

# Some empirical formulae for the degeneracy separation in the Clebsch-Gordan problem of $\mathfrak{su}(3)$

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**Abstract.** We propose an algorithmic prescription to isolate degenerate multiplets in the tensor product of irreducible  $\mathfrak{su}(3)$ -representations based on external labelling operators and illustrate how to obtain empirical formulae for their eigenvalue spectrum, allowing a degeneracy separation.

*Dedicated to the Centennial of I.E. Segal (1918-1998)*

## 1. Introduction

The decomposition of tensor products of irreducible representations of semisimple Lie algebras  $\mathfrak{s}$ , also known as the Clebsch-Gordan series for  $\mathfrak{s}$ , belongs nowadays to the standard techniques in group theoretical applications to physical phenomena, and has been widely studied in the literature [1–7]. For the classical Lie algebras, quite generic results on the decomposition of tensor products and the multiplicities of the representations have been obtained [8–12], which are unfortunately not always of easy practical use, as most of the formulae use the root system and the Weyl group of a simple Lie algebra. This has the inconvenience that for a non-root based basis of the Lie algebra (as usually encountered in physics) these results are quite unwieldy [13, 14]. Specially for the case of unitary groups, more explicit formulae suitable for practical computations have been obtained by various authors [15–18] by other methods.

Due to its structural simplicity, the Clebsch-Gordan series for  $\mathfrak{su}(3)$  can also be directly analyzed in terms of the so-called external labelling problem (see e.g. [19, 20]) corresponding to the non-canonical embedding of Lie algebras  $\mathfrak{su}(3) \subset \mathfrak{su}(3) \oplus \mathfrak{su}(3)$ . Using the trace method [21, 22], one admissible complete set of commuting labelling operators is obtained, from which appropriate linear combinations suitable for separating degeneracies in the Clebsch-Gordan series of  $\mathfrak{su}(3)$  are deduced. Using the explicit decomposition formula for the tensor product of  $\mathfrak{su}(3)$ -representations developed in [15, 23], various properties of the spectrum of labelling operators are analyzed, and explicit empirical formulae for their eigenvalues on tensor products of the form  $[1, \ell] \otimes [1, k]$  and  $[1, \ell] \otimes [k, 1]$  are obtained for arbitrary  $\ell, k \geq 1$ .



## 2. Missing label operators

We recall that for any semisimple Lie algebra  $\mathfrak{s}$  of rank  $\mathcal{N}(\mathfrak{s}) = l$  the total number of labels to specify (finite-dimensional) irreducible representations (IRs in short) unambiguously is given by

$$\Xi(\mathfrak{s}) = \frac{1}{2}(\dim \mathfrak{s} + \mathcal{N}(\mathfrak{s})). \quad (1)$$

From these,  $l$  labels correspond to the eigenvalues of the Casimir operators of  $\mathfrak{s}$  [24], so that within an IR, the number of internal labels separating the states is

$$\chi(\mathfrak{s}) = \frac{1}{2}(\dim \mathfrak{s} - \mathcal{N}(\mathfrak{s})). \quad (2)$$

It is often convenient to label irreducible representations  $\Gamma$  of  $\mathfrak{s}$  using some (semisimple) subalgebra  $\mathfrak{s}'$  corresponding to an internal symmetry [20]. In this situation, the subalgebra will provide  $\frac{1}{2}(\dim \mathfrak{s}' + \mathcal{N}(\mathfrak{s}')) - l_0$  labels, where  $\mathcal{N}(\mathfrak{s}')$  is the number of invariants of  $\mathfrak{s}'$  and  $l_0$  denotes the number of common invariants of  $\mathfrak{s}$  and  $\mathfrak{s}'$  [24]. It follows that to separate the IRs of  $\mathfrak{s}'$  appearing with multiplicity greater than one in the decomposition of  $\Gamma$ , additional

$$n = \frac{1}{2}(\dim \mathfrak{s} - \mathcal{N}(\mathfrak{s}) - \dim \mathfrak{s}' - \mathcal{N}(\mathfrak{s}')) + l_0 \quad (3)$$

operators, called either missing label operators or subgroup scalars (MLO in short), are required. The total number of available operators of this kind is  $m = 2n$ . For  $n > 1$ , the labelling operators must commute with each other to avoid non-trivial interactions [19].

## 3. The non-canonical embedding $\mathfrak{su}(3) \subset \mathfrak{su}(3) \oplus \mathfrak{su}(3)$

In order to construct a basis of  $\mathfrak{su}(3)$ , we consider the three-dimensional quark representation  $[1, 0]$  defined by the constraint

$$\text{Tr } \mathbf{M} = 0, \quad (4)$$

where  $\mathbf{M}$  is a  $3 \times 3$  complex matrix. If  $E_{\alpha\beta}$  denotes the elementary matrix with entries  $(E_{\alpha\beta})_{jk} = \delta_\alpha^j \delta_\beta^k$ , we take  $T_{\alpha\beta}$  with  $1 \leq \alpha \neq \beta \leq 3$  and the diagonal matrices  $T_{11} = E_{11} - E_{33}$  and  $T_{22} = E_{22} - E_{33}$  spanning the Cartan subalgebra as basis generators. For this choice of generators, the commutator table is given by

