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Pearson equations for discrete orthogonal polynomials: I. Generalized hypergeometric functions and Toda equations

Manuel Mañas[1,2] | **Itsaso Fernández-Irisarri**[1] | **Omar F. González-Hernández**

[1] Departamento de Física Teórica, Universidad Complutense de Madrid, Plaza Ciencias 1, Madrid, Spain

[2] Instituto de Ciencias Matematicas (ICMAT), Campus de Cantoblanco UAM, Madrid, Spain

Correspondence
Manuel Mañas, Departamento de Física Teórica, Universidad Complutense de Madrid, Plaza Ciencias 1, 28040-Madrid, Spain, and Instituto de Ciencias Matematicas (ICMAT), Campus de Cantoblanco UAM, 28049-Madrid, Spain.
Email: manuel.manas@ucm.es

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Abstract

The Cholesky factorization of the moment matrix is applied to discrete orthogonal polynomials on the homogeneous lattice. In particular, semiclassical discrete orthogonal polynomials, which are built in terms of a discrete Pearson equation, are studied. The Laguerre–Freud structure semiinfinite matrix that models the shifts by ± 1 in the independent variable of the set of orthogonal polynomials is introduced. In the semiclassical case it is proven that this Laguerre–Freud matrix is banded. From the well-known fact that moments of the semiclassical weights are logarithmic derivatives of generalized hypergeometric functions, it is shown how the contiguous relations for these hypergeometric functions translate as symmetries for the corresponding moment matrix. It is found that the 3D Nijhoff–Capel discrete Toda lattice describes the corresponding contiguous shifts for the squared norms of the orthogonal polynomials. The continuous 1D Toda equation for these semiclassical discrete orthogonal polynomials is discussed and the compatibility equations are derived. It is also shown that the Kadomtesev–Petviashvilii equation is connected to an adequate deformed semiclassical discrete weight, but in this case, the deformation does not satisfy a Pearson equation.

KEYWORDS
3D Nijhoff–Capel discrete Toda equation, Cholesky factorization, contigous relations, discrete orthogonal polynomials, generalized hypergeometric functions, Pearson equations, Toda hierarchy

1 | INTRODUCTION

In this paper, the Gauss–Borel factorization of the moment matrix is applied to study discrete orthogonal polynomials subject to a Pearson equation.

Discrete orthogonal polynomials is nowadays a well-established subject, the classical case has been treated extensively in Ref. 1 and the the Riemann–Hilbert problem has been used to study asymptotics and applications, see Ref. 2. Semiclassical reductions, in where a discrete Pearson equation is fulfilled by the weight has also been treated in the literature. See, for example, Refs. 3, 4 and 5, 6 and references therein for an comprehensive account. For some specific type of weights of generalized Charlier and Meixner types, the corresponding Freud–Laguerre-type equations for the coefficients of the three term recurrence have been studied, see, for example, Refs. 7–11. For a general account, see Refs. 12–14, and for discrete orthogonal polynomials and Painlevé equations, see Ref. 15.

With this introduction, we give a fast briefing of introductory character, of orthogonal polynomials, its discrete version, and the Cholesky factorization of the moment matrix. We also discuss for the first time in this context, to the best of our knowledge, the Pascal matrices and its dressed versions.

Then, in Section 2, we discuss the discrete Pearson equation and give Theorem 1, the first main result of the paper, which describes a new symmetry of the moment matrix, direct consequence of the Pearson equation, in terms of the Pascal matrix. We then deduce the corresponding symmetry for the Jacobi matrix, see Proposition 6. Then, in Theorem 2, the second main result of the paper, we describe a banded semi-infinite matrix, that we called Laguerre–Freud structure matrix, which models the shift in the spectral variable. The name is rooted on the fact that this banded matrix encodes Laguerre–Freud equations for the recurrence coefficients of the orthogonal polynomial sequence, see, for example, Ref. 16. The number of nontrivial superdiagonals and subdiagonals of this matrix is determined by the Pearson equation. Several properties involving the Jacobi and Laguerre–Freud matrices are derived. Of particular interest are those of compatibility type.

As the Pearson equation leads to generalized hypergeometric moments, we study the contiguous relations of these functions and its description as further symmetries of the moment matrix, see Theorem 4. In Theorem 5, the squared norms of these discrete orthogonal polynomials are shown to be solutions of a well-known discrete integrable multidimensional lattice, known as the Nijhoff–Capel lattice, see Refs. 17, 18. We also discuss the Toda hierarchy deformations, with only the first flow preserving the Pearson reduction. The compatibility with this first Toda flow and the Pearson equation is discussed in Proposition 21. The connection of deformations of these weights with solutions of the Kadomtsev–Petviashvilli (KP) equation is discussed as well.

1.1 | Linear functionals and orthogonal polynomials

Given a linear functional $\rho_z \in \mathbb{C}^*[z]$, here $\mathbb{C}^*[z]$ denotes the dual of the linear space of complex polynomials $\mathbb{C}[z]$, the corresponding moment matrix is

$$G = (G_{n,m}), \quad G_{n,m} = \rho_{n+m}, \quad \rho_n = \langle \rho_z, z^n \rangle, \quad n, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}, \tag{1}$$

with ρ_n the nth moment of the linear functional ρ_z. If the moment matrix is such that all its truncations, which are Hankel matrices, $G_{i+1,j} = G_{i,j+1}$,

$$G^{[k]} = \begin{pmatrix} G_{0,0} & \cdots & G_{0,k-1} \\ \vdots & & \vdots \\ G_{k-1,0} & \cdots & G_{k-1,k-1} \end{pmatrix} = \begin{pmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & \rho_2 & & \iddots & \rho_k \\ \rho_2 & & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & \vdots \\ \rho_{k-1} & \rho_k & \cdots & \cdots & \rho_{2k-2} \end{pmatrix} \tag{2}$$

are nonsingular; that is, the Hankel determinants $\Delta_k := \det G^{[k]}$ do not cancel, $\Delta_k \neq 0$, $k \in \mathbb{N}_0$, then there exists monic polynomials

$$P_n(z) = z^n + p_n^1 z^{n-1} + \cdots + p_n^n, \quad n \in \mathbb{N}_0, \tag{3}$$

with $p_0^1 = 0$, such that the following orthogonality conditions are fulfilled:

$$\left\langle \rho, P_n(z) z^k \right\rangle = 0, \quad k \in \{0, \ldots, n-1\}, \quad \langle \rho, P_n(z) z^n \rangle = H_n \neq 0. \tag{4}$$

Moreover, the set $\{P_n(z)\}_{n \in \mathbb{N}_0}$ is an orthogonal set of polynomials $\langle \rho, P_n(z) P_m(z) \rangle = \delta_{n,m} H_n$, $n, m \in \mathbb{N}_0$. In this case, we have a symmetric bilinear form $\langle F, G \rangle_\rho := \langle \rho, FG \rangle$, such that the moment matrix is the Gram matrix of this bilinear form and $\langle P_n, P_m \rangle_\rho := \delta_{n,m} H_n$. In this paper, we will use both denominations, moment matrix, or Gram matrix, indistinctly to refer to the matrix G.

1.2 | The shift matrix Λ

In terms of the semi-infinite vector of monomials,

$$\chi(z) := \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \end{pmatrix} \tag{5}$$

the moment matrix is written as $G = \langle \rho, \chi\chi^\top \rangle$, and it becomes evident that this matrix is symmetric, that is, $G = G^\top$. The vector of monomials χ is an eigenvector of the *shift matrix*

$$\Lambda := \begin{pmatrix} 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix} \tag{6}$$

that is, $\Lambda\chi = x\chi$. From here, it immediately follows that $\Lambda G = G\Lambda^\top$, that is, the Gram matrix is a Hankel matrix, as we previously said. The transposed matrix

$$\Lambda^\top = \begin{pmatrix} 0 & \cdots & \cdots \\ 1 & \ddots & \\ 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix} \tag{7}$$

satisfies $\Lambda\Lambda^\top = I$ and $\Lambda^\top\Lambda = I - E_{0,0}$, with $(E_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}, i, j \in \mathbb{N}_0$.

1.3 | The Cholesky factorization of the Gram matrix

Being the Gram matrix symmetric its Gauss–Borel factorization reduces to a Cholesky factorization

$$G = S^{-1}HS^{-\top}, \tag{8}$$

where S is a lower unitriangular matrix that can be written as

$$S = \begin{pmatrix} 1 & 0 & \cdots & \\ S_{1,0} & 1 & \ddots & \\ S_{2,0} & S_{2,1} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{9}$$

and $H = \operatorname{diag}(H_0, H_1, \ldots)$ is a diagonal matrix, with $H_k \neq 0$, for $k \in \mathbb{N}_0$. The Cholesky factorization holds whenever the principal minors of the moment matrix; that is, the Hankel determinants Δ_k, do not cancel.

The components $P_n(z)$ of the semi-infinite vector of polynomials

$$P(z) := S\chi(z), \tag{10}$$

are the monic orthogonal polynomials of the functional ρ. Indeed, from the Cholesky factorization, we know that

$$\langle \rho, \chi\chi^\top \rangle = G = S^{-1} H S^{-\top} \tag{11}$$

so that $S\langle \rho, \chi\chi^\top \rangle S^\top = H$, and consequently, $\langle \rho, S\chi\chi^\top S^\top \rangle = H$ and we get $\langle \rho, PP^\top \rangle = H$, which recollects the orthogonality of the polynomials $\{P_n(z)\}_{n=0}^\infty$.

1.4 | The Jacobi matrix

Given this construction is natural to introduce the lower Hessenberg semi-infinite matrix

$$J = S\Lambda S^{-1} \tag{12}$$

that has the vector $P(z)$ as eigenvector with eigenvalue z, that is $JP(z) = zP(z)$. The Hankel condition of the Gram matrix $\Lambda G = G\Lambda^\top$ together with the Cholesky factorization leads to $\Lambda S^{-1} H S^{-\top} = S^{-1} H S^\top \Lambda^\top$, or, equivalently, $S\Lambda S^{-1} H = H S^{-\top} \Lambda^\top S^{-\top}$, that is,

$$JH = (JH)^\top = HJ^\top. \tag{13}$$

That is, the Hessenberg matrix JH is symmetric, thus being Hessenberg and symmetric we deduce that is tridiagonal. Hence, the Jacobi matrix J given in (12) reads

$$J = \begin{pmatrix} \beta_0 & 1 & 0 & \cdots \\ \gamma_1 & \beta_1 & 1 & \ddots \\ 0 & \gamma_2 & \beta_2 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{14}$$

and the eigenvalue equation $JP = zP$ is a three term recursion relation

$$zP_n(z) = P_{n+1}(z) + \beta_n P_n(z) + \gamma_n P_{n-1}(z), \tag{15}$$

that with the initial conditions $P_{-1} = 0$ and $P_0 = 1$ completely determines the sequence of orthogonal polynomials $\{P_n(z)\}_{n\in\mathbb{N}_0}$ in terms of the recursion coefficients β_n, γ_n. In terms of the Hankel and modified Hankel determinants

$$\Delta_k := \det \begin{pmatrix} \rho_0 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \vdots & & & \vdots \\ \rho_{k-2} & & & \rho_{2k-3} \\ \rho_{k-1} & \cdots & \rho_{2k-3} & \rho_{2k-2} \end{pmatrix}, \qquad \tilde{\Delta}_k := \det \begin{pmatrix} \rho_0 & \cdots & \rho_{k-2} & \rho_k \\ \vdots & & \rho_{k-1} & \vdots \\ \rho_{k-2} & & & \rho_{2k-2} \\ \rho_{k-1} & \cdots & \rho_{2k-3} & \rho_{2k-1} \end{pmatrix} \tag{16}$$

we find the following.

Proposition 1. *The recursion coefficients are given by*

$$\beta_n = p_n^1 - p_{n+1}^1 = -\frac{\tilde{\Delta}_n}{\Delta_n} + \frac{\tilde{\Delta}_{n+1}}{\Delta_{n+1}}, \quad \gamma_{n+1} = \frac{H_{n+1}}{H_n} = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \quad n \in \mathbb{N}_0, \tag{17}$$

Proof. For $n \in \mathbb{N} := \{1, 2, \dots\}$, from $J^\top = H^{-1}JH$, we get that $\gamma_n = \frac{H_n}{H_{n-1}}$. On the other hand, as $J = S\Lambda S^{-1}$ and recalling that for the coefficients $S_{n,n-1}$ of the first subdiagonal of S, we have $S_{n,n-1} = p_n^1$, the first nontrivial leading coefficient of the monic polynomial P_n, we get $\beta_n = p_n^1 - p_{n+1}^1$. ■

For future use, we introduce the following diagonal matrices:

$$\gamma := \operatorname{diag}(\gamma_1, \gamma_2, \dots), \quad \beta := \operatorname{diag}(\beta_0, \beta_1, \dots) \tag{18}$$

and

$$J_- := \Lambda^\top \gamma, \quad J_+ := \beta + \Lambda, \tag{19}$$

so that we have the splitting

$$J = \Lambda^\top \gamma + \beta + \Lambda = J_- + J_+. \tag{20}$$

In general, given any semi-infinite matrix A, we will write $A = A_- + A_+$, where A_- is a strictly lower triangular matrix and A_+ an upper triangular matrix. Moreover, A_0 will denote the diagonal part of A.

