S-functions, reductions and hodograph solutions of the r-th dispersionless modified KP and Dym hierarchies

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Abstract

We introduce an S-function formulation for the recently found r-th dispersionless modified KP and r-th dispersionless Dym hierarchies, giving also a connection of these S-functions with the Orlov functions of the hierarchies. Then, we discuss a reduction scheme for the hierarchies that together with the S-function formulation leads to hodograph systems for the associated solutions. We consider also the connection of these reductions with those of the dispersionless KP hierarchy and with hydrodynamic type systems. In particular, for the 1-component and 2-component reduction we derive, for both hierarchies, ample sets of examples of explicit solutions.

1 Introduction

Dispersionless integrable hierarchies, which is an active line of research in the theory of integrable systems, originates from several sources. Let us metion mention here the pioneering work of Kodama and Gibbons [11, 12] on the dispersionless KP, of Kupershimdt on the dispersionless modified KP [16] and the important work of Tsarev on the role of Riemann invariants and hodograph transformations [7]. Another approach is that of Takasaki and Takabe, [21], [20] and [22] which gave the Lax formalism, additional symmetries, twistor formulation of the dispersionless KP and dispersionless Toda hierarchies.

It is also worthwhile mentioning the role of dispersionless systems in topological field theories, see [15] and [3]. More recent progress appears in relation with the theory of conformal maps [8] and [23], quasiconformal maps and $\bar{\partial}$- formulation [13], additional symmetries [19] and twistor equations [10], on hodograph equations for the Boyer–Finley equation [5] and its applications in General Relativity, see also [4]. An new and interesting approach is presented in [6]. Finally, we remark the contribution on geometrical optics and the dispersionless Veselov–Novikov equation [14].

Recently, a new Poisson bracket and associated Lie algebra splitting was presented in [1] to construct new dispersionless integrable hierarchies, as the r-th modified dispersionless KP hierarchy (r-dmKP) and the r-th dispersionless Dym hierarchy (r-dDym), and latter on, see

[2], the theory was further extended. Moreover, we studied in [17] the factorization of canonical transformations in these Poisson algebras to get a new hierarchy, the r-th dispersionless Toda (r-dToda) hierarchy which contains the r-dmKP and r-dDym hierarchies as particular cases. For this new hierarchy we found additional symmetries and a new Miura map among the r-dmKP and the r-dDym hierarchies.

In this paper we extend our results on the theory of reductions of the dispersionless KP (dKP) hierarchy [18] and of the Whitham hierarchies [9], to the r-dmKP and r-dDym hierarchies introduced in [1]. The keystone is the S-function formulation of the integrable hierarchies and a reduction scheme that leads to hodograph systems characterizing solutions.

The layout of the paper is as follows. In §2 we briefly present the r-dmKP and r-dDym hierarchies. Then, in §3 we discuss the S functions for these hierarchies, firstly we present the S-functions as potentials of the hierarchies, then we analyze their relation with the Orlov functions we introduced for these hierarchies in [17] and finally we reformulate the integrable hierarchies in terms of S-functions. To end, in §4, we extend our results on the dKP and Whitham hierarchies to the r-dmKP and r-dDym hierarchies. For each hierarchy we discuss the reductions scheme, the associated hydrodynamic type systems, hodograph systems and explicit examples of solutions for the 1-component and 2-component reductions.

We must underline that the additional symmetries found in [17] may be applied to the described solutions getting in this manner even more general sets of solutions.

2 The integrable hierarchies

The hierarchies we shall consider in this article were introduced in [1] within the Lax formalism as follows. The setting needs of the Lie algebra $\mathfrak{g}$ of Laurent series $H(p,x) := \sum_{n\in\mathbb{Z}} u_n(x)p^n$ in the variable $p \in \mathbb{C}$ with coefficients depending on the variable $x \in \mathbb{R}$, with Lie commutator given by the following Poisson bracket

$$\{H_1, H_2\} = p^r\Big(\frac{\partial H_1}{\partial p}\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial p}\Big), \quad r \in \mathbb{Z}. \tag{1}$$

