Research Article

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The Poincaré–Birkhoff Theorem for a Class of Degenerate Planar Hamiltonian Systems

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Abstract: In this paper, we investigate the problem of the existence and multiplicity of periodic solutions to the planar Hamiltonian system $x' = -\lambda\alpha(t)f(y)$, $y' = \lambda\beta(t)g(x)$, where α, β are non-negative T-periodic coefficients and $\lambda > 0$. We focus our study to the so-called "degenerate" situation, namely when the set $Z := \operatorname{supp}\alpha \cap \operatorname{supp}\beta$ has Lebesgue measure zero. It is known that, in this case, for some choices of α and β, no nontrivial T-periodic solution exists. On the opposite, we show that, depending of some geometric configurations of α and β, the existence of a large number of T-periodic solutions (as well as subharmonic solutions) is guaranteed (for $\lambda > 0$ and large). Our proof is based on the Poincaré–Birkhoff twist theorem. Applications are given to Volterra's predator-prey model with seasonal effects.

Keywords: Periodic Predator-Prey Model of Volterra Type, Subharmonic Coexistence States, Poincaré–Birkhoff Twist Theorem, Degenerate Versus Non-Degenerate Models, Point-Wise Behavior of the Low-Order Subharmonics as the Model Degenerates

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1 Introduction

This paper studies the non-autonomous planar Hamiltonian system

$$\begin{cases} x' = -\lambda\alpha(t)f(y), \\ y' = \lambda\beta(t)g(x), \end{cases} \tag{1.1}$$

where $\lambda > 0$ is regarded as a real parameter, and given a real number $T > 0$, α and β are nonnegative T-periodic continuous functions such that

$$A := \int_0^T \alpha > 0, \quad B := \int_0^T \beta > 0,$$

for which the set

$$Z := \operatorname{supp}\alpha \cap \operatorname{supp}\beta \tag{1.2}$$

has Lebesgue measure zero, $|Z| = 0$. This is why model (1.1) is said to be *degenerate*.

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In (1.1), $f, g \in \mathcal{C}(\mathbb{R})$ are locally Lipschitz continuous functions such that $f, g \in \mathcal{C}^1(-\rho, \rho)$ for some $\rho > 0$ and

$$\begin{cases} f(0) = 0, & f(y)y > 0 \quad \text{for all } y \neq 0, \\ g(0) = 0, & g(x)x > 0 \quad \text{for all } x \neq 0, \\ f'(0) > 0, & g'(0) > 0. \end{cases} \tag{1.3}$$

Moreover, it is assumed that either f or g satisfies one of the following four conditions:

$$\begin{array}{llll} (f_-) & f \text{ is bounded in } \mathbb{R}^-, & (f_+) & f \text{ is bounded in } \mathbb{R}^+, \\ (g_-) & g \text{ is bounded in } \mathbb{R}^-, & (g_+) & g \text{ is bounded in } \mathbb{R}^+. \end{array} \tag{1.4}$$

This paper analyzes the existence of nT-periodic solutions of model (1.1) for any integer $n \geq 1$. Those with $n \geq 2$ (and n minimal) are often referred to as *subharmonics of order* n.

Besides $|Z| = 0$, through this paper, we assume that, given two positive integers $k, \ell \geq 1$ such that $|k - \ell| \leq 1$, there exist $k + \ell$ continuous functions in the interval $[0, T]$, $\alpha_i \gneq 0$, $1 \leq i \leq k$, and $\beta_j \gneq 0$, $1 \leq j \leq \ell$, such that $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k$, $\beta = \beta_1 + \beta_2 + \cdots + \beta_\ell$, with

$$\operatorname{supp} \alpha_i \subseteq [t_0^i, t_1^i] \quad \text{and} \quad \operatorname{supp} \beta_j \subseteq [t_2^j, t_3^j] \tag{1.5}$$

for some partition of $[0, T]$,

$$\begin{array}{ll} 0 \leq t_0^1 < t_1^1 \leq t_2^1 < t_3^1 \leq t_0^2 < t_1^2 \leq t_2^2 < t_3^2 \leq \cdots \leq t_0^k < t_1^k \leq t_2^k < t_3^k \leq T & \text{if } k = \ell, \text{ or} \\ 0 \leq t_0^1 < t_1^1 \leq t_2^1 < t_3^1 \leq t_0^2 < t_1^2 \leq t_2^2 < t_3^2 \leq \cdots \leq t_0^k < t_1^k \leq T & \text{if } k = \ell + 1. \end{array}$$

Similarly, we also consider the case when, instead of (1.5),

$$\operatorname{supp} \beta_j \subseteq [t_0^j, t_1^j] \quad \text{and} \quad \operatorname{supp} \alpha_i \subseteq [t_2^i, t_3^i] \tag{1.6}$$

for some partition of $[0, T]$,

$$\begin{array}{ll} 0 \leq t_0^1 < t_1^1 \leq t_2^1 < t_3^1 \leq t_0^2 < t_1^2 \leq t_2^2 < t_3^2 \leq \cdots \leq t_0^\ell < t_1^\ell \leq t_2^\ell < t_3^\ell \leq T & \text{if } \ell = k, \text{ or} \\ 0 \leq t_0^1 < t_1^1 \leq t_2^1 < t_3^1 \leq t_0^2 < t_1^2 \leq t_2^2 < t_3^2 \leq \cdots \leq t_0^\ell < t_1^\ell \leq T & \text{if } \ell = k + 1. \end{array}$$

