On some topological realizations of groups and homomorphisms

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Abstract

Let $f : G \to H$ be a homomorphism of groups. We construct a topological space X_f such that its group of homeomorphisms is isomorphic to G, its group of homotopy classes of self-homotopy equivalences is isomorphic to H and the natural map between the group of homeomorphisms of X_f and the group of homotopy classes of self-homotopy equivalences of X_f is f. In addition, we consider realization problems involving homology, homotopy groups and groups of automorphisms.

1 Introduction

Alexandroff spaces are topological spaces with the property that the arbitrary intersection of open sets is open. That sort of topological spaces was first studied by P.S. Alexandroff in [1], where it is shown that they can also be seen as partially ordered sets. This viewpoint can be used to express topological notions in combinatorial terms. A particular case of Alexandroff spaces are finite topological spaces. There are two foundational papers on this subject that were published independently in 1966 [15, 17]. In [17], R.E. Stong made an analysis of the homeomorphism classification of finite topological spaces using matrices and also introduced combinatorial techniques to study their homotopy type. In [15], M.C. McCord studied the singular homology groups and homotopy groups of Alexandroff spaces proving that for every Alexandroff space X there exists a simplicial complex $\mathcal{K}(X)$ sharing the same homotopy groups and singular homology groups. In fact, it is shown that there is a continuous map $f_X : |\mathcal{K}(X)| \to X$ inducing isomorphism on all homotopy groups. The converse result is also obtained, that is, given a simplicial complex L, there exists an Alexandroff space $\mathcal{X}(L)$ having the same singular homology groups and homotopy groups of L.

Finite topological spaces or Alexandroff spaces are a good tool to solve realization problems: given a category C and a group G, is there an object X in C such that the group of automorphisms of X is isomorphic to G? In [6, 18, 5], it is proved that for every finite group G there exists a finite topological space X_G such that its group of homeomorphisms is isomorphic to G. Recently, in [4], J.A. Barmak constructed a finite topological space X_G with $4|G|$ points and lower cardinality than the finite topological spaces obtained in [6, 18, 5]. On the other hand, L. Babai in [2] obtained a finite topological space X_G with $3|G|$ points realizing G as a group of homeomorphisms. This topological space has the disadvantage that

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it requires to find a "good" set of generators of G satisfying a list of non-trivial properties. A generalization of these results for non-finite groups was made in [7], where the realization problem for the homotopical category $HTop$ and the pointed homotopical category $HTop_*$ was also solved. To be precise, it was proved that for every group G there exists an Alexandroff space $\overline{X}^*_G$ such that its group of automorphisms in the topological category Top, $HTop$ and $HTop_*$ is isomorphic to G.

The restriction of $HTop$ to topological spaces with the homotopy type of a CW-complex is denoted by $HPol$. In $HPol$ and the pointed version $HPol_*$, the realizability problem has a long history, see for instance [12]. Recently, C. Costoya and A. Viruel solved the realization problem in $HPol_*$ for finite groups in [9].

We introduce a bit of notation. Let X be a topological space. Let us denote by $Aut(X)$ the group of homeomorphisms of X. Let $\mathcal{E}(X)$ denote the group of homotopy classes of self-homotopy equivalences of X. Let $\mathcal{E}_*(X)$ denote the group of pointed homotopy classes of pointed self-homotopy equivalences of X.

The following result answers a natural question posed by professor Jesús Antonio Álvarez López during a talk in the VIII Encuentro de Jóvenes Topólogos (A Coruña, 2019).

Lemma 1.1. *Let G and H be two groups. There exists a topological space X^G_H such that $Aut(X^G_H)$ is isomorphic to G and $\mathcal{E}(X^G_H)$ is isomorphic to H.*

An immediate consequence of Lemma 1.1 is the following: for a general topological space there is no relation between its group of homeomorphisms and its group of homotopy classes of self-homotopy equivalences. Moreover, if G and H are two groups, then we can produce infinitely many non-homeomorphic topological spaces having G as their group of homeomorphisms and having H as their group of homotopy classes of self-homotopy equivalences.

For every topological space X there is a natural homomorphism of groups $\tau : Aut(X) \to \mathcal{E}(X)$ sending each homeomorphism f to its homotopy class $[f]$. Given two groups G and H, we consider the topological space X^H_G obtained in Lemma 1.1. The kernel of $\tau : Aut(X^H_G) \to \mathcal{E}(X^H_G)$ corresponds precisely to $Aut(X^G_H)$. The image of τ is the homotopy class of the identity map. However, modifying the construction of the topological space X^G_H, we can obtain a stronger version of Lemma 1.1.

Theorem 1.2. *Let $f : G \to H$ be a homomorphism of groups. There exists a topological space X_f such that $Aut(X_f) = G$, $\mathcal{E}(X_f) = H$ and $\tau = f$.*

Theorem 1.2 clearly generalizes Lemma 1.1. We prefer to prove Lemma 1.1 first for the sake of exposition. Omitting Lemma 1.1, the proof of Theorem 1.2 becomes less intuitive. In addition, subsequent results are obtained using the topological space given in Lemma 1.1.

Given a topological space X and a non-negative integer number n, we can consider the n-th homology group $H_n(X)$ of X or the n-th homotopy group $\pi_n(X)$ of X. It is natural to consider more realization problems involving these groups, the group of homeomorphisms and the group of homotopy classes of self-homotopy equivalences.

Theorem 1.3. *Let G and H be finite groups and let X be a topological space with the homotopy type of a compact CW-complex. There exists an Alexandroff space $\overline{X}^G_H$ such that $Aut(\overline{X}^G_H)$ is isomorphic to G, $\mathcal{E}(\overline{X}^G_H)$ is isomorphic to H and X is weak homotopy equivalent to $\overline{X}^G_H$, which implies that $H_n(\overline{X}^G_H)$ is isomorphic to $H_n(X)$ and $\pi_n(\overline{X}^G_H)$ is isomorphic to $\pi_n(X)$ for every $n \in \mathbb{N}$.*

As an immediate consequence of Theorem 1.3, we can deduce the following corollaries.

Corollary 1.4. *Let H and G be finite groups and let $\{F_i\}_{i\in I}$ be a set of finitely generated Abelian groups, where $I \subset \mathbb{N}$ is a finite set. There exists a topological space X such that $Aut(X)$ is isomorphic to G, $\mathcal{E}(X)$ is isomorphic to H and $H_i(X)$ is isomorphic to F_i for every $i \in I$.*

Corollary 1.5. *Let H and G be finite groups, $n \in \mathbb{N}$ and let T be a finitely presented (Abelian) group (if $n > 1$). There exists a topological space X such that $Aut(X)$ is isomorphic G, $\mathcal{E}(X)$ is isomorphic to H and $\pi_n(X)$ is isomorphic to T.*

Roughly speaking, these corollaries say that for a general topological space X its group of automorphisms in Top or $HTop$ does not have any relation to its n-th homology or homotopy group and vice versa. In contrast, for the category $HPol$, the situation is completely different since $\mathcal{E}(X)$ contains normal subgroups that are nilpotent. For instance, given a topological space X, we denote by $\mathcal{E}_{\#}(X)$ ($\mathcal{E}_*(X)$) the set of self-homotopy equivalences that induce the identity map in homotopy (homology). It is trivial to check that $\mathcal{E}_{\#}(X)$ ($\mathcal{E}_*(X)$) is a normal subgroup of $\mathcal{E}(X)$ and if X is a finite CW-complex, then $\mathcal{E}_{\#}(X)$ ($\mathcal{E}_*(X)$) is a nilpotent group. See [10] for more details. Using the construction obtained in Theorem 1.3, we can find topological spaces that do not satisfy the previous properties. From this we get that some of the techniques used to study the group of self-homotopy equivalences for CW-complexes cannot be adapted in a natural way to general spaces.

The organization of the paper is as follows. In Section 2 we introduce basic concepts and results from the literature. In Section 3 we provide an example of one of the main results in order to motivate the main ideas of the proof of Lemma 1.1. In Section 4 we prove Lemma 1.1 and give some remarks. In Section 5 we prove Theorem 1.2. In Section 6 we define a sequence of topological spaces whose homotopy and homology groups are all trivial, their group of automorphisms in Top and $HTop$ are also trivial, but they are not homeomorphic or homotopy equivalent to a point. Then, we prove Theorem 1.3 as well as Corollary 1.4 and Corollary 1.5. Finally, we give examples of topological spaces satisfying that the groups $\mathcal{E}_*(\cdot)$ and $\mathcal{E}_{\#}(\cdot)$ are not nilpotent in general.

2 Preliminaries

The following definitions and results can be found with more detail in [1, 3, 15, 17, 14] .

Definition 2.1. *Let X and Y be topological spaces. A continuous function $f : X \to Y$ is said to be a weak homotopy equivalence if it induces isomorphisms on all the homotopy groups.*

Definition 2.2. *An Alexandroff space X is a topological space satisfying that the arbitrary intersection of open sets is open.*

If X is an Alexandroff space, then for every $x \in X$ there exists a minimal open neighbourhood U_x given by the intersection of every open set containing x. F_x denotes the set given by the intersection of every closed set containing x. Trivially, every finite topological space is an Alexandroff space. An Alexandroff space X is locally finite if for every $x \in X$ the set U_x is finite.

Let $(X, \leq)$ be a partially ordered set or poset. If $x, y \in X$, then we write $x \prec y$ ($x \succ y$) if and only if $x < y$ ($x > y$) and there is no $z \in X$ such that $x < z < y$ ($x > z > y$). We will denote by $max(X)$ the maximum of X if it exists. We denote by $P_x = (E_x, S_x)$ the cardinal numbers $E_x = |\{y \in X | y \prec x\}|$ and $S_x = |\{y \in X | x \prec y\}|$. A set $S \subseteq X$ is called lower (upper) if for every $x \in S$ and $y \leq x$ ($y \geq x$) we have $y \in S$.