$[\diamond, \diamond]$	$T_{11}$	$T_{22}$	$T_{12}$	$T_{13}$	$T_{23}$	$T_{21}$	$T_{31}$	$T_{32}$
$T_{11}$	0	0	$T_{12}$	$2T_{13}$	$T_{23}$	$-T_{21}$	$-2T_{31}$	$-T_{32}$
$T_{22}$		0	$-T_{12}$	$T_{13}$	$2T_{23}$	$T_{21}$	$-T_{31}$	$-2T_{32}$
$T_{12}$			0	0	$T_{13}$	$T_{11} - T_{22}$	$-T_{32}$	0
$T_{13}$				0	0	$-T_{23}$	$T_{11}$	$T_{12}$
$T_{23}$					0	0	$T_{21}$	$T_{22}$
$T_{21}$						0	0	$-T_{31}$
$T_{31}$							0	0
$T_{32}$								0

(5)

It is clear from this table that for any representation  $\Gamma$  of  $\mathfrak{su}(3)$ , it will be sufficient to determine the matrix elements for the generators  $\{T_{13}, T_{23}, T_{31}, T_{32}\}$ , as the remaining generators follow from the commutators. Indeed,

$$T_{jk} = [T_{j3}, T_{3k}], \quad 1 \leq j, k \leq 2. \quad (6)$$

The two primitive Casimir operators of  $\mathfrak{su}(3)$  are obtained from the (symmetrized) traces of the matrix polynomial in the generators

$$\mathbf{L}(T) = \begin{pmatrix} (2T_{11} - T_{22})/3 & T_{12} & T_{13} \\ T_{21} & (-T_{11} + 2T_{22})/3 & T_{23} \\ T_{31} & T_{32} & -(T_{11} + T_{22})/3 \end{pmatrix}. \quad (7)$$

Specifically, as invariants we consider the symmetrization of the polynomials

$$C_2 = \text{Tr } \mathbf{L}^2 = \frac{2}{3} (T_{11}^2 + T_{22}^2 + T_{11}T_{22}) + 2 (T_{12}T_{21} + T_{13}T_{31} + T_{23}T_{32}), \quad (8)$$

$$C_3 = \frac{-\text{Tr } \mathbf{L}^3}{3} = -\frac{2}{27} (T_{11}^3 + T_{22}^3) + \frac{1}{9} (T_{11}^2T_{22} + T_{11}T_{22}^2) - \frac{1}{3} T_{11} (T_{12}T_{21} + T_{13}T_{31} - 2T_{23}T_{32}) \\ - \frac{1}{3} T_{22} (T_{12}T_{21} - 2T_{13}T_{31} + T_{23}T_{32}) - T_{12}T_{23}T_{31} - T_{21}T_{13}T_{32}. \quad (9)$$

For any irreducible representation  $\Gamma = [\lambda, \mu]$  of  $\mathfrak{su}(3)$  with  $\lambda, \mu \geq 0$ , the eigenvalues of the Casimir operators are accordingly given by:

$$C_2(\Gamma) = \frac{2}{3} (\lambda^2 + \lambda\mu + \mu^2 + 3\lambda + 3\mu), \\ C_3(\Gamma) = \frac{1}{27} (\mu - \lambda) (3 + 2\lambda + \mu) (3 + 2\mu + \lambda). \quad (10)$$

We now consider the Lie algebra  $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$  with a basis  $\{T_{ij}^1, T_{ij}^2\}$  consisting of two copies of the preceding basis  $\{T_{ij}\}$ . To keep a unified notation, we formally write  $T_{33}^a = -(T_{11}^a + T_{22}^a)$ . Then the generators satisfy the commutators

$$[T_{ij}^a, T_{kl}^b] = \delta_a^b (\delta_j^k T_{il}^a - \delta_i^l T_{kj}^a), \quad i, j, k, l = 1, 2, 3; \quad a, b = 1, 2. \quad (11)$$

It is straightforward to verify that the four Casimir operators of  $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$  are obtained from a matrix of type (7) by replacing  $T_{ij}$  by  $T_{ij}^a$  for  $a = 1, 2$ . In order to describe the non-canonical embedding  $\mathfrak{su}(3) \subset \mathfrak{s} = \mathfrak{su}(3) \oplus \mathfrak{su}(3)$ , we consider the change of basis

$$X_{\alpha\beta} = T_{\alpha\beta}^1 + T_{\alpha\beta}^2, \quad Y_{\alpha\beta} = T_{\alpha\beta}^1 - T_{\alpha\beta}^2, \quad 1 \leq \alpha, \beta \leq 3. \quad (12)$$

