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Maximal domains for strategy-proof pairwise exchange[☆]

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ABSTRACT

We analyze centralized markets for indivisible objects without money through pairwise exchange when each agent initially owns a single object. We consider rules that for each profile of agents preferences select an assignment of the objects to the agents. We present a family of domains of preferences (*minimal reversal domains*) that are maximal rich domains for the existence of rules that satisfy *individual rationality*, *efficiency*, and *strategy-proofness*. Each minimal reversal domain is defined by a common ranking of the set of objects, and agents' preferences over admissible objects coincide with such common ranking but for a specific pair of objects.

1. Introduction

A house exchange problem is an indivisible goods allocation problem without money where a set of agents own a set of indivisible objects that should be reassigned to the agents. Each agent initially owns a single object and is entitled to keep that initial object. Monetary transfers among the agents are not allowed. Relevant examples of house exchange problems are assignment of roommates at university dorms, allocation of graduate housing with tenants, and kidney exchange.[1] Since agents have initial rights over the objects, final assignments of objects to agents are obtained through exchange cycles, where a group of agents swap their initial objects among themselves. In many real life applications of house exchange markets, as holiday-house swaps and kidney exchange, the number of agents involved in exchange cycles is restricted by logistic or legal constraints. Here, we focus on the most stringent feasibility constraint and restrict our attention to pairwise exchanges, and we consider the possibility of designing rules (centralized clearing houses) that select a final assignment of the objects to the agents taking into account agents' preferences.

We investigate the existence of domains of preferences for which there exist rules for pairwise house exchange problems that satisfy *individual rationality*,[2] *efficiency* (subject to the logistic constraints),[3] and *strategy-proofness*.[4] When preferences are strict and feasible exchanges are not restricted, for each profile of agents' preferences there is a unique core assignment. That is, there is a reallocation of the objects among the agents such that no group of agents can improve upon it by trading their initial endowed objects among themselves (Shapley and Scarf, 1974; Roth and Postlewaite, 1977; Roth, 1982). The unique core assignment can be obtained through the Gale's Top Trading Cycles algorithm and the rule which selects the core assignment for each agents' preference profile is the unique rule that satisfies *individual rationality*, *efficiency*, and *strategy-proofness*. Ma (1994). When the number of agents involved in each exchange is limited, there are profiles of agents' preferences for which core assignments either may fail to exist or may be not unique (Chung, 2000). This implies that when preferences are unrestricted, there are no rules that jointly satisfy the proposed axioms (Sönmez, 1999).[5] Nicolò and Rodríguez-Álvarez (2017) shows that those properties are compatible in the domain where given a fixed common ranking over the objects, each agent's preferences coincide with that common ranking on the set of objects that she ranks above her initial object. That result however leaves as an open question whether there are larger domains of preferences where the properties are still compatible.

In this paper, we introduce a family of strict preference domains –*minimal reversal domains*– that extend the domain restriction proposed

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[1] See Sönmez and Ünver (2011) for a survey on the literature.

[2] A rule satisfies *individual rationality* if each agent never prefers her initial object to the object assigned by the rule.

[3] A rule satisfies *efficiency* if it never selects an allocation that is Pareto dominated by another feasible allocation.

[4] A rule satisfies *strategy-proofness* if agents never have incentives to misrepresent their preferences.

[5] The negative result extends to some restricted domains and to incentive compatibility properties weaker than *strategy-proofness*. Nicolò and Rodríguez-Álvarez (2012, 2013).

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by Nicolò and Rodríguez-Álvarez (2017). Given an initial common ranking of objects and a specific pair of objects that occupy adjacent positions in the common ranking, the associated minimal reversal domain is such that agents' preferences over objects that are ranked above their initially owned object coincide with the common ranking except for the specified pair of objects. Minimal reversal domains are maximal for *individual rationality*, *efficiency*, and *strategy-proofness*; that is, for any rich domain that strictly contains a minimal reversal domain, there is no rule that jointly satisfies the three conditions (Theorem 1). For each minimal reversal domain we present a rule – the associated reversal adjusted priority rule– that is characterized by *individual rationality*, *efficiency*, and *strategy-proofness* in that minimal reversal domain (Theorem 2). Reversal adjusted priority rules extend the idea of priority selection from the set of assignments that all agents prefer to the initial assignment of objects.[6] Under additional conditions on the domains of agents' preferences that require the existence of preferences under which either all the objects or no object is preferred to each agent's initially owned object (*sovereignty*), and that the ranking over a pair of objects can be preserved when that pair of objects are the only objects preferred to the initial owned object (*pairwise individuality*), every maximal rich domain for *individual rationality*, *efficiency*, and *strategy-proofness* is a minimal reversal domain (Theorem 3).

The remainder of the paper is organized as follow. In Section 2, we introduce notation and basic definitions. In Section 3, we present the concept of maximal rich domains of preferences. In Section 4, we collect the main results for strict preferences. In Section 5, we discuss the extension to domains with indifferences. In Section 6, we conclude and briefly discuss related works on maximal domains and allocation of indivisible objects under restricted domains of preferences. Finally, in Section 7, we provide all the proofs and ancillary results.

2. Notation and basic definitions

Let $N = \{1, \dots, n\}$ be a finite society consisting of a set of agents ($n \geq 3$). Each agent i initially owns an object ω_i, and let $\Omega = \{\omega_1, \dots, \omega_n\}$ be the set of available objects.

A preference relation is a complete and transitive binary relation on Ω. Each agent $i \in N$ is equipped with a preference $\succsim_i$. We denote by $\mathcal{R}$ the set of all preferences on Ω. For each agent $i \in N$ and each $\succsim_i \in \mathcal{R}$, $\succ_i$ is the associated strict preference, and $\sim_i$ is the associated indifference relation defined in the standard way. Let $\mathcal{P} \subset \mathcal{R}$ be the set of all strict (antisymmetric) preferences on Ω. A preference profile $\succsim$ is an n-tuple of preferences $\succsim = (\succsim_1, \dots, \succsim_n)$. For each $i \in N$ and each preference profile $\succsim$, $\succsim_{-i}$ is the complementary profile of preferences of all agents in $N \setminus \{i\}$. For each $i \in N$, let $\mathcal{D}^i$ denote the domain of admissible preferences for agent i, and $\mathcal{D}^N = \times_{i \in N} \mathcal{D}^i$ be the domain of admissible preference profiles.[7] Let $\mathcal{R}^N$ denote the domain of all the preference profiles and $\mathcal{P}^N \subset \mathcal{R}^N$ the domain of all strict preference profiles.

An ***assignment*** is a bijection from the set of agents to the set of objects. For each $i \in N$ and each assignment a, a_i is the object assigned to i according to a.

In every assignment, objects are allocated by forming exchange cycles. In each exchange cycle, every agent receives an object from some other agent in the cycle and her object is assigned to another agent in the cycle. In this paper, we focus on the most binding feasibility constraints, and consider pairwise assignments such that only exchanges between two agents are admitted. That is, an assignment a is a pairwise assignment if for each $i, j \in N$, $a_i = \omega_j$ implies $a_j = \omega_i$. Let $\mathcal{A}$ be the set of all pairwise assignments.

We are interested in rules that select an assignment for each preference profile. A ***(pairwise exchange) rule*** is a mapping $\varphi : \mathcal{D}^N \to \mathcal{A}$. For each agent $i \in N$ and each preference profile $\succsim$, we denote by $\varphi_i(\succsim)$ the object assigned to i by φ at profile $\succsim$.

The assignment selected by a rule can be interpreted as an optimal recommendation that takes into account the preferences of the agents over objects and that tries to find a compromise between their (perhaps conflicting) interests.

We present a formal definition of the standard conditions for desirable rules.

Individual rationality. For each $i \in N$ and each $\succsim \in \mathcal{D}^N$, $\varphi_i(\succsim) \succsim_i \omega_i$.

Efficiency. For each $\succsim \in \mathcal{D}^N$, there is no $a \in \mathcal{A}$ such that for each $i \in N$, $a_i \succsim_i \varphi_i(\succsim)$ and for some $j \in N$, $a_j \succ_j \varphi_j(\succsim)$.

Strategy-proofness. For each $i \in N$, each $\succsim \in \mathcal{D}^N$, and each $\succsim'_i \in \mathcal{D}^i$, $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim'_i, \succsim_{-i})$.

Individual rationality takes into account an agent's right to refuse any assignment that is worse than her own object. *Efficiency* is the natural version of constrained efficiency that considers Pareto domination only by feasible pairwise assignments. *Strategy-Proofness* implies that agents never improve by reporting a false preference.

3. Maximal rich domains

The objective of this paper is to provide a systematic analysis of the features of preference domains for which *individual rationality*, *efficiency*, and *strategy-proofness* are compatible. It is possible to construct examples of highly restricted domains that impose direct constraints on preference profiles. However, we are interested in finding the more general restrictions involving individual preference domains that align with the interpretation of the model of centralized exchange of indivisible goods without money. Therefore, we examine domains that contain a rich variety of preferences.

A domain of preferences $\mathcal{D}^N = \times_{i \in N} \mathcal{D}^i \subset \mathcal{R}^N$ is ***rich*** if it satisfies:

- ***Anonymity***: for each $i, j \in N$ and $\omega, \omega' \in \Omega$ with $\{\omega, \omega'\} \cap \{\omega_i, \omega_j\} = \emptyset$, there is $\succsim_i \in \mathcal{D}^i$ such that $\omega \succ_i \omega' \succ_i \omega_i$ if and only if for some $\succsim_j \in \mathcal{D}^j$, $\omega \succ_j \omega' \succ_j \omega_j$.
- ***Individuality***: for each $i \in N$, $\succsim_i \in \mathcal{D}^i$, and $\omega \in \Omega$ with $\omega \succsim_i \omega_i$, there is $\succsim'_i \in \mathcal{D}^i$ such that
  (i) for each $\omega' \in \Omega \setminus \{\omega\}$, $\omega' \succsim_i \omega$ if and only if $\omega' \succsim'_i \omega$,
  (ii) for each $\omega' \in \Omega \setminus \{\omega\}$, $\omega \succsim_i \omega'$ if and only if $\omega \succsim'_i \omega'$,
  (iii) and for each $\omega'' \in \Omega \setminus \{\omega\}$, $\omega \succ_i \omega''$ if and only if $\omega \succ'_i \omega''$ and $\omega \succsim'_i \omega_i \succsim'_i \omega''$.

Anonymity is a new condition that relates agents' individual preference domains. If there is an admissible preference for an agent with a specific pairwise comparison between two objects that are as good as her initially owned object, then the individual domains of the other agents also admit a preference where the ranking of those two objects aligns with the aforementioned pairwise comparison and both objects are at least as good as the agent initial object. *Anonymity* captures the idea of symmetry among individual preference domains. The literature on maximal domains compatible with *strategy-proofness* in public good economies has assumed the complete symmetry among the preference domains of the agents. However, complete symmetry of preference domains is not appropriate when the interpretation of an agent assigned to her initially owned object is not compatible with the idea of finally consuming that object. Examples of such applications include roommate problems or kidney exchange. Note that *anonymity* is immediately satisfied when $n = 3$.

Individuality implies that if there is a preference for agent i in the domain such that ω is at least as good as ω_i, then there is an admissible preference such that all the objects that are better than ω remain better than ω, all that are worse remain worse, and ω_i ranks right after object

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[6] The extension of the analysis to environments allowing for larger cycles of exchange or multiple objects as initial endowment for each agent would lead to impossibility results. See Nicolò and Rodríguez-Álvarez (2017, Theorem 4, Theorem 6). We address these extensions in Section 6.

[7] In this paper we only consider Cartesian domains of preference profiles.

ω. Under the interpretation of each agent's initially owned object as an outside option, *individuality* implies that each agent has the autonomy to determine which objects would justify their participation in the centralized exchange. In our framework, *individuality* is equivalent to Assumption B in Sönmez (1999). This property is satisfied in general assignment markets problems without money where agents have single-unit demand as matching, house allocation, or coalition formation problems.[8]

Our objective of finding the largest rich domains where *individual rationality*, *efficiency*, and *strategy-proofness* are compatible motivates the following definition.

A Cartesian domain of preference profiles $\mathcal{D}^N \subseteq \mathcal{R}^N$ is ***maximal for a set of axioms*** $\mathcal{S}$, if there exists a rule that satisfies the axioms $\mathcal{S}$ in the domain $\mathcal{D}^N$, and for each rich Cartesian domain $\bar{\mathcal{D}}^N \neq \mathcal{D}^N$ such that $\mathcal{D}^N \subseteq \bar{\mathcal{D}}^N$, there is no rule that satisfies the axioms $\mathcal{S}$ in the domain $\bar{\mathcal{D}}^N$.

4. Main results: Maximal domains of strict preferences

We start our analysis by presenting two examples of rich domains of strict preferences for which the relevant axioms are compatible. To introduce the rules that satisfy the axioms in those domains we require additional notation.

For each $i \in N$, $\succsim_i \in \mathcal{R}$, and $A \subseteq \mathcal{A}$, the ***top choice set*** for agent i from the assignments in A is the set

$$\text{top}(A, \succsim_i) \equiv \{\omega \in \Omega \mid \text{there is } a \in A, a_i = \omega, \text{ and for each } b \in A,\ \omega \succsim_i b_i\}.$$

The top choice set consists of the best objects that agent i can obtain in any assignment from the set A.

For each $\succsim \in \mathcal{R}^N$, the ***set of individually rational assignments*** is

$$I(\succsim) \equiv \left\{a \in \mathcal{A} \mid \text{for each } i \in N,\ a_i \succsim_i \omega_i\right\}.$$

The set of individually rational assignments consists of all the assignments such that each agent receives an object at least as good as her initially owned object.

Priority algorithm: Let $\succsim \in \mathcal{P}^N$,

- $\mathcal{M}_0 \equiv I(\succsim)$.
- For each $t \leq n$, let $\mathcal{M}_t \subseteq \mathcal{M}_{t-1}$ be such that:

$$\mathcal{M}_t \equiv \left\{a \in \mathcal{M}_{t-1} \mid a_t \in \text{top}(\mathcal{M}_{t-1}, \succsim_t)\right\}.$$

Starting with $\mathcal{M}_0$ as the set of individually rational assignments, the set $\mathcal{M}_1$ selects the best preferred assignments for agent 1 from the set of individually rational assignments. The selection proceeds iteratively, and at each step t agent t selects her preferred assignments among those that have been selected in the previous steps. The set $\mathcal{M}_n$ is non-empty and single-valued for every $\succsim \in \mathcal{P}^N$.[9] In the definition of the priority algorithm, we choose to drop the obvious reference to the preference profile of the sets $\mathcal{M}_t$ whenever it does not induce to confusion.[10]

Example 1 (*Common Ranking Preferences*). Let $\mathcal{D}^N_C = \times_{i \in N} \mathcal{D}^i_C \subset \mathcal{P}^N$ be the set of all preference profiles such that for each $i \in N$ and $\succsim_i \in \mathcal{D}^i_C$, for each $\omega_j, \omega_{j'} \in \Omega \setminus \{\omega_i\}$ such that $\omega_j \succ_i \omega_i$ and $\omega_{j'} \succ_i \omega_i$, $\omega_j \succsim_i \omega_{j'}$ if and only if $j \leq j'$.