It is not difficult to verify the following two properties:

- For a partially ordered set $(X, \leq)$ the family of lower (upper) sets of $\leq$ is a T_0 topology on X, that makes X a T_0 Alexandroff space.
- For a T_0 Alexandroff space, the relation $x \leq_\tau y$ if and only if $U_x \subset U_y$ ($U_y \subset U_x$) is a partial order on X.

In addition, for a set X, the T_0 Alexandroff space topologies on X are in bijective correspondence with the partial orders on X.

From now on, every Alexandroff space satisfies the T_0 separation axiom. The following results can be found, for instance, in [3, 14].

Proposition 2.3. *If $f : X \to Y$ is a map between Alexandroff spaces, then f is continuous if and only if f preserves the order.*

From this and previous properties, the following result can be deduced.

Theorem 2.4. *The category of T_0 Alexandroff spaces is isomorphic to the category of partially ordered sets.*

Hence, partially ordered sets and T_0 Alexandroff spaces can be treated as the same object.

Proposition 2.5. *Let $f, g : X \to Y$ be continuous maps between Alexandroff spaces. If $f(x) \leq g(x)$ ($f(x) \geq g(x)$) for every $x \in X$, then f and g are homotopic.*

Remark 2.6. *If X is a T_0 Alexandroff space with a minimum (maximum) $x*$, then X is contractible to $x*$. This follows from the previous proposition and the fact that the constant map $c : X \to X$ given by $c(x) = x*$ satisfies that $c(x) \leq x$ ($c(x) \geq x$) for every $x \in X$.*

Definition 2.7. *Given a finite poset $(X, \leq)$, the height $ht(X)$ of X is one less than the maximum number of elements in a chain of X. The height of a point x in a locally finite Alexandroff space is given by $ht(U_x)$. For a general Alexandroff space X, the height of a point $x \in X$ is defined as ∞ if U_x contains a chain without a minimum and $ht(U_x)$ otherwise.*

Example 2.8. Let us consider the real numbers with the usual order. For every $x \in \mathbb{R}$ the height of x is ∞ because the chain $... < x - n < ... < x - 2 < x - 1 < x$ does not have a minimum. Moreover, $P_x = (0, 0)$ since there is no $y \in \mathbb{R}$ satisfying that $x \prec y$ or $x \succ y$. Let us consider $X = \mathbb{N} \cup \{*\}$, where we consider the partial order defined as follows: $n < *$ for every $n \in \mathbb{N}$. It is clear that U_* is not a finite set but the height of $*$ is 1. Furthermore, $P_* = (|\mathbb{N}|, 0)$.

Proposition 2.9. *Let X and Y be Alexandroff spaces. If $f : X \to Y$ is a homeomorphism and $x \prec y$, then $f(x) \prec f(y)$ ($f(x) \succ f(y)$). Furthermore, for every $x \in X$ the height of x is equal to the height of $f(x)$ and $P_x = P_{f(x)}$.*

The following results provide a combinatorial way to study the homotopy and weak homotopy type of finite topological spaces.

Definition 2.10. *Let X be an Alexandroff space. A point x in X is a down beat point (resp. up beat point) if $U_x \setminus \{x\}$ has a maximum (resp. $F_x \setminus \{x\}$ has a minimum). A finite T_0 topological space X is a minimal finite space if it has no beat points. A core of a finite topological space is a strong deformation retract which is a minimal finite space.*

Proposition 2.11 ([17])**.** *Let X be an Alexandroff space and let $x \in X$ be a beat point. Then $X \setminus \{x\}$ is a strong deformation retract of X.*

If X is a finite T_0 topological space, then X has a core. We only need to remove beat points one by one to obtain a minimal finite space.

Theorem 2.12 ([17])**.** *If X is a minimal finite space, then $f : X \to X$ is a homeomorphism if and only if f is a homotopy equivalence.*

Corollary 2.13. *If X is a minimal finite space, then $Aut(X)$ is isomorphic to $\mathcal{E}(X)$.*

Remark 2.14. *The result of Corollary 2.13 can be stated in a stronger way. Let X be a minimal finite space. It is easy to check that $Aut(X) = \mathcal{E}(X)$. For every $[f]$ in $\mathcal{E}(X)$ there is only one element in the class $[f]$. We can identify every homeomorphism with its homotopy class.*

Remark 2.15. *Corollary 2.13 can be generalized to Alexandroff spaces. To do this, consider the notion of being locally a core introduced in [13]. This notion generalizes the notion of minimal finite space, that is, every minimal finite space is locally a core. Let X be locally a core. Then a continuous map $f : X \to X$ is a homeomorphism if and only if f is a homotopy equivalence. Moreover, we get that $Aut(X) \simeq \mathcal{E}(X)$.*

Definition 2.16. *Let X be a finite T_0 topological space. A point x in X is a down weak beat point (resp. up weak beat point) if $U_x \setminus \{x\}$ is contractible (resp. $F_x \setminus \{x\}$ is contractible).*

Proposition 2.17 ([3])**.** *Let X be a finite T_0 topological space and let $x \in X$ be a weak beat point. Then the inclusion $i : X \setminus \{x\} \to X$ is a weak homotopy equivalence.*

Definition 2.18 ([3])**.** *Let X be a finite T_0 topological space and let $Y \subset X$. It is said that X collapses to Y by an elementary collapse if Y is obtained from X by removing a weak beat point. Given two finite T_0 topological spaces X and Y, X collapses to Y if there is a sequence $X = X_1, X_2, ..., X_n = Y$ of finite T_0 topological spaces such that for each $1 \leq i < n$, X_i collapse to X_{i+1} by an elementary collapse.*

Lemma 2.19. *Let X and Y be Alexandroff spaces and let $f : X \to Y$ be a homeomorphism. Then $x \in X$ is a down (up) weak beat point if and only if $f(x)$ is a down (up) weak beat point.*

Proof. There is no loss of generality in assuming that x is a down weak beat point. If x is an up weak beat point, then the argument is similar. It is easy to see that $f(U_x) = U_{f(x)}$. Therefore, $f(U_x \setminus \{x\}) = U_{f(x)} \setminus \{f(x)\}$ and we get the desired result.

□

We recall the notion of a Hasse diagram for a locally finite Alexandroff space X. The Hasse diagram $H(X)$ of X is a directed graph. The vertices of $H(X)$ are the points of X. There is an edge between two vertices x and y if and only if $x \prec y$ and the orientation of the edge is from the lower element to the upper element. We omit the orientation of the subsequent Hasse diagrams and we assume an upward orientation.

Remark 2.20. *It is easy to identify beat points of a finite topological space X by looking at its Hasse diagram. A vertex x is a down beat point (resp. up beat point) if there is only one edge that enters (exits) it, i.e., $P_x = (a, b)$, where $a = 1$ ($b = 1$).*

The homotopy and singular homology groups of Alexandroff spaces were studied in [15].

Definition 2.21. *Let X be an Alexandroff space. Its McCord complex or order complex $\mathcal{K}(X)$ is the simplicial complex whose simplices are the non-empty chains of X. Let L be a simplicial complex. The face poset of L, denoted by $\mathcal{X}(L)$, is defined to be the poset of simplices of L ordered by inclusion.*

Remark 2.22. *A finite T_0 topological space X is said to be collapsible if it collapses to a point. If X is a collapsible finite T_0 topological space, then $\mathcal{K}(X)$ is also collapsible.*

The geometric realization of a simplicial complex K is denoted by $|K|$.

Theorem 2.23. *[15] Given an Alexandroff space X, there exists a weak homotopy equivalence $f : |\mathcal{K}(X)| \to X$.*

Theorem 2.24. *[15] Given a simplicial complex L, there exists a weak homotopy equivalence $f : |L| \to \mathcal{X}(L)$.*

Finally, we recall some remarks and a definition. For a more complete treatment we refer to the reader to [3].

Definition 2.25. *The non-Hausdorff join $X \circledast Y$ of two Alexandroff spaces X and Y is the disjoint union $X \sqcup Y$ keeping the given ordering within X and Y and setting $y \leq x$ for every $x \in X$ and $y \in Y$.*

Remark 2.26. *Given two Alexandroff spaces X and Y, $\mathcal{K}(X \circledast Y) = \mathcal{K}(X) * \mathcal{K}(Y)$, where $*$ denotes the usual join of simplicial complexes. If X and Y are finite topological spaces and one of them is collapsible, then $\mathcal{K}(X \circledast Y)$ is collapsible.*

Proposition 2.27. *Let X and Y be two Alexandroff spaces. Then $Aut(X \circledast Y) = Aut(X) \times Aut(Y)$.*

3 Examples and motivation of the proof of Lemma 1.1

We present an example to illustrate the idea of the construction given in the proof of Lemma 1.1.

Example 3.1. Let us consider the Klein four-group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, where we denote $g_1 = (0, 0), g_2 = (1, 0), g_3 = (0, 1)$ and $g_4 = (1, 1)$, and the cyclic group of two elements $\mathbb{Z}_2$, where we denote $h_1 = 0$ and $h_2 = 1$. We also denote $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $H = \mathbb{Z}_2$ for simplicity. Moreover, let

$S_G = \{g_2, g_3\}$, $S_H = \{h_2\}$ be generating sets of G, H respectively. We declare $g_2 < g_3$. Our goal is to find a finite T_0 topological space X^G_H such that $Aut(X^G_H) \simeq G$ and $\mathcal{E}(X^G_H) \simeq H$.

