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POSITIVE SEMIDEFINITE GERMS ON THE CONE

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We show that any positive semidefinite analytic function germ on the cone $z^2 = x^2 + y^2$ is a sum of two squares of analytic function germs.

1. Introduction and statement of the result.

The problem of representing a positive semidefinite function (=psd) as a sum of squares (=sos) is a very old matter in real algebra and real geometry. Still, it is a difficult question always appealing the specialists. Concerning real analytic germs we can summarize what is known in a few statements.

Let X be a irreducible real analytic set germ of dimension d. Any psd f of X is an sos of *meromorphic* germs (see [**AnBrRz**]) and the qualitative question is *whether every psd is an sos of* analytic *germs*. The quantitative question: *How many squares*, can refer either to meromorphic or analytic germs. We know the answer to the qualitative question in the regular case: YES *for the line and the plane,* NO *otherwise* (see [**Rz1**]). For singular curve germs we immediately realize that the answer is NO. For singular surfaces we have many examples where the answer is NO, and think there are very few where the answer is YES ([**Rz2**]). For higher dimensional germs we guess the answer is NO. Concerning the quantitative matter, in the regular case the sharp bound for the number of squares of analytic germs is d for $d = 1, 2$ (see [**Rz1**]) and *there are sos's of arbitrary length* for $d \geq 3$ ([**Ch *et al***]); bounds for the number of squares of meromorphic germs are $1, 2$ for $d = 1, 2$ (see [**Rz1**]) and 8 for $d = 3$ ([**Jw**]), (the general conjecture appears to be 2^{d-1}, but for $d > 3$ no bound is available yet). In the singular case the only systematic works concern curves ([**Or**], [**Qz**]): For sos's of meromorphic germs the bound is trivially 1, and for analytic germs the bound is the *multiplicity* (a conjecture suggested by Becker and proved by Quarez, and that seems very good as there can be no uniform bound for curve germs in $\mathbb{R}^3$). The next case are surface germs. Concerning this we know that *twice the multiplicity* is a bound for the number of squares of meromorphic germs, and that is all. We can add that for $d \geq 4$ there are always sos's of analytic germs of arbitrary length ([**Rz2**]), and notice the gap of knowledge for singular X of dimension 3.

With this state of affairs, the simplest possible question, putting together the quantitative and qualitative aspects is: *For which surface germs any psd*

is a sum of two squares of analytic germs? This was given a first answer in [**Rz1**]: For very few which are in a short list of candidates. However, these candidates are difficult to handle: Only *Brieskorn's surface* ($x^2+y^3+z^5=0$) and *Whitney's umbrella* ($x^2-zy^2=0$) are known to hold the property above. In this short note we settle another candidate of the list: The cone $z^2=x^2+y^2$. The result is:

Theorem 1.1. *Let $X: z^2=x^2+y^2$ be the cone singularity. Then any non-negative analytic germ on X is a sum of two squares of analytic function germs.*

We develop the proof in several steps in the next section.

2. Proof of the result.

The ring of analytic function germs on the cone $X:\ z^2=x^2+y^2$ is the ring $A=\mathbb{R}\{x,y,z\}/(z^2-x^2-y^2)$, which is a free module of rank 2 over $\mathbb{R}\{x,y\}$ with basis $1,z$. This follows by division by $z^2-x^2-y^2$, and describes the elements in A as $f(x,y)+zg(x,y)\in\mathbb{R}\{x,y\}[z]$, operations as usual with the condition that $z^2=x^2+y^2$. We will freely use this.