According to common ranking preferences, agents and objects are naturally ordered.[11] Each agent can rank any set of objects as more preferred than her initially owned object, but the order in which those objects are ranked is determined by the initial common ranking of the objects.

For each domain $\mathcal{D}^N \subseteq \mathcal{P}^N$, the rule $\varphi^P : \mathcal{D}^N \to \mathcal{A}$ such that for each $\succsim \in \mathcal{D}^N$, $\varphi^P(\succsim) \in \mathcal{M}_n$ is called the ***priority rule***.

The priority rule φ^P is the only rule that satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in the domain of common ranking preferences $\mathcal{D}^N_C$ (Nicolò and Rodríguez-Álvarez, 2012, Theorem 2).

Example 2 shows that the domain of common ranking preferences is not a maximal rich strict domain for our axioms.

Example 2 (*Free Pair Preferences*). Let $N = \{1, 2, 3\}$. Let $\mathcal{D}^N_F = \times_{i \in N} \mathcal{D}^i_F$ such that for each $i \in \{1, 2\}$, $\mathcal{D}^i_F = \mathcal{D}^i_C$, and $\mathcal{D}^3_F = \mathcal{P}$. Note that $\mathcal{D}^N_F$ is a rich domain.

In the domain $\mathcal{D}^N_F$, agents 1 and 2 have common ranking preferences according to the natural order, but agent 3 faces no restriction regarding how to rank ω_1 and ω_2. Hence, agents i and j may rank ω_3 as the best preferred object when it is the only object they prefer to keep their respective initially owned objects. Note that $\mathcal{D}^N_C \subset \mathcal{D}^N_F$.

Let φ^F be such that for each $\succsim \in \mathcal{D}^N_F$, $\varphi^F(\succsim) \in I(\succsim)$ and:

- If $\omega_2 \succ_1 \omega_1$ and $\omega_1 \succ_2 \omega_2$, then $\varphi^F_1(\succsim) = \omega_2$, $\varphi^F_2(\succsim) = \omega_1$, and $\varphi^F_3(\succsim) = \omega_3$,
- otherwise, $\varphi^F_3(\succsim) \in \text{top}(I(\succsim), \succsim_3)$, and for each $i \in \{1, 2\}$, if $\omega_i \notin \text{top}(I(\succsim), \succsim_3)$, $\varphi_i(\succsim) = \omega_i$.

The rule φ^F satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in the domain $\mathcal{D}^N_F$. We can think of the rule as a priority rule where 1 and 2 propose simultaneously for an exchange among the individually rational assignments. When they make proposal that are not compatible because both agent 1 and agent 2 propose an exchange with agent 3, then agent 3 chooses with whom she performs the exchange.

We present now a class of domains that combine the restrictions introduced by common ranking and free pair preferences for arbitrary sets of agents and objects.

Let $k, k+1 \in N$, with $k < n$. Let $\mathcal{D}^N_{\{k,k+1\}} = \times_{i \in N} \mathcal{D}^i_{\{k,k+1\}} \subset \mathcal{P}^N$ be the domain that contains all the preference profiles such that for each $i \in N$, $\succsim_i \in \mathcal{D}^i_{\{k,k+1\}}$, $\omega_j, \omega_{j'} \in \Omega \setminus \{\omega_i\}$ with $\{j, j'\} \neq \{k, k+1\}$, $\omega_j \succ_i \omega_i$, and $\omega_{j'} \succ_i \omega_i$, $\omega_j \succsim_i \omega_{j'}$ if and only if $j \leq j'$. We call $\mathcal{D}^N_{\{k,k+1\}}$ the ***minimal reversal domain for*** $\{k, k+1\}$. The domain $\mathcal{D}^N$ is a ***minimal reversal domain*** if there are $k, k+1 \in N$ such that $\mathcal{D}^N = \mathcal{D}^N_{\{k,k+1\}}$.

In a minimal reversal domain, each agent's preferences over objects that are preferred to her initially owned object are determined by the natural order (common ranking) but for a pair of objects. The way in which the agents may order that pair of objects is not restricted. There is no restriction neither on preferences over objects the agent considers worse than her initially owned object, nor on the set of objects that the agent prefers to her initial endowment object.[12]

Theorem 1. *Every minimal reversal domain is a maximal rich domain for individual rationality, efficiency, and strategy-proofness.*

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[8] In problems where agents may consume multiple objects and complementarities among objects are not admitted *individuality* does not necessary hold. This is the case for coalition formation problems where agents' preferences over coalitions they may join are additively separable (Rodríguez-Álvarez, 2009).

[9] The algorithm is also well-defined for weak preferences. In that case, $\mathcal{M}_n$ is also non-empty and essentially single-valued. That is, every agent is indifferent between every two possible elements of $\mathcal{M}_n$.

[10] We make the same notational choice in the definitions of the subsequent algorithms. We only need to make explicit reference to the specific profiles in the proof of Theorem 1 in Section 7.

[11] Each relabeling of the agents and objects may define a different common ranking domain. We have opted to consider a fixed order of the agents to keep notation simple.

[12] Therefore, all the preferences in which an agent's top choice is her initially owned object are admissible in a minimal reversal domain.

The proof follows a series of steps. First we prove that in rich preference domains that admit a preference cycle for three objects, or that admit that the ranking of two pairs of objects are reversed, there is no rule that satisfies our axioms (Lemmata 1–2). Then, we check that in any domain of preference that strictly contains a minimal reversal domain, there are preference profiles with such cycles or multiple preference reversals. Finally, for every minimal reversal domain we can construct a rule that satisfies our axioms in that domain. Note that Theorem 1 does not imply that minimal reversal domains are the unique maximal rich domains for *individual rationality*, *efficiency*, and *strategy-proofness*.

We next describe the rules that satisfy *individual rationality*, *efficiency*, and *strategy-proofness* in minimal reversal domains.

k-Reversal adjusted priority algorithm. Let $k, k+1 \in N$ and $\succsim \in \mathcal{P}^N$.

- $\mathcal{M}_0^* \equiv I(\succsim)$.
- For each $t \leq n$, to define $\mathcal{M}_t^* \subseteq \mathcal{M}_{t-1}^*$ we consider two cases:
  - If either $t \notin \{k, k+1\}$ or $t \in \{k, k+1\}$ and $\text{top}\left(\mathcal{M}_{k-1}^*, \succsim_k\right) \neq \text{top}\left(\mathcal{M}_{k-1}^*, \succsim_{k+1}\right)$, then

    $$\mathcal{M}_t^* \equiv \left\{a \in \mathcal{M}_{t-1}^* \mid a_t \in \text{top}(\mathcal{M}_{t-1}^*, \succsim_t)\right\}.$$

  - if $t \in \{k, k+1\}$ and there is $j \in N$ such that $\text{top}\left(\mathcal{M}_{k-1}^*, \succsim_k\right) = \text{top}\left(\mathcal{M}_{k-1}^*, \succsim_{k+1}\right) = \{\omega_j\}$, let us define $\bar{k} = k$ if $\omega_k \succ_j \omega_{k+1}$ and $\bar{k} = (k+1)$ if $\omega_{k+1} \succ_j \omega_k$, $\underline{k} \in \{k, k+1\} \setminus \{\bar{k}\}$, and

    $$\mathcal{M}_k^* \equiv \left\{a \in \mathcal{M}_{k-1}^* \mid a_{\bar{k}} \in \text{top}(\mathcal{M}_{k-1}^*, \succsim_{\bar{k}})\right\}, \textit{ and}$$
    $$\mathcal{M}_{k+1}^* \equiv \left\{a \in \mathcal{M}_k^* \mid a_{\underline{k}} \in \text{top}(\mathcal{M}_k^*, \succsim_{\underline{k}})\right\}.$$

For each pair of adjacent agents $\{k, k+1\}$, the rule $\varphi : \mathcal{D}_{\{k,k+1\}}^N \to \mathcal{A}$ such that for each $\succsim \in \mathcal{D}_{\{k,k+1\}}^N$, $\varphi(\succsim) \in \mathcal{M}_n^*$ is called the k- ***reversal adjusted priority rule***.

For each agent $k < n$, the associated k-reversal adjusted priority rule follows the logic of the priority rule but for the agents $k, k+1$. At the stage k of the k- reversal adjusted priority algorithm, agents k and $k+1$ propose an exchange to other agents. If they both propose an exchange to the same agent, this agent makes the choice of ω_k and ω_{k+1}.

Example 3 describes the process of assignment selection for reversal adjusted priority rules and compares those rules with the standard priority rule.

Example 3. Let $N = \{1, 2, 3, 4\}$, let $\mathcal{D}_{\{1,2\}}^N$ be the minimal reversal domain for $\{1, 2\}$, let φ be the 1-reversal adjusted priority and let φ^P be the priority rule. Consider the profile $\succsim \in \mathcal{D}_{\{1,2\}}^N$ be such that:

$$\omega_3 \succ_1 \omega_4 \succ_1 \omega_1 \succ_1 \omega_2,$$
$$\omega_1 \succ_2 \omega_3 \succ_2 \omega_4 \succ_2 \omega_2,$$
$$\omega_2 \succ_3 \omega_1 \succ_3 \omega_4 \succ_3 \omega_3,$$
$$\omega_1 \succ_4 \omega_2 \succ_4 \omega_3 \succ_4 \omega_4.$$

Note that $I(\succsim) = \{a \in \mathcal{A} | a_1 \neq \omega_2\} = \mathcal{M}_0 = \mathcal{M}_0^*$. Applying the priority algorithm, we obtain

$$\mathcal{M}_1 = \{a \in \mathcal{M}_0 \mid a_1 = \text{top}(\mathcal{M}_0^*, \succsim_1) = \omega_3\},$$
$$\mathcal{M}_2 = \{a \in \mathcal{M}_1 \mid a_2 = \text{top}(\mathcal{M}_1, \succsim_2) = \omega_4\} = \left\{a \in \mathcal{A} \mid a_1 = \omega_3,\ a_2 = \omega_4\right\}$$

Then, $\varphi^P(\succsim)$ is such that

$$\varphi_1^P(\succsim) = \omega_3,\ \varphi_2^P(\succsim) = \omega_4,\ \varphi_3^P(\succsim) = \omega_1,\ \varphi_4^P(\succsim) = \omega_2.$$

On the other hand, note that $\text{top}(\mathcal{M}_0^*, \succsim_1) = \text{top}(\mathcal{M}_0^*, \succsim_2) = \omega_3$, and since $\omega_2 \succ_3 \omega_1$, $\bar{k} = 2$, $\underline{k} = 1$, and

$$\mathcal{M}_1^* = \{a \in \mathcal{M}_0^* \mid a_2 = \text{top}(\mathcal{M}_0^*, \succsim_2) = \omega_3\},$$
$$\mathcal{M}_2^* = \{a \in \mathcal{M}_1^* \mid a_1 = \text{top}(\mathcal{M}_1^*, \succsim_1) = \omega_4\} = \left\{a \in \mathcal{A} \mid a_1 = \omega_4,\ a_2 = \omega_3\right\}$$

This suffices to check that,

$$\varphi_1(\succsim) = \omega_4,\ \varphi_2(\succsim) = \omega_3,\ \varphi_3(\succsim) = \omega_2,\ \varphi_4(\succsim) = \omega_1.$$

Consider the profile $\succsim' \in \mathcal{D}_{\{1,2\}}^N$ such that for each $i \neq 3$, $\succsim_i = \succsim_i'$ and

$$\omega_2 \succ_3' \omega_4 \succ_3' \omega_3 \succ_3' \omega_1.$$

Note that $I(\succsim') = \{a \in \mathcal{A} | a_1 \neq \omega_2, a_3 \neq \omega_1\}$, and therefore $\varphi^P(\succsim') = \varphi(\succsim') = \varphi(\succsim)$. Hence, since $\varphi_3^P(\succsim') \succ_3 \varphi_3^P(\succsim)$, φ^P violates *strategy-proofness* in $\mathcal{D}_{\{1,2\}}^N$.

We present now a characterization of k-reversal adjusted priority rules in minimal reversal domains. In any minimal reversal domain, the associated reversal adjusted priority rule is the only rule that satisfies *individual rationality*, *efficiency*, and *strategy-proofness*.

Theorem 2. *Let $k, k+1 \in N$. A rule $\varphi : \mathcal{D}_{\{k,k+1\}}^N \to \mathcal{A}$ satisfies individual rationality, efficiency, and strategy-proofness if and only if φ is the k-reversal adjusted priority rule.*

Individual rationality, *efficiency*, and *strategy-proofness* are independent axioms in minimal reversal domains. There are rules that satisfy each pair of axioms in every minimal reversal domain.

Example 4. Let $k, k+1 \in N$.

- Let $\{k, k+1\} \neq \{n-1, n\}$. The priority rule φ^P satisfies *individual rationality* and *efficiency*, but φ^P violates *strategy-proofness* in $\mathcal{D}_{\{k,k+1\}}^N$.
- Let $\bar{\varphi} : \mathcal{D}_{\{k,k+1\}}^N \to \mathcal{A}$ be such for each $i \in N$ and each $\succsim \in \mathcal{D}^N$, $\bar{\varphi}_i(\succsim) = \omega_i$. The rule $\bar{\varphi}$ satisfies *individual rationality* and *strategy-proofness* in $\mathcal{D}_{\{k,k+1\}}^N$, but $\bar{\varphi}$ violates *efficiency*.
- For each $\succsim \in \mathcal{D}_{\{k,k+1\}}^N$, let $\mathcal{S}_0 \equiv \mathcal{A}$ and for each $t \leq n$, let $\mathcal{S}_t \equiv \left\{a \in \mathcal{S}_{t-1} \mid a_i \in \text{top}\left(\mathcal{S}_{t-1}, \succsim_t\right)\right\}$. The rule φ^S is the ***serial dictatorship*** if for each $\succsim \in \mathcal{D}_{\{k,k+1\}}^N$, $\varphi(\succsim) \in \mathcal{S}_n$. The serial dictatorship φ^S satisfies *efficiency* and *strategy-proofness* in $\mathcal{D}_{\{k,k+1\}}^N$, but φ^S violates *individual rationality*.