Over this basis, the brackets of  $\mathfrak{s}$  are given by

$$[X_{\alpha\beta}, X_{\gamma\delta}] = [T_{\alpha\beta}^1, T_{\gamma\delta}^1] + [T_{\alpha\beta}^2, T_{\gamma\delta}^2] = \delta_\beta^\gamma X_{\alpha\delta} - \delta_\alpha^\delta X_{\gamma\beta}, \\ [X_{\alpha\beta}, Y_{\gamma\delta}] = [T_{\alpha\beta}^1, T_{\gamma\delta}^1] - [T_{\alpha\beta}^2, T_{\gamma\delta}^2] = \delta_\beta^\gamma Y_{\alpha\delta} - \delta_\alpha^\delta Y_{\gamma\beta}, \\ [Y_{\alpha\beta}, Y_{\gamma\delta}] = [T_{\alpha\beta}^1, T_{\gamma\delta}^1] - [T_{\alpha\beta}^2, T_{\gamma\delta}^2] = \delta_\beta^\gamma X_{\alpha\delta} - \delta_\alpha^\delta X_{\gamma\beta}. \quad (13)$$

We observe that the operators  $\{X_{\alpha\beta}\}$  generate a subalgebra  $\mathfrak{s}'$  isomorphic to  $\mathfrak{su}(3)$ , while the generators  $\{Y_{\alpha\beta}\}$  transform as a representation of  $\mathfrak{s}'$ . This means in particular that the subalgebra has no invariant in common with  $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$  (see equation (3)).

States within irreducible representations of  $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$  are characterized by  $\frac{16+4}{2} = 10$  labels according to formula (1), from which the values of the four Casimir operators determine the representation. As we are using the subalgebra  $\mathfrak{su}(3)$  generated by the  $X_{\alpha,\beta}$ , and this subalgebra

provides five labels, an additional subgroup scalar must be determined in order to complete the six internal labels required for  $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ -representations (see formulae (2) and (3))

Using the change of basis (12), the invariants of  $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$  in the  $\{X_{\alpha\beta}, Y_{\alpha\beta}\}$  basis are easily obtained using the functional matrix (see also [22])

$$\mathbf{L}^\pm(X, Y) = \begin{pmatrix} \frac{2(X_{11} \pm Y_{11}) - (X_{22} \pm Y_{22})}{3} & (X_{12} \pm Y_{12}) & (X_{13} \pm Y_{13}) \\ (X_{21} \pm Y_{21}) & \frac{-(X_{11} \pm Y_{11}) + 2(X_{22} \pm Y_{22})}{3} & (X_{23} \pm Y_{23}) \\ (X_{13} \pm Y_{13}) & (X_{32} \pm Y_{32}) & -\frac{(X_{11} \pm Y_{11}) + (X_{22} \pm Y_{22})}{3} \end{pmatrix}. \quad (14)$$

Like before, we extract the Casimir operators from the (symmetrized) traces of the matrix, for both signs:

$$C_2^{(1)} = \text{Tr}(\mathbf{L}^+)^2, \quad C_2^{(2)} = \text{Tr}(\mathbf{L}^-)^2, \quad C_3^{(1)} = -\frac{1}{3} \text{Tr}(\mathbf{L}^+)^3, \quad C_3^{(2)} = -\frac{1}{3} \text{Tr}(\mathbf{L}^-)^3. \quad (15)$$

The Casimir operators can thus be seen as homogeneous polynomials in the generators  $\{X_{\alpha\beta}, Y_{\alpha\beta}\}$ . Denoting by  $\Theta^{[p,q]}$  a homogeneous polynomial of degree  $p$  in  $X_{\alpha\beta}$  and degree  $q$  in  $Y_{\alpha\beta}$ , it is immediate to verify that the following decomposition holds ( $a = 1, 2$ ):

$$C_2^{(a)} = \frac{1}{4} \left( \Theta^{[2,0]} \pm \Theta^{[1,1]} + \Theta^{[0,2]} \right), \quad (16)$$

$$C_3^{(a)} = \frac{1}{8} \left( \Theta^{[3,0]} \pm \Theta^{[2,1]} + \Theta^{[1,2]} \pm \Theta^{[0,3]} \right), \quad (17)$$

where the operators  $\Theta^{[2,0]}$  and  $\Theta^{[3,0]}$  can be identified with the Casimir operators of the subalgebra  $\mathfrak{su}(3)$  [25]. The expressions (16)-(17) provide the two further relations

$$C_2^{(1)} + C_2^{(2)} = \frac{1}{2} \left( \Theta^{[2,0]} + \Theta^{[0,2]} \right); \quad C_2^{(1)} - C_2^{(2)} = \frac{1}{2} \Theta^{[1,1]}, \quad (18)$$

$$C_3^{(1)} + C_3^{(2)} = \frac{1}{4} \left( \Theta^{[3,0]} + \Theta^{[1,2]} \right); \quad C_3^{(1)} - C_3^{(2)} = \frac{1}{4} \left( \Theta^{[2,1]} + \Theta^{[0,3]} \right). \quad (19)$$

This implies that the Casimir operators can be expressed in terms of the seven subgroup scalars

$$\mathfrak{S} = \left\{ \Theta^{[2,0]}, \Theta^{[0,2]}, \Theta^{[1,1]}, \Theta^{[3,0]}, \Theta^{[2,1]}, \Theta^{[1,2]}, \Theta^{[0,3]} \right\}. \quad (20)$$