We start the proof of 1.1 by a particular case:

Proposition 2.1. *Let $f,g\in\mathbb{R}[x,y]$ be such that $f+zg$ is psd on X. Then for some integer $m\geq 0$, $z^m(f+zg)$ is a sum of two squares of analytic germs on X.*

Proof. First we estimate the orders of the series f,g. To that end, note that a planar curve germ (at,bt) lifts to two curve germs in the cone, namely $(at,bt,\pm ct)$ with $c=\sqrt{a^2+b^2}$. Since $f+zg\geq 0$ on the cone near the origin, we get

$$0\leq f(at,bt)\pm ctg(at,bt)=(f_p(a,b)t^p+\cdots)\pm(cg_q(a,b)t^{q+1}+\cdots),$$

where f_p is the initial form of f and g_q that of g. For a good choice of (a,b) we have $f_p(a,b)\neq 0$ and $g_q(a,b)\neq 0$, and it is clear then that the above inequality implies $p\leq q+1$ and p even, say $p=2n$. Thus we can expand f and g into homogeneous components in the form

$$f=f_{2n}+\cdots+f_r,\ \ g=g_{2n-1}+\cdots+g_{r-1}.$$

In particular, the initial form of $f+zg$ is $f_{2n}+zg_{2n-1}$, which we can suppose does not vanish on the cone (otherwise, we replace f by $f-f_{2n}$ and g by $g-g_{2n-1}$, getting the same function on the cone).

We consider the blowing-up $\varphi: x=ut, y=vt, z=t$, which maps the cilinder $Y:\ u^2+v^2=1$ onto the cone. In fact, φ maps the germ of Y along the circle $t=0$, $u^2+v^2=1$ onto the germ of X at the origin. We obtain

$$f\circ\varphi=f(tu,tv)+tg(tu,tv)=t^{2n}h(u,v,t),$$

where

$$h(u,v,t) = \big(f_{2n}(u,v) + g_{2n-1}(u,v)\big) + t\big(f_{2n+1}(u,v) + g_{2n}(u,v)\big) + \cdots \\ + t^{r-2n}\big(f_r(u,v) + g_{r-1}(u,v)\big).$$

Notice here that since $f_{2n}(x,y)+zg_{2n-1}(x,y)$ does not vanish on $z^2 = x^2+y^2$, then $f_{2n}(u,v) + g_{2n-1}(u,v)$ does not vanish on $u^2+v^2 = 1$. Also, as $f + zg$ is psd on $z^2 = x^2 + y^2$ near the origin, we have $h_t(u,v) \geq 0$ for $u^2+v^2 = 1$, and $|t|$ small. From now on, our argument will be a kind of parametrized remake of the classical proof that any psd on the circle is a sum of two squares of polynomials, as it can be found in [**PoSz**]. Of course, this naive presentation becomes quite involved along the way.

We consider the trigonometric polynomials for $u = \cos\theta$, $v = \sin\theta$:

$$\cos(k\theta) = P_k(u) \text{ where } P_k \text{ has degree } k, \qquad \text{and}$$
$$\sin(k\theta) = vQ_{k-1}(u) \text{ where } Q_{k-1} \text{ has degree } k-1.$$

Then, for $u^2+v^2 = 1$ we have an expression:

$$h(u,v,t) = \gamma_0(t) + \lambda_1(t)P_1(u) + \cdots + \lambda_r(t)P_r(u) \\ + v\big(\mu_1(t) + \mu_2(t)Q_1(u) + \cdots + \mu_r(t)Q_{r-1}(u)\big),$$

where $\gamma_0, \lambda_k, \mu_k \in \mathbb{R}[t]$ have degree $\leq r-2n$, and λ_r, μ_r are not both zero. On the other hand, let $\Re$ stand for *real part*, and $\Im$ for *imaginary part* (see [**Rz1**]). From the relations

$$P_k(u) = \Re\big((u+\sqrt{-1}v)^k\big),\ vQ_{k-1}(u) = \Im\big((u+\sqrt{-1}v)^k\big),$$

we deduce

$$\begin{aligned} h(u,v,t) &= \gamma_0 + \sum_k \lambda_k \Re\big((u+\sqrt{-1}v)^k\big) + \mu_k \Im\big((u+\sqrt{-1}v)^k\big) \\ &= \gamma_0 + \sum_k \tfrac{1}{2}(\lambda_k + \sqrt{-1}\mu_k)(u-\sqrt{-1}v)^k \\ &\quad + \tfrac{1}{2}(\lambda_k - \sqrt{-1}\mu_k)(u+\sqrt{-1}v)^k. \end{aligned}$$