1.5 | The Pascal matrices

The lower Pascal matrix, built up of binomial numbers, is defined by

$$B = (B_{n,m}), \quad B_{n,m} := \begin{cases} \binom{n}{m}, & n \geq m, \\ 0, & n < m, \end{cases} \tag{21}$$

so that

$$\chi(z+1) = B\chi(z). \tag{22}$$

Moreover,

$$B^{-1} = (\tilde{B}_{n,m}), \quad \tilde{B}_{n,m} := \begin{cases} (-1)^{n+m}\binom{n}{m}, & n \geq m, \\ 0, & n < m, \end{cases} \tag{23}$$

and

$$\chi(z-1) = B^{-1}\chi(z). \tag{24}$$

The lower Pascal matrix and its inverse are explicitly given by

$$B=\begin{pmatrix}1 & 0 & \cdots & & & & & \\ 1 & 1 & 0 & \cdots & & & & \\ 1 & 2 & 1 & 0 & \cdots & & & \\ 1 & 3 & 3 & 1 & 0 & \cdots & & \\ 1 & 4 & 6 & 4 & 1 & 0 & \cdots & \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & \cdots \\ \vdots & & & & & & \ddots & \ddots\end{pmatrix},\quad B^{-1}=\begin{pmatrix}1 & 0 & \cdots & & & & & \\ -1 & 1 & 0 & \cdots & & & & \\ 1 & -2 & 1 & 0 & \cdots & & & \\ -1 & 3 & -3 & 1 & 0 & \cdots & & \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdots & \\ -1 & 5 & -10 & 10 & -5 & 1 & 0 & \cdots \\ \vdots & & & & & & \ddots & \ddots\end{pmatrix}. \tag{25}$$

Let us introduce the lower unitriangular semi-infinite matrices, and let us refer to them as *dressed Pascal matrices,*

$$\Pi := SBS^{-1},\quad \Pi^{-1} := SB^{-1}S^{-1}, \tag{26}$$

which are connection matrices; that is,

$$P(z+1)=\Pi P(z),\quad P(z-1)=\Pi^{-1}P(z). \tag{27}$$

The lower Pascal matrix can be expressed in terms of its subdiagonal structure as follows:

$$B^{\pm1}=I\pm\Lambda^\top D+(\Lambda^\top)^2D^{[2]}\pm(\Lambda^\top)^3D^{[3]}+\cdots, \tag{28}$$

where the diagonal matrices $D, D^{[k]}$, with $k\in\mathbb{N}$, $(D=D^{[1]})$, are given by

$$D=\operatorname{diag}(1,2,3,\ldots),\quad D^{[k]}=\frac{1}{k}\operatorname{diag}(k^{(k)},(k+1)^{(k)},(k+2)^{(k)}\cdots), \tag{29}$$

in terms of the falling factorials $x^{(k)}=x(x-1)(x-2)\cdots(x-k+1)$. That is,

$$D^{[k]}_n=\frac{(n+k)\cdots(n+1)}{k},\quad k\in\mathbb{N},\quad n\in\mathbb{N}_0. \tag{30}$$

The lower unitriangular factor can also be written in terms of its subdiagonals

$$S=I+\Lambda^\top S^{[1]}+(\Lambda^\top)^2S^{[2]}+\cdots \tag{31}$$

with $S^{[k]}=\operatorname{diag}(S^{[k]}_0,S^{[k]}_1,\ldots)$ diagonal matrices. From (10), it is clear the following connection between these subdiagonals coefficients and the coefficients of the orthogonal polynomials, given in (3):

$$S^{[k]}_k=p^k_{n+k}. \tag{32}$$

We will use the *shift operators* $T_\pm$ with action on the diagonal matrices given by

$$T_-\operatorname{diag}(a_0,a_1,\ldots):=\operatorname{diag}(a_1,a_2,\ldots),\quad T_+\operatorname{diag}(a_0,a_1,\ldots):=\operatorname{diag}(0,a_0,a_1,\ldots), \tag{33}$$

where T_- is the lowering shift operator and T_+ the raising shift operator over the diagonal matrices. These shift operators have the following important properties, for any diagonal matrix $A = \operatorname{diag}(A_0, A_1, \dots)$:

$$\Lambda A = (T_-A)\Lambda, \quad A\Lambda = \Lambda(T_+A), \quad A\Lambda^\top = \Lambda^\top(T_-A), \quad \Lambda^\top A = (T_+A)\Lambda^\top. \tag{34}$$

Remark 1. Notice that the standard notation, see Ref. 1, for the differences of a sequence $\{f_n\}_{n\in\mathbb{N}_0}$,

$$\begin{aligned} \Delta f_n &:= f_{n+1} - f_n, \quad n \in \mathbb{N}_0, \\ \nabla f_n &= f_n - f_{n-1}, \quad n \in \mathbb{N}, \end{aligned} \tag{35}$$

and $\nabla f_0 = f_0$, connects with the shift operators by means of

$$T_- = I + \Delta, \quad T_+ = I - \nabla. \tag{36}$$

In terms of these shift operators, we find

$$2D^{[2]} = (T_-D)D, \quad 3D^{[3]} = (T_-^2D)(T_-D)D = 2(T_-D^{[2]})D = 2D^{[2]}(T_-^2D). \tag{37}$$

With these tools, we are ready to study the expansion in subdiagonals of S^{-1} in terms of the corresponding expansion of S.

Proposition 2. *The inverse matrix S^{-1} of the matrix S expands in terms of subdiagonals as follows:*

$$S^{-1} = I + \Lambda^\top S^{[-1]} + (\Lambda^\top)^2 S^{[-2]} + \cdots. \tag{38}$$

The subdiagonals $S^{[-k]}$ are explicitly given in terms of the subdiagonals of S, the first few are

$$\begin{aligned} S^{[-1]} &= -S^{[1]}, \\ S^{[-2]} &= -S^{[2]} + (T_-S^{[1]})S^{[1]}, \\ S^{[-3]} &= -S^{[3]} + (T_-S^{[2]})S^{[1]} + (T_-^2S^{[1]})S^{[2]} - (T_-^2S^{[1]})(T_-S^{[1]})S^{[1]}, \\ S^{[-4]} &= -S^{[4]} + (T_-S^{[3]})S^{[1]} + (T_-^2S^{[2]})S^{[2]} - (T_-^2S^{[2]})(T_-S^{[1])})S^{[1]} + (T_-^3S^{[1]})S^{[3]} \\ &\quad -(T_-^3S^{[1]})(T_-S^{[2])})S^{[1]} - (T_-^3S^{[1]})(T_-^2S^{[1])})S^{[2]} + (T_-^3S^{[1]})(T_-^2S^{[1]})(T_-S^{[1])})S^{[1]}. \end{aligned} \tag{39}$$

With all these at hand, we can also get expressions for the nontrivial coefficients corresponding to the highest powers of the monic orthogonal polynomials.

Proposition 3. *The following* nonlocal *expressions for the polynomial coefficients in terms of the recursion coefficients hold true:*

$$p^1_{n+1} = -\sum_{k=0}^{n} \beta_k, \quad p^2_{n+1} = -\sum_{k=1}^{n} \gamma_k + \sum_{0\leq l<k<n} \beta_k\beta_l. \tag{40}$$

Moreover, it is also true that

$$p^3_{n+2} - p^3_{n+3} = \gamma_{n+2} p^1_{n+1} + (\beta_{n+2} + \beta_{n+1} + \beta_n) p^2_{n+2} - (\beta_{n+1} + \beta_n) p^1_{n+2} p^1_{n+1}. \tag{41}$$

Proof. We start by noticing that

$$\begin{aligned} J = S\Lambda S^{-1} &= (I + \Lambda^\top S^{[1]} + (\Lambda^\top)^2 S^{[2]} + \cdots)\Lambda(I + \Lambda^\top S^{[-1]} + (\Lambda^\top)^2 S^{[-2]} + \cdots) \\ &= \Lambda + T_+ S^{[1]} + S^{[-1]} + \Lambda^\top (T_+ S^{[2]} + S^{[-2]} + S^{[1]} S^{[-1]}) \\ &\quad + (\Lambda^\top)^2 (T_+ S^{[3]} + S^{[-3]} + S^{[2]} S^{[-1]} + (T_+ S^{[1]}) S^{[-2]}) + \cdots . \end{aligned} \tag{42}$$

Thus, we obtain

$$\beta = T_+ S^{[1]} - S^{[1]}, \tag{43}$$

$$\gamma = T_+ S^{[2]} - S^{[2]} + (T_- S^{[1]} - S^{[1]}) S^{[1]}, \tag{44}$$

and an infinite set of relations among subdiagonals of S, being the first

$$T_+ S^{[3]} + S^{[-3]} + S^{[2]} S^{[-1]} + (T_+ S^{[1]}) S^{[-2]} = 0. \tag{45}$$

The relation (43) componentwise is (17). Equation (44) reads componentwise as follows:

$$\gamma_{n+1} = S^{[2]}_{n-1} - S^{[2]}_n + \left(S^{[1]}_{n+1} - S^{[1]}_n \right) S^{[1]}_n = p^2_{n+1} - p^2_{n+2} - \beta_{n+1} p^1_{n+1}. \tag{46}$$

Hence, using telescoping series again, we find (40). Finally, Equation (45) reads

$$\begin{aligned} T_+ S^{[3]} - S^{[3]} + (T_- S^{[2]}) S^{[1]} + (T_-^2 S^{[1]}) S^{[2]} - (T_-^2 S^{[1]})(T_- S^{[1]}) S^{[1]} - S^{[2]} S^{[1]} \\ + (T_+ S^{[1]}) \big(-S^{[2]} + (T_- S^{[1]}) S^{[1]} \big) = 0, \end{aligned} \tag{47}$$

that we can write

$$\begin{aligned} T_+ S^{[3]} - S^{[3]} + T_- \big(S^{[2]} - T_+ S^{[2]} - (T_- S^{[1]} - S^{[1]}) S^{[1]} \big) S^{[1]} + (T_-^2 S^{[1]} - T_+ S^{[1]}) S^{[2]} \\ + (T_+ S^{[1]} - T_- S^{[1]})(T_- S^{[1]}) S^{[1]} = 0, \end{aligned} \tag{48}$$

so that

$$T_+ S^{[3]} - S^{[3]} = (T_- \gamma) S^{[1]} + (T_-^2 \beta + T_- \beta + \beta) S^{[2]} - (T_- \beta + \beta)(T_- S^{[1]}) S^{[1]} = 0, \tag{49}$$

and componentwise, we have (41). ■

Remark 2. In (41), we can sum up on the LHS, observe that is a telescoping series, to get a nonlocal nonlinear expression, in terms of the recursion coefficients, for p^3_n. A similar statement holds for every p^k_n with $k = 4, 5, \ldots$.

Now, we turn our attention to the dressed Pascal matrix. We expand the dressed Pascal matrices into subdiagonals

$$\Pi^{\pm 1} = I + \Lambda^\top \pi^{[\pm 1]} + (\Lambda^\top)^2 \pi^{[\pm 2]} + \cdots \tag{50}$$

with $\pi^{[\pm n]} = \operatorname{diag}(\pi_0^{[\pm n]}, \pi_1^{[\pm n]}, \ldots)$. Then, for these subdiagonals we find the following.

Proposition 4 (The dressed Pascal matrix coefficients)**.** *The subdiagonal coefficients of the Pascal matrices satisfy*

$$\begin{aligned}\pi_n^{[\pm 1]} = \pm\,(n+1), \quad \pi_n^{[\pm 2]} &= \frac{(n+2)(n+1)}{2} \pm p^1_{n+2}(n+1) \mp (n+2)p^1_{n+1}\\ &= \frac{(n+2)(n+1)}{2} \mp (n+1)\beta_{n+1} \mp p^1_{n+1},\end{aligned} \tag{51}$$

$$\begin{aligned}\pi_n^{[\pm 3]} = &\pm \frac{(n+3)(n+2)(n+1)}{3} + \frac{(n+2)(n+1)}{2}p^1_{n+3} - \frac{(n+3)(n+2)}{2}p^1_{n+1}\\ &\pm (n+1)p^2_{n+3} \mp (n+3)p^2_{n+2} \pm (n+3)p^1_{n+2}p^1_{n+1} \mp (n+2)p^1_{n+3}p^1_{n+1}.\end{aligned} \tag{52}$$

Moreover, the following relations fulfill:

$$\begin{aligned}&\pi^{[1]} + \pi^{[-1]} = 0, \quad \pi^{[2]} + \pi^{[-2]} = 2D^{[2]}, \quad \pi^{[3]} + \pi^{[-3]} = 2((T_-^2 S^{[1]})D^{[2]} - (T_- D^{[2]})S^{[1]}),\\ &\pi^{[1]} - \pi^{[-1]} = 2D, \quad \pi^{[2]} - \pi^{[-2]} = 2((T_- S^{[1]})D - (T_- D)S^{[1]}),\\ &\pi^{[3]} - \pi^{[-3]} = 2\big(D^{[3]} + (T_- S^{[2]})D - (T_-^2 D)S^{[2]} + (T_-^2 D)(T_- S^{[1]})S^{[1]} - (T_-^2 S^{[1]})(T_- D)S^{[1]}\big).\end{aligned} \tag{53}$$