Using the analytical counterpart  $O^{[p,q]}$  of these operators (see e.g. [26]), their independence can be shown by means of the Jacobian matrix

$$\det(J) = \frac{\partial \{O^{[2,0]}, O^{[1,1]}, O^{[0,2]}, O^{[3,0]}, O^{[2,1]}, O^{[1,2]}, O^{[0,3]}\}}{\partial \{x_{11}, x_{22}, x_{12}, x_{13}, y_{12}, y_{13}, y_{23}\}} = x_{11}^2 x_{23}^2 y_{23} x_{31} y_{31}^3 y_{32}^2 + \dots \neq 0, \quad (21)$$

where  $\{x_{ij}, y_{ij}\}$  are the coordinates in  $(\mathfrak{su}(3) \oplus \mathfrak{su}(3))^*$  corresponding to the generators. As can be further easily shown either from the Berezin bracket for the operators  $O^{[p,q]}$  or directly using the commutation relations (5), the operators  $\Theta^{[p,q]}$  commute with each other. As a consequence, one admissible complete set of labelling operators for the multiplicity separation of the components in  $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ -representations can be extracted from  $\mathfrak{S}$ . Discarding the invariants of the Lie algebra and the subalgebra  $\mathfrak{su}(3)$ , we can choose among  $\Theta^{[2,1]}$  and  $\Theta^{[0,3]}$  as the missing label operator [22]. For computational purposes, it is convenient to take the latter, so that a possible set of labelling operators is given by

$$\mathfrak{S}_0 = \left\{ C_2^{(1)}, C_2^{(2)}, C_3^{(1)}, C_3^{(2)}, \Theta^{[2,0]}, \Theta^{[3,0]}, \Theta^{[0,3]} \right\}. \quad (22)$$

To these, three additional inner operators that are taken in the subalgebra  $\mathfrak{su}(3)$  are required, used for the separation of states within each  $\mathfrak{su}(3)$ -representation (see e.g. [19, 26]).

#### 4. Clebsch-Gordan series for $\mathfrak{su}(3)$

As irreducible representations of  $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$  correspond to the tensor products of IRs of  $\mathfrak{su}(3)$  [27], it follows that the operators (22) will enable us to separate the appearing degeneracies, i.e., the irreducible representations in a tensor product with multiplicity greater than one.

For the description of tensor products of (complex) IRs of  $\mathfrak{su}(3)$  there exists an explicit formula due to O'Reilly [15] that also allows to determine the multiplicity of the  $\mathfrak{su}(3)$ -representations intervening in the decomposition. We recall that, given two IRs  $[\lambda, \mu]$  and  $[\rho, \sigma]$ , the tensor product  $R = [\lambda, \mu] \otimes [\rho, \sigma]$  decomposes as

$$R = \sum_{k=0}^{\min\{\lambda+\mu, \sigma\}} \sum_{j=0}^{\min\{\mu, \lambda+\mu-k, \rho\}} \sum_{\max\{0, j+k-\mu\}}^{\min\{\lambda, k+\rho-j\}} [\lambda + \rho + k - j - 2i, \mu + \sigma + i - j - 2k]. \quad (23)$$

This expression can be briefly summarized as

$$[\lambda, \mu] \otimes [\rho, \sigma] = \sum_{k=1}^{s_0} \alpha_k \Gamma_k, \quad (24)$$

where the  $\Gamma_k = [\eta_k, \xi_k]$  are mutually non-isomorphic  $\mathfrak{su}(3)$ -representations for  $k \neq k'$  and the scalar  $\alpha \geq 1$  denotes the multiplicity of  $\Gamma_k$ . Now, using a well-known property of  $\mathfrak{su}(3)$ -tensor products (see [27])

$$\overline{[\lambda, \mu] \otimes [\rho, \sigma]} \simeq \overline{[\lambda, \mu]} \otimes \overline{[\rho, \sigma]} = [\mu, \lambda] \otimes [\sigma, \rho],$$

it follows from (24) that

$$\overline{[\lambda, \mu] \otimes [\rho, \sigma]} = \sum_{k=1}^{s_0} \alpha_k \bar{\Gamma}_k, \quad (25)$$

and thus the multiplicities of  $\Gamma_k$  and  $\bar{\Gamma}_k$  are the same. Further, as can be easily deduced from formula (23), a necessary and sufficient condition for the tensor product  $[\lambda, \mu] \otimes [\rho, \sigma]$  to be multiplicity free is that the constraint  $\lambda\mu\rho\sigma = 0$  is satisfied (see e.g. [17, 22] and references therein). Multiplicities greater than one thus appear for any product such that  $\lambda, \mu, \rho, \sigma \geq 1$ .