We next analyze the existence of maximal rich domains for *individual rationality*, *efficiency*, and *strategy-proofness* that are not minimal reversal domains. Example 5 shows that the minimal reversal structure is not required for domains satisfying *individual rationality*, *efficiency*, and *strategy-proofness*.

Example 5. Let $N = \{1, 2, 3\}$. Let $\widehat{\mathcal{D}}^N = \times_{i \in N} \widehat{\mathcal{D}}^i$ such that $\widehat{\mathcal{D}}^1 = \widehat{\mathcal{D}}^2 = \mathcal{P}$, and for each $\succsim_3 \in \mathcal{P}$, $\succsim_3 \in \widehat{\mathcal{D}}^3$ if and only if $\#\left\{\omega \in \Omega \,|\, \omega \succ_3 \omega_3\right\} \leq 1$.[13] That is, preferences of agent 1 and agent 2 are unrestricted but agent 3 can only consider either ω_1 or ω_2 as preferred to ω_3 The domain $\widehat{\mathcal{D}}^N$ is rich but it is not contained in any minimal reversal domain. There is a rule that satisfies *individual rationality*, *efficiency*, and *strategy-proofness* on $\widehat{\mathcal{D}}^N$. Let $\bar{a} \in \mathcal{A}$ be such that for each $i \in N$, $\bar{a}_i = \omega_i$. Since for each $\succsim \in \widehat{\mathcal{D}}^N$ agent 3 admits the exchange with at most one other agent, there are at most two assignments in $I(\succsim)$ besides $\bar{a}$. Let the rule $\widehat{\varphi} : \widehat{\mathcal{D}}^N \to \mathcal{A}$ be defined by:

- If $I(\succsim) = \{\bar{a}\}$, then $\widehat{\varphi}(\succsim) = \bar{a}$.
- If there is $a \in \mathcal{A}$ such that $I(\succsim) = \{\bar{a}, a\}$, then $\widehat{\varphi}(\succsim) = a$.
- If there are $a, a' \in \mathcal{A}$ such that $I(\succsim) = \{\bar{a}, a, a'\}$, let $i^* \in N$ be the unique agent such that $a_{i^*} \neq a'_{i^*}$ and $a_{i^*} \neq \omega_{i^*}$ and $a'_{i^*} \neq \omega_{i^*}$. In this case, $\widehat{\varphi}(\succsim) = a$ whenever $a_{i^*} \succ_{i^*} a'_{i^*}$ and $\widehat{\varphi}(\succsim) = a'$ otherwise.

The rule $\widehat{\varphi}$ satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in the domain $\widehat{\mathcal{D}}^N$. To check this fact, we only have to consider the possibility of an agent i such that $\widehat{\varphi}_i(\succsim) \neq \text{top}(\mathcal{A}, \succsim_i)$. Let $\text{top}(\mathcal{A}, \succsim_i) = \omega_j$. Since $\widehat{\varphi}(\succsim) \in I(\succsim)$, we have that $\omega_j \succ_j \omega_i$. Therefore, there is no $\succsim_i' \in \widehat{\mathcal{D}}^i$, $\widehat{\varphi}_i(\succsim_i', \succsim_{-i}) = \omega_j$.

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[13] For each set A, $\#A$ stands for the cardinality of the set A.

Example 5 shows that the minimal reversal structure is not implied by *individual rationality*, *efficiency*, and *strategy-proofness* in rich domains that do not contain preferences where an agent (weakly) prefers each pair of objects to her initially owned object. To clarify the role of minimal reversal domains on the family of maximal rich domains for *individual rationality*, *efficiency*, and *strategy-proofness*, we consider two additional conditions on preference domains. These new properties are consistent with the idea than the initially owned object represents an outside option available for each agent. Basically, *sovereignty* implies that for each agent, there are admissible preferences such that either all the objects are preferred or no object is preferred to her initially owned object. *Pairwise individuality* implies that each agent is allowed to express preferences for just two objects as preferred to her initially owned object.

Sovereignty. For each $i \in N$, there are $\succsim_i, \succsim'_i \in \mathcal{D}^i$ such that for each $\omega \in \Omega \setminus \{\omega_i\}$, $\omega \succ_i \omega_i$ and $\omega_i \succ'_i \omega$.

Sovereignty requires the individual domains of each agent to include a preference for which all the objects may be at least as good as the initially owned object and a preference for which no object is preferred to her initially owned object.

Pairwise individuality. For each $i \in N$, and $X \subset \Omega \setminus \{\omega_i\}$ with $\#X = 2$, if there is $\succsim_i \in \mathcal{D}^i$ such that for each $\omega \in X$, $\omega \succ_i \omega_i$, then there is $\succsim'_i \in \mathcal{D}^i$ such that

- $\omega \succ'_i \omega_i$ if and only if $\omega \in X$.
- for each $\omega, \omega' \in X$, $\omega \succ'_i \omega'$ if and only if $\omega \succ_i \omega'$.

Pairwise individuality requires that for each preference and every pair of objects that are strictly preferred to the initially owned object, there is an admissible preference such that those two objects are the only objects preferred to the initially owned object, and the ranking of those two objects coincides with the ranking under the initial preference. Note that *pairwise individuality* is immediately satisfied when $n = 3$. *Pairwise individuality* extends agents' freedom of choice with respect to an outside option in the same line of *individuality*. Agents are allowed to express any pair of objects as the only objects preferred to the initially owned object. Therefore, each agent may be interested in participating in the centralized exchange if she receives an object in that pair.

We conclude this section with Theorem 3 that states that every maximal rich domain for *individual rationality*, *efficiency*, and *strategy-proofness* that satisfies *sovereignty* and *pairwise individuality* is a minimal reversal domain.

Theorem 3. *The domain $\mathcal{D}^N = \times_{i \in N} \mathcal{D}^i$ is a maximal rich domain for individual rationality, efficiency, and strategy-proofness, and satisfies sovereignty and pairwise individuality if and only if $\mathcal{D}^N$ is a minimal reversal domain.*

Example 6 presents domains that are not included in any minimal reversal domain but still satisfy each combination of three conditions on preference domains. These domains allow for the existence of rules that satisfy *individual rationality*, *efficiency*, and *strategy-proofness*.

Example 6. Anonymity, *individuality*, *sovereignty*, and *pairwise individuality* are independent conditions.

- Let $\mathcal{D}^N_{NA} = \times_{i \in N} \mathcal{D}^i_{NA} \subset \mathcal{P}^N$ be such that $\mathcal{D}^1_{NA} = \mathcal{P}$ and for each $i \in N \setminus \{1\}$, let $\mathcal{D}^i_{NA} = \mathcal{D}^i_C$. The domain $\mathcal{D}^N_{NA}$ satisfies *sovereignty*, and *pairwise individuality*, but if $n \geq 4$ it does not satisfy *anonymity*. The priority rule φ^P satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in $\mathcal{D}^N_{NA}$.[14]
- Let n be even and $n \geq 4$.[15] Let $\Omega = \cup_{\tau=1}^{\frac{n}{2}} \Pi_\tau$, where for each $\tau = 1, \ldots, \frac{n}{2}$, $\Pi_\tau = \{\omega_{2\tau-1}, \omega_{2\tau}\}$. That is, the set of objects are ordered in pairs that are indexed. Let the preferences $\succsim^+_i, \succsim^-_i \in \mathcal{P}$ be such that for each $\omega \in \Omega \setminus \{\omega_i\}$, $\omega \succ^+_i \omega_i$, $\omega_i \succ^-_i \omega$, and for each $\omega_j, \omega_{j'} \in \Omega \setminus \{\omega_i\}$, $\omega_j \succsim^+_i \omega_{j'}$ if and only if $j \leq j'$. That is $\succsim^+_i$ and $\succsim^-_i$ correspond to common ranking preferences when either all the objects are preferred to the initially owned object, or no object is preferred to the initially owned object, respectively. For each $i \in N$, define the domain $\bar{\mathcal{D}}^i \subset \mathcal{P}$ by

  $$\succsim_i \in \bar{\mathcal{D}}^i \Leftrightarrow \begin{cases} \#\{\omega | \omega \succ_i \omega_i\} = 2, \text{ and} \\ \omega \succ_i \omega_i, \omega' \succ_i \omega_i, \omega \in \Pi_\tau, \omega' \in \Pi_{\tau'}, \tau < \tau' \Rightarrow \omega \succ'_i \omega'. \end{cases}$$

  According to each preference in $\succsim_i \in \bar{\mathcal{D}}^i$ exactly two objects are preferred to the initially owned objects. Provided that a pair of objects is preferred to the initially owned object, an object from a pair with a lower index is preferred to an object from a pair with higher index. Agents' preferences over objects in the same pair can be reversed. For each $i \in N$, let

  $$\mathcal{D}^i_{NI} = \bar{\mathcal{D}}^i \cup \{\succsim^+_i\} \cup \{\succsim^-_i\}.$$

  and $\mathcal{D}^N_{NI} = \times_{i \in N} \mathcal{D}^i_{NI}$. The domain $\mathcal{D}^N_{NI}$ satisfies *anonymity*, *sovereignty*, and *pairwise individuality*, but it does not satisfy *individuality*. For each $\succsim \in \mathcal{D}^N_{NI}$, let $\mathcal{M}^+_0 = I(\succsim)$. For each $\tau = 1, \ldots, \frac{n}{2}$ and each $t \in \{2\tau - 1, 2\tau\}$ let us define $\mathcal{M}^+_t$ by:

  - If $\text{top}(\mathcal{M}^+_{t-1}, \succsim_{2\tau-1}) \neq \text{top}(\mathcal{M}^+_{t-1}, \succsim_{2\tau})$, then

    $$\mathcal{M}^+_t \equiv \left\{a \in \mathcal{M}^+_{t-1} \mid a_t \in \text{top}(\mathcal{M}^+_{t-1}, \succsim_t)\right\}.$$

  - If there is $j \in N$ such that $\text{top}\left(\mathcal{M}^+_{t-1}, \succsim_{2\tau-1}\right) = \text{top}\left(\mathcal{M}^+_{t-1}, \succsim_{2\tau}\right) = \{\omega_j\}$, let us define $\bar{t}^+ \equiv 2\tau - 1$ if $\omega_{2\tau-1} \succ_j \omega_{2\tau}$ and $\bar{t}^+ = 2\tau$ if $\omega_{2\tau} \succ_j \omega_{2\tau-1}$, $\underline{t}^- \in \{2\tau - 1, 2\tau\} \setminus \{\bar{t}^+\}$, and

    $$\mathcal{M}^+_{2\tau-1} = \mathcal{M}^+_t \equiv \left\{a \in \mathcal{M}^+_{t-1} \mid a_{\bar{t}^+} \in \text{top}(\mathcal{M}^+_{t-1}, \succsim_{\bar{t}^+})\right\}, \textit{ and}$$
    $$\mathcal{M}^+_{2\tau} = \mathcal{M}^+_{t+1} \equiv \left\{a \in \mathcal{M}^+_t \mid a_{\underline{t}} \in \text{top}(\mathcal{M}^+_t, \succsim_{\underline{t}^-})\right\}.$$

  Let the rule $\varphi^{NI} : \mathcal{D}^N_{NI} \to \mathcal{A}$ be such that for each $\succsim \in \mathcal{D}^N_{NI}$, $\varphi^{NI}(\succsim) \in \mathcal{M}^+_n$. The rule φ^{NI} satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in $\mathcal{D}^N_{NI}$.[16]
- Let $\widehat{\mathcal{D}}^N \subset \mathcal{P}^N$ and the rule $\widehat{\varphi}$ be defined as in Example 5. The domain $\widehat{\mathcal{D}}^N$ satisfies *anonymity*, *individuality*, and *pairwise individuality*, but it does not satisfy *sovereignty*. The rule $\widehat{\varphi}$ satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in $\mathcal{D}^N$.
- Let n be even and $n \geq 4$. Let $\Omega = \cup_{\tau=1}^{\frac{n}{2}} \Pi_i$, where for each $\tau = 1, \ldots, \frac{n}{2}$, $\Pi_\tau = \{\omega_{2\tau-1}, \omega_{2\tau}\}$. For each $i \in N$, define the domain $\mathcal{D}^i_{NP} \subset \mathcal{P}$ as the set of all preferences such that for each $\omega, \omega' \in \Omega \setminus \{\omega_i\}$, $\omega \in \Pi_\tau$, $\omega' \in \Pi_{\tau'}$ and $\tau < \tau'$ implies $\omega \succ_i \omega'$. That is, the set of objects are ordered in pairs that are indexed. Objects from a pair with a lower index are preferred to objects from a pair with higher index. Agents' preferences over objects in the same pair can be reversed. Note that for each $i \in N$, $\succsim_i \in \mathcal{D}^i_{NP}$ and $\omega \in \Pi_\tau$, $\omega' \in \Pi_{\tau'}$, $\omega \succ_i \omega_i$ and $\tau' < \tau$ implies $\omega' \succ_i \omega_i$. The domain $\mathcal{D}^N_{NP} \equiv \times_{i \in N} \mathcal{D}^i_{NP}$ satisfies *anonymity*, *individuality* and *sovereignty*, but it does not satisfy *pairwise individuality*. The

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[14] When $n = 3$, *anonymity* is trivially fulfilled. Note that when $n = 3$, $\mathcal{D}^N_{NA} = \mathcal{D}^N_{\{2,3\}}$ and for each $\succsim \in \mathcal{D}^N_{NA}$, φ^P selects the same assignment than the 2-reversal adjusted priority rule.

[15] For the case $n = 3$, consider the domain of preferences $\mathcal{D}^N_{NI+}$ such that for each $i \in N$ and each preference $\succsim_i \in \mathcal{P}$, $\succsim_i \in \mathcal{D}^i_{NI+}$ if and only if $\#\{\omega | \omega \succ_i \omega_i\} \in \{0, 2\}$. The domain $\mathcal{D}^N_{NI+}$ is not contained in any minimal reversal domain, and it satisfies *anonymity*, *sovereignty*, and *pairwise individuality*, but it does not satisfy *individuality*. The priority rule φ^P satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in $\mathcal{D}^N_{NI+}$.

[16] Note that for each $\tau \leq \frac{n}{2}$ and each step $t \in \{2\tau - 1, 2\tau\}$ of the algorithm, the proposing agents $2\tau - 1$ and 2τ cannot improve by reporting a false preference. Moreover, the agents that receive the proposals cannot receive a preferred object in subsequent stages of the algorithm, which suffices to prove that φ^{NI} satisfies *strategy-proofness* in $\mathcal{D}^N_{NI}$.

priority rule φ^P satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in $\mathcal{D}^N_{NP}$.[17]

5. Discussion: Weak preferences

In environments with indifferences, the extension of reversal adjusted priority rules is not immediate because agents may have multi-valued top choices sets. Nicolò and Rodríguez-Álvarez (2017) analyzes domains with indifferences that are naturally related to common ranking preference domains – age based domains – where rules that satisfy *individual rationality*, *efficiency*, and *strategy-proofness* exist. In age based domains, there is an ordered partition of the set of objects, and each agent ranks the objects she prefers to her initially owned object according to the order over the elements of the partition, being indifferent among objects belonging to the same element of the partition. Common ranking preferences correspond to age based preferences according the finest partition of the set of objects. Age based preferences do not extend minimal reversal domains because they do not admit any preference reversal. Moreover, age based preferences do not admit strict preferences between objects in the same element of the partition.