Now, setting $w = u + \sqrt{-1}v$, we have $w\overline{w} = 1$ and get

$$\begin{aligned} w^r h(u,v,t) &= w^r(\gamma_r\overline{w}^r + \cdots + \gamma_1\overline{w} + \gamma_0 + \overline{\gamma}_1 w + \cdots + \overline{\gamma}_r w^r) \\ &= \gamma_r + \cdots + \gamma_1 w^{r-1} + \gamma_0 w^r + \overline{\gamma}_1 w^{r+1} + \cdots + \overline{\gamma}_r w^{2r} \\ &= G(t,w), \end{aligned}$$

where $\gamma_k = \frac{1}{2}(\lambda_k + \sqrt{-1}\mu_k)$ for $k \geq 1$. Recall that all this holds for $u = \cos\theta$, $v = \sin\theta$, but anyway we have this polynomial $G(t,w) \in \mathbb{C}[t,w]$. We remark here that the construction gives

$$G(0,w) = w^r\big(f_{2n}(u,v) + g_{2n-1}(u,v)\big)$$

for $u^2+v^2=1$, which says that $G(0,w)\neq 0$, and viewed as a series in t, this polynomial has order 0. We will study $G(t,w)$ as a polynomial in w, whose roots are Puiseux series in the variable t.

First of all those roots are associated by the property that:

(1) *No root of G is zero, and if ζ is a root, then $1/\overline{\zeta}$ is also a root of the same multiplicity.*

Let ζ be a root of G. Since $G(t,0)=\gamma_r=\frac{1}{2}(\lambda_r+\sqrt{-1}\mu_r)$, and λ_r,μ_r are not both zero, we have $\zeta\neq 0$. Moreover, we have

$$w^{2r}\overline{G}(t,1/w)=G(w),$$

and the substitution $w=1/\overline{\zeta}$ gives $G(t,1/\overline{\zeta})=0$. The assertion about the multiplicity follows from the same substitution after derivation of the above equality with respect to w.

We distinguish a special case for these roots, studying *conjugation over the quotient field of* $\mathbb{C}\{t\}$.

(2) *If ζ and $1/\overline{\zeta}$ are conjugated, then both roots have order* 0 *and even multiplicity, and their irreducible polynomial belongs to* $\mathbb{C}\{t\}[w]$.

Interchanging ζ and $1/\overline{\zeta}$ we can suppose $\zeta=\xi(t^{1/q})$ for some convergent series ξ. We recall that the conjugates of $\zeta=\xi(t^{1/q})$ are the series $\zeta^k=\xi\left(e^{\frac{2\pi i}{q}k}t^{1/q}\right)$, $1\leq k\leq q$, and the irreducible polynomial of ζ is the product $\prod_k(w-\zeta^k)$ (see [**Rz1**]), which already gives the last assertion of the statement. Now, suppose $\zeta^k=1/\overline{\zeta}$. This implies that

$$1=\xi\left(e^{\frac{2\pi i}{q}k}t^{1/q}\right)\overline{\xi}(t^{1/q})=\xi\left(e^{\frac{\pi i}{q}k}t^{1/q}\right)\overline{\xi}\left(e^{-\frac{\pi i}{q}k}t^{1/q}\right)$$

(after the substitution $t^{1/q}\mapsto e^{-\frac{\pi i}{q}k}t^{1/q}$), and setting $\eta=\xi\left(e^{\frac{\pi i}{q}k}t^{1/q}\right)$, we get $\overline{\eta}=\overline{\xi}\left(e^{-\frac{\pi i}{q}k}t^{1/q}\right)$ and $1=\eta\overline{\eta}$. Also note that ξ must have order 0.

Let us take a closer look at the new Puiseux series η. Firstly, since $\left(e^{-\frac{\pi i}{q}k}\right)^q=(-1)^k$, the substitution $t^{1/q}\mapsto e^{\frac{\pi i}{q}k}t^{1/q}$ transforms the polynomial $G(t,w)$ in the polynomial $G((-1)^kt,w)$, and ζ in η. Thus, η is a root of $G((-1)^kt,w)$ and we are reduced to show that η has even multiplicity.