By [5, 7], there exists a finite T_0 topological space X^G satisfying that $Aut(X^G)$ is isomorphic to G. In Figure 1 we have represented in blue the Hasse diagram of X^G. It is clear that adding to X^G a minimum $*$, i.e., $X^G_* = X^G \cup \{*\}$ with $* < x$ for every $x \in X_G$, we get that $\mathcal{E}(X^G_*)$ is trivial since X^G_* is contractible. On the other hand, if $f \in Aut(X^G_*)$, then we have that $f(*) = *$ because $*$ is a minimum. Thus we deduce that $Aut(X^G_*)$ is isomorphic to $Aut(X^G)$. Our next goal is to find a topological space X^*_H satisfying that $Aut(X^*_H)$ is trivial and $\mathcal{E}(X^*_H)$ is isomorphic to H. Again, by [7], there exists a finite T_0 topological space X_H satisfying that $\mathcal{E}(X_H) \simeq Aut(X_H) \simeq H$. In Figure 1, the Hasse diagram of X_H corresponds to the red and black parts of the diagram on the right. We modify X_H in order to reduce the number of self-homeomorphisms without changing the number of self-homotopy equivalences. For this purpose we add some points to X_H. We consider $X^*_H = X_H \cup \{w_{h_1}, w_{h_2}, a\}$, where we have the following relations: $A_{(h_1,0)} \prec w_{h_1}$ and $A_{(h_2,0)} \prec w_{h_2} \prec a$. The Hasse diagram of X^*_H can be seen on the right on Figure 1, where the new points are pictured in orange. It is easy to check that $\mathcal{E}(X^*_H) \simeq \mathcal{E}(X_H) \simeq H$. The new points are beat points so we can remove them without changing the homotopy type of X^*_H. Hence, we have that X^*_H and X_H have the same homotopy type. We prove that $Aut(X^*_H)$ is trivial. If $f \in Aut(X^*_H)$, then $f(M) = M$ and $f(N) = N$, where M and N denote the set of maximal and minimal elements of X^*_H respectively. From this, using Proposition 2.9 and the fact that f preserves heights, we deduce that f is the identity.

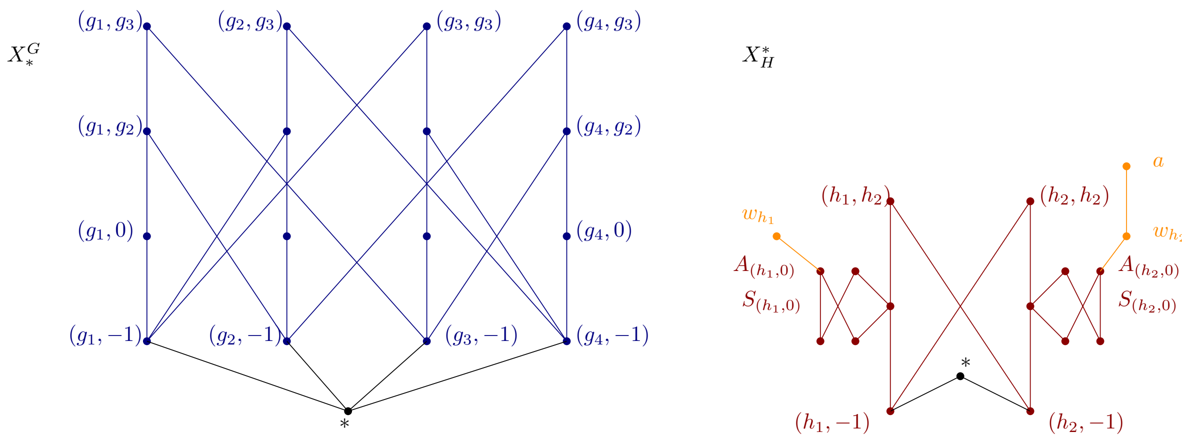

Figure 1: Hasse diagrams of X^G_* and X^*_H.

Combining X^*_H with X^G_* we obtain X^G_H. We identify the point $*$ of X^*_H and X^G_*. We also extend the partial order of the two previous posets using transitivity, that is, if $x \in X^*_H$ and $y \in X^G_*$, then we have that $x < y$ if and only if $x < * < y$. It is not difficult to check that X^G_H satisfies the properties required at the beginning. X^G_H and X_H have the same homotopy type because we can collapse X^G_* to $*$. We thus get that $\mathcal{E}(X^G_H) \simeq \mathcal{E}(X_H) \simeq H$. Since $*$ is the only point with height 1 and $P_* = (2, 4)$, it follows that $*$ is a fixed point for every $f \in Aut(X^G_H)$. By the continuity of f, it is easily seen that $f(X^G_*) = X^G_*$ and $f(X^*_H) = X^*_H$. From this, we conclude that $Aut(X^G_H)$ is isomorphic to G.

We can also consider what we call the dual case, that is, X^H_G, where we have that

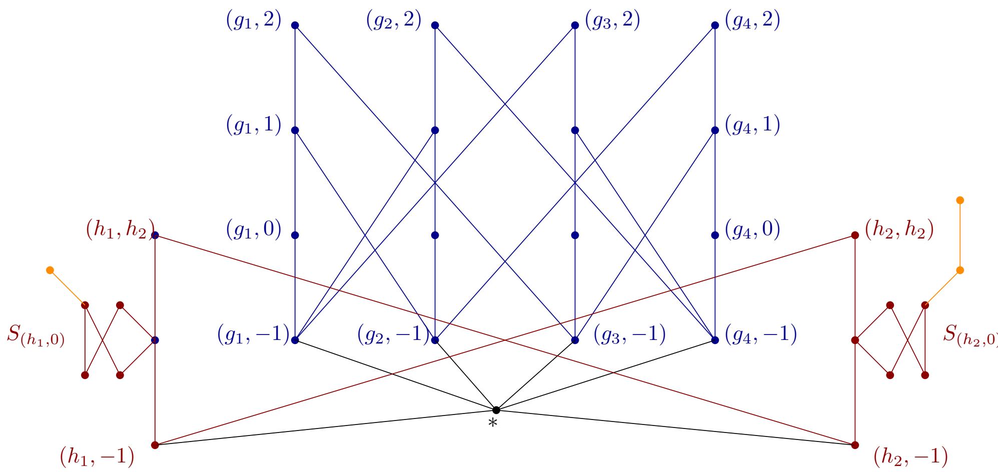

Figure 2: Hasse diagram of X_H^G.

$Aut(X_G^H) \simeq H$ and $\mathcal{E}(X_G^H) \simeq G$. We can now proceed analogously to the previous arguments, i.e., we find X_G^* and X_*^H. In Figure 3 we have the Hasse diagrams of X_*^H and X_G^*.

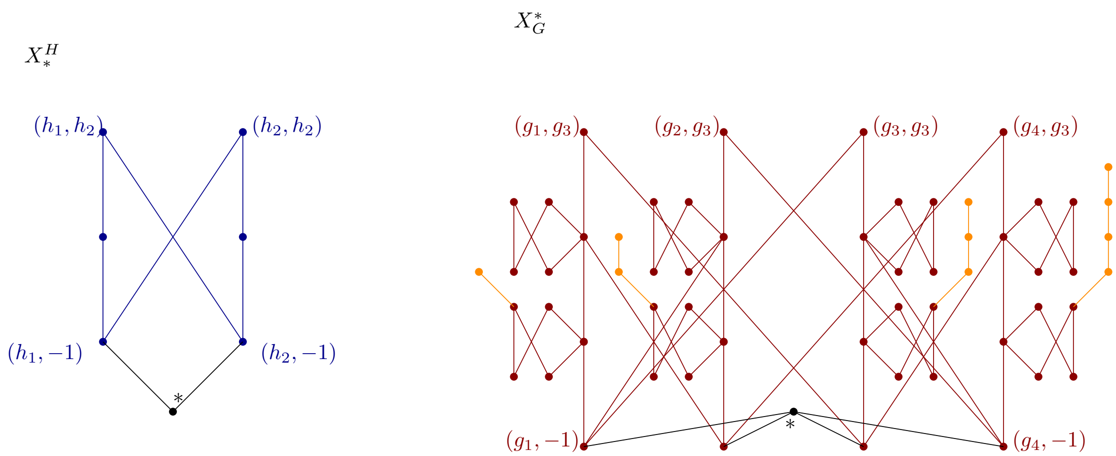

Figure 3: Hasse diagrams of X_*^H and X_G^*.

Again, combining X^H_* with X^*_G we get X^H_G, that is, the finite topological space given by the Hasse diagram of Figure 4. It is trivial to verify that X^H_G is not homeomorphic to X^G_H because of their different cardinality. Furthermore, X^G_H and X^H_G are not homotopy equivalent since $\mathcal{E}(X^G_H)$ is not isomorphic to $\mathcal{E}(X^H_G)$. Another way to prove the last assertion is the following. After removing one by one the beat points of X^G_H we get X_H; after removing one by one the beat points of X^H_G we get X_G. However, X_G is not homeomorphic to X_H because of their different cardinality. By Theorem 2.12 we conclude that X^G_H and X^H_G are not homotopy equivalent. Moreover, studying their McCord complexes it can be shown that X^G_H and X^H_G are not weak homotopy equivalent.