We shall use the following triangular type splitting of $\mathfrak{g}$ into Lie subalgebras

$$\mathfrak{g} = \mathfrak{g}_> \oplus \mathfrak{g}_{1-r} \oplus \mathfrak{g}_< \tag{2}$$

where

$$\mathfrak{g}_{\gtrless} := \mathbb{C}\{u_n(x)p^n\}_{n\gtrless(1-r)}, \quad \mathfrak{g}_{1-r} := \mathbb{C}\{u(x)p^{1-r}\},$$

and therefore fulfil the following property

$$\{\mathfrak{g}_{\gtrless}, \mathfrak{g}_{1-r}\} = \mathfrak{g}_{\gtrless}.$$

2.1 The r-th dispersionless modified KP hierarchy

If we define the Lie subalgebra $\mathfrak{g}_{\geqslant}$ as

$$\mathfrak{g}_{\geqslant} := \mathfrak{g}_{1-r} \oplus \mathfrak{g}_>$$

we have the direct sum decomposition of the Lie algebra $\mathfrak{g}$ given by

$$\mathfrak{g} = \mathfrak{g}_< \oplus \mathfrak{g}_\geqslant, \tag{3}$$

and the associated resolution of the identity into projectors

$$1 = P_\geqslant + P_<.$$

Given a Lax function L of the form

$$L = p + u_0(x) + u_1(x)p^{-1} + u_2(x)p^{-2} + \cdots, \quad p \to \infty, \tag{4}$$

we introduce

$$\Omega_n := P_\geqslant L^{n+1-r}, \quad n = 1, 2, \ldots. \tag{5}$$

The r-dmKP hierachy is the following set of Lax equations

$$\frac{\partial L}{\partial t_n} = \{\Omega_n, L\}, \quad n = 1, 2, \ldots, \tag{6}$$

where we have introduced an infinite set of time variables $\{t_n\}_{n=1}^{\infty}$. One easily deduce that the first equations of this hierarchies are

$$\begin{aligned} u_{0,t_1} &= (2-r)u_{1,x} + (2-r)(1-r)u_0 u_{0,x}, \\ u_{1,t_1} &= (2-r)u_{2,x} + (2-r)(1-r)u_0 u_{1,x} + (2-r)u_1 u_{0,x}, \\ u_{0,t_2} &= (3-r)u_{2,x} + (3-r)(2-r)(u_0 u_{1,x} + u_{0,x}u_1) + \frac{1}{2}(3-r)(2-r)(1-r)u_0^2 u_{0,x}. \end{aligned} \tag{7}$$

This system, once u_1 and u_2 are expressed in terms of u_0 leads to the r-dmKP equation for $u := u_0$

$$u_{t_2} = \frac{3-r}{(2-r)^2}(\partial_x^{-1}u)_{t_1 t_1} + \frac{(3-r)(1-r)}{2-r}u_x(\partial_x^{-1}u)_{t_1} + \frac{r(3-r)}{2-r}uu_{t_1} - \frac{(3-r)(1-r)}{2}u^2 u_x. \tag{8}$$

2.2 The r-th dispersionless Dym hierarchy

We now take the Lie subalgebra $\mathfrak{g}_\leqslant$ as

$$\mathfrak{g}_\leqslant := \mathfrak{g}_{1-r} \oplus \mathfrak{g}_<$$

and consider

$$\mathfrak{g} = \mathfrak{g}_> \oplus \mathfrak{g}_\leqslant, \tag{9}$$

and the corresponding resolution of the identity into projectors

$$1 = P_> + P_\leqslant.$$

Given the Lax function $\tilde{L}$ as follows

$$\tilde{L} = vp + v_0(x) + v_1(x)p^{-1} + \cdots, \quad p \to \infty, \tag{10}$$

we introduce

$$\tilde{\Omega}_n := P_{>}\tilde{L}^{n+1-r}, \quad n = 1, 2, \dots \tag{11}$$

The r-dDym hierarchy is defined by

$$\frac{\partial \tilde{L}}{\partial t_n} = \{\tilde{\Omega}_n, \tilde{L}\}, \quad n = 1, 2, \dots . \tag{12}$$

