi}) = (G_0 \circ G) \circ q$. By Lemma 2.19, we have $G_0 \circ G(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$. Therefore, $G(c_1, c_2) = (e_1 c_1, e_2 c_2 + \mathcal{O}(c^\infty))$, with $e_i = \pm 1$, $i = 1, 2$.

Conversely, if $G(c_1, c_2) = (e_1 c_1, e_2 c_2 + \mathcal{O}(c^\infty))$ we assume, without loss of generality, that $e_1 = e_2 = 1$. Otherwise, we can apply a pair of maps in (2.13). Let $\varphi = \widetilde{\varphi}_G\colon q^{-1}(U) \to q^{-1}(U')$ be the symplectomorphism defined as in Lemma 2.20. Then we have $q \circ \varphi = G \circ q$. □

For the sake of making this paper self-contained, we now recall the known results on the topological structure near the focus-focus fiber up to fiber-preserving diffeomorphisms; see [5, Chapter 9.8 and Lemma 9.8] and [23, Section 3] for reference.

Definition 2.22. Let $\mathcal{IS}_{\mathrm{ff}}$ be the collection of 4-dimensional integrable systems $(M, \omega, F) \in \mathcal{IS}$ such that F is proper onto its image and has connected fibers[1] one of which is a singular fiber $\mathcal{F} = F^{-1}(0)$ that contains focus-focus singular points of F. We further assume that there are no other singular points in a saturated neighborhood of $\mathcal{F}$.

The assumptions in Definition 2.22 are not too restrictive since, by the local normal form, focus-focus singular points are isolated. In fact, by [11, Proposition 3.21] an almost-toric integrable system (M, ω, F) (in the sense that F is proper onto its image and all of its singular orbits are compact and non-degenerate without hyperbolic blocks) satisfies the assumptions of $\mathcal{IS}_{\mathrm{ff}}$ if the fibers of F are connected. A semitoric integrable system also belongs to $\mathcal{IS}_{\mathrm{ff}}$, as defined in [15, Definition 2.1] in the sense that the first component of the momentum map is proper and generates a Hamiltonian circle action on M, and the momentum map has only non-degenerate singular points without hyperbolic blocks.

Let $(M, \omega, F) \in \mathcal{IS}_{\mathrm{ff}}$. Let $\mathrm{Crit}(\mathcal{F})$ denote the finite set of singular points of F in $\mathcal{F} = F^{-1}(0)$ whose cardinality is called the *multiplicity* of $\mathcal{F}$. For $k \in \mathbb{N}$, let $\mathcal{IS}^k_{\mathrm{ff}}$ be the collection of $(M, \omega, F) \in \mathcal{IS}_{\mathrm{ff}}$ where $\mathcal{F}$ has multiplicity k. For $(M, \omega, F) \in \mathcal{IS}^k_{\mathrm{ff}}$, as shown in Zung [22, Theorem 5.1], $\mathcal{F}$ is homeomorphic to a 2-torus pinched k times along k homologous 1-cycles with an infinite cyclic group as the isotropy group of the $T^*_0 B$-action on $\mathcal{F} \setminus \mathrm{Crit}(\mathcal{F})$ by Ψ, where $B = F(M)$.

Throughout this paper, we denote by $\mathbb{Z}_k$, $k \in \mathbb{N}$, the quotient group $\mathbb{Z}/k\mathbb{Z}$ of residue classes modulo k with the induced operation from the addition on $\mathbb{Z}$. Implied by [22, Theorem 5.1], there is a line in $T^*_0 B$ acting on $\mathcal{F} \setminus \mathrm{Crit}(\mathcal{F})$ as $\mathbb{S}^1$. We show that such an $\mathbb{S}^1$-action can be extended onto a saturated neighborhood of $\mathcal{F}$, which follows from [23, Proposition 3 and Corollary 1].

Lemma 2.23. *Let $(M, \omega, F) \in \mathcal{IS}^k_{\mathrm{ff}}$ and $B = F(M)$. Then the section space $\Lambda(B)$ of the sheaf Λ is an infinite cyclic group in $Z^1(B)$, so it can be viewed as a constant sheaf associated to $\mathbb{Z}$ over B. The quotient sheaf restricted to B_{r}, $(\Lambda/\Lambda(B))|_{B_{\mathrm{r}}}$, is also a constant sheaf associated to $\mathbb{Z}$ over B. In fact, to any simply connected open set $U \subseteq B_{\mathrm{r}}$ we assign a generator α_U of the infinite cyclic group $\Lambda(U)/\Lambda(B)|_U$, such that, for any such open sets U_1 and U_2, the restrictions of α_{U_1} and α_{U_2} to $U_1 \cap U_2$ coincide.*

[1] In this case, by the local models, F is an open map and the fibers of F being connected implies that the preimage of any connected set under F is connected.

There are some further results for the smooth structure of the neighborhoods of singular fibers in [4].

3. Symplectic classification theorem

In this section we state the main theorem of the paper (Theorem 3.9) formulated by the results from Section 2. As the multiple singular points on a fiber lie in no particular order, one must make a number of choices in the construction of the invariant. For a precise formulation of our result we need to first deal with issues of orientations and directions. The result announced in the introduction of the paper is stated precisely as a consequence of our main theorem (Corollary 3.13).

3.1. **Directions and singularity atlas.** Fix a $k \in \mathbb{N}$ and recall Definition 2.22. Let $(M, \omega, F) \in \mathcal{IS}^k_{\mathrm{ff}}$, $B = F(M)$, and $\mathcal{F}$ be the singular fiber. Shrink M to a saturated neighborhood of $\mathcal{F}$ if necessary. Then by Lemma 2.23 $\Lambda(B)$ and $(\Lambda/\Lambda(B))|_{B_{\mathrm{r}}}$ can be viewed as infinite cyclic groups.

Definition 3.1. A pair (α_1, α_2) is a *direction* of (M, ω, F) if α_1 is a generator of $\Lambda(B)$ and α_2 is an generator of $(\Lambda/\Lambda(B))|_{B_{\mathrm{r}}}$. We call α_1 the *J-direction* and α_2 the *H-direction*. We denote by $\mathrm{Dir}(M, \omega, F)$ the set of directions of (M, ω, F).

Given a direction as in Definition 3.1 we have

$$\Lambda(B) = \mathbb{Z}\alpha_1, \qquad (\Lambda/\Lambda(B))|_{B_{\mathrm{r}}} = \mathbb{Z}\alpha_2, \qquad \Lambda(U) = \mathbb{Z}\alpha_1|_U \oplus \mathbb{Z}\widetilde{\alpha}_2|_U.$$

for any simply connected open set $U \subseteq B_{\mathrm{r}}$ and note that α_1 is a 1-form on B while $\alpha_2|_U$ is a 1-form on U modulo integer multiples of $\alpha_1|_U$ and $\widetilde{\alpha}_2|_U$ is a representative of $\alpha_2|_U$.

Remark 3.2. The use of the letters J and H is inspired by the notations in semitoric systems where the momentum maps are usually written as (J, H) such that the flow of $\mathcal{X}_J$ is 2π-periodic.

The set $\mathrm{Dir}(M, \omega, F)$ contains 4 different directions. Recall from Section 2.2 that near any focus-focus singular point $m_j \in M$ there is an Eliasson local chart.

Definition 3.3. Let $(\alpha_1, \alpha_2) \in \mathrm{Dir}(M, \omega, F)$ and let $(m_j)_{j \in \mathbb{Z}_k} = \mathrm{Crit}(\mathcal{F})$. An Eliasson local chart (ψ_j, E_j) near $m_j \in M$ is *compatible with the direction* (α_1, α_2) if $E_j^* \,\mathrm{d}c_1 = \alpha_1$ and $E_j^* \,\mathrm{d}c_2$ at non-origin points are linear combinations of α_1 and (on an open subset any representative of) α_2 with positive α_2-coefficients. A collection $((\psi_j, E_j))_{j \in \mathbb{Z}_k}$ where (ψ_j, E_j) is an Eliasson local chart near m_j, $j \in \mathbb{Z}_k$, is called a *singularity atlas* of (M, ω, F).

A singularity atlas is *compatible with the direction* (α_1, α_2) if for every $j \in \mathbb{Z}_k$, (ψ_j, E_j) is compatible with (α_1, α_2), and for any flow line of $\mathcal{X}_{E_{\bar{0}}^* \mathrm{d}c_2}$ in $\mathcal{F}$, whenever the $\boldsymbol{\alpha}$-limit point is labeled m_j, $j \in \mathbb{Z}_k$, its $\boldsymbol{\omega}$-limit point is labeled $m_{j+\bar{1}}$.

Lemma 3.4. *Given $(\alpha_1, \alpha_2) \in \mathrm{Dir}(M, \omega, F)$ and $m_{\bar{0}} \in \mathrm{Crit}(\mathcal{F})$, there is a unique way to label $\mathrm{Crit}(\mathcal{F}) = (m_j)_{j \in \mathbb{Z}_k}$ such that (M, ω, F) has a singularity atlas $((\psi_j, E_j))_{j \in \mathbb{Z}_k}$ compatible with (α_1, α_2).*

Proof. By possibly composing with one of the pairs of maps in (2.13), we obtain a chart (ψ_j, E_j) near any m_j compatible with (α_1, α_2). By [22, Theorem 5.1], the trajectories of $\mathcal{X}_{E_{\bar{0}}^* \mathrm{d}c_2}$ in $\mathcal{F}$ away from $\mathrm{Crit}(\mathcal{F})$ have limits at different critical points (except when $k = 1$). On two spheres in $\mathcal{F}$ intersecting at a critical point, the trajectories go to opposite directions

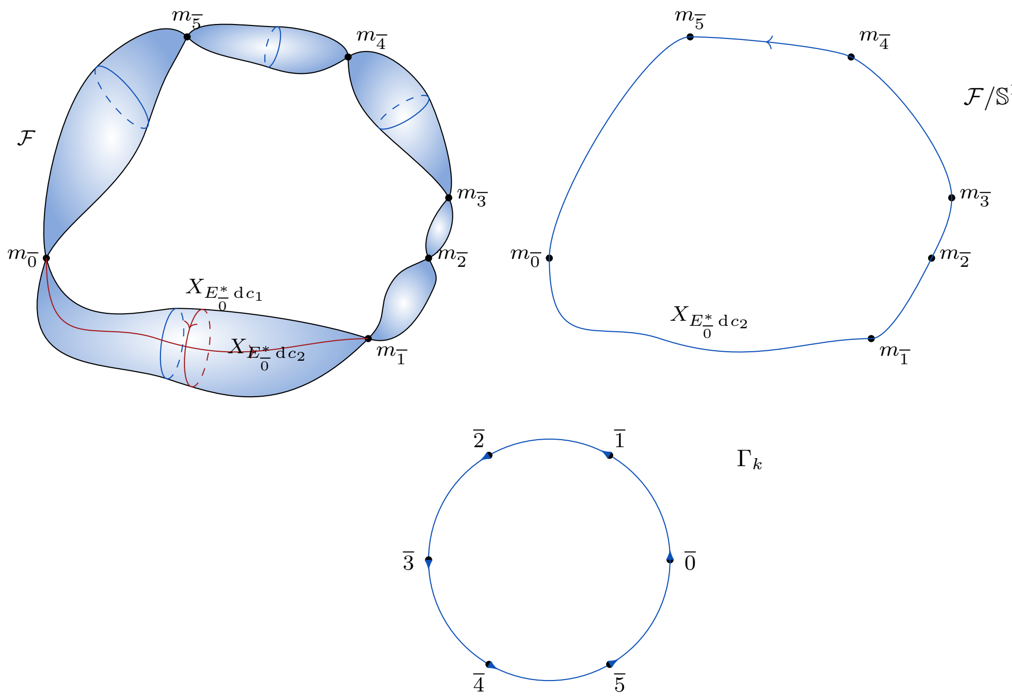

FIGURE 1. Reduction of the singular fiber. The quotient space $\mathcal{F}/\mathbb{S}^1$, is a circle with k marked points. Compare $\mathcal{F}/\mathbb{S}^1$ with Γ_k, which is the cycle graph with vertex set $\mathbb{Z}_k$ and set of edges $\{(j, j+\bar{1}) \mid j \in \mathbb{Z}_k\}$. The automorphism group D_k of Γ_k is isomorphic to the group generated by γ_Y and θ_p (see (3.4) to (3.6)). That is not a coincidence.

relative to that point. Hence we can sort $(m_j)_{j\in\mathbb{Z}_k}$ one by one such that $((\psi_j, E_j))_{j\in\mathbb{Z}_k}$ is compatible with (α_1, α_2). □

3.2. **Construction of the invariants.** Let $(M, \omega, F, (\alpha_1, \alpha_2), m_{\bar{0}})$ be a *marked directed integrable system*, which means that $(M, \omega, F) \in \mathcal{IS}^k_{\mathrm{ff}}$ with *direction* $(\alpha_1, \alpha_2) \in \mathrm{Dir}(M, \omega, F)$ and *mark* $m_{\bar{0}} \in \mathrm{Crit}(\mathcal{F})$. Let $\mathcal{F}$ be the singular fiber. Let $(m_j)_{j\in\mathbb{Z}_k} = \mathrm{Crit}(\mathcal{F})$ and let $((\psi_j : V_j \to \psi_j(V_j), E_j : U \to U_j))_{j\in\mathbb{Z}_k}$ be a singularity atlas[2] compatible with the direction (α_1, α_2). Let $W = F^{-1}(U)$.