We will refer to an *α-interval* (resp. *β-interval*) as the maximal interval I, where $|\operatorname{supp} \beta|_I| = 0$ and $|\operatorname{supp} \alpha|_I| > 0$ (resp. $|\operatorname{supp} \alpha|_I| = 0$ and $|\operatorname{supp} \beta|_I| > 0$). So the total number of α-intervals and β-intervals in $[0, T]$ is $k + \ell$. However, ascertaining the total number of α and β-intervals in $[0, nT]$ when $n \geq 2$ is slightly more subtle, as it depends on whether $k = \ell$ or $|k - \ell| = 1$. If $k = \ell$, it is apparent that the number of α-intervals is nk, whereas the number of β-intervals is $n\ell$. Thus, the total number of α and β-intervals in this case equals

$$n(k + \ell) = 2nk. \tag{1.7}$$

Now, assume that $|k - \ell| = 1$. Then, in case $k = \ell + 1$, there are $n\ell + 1$ α-intervals and $n\ell$ β-intervals in $[0, nT]$. Indeed, as for every $i \in \{0, 1, \dots, n - 2\}$ the last α-interval of $[iT, (i + 1)T]$ and the first one of $[(i + 1)T, (i + 2)T]$ produce a unique α-interval in $[0, nT]$, the total number of α-intervals in $[0, nT]$ is given by $n(\ell + 1) - (n - 1) = n\ell + 1$. Obviously, the number of β-intervals in $[0, nT]$ is $n\ell$. Thus, the total number of α and β-intervals equals $n\ell + 1 + n\ell = 2n\ell + 1$. Similarly, in case $\ell = k + 1$, the total number of α and β-intervals in $[0, nT]$ is $nk + 1 + nk = 2nk + 1$. Therefore, setting $m := \min\{k, \ell\}$, the total number of α and β-intervals in $[0, nT]$ in case $|k - \ell| = 1$ is

$$2nm + 1. \tag{1.8}$$

Figure 1 shows a series of examples satisfying the previous requirements. Note that the support of α_i and β_j on each of the intervals $[t_r^i, t_{r+1}^i]$, $1 \leq i \leq k$, and $[t_r^j, t_{r+1}^j]$, $1 \leq j \leq \ell$, might not be connected.

On each of the intervals $[t_r^s, t_{r+1}^s]$, $s \in \{i, j\}$, $r \in \{0, 2\}$, the structure of the support of α_i or β_j might be rather involved topologically, as illustrated by Figure 2, where we have plotted a sketch of the graph of a function α_i or β_j, vanishing on the tertiary Cantor set of the interval $[t_r^s, t_{r+1}^s]$ and being positive on the interior of its complement.

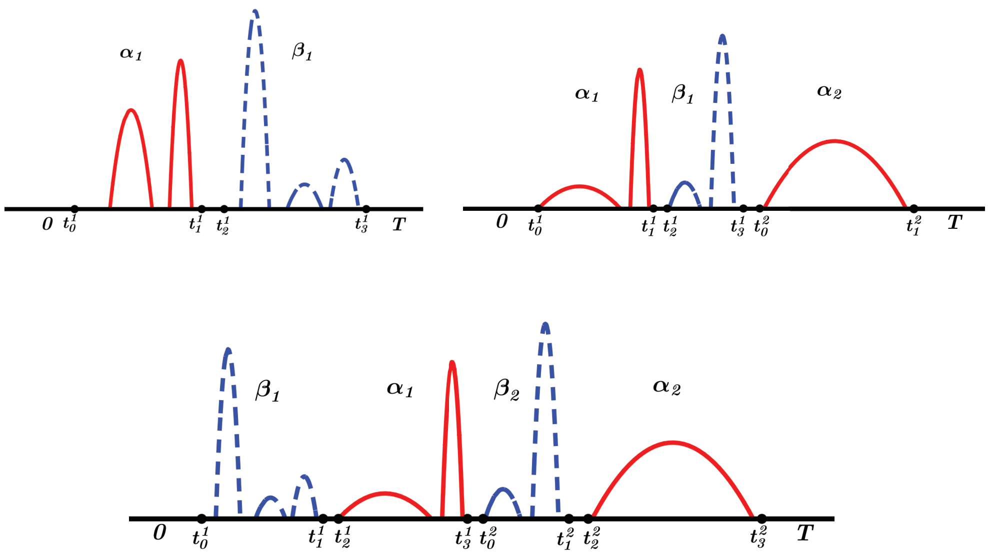

Figure 1: Some admissible distributions of α and β.

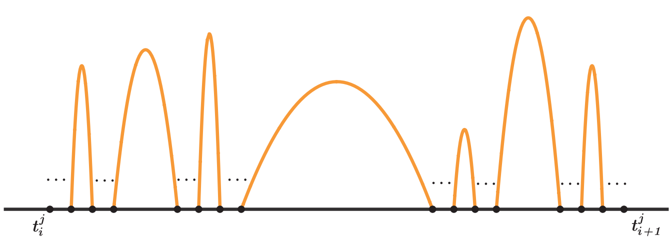

Figure 2: The internal complexity of the weights on each of their support intervals.

Note that the cases when $k + \ell \neq 2$, or $k + \ell = 2$ under condition (1.6), were not dealt with in any previous references. In particular, they stay outside the general scope of [13, 14, 16]. The main result of this paper can be stated as follows. (Subsequently, for every $r \in \mathbb{R}$, we are denoting by $[r]$ the integer part of r.)

Theorem 1. *Assume $nm \geq 3$ for some integer $n = 3h + i$, with $i \in \{0, 1, 2\}$, where $m := \min\{k, \ell\}$. Then there exists $\lambda_n > 0$ such that, for every $\lambda > \lambda_n$, (1.1) possesses at least*

$$\sigma(n) := 2\Big(hm + \Big[\frac{im}{3}\Big]\Big)$$

periodic solutions with period nT. Moreover, setting

$$\gamma(n) := \min\Big\{\gamma \geq 0 : \gcd\Big(n, \frac{\sigma(n)}{2} - \gamma\Big) = 1\Big\},$$

it turns out that, for every $\lambda > \lambda_n$, (1.1) has at least $\sigma(n) - 2\gamma(n)$ periodic solutions with minimal period nT.

The main technical device to prove Theorem 1 is the Poincaré–Birkhoff *twist* theorem collected in Theorem 2. Theorem 1 deals with a degenerate case in the context of Hamiltonian systems not previously studied in the literature, because neither the monotonicity of $\alpha(t)f(y)$ or $\beta(t)g(x)$ *for all* t, nor the non-degeneration of $\alpha(t)$ and $\beta(t)$ are required (see [5, 10, 12, 18], and [15, § 1]).