The first equations of this hierarchy are

$$\begin{aligned} v_{t_1} &= (2-r)v^{2-r}v_{0,x}, \\ v_{0,t_1} &= (2-r)v^{1-r}(vv_1)_x, \\ v_{t_2} &= (3-r)v^{2-r}(vv_1)_x + (3-r)(2-r)v^{2-r}v_0v_{0,x}. \end{aligned} \tag{13}$$

Eliminating v_0 and v_1 in terms of v we obtain the r-dDym equation

$$v_{t_2} = \frac{3-r}{(2-r)^2}v^{r-1}\big(v^{2-r}\partial_x^{-1}(v^{r-2}v_{t_1})\big)_{t_1}. \tag{14}$$

3 The S-functions for the r-dmKP and r-dDym hierarchies

In this section we shall consider the relation (4) as a univalent map $p \mapsto L = L(p)$, we shall also use its inverse $L \mapsto p = p(L)$. We use the notation $\boldsymbol{t} := (t_1, t_2, \dots)$.

3.1 The S-function as a potential

We introduce potential functions, $S(L, x, \boldsymbol{t})$ and $\tilde{S}(\tilde{L}, x, \boldsymbol{t})$, for

$$\omega_n(L, x, \boldsymbol{t}) := \Omega_n(p(L, x, \boldsymbol{t}), x, \boldsymbol{t}) \text{ and } \tilde{\omega}_n(L, x, \boldsymbol{t}) := \tilde{\Omega}_n(p(\tilde{L}, x, \boldsymbol{t}), x, \boldsymbol{t}).$$

First, we show that

Proposition 1. *The following identities*

$$\begin{aligned} \frac{\partial \omega_n}{\partial t_m} &= \frac{\partial \omega_m}{\partial t_n}, & \frac{\partial \omega_n}{\partial x} &= p^{-r}\frac{\partial p}{\partial t_n} \\ \frac{\partial \tilde{\omega}_n}{\partial t_m} &= \frac{\partial \tilde{\omega}_m}{\partial t_n}, & \frac{\partial \tilde{\omega}_n}{\partial x} &= p^{-r}\frac{\partial p}{\partial t_n} \end{aligned}$$

hold.

Proof. Let us compute the t_m-derivative of ω_n:

$$\frac{\partial \omega_n}{\partial t_m} = \frac{\partial \Omega_n(p(L, x, \boldsymbol{t}), x, \boldsymbol{t})}{\partial t_m} = \frac{\partial \Omega_n}{\partial p}(p(L, x, \boldsymbol{t}), x, \boldsymbol{t})\frac{\partial p}{\partial t_m}(L, x, \boldsymbol{t}) + \frac{\partial \Omega_n}{\partial t_m}(p(L, x, \boldsymbol{t}), x, \boldsymbol{t})$$

and of

$$p = p(L(p, x, \boldsymbol{t}), x, \boldsymbol{t})$$

to get

$$\frac{\partial p}{\partial t_m}=-\frac{\partial p}{\partial L}\frac{\partial L}{\partial t_m}=-\frac{\partial p}{\partial L}\{\Omega_m,L\}=-p^r\frac{\partial p}{\partial L}\Big(\frac{\partial \Omega_m}{\partial p}\frac{\partial L}{\partial x}-\frac{\partial \Omega_m}{\partial x}\frac{\partial L}{\partial p}\Big)=p^r\Big(\frac{\partial \Omega_m}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial \Omega_m}{\partial x}\Big).$$

Thus, we deduce

$$\frac{\partial \omega_n}{\partial t_m}=p^r\Big(\frac{\partial \Omega_n}{\partial p}\frac{\partial \Omega_m}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial \Omega_n}{\partial p}\frac{\partial \Omega_m}{\partial x}\Big)+\frac{\partial \Omega_n}{\partial t_m}$$

and

$$\frac{\partial \omega_n}{\partial t_m}-\frac{\partial \omega_m}{\partial t_n}=\{\Omega_n,\Omega_m\}+\frac{\partial \Omega_n}{\partial t_m}-\frac{\partial \Omega_m}{\partial t_n}=0$$

as Ω_n has zero curvature. Observe that

$$p^{-r}\frac{\partial p}{\partial t_n}=\frac{\partial \Omega_n}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial \Omega_n}{\partial x}=\frac{\partial \omega_n}{\partial x}.$$

The proof for the remaining cases is performed as above.