To zoom out from local to semiglobal, we will extend the isomorphism (ψ_j^{-1}, E_j^{-1}) along the trajectories. For $j \in \mathbb{Z}_k$, let W_j be the minimal invariant subset of M by Ψ containing $V_j \in \mathcal{N}(M, m_j)$; in other words,

$$W_j = \bigcup_{\alpha\in\Omega^1(F(V_j))} \Psi_\alpha(V_j); \tag{3.1}$$

see Figure 2. Let $W_{j,j+\bar{1}} = W_j \cap W_{j+\bar{1}}$; then $W_{j,j+\bar{1}}$ is a neighborhood of the orbit of Ψ in $\mathcal{F}$ where $\mathcal{X}_{E_j^*\,\mathrm{d}c_2}$ flows from m_j to $m_{j+\bar{1}}$. It is important to note that both $(W_j, \omega_0, F|_{W_j})$ and $(W_{j,j+\bar{1}}, \omega_0, F|_{W_{j,j+\bar{1}}})$ are complete but only the latter one is vertically transitive.

[2]The domain of E_j is not *a priori* independent of $j \in \mathbb{Z}_k$, but then we can replace it by the intersection U of all these domains.

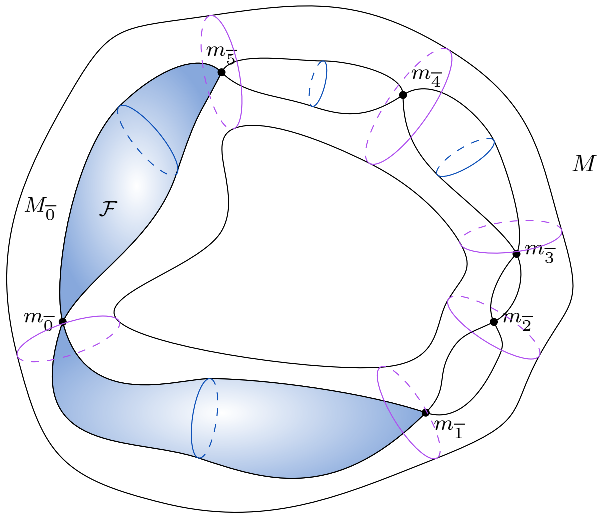

FIGURE 2. The set $W_{\bar 0}$ consists of the shaded region in $\mathcal{F}$ and whole regular fibers nearby.

We construct the first set of the invariants: we split the period form α_2 into the singular part "across singular points" and the regular part. The regular part is an invariant. Recall that local sections of Λ consist of closed 1-forms and then it makes sense to say whether a local section of $\Omega^1/2\pi\Lambda$ is closed, the subpresheaf of which is denoted by $Z^1/2\pi\Lambda$.

Lemma 3.5. *The closed section*

$$\sigma = -\sum_{j\in\mathbb{Z}_k} E_j^*\kappa = 2\pi\alpha_2 - \sum_{j\in\mathbb{Z}_k} E_j^*\kappa \in (Z^1/2\pi\Lambda)(U\cap B_{\mathrm{r}}) \tag{3.2}$$

has a representative in $Z^1(U\cap B_{\mathrm{r}})$ with a smooth extension in $Z^1(U)$, where κ is as defined in (2.4). In other words, there is a $\tilde\sigma \in Z^1(U)$ such that for any $b\in U\cap B_{\mathrm{r}}$ there is a $U_b \in \mathcal{N}(B_{\mathrm{r}}, b)$ such that $\tilde\sigma|_{U_b} \in \sigma|_{U_b} + 2\pi\Lambda(U_b)$.

Proof. The statement is valid since $E_j^*\kappa$ is an element of $(\Omega^1/2\pi\Lambda)(U\cap\mathbb{R}^2_{\mathrm{r}})$ by Lemma 2.18.

For $j\in\mathbb{Z}_k$, we can extend (ψ_j^{-1}, E_j^{-1}) by Lemma 2.10 to morphisms between vertically transitive integrable systems

$$(\lambda_j^+, E_j^{-1})\colon (W\setminus D^0_{\mathrm{s}}, \omega_0, q|_{W\setminus D^0_{\mathrm{s}}}) \to (W_{j,j+\bar 1}, \omega, F|_{W_{j,j+\bar 1}}),$$
$$(\lambda_j^-, E_j^{-1})\colon (W\setminus D^0_{\mathrm{u}}, \omega_0, q|_{W\setminus D^0_{\mathrm{u}}}) \to (W_{j-\bar 1,j}, \omega, F|_{W_{j-\bar 1,j}}),$$

respectively which must coincide in there intersection $W\cap M_{\mathrm{r}}$, and then glue them to a morphism

$$(\lambda_j, E_j^{-1})\colon (W,\omega_0,q|_W) \to (W_j, \omega, F|_{W_j}).$$

For $j\in\mathbb{Z}_k$, let $P_j, Q_j\colon U\to M$ be Lagrangian sections of F given by $P_j(c) = \lambda_j(1,c)$, $Q_j(c) = \lambda_j(c,1)$. Since φ_j is a symplectomorphism, by Lemma 2.9 and the definition of κ in (2.4), the translation form $\tau^{P_jQ_j} = E_j^*\kappa$ is a section in $(Z^1/2\pi\Lambda)(U\cap B_{\mathrm{r}})$.

Note that the images of Q_j and $P_{j+\bar 1}$ lie in W_j. Then there are two smooth sections of q in $\mathbb{R}^4\setminus D^0_{\mathrm{s}}$ mapped to Q_j and $P_{j+\bar 1}$ by λ_j and let $\tau\in(\Omega^1/2\pi\Lambda^{(\mathbb{R}^4,\omega_0,q)})(\mathbb{R}^2)$ be the translation form between them. By explicitly calculating τ using (2.3), it is represented by a smooth form on $\mathbb{R}^2$. So, $\tau^{Q_jP_{j+\bar{1}}} = G^*\tau \in (\Omega^1/2\pi\Lambda^{(M,\omega,F)})(U\cap B_{\mathrm{r}})$ is represented by a smooth form on U, too.

Since α_2 is a section in $\Lambda(B)$ and $\kappa \in (Z^1/2\pi\Lambda)(\mathbb{R}^2_{\mathrm{r}})$ (hence are closed), the section

$$\sigma = 2\pi\alpha_2 - \sum_{j\in\mathbb{Z}_k} E_j^*\kappa = 2\pi\alpha_2 - \sum_{j\in\mathbb{Z}_k} \tau^{P_jQ_j} = \sum_{j\in\mathbb{Z}_k} \tau^{Q_jP_{j+\bar{1}}} \in (\Omega^1/2\pi\Lambda)(U\cap B_{\mathrm{r}})$$

extends to a smooth and closed representative on U (hence is in $(Z^1/2\pi\Lambda)(U\cap B_{\mathrm{r}})$). □

Let $\mathbb{R}[[T_0^*U]]$ be the space of formal power series generated by the elements of a basis of T_0^*U, or equivalently, $\mathbb{R}[[T_0^*U]]$ is the direct sum of symmetric tensor products of T_0^*U. The *Taylor series* $\mathrm{Taylor}_0[f]$ of a smooth function $f\colon U\to\mathbb{R}$ at 0 may be viewed as an element in $\mathbb{R}[[T_0^*U]]$. Recall $\mathbb{R}[[X,Y]]$, the $\mathbb{R}$-algebra of formal power series in two variables X and Y. Let

$$\mathsf{R} = \left\{ \mathsf{u}(X,Y) = \sum_{p+q\geqslant 1} \mathsf{u}^{(p,q)}X^pY^q \;\middle|\; \mathsf{u}^{(p,q)}\in\mathbb{R} \right\}$$

be the ideal of $\mathbb{R}[[X,Y]]$ consisting of those with no constant term. Let $X = \mathrm{d}c_1$, $Y = \mathrm{d}c_2$ be the variables of the formal power series. Then the diffeomorphism $E_j\colon U\to U_j$ identifies two algebras by $E_j^*\colon \mathbb{R}[[X,Y]]\to\mathbb{R}[[T_0^*U]]$.

Let A_1 be the action integral with $\mathrm{d}A_1 = \alpha_1$ and $A_1(0) = 0$. Let $\mathsf{A}_1 = \mathrm{Taylor}_0[A_1] \in \mathbb{R}[[T_0^*U]]$ and we have $(E_j^{-1})^*\mathsf{A}_1 = X$ for any $j\in\mathbb{Z}_k$. Since σ defined in (3.2) is closed, by possibly shrinking U so as to be simply connected, there is a smooth $S\colon U\to\mathbb{R}$ such that $S(0) = 0$ and $\mathrm{d}S$ represents σ in $\Omega^1(U)$. The germ of S is unique up to adding integer multiples of $2\pi A_1$.

Definition 3.6. We call $\sigma\in(\Omega^1/2\pi\Lambda)(U\cap B_{\mathrm{r}})$ the *desingularized period form.* We call the coset $S + 2\pi A_1\mathbb{Z}$ the *desingularized action integral.* Let $\mathsf{R}_{2\pi X} = \mathsf{R}/(2\pi X)\mathbb{Z}$ and let $\mathsf{S} = \mathrm{Taylor}_0[S] + 2\pi\mathsf{A}_1\mathbb{Z} \in \mathbb{R}[[T_0^*U]]_0/(2\pi\mathsf{A}_1)\mathbb{Z}$. For any m_j, $j\in\mathbb{Z}_k$, let

$$\mathsf{s}_j(X,Y) \overset{\mathrm{def}}{=} (E_j^{-1})^*\mathsf{S}.$$

We call $\mathsf{s}_j\in\mathsf{R}_{2\pi X}$ the *action Taylor series at m_j.*

We construct the second set of the invariants: these invariants are Taylor series reflecting the difference between the Eliasson local charts at different singular points.

Definition 3.7. For $j,\ell\in\mathbb{Z}_k$ let $G_{j,\ell} = E_\ell\circ E_j^{-1}$ and $g_{j,\ell} = \mathrm{pr}_2\circ G_{j,\ell}$. Then we have $(G_{j,\ell})(c_1,c_2) = (c_1, g_{j,\ell}(c_1,c_2))$. We call $(g_{j,\ell})_{j,\ell\in\mathbb{Z}_k}$ the set of *momentum transitions.* Let $\mathsf{R}_+ = \{\mathsf{g}\in\mathsf{R} \mid \mathsf{g}^{(0,1)} > 0\}$ be a group with the product $(\mathsf{g}_1\cdot\mathsf{g}_2)(X,Y) = \mathsf{g}_1(X,\mathsf{g}_2(X,Y))$ for any $\mathsf{g}_1,\mathsf{g}_2\in\mathsf{R}_+$. Let $\mathsf{g}_{j,\ell} = \mathrm{Taylor}_0[g_{j,\ell}]\in\mathsf{R}_+$. They satisfy the cocycle condition $\mathsf{g}_{\ell,p}\cdot\mathsf{g}_{j,\ell} = \mathsf{g}_{j,p}$. We call $\mathsf{g}_{j,\ell}$ the *transition Taylor series* from m_j to m_ℓ. We call $(\mathsf{g}_{j,\ell})_{j,\ell\in\mathbb{Z}_k}$ the *transition cocycle.*

The group structure of R_+ is explicitly verified as follows: the multiplicative identity is Y, for $\mathsf{g}_1\in\mathsf{R}_+$, we solve for $\mathsf{g}_1^{-1}\in\mathsf{R}_+$ uniquely in

$$\mathsf{g}_1(X,\mathsf{g}_1^{-1}(X,Y)) = Y,$$

and for $\mathsf{g}_1,\mathsf{g}_2,\mathsf{g}_3\in\mathsf{R}_+$, we have

$$((\mathsf{g}_1\cdot\mathsf{g}_2)\cdot\mathsf{g}_3)(X,Y) = \mathsf{g}_1(X,\mathsf{g}_2(X,\mathsf{g}_3(X,Y))) = (\mathsf{g}_1\cdot(\mathsf{g}_2\cdot\mathsf{g}_3))(X,Y).$$

3.3. **Moduli spaces and main theorem.**

Definition 3.8. Let $\mathcal{M}^k_{\mathrm{ff}}$ be the *collection of integrable systems* in $\mathcal{IS}^k_{\mathrm{ff}}$ modulo isomorphisms of saturated neighborhoods of the singular fiber. Given two marked directed integrable systems $(M, \omega, F, (\alpha_1, \alpha_2), m_{\overline{0}})$ and $(M', \omega', F', (\alpha'_1, \alpha'_2), m'_{\overline{0}})$, an isomorphism (φ, G) from (M, ω, F) to (M', ω', F') is said to *preserve the direction and the mark* if $\varphi(m_{\overline{0}}) = m'_{\overline{0}}$ and $(G^*\alpha'_1, G^*\alpha'_2) = (\alpha_1, \alpha_2)$. Let $\mathcal{M}^k_{\mathrm{ff}\circledast}$ be the *collection of marked directed integrable systems* modulo isomorphisms of saturated neighborhoods of the singular fiber preserving the direction and the mark.