To that end, we consider a real variable θ and write:

$$\Gamma(t,\theta)=h(\cos\theta,\sin\theta,(-1)^kt)=e^{-ir\theta}G((-1)^kt,e^{i\theta}).$$

For $s\in\mathbb{R}$ we have the analytic function $\Gamma_s:\theta\mapsto\Gamma((-1)^ks,\theta)$. By succesive derivations we obtain

$$\tfrac{\partial^k}{\partial\theta^k}\Gamma(s,\theta)=\sum_{\ell=0}^{k}c_{k\ell}\tfrac{\partial^\ell}{\partial w^\ell}G((-1)^ks,e^{i\theta}),$$

with $c_{kk} \neq 0$. From these formulae one sees that if $e^{i\theta} \in \mathbb{C}$ is a root of the polynomial $G_s = G((-1)^k s, w) \in \mathbb{C}[w]$, then $\theta \in \mathbb{R}$ is a root of the same multiplicity of Γ_s. But by the construction, the real function Γ_s is ≥ 0 for any $\theta \in \mathbb{R}$, which readily implies that all of its real roots have even multiplicity. Now, if η has multiplicity p, then a suitable specialization gives a root $e^{i\theta} \in \mathbb{C}$ of multiplicity p of G_s for some s, and we conclude that p is even.

Thus the proof of (2) is complete, and we factorize:

$$G(t,w) = cQ^2 \prod_\ell (w - \zeta_\ell)\left(w - \frac{1}{\overline{\zeta}_\ell}\right), \tag{3}$$

where $c = \overline{\gamma}_r \in \mathbb{C}[t]$, and the factor $Q(t,w) \in \mathbb{C}\{t\}[w]$ corresponds to all roots of order 0 and even multiplicity of G, so that Q has order 0. Also, maybe interchanging some ζ_ℓ and $1/\overline{\zeta}_\ell$, we can suppose all ζ_ℓ's have order ≥ 0. What is important is that, by (2), the roots ζ_ℓ and $1/\overline{\zeta}_\ell$ are not conjugated.

After this preparation we claim:

(4) *The product* $\prod_\ell (w - \zeta_\ell)$ *is in fact a polynomial in* $\mathbb{C}\{t\}[w]$.

Indeed, the claim will follow from a close inspection of the conjugated roots of the ζ_ℓ's. Let ζ_ℓ and ζ' be conjugated. If ζ' is conjugated of some $1/\overline{\zeta}_{\ell'}$, then $\ell' \neq \ell$ and the order of $\zeta_{\ell'}$ is zero, so that we can interchange $\zeta_{\ell'}$ and $1/\overline{\zeta}_{\ell'}$. By this means, we guarantee that no root of $\prod_\ell (w - \zeta_\ell)$ is conjugated of a roots of $\prod_\ell \left(w - \frac{1}{\overline{\zeta}_\ell}\right)$. But by the form of our factorization, no root of $\prod_\ell (w - \zeta_\ell)$ is conjugated of a root of Q. Whence, we conclude that $\prod_\ell (w - \zeta_\ell)$ is a product of irreducible polynomials of ζ_ℓ's, all of them of order ≥ 0, and consequently a product of polynomials in $\mathbb{C}\{t\}[w]$, so in this ring as well.

Coming back to the starting point that $h(u,v,t) \geq 0$ for $|w| = u^2 + v^2 = 1$ and t real near 0, we can write:

$$\begin{aligned} h(u,v,t) = |h(u,v,t)| = |w^r h(u,v,t)| &= |G(t,w)| \\ &= |c|\,|Q^2| \prod_\ell |w - \zeta_\ell| \left|w - \frac{1}{\overline{\zeta}_\ell}\right| \\ &= \left|\frac{c}{\prod_\ell \overline{\zeta}_\ell}\right| |Q^2| \prod_\ell |w - \zeta_\ell||\overline{\zeta}_\ell w - 1| \\ &= \left|\frac{c}{\prod_\ell \zeta_\ell}\right| |Q^2| \prod_\ell |w - \zeta_\ell|^2, \end{aligned}$$

the last equality because $|w-\zeta_\ell| = |\overline{\zeta}_\ell w - 1|$ for $|w|=1$ and t real. Thus we have obtained

$$h(u,v,t) = \left|\frac{c}{\prod_\ell \zeta_\ell}\right| \left|Q\prod_\ell (w-\zeta_\ell)\right|^2. \tag{5}$$