Proof. From (34), we get

$$\begin{aligned}\Pi^{\pm 1} &= (I + \Lambda^\top S^{[1]} + (\Lambda^\top)^2 S^{[2]} + \cdots)(I \pm \Lambda^\top D + (\Lambda^\top)^2 D^{[2]} \pm (\Lambda^\top)^3 D^{[3]} + \cdots)(I + \Lambda^\top S^{[-1]} + (\Lambda^\top)^2 S^{[-2]} + \cdots)\\ &= I + (I + \Lambda^\top S^{[1]} + (\Lambda^\top)^2 S^{[2]} + \cdots)\big(\pm\Lambda^\top D + (\Lambda^\top)^2 D^{[2]} \pm (\Lambda^\top)^3 D^{[3]} + \cdots\big)(I + \Lambda^\top S^{[-1]} + (\Lambda^\top)^2 S^{[-2]} + \cdots)\end{aligned} \tag{54}$$

so that

$$\begin{aligned}\Pi^{\pm 1} = I &\pm \Lambda^\top D + (\Lambda^\top)^2\big(D^{[2]} \pm (T_- S^{[1]})D \pm (T_- D)S^{[-1]}\big) + (\Lambda^\top)^3(\pm D^{[3]} + (T_-^2 S^{[1]})D^{[2]} + (T_- D^{[2]})S^{[-1]}\\ &\pm (T_- S^{[2]})D \pm (T_-^2 D)S^{[-2]} \pm (T_-^2 S^{[1]})(T_- D)S^{[-1]}) + \cdots.\end{aligned} \tag{55}$$

From (55) and Proposition 2, we obtain

$$\pi^{[\pm 1]} = \pm D, \tag{56}$$

$$\pi^{[\pm 2]} = D^{[2]} \pm (T_- S^{[1]})D \mp (T_- D)S^{[1]}, \tag{57}$$

$$\begin{aligned}\pi^{[\pm 3]} = &\pm D^{[3]} + (T_-^2 S^{[1]})D^{[2]} - (T_- D^{[2]})S^{[1]} \pm (T_- S^{[2]})D \mp (T_-^2 D)S^{[2]} \\ &\pm (T_-^2 D)(T_- S^{[1]})S^{[1]} \mp (T_-^2 S^{[1]})(T_- D)S^{[1]}.\end{aligned} \tag{58}$$

That component wise gives the desired result once we use the expressions for the β's in (17). ■

Proposition 5. *For any polynomial $R(z)$, we have*

$$\begin{aligned} R(\Lambda)B^{\pm 1} &= B^{\pm 1}R(\Lambda \pm I), \quad B^{\pm 1}R(\Lambda) = R(\Lambda \mp I)B^{\pm 1}, \\ R(J)\Pi^{\pm 1} &= \Pi^{\pm 1}R(J \pm I), \quad \Pi^{\pm 1}R(J) = R(J \mp I)\Pi^{\pm 1}. \end{aligned} \tag{59}$$

Proof. The compatibility of the couple of equations

$$\begin{cases} B^{\pm 1}\chi(z) = \chi(z \pm 1), \\ R(\Lambda)\chi(z) = R(z)\chi(z), \end{cases} \tag{60}$$

leads to

$$R(\Lambda)B^{\pm 1}\chi(z) = R(\Lambda)\chi(z \pm 1) = R(z \pm 1)\chi(z \pm 1) = R(z \pm 1)B^{\pm 1}\chi(z) = B^{\pm 1}R(\Lambda \pm I)\chi(z), \tag{61}$$

so that $R(\Lambda)B^{\pm 1} = B^{\pm 1}R(\Lambda \pm I)$, and consequently, $B^{\pm 1}R(\Lambda) = R(\Lambda \mp I)B^{\pm 1}$. Finally, a similarity transformation $\Lambda = S^{-1}JS$ gives the result. ■

2 | DISCRETE ORTHOGONAL POLYNOMIALS AND DISCRETE PEARSON EQUATIONS

2.1 | Discrete Pearson equation

We now assume that the functional is a sum of Dirac delta functions supported $\mathbb{N}_0$,

$$\rho = \sum_{k=0}^{\infty} \delta(z-k)w(k). \tag{62}$$

for some weight function $w(z)$ with finite values $w(k)$ for $k \in \mathbb{N}_0$. Hence, the bilinear form is $\langle F, G\rangle = \sum_{k=0}^{\infty} F(k)G(k)w(k)$. The moments are

$$\rho_n = \sum_{k=0}^{\infty} k^n w(k), \tag{63}$$

and, in particular, the 0th moment reads as follows:

$$\rho_0 = \sum_{k=0}^{\infty} w(k). \tag{64}$$

We will study families of weights that satisfy the following *discrete Pearson equation*:

$$\nabla(\sigma w) = \tau w, \tag{65}$$

that is, $\sigma(k)w(k) - \sigma(k-1)w(k-1) = \tau(k)w(k)$, for $k \in \{1, 2, \ldots\}$, with $\sigma(z), \tau(z) \in \mathbb{R}[z]$ polynomials. If we write $\theta := \sigma - \tau$, the previous Pearson equation reads

$$\theta(k+1)w(k+1) = \sigma(k)w(k), \quad k \in \mathbb{N}_0. \tag{66}$$

Theorem 1 (Hypergeometric symmetry of the moment matrix)**.** *Let the weight w be subject to the discrete Pearson equation (66), where θ, σ are polynomials with $\theta(0) = 0$. Then, the corresponding moment matrix fulfills*

$$\theta(\Lambda)G = B\sigma(\Lambda)GB^{\top}. \tag{67}$$

Proof. The moment matrix is

$$G = \sum_{k=0}^{\infty} \chi(k)\chi(k)^{\top} w(k). \tag{68}$$

Thus,

$$\begin{aligned}
\theta(\Lambda)G &= \sum_{k=0}^{\infty} \theta(\Lambda)\chi(k)\chi(k)^{\top} w(k) \quad \text{use (68)} \\
&= \sum_{k=1}^{\infty} \chi(k)\chi(k)^{\top}\theta(k)w(k) \quad \text{use } \Lambda\chi = z\chi \quad \textit{and} \quad \theta(0) = 0 \\
&= \sum_{k=0}^{\infty} \chi(k+1)\chi(k+1)^{\top}\theta(k+1)w(k+1) \quad \text{shift the summation variable} \\
&= \sum_{k=0}^{\infty} \chi(k+1)\chi(k+1)^{\top}\sigma(k)w(k) \quad \text{use Pearson equation (66)} \\
&= \sum_{k=0}^{\infty} B\chi(k)\chi(k)^{\top}B^{\top}\sigma(k)w(k) \quad \text{use (22)} \\
&= \sum_{k=0}^{\infty} B\sigma(\Lambda)\chi(k)\chi(k)^{\top}w(k)B^{\top} \quad \text{use } \Lambda\chi = z\chi \text{ again} \\
&= B\sigma(\Lambda)GB^{\top} \quad \text{use (68).}
\end{aligned} \tag{69}$$

■

Remark 3. This result extends to the case when θ and σ are entire functions, not necessarily polynomials, and we can ensure some meaning for $\theta(\Lambda)$ and $\sigma(\Lambda)$. Later on, see Section 2.5, we will show that this symmetry of the Gram matrix is a direct consequence of the generalized hypergeometric ODE satisfied by the the 0th moment.

2.2 | Consequences for the Jacobi matrix

We can use the Cholesky factorization (8) and the Jacobi matrix (12) to get the symmetrization matrix for the Jacobi matrix and an important constraint fulfilled by the dressed Pascal and Jacobi matrices.

Proposition 6. *Let the weight w be subject to the discrete Pearson equation (66), where the functions θ, σ are entire functions, not necessarily polynomials, with $\theta(0) = 0$. Then,*

(i) The matrices $H\theta(J^\top)$ and $\sigma(J)H$ are symmetric.
(ii) The following relation

$$\Pi^{-1}H\theta(J^\top) = \sigma(J)H\Pi^\top \tag{70}$$

holds.

Proof. Given the Hankel property, $\Lambda G = G\Lambda^\top$, we write (67) as $G\theta(\Lambda^\top) = B\sigma(\Lambda)GB^\top$, and using the Cholesky factorization (8), we get $S^{-1}HS^{-\top}\theta(\Lambda^\top) = B\sigma(\Lambda)S^{-1}HS^{-\top}B^\top$, so that $HS^{-\top}\theta(\Lambda^\top)S^\top = SBS^{-1}S\sigma(\Lambda)S^{-1}HS^{-\top}B^\top S^\top$, and we get Equation (70).

Let us prove that the matrices $H\theta(J^\top)$ and $\sigma(J)H$ are symmetric. This fact follows from (13), $JH = HJ^\top$. Indeed,

$$\begin{aligned}(H\theta(J^\top))^\top &= \theta(J)H = \theta(HJ^\top H^{-1})H = H\theta(J^\top)H^{-1}H = H\theta(J^\top),\\ (\sigma(J)H)^\top &= H\sigma(J^\top) = H\sigma(H^{-1}JH) = HH^{-1}\sigma(J)H = \sigma(J)H.\end{aligned} \tag{71}$$

■

2.3 | Generalized hypergeometric functions and the Pearson equation

Let us assume that θ, σ are polynomials, and denote their respective degrees by $N + 1 := \deg\theta(z)$ and $M := \deg\sigma(z)$. The roots of these polynomials are denoted by $\{-b_i + 1\}_{i=1}^N$ and $\{-a_i\}_{i=1}^M$, respectively. Following Ref. 3, we choose

$$\theta(z) = z(z + b_1 - 1)\cdots(z + b_N - 1), \quad \sigma(z) = \eta(z + a_1)\cdots(z + a_M). \tag{72}$$

Notice that we have normalized θ to be a monic polynomial, whereas σ is not monic, being the coefficient of the leading power denoted by η. This parameter η will be instrumental in what follows. Therefore, see Ref. 3, we have that the weight satisfying the Pearson equation (66) is proportional to

$$w(z)=\frac{(a_1)_z\cdots(a_M)_z}{\Gamma(z+1)(b_1)_z\cdots(b_N)_z}\eta^z, \tag{73}$$

here we use the Pochhammer symbol $(\alpha)_z=\frac{\Gamma(\alpha+z)}{\Gamma(\alpha)}$, that for $z\in\mathbb{N}$ reduces to the standard expression

$$(\alpha)_n=\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1),\quad(\alpha)_0=1. \tag{74}$$

The moments of this weight are finite whenever, see Ref. 3 and references therein,

(i) $M\leq N$ and $\eta\in\mathbb{C}$.
(ii) $M>N$, $\eta\in\mathbb{C}$ and one or more of the parameters a_i is a nonpositive integer.
(iii) $M=N+1$ and $|\eta|<1$.
(iv) $M=N+1$, $|\eta|=1$ and $\operatorname{Re}(b_1+\cdots+b_{N-1}-a_1-\cdots-a_M)>0$.

According to (64), the 0th moment

$$\begin{aligned}\rho_0&=\sum_{k=0}^{\infty}w(k)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_M)_k}{(b_1+1)_k\cdots(b_N+1)_k}\frac{\eta^k}{k!}\\&={}_MF_N(a_1,\dots,a_M;b_1,\dots,b_N;\eta)={}_MF_N\left[\begin{matrix}a_1&\cdots&a_M\\b_1&\cdots&b_N\end{matrix};\eta\right]\end{aligned} \tag{75}$$

is the generalized hypergeometric function, where we are using the two standard notations, see Ref. 19. Then, according to (63), for $n\in\mathbb{N}$, the corresponding higher moments, $\rho_n=\sum_{k=0}^{\infty}k^nw(k)$, are

$$\rho_n=\vartheta_\eta^n\rho_0=\vartheta_\eta^n\left({}_MF_N\left[\begin{matrix}a_1&\cdots&a_M\\b_1&\cdots&b_N\end{matrix};\eta\right]\right),\quad\vartheta_\eta:=\eta\frac{\partial}{\partial\eta}. \tag{76}$$

Given a function $f(\eta)$, for $n\in\mathbb{N}_0$, we consider the Wronskian

$$\mathscr{W}_{n+1}(f)=\det\begin{pmatrix}f&\vartheta_\eta f&\vartheta_\eta^2 f&\cdots&\vartheta_\eta^n f\\\vartheta_\eta f&\vartheta_\eta^2 f&&\iddots&\vartheta_\eta^{n+1}f\\\vartheta_\eta^2 f&&\iddots&&\\\vdots&\iddots&&&\\\vartheta_\eta^n f&\vartheta_\eta^{n+1}f&&&\vartheta_\eta^{2n}f\end{pmatrix}. \tag{77}$$

Then, for $n\in\mathbb{N}$, we have that the Hankel determinants $\Delta_n=\det G^{[n]}$, determined by the truncations of the corresponding moment matrix, are Wronskians of generalized hypergeometric functions

$$\Delta_n=\tau_n,\quad\tau_n:=\mathscr{W}_n\left({}_MF_N\left[\begin{matrix}a_1&\cdots&a_M\\b_1&\cdots&b_N\end{matrix};\eta\right]\right), \tag{78}$$

$$\tilde{\Delta}_n = \vartheta_\eta \tau_n. \tag{79}$$

Moreover, we also have for $n \in \mathbb{N}_0$ and taking $\tau_0 = 1$ that

$$H_n = \frac{\tau_{n+1}}{\tau_n}, \quad p^1_n = -\vartheta_\eta \log \tau_n, \tag{80}$$

as well as

$$\tau_{n+1} = H_n H_{n-1} \cdots H_0. \tag{81}$$

Remark 4. The functions τ_k are knwon in the theory of integrable systems as τ-functions.

2.4 | The Laguerre–Freud structure matrix and the Cholesky factorization

We are ready to prove one of the main results in the paper. We will get a banded semi-infinite matrix describing the action of multiplication by the polynomials $\theta(x)$ and $\sigma(x)$ the sequence of orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$.