Let $(M, \omega, F, (\alpha_1, \alpha_2), m_{\overline{0}})$ be a marked directed integrable system with $(M, \omega, F) \in \mathcal{IS}^k_{\mathrm{ff}}$. Let $(m_j)_{j\in\mathbb{Z}_k} = \mathrm{Crit}(\mathcal{F})$ and suppose $((\psi_j, E_j))_{j\in\mathbb{Z}_k}$ is a singularity atlas compatible with (α_1, α_2). Let $(\mathsf{s}_j)_{j\in\mathbb{Z}_k}$ be the k-tuple of action Taylor series and let $(\mathsf{g}_{j,\ell})_{j,\ell\in\mathbb{Z}_k}$ be the transition cocycle. These series are constrained by the following relations:

$$\left.\begin{aligned} \mathsf{s}_j &= \mathsf{s}_\ell \cdot \mathsf{g}_{j,\ell} \text{ for } j, \ell \in \mathbb{Z}_k;\\ \mathsf{g}_{j,j}(X, Y) &= Y \text{ for } j \in \mathbb{Z}_k;\\ \mathsf{g}_{\ell,p} \cdot \mathsf{g}_{j,\ell} &= \mathsf{g}_{j,p} \text{ for } j, \ell, p \in \mathbb{Z}_k. \end{aligned}\right\} \tag{3.3}$$

Theorem 3.9 (Main Theorem)**.** *There is a bijection*

$$\Phi\colon \mathcal{M}^k_{\mathrm{ff}\circledast} \to \mathcal{I}^k_{\mathrm{ff}\circledast} \overset{\mathrm{def}}{=} \left\{((\mathsf{s}_j)_{j\in\mathbb{Z}_k}, (\mathsf{g}_{j,\ell})_{j,\ell\in\mathbb{Z}_k}) \in \mathsf{R}^k_{2\pi X} \times \mathsf{R}^{k^2}_+ \;\middle|\; (3.3)\right\}$$
$$[(M, \omega, F, (\alpha_1, \alpha_2), m_{\overline{0}})] \mapsto \left(\mathsf{s}_{\overline{0}}, \ldots, \mathsf{s}_{\overline{-1}}, \mathsf{g}_{\overline{0},\overline{0}}, \ldots, \mathsf{g}_{\overline{0},\overline{-1}}, \ldots, \mathsf{g}_{\overline{-1},\overline{0}}, \ldots, \mathsf{g}_{\overline{-1},\overline{-1}}\right).$$

Since the projection

$$\mathcal{I}^k_{\mathrm{ff}\circledast} \to \mathsf{R}_{2\pi X} \times \mathsf{R}^{k-1}_+,$$
$$((\mathsf{s}_j)_{j\in\mathbb{Z}_k}, (\mathsf{g}_{j,\ell})_{j,\ell\in\mathbb{Z}_k}) \mapsto \left(\mathsf{s}_{\overline{0}}, \mathsf{g}_{\overline{0},\overline{1}}, \mathsf{g}_{\overline{1},\overline{2}}, \ldots, \mathsf{g}_{\overline{k-2},\overline{k-1}}\right)$$

is one-to-one, equivalently, we have the following two statements:

- *Injectivity/Uniqueness*: two equivalence classes of semiglobal models of integrable systems $[(M, \omega, F, (\alpha_1, \alpha_2), m_{\overline{0}})]$ near a focus-focus fiber are isomorphic preserving the direction and the mark if and only if they have the same Taylor series invariant $(\mathsf{s}_{\overline{0}}, \mathsf{g}_{\overline{0},\overline{1}}, \mathsf{g}_{\overline{1},\overline{2}}, \ldots, \mathsf{g}_{\overline{k-2},\overline{k-1}})$.
- *Surjectivity/Existence*: given the abstract ingredient of a k-tuple of formal power series $(\mathsf{s}_{\overline{0}}, \mathsf{g}_{\overline{0},\overline{1}}, \mathsf{g}_{\overline{1},\overline{2}}, \ldots, \mathsf{g}_{\overline{k-2},\overline{k-1}}) \in \mathsf{R}_{2\pi X} \times \mathsf{R}^{k-1}_+$ there exists a unique equivalence class of semiglobal models of integrable systems $[(M, \omega, F, (\alpha_1, \alpha_2), m_{\overline{0}})]$ near a focus-focus fiber which has Taylor series invariants precisely the one we started with.

We prove Theorem 3.9 in the remaining sections. In the rest of this section, we are going to prove that Φ is a well-defined map.

Remark 3.10. When $\mathcal{F}$ is single pinched, we recover the *Taylor series invariant* $(S)^\infty$ from [20] by the following relation:

$$\mathsf{s}_{\overline{0}}(X, Y) = (S)^\infty(Y, X) + \frac{\pi}{2}X.$$

The addition of $\frac{\pi}{2}X$ is due to a change in convention. In this paper, it is the first component of the momentum map which has periodic Hamiltonian vector fields, while in [20] it is the second one.

3.4. **The invariants are well-defined.** In this subsection, we are going to show that the output of Φ does not depend on the choice of the singularity atlas, and we also want to know how the Taylor series will change if the direction and the base point change.

Define bijections γ_X and γ_Y of $\mathcal{I}^k_{\mathrm{ff}\circledast}$ by

$$\begin{aligned}\gamma_X(\dots,\mathsf{s}_j,\dots,\mathsf{g}_{j,\ell},\dots) &= \big(\dots,\mathsf{s}'_j,\dots,\mathsf{g}'_{j,\ell},\dots\big),\\ \mathsf{s}'_j(X,Y) &= \mathsf{s}_j(-X,Y)+k\pi X,\\ \mathsf{g}'_{j,\ell}(X,Y) &= \mathsf{g}_{j,\ell}(-X,Y);\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}\gamma_Y(\dots,\mathsf{s}_j,\dots,\mathsf{g}_{j,\ell},\dots) &= \big(\dots,\mathsf{s}''_j,\dots,\mathsf{g}''_{j,\ell},\dots\big),\\ \mathsf{s}''_j(X,Y) &= -\mathsf{s}_{-j}(X,-Y),\\ \mathsf{g}''_{j,\ell}(X,Y) &= -\mathsf{g}_{-j,-\ell}(X,-Y).\end{aligned}\tag{3.5}$$

Define a bijection θ_p of $\mathcal{I}^k_{\mathrm{ff}\circledast}$, $p\in\mathbb{Z}_k$, by

$$\theta_p(\dots,\mathsf{s}_j,\dots,\mathsf{g}_{j,\ell},\dots) = (\dots,\mathsf{s}_{j+p},\dots,\mathsf{g}_{j+p,\ell+p},\dots).\tag{3.6}$$

The proof of the well-definedness lemma follows from [20, Section 4] and [18, Lemma 4.55].

Lemma 3.11. *The map $\Phi\colon \mathcal{M}^k_{\mathrm{ff}\circledast}\to\mathcal{I}^k_{\mathrm{ff}\circledast}$ is well defined.*

Proof. Independence of the choice of the singularity atlas.

Let $[(M,\omega,F,(\alpha_1,\alpha_2),m_{\bar 0})]\in\mathcal{M}^k_{\mathrm{ff}\circledast}$ and let $((\psi_j,E_j))_{j\in\mathbb{Z}_k}$ and $((\psi'_j,E'_j))_{j\in\mathbb{Z}_k}$ be two singularity atlases compatible with (α_1,α_2). Let $(\dots,\mathsf{s}_j,\dots,\mathsf{g}_{j,\ell},\dots)$ and $(\dots,\mathsf{s}'_j,\dots,\mathsf{g}'_{j,\ell},\dots)$ be the outputs of Φ, let σ and σ' be the desingularized period forms, and let $(g_{j,\ell})_{j,\ell\in\mathbb{Z}_k}$ and $(g'_{j,\ell})_{j,\ell\in\mathbb{Z}_k}$ be the set of momentum transitions, respectively, of $((\psi_j,E_j))_{j\in\mathbb{Z}_k}$ and $((\psi'_j,E'_j))_{j\in\mathbb{Z}_k}$. We aim to show that $\mathsf{s}'_j=\mathsf{s}_j$ and $\mathsf{g}'_{j,\ell}=\mathsf{g}_{j,\ell}$ for $j,\ell\in\mathbb{Z}_k$.

Since $G_j\overset{\mathrm{def}}{=}E'_j\circ E_j^{-1}$ is a diffeomorphism of neighborhoods of 0 in $\mathbb{R}^2$ with properties that $\mathrm{d}(\mathrm{pr}_1\circ G_j)=\mathrm{d}c_1$, $\frac{\partial(\mathrm{pr}_2\circ G_j)}{\partial c_2}>0$, and $q\circ\psi'_j\circ\psi_j^{-1}=G_j\circ q$, by Lemma 2.21 we have $G_j(c_1,c_2)=(c_1,c_2+\mathcal{O}(c^\infty))$. Then Lemma 2.18, we have $G_j^*\kappa=\kappa+\mathcal{O}(c^\infty)\,\mathrm{d}c_1+\mathcal{O}(c^\infty)\,\mathrm{d}c_2$, and we have

$$\begin{aligned}\mathsf{g}'_{j,\ell} &= \mathrm{Taylor}_0[\mathrm{pr}_2\circ G_\ell\circ G_{j,\ell}\circ G_j^{-1}] = \mathrm{Taylor}_0[\mathrm{pr}_2\circ G_{j,\ell}] = \mathsf{g}_{j,\ell},\\ \sigma'-\sigma &= \left(2\pi\alpha_2-\sum_{j\in\mathbb{Z}_k}(E'_j)^*\kappa\right)-\left(2\pi\alpha_2-\sum_{j\in\mathbb{Z}_k}E_j^*\kappa\right)\\ &= \sum_{j\in\mathbb{Z}_k}E_j^*\big(\kappa-G_j^*\kappa\big) = \mathcal{O}(c^\infty)\,\mathrm{d}c_1+\mathcal{O}(c^\infty)\,\mathrm{d}c_2+2\pi\alpha_2\mathbb{Z},\end{aligned}$$

which implies that $\mathsf{S}'=\mathsf{S},\mathsf{s}'_j=\mathsf{s}_j$ for $j\in\mathbb{Z}_k$.

Independence of the choice of the representative.

Let $(M,\omega,F,(\alpha_1,\alpha_2),m_{\bar 0})$ and $(M',\omega',F',(\alpha'_1,\alpha'_2),m'_{\bar 0})$ be two marked directed integrable systems related by an isomorphism (φ,G) between saturated neighborhoods of $\mathcal{F}$ and $\mathcal{F}'$ preserving the direction and the mark. That is to say that $\varphi(m_{\bar 0})=m'_{\bar 0}$ and $(G^*\alpha'_1,G^*\alpha'_2)=$

(α_1, α_2). Let

$$(\dots, \mathsf{s}_j, \dots, \mathsf{g}_{j,\ell}, \dots) = \Phi\Big([(M, \omega, F, (\alpha_1, \alpha_2), m_{\overline{0}})]\Big),$$
$$\Big(\dots, \mathsf{s}'_j, \dots, \mathsf{g}'_{j,\ell}, \dots\Big) = \Phi\Big([(M', \omega', F', (\alpha'_1, \alpha'_2), m'_{\overline{0}})]\Big).$$

According to the first part, the above two lines do not depend on the choices of compatible singularity atlases and we aim to show that the above two lines are equal. Let $((\psi_j, E_j))_{j\in\mathbb{Z}_k}$ be a singularity atlas of (M, ω, F) compatible with (α_1, α_2) and then $((\psi'_j, E'_j))_{j\in\mathbb{Z}_k} \overset{\text{def}}{=} ((\psi_j \circ \varphi^{-1}, E_j \circ G^{-1}))_{j\in\mathbb{Z}_k}$ is a singularity atlas of (M', ω', F') compatible with (α'_1, α'_2). Let σ and σ' be the desingularized period forms, and then

$$G^*\sigma' = 2\pi G^*\alpha'_2 - \sum_{j\in\mathbb{Z}_k} (E'_j \circ G)^*\kappa = 2\pi\alpha_2 - \sum_{j\in\mathbb{Z}_k} E_j^*\kappa = \sigma,$$
$$E'_\ell \circ (E'_j)^{-1} = (E_\ell \circ G^{-1}) \circ (E_j \circ G^{-1})^{-1} = E_\ell \circ E_j^{-1}.$$

Hence we have $G \circ \mathsf{S}' = \mathsf{S}$, $\mathsf{s}'_j = \mathsf{s}_j$, and $\mathsf{g}'_{j,\ell} = \mathsf{g}_{j,\ell}$ for $j, \ell \in \mathbb{Z}_k$. □

Lemma 3.12. *The map* $\Phi\colon \mathcal{M}^k_{\mathrm{ff}\circledast} \to \mathcal{I}^k_{\mathrm{ff}\circledast}$ *satisfies the relations:*

$$\Phi\Big([(M, \omega, F, (-\alpha_1, \alpha_2), m_{\overline{0}})]\Big) = \gamma_X\Big(\Phi\Big([(M, \omega, F, (\alpha_1, \alpha_2), m_{\overline{0}})]\Big)\Big),$$
$$\Phi\Big([(M, \omega, F, (\alpha_1, -\alpha_2), m_{\overline{0}})]\Big) = \gamma_Y\Big(\Phi\Big([(M, \omega, F, (\alpha_1, \alpha_2), m_{\overline{0}})]\Big)\Big),$$
$$\Phi\Big([(M, \omega, F, (\alpha_1, \alpha_2), m_p)]\Big) = \theta_p\Big(\Phi\Big([(M, \omega, F, (\alpha_1, \alpha_2), m_{\overline{0}})]\Big)\Big), \textit{ for } p \in \mathbb{Z}_k.$$

Proof. Let $[(M, \omega, F, (\alpha_1, \alpha_2), m_{\overline{0}})] \in \mathcal{M}^k_{\mathrm{ff}\circledast}$ and let $((\psi_j, E_j))_{j\in\mathbb{Z}_k}$ be a singularity atlas compatible with (α_1, α_2). Let $((\psi'_j, E'_j))_{j\in\mathbb{Z}_k}$ be another singularity atlas and let $m'_{\overline{0}}$ be another mark. We may need to reorder $\mathrm{Crit}(\mathcal{F})$ to $(m'_j)_{j\in\mathbb{Z}_k}$ by Lemma 3.4 so that (ψ'_j, E'_j) is a chart near m'_j for $j \in \mathbb{Z}_k$ and $((\psi'_j, E'_j))_{j\in\mathbb{Z}_k}$ is compatible with some direction $(\alpha'_1, \alpha'_2) \in \mathrm{Dir}(M, \omega, F)$.