This formula explains itself, but we still have to look at the first factor. To that end, we analyze the product

$$c\prod_\ell (w-\zeta_\ell)\left(w-\frac{1}{\overline{\zeta}_\ell}\right) \in \mathbb{C}\{t\}[w].$$

Setting

$$\prod_\ell (w-\zeta_\ell) = w^p + \alpha_{p-1}w^{p-1} + \cdots + \alpha_1 w + \alpha_0,$$

a straightforward computation shows that

$$\prod_\ell \left(w-\frac{1}{\overline{\zeta}_\ell}\right) = w^p + \frac{\overline{\alpha}_1}{\overline{\alpha}_0}w^{p-1} + \cdots + \frac{\overline{\alpha}_{p-1}}{\overline{\alpha}_0}w + \frac{1}{\overline{\alpha}_0}.$$

This will give us explicit expressions for the coefficients $\beta_k = \eta_k/\overline{\alpha}_0$

$$P(t,w) = \prod_\ell (w-\zeta_\ell)\left(w-\frac{1}{\overline{\zeta}_\ell}\right) = w^{2p} + \beta_{2p-1}w^{2p-1} + \cdots + \beta_1 w + \beta_0.$$

We recall that $G(t,w) = cQ^2P$ is a series of order 0 in t, hence the same is true for cP. Consequently, the order in t of all the products $c\beta_k$ must be ≥ 0, and some of them exactly 0. For a few lines, let ord_t denote order with respect to t. We have:

$$\beta_p = \frac{1}{\overline{\alpha}_0}(|\alpha_0|^2 + \cdots + |\alpha_{p-1}|^2 + 1),$$

so that $\mathrm{ord}_t(\overline{\alpha}_0) \leq \mathrm{ord}_t(c)$. Then

$$0 \leq \mathrm{ord}_t(c\beta_k) = \big(\mathrm{ord}_t(c) - \mathrm{ord}_t(\overline{\alpha}_0)\big) + \mathrm{ord}_t(\eta_k),$$

and this is 0 only if $\mathrm{ord}_t(c) = \mathrm{ord}_t(\overline{\alpha}_0)$. As $\mathrm{ord}_t(\overline{\alpha}_0) = \mathrm{ord}_t(\alpha_0)$, we deduce that $c/\alpha_0 = c/\prod_\ell \zeta_\ell$ is a unit $a \in \mathbb{C}\{t\}$, so that $|a|^2$ is a unit of $\mathbb{R}\{t\}$, hence a square, say $|a|^2 = b^2$, $b \in \mathbb{R}\{t\}$. Now, as $w = u + \sqrt{-1}v$, we can write

$$H(u,v,t) = Q\prod_\ell (w-\zeta_\ell) \in \mathbb{C}\{t\}[u,v]$$

in the form $H = H_1 + \sqrt{-1}H_2$, $H_1, H_2 \in \mathbb{R}\{t\}[u,v]$, and $|H|^2 = H_1^2 + H_2^2$. The conclusion is that

$$h(u,v,t) = b(t)^2(H_1(u,v,t)^2 + H_2(u,v,t)^2)$$

all this for $u^2+v^2=1$, t small. Hence we change $u=x/z, v=y/z, t=z$, to find

$$\begin{aligned} f(x,y)+zg(x,y) &= t^{2n}h(u,v,t) \\ &= z^{2n}b(z)^2(H_1(x/z,y/z,t)^2+H_2(x/z,y/z,t)^2), \end{aligned}$$

and finally note that H_1 and H_2 being polynomials in the variables u,v, multiplying by a large power m of z, we get $z^mH_1(x/z,y/z,t), z^mH_2(x/z,y/z,t) \in \mathbb{R}\{t\}[x,y]$, and so

$$\begin{aligned} z^{2m}(f(x,y)+zg(x,y)) = z^{2n}b(z)^2\Big(&\big(z^mH_1(x/z,y/z,t)\big)^2 \\ &+\big(z^mH_2(x/z,y/z,t)\big)^2\Big) \end{aligned}$$

is a sum of two squares of analytic functions on the cone $z^2=x^2+y^2$. $\square$