Theorem 2 (Laguerre–Freud structure matrix)**.** *Let us assume that the weight w is subject to the discrete Pearson equation (66) with θ, σ polynomials such that $\theta(0) = 0$, $\deg \theta(z) = N + 1$, $\deg \sigma(z) = M$. Then, the Laguerre–Freud structure matrix*

$$\Psi := \Pi^{-1} H \theta(J^\top) = \sigma(J) H \Pi^\top = \Pi^{-1}\theta(J) H = H\sigma(J^\top)\Pi^\top \tag{82}$$

$$= \theta(J + I)\Pi^{-1} H = H\Pi^\top \sigma(J^\top - I), \tag{83}$$

has only $N + M + 2$ possibly nonzero diagonals ($N + 1$ superdiagonals, the main diagonal and M subdiagonals)

$$\Psi = (\Lambda^\top)^M \psi^{(-M)} + \cdots + \Lambda^\top \psi^{(-1)} + \psi^{(0)} + \psi^{(1)}\Lambda + \cdots + \psi^{(N+1)}\Lambda^{N+1}, \tag{84}$$

for some diagonal matrices $\psi^{(k)}$. In particular, the lowest subdiagonal and highest superdiagonal are given by

$$\begin{cases} (\Lambda^\top)^M \psi^{(-M)} = \eta (J_-)^M H, \quad \psi^{(-M)} = \eta H \prod_{k=0}^{M-1} T_-^k \gamma = \eta \operatorname{diag}\left(H_0 \prod_{k=1}^{M} \gamma_k, H_1 \prod_{k=2}^{M+1} \gamma_k, \dots\right), \\ \psi^{(N+1)}\Lambda^{N+1} = H(J_-^\top)^{N+1}, \quad \psi^{(N+1)} = H \prod_{k=0}^{N} T_-^k \gamma = \operatorname{diag}\left(H_0 \prod_{k=1}^{N+1} \gamma_k, H_1 \prod_{k=2}^{N+2} \gamma_k, \dots\right). \end{cases} \tag{85}$$

The vector $P(z)$ of orthogonal polynomials fulfills the following structure equations:

$$\theta(z) P(z-1) = \Psi H^{-1} P(z), \quad \sigma(z) P(z+1) = \Psi^\top H^{-1} P(z). \tag{86}$$

Proof. To get (83) just use (59). Now, notice that the matrices $H\theta(J^\top)$ and $\sigma(J)H$ are banded matrices with $2N+3$ diagonals ($N+1$ superdiagonals and subdiagonals) and $2M+1$ (M superdiagonals and subdiagonals), respectively. Thus, $\Pi^{-1}H\theta(J^\top)$ has at most $N+1$ superdiagonals, whereas $\sigma(J)H\Pi^\top$ has at most M subdiagonals. Consequently, Ψ given by (82) is a banded semi-infinite matrix as described. Then, (85) follow from (82) and the mentioned banded structure. To get the lowest subdiagonal, we use $\Psi=\sigma(J)H\Pi^\top$, so that the lowest subdiagonal will come from the lowest subdiagonal of $\sigma(J)$, namely, $\eta\,(J_-)^M$ right multiplied by the diagonal matrix H and then right multiplied the main diagonal of $\Pi^\top$, which happens to be the identity. To get the highest superdiagonal, we proceed similarly and use $\Psi=\Pi^{-1}H\theta(J^\top)$, so that the main superdiagonal will come from the product of the three factors I, H, and $\theta_N(J_-^\top)^{N+1}$, in this order. The componentwise expressions are direct computations.

Finally, we have

$$\begin{aligned}\Psi H^{-1}P(z)&=\Pi^{-1}\theta(J)HH^{-1}P(z) & \Psi^\top H^{-1}P(z)&=\Pi\sigma(J)HH^{-1}P(z)\\&=\Pi^{-1}\theta(J)P(z) & &=\Pi\sigma(J)P(z)\\&=\Pi^{-1}\theta(z)P(z) & &=\Pi\sigma(z)P(z)\\&=\theta(z)P(z-1), & &=\sigma(z)P(z+1).\end{aligned}\tag{87}$$

■

Remark 5 (Laguerre–Freud equations). Equation(82) is instrumental in obtaining nonlinear recurrences for the recursion coefficients, see Ref. 16

$$\gamma_{n+1}=F_1(n,\gamma_n,\gamma_{n-1},\dots,\beta_n,\beta_{n-1}\dots),\quad \beta_{n+1}=F_2(n,\gamma_{n+1},\gamma_n,\dots,\beta_n,\beta_{n-1},\dots),\tag{88}$$

for some functions F_1,F_2. These recurrences were named by Alphonse Magnus, attending to Refs. 20, 21, as Laguerre–Freud, see Refs. 22–25. This is the reason for the given name to Ψ.

The previous result extends to the Jacobi matrix, giving nontrivial nonlinear relations that link the Jacobi and Laguerre–Freud matrices.

Proposition 7. *The Laguerre–Freud and Jacobi matrices fulfill*

$$\sigma(J)\theta(J+I)=\Psi H^{-1}\Psi^\top H^{-1},\quad \theta(J)\sigma(J-I)=\Psi^\top H^{-1}\Psi H^{-1}.\tag{89}$$

Proof. We have

$$\begin{aligned}\sigma(J)\theta(J+I)P(z)&=\sigma(z)\theta(z+1)P(z)=\sigma(z)\Psi H^{-1}P(z+1)=\Psi H^{-1}\sigma(z)P(z+1)\\&=\Psi H^{-1}\Psi^\top H^{-1}P(z),\\ \theta(J)\sigma(J-I)P(z)&=\theta(z)\sigma(z-1)P(z)=\theta(z)\Psi^\top H^{-1}P(z-1)=\Psi^\top H^{-1}\theta(z)P(z-1)\\&=\Psi^\top H^{-1}\Psi H^{-1}P(z),\end{aligned}\tag{90}$$

which hold for all $z\in\mathbb{C}$ and, consequently, imply the desired result. ■

Theorem 3 (Cholesky factorization). *Let us assume that $H\theta(J^\top)$ and $\sigma(J)H$ have the following Cholesky factorizations:*

$$H\theta(J^\top) = \Theta^{-1}H_\theta\Theta^{-\top}, \quad \sigma(J)H = \Sigma^{-1}H_\sigma\Sigma^{-\top}, \tag{91}$$

with Θ and Σ lower unitriangular matrices and H_θ and H_σ diagonals matrices. Then,

(i) Θ^{-1} and Σ^{-1} have only first $N+1$ and M subdiagonals possibly nonzero, respectively.
(ii) We have

$$\Pi = \Theta^{-1}\Sigma, \quad H_\theta = H_\sigma =: h. \tag{92}$$

(iii) The Laguerre–Freud matrix has the following Gauss–Borel factorization:

$$\Psi = \Sigma^{-1}h\Theta^{-\top}. \tag{93}$$

Proof. Recall that according to Proposition 6, the matrices $H\theta(J^\top)$ and $\sigma(J)H$ are symmetric, and consequently, the corresponding Gauss–Borel factorizations, when they exist, will be Cholesky factorizations.

(i) If follows from the fact that J is tridiagonal and θ and σ polynomials of degrees $N+1$ and M, respectively.
(ii) The symmetry (70) yields

$$\Pi^{-1}\Theta^{-1}H_\theta\Theta^{-\top} = \Sigma^{-1}H_\sigma\Sigma^{-\top}\Pi^\top \tag{94}$$

and given the uniqueness of the Gauss–Borel factorization, the result follows.
(iii) From (82), we have different alternatives to show the result, let us see two of them

$$\begin{aligned}\Psi &= \Pi^{-1}H\theta(J^\top) = \Sigma^{-1}\Theta\Theta^{-1}H_\theta\Theta^{-\top} = \Sigma^{-1}h\Theta^{-\top},\\ \Psi &= \sigma(J)H\Pi^\top = \Sigma^{-1}H_\sigma\Sigma^{-\top}\Sigma^\top\Theta^{-1} = \Sigma^{-1}h\Theta^{-\top}.\end{aligned} \tag{95}$$

■

The compatibility with the recursion relation, that is, eigenfunctions of the Jacobi matrix, and the recursion matrix leads to some interesting equations.

Proposition 8. *The following compatibility conditions for the Laguerre–Freud and Jacobi matrices hold:*

$$[\Psi H^{-1}, J] = \Psi H^{-1}, \tag{96a}$$

$$[J, \Psi^\top H^{-1}] = \Psi^\top H^{-1}. \tag{96b}$$

Proof. To prove (96a), we evaluate the eigenvalue equation $JP(z) = zP(z)$ in $z-1$ to get $JP(z-1) = (z-1)P(z-1)$. Now multiply it by $\theta(z)$ to obtain $J\theta(z)P(z-1) = (z-1)\theta(z)P(z-1)$. Therefore, recalling (86), $J\Psi H^{-1}P(z) = (z-1)\Psi H^{-1}P(z) = \Psi H^{-1}(J-I)P(z)$ from where the relation follows.

Alternatively, observe that

$$\Psi H^{-1} = SB^{-1}\theta(\Lambda)S^{-1} \tag{97}$$

from where

$$[\Psi H^{-1}, J] = S[B^{-1}\theta(\Lambda), \Lambda]S^{-1} = S[B^{-1}, \Lambda]\theta(\Lambda)S^{-1} = SB^{-1}\theta(\Lambda)S^{-1} = \Psi H^{-1}. \tag{98}$$

To show (96b), we take the transpose of the proved one to get $[J^\top, H^{-1}\Psi^\top] = H^{-1}\Psi^\top$ so that

$$[HJ^\top H^{-1}, HH^{-1}\Psi^\top H^{-1}] = HH^{-1}\Psi^\top H^{-1} \tag{99}$$

and recalling that $HJ^\top H^{-1} = J$, see (13), we get the desired result. ■

2.5 | Contiguous hypergeometric relations

As we have seen, the polynomial discrete Pearson equation leads to (76), and the Hankel determinants are Wronskians of a generalized hypergeometric function, see (78) and (79). Hence, we expect that some properties of generalized hypergeometric functions may translate to the Gram matrix. To write this dictionary, we require of Proposition 10, discussed latter, which ensures

$$\vartheta_\eta G = \Lambda G = G\Lambda^\top. \tag{100}$$

We now study three important relations fulfilled by the generalized hypergeometric function, namely:

$$\left(\vartheta_\eta + a_i\right){}_MF_N\left[\begin{matrix} a_1\cdots a_i\cdots a_M \\ b_1\cdots b_N\end{matrix};\eta\right] = a_i\, {}_MF_N\left[\begin{matrix} a_1\cdots a_i+1\cdots a_M \\ b_1\cdots b_N\end{matrix};\eta\right], \tag{101}$$

$$(\vartheta_\eta + b_j - 1){}_MF_N\left[\begin{matrix} a_1\cdots a_M \\ b_1\cdots b_j\cdots b_N\end{matrix};\eta\right] = (b_j-1)\, {}_MF_N\left[\begin{matrix} a_1\cdots a_M \\ b_1\cdots b_j-1\cdots b_N\end{matrix};\eta\right], \tag{102}$$

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\, {}_MF_N\left[\begin{matrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N\end{matrix};\eta\right] = \kappa\, {}_MF_N\left[\begin{matrix} a_1+1 & \cdots & a_M+1 \\ b_1+1 & \cdots & b_N+1\end{matrix};\eta\right], \quad \kappa := \frac{\prod_{i=1}^M a_i}{\prod_{j=1}^N b_j}. \tag{103}$$

From these three equations, we also derive

$$\eta\prod_{n=1}^M\left(\eta\frac{\mathrm{d}}{\mathrm{d}\eta} + a_n\right)u = \eta\frac{\mathrm{d}}{\mathrm{d}\eta}\prod_{n=1}^N\left(\eta\frac{\mathrm{d}}{\mathrm{d}\eta} + b_n - 1\right)u, \quad u := {}_MF_N\left[\begin{matrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N\end{matrix};\eta\right]. \tag{104}$$

In (101) and (102), we have basic relations between contiguous generalized hypergeometric functions and its derivatives.

For the analysis of these equations, let us introduce the shift operators in the parameters $\{a_i\}_{i=1}^M$ and $\{b_j\}_{j=1}^N$. Thus, given a function $f[\begin{smallmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{smallmatrix}]$ of these parameters, we introduce the shifts ${}_iT$ and T_j as follows:

$$
\begin{aligned}
&{}_iTf\begin{bmatrix} a_1 \cdots a_i \cdots a_M \\ b_1 \cdots b_N \end{bmatrix} = f\begin{bmatrix} a_1 \cdots a_i + 1 \cdots a_M \\ b_1 \cdots b_N \end{bmatrix}, \quad T_jf\begin{bmatrix} a_1 \cdots a_M \\ b_1 \cdots b_j \cdots b_N \end{bmatrix} \\
&= f\begin{bmatrix} a_1 \cdots a_M \\ b_1 \cdots b_j - 1 \cdots b_N \end{bmatrix},
\end{aligned} \tag{105}
$$

and a total shift $T = {}_1T \cdots {}_MT\, T_1^{-1} \cdots T_N^{-1}$; that is,

$$
Tf\begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix} := f\begin{bmatrix} a_1 + 1 & \cdots & a_M + 1 \\ b_1 + 1 & \cdots & b_N + 1 \end{bmatrix}. \tag{106}
$$

Then, we find the following.