Let

$$(\dots, \mathsf{s}_j, \dots, \mathsf{g}_{j,\ell}, \dots) = \Phi\Big([(M, \omega, F, (\alpha_1, \alpha_2), m_{\overline{0}})]\Big),$$
$$\Big(\dots, \mathsf{s}'_j, \dots, \mathsf{g}'_{j,\ell}, \dots\Big) = \Phi\Big([(M, \omega, F, (\alpha'_1, \alpha'_2), m'_{\overline{0}})]\Big).$$

Let σ, σ' be the desingularized period forms, and $(g_{j,\ell})_{j,\ell\in\mathbb{Z}_k}, (g'_{j,\ell})_{j,\ell\in\mathbb{Z}_k}$ respectively be the set of momentum transitions of $((\psi_j, E_j))_{j\in\mathbb{Z}_k}, ((\psi'_j, E'_j))_{j\in\mathbb{Z}_k}$.

Case 1: If $(\alpha'_1, \alpha'_2) = (-\alpha_1, \alpha_2)$ and $m'_{\overline{0}} = m_{\overline{0}}$, then $m'_j = m_j$ for $j \in \mathbb{Z}_k$. We have $G_X^*\kappa = \kappa + \pi\,\mathrm{d}c_1$ (recall G_X defined in (2.5)). Since $((\psi'_j, G_X \circ E_j))_{j\in\mathbb{Z}_k}$ is compatible with (α'_1, α'_2), by Lemma 3.11, it is sufficient to assume that $E'_j = G_X \circ E_j$. Then

$$\begin{aligned}\sigma' &= 2\pi\alpha'_2 - \sum_{j\in\mathbb{Z}_k} (E'_j)^*\kappa = 2\pi\alpha_2 - \sum_{j\in\mathbb{Z}_k} E_j^*(\kappa + \pi\,\mathrm{d}c_1)\\ &= \sigma - \pi \sum_{j\in\mathbb{Z}_k} E_j^*\,\mathrm{d}c_1 = \sigma - k\pi\,\mathrm{d}c_1,\end{aligned}$$

and $g'_{j,\ell}(c) = \mathrm{pr}_2 \circ G_X \circ G_{j,\ell} \circ G_X^{-1}(c) = g_{j,\ell}(-\bar{c})$. In this case,

$$\begin{aligned} \mathsf{S}' &= \mathsf{S} - k\pi[c_1], \\ \mathsf{s}'_j(X,Y) &= \mathsf{s}_j(-X,Y) + k\pi X, \\ \mathsf{g}'_{j,\ell}(X,Y) &= \mathsf{g}_{j,\ell}(-X,Y). \end{aligned}$$

Case 2: If $(\alpha'_1, \alpha'_2) = (\alpha_1, -\alpha_2)$ and $m'_{\bar{0}} = m_{\bar{0}}$, then $m'_j = m_{-j}$ for $j \in \mathbb{Z}_k$ (since the direction of $\mathcal{X}_{E^*_{\bar{0}} \mathrm{d}c_2}$ is reversed). We have $G_Y^* \kappa = -\kappa$ (recall G_Y defined in (2.5)). By Lemma 3.11, it is sufficient to assume that $E'_j = G_Y \circ E_{-j}$. Then

$$\sigma' = 2\pi\alpha'_2 - \sum_{j \in \mathbb{Z}_k} (E'_j)^* \kappa = -2\pi\alpha_2 + \sum_{j \in \mathbb{Z}_k} E^*_{-j} \kappa = -\sigma,$$

and $g'_{j,\ell}(c) = \mathrm{pr}_2 \circ G_Y \circ G_{-j,-\ell} \circ G_Y^{-1}(c) = -g_{-j,-\ell}(\bar{c})$. In this case,

$$\begin{aligned} \mathsf{S}' &= -\mathsf{S}, \\ \mathsf{s}'_j(X,Y) &= -\mathsf{s}_{-j}(X,-Y), \\ \mathsf{g}'_{j,\ell}(X,Y) &= -\mathsf{g}_{-j,-\ell}(X,-Y). \end{aligned}$$

Case 3: If $(\alpha'_1, \alpha'_2) = (\alpha_1, \alpha_2)$ and $m'_{\bar{0}} = m_p$ for some $p \in \mathbb{Z}_k$, then $m'_j = m_{j+p}$ for $j \in \mathbb{Z}_k$. By Lemma 3.11, it is sufficient to assume that $E'_j = E_{j+p}$. Then $\sigma' = \sigma$ and $g'_{j,\ell} = g_{j+p,\ell+p}$. Hence $\mathsf{S}' = \mathsf{S}, \mathsf{s}'_j = \mathsf{s}_{j+p}, \mathsf{g}'_{j,\ell} = \mathsf{g}_{j+p,\ell+p}$. □

3.5. **Corollary of the main theorem.** The bijections γ_X, γ_Y, and θ_p are subject to the relations

$$\gamma_X^2 = \gamma_Y^2 = \theta_p^p = (\gamma_Y \circ \theta_p)^2 = \mathrm{id}\,.$$

So they generate a $(\mathbb{Z}_2 \times D_k)$-action on $\mathcal{I}^k_{\mathrm{ff}\circledast}$; γ_X generates $\mathbb{Z}_2$, γ_Y and θ_p generate D_k. We have:

Corollary 3.13 (Corollary of Theorem 3.9)**.** *There is a bijection*

$$\begin{aligned} \widetilde{\Phi}\colon \mathcal{M}^k_{\mathrm{ff}} &\to \mathcal{I}^k_{\mathrm{ff}} \stackrel{\mathrm{def}}{=} \mathcal{I}^k_{\mathrm{ff}\circledast}/(\mathbb{Z}_2 \times D_k) \\ [(M,\omega,F)] &\mapsto [\Phi([(M,\omega,F,(\alpha_1,\alpha_2),m_{\bar{0}})])] \end{aligned}$$

where $(\alpha_1, \alpha_2) \in \mathrm{Dir}(M,\omega,F)$, $m_{\bar{0}}$ is a singular point of F, and the $(\mathbb{Z}_2 \times D_k)$-action is generated by γ_X, γ_Y, and θ_p.

Remark 3.14. As pointed out in [18] the Taylor series invariant in the case that the singular fiber contains exactly one critical point of focus-focus is defined up to a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action, which accounts for the choices of Eliasson local charts in its construction. It becomes unique in the presence of a global $\mathbb{S}^1$-action (i.e., semitoric systems) provided one assumes everywhere that the Eliasson local charts preserve the $\mathbb{S}^1$-action and the $\mathbb{R}^2$-direction. In Corollary 3.13, we have the $(\mathbb{Z}_2 \times D_k)$-action instead. When $k = 1$, $(\mathbb{Z}_2 \times D_k) \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Remark 3.15. We can recover the smooth invariant in [4, Theorem 3.7] from our symplectic invariant $(\mathbb{Z}_2 \times D_k) \cdot ((\mathsf{s}_j)_{j\in\mathbb{Z}_k}, (\mathsf{g}_{j,\ell})_{j,\ell\in\mathbb{Z}_k})$. To any $\mathsf{w} \in \mathsf{R}_+$ we assign a complex formal series $\mathsf{w}_{\mathbb{C}}(Z, \overline{Z}) = X + \mathrm{i}\mathsf{w}(X, Y)$ by setting $Z = X + \mathrm{i}Y$ and then let

$$\mathsf{LR}_+ = \left\{ \mathsf{w} \in \mathsf{R}_+ \;\middle|\; \mathsf{w}_{\mathbb{C}}(Z, \overline{Z}) = \sum_{r=1}^{\infty}\sum_{s=0}^{\infty} \mathsf{w}_{\mathbb{C}}^{(r,\overline{s})} Z^r \overline{Z}^s, \mathsf{w}_{\mathbb{C}}^{(r,\overline{s})} \in \mathbb{C} \right\}.$$

Then LR_+ is the space of the Taylor series of the second components of orientation-preserving liftable diffeomorphisms characterized by [4, Theorem 3.4] that preserves the first coordinate. Consider the action of LR_+^{k-1} on $\mathsf{R}_+^{k^2}$ by

$$(\mathsf{w}_j)_{j\in\mathbb{Z}_k} \cdot (\mathsf{g}_{j,\ell})_{j,\ell\in\mathbb{Z}_k} = \left(\mathsf{w}_\ell \cdot \mathsf{g}_{j,\ell} \cdot \mathsf{w}_j^{-1}\right)_{j,\ell\in\mathbb{Z}_k}$$

for $(\mathsf{w}_j)_{j\in\mathbb{Z}_k\setminus\{\overline{0}\}} \in \mathsf{LR}_+^{k-1}$ and $\mathsf{w}_{\overline{0}}(X, Y) = Y$. The smooth invariant is equivalent to the orbit of

$$(\mathsf{LR}_+^{k-1} \rtimes (\mathbb{Z}_2 \times D_k)) \cdot (\mathsf{g}_{j,\ell})_{j,\ell\in\mathbb{Z}_k}.$$

The C^1-invariant is equivalent to the tuple of numbers given by

$$\mu_j = \frac{(\mathsf{g}_{\overline{0},j})_{\mathbb{C}}^{(0,\overline{1})}}{\overline{(\mathsf{g}_{\overline{0},j})_{\mathbb{C}}}^{(1,\overline{0})}} = \frac{1 - \left(\mathsf{g}_{\overline{0},j}^{(1,0)}\right)^2 - \left(\mathsf{g}_{\overline{0},j}^{(0,1)}\right)^2 + 2\mathrm{i}\mathsf{g}_{\overline{0},j}^{(1,0)}}{\left(\mathsf{g}_{\overline{0},j}^{(1,0)}\right)^2 + \left(1 + \mathsf{g}_{\overline{0},j}^{(0,1)}\right)^2}$$

for $j \in \mathbb{Z}_k \setminus \{\overline{0}\}$, up to simultaneously multiplying a complex number of unit norm or taking complex conjugations. Note that there are convention changes to the definitions so we said "is equivalent to" instead of "is". Note also that the realization theorem [4, Theorem 3.7] ensured that every smooth focus-focus singularity is diffeomorphic to a symplectic one, so we have quotient maps from symplectic invariants to smooth and C^1-invariants.

4. Proof of uniqueness

The goal of this section is to show Lemma 4.2, the injectivity claim of Theorem 3.9. Let $[(M, \omega, F, (\alpha_1, \alpha_2), m_{\overline{0}})], [(M', \omega', F', (\alpha_1', \alpha_2'), m_{\overline{0}}')] \in \mathcal{M}^k_{\mathrm{ff}\circledast}$ such that

$$\Phi\Big([(M, \omega, F, (\alpha_1, \alpha_2), m_{\overline{0}})]\Big) = \Phi\Big([(M', \omega', F', (\alpha_1', \alpha_2'), m_{\overline{0}}')]\Big).$$

Let $B = F(M)$, $B' = F'(M')$. Let $\mathcal{F} = F^{-1}(0)$ and $\mathcal{F}' = (F')^{-1}(0)$ be the singular fibers. Let $(m_j)_{j\in\mathbb{Z}_k} = \mathrm{Crit}(\mathcal{F})$ and $(m_j')_{j\in\mathbb{Z}_k} = \mathrm{Crit}(\mathcal{F}')$. We aim to show that, there are $U \in \mathcal{N}(B, 0)$ and $U' \in \mathcal{N}(B', 0)$, a symplectomorphism $\varphi\colon (F^{-1}(U), \omega, \mathcal{F}) \to ((F')^{-1}(U'), \omega', \mathcal{F}')$ lifting a diffeomorphism $G\colon U \to U'$ such that $(G^*\alpha_1', G^*\alpha_2') = (\alpha_1, \alpha_2)$ and $\varphi(m_j) = m_j'$.

The following lemma is analogous to Vũ Ngọc [20, Lemma 5.1].