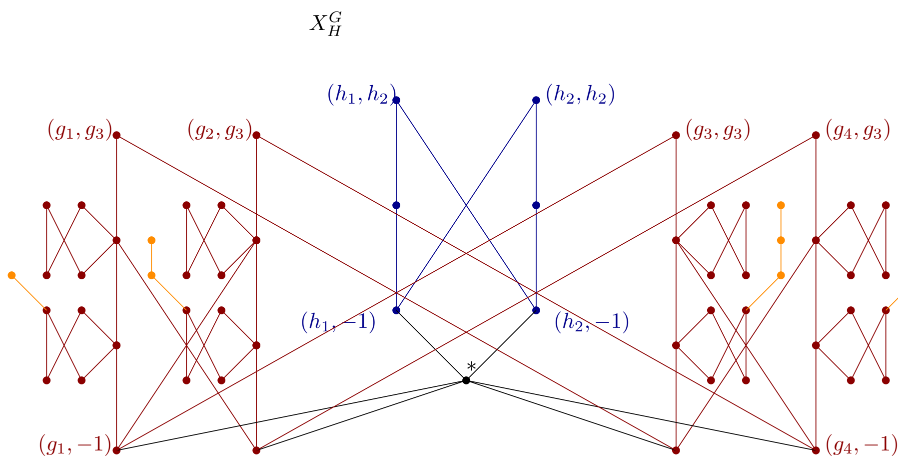

Figure 4: Hasse diagram of X^H_G.

4 Proof of Lemma 1.1 and remarks

Proof of Lemma 1.1. We follow the same strategy as in Example 3.1. Firstly, we find a topological space X^*_H such that $\mathcal{E}(X^*_H) \simeq H$ and $Aut(X^*_H)$ is trivial. Secondly, we find a topological space X^G_* such that $Aut(X^G_*) \simeq G$ and $\mathcal{E}(X^G_*)$ is trivial. Finally, we combine properly both topological spaces to obtain a topological space X^G_H satisfying that $Aut(X^G_H) \simeq G$ and $\mathcal{E}(X^G_H) \simeq H$.

We first assume that G and H are non-trivial groups. The trivial case will be considered later.

Construction of X^G_* and properties. We consider a set of non-trivial generators S'_G for G. Without loss of generality we can consider a well-order on S'_G satisfying that if $max(S'_G)$ exists and $|S'_G| > 1$, then there exists $\alpha \in S'_G$ satisfying $\alpha \prec max(S'_G)$. If $max(S'_G)$ exists and there is no $\alpha \in S'_G$ satisfying $\alpha \prec max(S'_G)$, then the well-order defined on S'_G can be modified as follows: $max(S'_G) < \alpha$ for every $\alpha \in S'_G \setminus \{max(S'_G)\}$ and the rest of the relations defined on $S'_G \setminus \{max(S'_G)\}$ unaltered. It is obvious that the new partial order defined on S'_G is indeed a well-order and $max(S'_G)$ does not exist. We consider $S_G = S'_G \cup \{0, -1\}$, where we assume that $-1, 0 \notin S'_G$, and extend the well-order defined on S'_G to S_G as follows: $-1 < 0 < \alpha$ for every $\alpha \in S'_G$. We consider

$$X^G_* = (G \times S_G) \cup \{*\},$$

where we have the following relations:

- $(g,\alpha) < (g,\delta)$ if $1 \leq \alpha < \delta$, where $g \in G$ and $\alpha, \delta \in S_G$.
- $(g\alpha, -1) \prec (g, \alpha)$, where $g \in G$ and $\alpha \in S_G \setminus \{-1, 0\}$.
- $* \prec (g, -1)$, where $g \in G$.

The rest of the relations can be deduced from the above relations using transitivity. It is easy to check that X_*^G is a partially ordered set.

We prove that $Aut(X_*^G) \simeq G$ and $\mathcal{E}(X_*^G)$ is the trivial group. We have that $\mathcal{E}(X_*^G)$ is the trivial group because X_*^G is contractible to $*$, which is a minimum. Since $*$ is a minimum, it follows that every self-homeomorphism must fix this point. From this we deduce that $Aut(X_*^G) \simeq Aut(X_*^G \setminus \{*\})$. In addition, $X_*^G \setminus \{*\}$ is the same topological space considered in [7, Section 3] and denoted by X_G. Hence, we know that $Aut(X_*^G) \simeq Aut(X_G) \simeq G$, where $\varphi : G \to Aut(X_G)$ is given by $\varphi(s)(g,\alpha) = (sg, \alpha)$ and is an isomorphism of groups.

Construction of X_H^* and properties. We consider a set of non-trivial generators S'_H for H. There is no loss of generality in assuming that there exists a well-order on S'_H satisfying that if $max(S'_H)$ exists and $|S'_H| > 1$, then there exists $\alpha \in S_H$ satisfying $\alpha \prec max(S'_H)$. We repeat the same construction made before, that is, we consider $S_H = S'_H \cup \{0, -1\}$, where we assume that $-1, 0 \notin S'_H$, and extend the well-order defined on S'_H to S_H as follows: $-1 < 0 < \beta$ for every $\beta \in S'_H$.

For every $h \in H$ we take a well-ordered non-empty set W_h such that W_h is isomorphic to W_t if and only if $h = t$. For every $h \in H$ let $w_h \in W_h$ denote the first element or minimum of W_h. We consider

$$X_H^* = (H \times S_H) \cup \Big(\bigcup_{\substack{(h,\beta) \in G \times S_H \\ 0 \leq \beta < max(S_H)}} (S_{(h,\beta)} \cup T_{(h,\beta)}) \cup \big(\bigcup_{h \in H} W_h \big) \Big) \cup \{*\},$$

where

$$S_{(h,\beta)} = \{A_{(h,\beta)}, B_{(h,\beta)}, C_{(h,\beta)}, D_{(h,\beta)}\}, T_{(h,\beta)} = \{E_{(h,\beta)}, F_{(h,\beta)}, G_{(h,\beta)}, H_{(h,\beta)}, I_{(h,\beta)}, J_{(h,\beta)}\},$$

and we have the following relations:

1. $(h,\beta) < (h,\gamma)$ if $-1 \leq \alpha < \gamma$, where $h \in H$ and $\beta, \gamma \in S_H$.
2. $(h\beta, -1) \prec (h, \beta)$, where $h \in H$ and $\beta \in S_H \setminus \{-1, 0\}$.
3. $A_{(h,\beta)} \succ C_{(h,\beta)}, D_{(h,\beta)}$; $B_{(h,\beta)} \succ (h,\beta), C_{(h,\beta)}$ and $(h,\beta) \succ D_{(h,\beta)}$, where $h \in H$ and $\beta \in S_H \setminus \{-1\}$.
4. $E_{(h,\beta)} \succ (h,\beta), I_{(h,\beta)}$; $F_{(h,\beta)} \succ H_{(h,\beta)}, J_{(h,\beta)}$; $G_{(h,\beta)} \succ I_{(h,\beta)}, J_{(h,\beta)}$ and $(h,\beta) \succ H_{(h,\beta)}$, where $h \in H$ and $\beta \in S_H \setminus \{-1\}$.
5. $* \succ (h, -1)$, where $h \in H$.
6. We extend the partial order defined on W_h to X_H declaring that $A_{(h,0)} \prec w_h$, where $h \in H$.

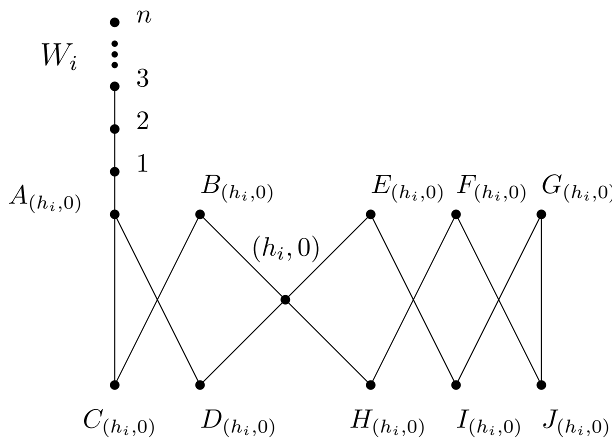

Figure 5: Hasse diagram of $S_{(h,0)} \cup T_{(h,0)} \cup W_h$, where W_h is a finite well-ordered set.

The remaining relations can be deduced from the above using transitivity. It is routine to verify that X^*_H with the previous relations is a partially ordered set.

We proceed to show that $Aut(X^*_H)$ is the trivial group and $\mathcal{E}(X^*_H) \simeq H$. It is clear that X^*_H and $X^*_H \setminus \{W_h | h \in H\}$ have the same homotopy type. We define $r : X^*_H \to X^*_H \setminus \{W_h | h \in H\}$ given by

$$r(x) = \begin{cases} A_{(h,0)} & x \in W_h \\ x & x \in X^*_H \setminus \{W_h | h \in H\}. \end{cases}$$

It is trivial to show that r is continuous and satisfies that $r(x) \leq id(x)$ for every $x \in X^*_H$, where $id : X^*_H \to X^*_H$ denotes the identity map. This implies that $X^*_H \setminus \{W_h | h \in H\}$ is a strong deformation retract of X^*_H. On the other hand, $X^*_H \setminus \{W_h | h \in H\}$ is the same topological space considered in [7, Section 3] and denoted by $\overline{X}^*_H$. Therefore we know that $\mathcal{E}(X^*_H) \simeq \mathcal{E}(\overline{X}^*_H) \simeq Aut(\overline{X}^*_H) \simeq H$, where $\phi : H \to Aut(\overline{X}^*_H)$ given by $\phi(t)(h, \beta) = (th, \beta)$ and $\phi(t)(S_{(h,\beta)} \cup T_{(h,\beta)}) = S_{(th,\beta)} \cup T_{(th,\beta)}$ is an isomorphism of groups.