Theorem 4 (Hypergeometric relations). *The moment matrix $G = (\rho_{n+m})_{n,n\in\mathbb{N}_0}$ of a weight satisfying (66) fulfills the following hypergeometric relations:*

$$
(\Lambda + a_i I)G = a_i\, {}_iTG, \tag{107a}
$$

$$
(\Lambda + (b_j - 1)I)G = (b_j - 1)T_jG, \tag{107b}
$$

$$
\Lambda G = \kappa B(TG)B^\top \tag{107c}
$$

with B given in (22).

Proof. We first prove (107a). For that aim, we use that $G_{n,m} = \vartheta_\eta^{n+m}({}_MF_N[\begin{smallmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{smallmatrix}; \eta])$ so that

$$
\begin{aligned}
(\vartheta_\eta + a_i)G_{n,m} &= (\vartheta_\eta + a_i)\vartheta_\eta^{n+m}\left({}_MF_N\begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix}; \eta\right) \\
&= \vartheta_\eta^{n+m}(\vartheta_\eta + a_i)\left({}_MF_N\begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix}; \eta\right) \\
&= a_i\vartheta_\eta^{n+m}\left({}_MF_N\begin{bmatrix} a_1 \cdots a_i + 1 \cdots a_M \\ b_1 \cdots b_N \end{bmatrix}; \eta\right) \\
&= a_i\, {}_iTG_{n,m}.
\end{aligned} \tag{108}
$$

Thus, $(\vartheta_\eta + a_i)G = a_i\, {}_iTG$, and recalling $\vartheta_\eta G = \Lambda G$, we get the result. Relation (107b) is proved similarly. Finally, (107c) follows from (103), the novelty is that in (103), we have $\frac{\mathrm{d}}{\mathrm{d}\eta} = \eta^{-1}\vartheta_\eta$, which do not commute with ϑ_η. To overcome this difficulty, observe that the relation

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\eta - \eta\frac{\mathrm{d}}{\mathrm{d}\eta} = 1 \tag{109}$$

can be written $\eta^{-1}\vartheta_\eta\eta = \vartheta_\eta + 1$. Hence, $\eta^{-1}\vartheta_\eta^n\eta = (\eta^{-1}\vartheta_\eta\eta)^n = (\vartheta_\eta+1)^n$, that, in turn, implies

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\vartheta_\eta^n = (\vartheta_\eta + 1)^n\frac{\mathrm{d}}{\mathrm{d}\eta}. \tag{110}$$

Thus, we get the equations

$$\begin{aligned}\frac{\mathrm{d}\,G_{n,m}}{\mathrm{d}\eta} &= \frac{\mathrm{d}}{\mathrm{d}\eta}\vartheta_\eta^{n+m}\left({}_MF_N\begin{bmatrix}a_1 & \cdots & a_M\\ b_1 & \cdots & b_N\end{bmatrix};\eta\right) = (\vartheta_\eta+1)^{n+m}\frac{\mathrm{d}}{\mathrm{d}\eta}\left({}_MF_N\begin{bmatrix}a_1 & \cdots & a_M\\ b_1 & \cdots & b_N\end{bmatrix};\eta\right)\\ &= (\vartheta_\eta+1)^{n+m}\kappa\,{}_MF_N\begin{bmatrix}a_1+1 & \cdots & a_M+1\\ b_1+1 & \cdots & b_N+1\end{bmatrix};\eta\\ &= \kappa\sum_{k=0}^{n+m}\binom{n+m}{k}\vartheta_\eta^k\,{}_MF_N\begin{bmatrix}a_1+1 & \cdots & a_M+1\\ b_1+1 & \cdots & b_N+1\end{bmatrix};\eta.\end{aligned} \tag{111}$$

Using the Chu–Vandermonde identity,

$$\binom{n+m}{k} = \sum_{l=0}^{k}\binom{n}{l}\binom{m}{k-l}, \tag{112}$$

we find

$$\begin{aligned}\frac{\mathrm{d}\,G_{n,m}}{\mathrm{d}\eta} &= \kappa\sum_{k=0}^{n+m}\sum_{l=0}^{k}\binom{n}{l}\binom{m}{k-l}\vartheta_\eta^{l+k-l}\,{}_MF_N\begin{bmatrix}a_1+1 & \cdots & a_M+1\\ b_1+1 & \cdots & b_N+1\end{bmatrix};\eta\\ &= \kappa\sum_{k=0}^{n+m}\sum_{l=0}^{k}\binom{n}{l}\binom{m}{k-l}TG_{l,k-l}\\ &= \kappa\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}TG_{l,k}\binom{m}{k}\end{aligned} \tag{113}$$

from where the result follows. ■

Remark 6. From (104), we derive, in an alternative manner, the relation (67). Indeed, using $\vartheta_\eta - 1 = \eta\vartheta_\eta\eta^{-1}$, we get $(\vartheta_\eta - 1)^n\eta = \eta(\vartheta_\eta)^n$ and we workout (104) as follows:

$$
\begin{aligned}
\eta\prod_{n=1}^{M}(\vartheta_\eta+a_n)G_{n,m}&=\eta\prod_{n=1}^{M}(\vartheta_\eta+a_n)\vartheta_\eta^{n+m}\left({}_MF_N\left[\begin{matrix}a_1 & \cdots & a_M\\ b_1 & \cdots & b_N\end{matrix};\eta\right]\right)\\
&=(\vartheta_\eta-1)^{n+m}\eta\prod_{n=1}^{M}(\vartheta_\eta+a_n)\left({}_MF_N\left[\begin{matrix}a_1 & \cdots & a_M\\ b_1 & \cdots & b_N\end{matrix};\eta\right]\right)\\
&=\vartheta_\eta\prod_{n=1}^{N}(\vartheta_\eta+b_n-1)(\vartheta_\eta-1)^{n+m}\left({}_MF_N\left[\begin{matrix}a_1 & \cdots & a_M\\ b_1 & \cdots & b_N\end{matrix};\eta\right]\right)\\
&=\vartheta_\eta\prod_{n=1}^{N}(\vartheta_\eta+b_n-1)\sum_{k=0}^{n+m}\binom{n+m}{k}(-1)^k\vartheta_\eta^k\left({}_MF_N\left[\begin{matrix}a_1 & \cdots & a_M\\ b_1 & \cdots & b_N\end{matrix};\eta\right]\right)\\
&=\vartheta_\eta\prod_{n=1}^{N}(\vartheta_\eta+b_n-1)\sum_{l=0}^{n}\sum_{k=0}^{m}(-1)^l\binom{n}{l}G_{l,k}(-1)^k\binom{m}{k},
\end{aligned}
\tag{114}
$$

and consequently, we finally deduce (67).

2.6 | Discrete lattice Toda equation

The shifts in the hypergeometric parameters induce corresponding transformations on the discrete orthogonal polynomials. To describe them, we introduce the following semi-infinite matrices:

$$
{}_i\Omega := S({}_iTS)^{-1},\quad i\in\{1,\dots,M\},\quad \Omega_k := S\,(T_kS)^{-1},\quad k\in\{1,\dots,N\},
\tag{115}
$$

so that the following connection formulas are fulfilled:

$$
{}_i\Omega\,{}_iTP(z)=P(z),\quad i\in\{1,\dots,M\},
\tag{116}
$$

$$
\Omega_j\,T_jP(z)=P(z),\quad j\in\{1,\dots,N\}.
\tag{117}
$$

In Ref. 26, we proved that

$$
{}_i\Omega=\begin{pmatrix}1 & 0 & \cdots & \\ \frac{1}{a_i}\frac{H_1}{{}_iTH_0} & 1 & \ddots & \\ 0 & \frac{1}{a_i}\frac{H_2}{{}_iTH_1} & \ddots & \ddots\\ \vdots & \ddots & \ddots & \ddots\end{pmatrix},\qquad
\Omega_j=\begin{pmatrix}1 & 0 & \cdots & \\ \frac{1}{b_j-1}\frac{H_1}{T_jH_0} & 1 & \ddots & \\ 0 & \frac{1}{b_j-1}\frac{H_2}{T_jH_1} & \ddots & \ddots\\ \vdots & \ddots & \ddots & \ddots\end{pmatrix}.
\tag{118}
$$

The connection formulas (116) and (117) for contiguous hypergeometric parameters imply a nonlinear compatibility equations that lead to a generalized lattice Toda equation found in Ref. 18. We use the following notation:

$$u_n = H_{n-1}\begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix}; \eta\Bigg]. \tag{119}$$

We now consider three variables (n, r, s), where r, s are any couples of variables taken from the set of hypergeometric parameters $\{a_1, \dots, a_M, b_1, \dots, b_N\}$ and denote the corresponding "shifts" in n, r, s as follows:

$$\bar{u}_n(r,s) = u_{n+1}(r,s), \quad \hat{u}_n(r,s) = \hat{a}u_n(r \pm 1, s), \quad \tilde{u}_n(r,s) = \tilde{a}u(r, s \pm 1), \tag{120}$$

where the $\pm$ signs is a $+$ if the corresponding variable belongs to $\{a_i\}_{i=1}^M$ or a $-$ if it belongs to $\{b_j\}_{j=1}^M$, and the constants $\hat{a}, \tilde{a}$ are taken from $\{a_i\}_{i=1}^N$ and $\{b_j - 1\}_{j=1}^N$ according to the choice selected for the variables r and s. We denote by $\Omega^{(r)}$ and $\Omega^{(s)}$, the corresponding matrices Ω's taken from $\{_i\Omega\}_{i=1}^M$ and $\{\Omega_j\}_{j=1}^N$, depending on the variables picked r and s.

Theorem 5 (Nijhoff–Capel discrete lattice Toda equations). *The squared norms satisfy the following nonlinear equations linking contiguous hypergeometric parameters:*

$$\frac{\hat{u} - \tilde{u}}{\bar{u}} = \tilde{\hat{u}}\left(\frac{1}{\tilde{u}} - \frac{1}{\hat{u}}\right). \tag{121}$$

Proof. The connection formulas are

$$\Omega^{(r)}\hat{P} = P, \quad \Omega^{(s)}\tilde{P} = P. \tag{122}$$

Then, we have

$$\tilde{\Omega}^{(r)}\tilde{\hat{P}} = \tilde{P}, \tag{123}$$

so that

$$\Omega^{(s)}\tilde{\Omega}^{(r)}\tilde{\hat{P}} = \Omega^{(s)}\tilde{P} = P, \tag{124}$$

and the compatibility $\tilde{\hat{P}} = \hat{\tilde{P}}$ leads to the nonlinear condition

$$\Omega^{(s)}\tilde{\Omega}^{(r)} = \Omega^{(r)}\hat{\Omega}^{(s)}. \tag{125}$$

Then, as we can write

$$\Omega^{(r)} = \begin{pmatrix} 1 & 0 & \cdots & \\ \frac{\bar{u}_1}{\hat{u}_1} & 1 & \ddots & \\ 0 & \frac{\bar{u}_2}{\hat{u}_2} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \Omega^{(s)} = \begin{pmatrix} 1 & 0 & \cdots & \\ \frac{\bar{u}_1}{\tilde{u}_1} & 1 & \ddots & \\ 0 & \frac{\bar{u}_2}{\tilde{u}_2} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{126}$$

Equation (125) is

$$
\begin{pmatrix} 1 & 0 & \cdots & \\ \frac{\bar{u}_1}{\tilde{u}_1} & 1 & \ddots & \\ 0 & \frac{\bar{u}_2}{\tilde{u}_2} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \end{pmatrix}
\begin{pmatrix} 1 & 0 & \cdots & \\ \frac{\tilde{\bar{u}}_1}{\hat{\tilde{u}}_1} & 1 & \ddots & \\ 0 & \frac{\tilde{\bar{u}}_2}{\hat{\tilde{u}}_2} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \end{pmatrix}
=
\begin{pmatrix} 1 & 0 & \cdots & \\ \frac{\bar{u}_1}{\hat{u}_1} & 1 & \ddots & \\ 0 & \frac{\bar{u}_2}{\hat{u}_2} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \end{pmatrix}
\begin{pmatrix} 1 & 0 & \cdots & \\ \frac{\hat{\bar{u}}_1}{\hat{\tilde{u}}_1} & 1 & \ddots & \\ 0 & \frac{\hat{\bar{u}}_2}{\hat{\tilde{u}}_2} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \end{pmatrix},
\tag{127}
$$

and consequently, we deduce

$$
\frac{\bar{u}}{\tilde{u}} + \frac{\tilde{\bar{u}}}{\hat{\tilde{u}}} = \frac{\bar{u}}{\hat{u}} + \frac{\hat{\bar{u}}}{\hat{\tilde{u}}}, \tag{128}
$$

and the result follows immediately. ■

Remark 7. Equation (121) is the generalized discrete lattice Toda equation described for the first time by Nijhoff and Capel,[18] that taking adequate continuous limits in the r and s variables (hypergeometric parameters) recovers the 2D Toda equation. This is a canonical equation among the difference equations in three variables involving up to second-order differences of octahedral type, consistent on the 4D lattice,[27] it appears as the type V of the octahedron-type integrable discrete equations in Section 3.9 of the book.[17]

More compatibility conditions appear in terms of the matrices, see Ref. 26

$$
{}_i\omega := ({}_jTS)(\Lambda + a_i I)S^{-1}, \quad i \in \{1, \dots, M\}, \quad \omega_j := (T_k S)(\Lambda + (b_j - 1)I)S^{-1}, \quad j \in \{1, \dots, N\}, \tag{129}
$$

related to the previous ones by

$$
{}_i\omega H = a_i({}_iTH)({}_i\Omega)^\top, \quad \omega_j H = (b_j - 1)(T_j H)(\Omega_j)^\top, \tag{130}
$$

as a consequence of the following connection formulas:

$$
{}_i\omega P(z) = (z + a_i){}_iTP(z), \quad i \in \{1, \dots, M\}, \quad \omega_j P(z) = (z + b_j - 1)T_j P(z), \quad j \in \{1, \dots, N\}. \tag{131}
$$