In Section 2, we state the version of the Poincaré–Birkhoff theorem invoked in the proof of Theorem 1 and make sure that it can be applied to deal with the degenerate model (1.1). Then, in Section 3, the proof of Theorem 1.1 is delivered. We refer to [3, 9] for a general discussion about the applications of the Poincaré–Birkhoff theorem to non-autonomous equations.

2 The Poincaré–Birkhoff Theorem in a Degenerate Setting

In this section, we adapt the Poincaré–Birkhoff theorem to deal with problem (1.1) in the *degenerate* case when $|Z| = 0$ (see (1.2), if necessary). The Poincaré–Birkhoff theorem has been applied, very successfully, to study some *non-degenerate* Volterra predator-prey models of type (1.1) (see, e.g., [1, 4, 5, 7, 10, 12, 18] and the recent paper by the authors [15]).

According to [15, Theorem 2.2], as soon as $|Z| > 0$, system (1.1) possesses at least two nT-periodic solutions for every integer $n \geq 1$ and sufficiently large $\lambda > 0$. The main result of this section, Theorem 3, provides with some general sufficient conditions on the coefficients $\alpha(t)$ and $\beta(t)$ for the validity of the same result in the case when $|Z| = 0$.

Subsequently, for any given nontrivial solution $(x(t), y(t))$ with initial data $z_0 := (x(0), y(0)) \neq (0, 0)$, we denote by $\theta(t)$ the angular polar coordinate so that, on any interval $[0, nT]$, the rotation number of the solution can be defined through

$$\operatorname{rot}(z_0; [0, nT]) := \frac{\theta(nT) - \theta(0)}{2\pi}.$$

To obtain the main result of this section, we need the following version of the Poincaré–Birkhoff twist theorem. It is, essentially, an application of Ding's version of the twist theorem for planar annuli (see [6]), as presented in [17, Theorem A] (see also [2] for another application).

Theorem 2. *Assume that, for some $0 < r_0 < R_0$ and an integer $\omega \geq 1$, the next twist condition holds:*

$$\operatorname{rot}(z_0; [0, nT]) > \omega \text{ if } \|z_0\| = r_0 \quad \text{and} \quad \operatorname{rot}(z_0; [0, nT]) < \omega \text{ if } \|z_0\| = R_0. \tag{2.1}$$

Then system (1.1) *has at least* 2 *nontrivial nT-periodic solutions belonging to different periodicity classes with rotation number ω.*

As observed in [5, § 3], given any nT-periodic solution (x, y) with $n \geq 2$, for every $j \in \{1, 2, \dots, n-1\}$, also $x_j(t) := x(t + jT)$, $y_j(t) := y(t + jT)$ is an nT-periodic solution. In Theorem 2, all these solutions are considered to be equivalent, and it is said that they belong to the same periodicity class.

Remark 1. The information on the rotation number provided by Theorem 2 is very relevant. First, because the solutions with different rotation numbers are essentially different since, as paths in $\mathbb{R}^2 \setminus \{(0, 0)\}$, they have a different fundamental group. Moreover, because Theorem 2, as stated, does not guarantee the minimality of the period nT, except in the special case when $\gcd(n, \omega) = 1$. Indeed, if $(x(t), y(t))$ is ℓT-periodic for some integer $\ell < n$, then the rotation number in the interval $[0, \ell T]$ must be an integer, say $\omega_1 \geq 1$. Thus, by the additivity property of the rotation numbers, it becomes apparent that $\operatorname{rot}(z_0; [0, n\ell T]) = \ell\omega = \omega_1 n$, where $z_0 = (x(0), y(0))$, which contradicts the fact that $\gcd(n, \omega) = 1$. Consequently, Theorem 2 is providing us with solutions of minimal period nT if $\omega = 1$.

Remark 2. By the continuous dependence of the solutions of (1.1) with respect to the initial conditions, for every $\varepsilon > 0$, $\lambda > 0$ and any integer $n \geq 1$, there exists $\delta \equiv \delta(n, \lambda, \varepsilon) > 0$ such that the unique solution of (1.1), $(x(t), y(t))$, satisfies $(x(t), y(t)) \in D_\varepsilon$ for all $t \in [0, nT]$ if $(x(0), y(0)) \in D_\delta$ (see [15, Proposition 2.1]). For every $R > 0$, we are denoting by D_R the disk of radius $R > 0$ centered at the origin.

According to (1.7) and (1.8) and recalling that $m = \min\{k, \ell\}$, throughout the rest of this paper, we will assume that $2nm \geq 6$ if $k = \ell$, and $2nm + 1 \geq 7$ if $|k - \ell| = 1$. Thus, unifying both conditions, throughout the rest of this paper, we will actually assume that

$$nm \geq 3. \tag{2.2}$$

This condition entails, essentially, at least five alternations between the components of the supports of $\alpha(t)$ and $\beta(t)$, as illustrated in Figure 3.

The main result of this paper reads as follows.

Theorem 3. *Assume $nm \geq 3$. Then there exists $\lambda_n > 0$ such that, for every $\lambda > \lambda_n$, the twist condition* (2.1) *in model* (1.1) *holds for $\omega \geq 1$.*

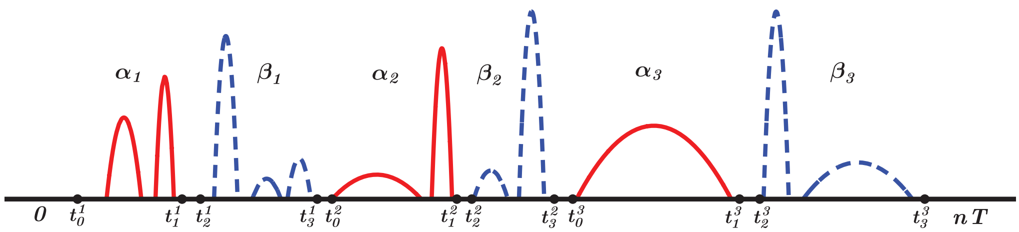

Figure 3: An example of five alternations between the supports of α and β.