We present an extension of minimal reversal domains that admit indifferences among objects that are adjacent in the natural order.

Let $k, k+1 \in N$. Let $\widetilde{\mathcal{D}}^N_{\{k,k+1\}} = \times_{i \in N} \widetilde{\mathcal{D}}^i_{\{k,k+1\}} \subset \mathcal{R}^N$ be the domain that contains all the preference profiles such that for each $i \in N$, $\succsim_i \in \widetilde{\mathcal{D}}^i_{\{k,k+1\}}$, there is no $\omega \in \Omega \setminus \{\omega_i\}$ such that $\omega \sim_i \omega_i$ and for each $\omega_j, \omega_{j'} \in \Omega \setminus \{\omega_i\}$ with $\{j, j'\} \neq \{k, k+1\}$, $\omega_j \succ_i \omega_i$, and $\omega_{j'} \succ_i \omega_i$,

- $j < j'$ implies $\omega_j \succsim_i \omega_{j'}$, and
- whenever there is l with $j < l < j'$, then $\omega_j \succ_i \omega_{j'}$.

We call $\widetilde{\mathcal{D}}^N_{\{k,k+1\}}$ the ***minimal reversal weak preference domain*** for $\{k, k+1\}$ We say that a domain $\mathcal{D}^N = \times_{i \in N} \mathcal{D}_i \subset \mathcal{R}^N$ is a ***minimal reversal domain with indifferences*** if there are $k, k+1 \in N$ such that $\mathcal{D}^N = \widetilde{\mathcal{D}}^N_{\{k,k+1\}}$.

In the previous section we have defined priority rules on domains of strict preferences. For preferences admitting indifferences, the priority algorithm obtains a non-empty set of assignments. By introducing arbitrary tie-breakers on the set of assignments selected by the priority algorithm, we can define different priority rules in domains with indifferences. It turns out that every such priority rule defined in that domain fails to satisfy *strategy-proofness* in age-based domains that admit indifferences. In fact, we can apply the same logic with k-reversal adjusted priority algorithms and k-reversal priority rules. Example 7 shows that in minimal reversal domain with indifferences such reversal adjusted priority rules do not satisfy *strategy-proofness*.[18]

Example 7. Let $N = \{1, 2, 3, 4\}$ and $\succsim \in \widetilde{\mathcal{D}}^N_{\{3,4\}}$ be such that

$\omega_3 \sim_1 \omega_4 \succ_1 \omega_1 \succ_1 \omega_2$,
$\omega_3 \succ_2 \omega_2 \succ_2 \omega_1 \succ_2 \omega_4$,
$\omega_1 \succ_3 \omega_2 \succ_3 \omega_3 \succ_3 \omega_4$,
$\omega_1 \succ_4 \omega_4 \succ_4 \omega_2 \succ_4 \omega_3$.

Let φ be a rule such that for each $\succsim \in \widetilde{\mathcal{D}}^N_{\{3,4\}}$, $\varphi(\succsim) \in \mathcal{M}^*_n$. To obtain $\varphi(\succsim)$, note that:

$$\mathcal{M}^*_0 = I(\succsim) = \left\{a \in \mathcal{A} \mid a_2 \in \{\omega_2, \omega_3\} \text{ and } a_3 \neq \omega_4\right\},$$
$$\mathcal{M}^*_1 = \left\{a \in \mathcal{A} \mid a_1 \in \{\omega_3, \omega_4\}, a_2 \in \{\omega_2, \omega_3\}\right\},$$
$$\mathcal{M}^*_2 = \left\{a \in \mathcal{A} \mid a_1 = \omega_4, a_2 = \omega_3\right\}.$$

Then, $\varphi(\succsim)$ is defined by

$$\varphi_1(\succsim) = \omega_4,\ \varphi_2(\succsim) = \omega_3,\ \varphi_3(\succsim) = \omega_2,\ \varphi_4(\succsim) = \omega_1.$$

Let $\succsim' \in \mathcal{D}^N_{\{3,4\}}$ be such that $\omega_1 \succ'_3 \omega_3 \succ'_3 \omega_2$ and $\succsim'_{-3} = \succsim_{-3}$. Then

$$\varphi_1(\succsim') = \omega_3,\ \varphi_2(\succsim') = \omega_2,\ \varphi_3(\succsim') = \omega_1,\ \varphi_4(\succsim') = \omega_4.$$

Since $\varphi_3(\succsim') \succ_3 \varphi_3(\succsim)$, φ violates *strategy-proofness*.

Example 7 highlights the challenges in extending reversal adjusted priority rules to domains with indifferences. Both according to the priority rule and reversal adjusted priority rules, the agent at the top of the priority order may be indifferent among two exchanges involving different agents. In such cases, the tie is resolved using the preferences of another agent who is not directly involved in those exchanges. This opens up possibilities for manipulation by an agent who would benefit from an exchange with the initial agent. An arbitrary tie-breaker at each step of the k-reversal adjusted priority algorithm would help to restore the incentives to reveal the true preferences for the agents, but further flexibility is required to satisfy *efficiency*. However, when indifferences among objects are limited to adjacent pairs in the natural order, it is possible to construct rules that meet the requirements of our axioms.

k-Reversal adjusted priority algorithm with indifferences. Let $k, k+1 \in N$ and $\succsim \in \mathcal{R}^N$.

- $\widetilde{\mathcal{M}}_0 \equiv I(\succsim)$.
- For each $t \leq n$, to define $\widetilde{\mathcal{M}}_t \subseteq \widetilde{\mathcal{M}}_{t-1}$ we consider two cases:
  - If either $t \notin \{k, k+1\}$, or $t \in \{k, k+1\}$ and $\min\{i \mid \omega_i \in \text{top}(\widetilde{\mathcal{M}}_{k-1}, \succsim_k)\} \neq \min\{i \mid \omega_i \in \text{top}(\widetilde{\mathcal{M}}_{k-1}, \succsim_{k+1})\}$, let
  $$\mu(t) \equiv \min\{i \mid \omega_i \in \text{top}(\widetilde{\mathcal{M}}_{t-1}, \succsim_t)\},$$
  $$\widetilde{\mathcal{M}}_t \equiv \left\{ a \in \widetilde{\mathcal{M}}_{t-1}(\succsim) \;\middle|\; \begin{array}{c} a_t \succsim_t \omega_{\mu(t)}, \\ a_{\mu(t)} \succsim_{\mu(t)} \omega_t \end{array} \right\}.$$
  - If $t \in \{k, k+1\}$ and there is $j \in N$ such that $j = \min\left\{i \mid \omega_i \in \text{top}(\widetilde{\mathcal{M}}_{k-1}, \succsim_k)\right\} = \min\left\{i \mid \omega_i \in \text{top}(\widetilde{\mathcal{M}}_{k-1}, \succsim_{k+1})\right\}$, then define $\bar{k}, \underline{k} \in \{k, k+1\}$ as
  $$\bar{k} \equiv \begin{cases} k & \text{if}\, \omega_k \succsim_j \omega_{k+1}, \\ k+1 & \text{if}\, \omega_{k+1} \succ_j \omega_k, \end{cases} \quad \text{and} \quad \underline{k} \in \{k, k+1\} \setminus \{\bar{k}\}.$$
  Let $\mu(\bar{k}) \equiv \min\{i \mid \omega_i \in \text{top}(\widetilde{\mathcal{M}}_{k-1}, \succsim_{\bar{k}})\}$ and
  $$\widetilde{\mathcal{M}}_k \equiv \left\{ a \in \widetilde{\mathcal{M}}_{k-1}(\succsim) \;\middle|\; \begin{array}{c} a_{\bar{k}} \succsim_{\bar{k}} \omega_{\mu(\bar{k})}, \\ a_{\mu(\bar{k})} \succsim_{\mu(\bar{k})} \omega_{\bar{\bar{k}}} \end{array} \right\}.$$
  Finally, let $\mu(\underline{k}) \equiv \min\{i \mid \omega_i \in \text{top}(\widetilde{\mathcal{M}}_k, \succsim_{\underline{k}})\}$, and
  $$\widetilde{\mathcal{M}}_{k+1} \equiv \left\{ a \in \widetilde{\mathcal{M}}_k \;\middle|\; \begin{array}{c} a_{\underline{k}} \succsim_{\underline{k}} \omega_{\mu(\underline{k})}, \\ a_{\mu(\underline{k})} \succsim_{\mu(\underline{k})} \omega_{\underline{k}} \end{array} \right\}.$$

Note that for each $\succsim \in \widetilde{\mathcal{D}}^N_{\{k,k+1\}}$, $\widetilde{\mathcal{M}}_n$ is non-empty and essentially single-valued.[19] For each pair of agents $\{k, k+1\}$, a rule $\varphi : \widetilde{\mathcal{D}}^N_{\{k,k+1\}} \to \mathcal{A}$ such that for each $\succsim \in \widetilde{\mathcal{D}}^N_{\{k,k+1\}}$, $\varphi(\succsim) \in \widetilde{\mathcal{M}}_n$ is called a ***k-reversal adjusted priority rule with indifferences***.

Every k-reversal adjusted priority rule with indifferences follows the same logic than the k-reversal adjusted priority rule in strict preference

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[17] To check that φ^P satisfies *strategy-proofness* in $\mathcal{D}^N_{NP}$, consider an arbitrary profile $\succsim \in \mathcal{D}^N_{NP}$. At each step $t \leq n$ of the priority algorithm, agent t cannot improve by reporting a false preference. Let $j \in N$ be such that $\varphi^P_j(\succsim) = \omega_t$ for some $t < j$. Let τ be such that $t \in \{2\tau - 1, 2\tau\}$. Note first that for each $t' < t$ and $\tau' < \tau$ such that $t' \in \{2\tau' - 1, 2\tau'\}$, since $\succsim_j \in \mathcal{D}^j_{NP}$, we have that $\omega_{t'} \succ_j \omega_t$ and $\varphi^P_j(\succsim) = \omega_t$ imply that $\varphi^P_{t'}(\succsim) \succ_{t'} \omega_j$. Hence, by unilaterally changing her preferences, agent j cannot obtain an object from an agent in a pair that precedes the pair to which agent t belongs. To conclude, assume that there is τ such that $\Pi_\tau = \{\omega_t, \omega_{t+1}\}$ and $\omega_{t+1} \succ_j \omega_t$. Since $\succsim \in \mathcal{D}^N_{NP}$, $t < j$ and $\varphi^P_t(\succsim) = \omega_j$ imply that $\omega_{t+1} \succ_t \omega_j \succ_t \omega_t$. Therefore, since $\varphi^P_t(\succsim) = \omega_j$, either $\omega_{t+1} \succ_{t+1} \omega_t$ and $\omega_{t+1} \succ_{t+1} \omega_j$, or there is $t' < t$ such that for each $a \in \mathcal{M}_{t-1}$, $a_{t'} = \omega_{t+1}$. Hence, there is no $\succsim'_j \in \mathcal{D}^j_{NP}$, such that $\varphi^P_j(\succsim'_j, \succsim_{-j}) = \omega_{t+1}$. This exhausts all the possibilities and suffices to prove that φ^P satisfies *strategy-proofness* in $\mathcal{D}^N_{NP}$.

[18] Example 7 is a convenient rephrasing of Example 2 in Nicolò and Rodríguez-Álvarez (2017).

[19] The set $\widetilde{\mathcal{M}}_n$ may be multi-valued only if there are two pairs of agents $j, j+1$ and $l, l+1$ such that $\omega_l \sim_j \omega_{l+1}$, $\omega_l \sim_{j+1} \omega_{l+1}$, $\omega_j \sim_l \omega_{j+1}$, and $\omega_j \sim_{l+1} \omega_{j+1}$.

domains, and in fact they coincide for profiles of strict preferences. To satisfy *strategy-proofness*, when the agent selecting at stage $t \notin \{k, k+1\}$ top choice set is multi-valued, k-reversal adjusted priority rules with indifferences break ties following the natural order independently of the preferences of the following agents and propose a tentative pairwise exchange for agent t and agent $\mu(t)$. In the next stage, agent $t+1$ selects among the assignments such that agent t obtains an object at least as preferred as $\omega_{\mu(t)}$ and agent $\mu(t)$ obtains an object at least as preferred as $\omega_{\mu(t)}$. The later condition is implicit in domains of strict preferences but it is needed in domains with indifferences. Hence, in the following stages of the process, the next agent selects the best assignment that is at least as good as that tentative assignment for all the agents involved in pairwise exchanges at previous stages. When indifferences are only admitted among adjacent objects, *efficiency* is also satisfied because Pareto improvements over the tentative selection obtained at a stage t can be performed only in the next stage $t+1$.[20] Finally, when the algorithm reaches the initial owners of the objects that admit preference reversals, the agents make simultaneous proposals, and in case that they coincide, the agent that receives the proposals selects her preferred tentative exchange.

Example 8. [Example 7 Continued.] Let φ^+ be a 3-reversal adjusted priority rule with indifferences applied to the preference profiles $\succsim, \succsim' \in \mathcal{D}^N_{\{3,4\}}$ defined in Example 7. Note that $\text{top}(I(\succsim), \succsim_1) = \{\omega_3, \omega_4\}$. Hence, $\mu(1) = \min\{i | \omega_i \in \text{top}(I(\succsim), \succsim_1)\} = 3$, and

$$\widetilde{\mathcal{M}}_1 = \{a \in I(\succsim) | a_1 \succsim_1 \omega_3,\ a_3 \succsim_3 \omega_1\} = \{a \in \mathcal{A} |\ a_1 = \omega_3,\ a_2 = \omega_2,\ a_3 = \omega_1,\ a_4 = \omega_4\}.$$

With the same arguments we obtain $\varphi^+(\succsim) = \varphi^+(\succsim')$, and agent 3 has no incentive to misrepresent her preferences.

Proposition 1. *Let $k, k+1 \in N$. Each k-reversal adjusted priority rule with indifferences satisfies individual rationality, efficiency, and strategy-proofness in the domain $\widetilde{\mathcal{D}}^N_{\{k,k+1\}}$.*

Proposition 1 demonstrates that minimal reversal domains cease to be maximal when indifferences are permitted. There exist rules that satisfy our axioms in minimal reversal weak preference domains, and each minimal reversal domain is strictly encompassed by a minimal reversal weak preference domain. Given the challenges in defining rules that adhere to our axioms within weak preference domains, we defer the examination of maximal domains of weak preferences to future research.

