

Global stability for convection when the viscosity has a maximum

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Abstract

When the viscosity depends on temperature in a quadratic manner such that the viscosity has a maximum, an unconditional nonlinear energy stability analysis for thermal convection according to Navier-Stokes theory has not yet been developed. We here analyse a model of non-Newtonian fluid behaviour which allows us to develop an unconditional analysis directly when the quadratic viscosity relation is allowed. The nonlinear stability boundaries are sharp when compared to the linear instability thresholds.

1. Introduction.

In everyday life the viscosity of a fluid is usually strongly dependent on temperature, cf. Richardson (1993), Capone & Gentile (1994,1995). Thus, a large change of viscosity with temperature may have a pronounced effect on Bénard convection which is the cellular motion which ensues when a layer of fluid is heated from below. In particular, the increasing or decreasing of viscosity with temperature was shown experimentally to give rise to decreasing or increasing, respectively, fluid motion in the centre of the Bénard cell. This important work is due to Tippelskirch (1956). While the theory of linearised instability of the problem of thermal convection with temperature dependent viscosity is well known, the corresponding *nonlinear stability theory* is incomplete. Using a generalised energy analysis Richardson (1993) established nonlinear stability bounds which are very sharp when compared to those of linearised instability theory when the viscosity is a linearly decreasing function of temperature. Richardson's work employs a non-trivial Lyapunov function and is an intricate analysis. This was extended by Capone & Gentile (1994,1995) to allow for a more general viscosity-temperature relationship of exponential form but essentially one which has a bounded derivative. These are important articles and are the first to derive nonlinear energy stability bounds for Bénard convection with temperature-dependent viscosity. However, they suffer from two drawbacks. The first is that the analysis holds for two stress free surfaces bounding the fluid layer. The second is that the stability thresholds derived are *conditional* in the sense that the initial data is restricted and tends to zero upon approaching the critical Rayleigh number. In this paper we propose a way to overcome the above obstacles when the viscosity is quadratic in temperature, by employing a non-Newtonian model of fluid behaviour.

For many fluids the viscosity-temperature relationship is essentially a decreasing exponential one. In this case, for temperature differences which are not too large one can approximate the viscosity by a function linear in the temperature field. However, not all fluids may be adequately modelled by a linear viscosity-temperature relationship. For example, liquid sulphur and bismuth are fluids where the viscosity achieves a maximum in temperature, cf. Lide (1991). For such fluids a quadratic relation of form

$$\nu(T) = \nu_0[1 - \gamma(T - T_0)^2] \quad (1.1)$$

is necessary to reflect the maximum behaviour in temperature. In (1.1), ν_0, T_0 and γ are constants. Richardson (1993) also developed a nonlinear, conditional energy method to handle the viscosity given by (1.1). The goal of this work is to incorporate relation (1.1) into an unconditional (for all initial data) nonlinear stability analysis of the Bénard

problem for a non-Newtonian fluid.

In the context of thermal convection in a porous medium nonlinear energy stability analyses were developed by Richardson (1993) and by Qin & Chadam (1996). Their work derived *conditional* (initial data dependent) stability thresholds. Payne & Straughan (2000) have employed L^p energy functionals to develop an *unconditional* nonlinear stability theory for the porous Bénard problem with temperature dependent viscosity provided one of the Forchheimer theories is used. Straughan (2002) has also developed an unconditional analysis with a linear viscosity by using Ladyzhenskaya's models of fluid behaviour.