This theorem analyzes a degenerate case for Hamiltonian systems of the form of (1.1), through the Poincaré–Birkhoff *twist* theorem, which had not been previously studied in this context, for as neither the monotonicity of $\alpha(t)f(y)$ or $\beta(t)g(x)$ *for all* t, nor the non-degeneration of $\alpha(t)$ and $\beta(t)$ are required (see [5, 10, 12, 18] and [15, § 1]).

The technical details of the proof will be given in the special case when α and β satisfy (1.5), as the case when (1.6) holds follows similarly. By (1.3),

$$\min\{f'(0), g'(0)\} > \eta$$

for some constant $\eta > 0$. Thus, for sufficiently small $|\zeta| \leq \varepsilon$,

$$f(\zeta)\zeta \geq \eta\zeta^2, \quad g(\zeta)\zeta \geq \eta\zeta^2. \tag{2.3}$$

Hence, since up to an additive constant,

$$\theta(t) = \arctan\frac{y(t)}{x(t)},$$

differentiating with respect to t and using the fact that $(x(t), y(t))$ solves (1.1) yields

$$\theta'(t) = \frac{y'(t)x(t) - x'(t)y(t)}{x^2(t) + y^2(t)} = \frac{\lambda\beta(t)g(x(t))x(t) + \lambda\alpha(t)f(y(t))y(t)}{x^2(t) + y^2(t)}$$

for every $t \in [0, nT]$. So, owing to (2.3), we have that

$$\theta'(t) \geq \lambda\eta\Big[\beta(t)\frac{x^2(t)}{x^2(t) + y^2(t)} + \alpha(t)\frac{y^2(t)}{x^2(t) + y^2(t)}\Big] = \lambda\eta[\beta(t)\cos^2\theta(t) + \alpha(t)\sin^2\theta(t)] \geq 0 \tag{2.4}$$

for every $t \in [0, nT]$. In particular, $\theta(t)$ is non-decreasing. Moreover, by Remark 2, for every integer $n \geq 1$, there exists $\delta > 0$ such that $(x(t), y(t)) \in D_\varepsilon$ for all $t \in [0, nT]$ if $(x_0, y_0) := (x(0), y(0)) \in D_\delta$. This condition will be kept throughout the next lemmas and the proof of Theorem 3 in order to guarantee that the solution cannot escape from D_ε. Naturally, the bigger λ is, the smaller is δ.

Based on (2.4) and Remark 2, the next result holds. Essentially, it establishes that each of the components of the support of α_i pushes the solutions of (1.1) from the first quadrant towards the second one, as well as from the third towards the fourth.

Lemma 1. *Assume that there exists* $(\rho_0, \rho_1) \subsetneq \operatorname{supp}\alpha$ *such that* $\theta(\rho_0) \in [\omega_0, \pi - \omega_0]$ *for some* $\omega_0 \in (0, \frac{\pi}{2})$. *Then there exists* $\lambda_1 > 0$ *such that* $\theta(\rho_1) > \pi - \omega_0$ *for all* $\lambda > \lambda_1$. *Similarly, if* $\theta(\rho_0) \in [\pi + \zeta_0, 2\pi - \zeta_0]$ *for some* $\zeta_0 \in (0, \frac{\pi}{2})$, *then* $\theta(\rho_1) > 2\pi - \zeta_0$ *for sufficiently large* λ.

Proof. Since we are dealing with small solutions, it is apparent from (2.4) that

$$\theta(\rho_1) = \theta(\rho_0) + \int_{\rho_0}^{\rho_1}\theta'(s)\,ds \geq \theta(\rho_0) + \lambda\eta\int_{\rho_0}^{\rho_1}\alpha(s)\frac{y^2(\rho_0)}{x^2(s) + y^2(\rho_0)}\,ds$$

because $\beta = 0$ on (ρ_0, ρ_1) and, hence, $y(s) \equiv y(\rho_0)$ therein. Thus, setting

$$\nu := \frac{\eta y^2(\rho_0)}{\varepsilon^2}\int_{\rho_0}^{\rho_1}\alpha(s)\,ds$$

and taking into account that $(x(s), y(s)) = (x(s), y(\rho_0)) \in D_\varepsilon$ for all $s \in (\rho_0, \rho_1)$, it follows that

$$\theta(\rho_1) \geq \theta(\rho_0) + \lambda\nu \geq \omega_0 + \lambda\nu > \pi - \omega_0 \quad \text{provided} \quad \lambda > \frac{\pi - 2\omega_0}{\nu} =: \lambda_1.$$

Note that the bound λ_1 remains invariant if either $\theta(\rho_0)$ or $y^2(\rho_0)$ increases. Moreover, $\theta(\rho_1)$ increases with λ for any given (fixed) $\theta(\rho_0)$ and $y^2(\rho_0)$. Similarly, we have $\theta(\rho_1) > 2\pi - \zeta_0$ for sufficiently large λ if $\theta(\rho_0) \in [\pi + \zeta_0, 2\pi - \zeta_0]$ for some $\zeta_0 \in (0, \frac{\pi}{2})$. This ends the proof. □

Analogously, the next result establishes that each of the components of the support of β_i pushes the solutions of (1.1) from the fourth quadrant towards the first, while it moves them from the second towards the third one.