The task is now to prove that $Aut(X^*_H)$ is the trivial group. Let us take $f \in Aut(X^*_H)$. We consider $A_{(h,0)}$ for some $h \in H$. Since $F_{A_{(h,0)}} \setminus \{A_{(h,0)}\}$ has a minimum w_h, it follows that $A_{(h,0)}$ is an up beat point. By Lemma 2.19, we know that $f(A_{(h_i,0)})$ is also an up beat point. Therefore, $f(A_{(h_i,0)})$ is of the form $A_{(t,0)}$ for some $t \in H$. By Proposition 2.9 we get that $f(w_h) = w_t$. It follows from the continuity of f that $f(W_h) \subseteq W_t$. Since f is a homeomorphism, we have that $f_{|W_h}$ is also a homeomorphism. Therefore we get that $h = t$; otherwise we would get a contradiction since W_h is homeomorphic to W_t if and only if $h = t$. Using Proposition 2.9 it is easy to verify that f fixes $S_{(h,0)}$ for every $h \in H$. On the other hand, [7, Remark 4.2] says that if a homeomorphism $g : X^*_H \setminus \{W_h | h \in H\} \to X^*_H \setminus \{W_h | h \in H\}$ coincides at one point with the identity map, then g is the identity map. Thus, we can deduce that f is the identity map and $Aut(X^*_H)$ is the trivial group.

<u>*Construction of X^G_H*</u>. We consider $X^G_H = X^*_H \cup X^G_*$, where we are identifying the point $*$ of both topological spaces, i.e., the partial order of X^G_H preserves the relations defined on X^*_H and X^G_*:

- If $x \in X^*_H$ and $y \in X^G_*$, then x is smaller than y if and only if $x \leq *$ and $* \leq y$.

- If $x, y \in X_H^*$, then x is smaller (greater) than y if and only if x is smaller (greater) than y with the partial order defined on X_H^*.
- If $x, y \in X_*^G$, then x is smaller (greater) than y if and only if x is smaller (greater) than y with the partial order defined on X_*^G.

It is evident that $\mathcal{E}(X_H^G)$ is isomorphic to H because X_*^G is contractible to $*$ and X_H^* is homotopy equivalent to $\overline{X}_H^*$. It suffices to show that $Aut(X_H^G)$ is isomorphic to G. We verify that every $f \in Aut(X_H^G)$ satisfies that $f(x) \in X_H^*$ for every $x \in X_H^*$ and $f(x) \in X_*^G$ for every $x \in X_*^G$. Firstly, we show that $*$ is a fixed point for every homeomorphism f. We have $ht(*) = 1$. Since for every $x \in X_*^G \setminus \{*\}$ the height of x is at least 2 or different from 1, it follows that $f(*) \notin X_*^G \setminus \{*\} \subset X_H^G$. The only elements of X_H^* that have height one are of the form $*$ or $(h, 0)$ or $A_{(s,\alpha)}$, $F_{(s,\alpha)}$, $E_{(s,\alpha)}$ for some $h, s \in H$ and $\alpha \in S_H \setminus \{-1\}$. We can discard the maximal elements, otherwise, f^{-1} would send a maximal element to a non-maximal element. If $f(*) = A_{(h,0)}$, then we get a contradiction since $A_{(h,0)}$ is an up beat point. If $f(*) = (h, 0)$ for some $h \in H$, then we get that $f^{-1}(E_{(h,0)}) \succ f^{-1}(h, 0) = *$ by Proposition 2.9. By Lemma 2.19 we have that $f^{-1}(E_{(h,0)}) \neq (g, -1)$ for every $g \in G$ because $E_{(h,0)}$ is not a down beat point. Hence, the only possibility is $f(*) = *$. Finally, by the continuity of f, we get that $f(x) \in X_*^G \subset X_H^G$ for every $x \in X_*^G \subset X_H^G$. This implies that $Aut(X_H^G)$ is isomorphic to G.

We prove the remaining case. If G is the trivial group, then it suffices to consider X_H^* to conclude. If H is the trivial group, then X_*^G satisfies the desired properties. □

Remark 4.1. *If $f, k \in Aut(X_H^G)$ are such that there exists $x \in X_*^G \setminus \{*\}$ satisfying $f(x) = k(x)$, then $f = k$. This is a consequence of the isomorphism of groups φ given in the proof of Lemma 1.1. Similarly, if $[f], [k] \in \mathcal{E}(X_H^G)$ are such that there exists $x \in X_H^G \setminus (\{X_*^G\} \cup \{W_h | h \in H\})$ satisfying that $f(x) = k(x)$, where $f \in [f]$ and $k \in [k]$, then $f = k$ and $[f] = [k]$. This is a consequence of the construction of X_H^G and ϕ given in the proof of Lemma 1.1.*

Proposition 4.2. *Let G and H be groups. Then the Alexandroff space X_H^G obtained in the proof of Lemma 1.1 has the weak homotopy type of the wedge sum of $3|H||S_H|$ circles when H is a finite group and the wedge sum of $|\mathbb{N}|$ circles when H is a non-finite countable set.*

Proof. We have that X_H^G has the same homotopy type of X_H^*. Repeating the same arguments used in [7, Proposition 6.1] the desired result follows. □

Remark 4.3. *Let G and H be finite groups and let X_H^G be the finite topological space obtained in the proof of Lemma 1.1. We can remove $\{T_{(h,\beta)} | h \in H, \beta \in S_H \setminus \{-1, 0\}\}$ from X_H^G. The resulting poset also satisfies that its group of homeomorphisms is isomorphic to G and its group of homotopy classes of self-homotopy equivalences is isomorphic to H. This finite topological space has $|G|(|S_G| + 2) + |H|(|S_H| + 2) + 4|S_H||H| + \frac{|H|(|H|+1)}{2} + 1$ points. The first term corresponds to $X_*^G \setminus \{*\}$, the second term corresponds to $X_H^* \setminus \{*\}$, the third term corresponds to the sets $S_{(h,\beta)}$, the fourth term corresponds to the points of the sets W_h and the last term corresponds to the point $*$.*

We can change the sets $T_{(h,\beta)}$ from the proof of Lemma 1.1 by $T^n_{(h,\beta)}$ as in [7, Section 5], where $T^n_{(h,\beta)} := \{x_1, x_2, ...x_{n+3}, y_1, y_2, ..., y_{n+3}\}$ with the following relations:

$$(h,\beta) < x_1 > y_2 < x_3 > y_4 < ... < x_{n+2} > y_{n+3} < x_{n+3} > y_{n+2} < x_{n+1} < ... < x_2 > y_1 < (h,\beta), \tag{1}$$

$$(h,\beta) < x_1 > y_2 < x_3 > y_4 < ... > y_{n+2} < x_{n+3} > y_{n+3} < x_{n+2} > y_{n+1} < ... < x_2 > y_1 < (h,\beta). \tag{2}$$

We consider (1) for n odd and (2) for n even. We denote this poset by X^G_{Hn}.

Corollary 4.4. *Given two finite groups G and H, there are infinitely many (non-homotopy-equivalent) topological spaces $\{X^G_{Hn}\}_{n\in\mathbb{N}}$ such that $Aut(X^G_{Hn})$ is isomorphic to G and $\mathcal{E}(X^G_{Hn})$ is isomorphic to H for every $n \in \mathbb{N}$.*

Proof. The proof is analogous to the proof of Lemma 1.1. By Theorem 2.12, we have that the topological spaces are not homotopy equivalent due to their different cardinality after removing all the beat points one by one. □

5 Examples, remarks and proof of Theorem 1.2

The idea of this section is to modify the topological space obtained in Lemma 1.1 to prove Theorem 1.2. Given a homomorphism of groups $f : G \to H$, we slightly modify the topological space X^*_H defined in the proof of Lemma 1.1 to get X_f. Adding new relations to X^*_H we can control the homomorphism of groups $\tau : Aut(X_f) \to \mathcal{E}(X_f)$ given by $\tau(f) = [f]$.

Example 5.1. Let us consider the cyclic group of two elements $\mathbb{Z}_2$ and the group of integer numbers $\mathbb{Z}$. We consider the homomorphism of groups $f : \mathbb{Z} \to \mathbb{Z}_2$ given by $f(n) = n \ mod\ 2$. We consider the topological space $X^{\mathbb{Z}}_{\mathbb{Z}_2}$ obtained in the proof of Lemma 1.1. We remove W_0 and W_1 from it. The resulting poset X_f corresponds to the Hasse diagram shown in black in Figure 6.

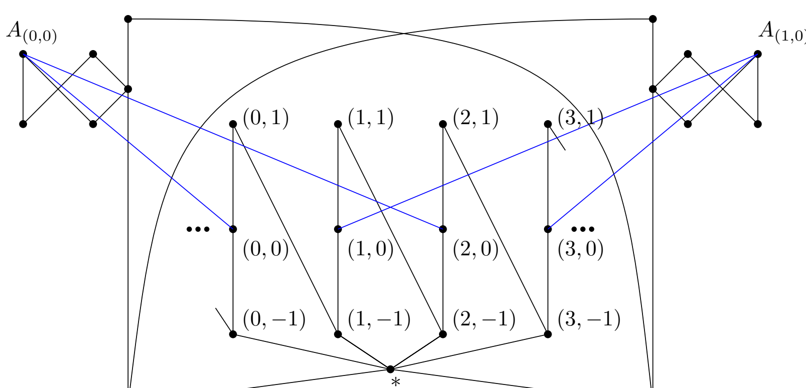

Figure 6: Hasse diagram of X_f.

Now we add the following relations to X_f: $(n, 0) \prec A_{(1,0)}$ if $f(n) = 1$ and $(n, 0) \prec A_{(0,0)}$ if $f(n) = 0$. In Figure 6 we have represented these relations in blue. It is easy to verify that X_f satisfies that $Aut(X_f) = \mathbb{Z}$, $\mathcal{E}(X_f) = \mathbb{Z}_2$ and $f = \tau$.