6. Conclusion and related literature

In this paper we have analyzed pairwise house exchange problems to find maximal extensions of common ranking domains where centralized rules that satisfy *individual rationality*, *efficiency*, and *strategy-proofness* exist. We show that those properties do not admit substantial departures from common ranking preferences. We identify a family of maximal domains, minimal reversal domains, that just admit preference reversals from a common ranking for two objects that are adjacent in the common ranking. In minimal reversal domains, reversal adjusted priority rules propose a sequential priority selection of individually rational pairwise exchanges and satisfy *individual rationality*, *efficiency*, and *strategy-proofness*.

Some remarks on possible extensions of the model are in order. In this paper we obtain positive results under the most stringent logistic constraint on the number of agents involved in exchange cycles. The positive results do not extend to the framework with less stringent logistic constraints. When cycles involving at least three agents are allowed, there is no rule defined in the common ranking domain that satisfies *individual rationality*, *efficiency*, and *strategy-proofness* (Nicolò and Rodríguez-Álvarez, 2017, Theorem 4).[21] Finally, throughout this paper we assume that all agents are initially endowed with a unique object. The main results of the paper would go through in minimal reversal domains where all the objects initially owned by each agent occupy adjacent positions in the common ranking. Unfortunately, the results do not extend to more general common rankings. When the different objects initially owned by an agent do not occupy adjacent positions in the remaining agents preferences, no rule satisfies *individual rationality*, *efficiency*, and *strategy-proofness* (Nicolò and Rodríguez-Álvarez, 2017, Theorem 6).

We close this concluding section examining the related literature. Many recent works analyze maximal domains for rules that satisfy *strategy-proofness* in different environments, like selection of multiple objects (Barberà et al., 1991; Le Breton and Sen, 1999; Hatsumi et al., 2014), public good provision (Ching and Serizawa, 1998; Berga and Serizawa, 2000), or division and allotment problems (Barberà et al., 1999; Massó and Neme, 2001; Mizobuchi and Serizawa, 2001; Wakayama, 2017; Chatterji et al., 2020). Beyond those papers that focus on maximal domains for general voting and public good provision settings, the most closely related papers analyze conditions on agents preferences that allow for rules that satisfy *strategy-proofness* in different problems of allocation of indivisible objects. Alcalde and Barberà (1994) considers two-sided marriage and college admission markets, and propose a necessary and sufficient condition (not related to minimal reversal domains) on domains of preferences for one side of the market that allow for rules that satisfy *individual rationality*, *efficiency*, and *strategy-proofness*. Ehlers (2002) considers the allocation of indivisible objects without property rights when indifference are admitted. In that context *individual rationality* is not relevant, and for the unrestricted domain of preferences hierarchically dictatorial rules satisfy *efficiency* and *strategy-proofness*. Ehlers (2002) finds the unique maximal domain extending the universal domain of strict preferences for *efficiency* and *group-strategy-proofness* when indifferences are admitted. Ergin (2002) considers school choice problems, that is two-sided matching markets where the preferences (priorities) of one side of the market (schools) are not taken into account to determine *efficiency*. Ergin (2002) shows that *individual rationality*, *efficiency*, and *strategy-proofness* are compatible if the preferences (priorities) of the members of the welfare irrelevant side of the market (schools) satisfy an acyclicity condition.[22]

Our paper is also related to the literature on the existence of single-valued core assignments. For general problems of allocation of indivisible objects where core assignments exist, the existence of single-valued core assignments is a necessary condition for the existence of satisfy *individual rationality*, *efficiency*, and *strategy-proofness*, only if the set of core assignments is (essentially) single-valued (Sönmez, 1999). In fact, under strict preferences when agents do not care about the objects received by other agents, it is also sufficient and the rule that selects the single-valued core assignment satisfies the proposed axioms (Takamiya, 2003). Hence, since we investigate domains of preferences where such rules and core assignments exist, as a by-product, our results may be interpreted as the identification of maximal domains of strict preferences for the existence of unique core assignments in pairwise house exchange problems. Alcalde (1995) and Abizada (2019) study roommate problems, which correspond to pairwise house exchange problems with

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[20] In this case, agent t obtains the initially owned object of agent $\mu(t)+1$ and agent $\mu(t)$ obtains the initially owned object of agent $t+1$.

[21] Nicolò and Rodríguez-Álvarez (2017) shows that rules that satisfy *individual rationality*, *strategy-proofness*, and an appropriate version of *efficiency* can be constructed in domains with common ranking with indifferences when large exchange cycles are restricted to involve objects in the same indifference class of the common ranking.

[22] In the school choice problem with retrade, Matsui and Murakami (2022) studies the implications of a restriction of school priorities consistent with a specific example of minimal reversal preferences (unreversed priorities).

additional feasibility restrictions. Specifically, agents are not allowed to keep their initial objects. Alcalde (1995) proposes a stability condition for roommate problems – exchange proofness – and a sufficient condition on preference profiles (α-reducible preferences) for the existence of exchange-proof assignments. Abizada (2019) shows that exchange-proof assignment always exist under weaker sufficient conditions on agents preferences (r-level mixed condition). Chung (2000) presents alternative sufficient conditions (no-odd-ring condition) on profiles of agents' preferences for general assignment problems for the existence of single-valued core assignments. Park (2017) analyzes the relation between the implications of conditions proposed by Alcalde (1995) and Chung (2000) for the existence of core assignments in pairwise house exchange and two-sided matching problems. Preferences in minimal reversal domains satisfy the proposed conditions in Chung (2000), Alcalde (1995), and Abizada (2019). Since we focus on maximality of Cartesian domains and consider different but related frameworks, our results and theirs are complementary.

7. Proofs

We start with a pair of auxiliary lemmata.

Lemma 1 (*No Cycle*). *Let $\mathcal{D}^N \subseteq \mathcal{R}^N$ be a domain that satisfies individuality. If there are $i, j, k \in N$ and $\succsim \in \mathcal{D}^N$ such that:*

$\omega_j \succ_i \omega_k \succ_i \omega_i$,
$\omega_k \succ_j \omega_i \succ_j \omega_j$,
$\omega_i \succ_k \omega_j \succ_k \omega_k$,

and for each $i' \in \{i, j, k\}$ and $i'' \notin \{i, j, k\}$, $\omega_{i''} \succ_{i'} \omega_{i'}$ only if $\omega_{i''} \succ_{i''} \omega_{i'}$, then there is no rule $\varphi : \mathcal{D}^N \to \mathcal{A}$ that satisfies individual rationality, efficiency, and strategy-proofness.

Proof. The proof replicates the arguments in Nicolò and Rodríguez-Álvarez (2012, Theorem 1). Assume, by way of contradiction, that there is a rule φ that satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in the domain $\mathcal{D}^N$. By *individual rationality*, for each $i' \in \{i, j, k\}$, $\varphi_{i'}(\succsim) \in \{\omega_i, \omega_j, \omega_k\}$. We focus on the restriction of $\varphi(\succsim)$ to agents $\{i, j, k\}$. By *individual rationality* and *efficiency*, φ selects a pairwise assignment in which two agents in $\{\omega_i, \omega_j, \omega_k\}$ exchange their objects while the remaining agent keeps her initially owned object. Without loss of generality let $\varphi_i(\succsim) = \omega_j$, $\varphi_j(\succsim) = \omega_i$, and $\varphi_k(\succsim) = \omega_k$.

By *individuality* of $\mathcal{D}^N$, there is $\succsim' \in \mathcal{D}^N$ such that:

(i) $\omega_j \succ'_i \omega_i \succ'_i \omega_k$,
(ii) for each $\omega \in \Omega$ such that $\omega \succ_i \omega_j$, $\omega \succ'_i \omega_j$,
(iii) for each $\omega' \in \Omega$ such that $\omega_j \succ_i \omega'$, $\omega_i \succ'_i \omega'$, and
(iv) $\succsim'_{-i} = \succsim_{-i}$.

By *strategy-proofness*, $\varphi_i(\succsim') \succsim'_i \varphi_i(\succsim) = \omega_j$. Hence, $\varphi_i(\succsim') = \omega_j$, $\varphi_j(\succsim') = \omega_i$, and $\varphi_k(\succsim') = \omega_k$. By *individuality* of $\mathcal{D}^N$, there is $\succsim'' \in \mathcal{D}^N$ such that

(i) $\omega_k \succ''_j \omega_j \succ''_j \omega_i$,
(ii) for each $\omega \in \Omega$ such that $\omega \succ'_j \omega_k$, $\omega \succ''_j \omega_k$,
(iii) for each $\omega' \in \Omega$ such that $\omega_k \succ'_j \omega'$, $\omega_j \succ'_j \omega'$, and
(iv) $\succsim''_{-j} = \succsim'_{-j}$.

By *individual rationality*, $\varphi_j(\succsim'') \in \{\omega_j, \omega_k\}$. By *strategy-proofness*, $\varphi_j(\succsim') = \omega_i \succsim'_j \varphi_j(\succsim'')$. Hence, $\varphi_j(\succsim'') = \omega_j$. By *individual rationality*, $\varphi_i(\succsim'') \in \{\omega_i, \omega_j\}$, $\varphi_i(\succsim'') = \omega_i$. Therefore, $\varphi_k(\succsim'') = \omega_k$. Consider the pairwise assignment $a \in \mathcal{A}$ such that $a_i = \omega_i$, $a_j = \omega_k$, $a_k = \omega_j$ and for each $i'' \notin \{i, j, k\}$, $a_{i''} = \varphi_{i''}(\succsim'')$. For each agent $l \in N$, $a_l \succsim''_l \varphi_l(\succsim'')$. Moreover, $a_j \succ''_j \varphi_j(\succsim'')$ and $a_k \succ''_k \varphi_k(\succsim'')$, which contradicts φ's *efficiency*. □

Lemma 2. *Let $\mathcal{D}^N \subseteq \mathcal{R}^N$ be a domain that satisfies individuality. If there are $i, j, k, l \in N$ such that $\{i, j\} \cap \{k, l\} = \emptyset$ and $\succsim \in \mathcal{D}^N$,*

$\omega_k \succ_i \omega_l \succ_i \omega_i$,
$\omega_l \succ_j \omega_k \succ_j \omega_j \succ_j \omega_i$,
$\omega_j \succ_k \omega_i \succ_k \omega_k \succ_k \omega_l$,
$\omega_i \succ_l \omega_j \succ_l \omega_l \succ_l \omega_k$,

and for each $i' \in \{i, j, k, l\}$ and $i'' \notin \{i, j, k, l\}$, $\omega_{i''} \succ_{i'} \omega_{i'}$ only if $\omega_{i''} \succ_{i''} \omega_{i'}$, then there is no rule $\varphi : \mathcal{D}^N \to \mathcal{A}$ that satisfies individual rationality, efficiency, and strategy-proofness.

Proof. The proof replicates the arguments in Alcalde and Barberà (1994, Theorem 1) on the impossibility of obtaining a rule that satisfies *individual rationality*, *efficiency*, and *strategy-proofness* for two-sided matching problems. Assume, by way of contradiction, that there is a rule φ that satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in the domain $\mathcal{D}^N$. By *individual rationality*, for each $i' \in \{i, j, k, l\}$, $\varphi_{i'}(\succsim) \in \{\omega_i, \omega_j, \omega_k, \omega_l\}$. Without loss of generality, we focus on the restriction of $\varphi(\succsim)$ to agents $\{i, j, k, l\}$. By *individual rationality* and *efficiency*, there are two cases.

Case (i). $\varphi_i(\succsim) = \omega_k$, $\varphi_j(\succsim) = \omega_l$, $\varphi_k(\succsim) = \omega_i$, and $\varphi_l(\succsim) = \omega_j$.

Case (ii). $\varphi_i(\succsim) = \omega_l$, $\varphi_j(\succsim) = \omega_k$, $\varphi_k(\succsim) = \omega_j$, and $\varphi_l(\succsim) = \omega_i$.

We focus on Case (i). A parallel argument applies to Case (ii) and we omit it. By *individuality* of $\mathcal{D}^N$, there is $\succsim' \in \mathcal{D}^N$ such that

(i) $\omega_j \succ'_k \omega_k \succ'_k \omega_i$,
(ii) for each $\omega \in \Omega$ such that $\omega \succ_k \omega_j$, $\omega \succ'_k \omega_j$,
(iii) for each $\omega' \in \Omega$ such that $\omega_j \succ_k \omega'$, $\omega_k \succ'_k \omega'$, and
(iv) $\succ'_{-k} = \succ_{-k}$.

By *individual rationality* and *strategy-proofness*, $\varphi_k(\succsim') = \omega_k$, and either $\varphi_i(\succsim') = \omega_l$ and $\varphi_j(\succsim') = \omega_j$, or $\varphi_i(\succsim') = \omega_i$ and $\varphi_j(\succsim') = \omega_l$. Note that if $\varphi_i(\succsim') = \omega_l$ and $\varphi_j(\succsim') = \omega_j$, then $\varphi_k(\succsim') = \omega_k$ and $\varphi_l(\succsim') = \omega_i$. In this case, there is $a \in \mathcal{A}$ such that $a_j = \omega_k \succ'_j \varphi_j(\succsim')$, $a_k = \omega_j \succ'_k \varphi_k(\succsim')$, and for each $i' \notin \{j, k\}$, $a_{i'} = \varphi_{i'}(\succsim')$, which contradicts *efficiency*. Finally assume that $\varphi_i(\succsim') = \omega_i$ and $\varphi_j(\succsim') = \omega_l$. By *individuality*, there is $\succsim'' \in \mathcal{D}^N$ such that

(i) $\omega_i \succ''_l \omega_l \succ''_l \omega_j$,
(ii) for each $\omega \in \Omega$ such that $\omega \succ'_l \omega_i$, $\omega \succ''_l \omega_i$,
(iii) for each $\omega' \in \Omega$ such that $\omega_i \succ'_l \omega'$, $\omega_i \succ''_l \omega'$, and
(iv) $\succsim''_{-l} = \succsim'_{-l}$.