□

Therefore, we have proven the local existence of functions $S(L,x,\boldsymbol{t})$ and $\tilde{S}(\tilde{L},x,\boldsymbol{t})$ such that

$$\frac{\partial S}{\partial t_n}=\omega_n,\quad \frac{\partial S}{\partial x}=\Pi_r,\quad \frac{\partial \tilde{S}}{\partial t_n}=\tilde{\omega}_n,\quad \frac{\partial \tilde{S}}{\partial x}=\Pi_r \tag{15}$$

where

$$\Pi_r:=\begin{cases}\dfrac{p^{1-r}}{1-r}, & r\neq 1,\\ \log p, & r=1.\end{cases} \tag{16}$$

Notice that

$$p^r\frac{\mathrm{d}\Pi_r}{\mathrm{d}p}=1.$$

We refer to functions satisfying the above equations (15) as S-functions.

3.2 The S-functions and its connection with the Orlov functions

The so called Orlov functions M and $\tilde{M}$ are characterized by the following properties

1. They have an expansion of the form

$$\begin{aligned}M&=\cdots+(3-r)t_2L^2+(2-r)t_1L+x+w_1(x)L^{-1}+w_2(x)L^{-2}+\cdots, && L\to\infty,\\ \tilde{M}&=\cdots+(3-r)t_2\tilde{L}^2+(2-r)t_1\tilde{L}+\tilde{w}_0(x)+\tilde{w}_1(x)\tilde{L}^{-1}+\tilde{w}_2(x)\tilde{L}^{-2}+\cdots, && \tilde{L}\to\infty.\end{aligned} \tag{17}$$

   Observe that when $r=1$ then $\tilde{w}_0=x$.

2. They are canonically conjugated to L and $\tilde{L}$, respectively; i.e.,

$$\{L,M\}=L^r \text{ and } \{\tilde{L},\tilde{M}\}=\tilde{L}^r. \tag{18}$$

3. Satisfy the Lax equations

$$\frac{\partial M}{\partial t_n} = \{\Omega_n, M\}, \quad \frac{\partial \tilde{M}}{\partial t_n} = \{\tilde{\Omega}_n, \tilde{M}\}. \tag{19}$$

For $r \neq 1$ weshall show that the following functions

$$S(L, x, \boldsymbol{t}) = \cdots + t_2 L^{3-r} + t_1 L^{2-r} + \frac{x}{1-r} L^{1-r} + \sum_{n=1}^{\infty} S_n(x, \boldsymbol{t}) L^{-n+1-r}, \quad S_n := \frac{1}{-n+1-r} w_n(x, \boldsymbol{t}) \tag{20}$$

$$\tilde{S}(\tilde{L}, x, \boldsymbol{t}) = \cdots + t_2 \tilde{L}^{3-r} + t_1 \tilde{L}^{2-r} + \sum_{n=0}^{\infty} \tilde{S}_n(x, \boldsymbol{t}) \tilde{L}^{-n+1-r}, \quad \tilde{S}_n := \frac{1}{-n+1-r} \tilde{w}_n(x, \boldsymbol{t}) \tag{21}$$

are S-functions. For $r = 1$ these functions are

$$S(L, x, \boldsymbol{t}) = \cdots + t_2 L^2 + t_1 L + x \log L + \sum_{n=1}^{\infty} S_n(x, \boldsymbol{t}) L^{-n}, \quad S_n := -\frac{1}{n} w_n(x, \boldsymbol{t}),$$
$$\tilde{S}(\tilde{L}, x, \boldsymbol{t}) = \cdots + t_2 \tilde{L}^2 + t_1 \tilde{L} + x \log \tilde{L} + \sum_{n=1}^{\infty} \tilde{S}_n(x, \boldsymbol{t}) \tilde{L}^{-n}, \quad \tilde{S}_n := -\frac{1}{n} \tilde{w}_n(x, \boldsymbol{t}).$$