Lemma 4.1. *Suppose $\beta, \beta' \in (\Omega^1/2\pi\Lambda^{(\mathbb{R}^4,\omega_0,q)})(U \cap \mathbb{R}^2_{\mathrm{r}})$ for some $U \in \mathcal{N}(\mathbb{R}^2, 0)$ such that $2\pi\beta = \tau + \sum_{j\in\mathbb{Z}_k} E_j^*\kappa$ and $2\pi\beta' = \tau' + \sum_{j\in\mathbb{Z}_k} E_j^*\kappa$ for some $\tau, \tau' \in \Omega^1(U)$ and diffeomorphisms $E_j\colon U \to E_j(U) \subseteq \mathbb{R}^2$, and $\tau' - \tau$ is a closed and flat. Then there are a diffeomorphism $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ isotopic to the identity and a $U' \in \mathcal{N}(\mathbb{R}^2, 0)$ such that $G(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$ and $G^*\beta' = \beta$ on U'.*

Proof. Let $\rho = \tau' - \tau \in Z^1(U)$. Throughout the proof t is a variable in $[0,1]$. Let $\beta_t = \beta + t\rho$, and let $R \in \mathcal{O}(c^\infty)$ be such that $\mathrm{d}R = \rho$. Define a family of functions $h_t\colon U \to \mathbb{R}$ as

$$h_t = \left\langle \beta_t, \frac{\partial}{\partial c_2} \right\rangle = \left\langle \frac{\tau}{2\pi} + t\rho, \frac{\partial}{\partial c_2} \right\rangle - \sum_{j\in\mathbb{Z}_k} \frac{\partial(\mathrm{pr}_2 \circ E_j)}{\partial c_2} \frac{\ln|E_j|}{2\pi}.$$

Since $\frac{\partial(\mathrm{pr}_2 \circ E_j)}{\partial c_2}(0) > 0$ for any $j \in \mathbb{Z}_k$, we have $h_t(c) \to \infty$ as $c \to 0$. Note that for any multi-index j the partial derivative $\partial^j(1/h_t)$ is a polynomial of $|E_j|^{-1}$ and $\ln|E_j|$, $j \in \mathbb{Z}_k$ with coefficients as smooth functions, divided by the $|j|$-th power of h_t. Thus $1/h_t$ satisfies (A.2) (in place of h). Since R is flat, by Lemma A.4 and using a bump function, for any $U'' \in \mathcal{N}(\mathbb{R}^2, 0)$ with $\overline{U''} \subset U$, the family of functions $f_t \overset{\mathrm{def}}{=} -R|_{U''}/h_t$ on $U'' \cap \mathbb{R}^2_{\mathrm{r}}$ has a smooth extension $\tilde{f}_t\colon \mathbb{R}^2 \to \mathbb{R}$ that is flat at the origin and has compact support. Take $G = G_1$ as G_t to be the flow of $\mathcal{Y}_t = \tilde{f}_t \frac{\partial}{\partial c_2}$ on $\mathbb{R}^2$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(G_t^*\beta_t) = G_t^*(\mathrm{d}\langle \beta_t, \mathcal{Y}_t\rangle + \rho) = G_t^*(\mathrm{d}(\tilde{f}_t\langle \beta_t, \tfrac{\partial}{\partial c_2}\rangle) + \rho) = G_t^*(-\mathrm{d}R + \rho) = 0$$

on $G_t^{-1}(U'')$. Hence $G_t^*\beta_t = \beta$ on U', the intersection of $G_t^{-1}(U'')$ for $t \in [0,1]$ and by the construction $G_t(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$. □

Lemma 4.2. *The map $\Phi\colon \mathcal{M}^k_{\mathrm{ff}\circledast} \to \mathcal{I}^k_{\mathrm{ff}\circledast}$ is injective.*

Proof. Initialization: Let $((\psi_j\colon V_j \to \psi_j(V_j), E_j\colon U \to U_j))_{j\in\mathbb{Z}_k}$ be a singularity atlas of (M, ω, F) compatible with (α_1, α_2), and let $((\psi'_j\colon V'_j \to \psi'_j(V'_j), E'_j\colon U' \to U'_j))_{j\in\mathbb{Z}_k}$ be a singularity atlas of (M', ω', F') compatible with (α'_1, α'_2). Suppose the two marked directed integrable systems share the same set of invariants $((\mathsf{s}_j)_{j\in\mathbb{Z}_k}, (\mathsf{g}_{j,\ell})_{j,\ell\in\mathbb{Z}_k})$. For $j, \ell \in \mathbb{Z}_k$ recall that $G_{j,\ell} = E_\ell \circ E_j^{-1}$ and let $G'_{j,\ell} = E'_\ell \circ (E'_j)^{-1}$; then both

$$((E'_j)^{-1})^*\sigma' - (E_j^{-1})^*\sigma, \qquad \mathrm{pr}_2 \circ (G'_{j,\ell} - G_{j,\ell})$$

are flat. We will construct a new system over U isomorphic to (M', ω', F') over U' with a singularity atlas of the system compatible with (α_1, α_2).

Calculating a smooth representative of the difference of two singular forms

$$\begin{aligned}
((E'_{\bar{0}})^{-1})^*\alpha'_2 - (E_{\bar{0}}^{-1})^*\alpha_2 &= \frac{1}{2\pi}\left(\left(((E'_{\bar{0}})^{-1})^*\sigma' - \sum_{j\in\mathbb{Z}_k} (G'_{\bar{0},j})^*\kappa\right) - \left((E_{\bar{0}}^{-1})^*\sigma - \sum_{j\in\mathbb{Z}_k} G^*_{\bar{0},j}\kappa\right)\right) \\
&= \Big(\big(((E'_j)^{-1})^*\sigma' - (E_j^{-1})^*\sigma\big) - \sum_{j\in\mathbb{Z}_k} E_j^*\Big((G'_{\bar{0},j})^*\kappa - G^*_{\bar{0},j}\kappa\Big) \\
&= \mathcal{O}(c^\infty)\,\mathrm{d}c_1 + \mathcal{O}(c^\infty)\,\mathrm{d}c_2.
\end{aligned}$$

We have used the fact that, by Lemma 2.18,

$$(G'_{\bar{0},j})^*\kappa - G^*_{\bar{0},j}\kappa = G^*_{\bar{0},j}\Big((G^{-1}_{\bar{0},j} \circ G'_{\bar{0},j})^*\kappa - \kappa\Big) = \mathcal{O}(c^\infty)\,\mathrm{d}c_1 + \mathcal{O}(c^\infty)\,\mathrm{d}c_2.$$

By Lemma 4.1 and shrinking $U \in \mathcal{N}(B, 0)$, and then $U_j = E_j(U)$ and $U'_j = E'_j(U)$ for $j \in \mathbb{Z}_k$ are shrunk accordingly, there is diffeomorphism G_1 of $\mathbb{R}^2$ such that

$$G_1^*((E'_{\bar{0}})^{-1})^*\alpha'_1 = (E_{\bar{0}}^{-1})^*\alpha_1, \quad G_1^*((E'_{\bar{0}})^{-1})^*\alpha'_2 = (E_{\bar{0}}^{-1})^*\alpha_2$$

on $U_{\bar{0}}$. Let $G = (E'_{\bar{0}})^{-1} \circ G_1 \circ E_{\bar{0}} \colon U \to U' = (E'_{\bar{0}})^{-1}(U'_{\bar{0}})$ and then we have $(G^*\alpha'_1, G^*\alpha'_2) = (\alpha_1, \alpha_2)$. We calculate the Taylor series of the diffeomorphism $G'_j = E'_j \circ G \circ E_j^{-1} \colon U_j \to U'_j$,

$$\begin{aligned} \mathrm{Taylor}_0[G'_j] &\stackrel{\text{def}}{=} (\mathrm{Taylor}_0[\mathrm{pr}_1 \circ G'_j], \mathrm{Taylor}_0[\mathrm{pr}_2 \circ G'_j]) \\ &= \mathrm{Taylor}_0[E'_j \circ G \circ E_j^{-1}] \\ &= \mathrm{Taylor}_0[E'_j \circ (E'_{\bar{0}})^{-1} \circ G_1 \circ E_{\bar{0}} \circ E_j^{-1}] \\ &= \mathrm{Taylor}_0[G'_{\bar{0},j} \circ G_1 \circ G_{\bar{0},j}^{-1}] \\ &= \mathrm{Taylor}_0[G_1] = (X, Y). \end{aligned}$$

Then $G'_j(c_1, c_2) = (c_1, c_2 + \mathcal{O}(c^\infty))$. Define symplectomorphisms

$$\widetilde{\varphi}_{G'_j} \colon (q^{-1}(U_j), \omega_0) \to (q^{-1}(U'_j), \omega_0)$$

as in Lemma 2.20, lifting G'_j. Therefore, $(\widetilde{\varphi}_{G'_j}^{-1} \circ \psi'_j, E_j)$ is an Eliasson local chart at m'_j for $j \in \mathbb{Z}_k$ and $((\widetilde{\varphi}_{G'_j}^{-1} \circ \psi'_j, E_j))_{j \in \mathbb{Z}_k}$ is a singularity atlas of $(M', \omega', G^{-1} \circ F')$, both compatible with (α_1, α_2).

By replacing $(M', \omega', F', (\alpha'_1, \alpha'_2), m'_{\bar{0}})$ with $(M', \omega', G^{-1} \circ F', (\alpha_1, \alpha_2), m'_{\bar{0}})$ and $((\psi'_j, E'_j))_{j \in \mathbb{Z}_k}$ with $((\widetilde{\varphi}_{G'_j}^{-1} \circ \psi'_j, E_j))_{j \in \mathbb{Z}_k}$ if necessary, we assume later, without loss of generality, that $(\alpha'_1, \alpha'_2) = (\alpha_1, \alpha_2)$ and $E'_\ell \circ (E'_j)^{-1} = E_\ell \circ E_j^{-1}$ for $j, \ell \in \mathbb{Z}_k$.

Construction of the semiglobal isomorphism: After the initialization we have marked directed systems $(M, \omega, F, (\alpha_1, \alpha_2), m_{\bar{0}})$ with a singularity atlas $((\psi_j \colon V_j \to \psi_j(V_j), E_j \colon U \to U_j))_{j \in \mathbb{Z}_k}$, and $(M', \omega', F', (\alpha_1, \alpha_2), m'_{\bar{0}})$ with a singularity atlas $((\psi'_j \colon V'_j \to \psi'_j(V'_j), E'_j \colon U' \to U'_j))_{j \in \mathbb{Z}_k}$ such that $E'_\ell \circ (E'_j)^{-1} = E_\ell \circ E_j^{-1}$ for $j, \ell \in \mathbb{Z}_k$. We aim to find a symplectomorphism $\varphi \colon (W, \omega) \to (W', \omega')$ with $F' \circ \varphi = F$, and $\varphi(m_j) = m'_j$ where $W = F^{-1}(U) \in \mathcal{N}_F(M, \mathcal{F})$ and $W' = (F')^{-1}(U') \in \mathcal{N}_{F'}(M', \mathcal{F}')$.

We construct φ by induction as follows. Define a symplectomorphism $\varphi_{\bar{0}} = (\psi'_{\bar{0}})^{-1} \circ \psi_{\bar{0}} \colon (V_{\bar{0}}, \omega) \to (V'_{\bar{0}}, \omega')$. Recall the definition of W_j in (3.1); $M'_j \subset M'$ is defined analogously. Analogous to the proof of Lemma 3.5, we can extend $\varphi_{\bar{0}}$ to a symplectomorphism $\widetilde{\varphi}_{\bar{0}} \colon (W_{\bar{0}}, \omega) \to (W'_{\bar{0}}, \omega')$.

For $j \in \mathbb{Z}_k \backslash \{\overline{-1}\}$, suppose we have defined the symplectomorphism $\widetilde{\varphi}_j \colon (W_j, \omega) \to (W'_j, \omega')$, and we want to define $\widetilde{\varphi}_{j+\bar{1}} \colon (W_{j+\bar{1}}, \omega) \to (W'_{j+\bar{1}}, \omega')$. Let $\mu_{j+\bar{1}}$ be a symplectomorphism determined by the following commutative diagram:

$$\begin{array}{ccc} (V_j, \omega) & \xrightarrow{\varphi_j} & (V'_j, \omega') \\ \Big\downarrow{\scriptstyle \psi_{j+\bar{1}}^{-1} \circ \psi_j} & & \Big\downarrow{\scriptstyle (\psi'_{j+\bar{1}})^{-1} \circ \psi'_j} \\ (V_{j+\bar{1}}, \omega) & \xrightarrow[\mu_{j+\bar{1}}]{} & (V'_{j+\bar{1}}, \omega') \end{array} .$$

By Lemma 2.10 and analogous to the proof of Lemma 3.5, we extend $\mu_{j+\bar{1}}$ to a symplectomorphism $\widetilde{\mu}_{j+\bar{1}} \colon (W_{j+\bar{1}}, \omega) \to (W'_{j+\bar{1}}, \omega')$. Recall $W_{j,j+\bar{1}} = W_j \cap W_{j+\bar{1}}$ and let $M'_{j,j+\bar{1}} = M'_j \cap M'_{j+\bar{1}}$.

Define $\mu_{j,j+\bar{1}}$ such that the diagram

$$\begin{array}{ccc} (W_{j,j+\bar{1}},\omega) & \xrightarrow{\widetilde{\varphi}_j} & (W'_{j,j+\bar{1}},\omega') \\ & \searrow^{\widetilde{\mu}_{j+\bar{1}}} & \downarrow{\scriptstyle \mu_{j,j+\bar{1}}} \\ & & (W'_{j,j+\bar{1}},\omega') \end{array}$$

commutes.

Note that $\widetilde{\mu}_{j+\bar{1}}(m_{j+\bar{1}}) = m'_{j+\bar{1}}$ and $\widetilde{\varphi}_j(x) \to m'_{j+\bar{1}}$ as $x \to m_{j+\bar{1}}$ in M, so we have $\mu_{j,j+\bar{1}}(x) \to m'_{j+\bar{1}}$. As a fiberwise translation by $\tau'_{j,j+\bar{1}} \in \Omega^1(U')$, in the sense that $\mu_{j,j+\bar{1}} = \Psi_{\tau'_{j,j+\bar{1}}}|_{W'_{j,j+\bar{1}}}$, we could extend $\mu_{j,j+\bar{1}}$ to a symplectomorphism $\widetilde{\mu}_{j,j+\bar{1}} = \Psi_{\tau'_{j,j+\bar{1}}}$ of (W',ω'). Now let

$$\widetilde{\varphi}_{j+\bar{1}} = \widetilde{\mu}^{-1}_{j,j+\bar{1}} \circ \widetilde{\mu}_{j+\bar{1}} \colon (W_{j+\bar{1}},\omega) \to (W'_{j+\bar{1}},\omega'),$$

and then $\widetilde{\varphi}_{j+\bar{1}} = \widetilde{\varphi}_j$ in their common domain $W_{j,j+\bar{1}}$.