Convection theory with temperature or concentration dependent viscosity is a highly active area and there has been much recent mathematical work, cf. Capone & Gentile (1994,1995), Diaz & Galiano (1997,1998), Flavin & Rionero (1998,1999a,b), Galiano (2000), Payne *et al.* (1999), Payne & Straughan (2000), Qin & Chadam (1996), Richardson (1993), Straughan (2002). These articles demonstrate the need for a well established theory for convection which takes into account the variation of fluid properties such as viscosity, or conductivity, with temperature. In particular, Diaz & Galiano (1997,1998), and Galiano (2000) establish existence of solutions to the fluid equations with temperature varying viscosity and thermal conductivity. The present paper is devoted to studying a well defined nonlinear stability problem for a non-Newtonian fluid for the case when the viscosity is a quadratic function of temperature. To the best of our knowledge, this is the first *unconditional* nonlinear energy stability analysis for Bénard convection in a *fluid* when the viscosity is a quadratic function of temperature. The nonlinear stability thresholds we derive are very sharp when compared with those of linearised instability theory.

The real strength of a nonlinear energy stability analysis is when it can be employed to yield sharp stability thresholds which are valid for *all* initial data, or at least for a large set of initial data. This is stressed by Straughan (1998), p. 157, and several recent articles have explicitly addressed this question in a variety of contexts, cf. Budu (2002), Flavin & Rionero (1998,1999a,b), Lombardo *et al.* (2000), Payne & Straughan (2000) and Straughan (2001,2002). Flavin & Rionero (1998,1999a,b), in particular, employ a “natural” transformation to cope with the situation in which the thermal conductivity is a nonlinear function of temperature, at least for a certain class of nonlinearities. We point out that energy methods in general for various problems in partial differential equations are addressed in the books of Antontsev *et al.* (2001), Doering & Gibbon (1995), Flavin & Rionero (1995), and Straughan (1992).

2. The non-Newtonian fluid model

We introduce a constitutive equation where the viscosity is composed of two parts, one involving the usual viscosity term as a function of temperature while the second part is a nonlinear function of the symmetric part of the velocity gradient. Such a model is a generalisation to the temperature dependent viscosity case of a well known equation in non-Newtonian fluid mechanics, cf. Antontsev *et al.* (2001), p. 228. Our constitutive theory also resembles the generalised second grade fluid model of Man & Sun (1987) and Massoudi & Phuoc (2001), although they were interested in other questions such as glacier flow.

Thus, our basic equations of momentum, continuity, and energy balance are

$$\begin{aligned} \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} &= -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_k} 2\nu_0 \left\{ [1 - \gamma(T - T_0)^2] D_{ik} \right\} \\ &\quad + 2\nu_1 \frac{\partial}{\partial x_k} (|\mathbf{D}|^{2\mu} D_{ik}) + g\alpha k_i T, \\ \frac{\partial v_i}{\partial x_i} &= 0, \quad \frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i} = \kappa \Delta T. \end{aligned} \quad (2.1)$$

Here v_i, T, p are velocity, temperature and pressure in the fluid, $\mathbf{k}=(0,0,1)$, g, α, ρ, κ are gravity, thermal expansion coefficient, density (constant), and thermal diffusivity, and $D_{ik} = (v_{i,k} + v_{k,i})/2$. Standard indicial notation is used throughout with a repeated index summing from 1 to 3. The coefficients ν_1 and μ are positive constants and throughout this work we select $\mu = 1$. This choice is consistent with other non-Newtonian fluid models, e.g. the equations for a fluid of third grade, cf. Budu (2002), Tigoiu (2000).

To study the Bénard problem we suppose the fluid occupies the layer $z \in (0, d)$, $(x, y) \in \mathbb{R}^2$. Gravity is in the negative z -direction and the planes $z = 0, d$ are, respectively, held at fixed temperatures T_L and T_U , with $T_L > T_U$.

Equations (2.1) admit the steady solution

$$\bar{T} = T_L - \beta z, \quad \bar{v}_i \equiv 0, \quad (2.2)$$

where

$$\beta = \frac{T_L - T_U}{d}, \quad (2.3)$$

and the steady pressure $\bar{p}(z)$ is found from the momentum equation. It is the nonlinear stability of this solution which is the object of this paper.