Lemma 2. *Assume that there exists $(\sigma_0, \sigma_1) \subsetneq \operatorname{supp}\beta$ such that $\theta(\sigma_0) \in [-\frac{\pi}{2} + \tau_0, \frac{\pi}{2} - \tau_0]$ for some $\tau_0 \in (0, \frac{\pi}{2})$. Then there exists μ_1 such that $\theta(\sigma_1) > \frac{\pi}{2} - \tau_0$ for all $\lambda > \mu_1$. Similarly, $\theta(\sigma_1) > \frac{3\pi}{2} - \xi_0$ for sufficiently large λ if $\theta(\sigma_0) \in [\frac{\pi}{2} + \xi_0, \frac{3\pi}{2} - \xi_0]$ for some $\xi_0 \in (0, \frac{\pi}{2})$.*

Proof. As in Lemma 1, from (2.4), it follows that

$$\theta(\sigma_1) = \theta(\sigma_0) + \int_{\sigma_0}^{\sigma_1} \theta'(s)\,ds \geq \theta(\sigma_0) + \int_{\sigma_0}^{\sigma_1} \beta(s)\frac{x^2(\sigma_0)}{x^2(\sigma_0) + y^2(s)}\,ds$$

because $\alpha = 0$ on (σ_0, σ_1) and, hence, $x(s) \equiv x(\sigma_0)$ therein. Thus, denoting

$$\varsigma := \frac{\eta x^2(\sigma_0)}{\varepsilon^2} \int_{\sigma_0}^{\sigma_1} \beta(s)\,ds$$

and arguing as in Lemma 1, it is apparent that

$$\theta(\sigma_1) \geq \theta(\sigma_0) + \lambda\varsigma \geq \tau_0 + \lambda\varsigma > \frac{\pi}{2} - \tau_0 \quad \text{provided} \quad \lambda > \frac{\pi - 4\tau_0}{2\varsigma} =: \mu_1.$$

As highlighted in the proof of Lemma 1, the value of μ_1 does not vary if either $\theta(\sigma_0)$ or $x^2(\sigma_0)$ increases. Similarly, $\theta(\sigma_1)$ increases with λ, and the second assertion of the lemma holds. This ends the proof. □

Now, we are ready to prove Theorem 3.

Proof of Theorem 3. The proof is based on the version of the Poincaré–Birkhoff theorem collected in Theorem 2. First, we will prove that all small solutions in the disk D_ε, where ε is chosen sufficiently small so that (2.3) holds, have a rotation number greater than one. To prove this feature, we will distinguish between three different cases according to the precise location of their initial values, (x_0, y_0).

Case 1: Assume that $x_0 y_0 > 0$. Then (x_0, y_0) lies either in the first or in the third quadrant. Both cases being similar, we will pay attention only to the case when $x_0 > 0$ and $y_0 > 0$. Then $\theta(t_0^1) \in (0, \frac{\pi}{2})$. Thus, by Lemma 1, there exists $\lambda_1 > 0$ such that $\theta(t_1^1) > \pi - \theta(t_0^1)$ for all $\lambda > \lambda_1$. Since $\alpha = \beta = 0$ in $[t_1^1, t_2^1]$, this implies that

$$\theta(t_2^1) = \theta(t_1^1) > \pi - \theta(t_0^1).$$

Thus, by Lemma 2, there exists $\lambda_2 > 0$ such that $\theta(t_3^1) > \pi + \theta(t_0^1)$ as soon as $\lambda > \max\{\lambda_1, \lambda_2\}$. Also by Lemma 1, there exists $\lambda_3 > 0$ such that $\theta(t_1^2) > 2\pi - \theta(t_0^1)$ for every $\lambda > \max\{\lambda_1, \lambda_2, \lambda_3\}$, and due to Lemma 2, there is $\lambda_4 > 0$ such that $\theta(t_3^2) > 2\pi + \theta(t_0^1)$ for all $\lambda > \max\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. Therefore, the solution with initial values (x_0, y_0) completes an entire turn in the interval $[0, t_3^2]$ for every $\lambda > \max\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. In order to apply Theorem 2, it remains to show the existence of a uniform bound, $\Lambda_1 > 0$, such that the solutions with initial data in the sector of the circumference of radius r_0 within the first quadrant,

$$S_1 := \{z = (x, y) \in \mathbb{R}^2 : \|z\| = r_0 \text{ and } x > 0,\ y > 0\},$$

have a rotation number greater than one for all $\lambda > \Lambda_1$ if $0 < r_0 < \varepsilon$. To prove it, we consider an angle $\tilde{\omega}_0 \in (0, \frac{\pi}{2})$ and the sectors of S_1 defined by

$$S_{1,\tilde{\omega}_0}^+ := \{z = (x, y) \in \mathbb{R}^2 : \|z\| = r_0,\ r_0 \cos\tilde{\omega}_0 \geq x > 0,\ y > 0\}, \quad S_{1,\tilde{\omega}_0}^- := S_1 \setminus S_{1,\tilde{\omega}_0}^+.$$

Using recursively Lemmas 1 and 2 as above, it becomes apparent that there exists $M_1 > 0$ such that the solutions with initial data in $S^+_{1,\tilde{\omega}_0}$ complete an entire turn in the interval $[0, t^2_3]$ for every $\lambda > M_1$. For every $t > 0$, let us denote by Φ_t the Poincaré map at time t of system (1.1), if defined. Then, by Lemma 2, there exists $\omega_1 > 0$ such that

$$\Phi_{t^1_3}(S^-_{1,\tilde{\omega}_0}) \subset \left\{(r,\theta) : 0 < r \le \varepsilon,\ 0 < \omega_1 \le \theta < \frac{3\pi}{2}\right\}.$$

Therefore, as for $S^+_{1,\tilde{\omega}_0}$, there is M_2 such that $S^-_{1,\tilde{\omega}_0}$ completes one turn in $[0, t^3_3]$ for all $\lambda > \max\{M_1, M_2\}$.

Adapting the previous argument, it readily follows the existence of $\tilde{M}_1, \tilde{M}_2 > 0$ such that the solutions of (1.1) with initial data in

$$S_3 := \{z = (x, y) \in \mathbb{R}^2 : \|z\| = r_0 \text{ and } x < 0,\ y < 0\}$$

have a rotation number greater than one for all $\lambda > \max\{\tilde{M}_1, \tilde{M}_2\}$. Consequently, for every $z_0 \in S_1 \cup S_3$, it is apparent that $\mathrm{rot}(z_0; [0, nT]) > 1$ for all $\lambda > \Lambda_1 := \max\{M_1, M_2, \tilde{M}_1, \tilde{M}_2\}$.