For $\varphi_{\overline{-1}}$ and $\varphi_{\bar{0}}$, they coincide on regular values of F near $\mathcal{F}$, so by continuity, they must coincide on their common domain $W_{\overline{-1},\bar{0}}$. Hence, we can glue $\widetilde{\varphi}_j$, $j \in \mathbb{Z}_k$ to get a symplectomorphism $\varphi\colon (W,\omega) \to (W',\omega')$ with the commuting diagram:

$$\begin{array}{ccc} (W,\omega) & \xrightarrow{\quad\varphi\quad} & (W',\omega') \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle F'} & \\ & U & \end{array} .$$

∎

5. Proof of existence

The goal of this section is to show Lemma 5.3, the surjectivity claim of Theorem 3.9. Let

$$(\dots,\mathsf{s}_j,\dots,\mathsf{g}_{j,\ell},\dots) \in \mathcal{I}^k_{\mathrm{ff}\circledast}.$$

We aim to show that there is $(M,\omega,F,(\alpha_1,\alpha_2),m_{\bar{0}})$ such that

$$\Phi\Big([(M,\omega,F,(\alpha_1,\alpha_2),m_{\bar{0}})]\Big) = (\dots,\mathsf{s}_j,\dots,\mathsf{g}_{j,\ell},\dots). \tag{5.1}$$

The local structures of the integrable system (M,ω,F) near the singular points m_j are isomorphic to the local normal form in Section 2.2. We are going to show that the isomorphism can be extended to a complete neighborhood W_j of m_j. Then, we use the symplectic gluing technique similar to [16, Section 3] to construct (M,ω,F).

By Borel's lemma, there are $U \in \mathcal{N}(\mathbb{R}^2,0)$ and smooth maps $s_{\bar{0}}\colon U \to \mathbb{R}$ and $G_{\bar{0},j}\colon U \to \mathbb{R}^2$ such that $\mathrm{Taylor}_0[s_{\bar{0}}] = \mathsf{s}_{\bar{0}}$ and $\mathrm{Taylor}_0[G_{\bar{0},j}] = (X,\mathsf{g}_{\bar{0},j})$. We choose $G_{\bar{0},\bar{0}} = \mathrm{id}$. Let $\mathsf{g}^Y_{\bar{0},j} > 0$ be the Y-coefficient of $\mathsf{g}_{\bar{0},j}$, and $\mathsf{s}^Y_{\bar{0}} \in \mathbb{R}$ the Y-coefficient of $\mathsf{s}_{\bar{0}}$. There are $\delta \in (0,\min_{j\in\mathbb{Z}_k}\{1,\mathsf{s}^Y_{\bar{0}},\mathsf{g}^Y_{\bar{0},j}\})$ and $U_{\bar{0}} \in \mathcal{N}(\mathbb{R}^2,0)$ contained in U such that for any $j \in \mathbb{Z}_k$ and for any $c \in U_j = G_{\bar{0},j}(U_{\bar{0}})$ we have

$$|c| < \delta < 1, \qquad \frac{\partial(\mathrm{pr}_2 \circ G_{\bar{0},j})}{\partial c_2}(c) > \mathsf{g}^Y_{\bar{0},j} - \delta > 0, \qquad \frac{\partial s_{\bar{0}}}{\partial c_2}(c) > \mathsf{s}^Y_{\bar{0}} - \delta > 0. \tag{5.2}$$

For $j \in \mathbb{Z}_k$, let $W_j = q^{-1}(U_j)$, $W_{j,\mathrm{r}} = W_j \cap \mathbb{R}^4_\mathrm{r}$, $U_{j,\mathrm{r}} = U_j \cap \mathbb{R}^2_\mathrm{r}$. Let $kW = \coprod_{j\in\mathbb{Z}_k} W_j$ and $kW_\mathrm{r} = \coprod_{j\in\mathbb{Z}_k} W_{j,\mathrm{r}}$. Let $kD = \coprod_{j\in\mathbb{Z}_k} D_j \subset kW$ where

$$D_{\overline{0}} = \left\{(z,\zeta) \in W_{\overline{0}} \;\middle|\; |z| \leqslant 1, |\zeta| \leqslant \exp \tfrac{\partial s_{\overline{0}}}{\partial c_2}(q(z,\zeta))\right\},$$
$$D_j = \{(z,\zeta) \in W_j \mid |z| \leqslant 1, |\zeta| \leqslant 1\}, \qquad \text{for } j \in \mathbb{Z}_k \setminus \{\overline{0}\}.$$

Note that these spaces depend on δ.

Define, for $j,\ell \in \mathbb{Z}_k$, diffeomorphisms $G_{j,\ell} = G_{\overline{0},\ell} \circ G^{-1}_{\overline{0},j} \colon U_j \to U_\ell$ and by Lemmas 2.7 and 2.9 symplectomorphisms $\varphi_{j,j+\overline{1}} \colon W_j \setminus D^0_\mathrm{u} \to W_{j+\overline{1}} \setminus D^0_\mathrm{s}$ as

$$\varphi_{j,j+\overline{1}} = \begin{cases} \varphi_{G_{j,j+\overline{1}}} \circ \Psi_{-\kappa}, & j \neq \overline{-1}; \\ \Psi_{-\,\mathrm{d}s_{\overline{0}}} \circ \varphi_{G_{\overline{-1},\overline{0}}} \circ \Psi_{-\kappa}, & j = \overline{-1}. \end{cases} \tag{5.3}$$

Let $\mathcal{G}$ be the groupoid generated by the restrictions of $\varphi_{j,j+\overline{1}}$ for $j \in \mathbb{Z}_k$ to open subsets. Recall Γ_k the cycle graph with k vertices. Consider its fundamental groupoid $\Pi(\Gamma_k)$ whose elements are of the form $[j,\ell]_p$, where $j,\ell \in \mathbb{Z}_k, p \in \mathbb{Z}$ and $j + \overline{p} = \ell$, The multiplication is given by concatenation $[\ell,j']_{p'} \cdot [j,\ell]_p = [j,j']_{p+p'}$. Any element $[j,\ell]_p$ of $\Pi(\Gamma_k)$ corresponds to an element of $\mathcal{G}$:

$$\begin{cases} \varphi_{[j,j]_{\overline{0}}} = \mathrm{id} \colon W_j \to W_j, & p = 0; \\ \varphi_{[j,j+\overline{1}]_1} = \varphi_{j,j+\overline{1}} \colon W_j \setminus D^0_\mathrm{u} \to W_{j+\overline{1}} \setminus D^0_\mathrm{s}, & p = 1; \\ \varphi_{[j,j-\overline{1}]_{-1}} = \varphi^{-1}_{j-\overline{1},j} \colon W_j \setminus D^0_\mathrm{s} \to W_{j-\overline{1}} \setminus D^0_\mathrm{u}, & p = -1; \\ \varphi_{[j,j+\overline{p}]_p} = \varphi_{j+\overline{p-1},j+\overline{p}} \circ \cdots \circ \varphi_{j+\overline{1},j+\overline{2}} \circ \varphi_{j,j+\overline{1}} \colon W_{j,\mathrm{r}} \to W_{j+\overline{p},\mathrm{r}}, & p \geqslant 2; \\ \varphi_{[j,j+\overline{p}]_p} = \varphi^{-1}_{j+\overline{p},j+\overline{p+1}} \circ \cdots \circ \varphi^{-1}_{j-\overline{2},j-\overline{1}} \circ \varphi^{-1}_{j-\overline{1},j} \colon W_{j,\mathrm{r}} \to W_{j+\overline{p},\mathrm{r}}, & p \leqslant -2. \end{cases}$$

Actually, $\mathcal{G}$ consists of restrictions of $\varphi_{[j,\ell]_p}$ for all $[j,\ell]_p \in \Pi(\Gamma_k)$ to open subsets, and $\mathcal{G}$ is a groupoid of symplectomorphisms. Note that for any $[j,\ell]_p \in \Pi(\Gamma_k)$, we have

$$q \circ \varphi_{[j,\ell]_p} = G_{j,\ell} \circ q.$$

Define a smooth function

$$f_L \colon kW_\mathrm{r} \to \mathbb{R},$$
$$W_{j,\mathrm{r}} \ni (z,\zeta) \mapsto \frac{\partial g_{\overline{0},j}}{\partial c_2}(G^{-1}_{\overline{0},j}(q(z,\zeta))) \ln|z|$$

for $j \in \mathbb{Z}_k$.

Lemma 5.1. *For any $j \in \mathbb{Z}_k$ and $(z,\zeta) \in W_j \subset kW$ there is a $p \in \mathbb{Z}$ such that $\varphi_{[j,j+\overline{p}]_p}(z,\zeta) \in D_{j+\overline{p}} \subset kD$. For any $j \in \mathbb{Z}_k$, we have $f_L \circ \varphi_{[j,j+\overline{p}]_p} - f_L \to \infty$ uniformly as $p \to \infty$ and $f_L \circ \varphi_{[j,j+\overline{p}]_p} - f_L \to -\infty$ uniformly as $p \to -\infty$, both on $W_{j,\mathrm{r}}$.*

Proof. Define functions $L_j \colon U_{\overline{0},\mathrm{r}} \to \mathbb{R}$, $j \in \mathbb{Z}_k$, as

$$L_{\overline{0}}(c) = -\ln|c| + \frac{\partial s_{\overline{0}}}{\partial c_2}(c) \geqslant (1-\delta)|\ln\delta| + (\mathsf{s}^Y_{\overline{0}} - \delta),$$
$$L_j(c) = -\frac{\partial g_{\overline{0},j}}{\partial c_2}(c) \ln\left|G_{\overline{0},j}(c)\right| \geqslant (\mathsf{g}^Y_{\overline{0},j} - \delta)|\ln\delta|, \qquad \text{for } j \in \mathbb{Z}_k \setminus \{\overline{0}\};$$

the inequalities hold by (5.2) for $\delta > 0$ and $U_{\overline{0}} \in \mathcal{N}(\mathbb{R}^2, 0)$ small enough.

Recall the definition of the function r in (2.6) and we have $f_L = (\frac{\partial g_{\overline{0},j}}{\partial c_2} \circ G_{\overline{0},j}^{-1} \circ q) \cdot r$ on $W_{j,\mathrm{r}}$. By (2.7), (2.8), (2.11) and (5.3) we have, for any $p \in \mathbb{Z}$,

$$r \circ \varphi_{[0,\overline{p}]_p} = \frac{r \circ \varphi_{[0,\overline{p-1}]_{p-1}} - \ln\left|G_{\overline{0},\overline{p-1}} \circ q\right|}{\frac{\partial g_{\overline{p-1},\overline{p}}}{\partial c_2} \circ q}, \qquad \text{for } \overline{p} \neq 0;$$

$$r \circ \varphi_{[0,\overline{p}]_p} = \frac{r - \ln\left|G_{\overline{0},\overline{-1}} \circ q\right|}{\frac{\partial g_{\overline{-1},\overline{0}}}{\partial c_2} \circ q} - \frac{\partial s_{\overline{0}}}{\partial c_2} \circ q, \qquad \text{for } \overline{p} = 0$$

on $W_{\overline{0},\mathrm{r}}$. Therefore, f_L and L_j are related by

$$\begin{aligned} & f_L \circ \varphi_{[0,\overline{p}]_p} - f_L \circ \varphi_{[0,\overline{p-1}]_{p-1}} \\ =& \left(\frac{\partial g_{\overline{0},\overline{p}}}{\partial c_2} \circ q\right) \cdot \left(r \circ \varphi_{[0,\overline{p}]_p}\right) - \left(\frac{\partial g_{\overline{0},\overline{p-1}}}{\partial c_2} \circ q\right) \cdot \left(r \circ \varphi_{[0,\overline{p-1}]_{p-1}}\right) \\ =& L_{\overline{p}} \circ q. \end{aligned}$$

on $W_{\overline{0},\mathrm{r}}$, and then we have

$$f_L \circ \varphi_{[\overline{0},\overline{p}]_p} = \begin{cases} f_L + \sum_{s=1}^{p} L_{\overline{s}} \circ q, & p \geqslant 0, \\ f_L - \sum_{s=p+1}^{0} L_{\overline{s}} \circ q, & p < 0. \end{cases}$$

Together with the fact that L_j, $j \in \mathbb{Z}_k$, are positive and bounded away from zero, we conclude that $f_L \circ \varphi_{[j,j+\overline{p}]_p} - f_L$ diverges to ∞ uniformly as $p \to \infty$ and diverges to $-\infty$ uniformly as $p \to -\infty$ for $j = \overline{0}$ and then for any $j \in \mathbb{Z}_k$.