We introduce perturbations (u_i, θ, π) by $v_i = \bar{v}_i + u_i$, $T = \bar{T} + \theta$, $p = \bar{p} + \pi$ to system (2.1), and rescale the equations for the perturbation quantities to make them non-dimensional.

The scalings we employ are those of Richardson (1993), p. 58, who refers everything to an average viscosity, $\nu_m = d^{-1} \int_0^d \nu(\bar{T}) dz$. The non-dimensional perturbation equations are, where ω is a non-dimensional form for ν_1 ,

$$\begin{aligned} \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} &= -\frac{\partial \pi}{\partial x_i} + R\theta k_i + (fd_{ij})_{,j} + h(z\theta d_{ij})_{,j} - \zeta(d_{ij}\theta^2)_{,j} + \omega(|\mathbf{d}|^{2\mu} d_{ij})_{,j}, \\ \frac{\partial u_i}{\partial x_i} &= 0, \quad Pr \left(\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} \right) = Rw + \Delta \theta, \end{aligned} \quad (2.4)$$

where

$$d_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad f(z) = \frac{6}{(3-\Gamma)}(1-\Gamma z^2), \quad h = \frac{12\Gamma Pr}{R(3-\Gamma)}, \quad \zeta = \frac{6\Gamma Pr^2}{R^2(3-\Gamma)}.$$

The coefficient Γ is such that $0 < \Gamma < 1$.

Equations (2.4) hold on the domain $\{(x, y) \in \mathbb{R}^2\} \times \{z \in (0, 1)\} \times \{t > 0\}$. The boundary conditions on the perturbations become

$$u_i = 0, \quad z = 0, 1, \quad \theta = 0, \quad z = 0, 1, \quad (2.5)$$

and u_i, θ, π satisfy a plane tiling periodic shape in the (x, y) plane, cf. Straughan (1992), p. 52. The cell which arises due to this plane tiling form is denoted by V .

3. Unconditional nonlinear stability

We develop two energy equations by multiplying (2.4)₁ by u_i , and (2.4)₃ by θ and integrating over V . This yields with $\mu = 1$,

$$\frac{d}{dt} \frac{1}{2} \|\mathbf{u}\|^2 = R(\theta, w) - \int_V f |\mathbf{d}|^2 dx - h \langle z\theta |\mathbf{d}|^2 \rangle - \omega \int_V |\mathbf{d}|^4 dx + \zeta \langle \theta^2 |\mathbf{d}|^2 \rangle, \quad (3.1)$$

$$\frac{d}{dt} \frac{1}{2} Pr \|\theta\|^2 = R(\theta, w) - \|\nabla \theta\|^2. \quad (3.2)$$

Here, $\|\cdot\|$ and (\cdot, \cdot) denote the norm and inner product on $L^2(V)$, $w = u_3$, and $\langle \cdot \rangle$ denotes integration over V .

The first step is to use the arithmetic - geometric mean inequality on the third and the last terms on the right in (3.1) to derive for $\alpha, \beta > 0$ numbers to be chosen,

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|\mathbf{u}\|^2 \leq & R(\theta, w) - \langle f|\mathbf{d}|^2 \rangle - \langle |\mathbf{d}|^4 \rangle \left(\omega - \frac{\alpha h}{2} - \frac{\zeta \beta}{2} \right) \\ & + \frac{h}{2\alpha} \langle z^2 \theta^2 \rangle + \frac{\zeta}{2\beta} \langle \theta^4 \rangle. \end{aligned} \quad (3.3)$$

Let now $\lambda > 0$ be a coupling parameter we may select optimally later and then from (3.2) and (3.3) we form