By *strategy-proofness*, $\varphi_l(\succsim'') \neq \omega_i$. By *individual rationality*, $\varphi_l(\succsim'') \neq \omega_j$. Therefore, $\varphi_l(\succsim'') = \omega_l$. By *efficiency*, $\varphi_k(\succsim'') = \omega_j$. Let $\succsim''' \in \mathcal{D}^N$ be such that $\succsim'''_k = \succsim_k$ and $\succsim'''_{-k} = \succsim''_k$. By *strategy-proofness*, $\varphi_k(\succsim''') = \omega_j$,. By *efficiency*, $\varphi_l(\succsim''') = \omega_i$. Note that $\succsim'''_{-l} = \succsim_{-l}$ and $\varphi_l(\succsim''') \succ_l \varphi_l(\succsim)$, which contradicts *strategy-proofness*. □

Proof of Theorem 1. Let $k, k+1 \in N$ and φ be the k-*reversal adjusted priority rule*. We start by showing that φ satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in the domain $\mathcal{D}^N_{\{k,k+1\}}$. In this proof, with a slight abuse of notation, for each $t \in N$ and $\succsim \in \mathcal{D}^N_{\{k,k+1\}}$, we denote the set $\mathcal{M}^*_t$ for $\mathcal{M}^*_0 = I(\succsim)$ by $\mathcal{M}^*_t(\succsim)$.

By construction, φ satisfies *individual rationality* and *efficiency*. Consequently, we only focus on checking that φ satisfies *strategy-proofness*. Let $i \in N$, and $\succsim, \succsim' \in \mathcal{D}^N_{\{k,k+1\}}$ be such that $\succsim'_{-i} = \succsim_{-i}$. Assume first that, $\varphi_i(\succsim) = \varphi_i(\succsim')$, then $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim')$. Assume now that $\varphi_i(\succsim) \neq \varphi_i(\succsim')$. Let $j, j' \in N$ be such that $\varphi_i(\succsim) = \omega_j$ and $\varphi_i(\succsim') = \omega_{j'}$. There are two cases:

Case (i). $i = j$. Assume first $i \notin \{k, k+1\}$. In this case, $\varphi_i(\succsim) = \omega_i$ implies that for each $a \in \mathcal{M}^*_{i-1}(\succsim)$, $a_i = \omega_i$. Since only agent i has changed her preferences, $\varphi_i(\succsim') \in \mathcal{M}^*_{i-1}(\succsim')$ and $\varphi_i(\succsim') = \omega_{j'}$ imply that $\omega_i \succ_i \omega_{j'}$. Hence, $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim')$. Finally, assume that $i \in \{k, k+1\}$. In this case, $\varphi_i(\succsim) = \omega_i$ implies that either for each $a \in \mathcal{M}^*_{k-1}(\succsim)$, $a_i = \omega_i$, or that $i = \underline{k}$ at profile $\succsim$ and for each $b \in \mathcal{M}^*_k(\succsim)$, $b_i = \omega_i$. In both cases, the argument for $i \notin \{k, k+1\}$ immediately applies and $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim')$.

Case (ii). $i \neq j$. Since $\varphi_i(\succsim) \neq \omega_i$, by *individual rationality* $\omega_j \succ_i \omega_i$. If $j' = i$, then $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim')$. Hence, we focus on the case $j' \neq i$. Now, we consider two subcases:

Subcase ii. (1). $\{j, j'\} \neq \{k, k+1\}$. Note first that If $j < j'$, then $\succsim_i \in \mathcal{D}^i_{\{k,k+1\}}$ and $\omega_j \succ_i \omega_i$ imply that $\omega_j \succ_i \omega_{j'}$ and $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim')$. If $j' < j$, then since only agent i changes her preferences, $\omega_i \in \text{top}(\mathcal{M}^*_{j'-1}(\succsim'), \succsim_{j'}) \setminus \text{top}(\mathcal{M}^*_{j'-1}(\succsim), \succsim_{j'})$ implies that $\omega_i \succ_i \omega_{j'}$ and $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim')$.

Subcase ii. (2). $\{j, j'\} = \{k, k+1\}$. If $i < k$, then, $\varphi_i(\succsim) = \text{top}(\mathcal{M}^*_{i-1}(\succsim), \succsim_i) = \omega_j$ implies that $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim')$. If $i > k+1$, note that since only agent i changes her preferences, $\varphi_i(\succsim') = \omega_{j'}$ implies that $\omega_i \succ_j \omega_j$ and $\omega_i \succ_{j'} \omega_{j'}$. Since $\varphi_i(\succsim) = \omega_j$ and $\varphi_i(\succsim') = \omega_{j'}$ we have three possibilities:

(i) $\text{top}\left(\mathcal{M}^*_{k-1}(\succsim), \succsim_j\right) \neq \text{top}\left(\mathcal{M}^*_{k-1}(\succsim), \succsim_{j'}\right)$. The result follows immediately with the same arguments of Subcase ii. (1).

(ii) $\text{top}\left(\mathcal{M}^*_{k-1}(\succsim), \succsim_j\right) = \text{top}\left(\mathcal{M}^*_{k-1}(\succsim), \succsim_{j'}\right) = \omega_i$. By the definition of the k-adjusted priority rule, $\omega_j \succ_i \omega_{j'}$ and $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim')$.

(iii) There is $l \in N$ such that $\text{top}\left(\mathcal{M}^*_{k-1}(\succsim), \succsim_j\right) = \text{top}\left(\mathcal{M}^*_{k-1}(\succsim), \succsim_{j'}\right) = \omega_l$. In this case, $\varphi_i(\succsim) = \omega_j$ implies that $\varphi_{j'}(\succsim) = \omega_l$ and $\omega_{j'} \succ_l \omega_j$. Since only agent i changes her preferences, there is $a \in \mathcal{M}^*_{k-1}(\succsim')$ such that $a_{j'} = \omega_l$. Moreover, $\varphi_{j'}(\succsim') = \omega_i$ implies that $\omega_i \succ_{j'} \omega_l$ and that there is no $b \in \mathcal{M}^*_{k-1}(\succsim)$, with $b_{j'} = \omega_i$. Again, since only agent i has changed her preferences, necessarily $\omega_i \succ_i \omega_{j'} = \varphi_i(\succsim')$, and $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim')$.

Once we have shown that the *k-reversal adjusted priority rule* satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in $\mathcal{D}^N_{\{k,k+1\}}$, we show that $\mathcal{D}^N_{\{k,k+1\}}$ is maximal. Assume there is a rich domain $\mathcal{D}^N = \times_{i \in N} \mathcal{D}^i$ such that $\mathcal{D}^N_{\{k,k+1\}} \subset \mathcal{D}^N \subsetneq \mathcal{P}^N$ and there is a rule $\varphi : \mathcal{D}^N \rightarrow \mathcal{A}$ that satisfies *individual rationality*, *efficiency*, and *strategy-proofness*. Hence, there are $i \in N$ and $\succsim_i \in \mathcal{D}^i \setminus \mathcal{D}^N_{\{k,k+1\}}$ such that for some $j, j' \in N$, $\{j, j'\} \neq \{k, k+1\}$, $j < j'$ and $\omega_{j'} \succ_i \omega_j \succ_i \omega_i$. We have to consider two exhaustive cases:

Case (i). $\{j, j'\} \cap \{k, k+1\} \neq \emptyset$. Consider first the case $j' \in \{k, k+1\}$. Note that $j < j'$ and $j' \in \{k, k+1\}$ imply that $j < k$. Without loss of generality, let $j' = k$. There are $i \in N$ and $\succsim^*_i \in \mathcal{D}^i$ such that $\omega_k \succ^*_i \omega_j \succ^*_i \omega_i$. By *anonymity* of $\mathcal{D}^N$, there is $\succsim^*_{k+1} \in \mathcal{D}^{k+1}$ such that $\omega_k \succ^*_{k+1} \omega_j \succ^*_{k+1} \omega_{k+1}$. Let $\succsim \in \mathcal{D}^N$ be such that

(i) $\succsim_{k+1} = \succsim^*_{k+1}$,

(ii) for each $i' \neq k+1$, $\succsim_{i'} \in \mathcal{D}^{i'}_{\{k,k+1\}}$,

(iii) $\omega_{k+1} \succ_j \omega_k \succ_j \omega_j$,

(iv) $\omega_j \succ_k \omega_{k+1} \succ_k \omega_k$, and

(v) for each $i'' \notin \{j, k, k+1\}$ and $\omega \in \{\omega_j, \omega_k, \omega_{k+1}\}$, $\omega_{i''} \succ_{i''} \omega$.

Since $\mathcal{D}^N$ is a rich domain, $\mathcal{D}^N$ satisfies *individuality*. Hence, $\mathcal{D}^N$ satisfies the conditions of Lemma 1, and there is no rule that satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in $\mathcal{D}^N$. Finally, the case $j \in \{k, k+1\}$ follows from a parallel argument. Note that $j < j'$ and $j \in \{k, k+1\}$ imply that $k+1 < j'$. Without loss of generality, let $j = k$. There are $i \in N$ and $\succsim^+_i \in \mathcal{D}^i$ such that $\omega_{j'} \succ^+_i \omega_j \succ^+_i \omega_i$. By *anonymity* of $\mathcal{D}^N$, there is $\succsim^+_{k+1} \in \mathcal{D}^{k+1}$ such that $\omega_{j'} \succ^+_{k+1} \omega_j \succ^+_{k+1} \omega_{k+1}$. Let $\succsim \in \mathcal{D}^N$ be such that

(i) $\succsim_{k+1} = \succsim^+_{k+1}$,

(ii) for each $i' \neq k+1$, $\succsim_{i'} \in \mathcal{D}^{i'}_{\{k,k+1\}}$,

(iii) $\omega_{k+1} \succ_j \omega_{j'} \succ_j \omega_j$,

(iv) $\omega_j \succ_{j'} \omega_{k+1} \succ_{j'} \omega_{j'}$, and

(v) for each $i'' \notin \{j, k, k+1\}$ and $\omega \in \{\omega_j, \omega_{j'}, \omega_{k+1}\}$, $\omega_{i''} \succ_{i''} \omega$.

Since $\mathcal{D}^N$ is a rich domain and $\mathcal{D}^N$ satisfies the conditions of Lemma 1, which is a contradiction.

Case (ii). $\{j, j'\} \cap \{k, k+1\} = \emptyset$. Assume that $j < j'$, and for some $i \in N$ and $\succsim^*_i \in \mathcal{D}^i$, $\omega_{j'} \succ^*_i \omega_j \succ^*_i \omega_i$. By *anonymity* of $\mathcal{D}^N$, there is $\succsim^*_{k+1} \in \mathcal{D}^{k+1}$ such that $\omega_{j'} \succ^*_{k+1} \omega_j \succ^*_{k+1} \omega_{k+1}$. Let $\succsim \in \mathcal{D}^N$ such that $\succsim_{k+1} = \succsim^*_{k+1}$, for each $i' \neq k+1$, $\succsim_{i'} \in \mathcal{D}^{i'}_{\{k,k+1\}}$,

$$\omega_j \succ_k \omega_{j'} \succ_k \omega_k \succ_k \omega_{k+1},$$
$$\omega_{k+1} \succ_j \omega_k \succ_j \omega_j \succ_j \omega_{j'},$$
$$\omega_k \succ_{j'} \omega_{k+1} \succ_{j'} \omega_{j'} \succ_{j'} \omega_j,$$

and for each $i'' \notin \{j, j', k, k+1\}$, $\omega \in \{\omega_j, \omega_{j'}, \omega_k, \omega_{k+1}\}$, $\omega_{i''} \succ_{i''} \omega$. Since $\mathcal{D}^N$ is a rich domain, $\mathcal{D}^N$ satisfies *individuality*. Hence, $\mathcal{D}^N$ satisfies the conditions of Lemma 2, and there is no rule that satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in $\mathcal{D}^N$. ☐

Before proceeding with the proof of Theorem 2, we need to introduce some useful properties and definitions. Consider a domain of preferences $\mathcal{D}^N$. For completeness, we present minimal properties of preference domains.

Assumption A (Sönmez, 1999). For each $i \in N$, each $\succsim_i \in \mathcal{D}^i$, and each $a \in \mathcal{A}$, $a_i \sim_i \omega_i$ if and only if $a_i = \omega_i$.

Assumption B (Sönmez, 1999). For each $i \in N$, each $\succsim_i \in \mathcal{D}^i$, and each $a \in \mathcal{A}$ with $a_i \succsim_i \omega_i$ there is $\succsim' \in \mathcal{D}^i$ such that

(i) For each $b \in \mathcal{A} \setminus \{a\}$, $b_i \succsim a_i$ if and only if $b_i \succsim'_i a_i$,

(ii) For each $b \in \mathcal{A} \setminus \{a\}$, $a_i \succsim b_i$ if and only if $a_i \succsim'_i b_i$,

(iii) For each $b \in \mathcal{A} \setminus \{a\}$, $a_i \succ_i b_i$ if and only if $a_i \succsim'_i b_i$, and $a_i \succsim'_i \omega_i \succsim'_i b_i$-

Let $a \in \mathcal{A}$ and $\succsim \in \mathcal{P}^N$. Let $C \subseteq N$, we say that the coalition C weakly blocks assignment $a \in \mathcal{A}$ at profile $\succsim$ if there is an assignment $b \in \mathcal{A}$ such that for each $i \in C$, there is $i' \in C$ with $b_i = \omega_{i'}$, for each $j \in C$, $b_j \succsim_j a_j$, and for some $j' \in C$, $b_{j'} \succ_{j'} a_{j'}$.

For each $\succsim \in \mathcal{R}^N$, the assignment $a \in \mathcal{A}$ is a ***core assignment*** at profile $\succsim$ if there is no coalition C that weakly blocks a at $\succsim$. The set of core assignments is ***essentially single-valued*** at profile $\succsim$ if for each two core assignments a, a' at profile $\succsim$ and each agent $i \in N$, $a_i \sim_i a'_i$.

Note that for pairwise exchange problems with strict preferences, if for some $a \in \mathcal{A}$ and $\succsim \in \mathcal{P}^N$ there is a coalition C such that C weakly blocks a, then there is pair of agents $i, j \in C$ such that $\{i, j\} \subseteq C$ that also weakly blocks a.

Proof of Theorem 2. By Theorem 1, the k-reversal adjusted priority rule satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in $\mathcal{D}^N_{\{k,k+1\}}$, which proves sufficiency. Thus, we focus on the converse statement.