Let  $\Gamma = [\lambda, \mu] \otimes [\rho, \sigma]$  be a given IR of  $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ . We now inspect the action of the operators in  $\mathfrak{S}_0$  on  $\Gamma$ . As  $\{C_2^{(1)}, C_2^{(2)}, C_3^{(1)}, C_3^{(2)}\}$  are the Casimir operators of  $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ , it follows from the relations in (18) and (19) that

$$C_k^{(1)}(\Gamma) = C_k([\lambda, \mu]), \quad C_k^{(2)}(\Gamma) = C_k([\rho, \sigma]), \quad k = 2, 3. \quad (26)$$

The operators  $\{\Theta^{[2,0]}, \Theta^{[3,0]}\}$  correspond to the Casimir operators of the subalgebra  $\mathfrak{su}(3)$ , implying that acting on  $\Gamma$ , the result is a block matrix

$$\Theta^{[p,0]}(\Gamma) = \begin{pmatrix} \Theta^{[p,0]}(\Gamma_1) \text{Id}_{\alpha_1 \dim \Gamma_1} & & \\ & \ddots & \\ & & \Theta^{[p,0]}(\Gamma_{s_0}) \text{Id}_{\alpha_{s_0} \dim \Gamma_{s_0}} \end{pmatrix}, \quad (27)$$

with  $p = 2, 3$  and the corresponding eigenvalue  $\Theta^{[p,0]}(\Gamma_k)$  is given by formula (10). So far, the operators  $\{C_2^{(1)}, C_2^{(2)}, C_3^{(1)}, C_3^{(2)}, \Theta^{[2,0]}, \Theta^{[3,0]}\}$  characterize the  $\mathfrak{su}(3)$ -representations appearing in the decomposition (24), but do not separate the degenerate representations with  $\alpha_k > 1$ . This will be the function of the missing label operator  $\Theta^{[0,3]}$ . Although this operator acts diagonally on  $\mathfrak{su}(3)$  representations, it is not an invariant of the subalgebra, and its eigenvalue for a representation will explicitly depend on the tensor product where it appears as a component.

#### 4.1. Computation of the eigenvalues of $\Theta^{[0,3]}$

Let  $R = [\lambda, \mu] \otimes [\rho, \sigma] = \sum_{k=1}^{s_0} \alpha_k \Gamma_k$  and suppose that the multiplet  $\Gamma_{k_0}$  has multiplicity (or degeneracy index)  $\alpha_{k_0} > 1$  in  $R$ . Let  $d = \dim [\lambda, \mu] \dim [\rho, \sigma]$  and denote by  $V$  the carrier space of the representation  $R$ . The algorithm to isolate such a degenerate multiplet and compute the eigenvalues of the labelling operators can be summarized in the following five steps:

- (i) Compute the eigenvalues  $\lambda_2 = \Theta^{[2,0]}(\Gamma_0)$  and  $\lambda_3 = \Theta^{[3,0]}(\Gamma_0)$  of the  $\mathfrak{su}(3)$ -Casimir operators on the representation  $\Gamma_0$  using formula (10).
- (ii) Set  $S_0 = V$  and define

$$S_1 = \left\{ \mathbf{w} \in S_0 : \left[ \Theta^{[2,0]} - \lambda_2 \text{Id}_d \right] \mathbf{w} = 0 \right\}, \quad (28)$$

$$S_2 = \left\{ \mathbf{w} \in S_1 : \left[ \Theta^{[3,0]} - \lambda_3 \text{Id}_d \right] \mathbf{w} = 0 \right\}. \quad (29)$$

The subspace  $S_2 \subset V$  is easily seen to correspond to the vectors in  $R$  having the eigenvalues  $(\lambda_2, \lambda_3)$ , thus  $S_2$  equals the representation  $\Gamma_0$  in  $R$  along with its multiplicity  $\alpha_0$ .

- (iii) Take an arbitrary basis  $B = \{w_i : 1 \leq i \leq d_0 = \alpha_0 \dim \Gamma_0\}$  of  $S_2$  and consider the following linear system:

$$\Theta^{[0,3]}.w_k = \sum_{j=1}^{d_0} \mu_{\gamma,k}^j w_j, \quad 1 \leq k \leq d_0. \quad (30)$$

- (iv) Solve the system (30) and define the coefficient matrix  $M_\Theta$  with entries

$$(M_\Theta)_{kj} = \mu_{\gamma,k}^j. \quad (31)$$

- (v) Compute the roots of the corresponding characteristic equation

$$\chi(T) = \det [M_\Theta - T \text{Id}_{d_0}] = 0. \quad (32)$$

We observe that this prescription is by no means exclusive of the Lie algebra  $\mathfrak{su}(3)$ , as it can be generalized to any rank  $l$  simple Lie algebra and any admissible labelling operator. This general approach to Clebsch-Gordan series will be reported elsewhere.