The role of S and $\tilde{S}$ as a generating functions is encoded in the following formulae

$$M = L^r \frac{\partial S}{\partial L}, \quad \tilde{M} = \tilde{L}^r \frac{\partial \tilde{S}}{\partial \tilde{L}}.$$

Proposition 2. *The functions $S(L, x, \boldsymbol{t})$ and $\tilde{S}(\tilde{L}, x, \boldsymbol{t})$ given by* (20) *and* (21) *are S-functions; i.e.,*

$$\begin{aligned} \frac{\partial S}{\partial x} &= \Pi_r, \quad \frac{\partial \tilde{S}}{\partial x} = \Pi_r, \\ \frac{\partial S}{\partial t_n} &= \omega_n, \quad \frac{\partial \tilde{S}}{\partial t_n} = \tilde{\omega}_n, \quad n = 1, 2, \dots . \end{aligned} \tag{22}$$

Proof. Let us prove that

$$\frac{\partial S}{\partial x} = \Pi_r.$$

Observe that according to (17)

$$\frac{\partial M}{\partial x} = \frac{\partial M}{\partial L} \frac{\partial L}{\partial x} + 1 + \sum_{n=1}^{\infty} \frac{\partial w_n}{\partial x} L^{-n}.$$

Thus, assuming that $p, L \in \mathbb{C}$ and taking a small circle γ centered at $L = 0$ in the complex L-plane we have,

$$\begin{aligned} \frac{\partial w_m}{\partial x} &= \int_\gamma \frac{\mathrm{d}L}{2\pi \mathrm{i}} \Big(L^{m-1} \frac{\partial M}{\partial x} - L^{m-1} \frac{\partial M}{\partial L} \frac{\partial L}{\partial x} \Big), \\ &= \int_\gamma \frac{\mathrm{d}p}{2\pi \mathrm{i}} L^{m-1} (M_x L_p - L_x M_p) && \text{change of variables } L = L(p) \\ & && \text{and } M_L L_p = M_p \\ &= \int_\gamma \frac{\mathrm{d}p}{2\pi \mathrm{i}} L^{m-1+r} p^{-r} && \text{in virtue of (18).} \end{aligned}$$

For $r \neq 1$ we have

$$\begin{aligned}\frac{\partial w_m}{\partial x} =&(-m+1-r)\int_\gamma \frac{\mathrm{d}p}{2\pi\mathrm{i}} L^{m-1+r-1} L_p \frac{p^{1-r}}{1-r} && \text{integration by parts}\\ =&(-m+1-r)\int_\gamma \frac{\mathrm{d}L}{2\pi\mathrm{i}} L^{m-1+r-1} \frac{p(L)^{1-r}}{1-r} && \text{change of variables } p=p(L).\end{aligned}$$

In the space $\mathcal{L}$ of Laurent series in L we have a resolution of the identity $1_{\mathcal{L}} = \varpi_{\geq} + \varpi_{<}$, associated with the splitting in powers of L of greater or equal degree than $1-r$, say $\mathcal{L}_{\geq}$, and or less order, $\mathcal{L}_{<}$

$$\varpi_{\geq}(p^{1-r}) = L^{1-r}, \quad \varpi_{<} f = \sum_{m<1-r} \Big(\int_\gamma \frac{\mathrm{d}L}{2\pi\mathrm{i}} L^{-m-1} f(L)\Big) L^m, \quad \forall f \in \mathcal{L}$$

we get

$$\begin{aligned}\frac{\partial S(L,x,\boldsymbol{t})}{\partial x} &= \frac{L^{1-r}}{1-r} + \sum_{n=1}^{\infty} \frac{1}{-n+1-r} w_{n,x}(x,\boldsymbol{t}) L^{-n+1-r}\\ &= \frac{L^{1-r}}{1-r} + \sum_{m<1-r} \Big(\int_\gamma \frac{\mathrm{d}L}{2\pi\mathrm{i}} L^{-m-1} \frac{p(L)^{1-r}}{1-r}\Big) L^m\\ &= \varpi_{\geq}\Big(\frac{p^{1-r}}{1-r}\Big) + \varpi_{<}\Big(\frac{p^{1-r}}{1-r}\Big) = \frac{p^{1-r}}{1-r}.\end{aligned}$$