Let φ be the k-reversal adjusted priority rule. First we check that for each $\succsim \in \mathcal{D}^N_{\{k,k+1\}}$, $\varphi(\succsim)$ is a core assignment. Consider agent 1, and let $i \in N$ be the agent such that $\varphi_1(\succsim) = \omega_i$. If $1 \notin \{k, k+1\}$, since 1 and i obtain their best preferred assignment in the set of individually rational assignments, then there is no $a \in \mathcal{A}$ such that $\{1, i\}$ blocks $\varphi(\succsim)$ with a. Moreover, agents 1 and agent i cannot belong to any coalition formed by two agents that blocks $\varphi(\succsim)$. If $1 \in \{k, k+1\}$, and either $\text{top}(\mathcal{M}^*_0, \succsim_1) \neq \text{top}(\mathcal{M}^*_0, \succsim_2)$, or $\text{top}(\mathcal{M}^*_0, \succsim_1) = \text{top}(\mathcal{M}^*_0, \succsim_2) = \{\omega_i\}$ and $\omega_1 \succ_i \omega_2$, the same argument applies. Finally, assume that $1 \in \{k, k+1\}$, and $\text{top}(\mathcal{M}^*_0, \succsim_1) = \text{top}(\mathcal{M}^*_0, \succsim_2) = \{\omega_j\}$, and $\omega_2 \succ_j \omega_1$. In this case, agent 2 and agent j cannot belong to any coalition formed by two agents that blocks $\varphi(\succsim)$ because they both obtain their best preferred assignment in the set of individually rational assignments. The argument can be iterated for every pair of agents matched along subsequent steps of the reversal adjusted priority algorithm, to show that no pair of agents blocks $\varphi(\succsim)$. Hence, $\varphi(\succsim)$ is a core assignment, and for each $\succsim \in \mathcal{D}^N_{\{k,k+1\}}$ there always exists at least one core assignment. Note that, for each $i \in N$ and $\succsim_i \in \mathcal{D}^i_{\{k,k+1\}}$, since $\mathcal{D}^i_{\{k,k+1\}} \subset \mathcal{P}$, for each $a \in \mathcal{A}$, $a_i \sim_i \omega_i$ if and only if $a_i = \omega_i$. Moreover, $\mathcal{D}^N_{\{k,k+1\}}$ satisfies *individuality*. Hence, $\mathcal{D}^N_{\{k,k+1\}}$ satisfies Assumptions A and B on domains of preferences proposed by Sönmez (1999). By Sönmez (1999, Theorem 1), if there is a rule that satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in $\mathcal{D}^N_{\{k,k+1\}}$, then the rule selects from the set of core assignments for each preference profile and the set of core assignments is essentially single-valued. Indeed, since preferences are strict and agents only care about the object they are assigned, the set of core assignments being essentially single-valued implies that this set is single-valued. Therefore, the rule that selects the unique core assignment is the only rule that satisfies *individual rationality*, *efficiency*, and *strategy-proofness*. Takamiya (2003, Theorem 1). By Theorem 1, the k-reversal adjusted priority rule satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in $\mathcal{D}^N_{\{k,k+1\}}$, which suffices to prove necessity. □

Proof of Theorem 3. By definition every minimal reversal domain satisfies *sovereignty* and *pairwise individuality*. By Theorem 1, every minimal reversal domain is a maximal rich domain for *individual rationality*, *efficiency*, and *strategy-proofness*. Hence, we focus on showing that every maximal rich domain for *individual rationality*, *efficiency*, and *strategy-proofness* that satisfies *sovereignty* and *pairwise individuality* is a minimal reversal domain.

Let $\mathcal{D}^N = \times_{i\in N}\mathcal{D}^i$ be a rich domain that satisfies sovereignty and *pairwise individuality* and such that there is a rule $\varphi : \mathcal{D}^N \rightarrow \mathcal{A}$ that satisfies *individual rationality*, *efficiency*, and *strategy-proofness*. We proceed by a series of steps that use the arguments in the proofs of Lemmata 1 and 2.

Step 1. Let $i, k, k' \in N$ such that for some $\succsim_i, \succsim'_i \in \mathcal{D}^i$, $\omega_k \succ_i \omega_{k'} \succ_i \omega_i$ and $\omega_{k'} \succ'_i \omega_k \succ'_i \omega_i$. If there is $j \in N \setminus \{i,k\}$ and $\succsim_j \in \mathcal{D}^j$, such that $\omega_k \succ_j \omega_i \succ_j \omega_j$, then there is no $\succsim'_j \in \mathcal{D}^j$ with $\omega_i \succ'_j \omega_k \succ'_j \omega_j$.

Assume to the contrary that there is $j \in N \setminus \{i,k\}$ and $\succsim_j, \succsim'_j \in \mathcal{D}^j$, such that $\omega_k \succ_j \omega_i \succ_j \omega_j$ and $\omega_i \succ'_j \omega_k \succ'_j \omega_j$. By *anonymity*, there are $\succsim_{k'}, \succsim'_{k'} \in \mathcal{D}^{k'}$ such that $\omega_k \succ_{k'} \omega_i \succ_{k'} \omega_{k'}$ and $\omega_i \succ'_{k'} \omega_k \succ'_{k'} \omega_{k'}$. By *sovereignty*, there is a $\succsim_k \in \mathcal{D}^k$ such that for each $\omega \in \Omega$, $\omega \succsim_k \omega_k$. We have two cases either $\omega_{k'} \succ_k \omega_i \succ_k \omega_k$ or $\omega_i \succ_k \omega_{k'} \succ_k \omega_k$. We assume that $\omega_{k'} \succ_k \omega_i \succ_k \omega_k$, the other case follows from a parallel argument. By *sovereignty*, for each $i' \in N \setminus \{i,k,k'\}$ there is $\succsim_{i'} \in \mathcal{D}^{i'}$ such that for each $\omega \in \{\omega_i, \omega_k, \omega_{k'}\}$, $\omega_{i'} \succ_{i'} \omega$. Hence, there is $\succsim^* \in \mathcal{D}^N$ such that

$$
\begin{aligned}
&\omega_k \succ^*_i \omega_{k'} \succ^*_i \omega_i,\\
&\omega_{k'} \succ^*_k \omega_i \succ^*_k \omega_k,\\
&\omega_i \succ^*_{k'} \omega_k \succ^*_{k'} \omega_{k'},
\end{aligned}
$$

and for each $i' \in N \setminus \{i,k,k'\}$ and each $\omega \in \{\omega_i, \omega_k, \omega_{k'}\}$, $\omega_{i'} \succ^*_{i'} \omega$. Since $\mathcal{D}^N$ is rich, $\mathcal{D}^N$ satisfies *individuality*. By Lemma 1, there is no rule defined on $\mathcal{D}^N$ that satisfies *individual rationality*, *efficiency*, and *strategy-proofness*, which is a contradiction.

A parallel argument proves the reverse step.

Step 2. Let $i, k, k' \in N$ such that for some $\succsim_i, \succsim'_i \in \mathcal{D}^i$, $\omega_k \succ_i \omega_{k'} \succ_i \omega_i$ and $\omega_{k'} \succ'_i \omega_k \succ'_i \omega_i$. If for some $j \in N \setminus \{i,k\}$ and $\succsim_j \in \mathcal{D}^j$, $\omega_i \succ_j \omega_k \succ_j \omega_j$, then there is no $\succsim'_j \in \mathcal{D}^j$ with $\omega_k \succ'_j \omega_i \succ'_j \omega_j$.

Next, we show that if for some agent i there are two objects ω_k, ω'_k and two preferences in the domain $\mathcal{D}^i$ such that both objects are preferred to ω_i but the order in which those objects is reversed, then there is no preference for which agent i ranks another object between ω_k and $\omega_{k'}$ and such object is preferred to ω_i.

Step 3. Let $i, k, k' \in N$ such that for some $\succsim_i, \succsim'_i \in \mathcal{D}^i$, $\omega_k \succ_i \omega_{k'} \succ_i \omega_i$ and $\omega_{k'} \succ'_i \omega_k \succ'_i \omega_i$. For no $j \in N \setminus \{i,k,k'\}$ there are $\hat{\succsim}_i, \hat{\succsim}'_i \in \mathcal{D}^i$ such that $\omega_k \hat{\succ}_i \omega_j \hat{\succ}_i \omega_i$ and $\omega_j \hat{\succ}' \omega_{k'} \hat{\succ}'_i \omega_i$.

Assume to the contrary that there are $j \in N \setminus \{i,k,k'\}$ and $\hat{\succsim}_i, \hat{\succsim}'_i \in \mathcal{D}^i$ such that $\omega_k \hat{\succ}_i \omega_j \hat{\succ}_i \omega_i$, and $\omega_j \hat{\succ}'_i \omega_{k'} \hat{\succ}'_i \omega_i$. By *anonymity*, there are $\succsim_j \in \mathcal{D}^j$, $\succsim_k \in \mathcal{D}^k$, and $\succsim_{k'} \in \mathcal{D}^{k'}$ such that

$$
\begin{aligned}
&\omega_{k'} \succ_j \omega_k \succ_j \omega_j,\\
&\omega_j \succ_k \omega_{k'} \succ_k \omega_k,\\
&\omega_k \succ_{k'} \omega_j \succ_{k'} \omega_{k'}.
\end{aligned}
$$

By *sovereignty*, for each $i' \in N \setminus \{j,k,k'\}$ there is $\succsim_{i'} \in \mathcal{D}^{i'}$ such that for each $\omega \in \{\omega_j, \omega_k, \omega_{k'}\}$, $\omega_{i'} \succ_{i'} \omega$. Since $\mathcal{D}^N$ is rich, $\mathcal{D}^N$ satisfies *individuality*. By Lemma 1, there is no rule defined on $\mathcal{D}^N$ that satisfies *individual rationality*, *efficiency*, and *strategy-proofness*, which is a contradiction.

Next, we show that each agent's domain may contain preferences that reverse the order of at most two objects that are preferred to the agent initially owned object.

Step 4. Assume there are $i, k, k' \in N$ such that for some $\succsim_i, \succsim'_i \in \mathcal{D}^i$, $\omega_k \succ_i \omega_{k'} \succ_i \omega_i$ and $\omega_{k'} \succ'_i \omega_k \succ'_i \omega_i$. If for some pair $\{j, j'\} \subseteq N \setminus \{i\}$ such that $\{j, j'\} \neq \{k, k'\}$, there is $\succsim^*_i \in \mathcal{D}^i$ such that $\omega_j \succ^*_i \omega_{j'} \succ^*_i \omega_i$, then there is no $\succsim''_i \in \mathcal{D}^i$ such that $\omega_{j'} \succ''_i \omega_j \succ''_i \omega_i$.

Assume to the contrary that there are $i, j, j', k, k' \in N$ with $\{j, j'\} \neq \{k, k'\}$ such that for some $\succsim_i, \succsim'_i \in \mathcal{D}^i$, $\omega_k \succ_i \omega_{k'} \succ_i \omega_i$ and $\omega_{k'} \succ'_i \omega_k \succ'_i \omega_i$, there are $\succsim^*_i, \succsim''_i \in \mathcal{D}^i$ such that $\omega_j \succ^*_i \omega_{j'} \succ^*_i \omega_i$ and $\omega_{j'} \succ''_i \omega_j \succ''_i \omega_i$.

Consider first the case $\{j, j'\} \cap \{k, k'\} \neq \varnothing$. Without loss of generality, let $j = k$. By *sovereignty*, there is $\hat{\succsim}_i \in \mathcal{D}^i$ such that for each $\omega \in \Omega \setminus \{\omega_i\}$, $\omega \hat{\succ}_i \omega_i$. Assume that $\omega_{j'} \hat{\succ}_i \omega_{k'}$, a parallel argument applies in the case $\omega_{k'} \hat{\succ}_i \omega_{j'}$. By *anonymity*, there are $\succsim_j \in \mathcal{D}^j$, $\succsim_{j'} \in \mathcal{D}^{j'}$, and $\succsim_{k'} \in \mathcal{D}^{k'}$ such that:

$$
\begin{aligned}
&\omega_{j'} \succ_j \omega_{k'} \succ_j \omega_j,\\
&\omega_{k'} \succ_{j'} \omega_j \succ_{j'} \omega_{j'},\\
&\omega_j \succ_{k'} \omega_{j'} \succ_{k'} \omega_{k'}.
\end{aligned}
$$

By *sovereignty*, for each $i' \in N \setminus \{j,k,k'\}$ there is $\succsim_{i'} \in \mathcal{D}^{i'}$ such that for each $\omega \in \{\omega_i, \omega_k, \omega_{k'}\}$, $\omega_{i'} \succ_{i'} \omega$. Since $\mathcal{D}^N$ is rich, $\mathcal{D}^N$ satisfies *individuality*. By Lemma 1, there is no rule defined on $\mathcal{D}^N$ that satisfies *individual rationality*, *efficiency*, and *strategy-proofness*, which is a contradiction.

Finally, assume that $\{j, j'\} \cap \{k, k'\} = \varnothing$. By *anonymity*, there are $\succsim_j \in \mathcal{D}^j$, $\succsim_{j'} \in \mathcal{D}^{j'}$, $\succsim_k \in \mathcal{D}^k$, and $\succsim_{k'} \in \mathcal{D}^{k'}$ such that

$$
\begin{aligned}
&\omega_k \succ_j \omega_{k'} \succ_j \omega_j,\\
&\omega_{k'} \succ_{j'} \omega_k \succ_{j'} \omega_{j'},\\
&\omega_j \succ_k \omega_{j'} \succ_k \omega_k,\\
&\omega_{j'} \succ_{k'} \omega_j \succ_{k'} \omega_{k'}.
\end{aligned}
$$

By *pairwise individuality*, there are $\bar{\succsim}_j \in \mathcal{D}^j$, $\bar{\succsim}_{j'} \in \mathcal{D}^{j'}$, $\bar{\succsim}_k \in \mathcal{D}^k$, and $\bar{\succsim}_{k'} \in \mathcal{D}^{k'}$, such that:

$$
\begin{aligned}
&\{\omega \mid \omega \bar{\succ}_j \omega_j\} = \{\omega \mid \omega \bar{\succ}_{j'} \omega_{j'}\} = \{\omega_k, \omega_{k'}\},\\
&\{\omega \mid \omega \bar{\succ}_k \omega_k\} = \{\omega \mid \omega \bar{\succ}_{k'} \omega_{k'}\} = \{\omega_j, \omega_{j'}\},
\end{aligned}
\quad \text{and} \quad
\begin{aligned}
&\omega_k \bar{\succ}_j \omega_{k'}, \quad \omega_{k'} \bar{\succ}_{j'} \omega_k,\\
&\omega_j \bar{\succ}_k \omega_{j'}, \quad \omega_{j'} \bar{\succ}_{k'} \omega_j.
\end{aligned}
$$

By *sovereignty*, there is $\succsim^* \in \mathcal{D}^N$ such that $\succsim^*_j = \bar{\succsim}_j$, $\succsim^*_{j'} = \bar{\succsim}_{j'}$, $\succsim^*_k = \bar{\succsim}_k$, and $\succsim^*_{k'} = \bar{\succsim}_{k'}$, and for each $i \in N \setminus \{j, j', k, k'\}$ and each $\omega \in \Omega \setminus \{\omega_j . \omega_{j'}, \omega_k, \omega_{k'}\}$, $\omega_i \succ^*_i \omega$. Since $\mathcal{D}^N$ is rich, $\mathcal{D}^N$ satisfies *individuality*. By Lemma 2, there is no rule defined on $\mathcal{D}^N$ that satisfies *individual rationality*, *efficiency*, and *strategy-proofness*, which is a contradiction.