Now we study compatibility of the recursion relation and the connection formula for contiguous parameters; that is, the compatibility of

$$
\begin{aligned}
zP &= JP, \\
\hat{P} &= \hat{a}\frac{\omega^{(r)}}{z + \hat{a}}P.
\end{aligned} \tag{132}
$$

We use the notation

$$
u_n = H_{n-1}, \quad v_n = \beta_{n-1} \tag{133}
$$

so that $\gamma_n = \frac{\bar{u}_n}{u_n}$ and we have the following expressions:

$$J = \begin{pmatrix} v_1 & 1 & 0 & \cdots & \\ \frac{\bar{u}_1}{u_1} & v_2 & 1 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \omega^{(r)} = \begin{pmatrix} \frac{\hat{u}_1}{u_1} & 1 & 0 & \cdots \\ 0 & \frac{\hat{u}_2}{u_2} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{134}$$

Let $\hat{J}$ stand for the Jacobi matrix J shifted in the corresponding hypergeometric parameter r, that is,

$$\hat{J} = \begin{pmatrix} \frac{\hat{v}_1}{\hat{a}} & 1 & 0 & \cdots & \\ \frac{\hat{\bar{u}}_1}{\hat{u}_1} & \frac{\hat{v}_1}{\hat{a}} & 1 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{135}$$

Then, the eigenvalue property $JP = zP$ when shifted reads $\hat{J}\hat{P}(z) = z\hat{P}(z)$. Hence,

$$\hat{J}\hat{a}\frac{\omega^{(r)}}{z+\hat{a}}P(z) = z\hat{a}\frac{\omega^{(r)}}{z+\hat{a}}P(z) = \hat{a}\frac{\omega^{(r)}}{z+\hat{a}}JP(z), \tag{136}$$

and therefore, we find the following compatibility equation:

$$\hat{J}\omega^{(r)} = \omega^{(r)}J. \tag{137}$$

the compatibility reads

$$\begin{pmatrix} \frac{\hat{v}_1}{\hat{a}} & 1 & 0 & \cdots & \\ \frac{\hat{\bar{u}}_1}{\hat{u}_1} & \frac{\hat{v}_2}{\hat{a}} & 1 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \frac{\hat{u}_1}{u_1} & 1 & 0 & \cdots \\ 0 & \frac{\hat{u}_2}{u_2} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \frac{\hat{u}_1}{u_1} & 1 & 0 & \cdots \\ 0 & \frac{\hat{u}_2}{u_2} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} v_1 & 1 & 0 & \cdots & \\ \frac{\bar{u}_1}{u_1} & v_2 & 1 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{138}$$

and we find $\frac{\hat{v}_1}{\hat{a}}\frac{\hat{u}_1}{u_1} = v_1\frac{\hat{u}_1}{u_1} + \frac{\bar{u}_1}{u_1}$ and

$$\begin{aligned} \frac{\hat{\bar{u}}_n}{\hat{u}_n} + \frac{\hat{v}_n}{\hat{a}}\frac{\hat{u}_n}{\bar{u}_n} &= \frac{\bar{\bar{u}}_n}{\bar{u}_n} + \bar{v}_n\frac{\hat{u}_n}{\bar{u}_n}, \\ \frac{\hat{v}_n}{\hat{a}} + \frac{\hat{\bar{u}}_n}{\bar{u}_n} &= \bar{v}_n + \frac{\hat{u}_n}{u_n} \end{aligned} \tag{139}$$

for $n \in \mathbb{N}$. Thus, we find the following.

Proposition 9. *The squared norms $u_n = H_{n-1}$ and the recursion coefficients, $v_n = \beta_{n-1} = p^1_{n-1} - p^1_n$, satisfy the following system of nonlinear difference equations:*

$$\frac{\hat{\bar{v}}}{\hat{a}} - \bar{v} = \frac{\bar{\bar{u}}}{\hat{u}} - \frac{\bar{u}}{\hat{u}}, \tag{140a}$$

$$\frac{\hat{v}}{\hat{a}} - \bar{v} = \frac{\hat{u}}{u} - \frac{\hat{\bar{u}}}{\bar{u}}. \tag{140b}$$

2.7 | The Toda flows

Given a semi-infinite vector $\boldsymbol{\eta} := \{\eta_l\}_{l=1}^{\infty} \in \mathbb{C}^{\infty}$, for each $z \in \mathbb{C}$, we define $\mathscr{E}(z; \boldsymbol{\eta}) := \prod_{l=1}^{\infty} \eta_l^{z^l}$, and we assume a weight of the form $w(z) = v(z)\mathscr{E}(z; \boldsymbol{\eta})$, with v η-independent, so that the corresponding moment matrix is

$$G = \sum_{k=0}^{\infty} \chi(k)\chi(k)^{\top} v(k)\mathscr{E}(k; \boldsymbol{\eta}). \tag{141}$$

Observe that only the first flow preserves the Pearson reduction. Notice also that $\mathscr{E} = \mathrm{e}^{\sum_{l=1}^{\infty} t_l z^l}$, with $t_l := \log \eta_l$ the standard time flows in an integrable hierarchy. We will write $\eta_1 = \eta$; normally, t_1 is denoted by x, $t_1 = x$. Also, the following notation

$$\vartheta_l := \eta_l \frac{\partial}{\partial \eta_l} = \frac{\partial}{\partial t_l} \tag{142}$$

will be used. We notice that, in particular, $\vartheta_1 = \vartheta_\eta$. To ensure the converge of the corresponding series for the moments, we require $|\eta_k| \leq 1$ for $k \in \{2, 3, \ldots\}$.

If we take $\eta_k = 1$, $k \in \mathbb{N}$, then the corresponding moment matrix is

$$G_0 = \sum_{k=0}^{\infty} \chi(k)\chi(k)^{\top} v(k). \tag{143}$$

Moreover, we can write for the deformed Gram matrix

$$G = \mathscr{E}(\Lambda; \boldsymbol{\eta})G_0 = G_0\mathscr{E}(\Lambda^{\top}; \boldsymbol{\eta}), \quad \mathscr{E}(\Lambda; \boldsymbol{\eta}) := \prod_{l=1}^{\infty} \eta_l^{\Lambda^l} = \exp\left(\sum_{l=1}^{\infty} t_l \Lambda^l\right). \tag{144}$$

Proposition 10. *For a linear functional of the form $\rho = \sum_{k=0}^{\infty} \delta(z-k)v(z)\mathscr{E}(z; \boldsymbol{\eta})$, given $l \in \mathbb{N}$, the moment matrix satisfies*

$$\vartheta_l G = (\Lambda)^l G = G(\Lambda^{\top})^l. \tag{145}$$

Proof. It follows form $\vartheta_l \mathscr{E}(z;\boldsymbol{\eta}) = z^l \mathscr{E}(z;\boldsymbol{\eta})$; indeed,

$$\vartheta_l G = \sum_{k=0}^{\infty} \chi(k)\chi(k)^\top k^l v(k)\mathscr{E}(k;\boldsymbol{\eta}) = \Lambda^l G = G(\Lambda^\top)^l. \tag{146}$$

■

For $k \in \mathbb{N}$, let us define the strictly lower triangular matrix (strictly because it has zeros on the main diagonal)

$$\Phi_k := (\vartheta_k S)S^{-1}. \tag{147}$$

In particular, we denote $\Phi := \Phi_1 = (\vartheta_\eta S)S^{-1}$.

Proposition 11. *The semi-infinite vector P fulfills*

$$\vartheta_k P = \Phi_k P. \tag{148}$$

Proof. As $P = S\chi$, then $\vartheta_k P = (\vartheta_k S)\chi = (\vartheta_k S)S^{-1}S\chi = \Phi P$. ■

Proposition 12 (Sato–Wilson equations)**.** *The following equations holds:*

$$-\Phi_k H + \vartheta_k H - H\Phi_k^\top = J^k H, \quad k \in \mathbb{N}. \tag{149}$$

Consequently, for $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we have

$$\Phi_k = -(J^k)_-, \quad \vartheta_k \log H_n = (J^k)_{n,n}. \tag{150}$$

Proof. Using the Cholesky factorization (8) for the moment matrix, we get

$$\vartheta_k G = \vartheta_k(S^{-1}HS^{-\top}) = -S^{-1}(\vartheta_k S)S^{-1}HS^{-\top} + S^{-1}(\vartheta_k H)S^{-\top} - S^{-1}HS^{-\top}(\vartheta_k S)^\top S^{-\top}. \tag{151}$$

Hence, from the symmetry relation (145), we deduce

$$-S^{-1}(\vartheta_\eta S)S^{-1}HS^{-\top} + S^{-1}(\vartheta_\eta H)S^{-\top} - S^{-1}HS^{-\top}(\vartheta_\eta S)^\top S^{-\top} = \Lambda^k S^{-1}HS^{-\top}, \tag{152}$$

from where

$$-(\vartheta_k S)S^{-1}H + \vartheta_k H - H\big((\vartheta_k S)S^{-1}\big)^\top = J^k H, \tag{153}$$

for $J = S\Lambda S^{-1}$, and the result follows. ■

For $k = 1$; that is for the first flow $\eta_1 = \eta$, we have formulas (78), (79), and (80) in terms of the τ-function. Moreover,

Proposition 13 (Toda)**.** *The following equations hold:*

$$\Phi = (\vartheta_\eta S)S^{-1} = -\Lambda^\top \gamma, \tag{154a}$$

$$(\vartheta_\eta H)H^{-1} = \beta. \tag{154b}$$

In particular, for $n, k-1 \in \mathbb{N}$, we have

$$\vartheta_\eta p^1_n = -\gamma_n, \quad \vartheta_\eta p^k_{n+k} = -\gamma_{n+k} p^{k-1}_{n+k-1}, \tag{155a}$$

$$\vartheta_\eta \log H_n = \beta_n. \tag{155b}$$

The functions $q_n := \log H_n$, $n \in \mathbb{N}$, satisfy the Toda equations

$$\vartheta^2_\eta q_n = \mathrm{e}^{q_{n+1}-q_n} - \mathrm{e}^{q_n - q_{n-1}}. \tag{156}$$

For $n \in \mathbb{N}$, we also have

$$\vartheta_\eta P_n(z) = -\gamma_n P_{n-1}(z). \tag{157}$$

Proof. Equation (154a) is a direct consequence of (149) for $k = 1$. To obtain (155), we we write $\vartheta_\eta S = -\Lambda^\top \gamma S$ in terms of its subdiagonals, that is,

$$\vartheta_\eta (I + \Lambda^\top S^{[1]} + (\Lambda^\top)^2 S^{[2]} + \cdots) = -\Lambda^\top \gamma (I + \Lambda^\top S^{[1]} + (\Lambda^\top)^2 S^{[2]} + \cdots) \tag{158}$$

which, for $k \in \mathbb{N}$, gives

$$\vartheta_\eta S^{[k]} = -(T^{k-1}_{-} \gamma) S^{[k-1]} \tag{159}$$

with $S^{[0]} = I$. Now, recalling (32) we get (155). Equation (155b) follows component wise from (154b).

As $\beta_n = p^1_n - p^1_{n+1}$ and $\gamma_n = \frac{H_n}{H_{n-1}}$, we deduce that

$$\vartheta^2_\eta \log H_n = \vartheta_\eta p^1_n - \vartheta_\eta p^1_{n+1} = -\frac{H_n}{H_{n-1}} + \frac{H_{n+1}}{H_n}, \tag{160}$$

that is, the Toda equation (156) for $H_n = \mathrm{e}^{q_n}$. The last equation follows form $\vartheta_\eta P = \Phi P = -\Lambda^\top \gamma P$. ■

Now, let us study the compatibility with the connection relations. From $\vartheta_\eta P = \Phi P$ and $\Omega \hat{P} = P$, we get

$$\vartheta_\eta \Omega = \Phi \Omega - \Omega \hat{\Phi}, \tag{161}$$

so that

Proposition 14. *The squared norms $u_n = H_{n-1}$ fulfill*

$$\vartheta_\eta\left(\frac{\bar{u}}{\hat{u}}\right) = \frac{\hat{u}}{\hat{u}} - \frac{\bar{u}}{u}. \tag{162}$$

Given the τ-function (78), we find the following.

Proposition 15 (τ-function expressions)**.** *In terms of the τ-function*

$$\tau_n := \mathscr{W}_n\left({}_MF_N\begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix}; \eta\right), \tag{163}$$

we find

$$H_n = \frac{\tau_{n+1}}{\tau_n}, \quad p_n^1 = -\vartheta_\eta \log \tau_n, \quad \gamma_n = \vartheta_\eta^2 \log \tau_n, \quad \beta_n = \vartheta_\eta \log \frac{\tau_{n+1}}{\tau_n}. \tag{164}$$

Proof. Collect together (80) and (155). ■

We now give the differential relations involving the operator $\vartheta_\eta \eta$ and the Jacobi matrix as a Lax equation. Then, the 1D Toda equation emerges naturally.