Case 2: Assume that $x_0 y_0 < 0$. Most of the attention will be focused to the special case when $x_0 < 0$ and $y_0 > 0$, as the case $x_0 > 0$ and $y_0 < 0$ is analogous. Obviously, in this case, $\theta(t^1_0) \in (\frac{\pi}{2}, \pi)$. As in case 1, it should be proved the existence of a uniform bound, Λ_2, such that the solutions with initial data in the quadrant sector

$$S_2 := \{z = (x, y) \in \mathbb{R}^2 : \|z\| = r_0 \text{ and } x < 0,\ y > 0\}$$

have rotation number greater than one for all $\lambda > \Lambda_2$ if $0 < r_0 < \varepsilon$. By Lemma 1, there exists $\omega_2 > \frac{\pi}{2}$ such that

$$\Phi_{t^1_1}(S_2) \subset \left\{(r,\theta) : 0 < r \le \varepsilon,\ \frac{\pi}{2} < \omega_2 \le \theta < \pi\right\}.$$

Thus, as in case 1, we have already proven that, once the solution reaches the second quadrant, being separated away from $\frac{\pi}{2}$, it must have a rotation number greater than one for sufficiently large λ (which was a direct consequence from Lemmas 1 and 2), there exists $\Lambda_2 > 0$ such that the solution with $\theta(t^1_1) = \omega_2 > \frac{\pi}{2}$ completes one turn for all $\lambda > \Lambda_2$. Moreover, by the monotonicity properties of Lemmas 1 and 2, the solutions with initial data in S_2 have rotation number greater than one for all $\lambda > \Lambda_2$. Since the previous argument can be easily adapted to deal with

$$S_4 := \{z = (x, y) \in \mathbb{R}^2 : \|z\| = r_0 \text{ and } x > 0,\ y < 0\},$$

it becomes apparent that, for every $z_0 \in S_2 \cup S_4$, $\mathrm{rot}(z_0; [0, nT]) > 1$ for all $\lambda > \Lambda_2$.

Case 3: Assume $x_0 y_0 = 0$, i.e., (x_0, y_0) lies on some coordinate axis. Without loss of generality, we can assume that $x_0 > 0$ and $y_0 = 0$, as the remaining cases can be treated similarly. Then, since $y_0 = 0$ and $\beta = 0$ on $[t^1_0, t^1_1]$, integrating (1.1) yields $\theta(t^1_0) = \theta(t^1_2) = 0$. Thus, by Lemma 2, for every $\omega \in (0, \frac{\pi}{2})$, there exists $\mu_1 := \mu_1(\omega)$ such that $\theta(t^1_3) > \omega$ for all $\lambda > \mu_1$. Thus, much like in case 1, owing to Lemmas 1 and 2, there exist $\mu_2, \mu_3, \mu_4, \mu_5 > 0$, depending on ω, such that

$$\begin{aligned}
&\theta(t^2_1) > \pi - \omega && \text{if } \lambda > \max\{\mu_1, \mu_2\},\\
&\theta(t^2_3) > \pi + \omega && \text{if } \lambda > \max\{\mu_1, \mu_2, \mu_3\},\\
&\theta(t^3_1) > 2\pi - \omega && \text{if } \lambda > \max\{\mu_1, \mu_2, \mu_3, \mu_4\},\\
&\theta(t^3_3) > 2\pi && \text{if } \lambda > \max\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\} =: \Lambda_{3,1}.
\end{aligned}$$

Therefore, the solution completes one turn in the time interval $[0, t^3_3]$ for all $\lambda > \Lambda_{3,1}$. Similarly, it can be easily shown that the solutions complete a turn in each of the remaining three cases when $x_0 = 0$ and $y_0 > 0$, $x_0 < 0$ and $y_0 = 0$, or $x_0 = 0$ and $y_0 < 0$, for $\lambda > \Lambda_{3,2}$, $\lambda > \Lambda_{3,3}$ and $\lambda > \Lambda_{3,4}$, respectively. Thus, taking

$$\Lambda_3 := \max\{\Lambda_{3,1}, \Lambda_{3,2}, \Lambda_{3,3}, \Lambda_{3,4}\},$$

it becomes apparent that $\mathrm{rot}(z_0; [0, nT]) > 1$ provided $\lambda > \Lambda_3$ and

$$z_0 \in S_0 := \{z = (x, y) \in \mathbb{R}^2 : \|z\| = r_0 \text{ and } xy = 0\}.$$

Subsequently, we set

$$\lambda_n := \max\{\Lambda_1, \Lambda_2, \Lambda_3\}.$$

By Remark 2 and the analysis already done in the proof of the theorem, it is apparent that, for every $\lambda > \lambda_n$, there exists $0 < r_0 < \delta(n, \lambda, \varepsilon)$ such that, for every $z_0 = (x_0, y_0)$ with $\|z_0\| = r_0$,

$$(x(t), y(t)) \in D_\varepsilon \quad \text{for all } t \in [0, nT]$$

and

$$\operatorname{rot}(z_0; [0, nT]) > 1. \tag{2.5}$$

In order to apply Theorem 2, it remains to prove that, for sufficiently large $\lambda > 0$, the large solutions do not rotate. As the proof of this feature follows the general scheme of the proof of [15, Theorem 2.1], we will simply sketch it here. Being analogous the remaining cases, the proof will be delivered in the special case when condition (g_-) holds in (1.4).