Example 5.2. Let $f : \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ be the homomorphism of groups given by $f(0) = (0,0)$ and $f(1) = (1,0)$. In Figure 7 we have the Hasse diagram of X_f, where we use the same notation introduced in Example 3.1. We have that $Aut(X_f) = \mathbb{Z}_2$, $\mathcal{E}(X_f) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\tau = f$.

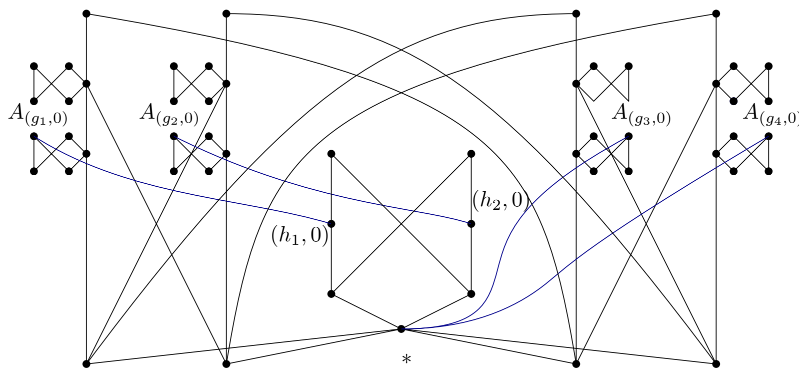

Figure 7: Hasse diagram of X_f.

Proof of Theorem 1.2. Suppose G and H are not trivial groups since otherwise the result follows from Lemma 1.1. Let X_H^G be the topological space obtained in the proof of Lemma 1.1. We consider $X_f = X_H^G \setminus \{W_h | h \in H\}$. We keep the same relations defined on X_f as a subspace of X_H^G and add the following relations:

- $A_{(h,0)} \succ (g,0)$ if $f(g) = h$, where $h \in H$ and $g \in G$.
- $A_{(h,0)} \succ *$ if $h \notin f(G)$.

It is easy to check that X_f is a partially ordered set with the above relations. The task is now to show that $\mathcal{E}(X_f) \simeq H$. We consider $r : X_f \to X_f$ given by

$$r(x) = \begin{cases} * & x \in X_*^G \\ x & x \in X_f \setminus \{X_*^G\}. \end{cases}$$

We have that r preserves the order so it is a continuous map. It is simple to verify that $r(x) \leq id(x)$ for every $x \in X_f$, where id denotes the identity map. From this it follows that X_f is homotopy equivalent to $r(X_f) = X_H^* \subset X_f$. On the other hand, repeating the same arguments used in [7], it can be proved that X_H^* is locally a core [13] or a minimal finite space in case H is a finite group, which implies that $\mathcal{E}(X_f) \simeq \mathcal{E}(X_H^*) = Aut(X_H^*)$. Since $P_* = (|H|, |H|)$ and $ht(*) = 1$, it follows that every homeomorphism $T : X_H^* \to X_H^*$ fixes $*$. Hence, the group of homeomorphisms of X_H^* as a subspace of X_f is isomorphic to the group of homeomorphisms of the topological space X_H^* obtained in the proof of Lemma 1.1. Thus, $\mathcal{E}(X_f) \simeq H$.

We proceed to show that $Aut(X_f) \simeq G$. We consider the following auxiliary sets: $Col_h = \{(h,\beta) | \beta \in S_H\} \cup \{S_{(h,\beta)} \cup T_{(h,\beta)} | \beta \in S_H \setminus \{-1, max(S_H)\}\}$, where $h \in H$, and $Col^g = \{(g,\alpha) | \alpha \in S_G\}$, where $g \in G$. If $x \in X_*^G \subset X_f$, then every homeomorphism $T : X_f \to X_f$ satisfies that $T(x) \in X_*^G$. We prove the last assertion. We know that $X_f \setminus \{X_*^G\}$ does not contain beat points. On the other hand, for every $g \in G$ we have that $(g,-1)$ is a beat point of height 2. Using Proposition 2.9 and the notion of continuity we deduce that for every $T \in Aut(X_f)$ and (g,α), where $g \in G$ and $\alpha \in S_G$, we have $T(g,\alpha) = (g',\alpha)$ for some $g' \in G$.

We consider $\varphi : G \to Aut(X_f)$ given by $\varphi(g)(g',\alpha) = (gg',\alpha)$ if $g' \in G$ and $\alpha \in S_G$, $\varphi(g)(h,\beta) = (f(g)h,\beta)$ if $h \in H$ and $\beta \in S_H$, $\varphi(g)(S_{(h,\beta)} \cup T_{(h,\beta)}) = S_{(f(g)h,\beta)} \cup T_{(f(g)h,\beta)}$ defined in the natural way if $h \in H$ and $\beta \in S_H \setminus \{-1, max(S_H)\}$ and $\varphi(g)(*) = *$. We prove that φ is well-defined. We verify the continuity of $\varphi(g)$. Suppose $(g',0) \prec A_{(h,0)}$ for some $g' \in G$ and $h \in H$. By hypothesis, $f(g') = h$. Therefore,

$$\varphi(g)(g',0) = (gg',0) \prec A_{(f(gg'),0)} = A_{(f(g)f(g'),0)} = A_{(f(g)h,0)} = \varphi(g)A_{(h,0)}.$$

It is easy to check that $\varphi(g)$ preserves the remaining relations. The inverse of $\varphi(g)$ is given by $\varphi(g^{-1})$. Hence, φ is well-defined. By construction, φ is a monomorphism of groups. Suppose $T \in Aut(X_f)$. Proposition 2.9, Remark 4.1 and the fact that $T_{|X_*^G} \in Aut(X_*^G)$ imply that every $T \in Aut(X_f)$ satisfies $T(Col_h) = Col_{h'}$ and $T(Col^g) = Col_{g'}$ for some $g' \in G$ and $h' \in H$, where $g \in G$ and $h \in H$. We consider $(g,0) \prec A_{(h,0)}$ for some $g \in G$ and $h \in H$. We get $T(Col^g) = Col^{g'}$ for some $g' \in G$ and $T(Col_h) = Col_{h'}$ for some $h' \in H$. By Remark 4.1, the proof of Lemma 1.1 and the fact that $T_{|X_G^*} \in Aut(X_*^G)$, there exists $t \in G$ such that $T(Col^s) = Col^{ts}$, where $s \in G$. Hence, $g' = tg$. By Proposition 2.9, $T(A_{(h,0)}) = A_{(h',0)} \succ (tg,0) = T(g,0)$, we have $h' = f(tg) = f(t)f(g) = f(t)h$. Thus, $T = \varphi(t)$ because of Remark 4.1 and the fact that $T_{|X_f \setminus \{X_*^G\}} \in Aut(X_H^* \setminus \{*\})$. By construction, for every $g \in G$ the equality $\tau(g) = f(g)$ holds. Since every $T \in Aut(X_f)$ can be seen as $T = \varphi(g)$ for some g, it follows that $\tau(T) = f(g)$, where $f(g) = \varphi(g)_{|X_H^*} \in \mathcal{E}(X_f)$. □

Remark 5.3. *Theorem 1.2 generalizes Lemma 1.1 and the results of realization obtained in [7]. Let G and H be two groups. Using Theorem 1.2, we obtain a family of topological spaces* $\{X_f\}_{f:G\to H}$ *satisfying that* $Aut(X_f) \simeq G$ *and* $\mathcal{E}(X_f) \simeq H$.

Proposition 5.4. *Let G and H be groups. If $g, f : G \to H$ are homomorphisms of groups, then X_f is homotopy equivalent to X_g.*

Proof. The result is an immediate consequence of the construction. We have that X_f is homotopy equivalent to X_H^* for every homomorphism of groups $f : G \to H$. Therefore, the homotopy type of the topological space obtained in the proof of Theorem 1.2 does not depend on the homomorphism chosen to construct it. Thus we deduce the desired result. □

Proposition 5.5. *Let G and H be groups and let $f, g : G \to H$ be homomorphisms of groups. Then $f = g$ if and only if X_f is homeomorphic to X_g.*

Proof. One of the implications is trivial. It suffices to show that if X_f is homeomorphic to X_g, then $f = g$. Since X_f is homeomorphic to X_g, it follows that there exists a homeomorphism $T' : X_f \to X_g$. From the construction of X_f and X_g in the proof of Theorem 1.2 it can be easily deduced that $T'_{|X_*^G} \in Aut(X_*^G) \simeq G$ and $T'_{|X_H^*} \in Aut(X_H^*) \simeq H$. This is due to the fact that X_*^G contains beat points while X_H^* does not have beat points. Therefore, $T'_{|X_*^G}$ can be related to the action of an element $T \in G$ and $T'_{|X_H^*}$ can be related to the action of an element $\overline{T} \in H$. We have $(e,1) \prec A_{(f(e),0)}$, where e denotes the identity element in G, and we also have

$$T'(e,1) = (T,1) \prec A_{(\overline{T}f(e),0)} = T'(A_{(f(e),0)}),$$

which implies that $g(T) = \overline{T}f(e)$. Thus, $g(T) = \overline{T}$ because f is a homomorphism of groups. In addition, for every $h \in G$, we know that there exists a relation in X_f of the following form $(h, 1) \prec A_{(f(h),0)}$. We have

$$T'(h, 1) = (Th, 1) \prec A_{(\overline{T}f(h),0)} = T'(A_{(f(h),0)}).$$

By the construction of X_g we get $g(Th) = g(T)g(h) = \overline{T}f(h)$. Earlier we prove that $g(T) = \overline{T}$, which implies that $g(h) = f(h)$ for every $h \in G$. □

6 Groups of homology, homotopy and automorphisms

We first prove that the groups studied previously do not determine neither the homotopy type nor the topological type of a topological space X in general. To do this we provide an example. However, if the topological space X satisfies some properties, namely, X is compact and a locally Euclidean manifold with or without boundary, then its group of homeomorphisms determines its topological type of it, see [19] for more details.