Proposition 16. *The following Lax equation holds:*

$$\vartheta_\eta J = [J_+, J]. \tag{165}$$

The recursion coefficients satisfy the following Toda system:

$$\vartheta_\eta \beta_n = \gamma_{n+1} - \gamma_n, \tag{166a}$$

$$\vartheta_\eta \log \gamma_n = \beta_n - \beta_{n-1}, \tag{166b}$$

for $n \in \mathbb{N}_0$ and $\beta_{-1} = 0$. Consequently, we get

$$\vartheta_\eta^2 \log \gamma_n + 2\gamma_n = \gamma_{n+1} + \gamma_{n-1}. \tag{167}$$

Moreover,

$$\vartheta_\eta^2 \log \tau_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}. \tag{168}$$

Proof. As $J = S\Lambda S^{-1}$ clears that

$$\vartheta_\eta J = [\Phi, J] = [-J_-, J] = [J_+ - J, J] = [J_+, J]. \tag{169}$$

From this Lax equation, one could derive, component-wise, the Toda system (166). Alternatively, the Toda system (166) could be derived as follows. From (17) and (155), we obtain

$$
\begin{aligned}
\vartheta_\eta \beta_n &= \vartheta_\eta p_n^1 - \vartheta_\eta p_{n+1}^1 = -\gamma_n + \gamma_{n+1}, \\
\vartheta_\eta \log \gamma_n &= \vartheta_\eta \log H_n - \vartheta_\eta \log H_{n-1} = \beta_n - \beta_{n-1}.
\end{aligned} \tag{170}
$$

Finally,

$$
\vartheta_\eta^2 \log \tau_n = \gamma_n = \frac{H_n}{H_{n-1}} = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n}. \tag{171}
$$

■

For higher Toda flows, which generically are not consistent with the hypergeometric reduction, we have the following.

Proposition 17 (Lax equations)**.** *The following Lax equation holds:*

$$
\vartheta_k J = [(J^k)_+, J]. \tag{172}
$$

Proof. As $J = S\Lambda S^{-1}$, we have

$$
\vartheta_k J = [\Phi_k, J] = [-(J)_-^k, J] = [(J^k)_+ - J^k, J] = [(J^k)_+, J]. \tag{173}
$$

■

A very important description of these integrable flows, and alternatively to the Lax equations previously discussed, are the zero-curvature conditions first discussed by Zakharov and Shabat in the context of the nonlinear Schrödinger equation.

Proposition 18 (Wave functions and Zakharov–Shabat equations)**.** *The wave function satisfies the linear system*

$$
\vartheta_k \Psi = (J^k)_+ \Psi. \tag{174}
$$

Moreover, the following Zakharov–Shabat equations hold:

$$
\vartheta_k (J^l)_+ - \vartheta_l (J^k)_+ + [(J^l)_+, (J^k)_+] = 0. \tag{175}
$$

Proof. The orthogonal polynomials we have

$$
\vartheta_k P = \Phi_k P = ((J^k)_+ - J^k) P = ((J^k)_+ - z^k) P. \tag{176}
$$

Thus, the wave function fulfills the following linear system:

$$
\vartheta_k \Psi = \vartheta_k(P)\mathscr{E} + P\vartheta_k(\mathscr{E}) = (J^k)_+ \Psi. \tag{177}
$$

Finally, the Zakharov–Shabat equations follow as the compatibility conditions for the previous linear system. ■

Finally, we connect the Toda equation with another very important soliton equation, the Kadomtsev–Petviashvili equation, or 2D KdV equation. For that aim, we require of the following technical result.

Proposition 19. *Let us assume that G_0, the moment matrix of the linear functional $\rho = \sum_{k=0}^{\infty} \delta(z-k)v(z)$, admits a Cholesky factorization, and that two semi-infinite matrices Z_1 and Z_2 are given, such that*

(i) $Z_1 \mathscr{E}(\Lambda, \boldsymbol{\eta})^{-1}$ *is strictly lower triangular,*
(ii) Z_2 *is upper triangular, and*
(iii) $Z_1(\boldsymbol{\eta}) G_0 = Z_2(\boldsymbol{\eta})$.

Then, we can ensure that $Z_1 = Z_2 = 0$.

Proof. We have

$$Z_1 G_0 = Z_1 \mathscr{E}(\Lambda; \boldsymbol{\eta})^{-1} G = Z_1 \mathscr{E}(\Lambda; \boldsymbol{\eta})^{-1} S^{-1} H S^{-\top}, \tag{178}$$

and we get

$$Z_1 \mathscr{E}(\Lambda; \boldsymbol{\eta})^{-1} S^{-1} H S^{-\top} = Z_2. \tag{179}$$

Consequently,

$$Z_1 \mathscr{E}(\Lambda; \boldsymbol{\eta})^{-1} S^{-1} = Z_2 S H^{-1}. \tag{180}$$

But $Z_1 \mathscr{E}(\Lambda; \boldsymbol{\eta})^{-1} S^{-1}$ is strictly lower triangular, whereas $Z_2 S H^{-1}$ is upper triangular. As both terms are equal, the only possibility is that both are the zero matrix

$$Z_1 \mathscr{E}(\Lambda; \boldsymbol{\eta})^{-1} S^{-1} = Z_2 S H^{-1} = 0, \tag{181}$$

and we get the result. ■

Using this results one proves that

Proposition 20 (KP equation)**.** *The wave function Ψ_n satisfies the linear equations*

$$\vartheta_m \Psi_n = \mathscr{P}_m(\vartheta_\eta) \Psi_n \tag{182}$$

with

$$\mathscr{P}_m(\vartheta_\eta) = \vartheta_\eta^m - m p_n^1 \vartheta_\eta^{m-2} + U_{n,m-3} \vartheta_\eta^{m-3} + \cdots + U_{n,0}, \tag{183}$$

where $\{U_{n,j}\}_{j=0}^{m-3}$ *are polynomials in* $\vartheta_\eta^l p_n^1, \vartheta_2 p_n^1, \dots, \vartheta_{m-1} p_n^1$. *In particular, the following linear equations hold:*

$$\begin{aligned}\vartheta_2\Psi_n &= \vartheta_\eta^2\Psi_n - 2\vartheta_\eta\big(p_n^1\big)\Psi_n,\\ \vartheta_3\Psi_n &= \vartheta_\eta^3\Psi_n - 3\vartheta_\eta\big(p_n^1\big)\vartheta_\eta\Psi_n - \frac{3}{2}\big(\vartheta_\eta^2\big(p_k^1\big) + \vartheta_2\big(p_k^1\big)\big)\Psi_n,\end{aligned} \tag{184}$$

and consequently, its compatibility condition, which is the KP equation,

$$\vartheta_\eta\Big(4\vartheta_3\big(p_n^1\big) + 6\big(\vartheta_\eta\big(p_n^1\big)\big)^2 - \vartheta_\eta^3\big(p_n^1\big)\Big) = \vartheta_2^2\big(p_n^1\big), \tag{185}$$

is fulfilled.

Remark 8. The weight w for which this KP equation appears is

$$w(z) = \frac{(a_1)_z\cdots(a_M)_z}{\Gamma(z+1)(b_1)_z\cdots(b_N)_z}\eta^z\eta_2^{z^2}\eta_3^{z^3}, \quad |\eta_2|, |\eta_3| < 1, \tag{186}$$

and the corresponding 0th moment is the series

$$\rho_0 = \sum_{k=0}^{\infty} w(k) = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_M)_k}{(b_1+1)_k\cdots(b_{N-1}+1)_k}\frac{\eta^k\eta_2^{k^2}\eta_3^{k^3}}{k!}. \tag{187}$$

We see that only for $\eta_2, \eta_3 \to 1$, we recover a hypergeometric function.

2.8 | Toda–Pearson compatibility

For the compatibility of (27) and (148), that is, for the compatibility of the systems,

$$\begin{cases} P(z+1) = \Pi P(z),\\ \vartheta_\eta(P(z)) = \Phi P(z),\end{cases} \tag{188}$$

we require

$$\begin{aligned}&\vartheta_\eta(P(z+1)) = \vartheta_\eta(\Pi P(z)) = \vartheta_\eta(\Pi)P(z) + \Pi\vartheta_\eta(P(z)) = (\vartheta_\eta(\Pi) + \Pi J\Phi)P(z),\\ &(\vartheta_\eta P)(z+1) = \Phi P(z+1) = \Phi\Pi P(z)\end{aligned} \tag{189}$$

to be equal, and consequently, we obtain

$$\vartheta_\eta(\Pi) = [\Phi, \Pi]. \tag{190}$$

In the general case, the dressed Pascal matrix Π is a lower unitriangular semi-infinite matrix, which possibly has an infinite number of subdiagonals. However, for the case when the weight

$w(z) = v(z)\eta^z$ satisfies the Pearson equation (66), with v independent of η, that is,

$$\theta(k+1)v(k+1)\eta^{k+1} = \sigma(k)v(k)\eta^k \tag{191}$$

for $k \in \mathbb{N}_0$; that is,

$$\theta(k+1)v(k+1)\eta = \sigma(k)v(k), \tag{192}$$

the situation improves as we have the banded semi-infinite matrix Ψ that models the shift in the z variable as in (86). From the previous discrete Pearson equation, we see that $\sigma(z) = \eta\kappa(z)$ with κ, θ η-independent polynomials in z

$$\theta(k+1)v(k+1) = \eta\kappa(k)v(k). \tag{193}$$

Proposition 21. *Let us assume a weight w satisfying the Pearson equation (66), and consequently, of the form (73). Then, the Laguerre–Freud structure matrix Ψ given in (82) satisfies*

$$\vartheta_\eta(\eta^{-1}\Psi^\top H^{-1}) = [\Phi, \eta^{-1}\Psi^\top H^{-1}], \tag{194a}$$

$$\vartheta_\eta(\Psi H^{-1}) = [\Phi, \Psi H^{-1}]. \tag{194b}$$

Alternatively, the above equations can be written as follows:

$$\vartheta_\eta(\Psi^\top H^{-1}) = [J_+, \Psi^\top H^{-1}], \tag{195a}$$

$$\vartheta_\eta(\eta^{-1}\Psi H^{-1}) = [J_+, \eta^{-1}\Psi H^{-1}]. \tag{195b}$$

Relations (194a) and (194b) are gauge *equivalent.*

Proof. We look at the compatibility of (86) and (148); that is, for (194a), we consider

$$\begin{cases} \sigma(z)P(z+1) = \Psi^\top H^{-1}P(z), \\ \vartheta_\eta(P(z)) = \Phi P(z), \end{cases} \tag{196}$$

while for (194b), we consider

$$\begin{cases} \theta(z)P(z-1) = \Psi H^{-1}P(z), \\ \vartheta_\eta(P(z)) = \Phi P(z). \end{cases} \tag{197}$$

From (196), observing that $\vartheta_\eta(\sigma) = \sigma$, we deduce

$$\vartheta_\eta(\sigma(z)P(z+1)) = \vartheta_\eta(\Psi^\top H^{-1}P(z)) = \vartheta_\eta(\Psi^\top H^{-1})P(z) + \Psi^\top H^{-1}\vartheta_\eta(P(z))$$

$$= (\vartheta_\eta(\Psi^\top H^{-1}) + \Psi^\top H^{-1}\Phi)P(z),$$

$$\vartheta_\eta(\sigma(z)P(z+1)) = \sigma(z)P(z+1) + \sigma(z)\Phi P(z+1) = (\Phi + I)\Psi^\top H^{-1}P(z), \tag{198}$$

so that

$$\vartheta_\eta(\Psi^\top H^{-1}) - \Psi^\top H^{-1} = [\Phi, \Psi^\top H^{-1}], \tag{199}$$

and recalling $\vartheta_\eta(\eta^{-1}) = -\eta^{-1}$, we obtain (194a).

From (197), we find ($\vartheta_\eta\theta = 0$)

$$\vartheta_\eta(\theta(z)P(z-1)) = \vartheta_\eta(\Psi H^{-1}P(z)) = \vartheta_\eta(\Psi H^{-1})P(z) + \Psi H^{-1}\vartheta_\eta(P(z))$$

$$= (\vartheta_\eta(\Psi H^{-1}) + \Psi H^{-1}\Phi)P(z),$$

$$\vartheta_\eta(\theta(z)P(z-1)) = \theta(z)\Phi P(z-1) = \Phi\Psi H^{-1}P(z), \tag{200}$$

and (194b) follows immediately.

Notice that from (97), this equation (194b) follows immediately:

$$\vartheta_\eta(\Psi H^{-1}) = [(\vartheta_\eta S)S^{-1}, SB^{-1}\theta(\Lambda)S^{-1}] = [\Phi, \Psi H^{-1}]. \tag{201}$$

This comment applies similarly to (194a).

Using $\Phi = -J_- = J_+ - J$ in (194a) and (96a), we easily get (195a), and similarly from (194b) and (96b), we deduce (195b).