We proceed by showing that the objects whose order may be reversed are the same for all agents.

Step 5. Assume there are $i, j \in N$, $\{k, k'\}$, $\{\hat{k}, \hat{k}'\}$, $\succsim_i, \succsim'_i \in \mathcal{D}^i$, $\succsim_j, \succsim'_j \in \mathcal{D}^i$, such that

$$
\begin{aligned}
&\omega_k \succ_i \omega_{k'} \succ_i \omega_i,\\
&\omega_{k'} \succ'_i \omega_k \succ'_i \omega_i,\\
&\omega_{\hat{k}} \succ_j \omega_{\hat{k}'} \succ_j \omega_j,\\
&\omega_{\hat{k}'} \succ'_j \omega_{\hat{k}} \succ'_j \omega_j,
\end{aligned}
$$

then $\{k, k'\} = \{\hat{k}, \hat{k}'\}$.

Assume to the contrary that $\{k, k'\} \neq \{\hat{k}, \hat{k}'\}$. Consider first the case $\{k, k'\} \cap \{\hat{k}, \hat{k}'\} \neq \varnothing$, and without loss of generality, let $k' = \hat{k}$. If $n = 3$, then we obtain an immediate contradiction with either Step 1 or Step 2. If $n \geq 4$, then by *sovereignty*, there is $\succsim_{k'} \in \mathcal{D}^{k'}$ such that for each $\omega \in \Omega \setminus \{\omega_{k'}\}$, $\omega \succ_{k'} \omega_{k'}$. Without loss of generality assume that $\omega_{\hat{k}'} \succ_{k'} \omega_k \succ_{k'} \omega_{k'}$. By *anonymity*, there are $\succsim_k \in \mathcal{D}^k$ and $\succsim_{\hat{k}'} \in \mathcal{D}^{\hat{k}'}$ such that $\omega_{k'} \succ_k \omega_{\hat{k}'} \succ_k \omega_k$, and $\omega_k \succ_{\hat{k}'} \omega_{k'} \succ_{\hat{k}'} \omega_{\hat{k}'}$. By *sovereignty* for every $i \in N \setminus \{k, k', \hat{k}'\}$, there is $\succsim_i \in \mathcal{D}^i$ such that for each $\omega \in \{\omega_k, \omega_{k'}, \omega_{\hat{k}'}\}$, $\omega_i \succ_i \omega$. Since $\mathcal{D}^N$ is rich, $\mathcal{D}^N$ satisfies *individuality*. By Lemma 1, there is no rule defined on $\mathcal{D}^N$ that satisfies *individual rationality*, *efficiency*, and *strategy-proofness*, which is a contradiction.

If $\{k,k'\}\cap\{\hat{k},\hat{k}'\}=\emptyset$, then similar arguments apply. If $n=4$, then we can replicate the arguments in case $\{k,k'\}\cap\{\hat{k},\hat{k}'\}=\emptyset$ of Step 4 and invoke Lemma 2 to reach the desired contradiction. If $n\geq 5$, then there is $i'\in N\setminus\left(\{k,k'\}\cup\{\hat{k},\hat{k}'\}\right)$, which by *anonymity* contradicts Step 4.

In the next steps, we conclude by showing that the preferences of the agents over objects preferred to each agent's initially owned object are consistent with the existence of a common ranking of the objects .

Step 6. If there is no agent $i\in N$ such that for each two objects $\omega,\omega'\in\Omega\setminus\{\omega_i\}$, and $\succsim_i,\succsim_i'\in\mathcal{D}^i$ such that $\omega\succ_i\omega'\succ_i\omega_i$ and $\omega'\succ_i'\omega\succ_i'\omega_i$ then there is a complete, transitive and antisymmetric binary relation on N, $\bar{\lesssim}$ such that for each $j\in N$, and each pair $\{j',j''\}\subset N$, with $j\notin\{j',j''\}$, for each $\succsim_j\in\mathcal{D}^j$, $j'\bar{\lesssim}j''$, $\omega_{j'}\succ_j\omega_j$, and $\omega_{j''}\succ_j\omega_j$ imply $\omega_{j'}\succ_j\omega_{j''}\succ_j\omega_j$.

By *sovereignty*, there is $\succsim\in\mathcal{D}^N$ such that for each $i\in N$, and each $\omega\in\Omega\setminus\{\omega_i\}$, $\omega\succ_i\omega_i$. We consider two cases:

Case (i). $n=3$. Let $N=\{i,j,k\}$. Without loss of generality, assume that $\omega_j\succ_i\omega_k\succ_i\omega_i$. Construct the binary relation $\bar{\lesssim}$ on N in such a way that $i\bar{\lesssim}j$ if and only if $\omega_i\succ_k\omega_j$, and for each $j',j''\in\{j,k\}$, $i\bar{\lesssim}j'$ if and only if $\omega_i\succ_{j''}\omega_{j'}$. By construction, the binary relation $\bar{\lesssim}$ is complete and antisymmetric. By assumption, $j\bar{\lesssim}k$. To check that $\bar{\lesssim}$ is transitive, we consider two exhaustive possibilities.

(i) If $\omega_k\succ_j\omega_i\succ_j\omega_j$, then necessarily $\omega_j\succ_k\omega_i\succ_k\omega_k$, otherwise we obtain a contradiction with Lemma 1. Hence, $k\bar{\lesssim}i$ and $j\bar{\lesssim}i$.

(ii) If $\omega_i\succ_j\omega_k\succ_j\omega_j$, then either $\omega_i\succ_k\omega_j\succ_k\omega_k$ or $\omega_j\succ_k\omega_i\succ_k\omega_k$. Hence, $i\bar{\lesssim}k$, and either $i\bar{\lesssim}j$ or $j\bar{\lesssim}i$.

In all the possible cases, $\bar{\lesssim}$ is transitive. Hence, for each $i'\in N$, and each $\omega_{j'},\omega_{k'}\in\Omega\setminus\{\omega_{i'}\}$, $\omega_{j'}\succ_{i'}\omega_{k'}\succ_{i'}\omega_{i'}$ if and only if $j'\bar{\lesssim}k'$.

Case (ii). $n\geq 4$. Let $i\in N$ and define the complete, antisymmetric, and transitive binary relation $\bar{\lesssim}_i$ on $N\setminus\{i\}$ by $j\bar{\lesssim}_i j'$ if and only if $\omega_j\succ_i\omega_{j'}$. For each $j,k\in N\setminus\{i\}$ construct analogously the binary relations $\bar{\lesssim}_j$ on $N\setminus\{j\}$ and $\bar{\lesssim}_k$ on $N\setminus\{k\}$. Since agents do not reverse the preference for no pair of objects and by *anonymity*, for each $\{k',k''\}\subset N\setminus\{i,j\}$ if there is $\succsim_i'\in\mathcal{D}^i$ such that $\omega_{k'}\succ_i'\omega_{k''}\succ_i'\omega_i$, then there is $\succsim_j'\in\mathcal{D}^j$ such that $\omega_{k'}\succ_j'\omega_{k''}\succ_j'\omega_j$, then $k'\bar{\lesssim}_i k''$ if and only if $k'\bar{\lesssim}_j k''$. Define the binary relation $\bar{\lesssim}$ on N by for each $\{i',i''\}\subseteq N\setminus\{i\}$, $i'\bar{\lesssim}i''$ if and only if $i'\bar{\lesssim}_i i''$, for each $\{j',j''\}\subseteq N\setminus\{j\}$, $j'\bar{\lesssim}j''$ if and only if $j'\bar{\lesssim}_j j''$, and for each $\{k',k''\}\subseteq N\setminus\{k\}$, $k'\bar{\lesssim}k''$ if and only if $k'\bar{\lesssim}_k k''$. Note that $\bar{\lesssim}$ is by construction complete, and antisymmetric. By agents' preferences transitivity and the arguments employed in Case (i) to determine the comparisons for i,j and k, $\bar{\lesssim}$ is transitive. Hence, for each $i'\in N$, and each $\omega_{j'},\omega_{k'}\in\Omega\setminus\{\omega_{i'}\}$, $\omega_{j'}\succ_{i'}\omega_{k'}\succ_{i'}\omega_{i'}$ if and only if $j'\bar{\lesssim}k'$.

Step 7. If there are $i,k,k'\in N$ and $\succsim_i,\succsim_i'\in\mathcal{D}^i$ such that

$$\omega_k\succ_i\omega_{k'}\succ_i\omega_i,$$
$$\omega_{k'}\succ_i'\omega_k\succ_i'\omega_i,$$

then there is a complete, transitive and antisymmetric binary relation $\bar{\lesssim}$ on N such that for each $j\in N$, and each pair $\{j',j''\}\neq\{k,k'\}$, with $j\notin\{j',j''\}$, for each $\succsim_j\in\mathcal{D}^j$, $j'\bar{\lesssim}j''$ implies $\omega_{j'}\succ_j\omega_{j''}\succ_j\omega_j$.

By *sovereignty*, there is $\succsim\in\mathcal{D}^N$ such that for each $i'\in N$, and each $\omega\in\Omega\setminus\{\omega_{i'}\}$, $\omega\succ_{i'}\omega_{i'}$. The arguments are similar to the arguments in Step 6. Again, we have two cases:

Case (i): $N=\{i,j,k\}$. Without loss of generality assume that $\omega_j\succ_i\omega_k\succ_i\omega_i$ and there is $\succsim_i'\in\mathcal{D}^i$ such that $\omega_k\succ_i'\omega_j\succ_i'\omega_i$. Define the binary relation $\bar{\lesssim}$ on N by $j\bar{\lesssim}k$ if and only if $\omega_j\succ_i\omega_k$, and for each $j',j''\in\{j,k\}$, $i\bar{\lesssim}j'$ if and only if $\omega_i\succ_{j''}\omega_{j'}$. We consider two possibilities:

(i) If $\omega_k\succ_j\omega_i\succ_j\omega_j$, then necessarily $\omega_j\succ_k\omega_i\succ_k\omega_k$, otherwise we obtain a contradiction with Lemma 1. Hence, $k\bar{\lesssim}i$ and $j\bar{\lesssim}i$.

(ii) If $\omega_i\succ_j\omega_k\succ_j\omega_j$, since there is $\succsim_i'\in\mathcal{D}^i$ such that $\omega_k\succ_i'\omega_j\succ_i'\omega_i$, by Lemma 1, then $\omega_i\succ_k\omega_j\succ_k\omega_k$. Hence, $i\bar{\lesssim}k$ and $i\bar{\lesssim}j$.

Thus, replicating the arguments in Case (i) of Step 6, we check that $\bar{\lesssim}$ is complete, antisymmetric, and transitive. Moreover, j and k occupy adjacent positions in $\bar{\lesssim}$.

Case (ii): $n\geq 4$. Note that by Step 4 and Step 5 there are at most two agents k', k'' such that $\omega_{k'}\succ_i\omega_{k''}\succ_i\omega_i$ and for some preference $\succsim_i'\in\mathcal{D}^i$, $\omega_{k''}\succ_i'\omega_{k'}\succ_i'\omega_i$. Construct the binary relation $\bar{\lesssim}_i$ on $N\setminus\{i\}$ in such a way that for each $j',j''\in N\setminus\{i\}$, $j'\bar{\lesssim}_i j''$ if and only if $\omega_{j'}\succ_i\omega_{j''}$. The binary relation $\bar{\lesssim}_i$ is complete, transitive, and antisymmetric. By Step 3, there is no k'' such that $k\bar{\lesssim}_i k''\bar{\lesssim}_i k'$. Consider $j,k\in N\setminus\{i\}$ and construct analogously the relation $\bar{\lesssim}_j$ defined on $N\setminus\{j\}$, and $\bar{\lesssim}_k$ defined on $N\setminus\{k\}$. By Step 5, for each $j',j''\notin\{i,j,k,k'\}$, $j'\bar{\lesssim}_i j''$ if and only if $j'\bar{\lesssim}_j j''$. Construct the binary relation $\bar{\lesssim}$ defined on N in such a way for each $j',j''\in N\setminus\{i\}$, $j'\bar{\lesssim}j''$ if and only if $j'\bar{\lesssim}_i j''$, for each $j'\in N\setminus\{j\}$, $j'\bar{\lesssim}i$ if and only if $j'\bar{\lesssim}_j i$, and $j\bar{\lesssim}i$ if and only if $j\bar{\lesssim}_k i$. Replicating the arguments in Case (ii) of Step 6; $\bar{\lesssim}$ is complete, antisymmetric, and transitive.

By construction, $\bar{\lesssim}$ defines agents' preferences over objects that are preferred to each agent initially owned object but objects k' and k''.

To conclude the proof, let $\succsim\in\mathcal{D}^N$. By Step 6 and 7 there is an order of the agents $\bar{\lesssim}$ and at most two agents k',k'' occupying adjacent positions in the order $\bar{\lesssim}$ such that for each $i\in N$, $\succsim_i\in\mathcal{D}^N$, and each $j',j''\in N\setminus\{i\}$, $\{j,j'\}\neq\{k',k''\}$, with $\omega_{j'}\succ_i\omega_i$ and $\omega_{j''}\succ_i\omega_i$, $j'\bar{\lesssim}j''$ if and only if $\omega_{j'}\succ_i\omega_{j''}$. Therefore, $\succsim\in\mathcal{D}^N_{\{k',k''\}}$. That is, $\mathcal{D}^N$ is contained in a minimal reversal domain. By Theorem 1, minimal reversal domain are maximal rich domains for *individual rationality*, *efficiency*, and *strategy-proofness*. Hence, if $\mathcal{D}^N$ is maximal, then $\mathcal{D}^N$ is a minimal reversal domain. ☐

Proof of Proposition 1. By construction, every *k-reversal adjusted priority rule with indifferences* satisfies *individual rationality*. Note that at each stage $t<n$ of the adjusted priority algorithm, a temporary match is proposed for agents t and $\mu(t)$ (the same argument applies for $t\in\{k,k+1\}$). Both t and $\mu(t)$ are tentatively assigned the best object available at stage t. This tentative match may change if there is another match under which they are both indifferent but allows an improvement for the next agent in the priority order. Since agents cannot be indifferent among three objects, the algorithm is well-defined and the *k-reversal adjusted priority rule with indifferences* satisfies *efficiency*. Since ties are broken using the natural order and do not depend on agents preferences, the arguments in the proof of Theorem 1 apply to check that by misrepresenting her preferences an agent cannot obtain a preferred object. Therefore, *k-reversal adjusted priority rule with indifferences* satisfies *strategy-proofness*. ☐

Data availability

No data was used for the research described in the article.

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