#### 4.2. An empirical formula for the products $[1, 1] \otimes [\ell, k]$

In the following, we will apply the preceding procedure to establish empirical formulae providing the eigenvalues of the missing label operator  $\Theta^{[0,3]}$  for some types of IRs of  $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ . In this context, the most elementary tensor product of IRs of  $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$  that presents a degenerate multiplet is given by  $R = [1, 1] \otimes [\ell, k]$  with  $\ell, k \geq 1$ , where in the decomposition into irreducible components only the multiplet  $[\ell, k]$  is degenerate with multiplicity two for each  $\ell, k \geq 1$ .<sup>1</sup> Tables 1 and 2 list the values of  $\alpha_{\ell,k}$  and  $\beta_{\ell,k}$  respectively for the range  $1 \leq \ell, k \leq 10$ , corresponding to the first hundred tensor products presenting degeneracy. Basing on these tables, the numerical analysis of the eigenvalues  $\lambda_{1,2}$  of  $\Theta^{[0,3]}$  on the degenerate multiplet  $[\ell, k]$  suggests to write  $\lambda_{1,2}$  generically in the following form:

$$\lambda_{1,2} = \frac{\alpha_{\ell,k}}{27} \pm \sqrt{\beta_{\ell,k}}. \quad (33)$$

Separation of the degeneracy is therefore equivalent to the condition  $\beta_{\ell,k} \neq 0$ . With the purpose of deriving an empiric formula for the eigenvalues of the MLO  $\Theta^{[0,3]}$  valid for all values  $\ell, k \geq 1$ , we now look for recurrence relations between these eigenvalues.

**Table 1.** Values of  $\alpha_{\ell,k}$  for  $1 \leq \ell \leq k \leq 10$  and  $R = [1, 1] \otimes [\ell, k]$ .

$[\ell, k]$	1	2	3	4	5	6	7	8	9	10
1	0	-56	-160	-324	-560	-880	-1296	-1820	-2464	-3240
2	56	0	-110	-286	-540	-884	-1330	-1890	-2576	-3400
3	160	110	0	-182	-448	-810	-1280	-1870	-2592	-3458
4	324	286	182	0	-272	-646	-1134	-1748	-2500	-3402
5	560	540	448	272	0	-380	-880	-1512	-2288	-3220
6	880	884	810	646	380	0	-506	-1150	-1944	-2900
7	1296	1330	1280	1134	880	506	0	-650	-1456	-2340
8	1820	1890	2491	1748	1512	1150	650	0	-812	-1798
9	2464	2576	2592	2500	2288	1944	1456	812	0	-992
10	3240	3400	3458	3402	3220	2900	2340	1798	992	0

**Table 2.** Values of  $\beta_{\ell,k}$  for  $1 \leq \ell \leq k \leq 10$  and  $R = [1, 1] \otimes [\ell, k]$ .

$[\ell, k]$	1	2	3	4	5	6	7	8	9	10
1	45	73	109	153	205	265	333	409	493	585
2	73	105	145	193	249	313	385	465	553	649
3	109	145	189	241	301	369	445	529	621	721
4	153	193	241	297	361	433	513	601	697	801
5	205	249	301	361	429	505	589	681	781	889
6	265	313	369	433	505	585	673	769	873	985
7	333	385	445	513	589	673	765	865	973	1089
8	409	465	529	601	681	769	865	969	1081	1201
9	493	553	621	697	781	873	973	1081	1197	1321
10	585	649	721	801	889	985	1089	1201	1321	1449

As follows from inspection of Table 1, the leading term  $\alpha_{\ell,k}$  is skew-symmetric. It is straightforward to verify that it actually coincides with  $-\Theta^{[3,0]}([\ell, k])$ , i.e., it is given by the eigenvalue of the  $\mathfrak{su}(3)$  cubic Casimir operator (9) on the representation  $[\ell, k]$ . Therefore

$$\alpha_{\ell,k} = (k - l)(3 + 2k + l)(3 + 2l + k), \quad \ell, k \geq 1. \quad (34)$$

Now, inspecting Table 2, we see that the discriminant  $\beta_{\ell,k}$  is symmetric, i.e., satisfies  $\beta_{\ell,k} = \beta_{k,\ell}$  for any values. Fixing the index  $\ell$ , we find the following relations

$$\beta_{1,k+1} - \beta_{1,k} = 20 + 8k, \quad \beta_{2,k+1} - \beta_{2,k} = 24 + 8k, \quad \beta_{3,k+1} - \beta_{3,k} = 28 + 8k, \dots \quad (35)$$

Due to the symmetry, an analogous relation holds when replacing the index  $\ell$  by  $k$ . A short computation shows that following identities hold:

$$\beta_{\ell,k+1} - \beta_{\ell,k} = 16 + 4\ell + 8k, \quad \beta_{\ell+1,k} - \beta_{\ell,k} = 16 + 8\ell + 4k, \quad \ell, k \geq 1. \quad (36)$$

This provides the pattern for the discriminant when moving in each row and column respectively. Moving now in the diagonals of Table 2, starting from the main diagonal, we get the differences

$$\beta_{\ell+1,\ell+1} - \beta_{\ell,\ell} = 36 + 24\ell, \quad \beta_{\ell+1,\ell+2} - \beta_{\ell,\ell+1} = 48 + 24\ell, \quad \beta_{\ell+1,\ell+3} - \beta_{\ell,\ell+2} = 60 + 24\ell, \dots \quad (37)$$

Taking into account that the indices are symmetric, solving the latter recurrence equation for  $k$  leads to the expression

$$\beta_{\ell+1,k+1} - \beta_{\ell,k} = 36 + 12(\ell + k), \quad \ell, k \geq 1. \quad (38)$$

<sup>1</sup> It can be shown that this is actually the only type of tensor product with only one degenerate representation.