The next step is to get rid of the denominator z^{2m} in the preceding proposition. For this we prove:

Proposition 2.2. *Suppose that the function $f(x,y)+zg(x,y)$ has the property that $z^{2m}(f+zg)$ is a sum of two squares of analytic germs on the cone. Then $f+zg$ itself is a sum of two squares of analytic germs.*

Proof. By hypothesis, there are series $a_1,a_2,b_1,b_2 \in \mathbb{R}\{x,y\}$ and $h \in \mathbb{R}\{x,y,z\}$ such that

$$(x^2+y^2)^m(f+zg) = (a_1+zb_1)^2+(a_2+zb_2)^2-(z^2-x^2-y^2)h.$$

By the Weierstrass Division Theorem (see [**Rz1**]), h must be a polynomial in z, and looking at the degrees in both sides we conclude that in fact $h \in \mathbb{R}\{x,y\}$, and in the end

$$\begin{gathered} h=b_1^2+b_2^2, \qquad (x^2+y^2)^mg = 2a_1b_1+2a_2b_2, \\ (x^2+y^2)^mf-(x^2+y^2)h = a_1^2+a_2^2. \end{gathered}$$

Now we compute a little:

$$\begin{aligned} \big((x^2+y^2)^mf-(x^2+y^2)h\big)h &= (a_1^2+a_2^2)(b_1^2+b_2^2) \\ &= (a_1b_1+a_2b_2)^2+(a_1b_2-a_2b_1)^2 \\ &= \tfrac{1}{4}(x^2+y^2)^{2m}g^2+(a_1b_2-a_2b_1)^2, \end{aligned}$$

so that x^2+y^2 divides $a_1b_2-a_2b_1$. Since it also divides $a_1b_1+a_2b_2$, we deduce it divides

$$\begin{gathered} (a_1b_2-a_2b_1)b_2+(a_1b_1+a_2b_2)b_1 = a_1(b_1^2+b_2^2) = a_1h \\ -(a_1b_2-a_2b_1)b_1+(a_1b_1+a_2b_2)b_2 = a_2(b_1^2+b_2^2) = a_2h. \end{gathered}$$

Suppose first $m \geq 2$, and that x^2+y^2 does not divide h. Then it divides both a_1 and a_2, hence $(x^2+y^2)^2$ divides $a_1^2+a_2^2=(x^2+y^2)^mf-(x^2+y^2)h$,

and x^2+y^2 divides h, a contradiction. Thus x^2+y^2 divides h, so that it divides $(a_1+zb_1)^2+(a_2+zb_2)^2=\alpha_1^2+\alpha_2^2$. Factorizing in $\mathbb{C}\{x,y,z\}$ we find that $x+\sqrt{-1}y$ divides $\alpha_1+\sqrt{-1}\alpha_2$ (or the conjugate, but then replace α_2 by $-\alpha_2$). We write this:

$$\alpha_1+\sqrt{-1}\alpha_2=(x+\sqrt{-1}y)(\beta_1+\sqrt{-1}\beta_2),$$
$$\alpha_1-\sqrt{-1}\alpha_2=(x-\sqrt{-1}y)(\beta_1-\sqrt{-1}\beta_2),$$

where $\beta_1,\beta_2\in\mathbb{R}\{x,y,z\}$. Whence on the cone $z^2=x^2+y^2$ we have

$$(x^2+y^2)^m(f+zg)=\alpha_1^2+\alpha_2^2=(x^2+y^2)(\beta_1^2+\beta_2^2),$$

and simplifying x^2+y^2 we see that $(x^2+y^2)^{m-1}(f+zg)$ is a sum of two squares of analytic germs.