For any fixed $(z,\zeta) \in W_{\overline{0}}$, we aim to find a $p \in \mathbb{Z}$ such that $\varphi_{[\overline{0},\overline{p}]_p}(z,\zeta) \in D_{\overline{p}}$. Let $c = q(z,\zeta)$. Suppose $(z,\zeta) \in W_{\overline{0},\mathrm{r}}$. If $f_L(z,\zeta) \leqslant 0$, there is a $p \in \mathbb{Z}$, $p \geqslant 1$ such that

$$-\sum_{s=1}^{p} L_{\overline{s}}(c) \leqslant f_L(z,\zeta) \leqslant -\sum_{s=1}^{p-1} L_{\overline{s}}(c)$$

so $-L_{\overline{p}}(c) \leqslant f_L \circ \varphi_{[\overline{0},\overline{p}]_p}(z,\zeta) \leqslant 0$ and $\varphi_{[\overline{0},\overline{p}]_p}(z,\zeta) \in D_{\overline{p}}$; if otherwise $f_L(z,\zeta) > 0$, there is a $p \in \mathbb{Z}$, $p \leqslant 0$ such that $\varphi_{[\overline{0},\overline{p}]_p}(z,\zeta) \in D_{\overline{p}}$ by a similar argument. Suppose otherwise $(z,\zeta) \in W_{\overline{0}} \setminus W_{\overline{0},\mathrm{r}}$. If $\zeta = 0$ and $|z| \leqslant 1$, or $z = 0$ and $|\zeta| \leqslant 1$, we already have $(z,\zeta) \in D_{\overline{0}}$. If $\zeta = 0$ and $|z| > 1$, then $\varphi_{[\overline{0},\overline{-1}]_{-1}}(z,\zeta) = (0,\zeta') \in D_{\overline{-1}}$ since $0 < |\zeta'| < 1$. If $z = 0$ and $|\zeta| > 1$, then $\varphi_{[\overline{0},\overline{1}]_1}(z,\zeta) = (z',0) \in D_{\overline{1}}$ since $0 < |z'| < 1$. Analogously, for any $(z,\zeta) \in W_j$, $j \in \mathbb{Z}_k$, there is a $p \in \mathbb{Z}$ such that $\varphi_{[j,j+\overline{p}]_p}(z,\zeta)$ is in $D_{j+\overline{p}}$. □

We define an equivalence equation $\sim_{\mathcal{G}}$ on kW as $x \sim_{\mathcal{G}} y$ if and only if there is a $\varphi \in \mathcal{G}$ such that $y = \varphi(x)$. Let $M = kW/\sim_{\mathcal{G}}$ be the quotient space, $\lambda \colon kW \to M$, $\lambda_j \colon W_j \to M$, $j \in \mathbb{Z}_k$ be the quotient maps. Let $\Delta_{\mathcal{G}} = \{(x,y) \in kW \times kW \mid x \sim_{\mathcal{G}} y\}$.

Lemma 5.2. *The topological space M can be uniquely realized as a symplectic manifold with a symplectic structure ω and a smooth function $F \colon M \to \mathbb{R}^2$ such that for every $j \in \mathbb{Z}_k$ the map $\lambda_j \colon (W_j, \omega_0) \to (\lambda_j(W_j), \omega)$ is a symplectomorphism and $G_{\overline{0},j} \circ F \circ \lambda_j = q|_{W_j}$.*

Proof. We want to prove that M is a topological manifold with the quotient topology.

The map λ_j is open: for any open set $V \subseteq W_j$, the preimage

$$\lambda_j^{-1}(\lambda_j(V)) = V \cup \varphi_{[j,j+\overline{1}]_1}(V \cap W_j \setminus D^0_{\mathrm{u}}) \cup \varphi_{[j,j-\overline{1}]_{-1}}(V \cap W_j \setminus D^0_{\mathrm{s}}) \cup \bigcup_{p \in \mathbb{Z}, |p| \geqslant 2} \varphi_{[j,j+\overline{p}]_p}(V \cap W_{j,\mathrm{r}})$$

is open, so λ_j is an open map.

The map λ_j is locally injective: we need to prove that, for any $x \in W_j$ there is $V \in \mathcal{N}(W_j, x)$ such that for any $p \in \mathbb{Z} \setminus \{0\}$, as long as x is in the domain, the map $\varphi_{[j,j]_{pk}}$ sends x outside of V. If $k \geqslant 2$, then $x \in W_{j,\mathrm{r}}$. This is a consequence of Lemma 5.1. If $k = 1$ and $x \in W_{\overline{0},\mathrm{s}} \setminus \{0\}$, we have $\varphi_{[\overline{0},\overline{0}]_1}(x) \in W_{\overline{0},\mathrm{u}}$ away from x. The case $k = 1$ and $x \in W_{\overline{0},\mathrm{u}} \setminus \{0\}$ is analogous.

The subset $\Delta_{\mathcal{G}}$ is closed in $kW \times kW$: suppose there is a sequence of points $((x_i, y_i))_{i=1}^{\infty} \subset \Delta_{\mathcal{G}}$ converging to $(x_\infty, y_\infty) \in kW \times kW$. Assume, without loss of generality, that $(x_\infty, y_\infty) \in W_{\overline{0}} \times W_j$ for some fixed $j \in \mathbb{Z}_k$. Since $W_{\overline{0}}, W_j$ are open in kW, we can assume $((x_i, y_i))_{i=1}^{\infty} \subset W_{\overline{0}} \times W_j$. There is $[0, \overline{p_i}]_{p_i} \in \Pi(\Gamma_k)$ such that $y_i = \varphi_{[0,\overline{p_i}]_{p_i}}(x_i)$. If there is a subsequence $\{p_{i_m}\}$ of p_i with $p_{i_m} = p_0 \in \mathbb{Z}$, then $y_{i_m} = \varphi_{[0,\overline{p_0}]_{p_0}}(x_{i_m})$. In this case, $y_\infty = \varphi_{[0,\overline{p_0}]_{p_0}}(x_\infty)$, so $(x_\infty, y_\infty) \in \Delta_{\mathcal{G}}$. Otherwise, by descending to a subsequence we can assume $|p_i| \to \infty$, so for i large, $x_i \in W_{\overline{0},\mathrm{r}}$ and $y_i \in W_{j,\mathrm{r}}$. By Lemma 5.1, we have $|f_L(x_i) - f_L(y_i)| \to \infty$, which contradicts $(x_i, y_i) \to (x_\infty, y_\infty)$.

Since λ_j, $j \in \mathbb{Z}_k$ are open and locally injective, we conclude that they are local homeomorphisms and M is locally Euclidean. Since λ_j, $j \in \mathbb{Z}_k$ are open and $\Delta_{\mathcal{G}} \subset kW \times kW$ is closed, M is Hausdorff. Since W_j, $j \in \mathbb{Z}_k$ are second countable, $M = \bigcup_{j \in \mathbb{Z}_k} \lambda_j(W_j)$ is second countable. We conclude that M is a topological manifold.

Noting that the maps $\varphi_{j,\ell}$, $j, \ell \in \mathbb{Z}_k$ are symplectomorphisms satisfying $q \circ \varphi_{j,\ell} = G_{j,\ell} \circ q$, there are a unique symplectic structure ω on M and a smooth function $F \colon M \to \mathbb{R}^2$ such that $\lambda_j^* \omega = \omega_0$ and $G_{\overline{0},j} \circ F \circ \lambda_j = q|_{W_j}$. □

Lemma 5.3. *The map $\Phi \colon \mathcal{M}^k_{\mathrm{ff}\circledast} \to \mathcal{I}^k_{\mathrm{ff}\circledast}$ is surjective.*

Proof. Let $m_j = \lambda_j(0)$ and $\mu_j = \lambda_j|^{-1}_{\lambda_j(W_j)} \colon (\lambda_j(W_j), \omega) \to (W_j, \omega_0)$ be a symplectomorphism for $j \in \mathbb{Z}_k$. Finally, we need to show that, the construction (M, ω, F) in Lemma 5.2 lies inside of $\mathcal{IS}^k_{\mathrm{ff}}$, has a singularity atlas $((\mu_j, G_{\overline{0},j}))_{j \in \mathbb{Z}_k}$ for singular points m_j, $j \in \mathbb{Z}_k$, compatible with some $(\alpha_1, \alpha_2) \in \mathrm{Dir}(M, \omega, F)$ such that (5.1) holds.

The triple (M, ω, F) is in $\mathcal{IS}^k_{\mathrm{ff}}$: The triple (M, ω, F) is an integrable system since it is locally an integrable system everywhere, and the only singular points of F are m_j on $\mathcal{F}$, $j \in \mathbb{Z}_k$, which are of focus-focus type. To show that F is proper, let $K \subset U_0$ be any compact subset. By Lemma 5.1, $\lambda(kW) = \lambda(kD)$. Since $q^{-1}(G_{\overline{0},j}(K)) \cap D_j$ is compact, $F^{-1}(K) = \bigcup_{j \in \mathbb{Z}_k} \lambda_j(q^{-1}(G_{\overline{0},j}(K)) \cap D_j)$ is compact. The fibers of F are connected since q has connected fibers.

Computation of $\Lambda^{(M,\omega,F)}$: Let $U \subsetneq U_{\overline{0},\mathrm{r}}$ be a simply connected open set. Note that $\kappa|_U \in (\Omega^1/2\pi\Lambda)(\mathbb{R}^2_{\mathrm{r}})$ and let $\kappa_U \in \Omega^1(U)$ be a representative of $\kappa|_U$. Let $\alpha_2|_U = \mathrm{d}s_{\overline{0}} - \sum_{j \in \mathbb{Z}_k} G^*_{\overline{0},j} \kappa_U \in Z^1(U)$. We have, in $F^{-1}(U)$,

$$
\begin{aligned}
\varphi_{[\overline{0},\overline{0}]_k}\big|_{F^{-1}(U)} &= \Psi_{-\mathrm{d}s_{\overline{0}}} \circ \varphi_{G_{\overline{-1},\overline{0}}} \circ \Psi_{-\kappa_U} \circ \cdots \circ \Psi_{-\kappa_U} \circ \varphi_{G_{\overline{1},\overline{2}}} \circ \Psi_{-\kappa_U} \circ \varphi_{G_{\overline{0},\overline{1}}} \circ \Psi_{-\kappa_U} \\
&= \Psi_{-\mathrm{d}s_{\overline{0}} - \sum_{j \in \mathbb{Z}_k} G^*_{\overline{0},j} \kappa_U} \circ \varphi_{G_{\overline{-1},\overline{0}}} \circ \cdots \circ \varphi_{G_{\overline{1},\overline{2}}} \circ \varphi_{G_{\overline{0},\overline{1}}} \\
&= \Psi_{-2\pi\alpha_2|_U}.
\end{aligned}
$$

So $\alpha_2|_U \in \Lambda^{(M,\omega,F)}(U)$. Let $\alpha_1 = dc_1 \in \Omega^1(U_{\overline{0}})$, then $\alpha_1|_U \in \Lambda^{(M,\omega,F)}(U)$. On the other hand, for any $\tau \in Z^1(U)$ to be a period form, it has to satisfy $\Psi_{2\pi\tau} = \varphi_{[\overline{0},\overline{0}]_{pk}}$ for some $p \in \mathbb{Z}$. Therefore, $\Lambda^{(M,\omega,F)}(U)$ is the abelian group generated by $\alpha_1|_U, \alpha_2|_U$. Similarly, we have $\Lambda^{(M,\omega,F)}(U) = \alpha_1\mathbb{Z}$ if U is an open neighborhood of 0.

Computation of the invariants: For each $j \in \mathbb{Z}_k$, $(\mu_j, G_{\overline{0},j})$ is an Eliasson local chart near m_j since $q \circ \mu_j = G_{\overline{0},j} \circ F$. For $j = \overline{0}$, note that $\alpha_1 = \mathrm{d}c_1$ and $\frac{\partial}{\partial c_2} \lrcorner \alpha_2 = L$, so $(\mu_{\overline{0}}, \mathrm{id})$ is compatible with (α_1, α_2). For $j \in \mathbb{Z}_k$, since $\mathrm{d}G_{\overline{0},j}$ has positive diagonal entries near the origin, $(\mu_j, G_{\overline{0},j})$ is compatible with (α_1, α_2). By the construction of M, any flow line of $\mathcal{X}_{\alpha_2}$ with α-limit m_j for some $j \in \mathbb{Z}_k$ has ω-limit $m_{j+\overline{1}}$, so $((\mu_j, G_{\overline{0},j}))_{j\in\mathbb{Z}_k}$ is a singularity atlas compatible with (α_1, α_2).

Now since $\mathrm{d}s_{\overline{0}} = 2\pi\alpha_2 - \sum_{j\in\mathbb{Z}_k} G^*_{\overline{0},j}\kappa$ and $s_{\overline{0}}(0) = 0$, the action Taylor series at $m_{\overline{0}}$ is $\mathrm{Taylor}_0[s_{\overline{0}}] = \mathsf{s}_{\overline{0}}$. The transition cocycle is $(\mathrm{Taylor}_0[\mathrm{pr}_2 \circ G_{j,\ell}])_{j,\ell\in\mathbb{Z}_k} = (\mathsf{g}_{j,\ell})_{j,\ell\in\mathbb{Z}_k}$. □

6. Proof of the main theorem

Theorem 3.9 follows from Lemmas 4.2, 5.3 and 3.11 put together.

Proof of Theorem 3.9. The map

$$\Phi\colon \mathcal{I}^k_{\mathrm{ff}\circledast} \to \mathsf{R}_{2\pi X} \times \mathsf{R}^{k-1}_{+}$$
$$\left(\mathsf{s}_{\overline{0}}, \ldots, \mathsf{s}_{\overline{-1}}, \mathsf{g}_{\overline{0},\overline{0}}, \ldots, \mathsf{g}_{\overline{0},\overline{-1}}, \ldots, \mathsf{g}_{\overline{-1},\overline{0}}, \ldots, \mathsf{g}_{\overline{-1},\overline{-1}}\right) \mapsto \left(\mathsf{s}_{\overline{0}}, \mathsf{g}_{\overline{0},\overline{1}}, \mathsf{g}_{\overline{1},\overline{2}}, \ldots, \mathsf{g}_{\overline{-1},\overline{0}}\right)$$

is a bijection. □

Appendix A. Technical results on flat functions

Recall that $\mathrm{Taylor}_0[f] \in \mathbb{R}[[T^*_0 U]]$ denotes the Taylor series of $f\colon U \to \mathbb{R}$ at the origin for $U \in \mathcal{N}(\mathbb{R}^2, 0)$. Note how the Taylor series, as a formal power series, depends on the choice of a basis of $T^*_0 U$. Note that the Taylor series only depends on the germ of the function.