The recurrence relations (36) and (38) contain the necessary information to find an explicit formula for  $\beta_{\ell,k}$ , that is found after some computation to be

$$\beta_{\ell,k} = 4(\ell^2 + k^2) + 12(\ell + k) + 4\ell k + 9, \quad \ell, k \geq 1. \quad (39)$$

In conclusion, the eigenvalues for the MLO  $\Theta^{[0,3]}$  in the degenerate multiplet  $[\ell, k]$  within the tensor product  $[1, 1] \otimes [\ell, k]$  are given by

$$\lambda_{1,2} = \frac{1}{27} (k - \ell) (3 + 2k + \ell) (3 + 2\ell + k) \pm \sqrt{4(\ell^2 + k^2) + 12(\ell + k) + 4\ell k + 9}. \quad (40)$$

We observe that self-conjugate degenerate representations  $[\ell, \ell]$  are separated by the roots of the discriminant, as the cubic Casimir operator has always eigenvalue zero.

#### 4.3. The tensor products $[1, \ell] \otimes [1, k]$

In contrast with the previous case, for the IRs  $[1, \ell] \otimes [1, k]$  with  $\ell, k > 1$  of  $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$  the number of degenerate multiplets in the tensor product is not fixed, but is determined by the index  $\ell$ . Using formula (23), it can be easily verified that degenerate multiplets can be described by

$$[1, \ell] \otimes [1, k] \supset \sum_{q=1}^{\ell} [q, k + \ell + 1 - 2q]^2, \quad (41)$$

with all of them having a degeneracy index two. Due to the latter relation, any formula describing the eigenvalues of  $\Theta^{[0,3]}$  must depend on the three indices  $q, \ell, k$  appearing in (41). In analogy with (33), we suppose without loss of generality that the eigenvalues of the labelling operator  $\Theta^{[0,3]}$  adopt the form

$$\lambda_{1,2} = \frac{\alpha_{q,\ell,k}}{27} \pm \sqrt{\beta_{q,\ell,k}}, \quad (42)$$

where the first subindex  $q$  makes reference to the degenerate multiplet  $[q, k + \ell + 1 - 2q]$  in formula (41). Hence, in order to derive an empirical formula, we have to compute numerical tables for varying values of  $\ell, k$  and a fixed value  $1 \leq q \leq \ell$ , and then looking for auxiliary relations satisfied by  $\alpha_{q,\ell,k}$  and  $\beta_{q,\ell,k}$  when varying the indices  $\ell, k$  and  $q$ . Proceeding like this, the following recurrence relations are found for the leading term and the discriminant when  $\ell, k, q \geq 1$ :

$$\alpha_{q,\ell+1,k+1} - \alpha_{q,\ell,k} = 18(k - \ell)(1 + q), \quad \alpha_{q+1,\ell,k} - \alpha_{q,\ell,k} = 9(k - \ell)(1 + k + \ell - 2q), \quad (43)$$

$$\beta_{q,\ell+1,k+1} - \beta_{q,\ell,k} = 4q(2 + q)(4 + k + \ell - q), \quad \beta_{q,\ell,k+1} - \beta_{q,\ell+1,k} = 4(k - \ell). \quad (44)$$

Solving these recurrence expressions, we find explicit expressions for  $\alpha_{q,\ell,k}$  and  $\beta_{q,\ell,k}$  as

$$\begin{aligned} \lambda_{1,2}(q, \ell, k) = & \frac{1}{27} (k - \ell) \left( 2(k^2 + \ell^2) + 9(k + \ell)(q + 1) - 4k\ell - 9(q - 1)^2 \right) \pm \left\{ q^4 - 4q^3 \right. \\ & \left. + 4q^2 + 16q + \left\{ (q + 1)^2 (k^2 + \ell^2) + 2q(6 + q - q^2)(k + \ell) + 2\ell k (q^2 + 2q - 1) \right\}^{\frac{1}{2}} \right\}. \end{aligned} \quad (45)$$

In this case, when  $k = \ell$  holds, the degenerate multiplet  $[q, 2\ell + 1 - 2q]$  is separated by the roots of the discriminant.



#### 4.4. The tensor products $[1, \ell] \otimes [k, 1]$

We finally consider the IRs  $[1, \ell] \otimes [k, 1]$  of  $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ , that exhibit a behavior similar to the previous type considered, in the sense that there are exactly  $\ell$  degenerate multiplets, all of them having multiplicity two. The degenerate  $\mathfrak{su}(3)$ -representations are in this case described by

$$[1, \ell] \otimes [k, 1] \supset \sum_{q=1}^{\ell} [k + q - \ell, q]^2. \quad (46)$$

Again, basing on tables of eigenvalues computed for all values  $1 \leq q, \ell, k \leq 10$ , the following recurrence relations are found:

$$\alpha_{q, \ell+1, k} - \alpha_{q, \ell, k+1} = 18(5 + k + \ell)(1 + q), \quad \alpha_{q+1, \ell, k} - \alpha_{q, \ell, k} = 9(4 + k + \ell)(3 + k - \ell + 2q), \quad (47)$$

$$\beta_{q, \ell+1, k+1} - \beta_{q, \ell, k} = 4(5 + k + \ell), \quad \beta_{q, \ell+1, k} - \beta_{q, \ell, k+1} = 4q(2 + q)(1 + k - \ell + q). \quad (48)$$