Example 6.1. Let us consider the Alexandroff space W_2 given by the Hasse diagram of Figure 8. It is the union of $L_1 = \{x_i\}_{i=1,\ldots,9}$ and $L_2 = \{x_j\}_{j=9,\ldots17}$, where we are identifying the point x_9 of L_i for $i = 1, 2$. For simplicity, W_1 denotes L_1. The topological space L_1 was introduced in [16, Figure 2] and has the weak homotopy type of a point. It is proved in [8] that W_1 is the smallest finite topological space having the same weak homotopy type of a point but not contractible. It is clear that W_2 does not have beat points, so $Aut(W_2)$ is isomorphic to $\mathcal{E}(W_2)$. On the other hand, it is easy to show that $Aut(W_2)$ is the trivial group. Since $P_x = (3, 5)$ only for $x \in \{x_5, x_{13}\}$ and the heights of these points are different, it follows that every homeomorphism must fix x_5 and x_{13}. Proposition 2.9 leads to the desired result. Furthermore, the homotopy and singular homology groups of W_2 are trivial. We can study the weak homotopy type of W_2 studying the McCord complex $\mathcal{K}(W_2)$ or removing beat and weak beat points. We have that x_{16} is a weak beat point. After removing this point, x_{12} and x_{14} are up beat points. If we remove them, then it is easy to check that the remaining space is homotopy equivalent to the space given by the points $\{x_i\}_{i=1,\ldots,9}$. We continue in this fashion. We have x_8 is a weak beat point. After removing this point, x_7 and x_9 are down beat points. Thus, X has the same weak homotopy type of a point. Therefore we have that W_2 is a topological space satisfying that $Aut(W_2) \simeq \mathcal{E}(W_2) \simeq \pi_n(W_2) \simeq H_n(W_2) \simeq 0$ for every $n > 0$ but it is not homeomorphic nor homotopy equivalent to a point. We can generalize this topological space by taking more copies of the topological space introduced in [16]. For instance, we can define W_3 just as $L_3 \cup W_2$, where $L_3 = \{x_i\}_{i=17,\ldots,25}$ and we are identifying the point x_{17} of W_3 and W_2. It is easy to prove that W_n has the weak homotopy type of a point for every $n \in \mathbb{N}$ because $x_i \in W_n$ with $i \equiv 0 \pmod 8$ is a weak beat point.

One possible consequence of the previous construction is the following result.

Proposition 6.2. *Let G and H be finite groups. There exists a topological space X such that $Aut(X)$ is isomorphic to G, $\mathcal{E}(X)$ is isomorphic to H and X is weak homotopy equivalent to a point.*

Proof. We consider the topological space X^G_H obtained in the proof of Lemma 1.1 and the finite topological space W_2 given in Example 6.1. Let X denote $X^G_H \circledast W_2$. By Proposition

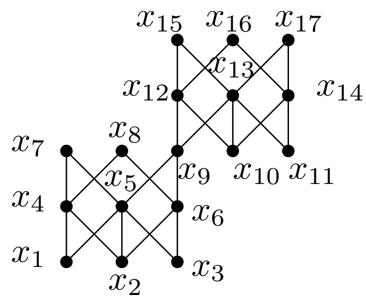

Figure 8: Hasse diagram of W_2.

2.27, Example 6.1 and the proof of Lemma 1.1 we have that $Aut(X) \simeq Aut(X^G_H) \times Aut(W_2) \simeq Aut(X^G_H) \simeq G$. We get that X is homotopy equivalent to $X^*_H \circledast W_2$ by removing its beat points one by one. Since $X^*_H \circledast W_2$ does not contain beat points by Corollary 2.13, it follows that $\mathcal{E}(X^*_H \circledast W_2) \simeq Aut(X^*_H \circledast W_2) \simeq Aut(X^*_H) \simeq H$. In addition, since W_2 is collapsible, it follows that X has the weak homotopy type of a point by Remark 2.26 □

We motivate the proof of Theorem 1.3 with one example.

Example 6.3. Let us consider $G = \mathbb{Z}_3$ and $H = \mathbb{Z}_2$. We consider the minimal finite model of the 2-dimensional sphere X, that is, $X = \{A, B, C, D, E, F\}$, where $A, B > C, D, E, F$ and $C, D > E, F$. Then $|\mathcal{K}(X)|$ is homeomorphic to S^2.

We want to find a finite T_0 topological space $\overline{X}^G_H$ such that $H_n(\overline{X}^G_H)$ and $\pi_n(\overline{X}^G_H)$ are isomorphic to $H_n(S^2)$ and $\pi_n(S^2)$ respectively for every non-negative integer n, $Aut(\overline{X}^G_H)$ is isomorphic to $\mathbb{Z}_3$ and $\mathcal{E}(\overline{X}^G_H)$ is isomorphic to $\mathbb{Z}_2$. The idea is to modify X to obtain a new space satisfying that its group of homeomorphisms is trivial. We enumerate the points of X. For each $i \in X$ with $i = 1, ..., |X|$, we add W_i to X as in Example 6.1. The Hasse diagram of the new topological space, denoted by X', can be seen in Figure 9. The Hasse diagram of X is painted black whereas the new part is blue and purple. In purple we have the weak beat points that are not beat points. It is clear that X' does not have beat points so $Aut(X')$ is isomorphic to $\mathcal{E}(X')$. A homeomorphism f sends weak beat points to weak beat points by Lemma 2.19. It is easy to deduce from this that $Aut(X')$ is the trivial group. On the other hand, the new structure added can be removed without changing the weak homotopy type of the space, see Example 6.1. Therefore, we have that $H_n(X')$ is isomorphic to $H_n(X)$ and $\pi_n(X')$ is isomorphic to $\pi_n(X)$ for every non-negative integer n.

Finally, we add a new point t that connects X' to $X^G_H \circledast W_2$, where X^G_H is the space obtained the proof of Lemma 1.1 for finite groups. In Figure 9 we have the Hasse diagram of the new topological space $\overline{X}^H_G$. The relations with the point t are shown in green. The Hasse diagram of $X^G_H \circledast W_2$ is painted read and orange. It is easy to check that $\overline{X}^G_H$ satisfies the desired properties.

Proof of Theorem 1.3. If X has the same homotopy type of a point, then the result can be deduced from Proposition 6.2. Therefore we can assume that X does not have the same homotopy type of a point. The idea of the proof is to follow techniques similar to the ones used in the proof of Lemma 1.1. From the simplicial approximation to CW-complexes, [11, Theorem 2C.5.] we get that there exists a finite simplicial complex that is homotopy equivalent

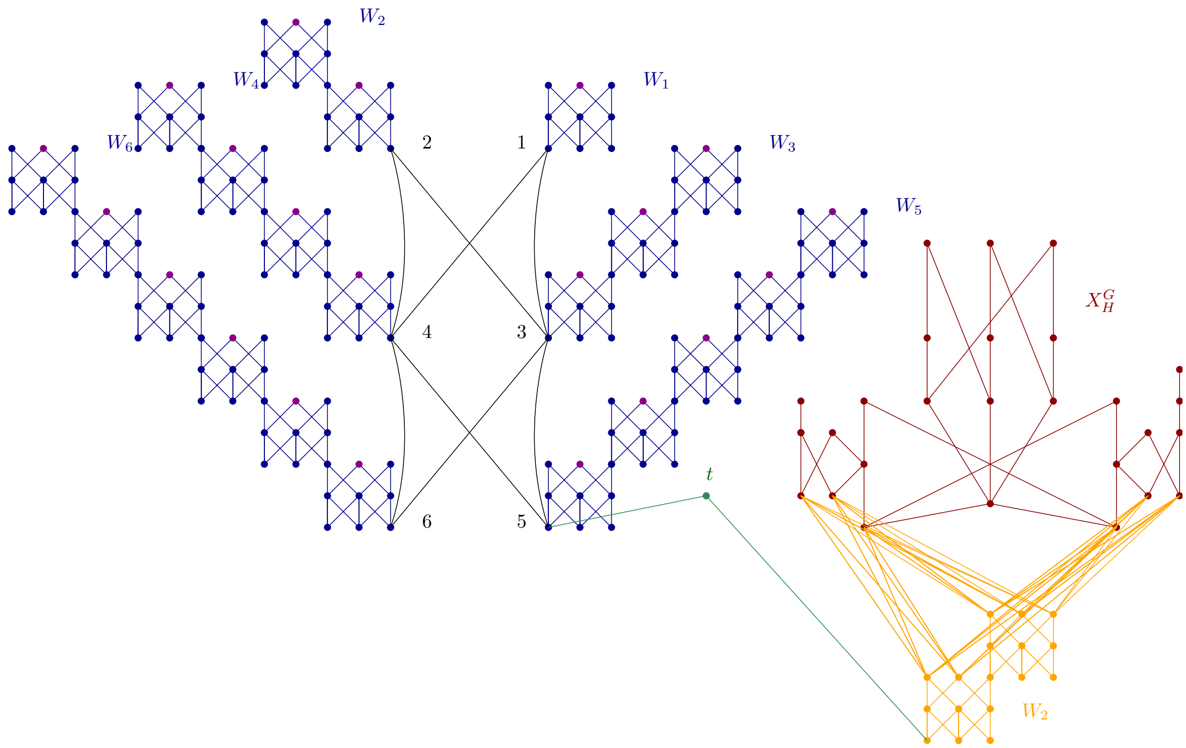

Figure 9: Hasse diagram of $\overline{X}_H^G$.

to X. By abuse of notation we continue to write X for the finite simplicial complex. We apply the McCord functor $\mathcal{X}$ to X in order to obtain a finite T_0 topological space $\mathcal{X}(X)$ such that $\mathcal{X}(X)$ is weak homotopy equivalent to X. We can suppose that $\mathcal{X}(X)$ does not have beat points or weak beat points; otherwise we can remove them one by one until there are none left. We denote by $n = |\mathcal{X}(X)|$ and label the points in $\mathcal{X}(X)$, that is, $\mathcal{X}(X) = \{y_i\}_{i=1...n}$. For each $y_i \in \mathcal{X}(X)$ we consider W_i, where W_i is the topological space obtained in Example 6.1. We consider $Z = \mathcal{X}(X) \cup \bigcup_{i=1,...,n} W_i$, where we are identifying the point y_i with $x_1 \in W_i$ for every $i = 1, ..., n$. We define the partial order on Z extending the already existing partial orders. To do that we use transitivity, i.e., for $x, y \in Z$, $x \geq y$ if and only if one of the following situations is satisfied:

- $x, y \in \mathcal{X}(X)$ and x is greater than y with the partial order defined on $\mathcal{X}(X)$.
- $x, y \in W_i$ for some i and x is greater than y with the partial order defined on W_i.
- $y \in \mathcal{X}(X)$, $x \in W_i$ for some i and $x \geq y_i (= x_1) \geq y$.