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|\mathbf{u}\|^2 + \frac{\lambda Pr}{2} \|\theta\|^2 \right) \leq & R(1 + \lambda)(\theta, w) - \langle f|\mathbf{d}|^2 \rangle - \lambda \|\nabla \theta\|^2 + \frac{h}{2\alpha} \langle z^2 \theta^2 \rangle \\ & + \frac{\zeta}{2\beta} \langle \theta^4 \rangle - \langle |\mathbf{d}|^4 \rangle \left(\omega - \frac{\alpha h}{2} - \frac{\zeta \beta}{2} \right). \end{aligned} \quad (3.4)$$

It would appear that we need extra dissipation on the right of (3.4) to deal with the $\langle \theta^4 \rangle$ term. Hence, we now derive an “energy” identity for $\langle \theta^4 \rangle$,

$$\frac{d}{dt} \frac{aPr}{4} \langle \theta^4 \rangle = aR(\theta^3, w) - \frac{3a}{4} \|\nabla \theta^2\|^2, \quad (3.5)$$

where $a > 0$ is another coupling parameter to be selected. We next use Young’s inequality on the first term on the right of (3.5) and Poincaré’s inequality on the last term to find for a constant $\epsilon > 0$,

$$\frac{d}{dt} \frac{aPr}{4} \langle \theta^4 \rangle \leq \langle \theta^4 \rangle \left(\frac{3aR\epsilon^{4/3}}{4} - \frac{3\pi^2 a}{4} \right) + \frac{aR}{4\epsilon^4} \langle w^4 \rangle. \quad (3.6)$$

Next, we estimate $\langle w^4 \rangle$ using Hölder’s inequality and the Sobolev inequality

$$\begin{aligned} \langle w^4 \rangle & \leq \langle |\mathbf{u}|^4 \rangle \leq m^{1/3} \langle |\mathbf{u}|^6 \rangle^{2/3} \\ & \leq m^{1/3} c_S^4 \|\nabla \mathbf{u}\|^4 \\ & \leq 4m^{1/3} c_S^4 \langle |\mathbf{d}|^2 \rangle^2 \\ & \leq 4m^{4/3} c_S^4 \int_V |\mathbf{d}|^4 dV, \end{aligned} \quad (3.7)$$

where $m = \text{volume of } \Omega$ and c_S is the constant in the Sobolev inequality $\|u\|_{L^6} \leq c_S \|\nabla u\|_{L^2}$. We now let $c_1 = 4m^{4/3} c_S^4$. Using (3.7) we may derive from (3.6) and (3.4),

$$\begin{aligned} \frac{dE}{dt} \leq & I - D - \|\theta\|_4^4 \left(\frac{3\pi^2 a}{4} - \frac{3aR\epsilon^{4/3}}{4} - \frac{\zeta}{2\beta} \right) \\ & - \langle |\mathbf{d}|^4 \rangle \left(\omega - \frac{\alpha h}{2} - \frac{\zeta \beta}{2} - \frac{ac_1 R}{4\epsilon^4} \right), \end{aligned} \quad (3.8)$$

where the functions, E , D and I , are defined by

$$E(t) = \frac{1}{2}\|\mathbf{u}\|^2 + \frac{1}{2}\lambda Pr\|\theta\|^2 + \frac{aPr}{4}\|\theta\|_4^4, \quad (3.9)$$

$$I(t) = R(1 + \lambda)(\theta, w) + \frac{h}{2\alpha} \langle z^2 \theta^2 \rangle, \quad (3.10)$$

$$D(t) = \langle f|\mathbf{d}|^2 \rangle + \lambda\|\nabla\theta\|^2. \quad (3.11)$$