We proceed by contradiction assuming that, regardless the size of the initial data (x_0, y_0), the solution $(x(t), y(t))$ completes at least one turn for sufficiently large λ. Thus, without loss of generality, changing the initial data if necessary, we can assume that $(x(t), y(t))$ goes across the entire third quadrant. In such a case, there is an interval $[s_0, s_1] \subset [0, nT]$ such that $y(s_0) = 0 = x(s_1)$ and $x(t) < 0$, $y(t) < 0$ for every $t \in (s_0, s_1)$. Thus, by (g_-), it becomes apparent that, setting $B := \int_0^T \beta(s)\,ds$,

$$|y(t)| = \lambda \left| \int_{s_0}^{t} \beta(s) g(x(s))\,ds \right| \le \lambda M \int_0^{nT} \beta(s)\,ds = \lambda M n B$$

for every $t \in [s_0, s_1]$. Hence, defining $N := \max\{|f(y)| : |y| \le \lambda M n B\}$, it follows that, for every $t \in [s_0, s_1]$,

$$|x(t)| = \left| \lambda \int_{t}^{s_1} \alpha(s) f(y(s))\,ds \right| \le \lambda N \int_0^{nT} \alpha(s)\,ds = \lambda N n A,$$

where $A := \int_0^T \alpha(s)\,ds$. Suppose that, for some $\tilde{t} \in [0, nT]$,

$$x^2(\tilde{t}) + y^2(\tilde{t}) > \lambda^2 n^2 (M^2 B^2 + N^2 A^2) \equiv R^2$$

with $x(\tilde{t}) < 0$ and $y(\tilde{t}) < 0$. Then the solution $(x(t), y(t))$ cannot cross entirely the third quadrant. At this stage, the proof follows almost *mutatis mutandis* the steps of the proof of [15, Theorem 2.1], where the reader is sent for any further details. According to it, there exists a radius $R_0 \ge R$ such that, for every solution with $z_0 = x_0^2 + y_0^2 \ge R_0$,

$$\operatorname{rot}(z_0; [0, nT]) < 1. \tag{2.6}$$

By (2.5) and (2.6), the twist condition holds, and hence, by Theorem 2, system (1.1) admits at least two nontrivial nT-periodic solutions belonging to different periodicity classes with rotation number $\omega \le n$ for sufficiently large λ. This concludes the proof. □

In order to apply Theorem 2, the distribution of the weight functions settled by (2.2) is *optimal*. Indeed, if $\alpha_i = 0$ or $\beta_i = 0$ for some $i \in \{1, 2, 3\}$, then each of the points $(-r_0, 0)$ and $(r_0, 0)$, for sufficiently small $r_0 > 0$, have rotation number less than one in the interval $[0, T]$.

Remark 3. As already observed in [15, Remark 3], without any significant change in the proof, a slightly more general version of Theorem 2 can be proven by assuming f, g only continuous (and not locally Lipschitz) and replacing the condition on the derivatives in (1.3) with the following one:

$$0 < \liminf_{|y| \to 0} \frac{f(y)}{y} \le \limsup_{|y| \to 0} \frac{f(y)}{y} < \infty, \quad 0 < \liminf_{|x| \to 0} \frac{g(x)}{x} \le \limsup_{|x| \to 0} \frac{g(x)}{x} < \infty.$$

To this aim, instead of Theorem 2, one can apply the generalized version of the Poincaré–Birkhoff theorem due to Fonda and Ureña [11] for Hamiltonian systems where the uniqueness of the solutions of the initial value problems is not required (see also [8, Theorem 10.6.1] for the precise statement).

3 Counting T-Periodic Solutions and Subharmonics of (1.1)

This section applies Theorem 3 to model (1.1) when condition (2.2) holds. Recall that either $k = \ell$, or $|k - \ell| = 1$ and $m = \min\{k, \ell\}$. Based on Theorem 3, the next result holds.

Theorem 4. *Assume that $nm \geq 3$ for some integer $n \geq 1$. Then there exists $\lambda_n > 0$ such that, for every $\lambda > \lambda_n$, (1.1) possesses at least $\sigma(n)$ periodic solutions with period nT, where*

$$\sigma(n) := \begin{cases} 2hm & \text{if } n = 3h, \\ 2(hm + [\frac{m}{3}]) & \text{if } n = 3h + 1, \\ 2(hm + [\frac{2m}{3}]) & \text{if } n = 3h + 2. \end{cases}$$

Moreover, setting

$$\gamma(n) := \min\Big\{\gamma \geq 0 : \gcd\Big(n, \frac{\sigma(n)}{2} - \gamma\Big) = 1\Big\},$$

it turns out that, for every $\lambda > \lambda_n$, (1.1) has at least $\sigma(n) - 2\gamma(n)$ periodic solutions with minimal period nT.

Proof. Suppose $k = \ell$. Then $m = k = \ell$. Hence, according to (1.7), the total number of α and β-intervals in $[0, nT]$ equals

$$2nk = 2nm. \tag{3.1}$$

Thus, if $n = 3h$ for some integer $h \geq 1$, the sum of α-intervals and β-intervals in $[0, nT]$ is $6hk$. Hence, by Theorem 3, there exists $\lambda_n > 0$ such that, for every $\lambda > \lambda_n$, the solutions of (1.1) with sufficiently small $z_0 = (x_0, y_0)$ complete hk turns, whereas the solutions with sufficiently large z_0 cannot complete any. Therefore, by Theorem 2, (1.1) has at least two nT-periodic coexistence states with rotation number $j \in \{1, 2, \ldots, hk\}$. Consequently, (1.1) possesses at least $2hk = \sigma(n)$ coexistence states with period nT.

Now, assume that $n = 3h + 1$ for some integer $h \geq 0$. Then there are a total of

$$2mn = 2k(3h + 1) = 6hk + 2k = 6\Big(hk + \frac{k}{3}\Big)$$

α and β-intervals in $[0, nT]$. Thus, by Theorem 3, there exists $\lambda_n > 0$ such that, for every $\lambda > \lambda_n$, the solutions of (1.1) with sufficiently small z_0 complete $hk + [\frac{k}{3}]$ turns, while the solutions with large initial data cannot rotate. Therefore, thanks to Theorem 2, (1.1) possesses at least

$$2\Big(hk + \Big[\frac{k}{3}\Big]\Big) = \sigma(n)$$

periodic coexistence states of period nT.

Similarly, according to Theorems 2 and 3, when $n = 3h + 2$ for some integer $h \geq 0$, there exists $\lambda_n > 0$ such that, for every $\lambda > \lambda_n$, (1.1) possesses at least

$$2\Big(hk + \Big[\frac{2k}{3}\Big]\Big) = \sigma(n)$$

coexistence states with period nT.