Definition A.1. We call f a *flat function* at 0, or the function f is flat at 0, if $\mathrm{Taylor}_0\, b[f] = 0$. Denote by $\mathcal{O}(c^\infty)$ the space of flat functions at 0.

Note that, by the Faà di Bruno's formula, the Taylor series of the composition of smooth maps is the composition of their Taylor series. This is why the definition of flat functions is independent of the choice of the basis of $T^*_0 U$.

We will use the multi-index notations in Lemmas A.2 and A.3. A multi-index j is a pair (j_1, j_2) where $j_1, j_2 \in \mathbb{Z}_{\geqslant 0}$. We use $|j| = j_1 + j_2$. If $c = (c_1, c_2) \in \mathbb{R}^2$ then $c^j = c_1^{j_1} c_2^{j_2}$. If f is a function on an open subset of $\mathbb{R}^2$ then $\partial^j f = \frac{\partial^{|j|} f}{\partial c_1^{j_1} \partial c_2^{j_2}}$.

Lemma A.2. *Let $m \in \mathbb{Z}_{\geqslant 0}$. Let $g_j\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth function for multi-index j with $|j| = m$ and let $g(c) = \sum_{|j|=m} g_j(c) c^j$. Then the function $g \ln|\cdot|\colon \mathbb{R}^2_{\mathrm{r}} \to \mathbb{R}$ can be extended to a C^{m-1}-function on $\mathbb{R}^2$ if $m \geqslant 1$. Furthermore, the extension is C^m if and only if $g_j(0) = 0$ for all j.*

Proof. Let L_m, $m \in \mathbb{Z}_{\geqslant 0}$, be the $\mathrm{C}^\infty(\mathbb{R}^2)$-vector space spanned by functions of the form $(\mathbb{R}^2_{\mathrm{r}} \ni c \mapsto c^j \ln|c|)$ for which j is a multi-index with $|j| \geqslant m$, extended onto $\mathbb{R}^2$ by zero. Let Q_m, $m \in \mathbb{Z}$, be the $\mathrm{C}^\infty(\mathbb{R}^2)$-vector space spanned by functions of the form $(\mathbb{R}^2_{\mathrm{r}} \ni c \mapsto \frac{c^j}{|c|^{m_0}})$ for which j is a multi-index, $m_0 \in \mathbb{Z}_{\geqslant 0}$, and $|j| \geqslant m_0 + m$, extended onto $\mathbb{R}^2$ by zero. By direct calculations,

$$\frac{\partial}{\partial c_1}\big(c^j \ln|c|\big) = j_1 c_1^{j_1-1}c_2^{j_2}\ln|c| + \frac{c_1^{j_1+1}c_2^{j_2}}{|c|^2},$$
$$\frac{\partial}{\partial c_1}\left(\frac{c^j}{|c|^{m_0}}\right) = j_1\frac{c_1^{j_1-1}c_2^{j_2}}{|c|^{m_0}} - m_0\frac{c_1^{j_1+1}c_2^{j_2}}{|c|^{m_0+2}},$$

which implies that if $f \in L_m$ then $\frac{\partial}{\partial c_1}f, \frac{\partial}{\partial c_2}f \in L_{m-1} + Q_{m-1}$ for $m \geqslant 1$, and if $f \in Q_m$ then $\frac{\partial}{\partial c_1}f, \frac{\partial}{\partial c_2}f \in Q_{m-1}$ for $m \in \mathbb{Z}$. Therefore, $Q_0 \subseteq \mathrm{B}_0$ the set of functions on $\mathbb{R}^2$ bounded in a neighborhood of the origin, $L_1, Q_1 \subseteq \mathrm{C}^0$ and then $L_m, Q_m \subseteq \mathrm{C}^{m-1}$ for $m \geqslant 1$.

We aim to find $L_0 \cap \mathrm{B}_0$ and let $f = h\ln|\cdot| \in L_0$ for some $h \in \mathrm{C}^\infty$. On one hand, if $f \in L_0 \cap \mathrm{B}_0$, then the boundedness requires that $h(0) = 0$. On the other hand, if $h(0) = 0$, then $h(c) = h_1c_1 + h_2c_2$ for some $h_1, h_2 \in \mathrm{C}^\infty$ and then $f \in L_1 \subseteq \mathrm{C}^0$. Therefore, $L_0 \cap \mathrm{B}_0 = L_1$.

In particular, given $g(c) = \sum_{|j|=m} g_j(c)c^j$, we have $g\ln|\cdot| \in L_m \subseteq \mathrm{C}^{m-1}$ if $m \geqslant 1$, and for any multi-index j with $|j| = m$, we have

$$\partial^j\big(g(c)\ln|c|\big) \in j!\, g_j(c)\ln|c| + L_1 + Q_0.$$

On one hand, $g\ln|\cdot| \in \mathrm{C}^m$ requires that $\partial^j(g(c)\ln|c|) \in \mathrm{B}_0$, and hence $g_j(0) = 0$. On the other hand, $g_j(0) = 0$ for every multi-index j with $|j| = m$ implies that $g\ln|\cdot| \in L_{m+1} \subseteq \mathrm{C}^m$. ☐

Lemma A.3. *For a smooth function f on $\mathbb{R}^2$, the function $f\ln|\cdot|$ on $\mathbb{R}^2_{\mathrm{r}}$ can be smoothly extended onto $\mathbb{R}^2$ only when f is flat. If f is flat, the extension of $f\ln|\cdot|$ is also flat.*

Proof. Using Taylor expansion of f, for any $m \in \mathbb{N}$, there exist smooth functions $g_j \colon \mathbb{R}^2 \to \mathbb{R}$ for all multi-indices j with $|j| = m + 1$ such that

$$f(c) = \sum_{|j|=0}^{m} \frac{1}{j!}\partial^j f(0)c^j + \sum_{|j|=m+1} \frac{1}{j!}g_j(c)c^j. \tag{A.1}$$

By Lemma A.2 and (A.1), $f\ln|\cdot| \in \mathrm{C}^m$ if and only if $\partial^j f(0) = 0$ for any multi-index j with $|j| \leqslant m$. Therefore, $f\ln|\cdot| \in \mathrm{C}^\infty$ if and only if $f \in \mathcal{O}(c^\infty)$.

Note that $\ln|c| \in \mathcal{O}(|c|^{-1})$. If $f \in \mathcal{O}(c^\infty)$, then for any $m \in \mathbb{N}$, there exist smooth functions $g_j \colon \mathbb{R}^2 \to \mathbb{R}$ for all multi-index j with $|j| = m + 1$ such that

$$f(c)\ln|c| = \sum_{|j|=m+1} \frac{1}{j!}g_j(c)c^j\ln|c| \in \mathcal{O}(|c|^m).$$

Hence $f\ln|\cdot| \in \mathcal{O}(c^\infty)$. ☐

Lemma A.4. *If g is a flat function on $\mathbb{R}^2$ and h is a smooth function on $\mathbb{R}^2_{\mathrm{r}}$ that satisfies*

$$\forall \textit{multi-index } j\ \exists m_j \in \mathbb{Z} \textit{ such that } \lim_{c\to 0}|c|^{m_j}\big|\partial^j h(c)\big| = 0, \tag{A.2}$$

then $f = gh$ on $\mathbb{R}^2_{\mathrm{r}}$ has a smooth extension $\tilde{f}$ on $\mathbb{R}^2$.

Proof. We calculate the partial derivatives of f for a multi-index j and $m \in \mathbb{Z}$:

$$|c|^m\big|\partial^j f(c)\big| \leqslant \sum_{0\leqslant \ell\leqslant j} \binom{j}{\ell}|c|^{m-m_\ell}\big|\partial^{j-\ell}g(c)\big| \cdot |c|^{m_\ell}\big|\partial^\ell h(c)\big| \to 0 \tag{A.3}$$

as $c \to 0$. Here we use the fact that $\partial^{j-\ell} g$ is a flat function so it is dominated by any power of $|c|$. Now let $\tilde{f}$ be the extension of f by $\tilde{f}(0) = 0$. Then by (A.3), $\frac{\partial \tilde{f}}{\partial c_1}(0) = \lim_{\delta \to 0} \frac{f(\delta,0)}{\delta} = 0$ and $\frac{\partial \tilde{f}}{\partial c_2}(0) = \lim_{\delta \to 0} \frac{f(0,\delta)}{\delta} = 0$, and then $\tilde{f} \in \mathrm{C}^1$. Inductively, we can show that $\tilde{f} \in \mathrm{C}^\infty$ and any higher order derivative of $\tilde{f}$ vanishes at the origin. □

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References

[1] Jaume Alonso, Holger R. Dullin, and Sonja Hohloch. Taylor series and twisting-index invariants of coupled spin-oscillators. *J. Geom. Phys.*, 140:131–151, 2019.

[2] Jaume Alonso, Holger R. Dullin, and Sonja Hohloch. Symplectic classification of coupled angular momenta. *Nonlinearity*, 33(1):417–468, 2020.

[3] Vladimir I. Arnold. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1989. Translated from the Russian by K. Vogtmann and A. Weinstein.

[4] Alexey Bolsinov and Anton Izosimov. Smooth invariants of focus-focus singularities and obstructions to product decomposition. *J. Symplectic Geom.*, 17(6):1613–1648, 2019.

[5] Alexey V. Bolsinov and Anatoly T. Fomenko. *Integrable Hamiltonian systems*. Chapman & Hall/CRC, Boca Raton, FL, 2004. Geometry, topology, classification, Translated from the 1999 Russian original.

[6] Marc Chaperon. Normalisation of the smooth focus-focus: a simple proof. *Acta Math. Vietnam.*, 38(1):3–9, 2013.

[7] Johannes J. Duistermaat. On global action-angle coordinates. *Comm. Pure Appl. Math.*, 33(6):687–706, 1980.

[8] Holger R. Dullin. Semi-global symplectic invariants of the spherical pendulum. *J. Differential Equations*, 254(7):2942–2963, 2013.

[9] Håkan Eliasson. *Hamiltonian systems with Poisson commuting integrals*. PhD thesis, 1984. Diss. Stockholm : Univ.

[10] S. Hohloch and J. Palmer. A family of compact semitoric systems with two focus-focus singularities. *Journal of Geometric Mechanics*, 10(3):331–357, 2018.

[11] Sonja Hohloch, Silvia Sabatini, Daniele Sepe, and Margaret Symington. Faithful semitoric systems. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 14:Paper No. 084, 66, 2018.

[12] Yohann Le Floch and Álvaro Pelayo. Symplectic geometry and spectral properties of classical and quantum coupled angular momenta. *J. Nonlinear Sci.*, 29(2):655–708, 2019.

[13] Henri Mineur. Sur les systèmes mécaniques dans lesquels figurent des paramètres fonctions du temps. etude des systemes admettant n intégrales premies uniformes en involution. extension à ces systèmes

des conditions de quantification de bohr-sommerfeld. *Journal de l'Ecole Polytechnique, Série III*, pages 173–191, 237–270, 1937.
[14] Álvaro Pelayo. Symplectic invariants of semitoric systems and the inverse problem for quantum systems. *Indag. Math. (N.S.)*, 32(1):246–274, 2021.
[15] Álvaro Pelayo and San Vũ Ngọc. Semitoric integrable systems on symplectic 4-manifolds. *Invent. Math.*, 177(3):571–597, 2009.
[16] Álvaro Pelayo and San Vũ Ngọc. Constructing integrable systems of semitoric type. *Acta Math.*, 206(1):93–125, 2011.
[17] Álvaro Pelayo and San Vũ Ngọc. Hamiltonian dynamics and spectral theory for spin-oscillators. *Comm. Math. Phys.*, 309(1):123–154, 2012.
[18] Daniele Sepe and San Vũ Ngọc. Integrable systems, symmetries, and quantization. *Lett. Math. Phys.*, 108(3):499–571, 2018.
[19] Margaret Symington. Four dimensions from two in symplectic topology. In *Topology and geometry of manifolds (Athens, GA, 2001)*, volume 71 of *Proc. Sympos. Pure Math.*, pages 153–208. Amer. Math. Soc., Providence, RI, 2003.
[20] San Vũ Ngọc. On semi-global invariants for focus-focus singularities. *Topology*, 42(2):365–380, 2003.
[21] San Vũ Ngọc and Christophe Wacheux. Smooth normal forms for integrable Hamiltonian systems near a focus-focus singularity. *Acta Math. Vietnam.*, 38(1):107–122, 2013.
[22] Nguyen Tien Zung. Symplectic topology of integrable Hamiltonian systems. I. Arnold-Liouville with singularities. *Compositio Math.*, 101(2):179–215, 1996.
[23] Nguyen Tien Zung. A note on focus-focus singularities. *Differential Geom. Appl.*, 7(2):123–130, 1997.
[24] Nguyen Tien Zung. Another note on focus-focus singularities. *Lett. Math. Phys.*, 60(1):87–99, 2002.

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