Consider $\overline{X}_H^G = Z \cup X_H^G \circledast W_2 \cup \{t\}$, where X_H^G is the space obtained in the proof of Lemma 1.1 and W_2 the space given in Example 6.1. We extend the partial order defined on Z and $X_H^G \circledast W_2$ to $\overline{X}_H^G$ by declaring that $y_j < t > x_1$ for some $h \in H$, where y_j is a minimal point in $\mathcal{X}(X)$ and $x_1 \in W_2$. We prove that $f \in Aut(\overline{X}_H^G)$ restricted to Z is the identity. In W_i there are i weak beat points that we will denote by z_j^i with $j = 1, ..., i$. In fact, we have that the only weak beat points that are not beat points or do not have a bigger beat point are in

Z. Hence, if $z_j^i \in W_i$ is a weak beat point, we have $f(z_j^i) = z_l^k \in W_k$ for some $l \leq k \leq n$. By Proposition 2.9 and Lemma 2.19, we have that $f(W_i) = W_k$. But W_i is homeomorphic to W_k if and only if $i = k$. By the continuity of f, $f(W_i) = id(W_i)$, so $f_{|Z} = id(Z)$, as we wanted. It is easy to check that t is also a fixed point for every homeomorphism since $y_j \prec t$ and $f(y_j) = y_j$. We get $Aut(\overline{X}_H^G) \simeq Aut(X_H^G \circledast W_2) \simeq Aut(X_H^G) \times Aut(W_2)$. By Proposition 2.27, Example 6.1 and the Proof of Lemma 1.1, $Aut(\overline{X}_H^G) \simeq Aut(X_H^G) \simeq Aut(X_*^G) \simeq G$. On the other hand, $\mathcal{E}(\overline{X}_H^G) \simeq \mathcal{E}(Z \cup \{t\} \cup X_H^* \circledast W_2)$, but $Z \cup \{t\} \cup X_H^* \circledast W_2$ does not contain beat points. Therefore, by Corollary 2.13, $\mathcal{E}(Z \cup \{t\} \cup X_H^* \circledast W_2) \simeq Aut(Z \cup \{t\} \cup X_H^* \circledast W_2)$. From here, repeating similar arguments than the ones used before, it can be deduced that $Aut(Z \cup \{t\} \cup X_H^* \circledast W_2) \simeq Aut(X_H^*) \simeq H$.

Finally, $|\mathcal{K}(\overline{X}_H^G)|$ is clearly the wedge sum of $|\mathcal{K}(Z)|$ and $|\mathcal{K}(X_H^G \circledast W_2)|$. From Remark 2.26 we obtain that $\mathcal{K}(X_H^G \circledast W_2)$ is homotopy equivalent to a point since W_2 is collapsible, which implies that $\mathcal{K}(W_2)$ is also collapsible, and X_H^G is homotopy equivalent to X_H^*. We also get that $|\mathcal{K}(Z)|$ is homotopy equivalent to X because every W_i can be removed following the steps of Example 6.1 without changing the weak homotopy type of Z. Therefore, for every $n \in \mathbb{N}$, we have $\pi_n(X) \simeq \pi_n(Z)$ and $H_n(X) \simeq H_n(Z)$. □

Proof of Corollary 1.4. It is an immediate consequence of Theorem 1.3. We only need to consider the wedge sum of Moore spaces and then apply Theorem 1.3. □

Proof of Corollary 1.5. We only need to use the beginning of the construction of Eilenberg-Maclane spaces to obtain a compact CW-complex X with $\pi_n(X) \simeq H$ and possibly non-trivial higher homotopy groups. Therefore, the result is an immediate consequence of Theorem 1.3. □

Remark 6.4. *The results obtained in this section are stated in terms of finite groups but it may be possible to get the same results for general groups. The idea could be to use the same constructions described in this section and the theory of [13], which is a generalization of the theory of R.E. Stong [17].*

For a compact CW-complex X, $\mathcal{E}_*(X)$ and $\mathcal{E}_\#(X)$ are nilpotent groups, see for instance [10, Section 4]. $\mathcal{E}_\#(X)$ ($\mathcal{E}_*(X)$) can be seen as the kernel of a homomorphism of groups. We consider the functor π (H_*) between $HPol$ and the category of groups given by $\pi(X) = \bigoplus_{i=1}^{dim(X)} \pi_i(X)$ ($H_*(X) = \bigoplus_{i=1}^{\infty} H_i(X)$). It is easy to check that π (H_*) induces a homomorphism of groups, $\overline{\pi} : \mathcal{E}(X) \to Aut(\pi(X))$ ($\overline{H}_* : \mathcal{E}(X) \to Aut(H_*(X))$). This sends each self-homotopy equivalence to its induced morphism in the homotopy groups (homology groups), where $Aut(\cdot)$ denotes here the group of automorphisms of a group in the category of groups. Then, $\mathcal{E}_\#(X)$ ($\mathcal{E}_*(X)$) can be seen as the kernel of $\overline{\pi}$ ($\overline{H}_*$) and so it is a normal subgroup of $\mathcal{E}(X)$. With the following example we prove that for a general topological space we cannot expect the same result.

Example 6.5. Applying the construction obtained in the proof of Theorem 1.3 we can get a topological space X such that $Aut(X)$ is trivial, $\mathcal{E}(X) = S_3$ and X is weak homotopy equivalent to a circle, where S_3 denotes the symmetric group on a set of 3 elements. In Figure 10 we present the Hasse diagram of X. By construction, every self-homotopy equivalence of X fixes the blue, black, green and orange parts of the Hasse diagram. On the other hand, the only part that contributes to the homotopy groups or homology groups is the black one.

Therefore we can deduce that $\mathcal{E}_{\#}(X) = \mathcal{E}_{*}(X) = \mathcal{E}(X) = S_3$, which implies that $\mathcal{E}_{\#}(X)$ and $\mathcal{E}_{*}(X)$ are not nilpotent groups.

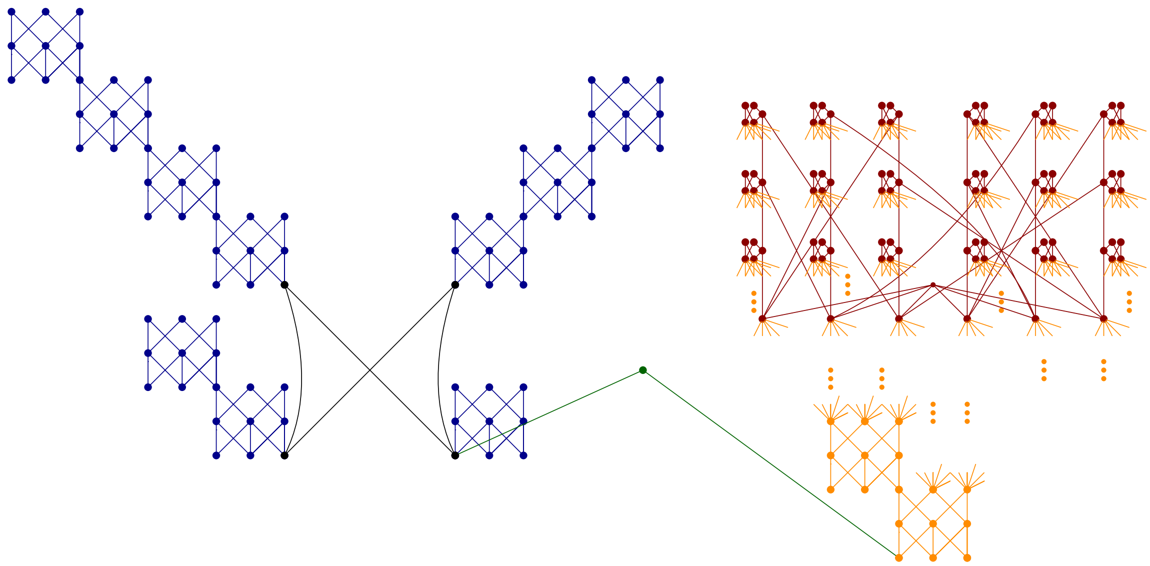

Figure 10: Hasse diagram of X.

Remark 6.6. *It is not difficult to show that the topological spaces obtained in the proof of Theorem 1.3 satisfy that $\mathcal{E}_{*}(X) = \mathcal{E}_{\#}(X) = \mathcal{E}(X)$. Then it is easy to find more examples such as Example 6.5.*

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