The explicit eigenvalue formula obtained from these recurrence relations is given by:

$$\begin{aligned} \lambda_{1,2}(q, \ell, k) &= \frac{1}{27}(k + \ell + 4)(2(\ell^2 + k^2) + k(4\ell - 9q + 7) + (25 + 9q)\ell - (9q^2 + 18q + 13)) \\ &\pm \left\{ (q+1)^2(k^2 + \ell^2) + (2q^3 + 6q^2 + 4q)(k - \ell) + 8(k + \ell) - \ell k(2q^2 + 4q - 2) \right. \\ &\quad \left. + q^4 + 4q^3 + 4q^2 + 16 \right\}^{\frac{1}{2}}. \end{aligned} \quad (49)$$

In contrast with the other types inspected, it is a noticeable fact that the leading term  $\alpha_{q, \ell, k}$  appearing in (49) does not vanish for indices  $\ell, k \geq 2$ . This clearly indicates that a given representation can have radically different eigenvalues for the MLO  $\Theta^{[0,3]}$ , depending on the tensor product where it appears as a component in the decomposition (24).

## 5. Conclusions

Explicit formulae for the eigenvalues of the missing label operator  $\Theta^{[0,3]}$  to separate the degenerate representations of the  $\mathfrak{su}(3)$ -tensor products  $[1, 1] \otimes [k, \ell]$ ,  $[1, \ell] \otimes [1, k]$  and  $[1, \ell] \otimes [k, 1]$  for  $\ell, k \geq 1$  have been obtained. All the cases treated have the same constant degeneracy index equal to two and a number of degenerate multiplets that always coincides with  $\ell$ . As a consequence, the degenerate multiplets can be described adding a third continuously increasing index  $q$ , a fact that allows to deduce recurrence relations between the eigenvalues of the MLO depending exclusively on  $q, \ell$  and  $k$ , and from which the empirical formulae are obtained. We observe that for all the considered types of tensor products, the eigenvalues for the other admissible missing label operator  $\Theta^{[2,1]}$  are immediately deduced from those of  $\Theta^{[0,3]}$  using formula (19). As a consequence, the procedure described allows to compute the spectrum for an arbitrary linear combination  $a_1\Theta^{[2,1]} + a_2\Theta^{[0,3]}$  that may be the appropriate labelling operator in some application [22, 28].

Some of the patterns observed suggest that similar rules should work for the generic case, although the computational difficulties increase considerably due to the dimension of the resulting representation of  $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$  and the number of degenerate multiplets in the decomposition (23). Formally the same procedure based on finding recurrence relations and deriving explicit formulae for the eigenvalues of  $\Theta^{[0,3]}$  in other tensor products can be established, in spite of the various computational complications arising when looking for generic formulae. The first difficulty concerns a unified description of degenerate multiplets, as its number in the product  $[j, \ell] \otimes [k, m]$  depends on the specific values of  $j, \ell, k, m$ , and thus a generic formula

must contain these four indices, as well as those specifying the degenerate representation. In addition, within a given tensor product  $[j, \ell] \otimes [k, m]$ , the multiplicities are varying, implying some difficulties in establishing a unified formula for the eigenvalues and all possible indices.

However, basing on further numerical computations for other types of tensor products for which an empirical formulae for the  $\Theta^{[0,3]}$  eigenvalues has still not been fully established, some additional patterns concerning the distribution of these eigenvalues have been observed. So, for example, if a representation  $[p, q]$  has an odd degeneracy index  $\nu_0$  in the tensor product  $[j, \ell] \otimes [k, m]$ , then the eigenvalue  $\xi_0 = 0$  always appears. From this fact it can be inferred that the characteristic polynomial of  $\Theta^{[0,3]}$  restricted to a degenerate multiplet  $[p, q]$  has the following generic structure

$$\chi(T) = \prod_{s=1}^{\left[\frac{\nu_0}{2}\right]} \left( T^2 - \frac{2}{27} \alpha_s T + \frac{\alpha_s^2 - 729 \beta_s}{729} \right) T^\varepsilon, \quad (50)$$

where  $\varepsilon = \frac{1-(-1)^{\nu_0}}{2}$  and  $\lambda_s = \frac{1}{27} \alpha_s \pm \sqrt{\beta_s}$  for  $s = 0, \dots, \left[\frac{\nu_0}{2}\right]$  are the nonvanishing eigenvalues of  $\Theta^{[0,3]}$  on  $[p, q]$ . A currently unsolved problem is how to incorporate the degeneracy index to the eigenvalue formula, in order to obtain a unified expression for all degenerate representations within a tensor product. Progress in this direction is expected to be reported in future work.

### Acknowledgements

During the preparation of this work, the author was financially supported by research grant MTM2016-79422-P (AEI/FEDER, EU).

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