Finally, we study the relation between (194a) and (194b). Assume that (194b), $\vartheta_\eta(\Psi H^{-1}) = [\Phi, \Psi H^{-1}]$, holds and take its transpose to obtain $\vartheta_\eta(H^{-1}\Psi^\top) = -[\Phi^\top, H^{-1}\Psi^\top]$, which is not (194a). Observe that

$$\Psi^\top H^{-1} = H(H^{-1}\Psi^\top)H^{-1} \tag{202}$$

so that the following *gauge*-type transformation equation arises:

$$\vartheta_\eta(\Psi^\top H^{-1}) = \vartheta_\eta(H)(H^{-1}\Psi^\top)H^{-1} + H\vartheta_\eta(H^{-1}\Psi^\top)H^{-1} - H(H^{-1}\Psi^\top)H^{-1}\vartheta_\eta(H)H^{-1}$$

$$= \left[\vartheta_\eta(H)H^{-1}, H(H^{-1}\Psi^\top)H^{-1}\right] - H[\Phi^\top, H^{-1}\Psi^\top]H^{-1}$$

$$= \left[\vartheta_\eta(H)H^{-1} - H\Phi^\top H^{-1}, \Psi^\top H^{-1}\right]. \tag{203}$$

Now, attending to (149), we have $\vartheta_\eta(H)H^{-1} - H\Phi^\top H^{-1} = J + \Phi$ so that

$$\vartheta_\eta(\Psi^\top H^{-1}) = [J + \Phi, \Psi^\top H^{-1}] = [\Phi, \Psi^\top H^{-1}] + \Psi^\top H^{-1}, \tag{204}$$

and (194a) follows immediately. ■

2.9 | An example: The Meixner polynomials

For the Meixner polynomials, we have

$$w(z)=\frac{(a)_z}{\Gamma(z+1)}\eta^z,\quad \theta=z,\quad \sigma=\eta(z+a),\tag{205}$$

with zero-order moment given by

$$\rho_0={}_1F_0\left[\begin{matrix}a\\ -\end{matrix};\eta\right]=\frac{1}{(1-\eta)^a},\tag{206}$$

and the successive moments by, see Ref. 28,

$$\rho_n=\frac{1}{(1-\eta)^a}\sum_{m=0}^{n}\left\{\begin{matrix}n\\ m\end{matrix}\right\}(a)_m\frac{\eta^m}{(1-\eta)^m},\tag{207}$$

with $\left\{\begin{smallmatrix}n\\ m\end{smallmatrix}\right\}$ the Stirling numbers of the second kind, $\left\{\begin{smallmatrix}n\\ m\end{smallmatrix}\right\}$ is the number of partitions of $\{1,\dots,n\}$ into exactly k parts, $1\leq k\leq n$. The moment matrix in this Meixner case satisfies

$$\Lambda G=B(\Lambda G+aG)B^{-\top},\tag{208}$$

which implies for the moments ρ_n in (207) the following relation:

$$\rho_{n+m+1}=\sum_{k,l=0}^{n}(\rho_{k+l+1}+a\rho_{k+l})(-1)^{k+l}\binom{n}{k}\binom{m}{l}.\tag{209}$$

Alternatively, for $n,m\in\mathbb{N}_0$, we can write

$$\sum_{k=0}^{m}\binom{m}{k}\rho_{n+k+1}=\sum_{k=0}^{n}\binom{n}{k}(\rho_{m+k+1}+a\rho_{m+k}).\tag{210}$$

Hence, using the explicit expressions in terms of Stirling numbers (207), we get the following identities for binomial and Stirling numbers that hold for any a,η:

$$\sum_{k=0}^{m}\sum_{l=0}^{n+k+1}\binom{m}{k}\left\{\begin{matrix}n+k+1\\ l\end{matrix}\right\}(a)_l\frac{\eta^l}{(1-\eta)^l}-\sum_{k=0}^{n}\sum_{l=0}^{m+k+1}\binom{n}{k}\left\{\begin{matrix}m+k+1\\ l\end{matrix}\right\}(a)_l\frac{\eta^l}{(1-\eta)^l}$$
$$=a\sum_{k=0}^{n}\sum_{l=0}^{n+k}\binom{n}{k}\left\{\begin{matrix}m\\ l\end{matrix}\right\}(a)_l\frac{\eta^l}{(1-\eta)^l}.\tag{211}$$

According to Ref. 5, we have the recursion coefficients

$$\beta_n=\frac{n+(n+a)\eta}{1-\eta},\quad \gamma_n=\frac{n(n+a-1)\eta}{(1-\eta)^2}.\tag{212}$$

It is easily checked, we used SageMath 9.4, that these functions are, for every value of a, solutions of the Toda system (166). The monic orthogonal polynomials are expressed in terms of the Gaussian hypergeometric function as follows:

$$P_n(x) = (a)_n \frac{\eta^n}{(\eta-1)^n} M_n(x;a,\eta), \quad M_n(x;a,\eta) := {}_2F_1\left[\begin{matrix} -n, -x \\ a \end{matrix}; \frac{\eta-1}{\eta}\right]. \tag{213}$$

Moreover, as we know that (Ref. 13, Theorem 6.1.1)

$$\sum_{k=0}^{\infty} M_n^2(k;a,\eta)w(k) = \frac{n!(1-\eta)^{-a}}{\eta^n (a)_n}, \tag{214}$$

we find

$$H_n = \sum_{k=0}^{\infty} P_n^2(k)w(k) = (a)_n^2 \frac{\eta^{2n}}{(\eta-1)^{2n}} \frac{n!(1-\eta)^{-a}}{\eta^n (a)_n} = n!(a)_n \frac{\eta^n}{(1-\eta)^{2n+a}}. \tag{215}$$

Observe that, as $\gamma_n = \frac{H_n}{H_{n-1}}$ and $\beta_n = \vartheta_\eta \log H_n$, we recover the previous expressions (212). Consequently, the function

$$q_n = \log n! + \log(a)_n + n\log\eta - (2n+a)\log(1-\eta) \tag{216}$$

satisfies the Toda equation (156). Now, for the Hankel τ-functions, we have

$$\tau_n = \frac{\eta^{\frac{n(n-1)}{2}}}{(1-\eta)^{n(n+a-1)}} \prod_{k=1}^{n-1} k!(a+k)^{n-k-1}, \tag{217}$$

see Ref. 7.

The Nijhoff–Capel equation requires of two or more hypergeometric parameters. Therefore, the classical discrete polynomials, as the Charlier or Meixner orthogonal polynomials, having 0 and 1 parameters, do not give solutions to that integrable discrete equation. Cases as the generalized Meixner or generalized Hahn[3] do have such number of parameters and do describe solutions of the Nijhoff–Capel equation.

The compatibility conditions (140) and (162) are fulfilled as can be checked using SageMath 9.4. However, we will compute them explicitly to show that they hold. For that aim, we consider

$$u_n = H_{n-1} = (n-1)!(a)_{n-1} \frac{\eta^{n-1}}{(1-\eta)^{2n+a-2}}, \quad v_n = \beta_{n-1} = \frac{n-1+(n+a-1)\eta}{1-\eta}, \tag{218}$$

Equation (140a) is fullfilled as a direct computation shows

$$\frac{\hat{\bar{v}}}{a} - \bar{v} = \frac{n+(n+a+1)\eta}{1-\eta} - \frac{n+(n+a)\eta}{1-\eta} = \frac{\eta}{1-\eta},$$

$$\frac{\bar{\bar{u}}}{\hat{\bar{u}}}-\frac{\bar{u}}{\hat{u}}=\frac{(n+1)!(a)_{n+1}\frac{\eta^{n+1}}{(1-\eta)^{2n+a+2}}}{an!(a+1)_n\frac{\eta^n}{(1-\eta)^{2n+a+1}}}-\frac{n!(a)_n\frac{\eta^n}{(1-\eta)^{2n+a}}}{a(n-1)!(a+1)_{n-1}\frac{\eta^{n-1}}{(1-\eta)^{2n+a-1}}}$$

$$=(n+1)\frac{\eta}{1-\eta}-n\frac{\eta}{1-\eta}=\frac{\eta}{1-\eta}. \tag{219}$$

Also (140b) follows from a direct computation

$$\frac{\hat{v}}{a}-\bar{v}=\frac{n-1+(n+a)\eta}{1-\eta}-\frac{n+(n+a)\eta}{1-\eta}=\frac{1}{\eta-1},$$

$$\frac{\hat{u}}{u}-\frac{\hat{\bar{u}}}{\bar{u}}=\frac{a(n-1)!(a+1)_{n-1}\frac{\eta^{n-1}}{(1-\eta)^{2n+a-1}}}{(n-1)!(a)_{n-1}\frac{\eta^{n-1}}{(1-\eta)^{2n+a-2}}}-\frac{a\,n!(a+1)_n\frac{\eta^n}{(1-\eta)^{2n+a+1}}}{n!(a)_n\frac{\eta^n}{(1-\eta)^{2n+a+1}}}$$

$$=\frac{a+n-1}{1-\eta}-\frac{a+n}{1-\eta}=\frac{1}{\eta-1}. \tag{220}$$

Now, we check (162)

$$\vartheta_\eta\left(\frac{\bar{u}}{\hat{u}}\right)=\vartheta_\eta\left(n\frac{\eta}{1-\eta}\right)=\frac{n\eta}{(1-\eta)^2}$$

$$\frac{\hat{\bar{u}}}{\hat{u}}-\frac{\bar{u}}{u}=\frac{n!(a+1)_n\frac{\eta^n}{(1-\eta)^{2n+a+1}}}{(n-1)!(a+1)_{n-1}\frac{\eta^{n-1}}{(1-\eta)^{2n+a-1}}}-\frac{n!(a)_n\frac{\eta^n}{(1-\eta)^{2n+a}}}{(n-1)!(a)_{n-1}\frac{\eta^{n-1}}{(1-\eta)^{2n+a-2}}}$$

$$=\frac{n(a+n)\eta}{(1-\eta)^2}-\frac{n(a+n-1)\eta}{(1-\eta)^2}=\frac{n\eta}{(1-\eta)^2}. \tag{221}$$

The Laguerre–Freud structure matrix is a band matrix, see (85), $\Psi=\Lambda^\top\psi^{(-1)}+\psi^{(0)}+\psi^{(1)}\Lambda$. As $\Psi=\eta(J+aI)H\Pi^\top$, we find

$$\begin{aligned}\Psi&=\eta(\Lambda^\top\gamma+\beta+aI+\Lambda)H(I+D\Lambda+\pi^{[2]}\Lambda^2+\pi^{[3]}\Lambda^3+\cdots)\\&=\underbrace{\eta\Lambda^\top\gamma H}_{\text{first subdiagonal}}+\underbrace{\eta(\Lambda^\top\gamma HD\Lambda+(\beta+aI)H)}_{\text{main diagonal}}+\underbrace{\eta(\Lambda^\top\gamma H\pi^{[2]}\Lambda^2+(\beta+aI)HD\Lambda+\Lambda H)}_{\text{first superdiagonal}}+\cdots\end{aligned} \tag{222}$$

and we get $\psi^{(-1)}=\eta\gamma H$. From the alternative expression $\Psi=\Pi^{-1}HJ^\top$, we find

$$\begin{aligned}\Psi&=(I-\Lambda^\top D+(\Lambda^\top)^2\pi^{[-2]}-(\Lambda^\top)^3\pi^{[-3]}+\cdots)H(\Lambda^\top+\beta+\gamma\Lambda)\\&=\underbrace{H\gamma\Lambda}_{\text{first superdiagonal}}\quad\underbrace{-\Lambda^\top DH\gamma\Lambda+H\beta}_{\text{main diagonal}}+\underbrace{H\Lambda^\top-\Lambda^\top DH\beta+(\Lambda^\top)^2\pi^{[-2]}H\gamma\Lambda}_{\text{first subdiagonal}}+\cdots.\end{aligned} \tag{223}$$

Hence, $\psi^{(1)} = H\gamma$, and there two alternative expressions for the main diagonal

$$\psi^{(0)} = \eta H T_+ D + \eta(\beta + aI)H = -HT_+D + H\beta, \tag{224}$$

which hold given the form of the β's. We finally get the following Laguerre–Freud structure matrix:

$$\Psi = \begin{pmatrix} \beta_0 H_0 & H_1 & 0 & \cdots & \\ \eta H_1 & (\beta_1 - 1)H_1 & H_2 & \ddots & \\ 0 & \eta H_2 & (\beta_2 - 2)H_2 & H_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{225}$$

The connection formulas (86) are, in this case,

$$zP(z-1) = \Psi H^{-1} P(z), \quad (z+a)P(z+1) = \Psi^\top H^{-1} P(z). \tag{226}$$

3 | CONCLUSIONS AND OUTLOOK

The use of the Gauss–Borel factorization of the moment matrix has been throughly used by Adler and van Moerbeke in their studies of integrable systems and orthogonal polynomials, see Refs. 29–31. We have extended these ideas and applied them in different contexts, CMV orthogonal polynomials, matrix orthogonal polynomials, multiple orthogonal polynomials, and multivariate orthogonal.[32–40] For an general overview, see Ref. 41.

In this paper, we extended those ideas to the discrete world. In particular, we applied this approach to the study of the consequences of the Pearson equation on the moment matrix and Jacobi matrices. For that description, a new banded matrix is required, the Laguerre–Freud structure matrix that encodes the Laguerre–Freud relations for the recurrence coefficients. We have also found that the contiguous relations fulfilled generalized hypergeometric functions determining the moments of the weight described for the squared norms of the orthogonal polynomials a discrete Toda hierarchy known as Nijhoff–Capel equation, see Ref. 18.

Further work in this direction is the study of the role of Christoffel and Geronimus transformations for the description of the mentioned contiguous relations, and the use of the Geronimus–Christoffel transformations to characterize the shifts in the spectral independent variable of the orthogonal polynomials.[26] In Ref. 16, these ideas are applied to generalized Charlier, Meixner, and type I Hahn discrete orthogonal polynomials extending the results of Refs. 5, 8–11.

For the future, we will study the type II generalized Hahn polynomials, and extend these techniques to multiple discrete orthogonal polynomials[42] and its relations with the transformations presented in Ref. 43 and quadrilateral lattices.[44,45]

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