Repeating the argument above we end up with $f+zg$ being a sum of two squares of analytic germs or with $m=1$ and x^2+y^2 dividing both a_1 and a_2. If that is the case, we can write $a_1=(x^2+y^2)a_1'$, $a_2=(x^2+y^2)a_2'$, and on the cone

$$\begin{aligned}z^2(f+zg)=(a_1+zb_1)^2+(a_2+zb_2)^2&=(z^2a_1'+zb_1)^2+(z^2a_2'+zb_2)^2\\&=z^2\Big((za_1'+b_1)^2+(za_2'+b_2)^2\Big),\end{aligned}$$

so that in fact $f+zg$ is a sum of two squares of analytic functions. □

Combining the two preceding results, we have solved the problem for functions $f+zg$ where f,g are polynomials. For the general case we only need the following approximation lemma:

Lemma 2.3. *Let H be a non-negative function germ on the cone. Then, for every $k\geq 0$ there is a non-negative function germ on the cone $f+zg$, with f,g polynomials, such that $H\equiv f+zg \mod (x,y,z)^k$.*

For, let us assume this. What we must show is that the equation

$$H=\mathtt{h}_1^2+\mathtt{h}_2^2+(z^2-x^2-y^2)\mathtt{h}$$

has a solution $\mathtt{h}_1=h_1,\mathtt{h}_2=h_2,\mathtt{h}=h$ in $\mathbb{R}\{x,y,z\}$. By M. Artin's Approximation Theorem (see [**Rz1**] and [**Bb *et al***]), a solution exists if it exists up to order k large enough, that is, if the congruence

$$H\equiv\mathtt{h}_1^2+\mathtt{h}_2^2+(z^2-x^2-y^2)\mathtt{h} \mod (x,y,z)^k$$

has a solution for k large. But the psd $f+zg$ provided by 2.3 is a sum of two squares by the polynomial case 2.1: There are $h_1,h_2,h\in\mathbb{R}\{x,y,z\}$ such that $f+zg=h_1^2+h_2^2+(z^2-x^2-y^2)h$. Clearly h_1,h_2,h are the solution up to order k we sought, and we are done.

To finish, let us prove the lemma.

Proof of 2.3. First, replacing H by $(x^2+y^2)^k + H$ we can assume that H vanishes only at the origin. Then we write $H = f + zg$ with $f, g \in \mathbb{R}\{x, y\}$. If $f = 0$ contains some planar curve germ Y, this germ Y lifts to two curve germs Y_+ and Y_- in the cone corresponding to $z = \pm\sqrt{x^2+y^2}$. On these curve germs $H = zg \geq 0$, and, z changing sign from Y_+ to Y_-, this is only possible if g vanishes on Y, and then H vanishes on Y_+ and Y_-, against the assumption that it only vanishes at the origin. All this means that f only vanishes at the origin. Next we use again the trick that the change $z = -z$ leaves the cone invariant, so that both $f + zg$ and $f - zg$ vanish only at the origin, hence so does their product $f^2 - z^2g^2 = f^2 - (x^2+y^2)g^2 \in \mathbb{R}\{x, y\}$. This means that all the complex branches of f and $f^2 - (x^2+y^2)g^2$ are imaginary. Now, we recall an easy consequence of the Newton algorithm (see [**Wk**]) for the computation of the complex branches of a planar curve germ: The jets of order r of those branches depend on the jets of order s of the given germ, or in other words, if we approximate the curve germ till order s, the branches of the approximation approximate the branches of the given germ till order r. Hence if we have a germ whose branches are all imaginary, any close approximation of the germ has also this property. Applying this to f and $f^2 - (x^2+y^2)g^2$ we find that replacing f and g by their jets of order high enough (and $\geq k$), f and $f^2 - (x^2+y^2)g^2$ have only imaginary branches, and are ≥ 0. Consequently, after the replacement by those jets we still have

$$(f + zg) + (f - zg) = 2f \geq 0,$$
$$(f + zg)(f - zg) = f^2 - (x^2+y^2)g^2 \geq 0.$$

Whence, $f + zg \geq 0$, and we are done. □

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