The last assertion of the theorem will be derived from the fact that, owing to Remark 1, any nT-periodic coexistence state of (1.1) such that, for some $0 < r_0 < R_0$, it satisfies

$$\begin{cases} \operatorname{rot}(z_0; [0, nT]) > \omega & \text{if } \|z_0\| = r_0, \\ \operatorname{rot}(z_0; [0, nT]) < \omega & \text{if } \|z_0\| = R_0, \end{cases} \tag{3.2}$$

has minimal period nT if $\gcd(n, \omega) = 1$. In all the cases covered by Theorem 4, we have actually proven the existence of $0 < r_0 < R_0$ such that

$$\begin{cases} \operatorname{rot}(z_0; [0, nT]) > \frac{\sigma(n)}{2} & \text{if } \|z_0\| = r_0, \\ \operatorname{rot}(z_0; [0, nT]) < 1 & \text{if } \|z_0\| = R_0, \end{cases}$$

by the definition of $\sigma(n)$. Thus, (3.2) holds for the choice $\omega = \frac{\sigma(n)}{2}$. In the case $\gcd(n, \frac{\sigma(n)}{2}) = 1$, by Remark 1, problem (1.1) possesses at least $\sigma(n)$ coexistence states with minimal period nT. This ends the proof in this case because we can take $\gamma = 0$ in (3.1), and hence, $\gamma(n) = 0$.

Subsequently, we assume that $\gcd(n, \frac{\sigma(n)}{2}) \neq 1$ and consider the unique integer $j \geq 1$ such that

$$\gcd\Big(n, \frac{\sigma(n)}{2} - j\Big) = 1 \quad \text{and} \quad \gcd\Big(n, \frac{\sigma(n)}{2} - i\Big) \neq 1 \quad \text{for all } 0 \leq i < j. \tag{3.3}$$

In such a case, we can make the choice $\omega = \frac{\sigma(n)}{2} - j$. By (3.3), $\gcd(n, \omega) = 1$. Moreover, as soon as $\|z_0\| = r_0$, we have that

$$\operatorname{rot}(z_0; [0, nT]) > \frac{\sigma(n)}{2} > \frac{\sigma(n)}{2} - j = \omega.$$

And due to (3.3), it is apparent that, whenever $\|z_0\| = R_0$,

$$\operatorname{rot}(z_0; [0, nT]) < 1 \leq \frac{\sigma(n)}{2} - j = \omega.$$

Indeed, if $\frac{\sigma(n)}{2} - j < 1$, then there exists $0 \leq i < j$ such that $\frac{\sigma(n)}{2} - i = 1$, and hence,

$$\gcd\Big(n, \frac{\sigma(n)}{2} - i\Big) = \gcd(n, 1) = 1,$$

contradicting the minimality of j. Therefore, by Remark 1, it becomes apparent that (1.1) has at least

$$2\omega = 2\Big(\frac{\sigma(n)}{2} - j\Big) = \sigma(n) - 2j = \sigma(n) - 2\gamma(n)$$

coexistence states with minimal period nT. The proof is complete when $k = \ell$.

Now, assume that $k = \ell + 1$. Then $m = \ell$. Thus, according to (1.8), the total number of the α and β-intervals in $[0, nT]$ is $nm + 1 + nm = 2nm + 1$. As the integers $2nm + 1$ and $2nm$, going back to (3.1), have the same divisibility properties by 6, the result when $k = \ell + 1$ follows the same patterns as for $k = \ell$. Similarly, the same result holds when $\ell = k + 1$. This concludes the proof. □

Remark 4. As far as it concerns the cases not treated in this paper when $n(k + \ell) \leq 5$, so far, it is known that if $n(k + \ell) \leq 3$, then (1.1) does not admit any nT-periodic solutions because the condition $|Z| = 0$ ensures that no solution of (1.1) different from $(0, 0)$, say $(x(t), y(t))$, can complete one turn around the origin. Thus, it cannot satisfy $(x(0), y(0)) = (x(nT), y(nT))$ for some $n \geq 1$. The cases when $n(k + \ell) = 4, 5$ remain outside the general scope of this paper and will be analyzed elsewhere.

4 An Application to a Class of Predator-Prey Models

The non-autonomous planar Hamiltonian system (1.1) covers a large number of mathematical models of physical and biological nature. In particular, for the special choice $f(y) = e^y - 1$ and $g(x) = e^x - 1$, system (1.1) can be written, through the change of variables $x = \log u$ and $y = \log v$, as

$$\begin{cases} u' = \lambda\alpha(t)u(1 - v), \\ v' = \lambda\beta(t)v(-1 + u), \end{cases} \tag{4.1}$$

which is a non-autonomous T-periodic predator-prey model of Volterra type. As shown in [1, Section 5] and in [15, Introduction], system (4.1) can be obtained from the Volterra system with periodic coefficients

$$\begin{cases} p' = \lambda p(a(t)p - b(t)q), \\ q' = \lambda q(-c(t) + d(t)p), \end{cases}$$

after a suitable change of variables. It is clear that the (nontrivial) nT-periodic solutions of (1.1) are the nT-periodic coexistence states of (4.1). By a coexistence state, it is meant a component-wise positive solution pair.

This model was introduced in a degenerate setting in [13, 16] and later analyzed in [14] in the very special case when supp $\alpha \subset [0, \frac{T}{2}]$ and supp $\beta \subset [\frac{T}{2}, T]$. Since the functions $f(y) = e^y - 1$, $g(x) = e^x - 1$ satisfy (1.3) and (1.4), according to Theorems 2, 3 and 4, system (4.1) has at least $\sigma(n)$ coexistence states with period nT provided $n(k + \ell) \geq 6$, among them, $\sigma(n) - 2\gamma(n)$ with minimal period nT. By Remark 4, system (4.1) cannot admit any nT-periodic coexistence state if $n(k + \ell) \leq 3$.

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