We now select $\alpha = 16RPr\sqrt{c_1}/9\pi^4$. We then choose the coefficients a , β and ϵ optimally. To do this we put $a = a' + k\delta$ and pick

$$\frac{3a'}{4}(\pi^2 - R\epsilon^{4/3}) - \frac{\zeta}{2\beta} = 0, \quad (3.12)$$

and then minimize the right hand side of

$$\omega > \frac{\alpha h}{2} + \frac{k\delta c_1 R}{4\epsilon^4} + f(\beta, \epsilon),$$

where $f(\beta, \epsilon) = \zeta\beta/2 + a'c_1R/4\epsilon^4$. We solve (3.12) for a' and substitute in f and minimize f in β to find $\beta = \beta(\epsilon)$. Then we use this value in f and minimise the result in ϵ . This yields the values

$$\epsilon = \left(\frac{3\pi^2}{4R}\right)^{3/4}, \quad \beta = \frac{16\sqrt{c_1}R^2}{9\pi^4}, \quad a' = \frac{3\zeta\pi^2}{2\sqrt{c_1}R^2}.$$

This leads to the restriction on ω

$$\omega > \frac{\alpha h}{2} + \frac{k\delta c_1 R}{4\epsilon^4} + \frac{16\zeta\sqrt{c_1}R^2}{9\pi^4}.$$

The number δ is arbitrarily small and in the limit this yields the restriction ω must satisfy

$$\omega > \frac{64\sqrt{c_1}Pr^2}{3\pi^4(3 - \Gamma)} \Gamma. \quad (3.13)$$

From inequality (3.8) we now pick $k = 4/3\pi^2$ and then derive

$$\frac{dE}{dt} \leq -D\left(1 - \frac{1}{R_E}\right) - \delta\|\theta\|_4^4, \quad (3.14)$$

where

$$\frac{1}{R_E} = \max_H \frac{I}{D}, \quad (3.15)$$

H being the space of admissible solutions.

We require $R_E > 1$ and then use the bound $f \geq 6(1 - \Gamma)/(3 - \Gamma) \equiv f_0 > 0$ so that

$$D \geq \min\left\{8f_0\pi^2, \frac{2\pi^2}{Pr}\right\} \left(\frac{1}{2}\|\mathbf{u}\|^2 + \frac{\lambda Pr}{2}\|\theta\|^2\right).$$

Then from (3.14) we may show

$$\frac{dE}{dt} \leq -KE, \quad (3.16)$$

where

$$K = \left(\frac{R_E - 1}{R_E}\right) \min\left\{\frac{48\pi^2(1 - \Gamma)}{(3 - \Gamma)}, \frac{2\pi^2}{Pr}, \frac{4\delta}{aPr}\right\}.$$

Inequality (3.16) leads to exponential decay and hence global nonlinear stability (for all initial data). The only requirement we make is that the non-Newtonian coefficient satisfies (3.13). Since in practice Γ is relatively small this is a weak requirement.

The nonlinear stability critical Rayleigh number is calculated from (3.15) and we put $R_E = 1$ to obtain the sharpest value. This leads to the following Euler-Lagrange equations for the determination of the nonlinear critical Rayleigh number R^2 ,

$$\begin{aligned} R(1 + \lambda)\theta k_i + 2(fd_{ij})_{,j} &= -\pi_{,i}, \\ u_{i,i} &= 0, \quad R(1 + \lambda)w + \frac{27\pi^4\Gamma}{4R^2(3 - \Gamma)\sqrt{c_1}}z^2\theta + 2\lambda\Delta\theta. \end{aligned} \quad (3.17)$$

Since the linearised equations we obtain from (2.4) are symmetric the relevant equations for the linear instability boundary are

$$\begin{aligned} R\theta k_i + (fd_{ij})_{,j} &= -\pi_{,i}, \\ u_{i,i} &= 0, \quad \Delta\theta + Rw = 0. \end{aligned} \quad (3.18)$$

When the Γ term in (3.17)₃ is absent (3.17) and (3.18) are identical if we set $\lambda = 1$. The flexibility of λ allows us to optimize the nonlinear stability threshold and for Γ small the critical Rayleigh numbers obtained from (3.17) and (3.18) are very close. The calculations of Payne & Straughan (2000) in porous convection indicate that the Rayleigh number values obtained from (3.17) and (3.18) will be close even for Γ not small.

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