For $r=1$ we have

$$\frac{\partial w_m}{\partial x} = \int_\gamma \frac{\mathrm{d}\log p}{2\pi\mathrm{i}} L^m,$$

that together with the assumption, suggested by (4),

$$\log p = \log L - \Lambda, \quad \Lambda = u_0 L^{-1} + \big(u_1 + \frac{u_0^2}{2}\big) L^{-2} + \cdots$$

allows us to deduce the relations

$$\frac{\partial w_m}{\partial x} = \int_\gamma \frac{\mathrm{d}L}{2\pi\mathrm{i}} L^{m-1} - \int_\gamma \frac{\mathrm{d}L}{2\pi\mathrm{i}} \Lambda_L L^m.$$

Notice that the first term in the r.h.s cancels as $m \geq 1$, then an integration by parts of the second term in the r.h.s leads to

$$\frac{\partial w_m}{\partial x} = m\int_\gamma \frac{\mathrm{d}L}{2\pi\mathrm{i}} \Lambda(L) L^{m-1}.$$

Hence, we compute

$$\begin{aligned}\frac{\partial S(L,x,\boldsymbol{t})}{\partial x} &= \log L - \sum_{m=1}^{\infty} \frac{1}{m} w_{m,x}(x,\boldsymbol{t}) L^{-m}\\ &= \log L - \sum_{m<0} \Big(\int_\gamma \frac{\mathrm{d}L}{2\pi\mathrm{i}} L^{-m-1} \Lambda(L)\Big) L^m\\ &= \log L - \Lambda = \log p.\end{aligned}$$

Let us prove the relation $\dfrac{\partial S}{\partial t_n}=\omega_n$. Observe that, according to (20) we have

$$\frac{\partial S}{\partial t_n}=L^{n+1-r}+\sum_{m=1}^{\infty}\frac{1}{-m+1-r}\frac{\partial w_m}{\partial t_n}L^{-m+1-r},$$

now, from (17) we deduce

$$\frac{\partial M}{\partial t_n}=\frac{\partial M}{\partial L}\frac{\partial L}{\partial t_n}+(n+1-r)L^n+\sum_{m=1}^{\infty}\frac{\partial w_m}{\partial t_n}L^{-m}. \tag{23}$$

Thus

$$\begin{aligned}
\frac{\partial w_m}{\partial t_n}&=\int_\gamma\frac{\mathrm{d}L}{2\pi\mathrm{i}}\Big(L^{m-1}\frac{\partial M}{\partial t_n}-L^{m-1}\frac{\partial M}{\partial L}\frac{\partial L}{\partial t_n}\Big), && \text{from (23)}\\
&=\int_\gamma\frac{\mathrm{d}L}{2\pi\mathrm{i}}\Big(L^{m-1}\{\Omega_n,M\}-L^{m-1}\frac{\partial M}{\partial L}\{\Omega_n,L\}\Big), && \text{derived from (6) and (19)}\\
&=\int_\gamma\frac{\mathrm{d}L}{2\pi\mathrm{i}}\Big(L^{m-1}p^r(\Omega_{n,p}M_x-\Omega_{n,x}M_p)\\
&\qquad -L^{m-1}\frac{\partial M}{\partial L}p^r(\Omega_{n,p}L_x-\Omega_{n,x}L_p)\Big) && \text{see (1)}\\
&=\int_\gamma\frac{\mathrm{d}p}{2\pi\mathrm{i}}L^{m-1}p^r\Omega_{n,p}(M_xL_p-L_xM_p) && \text{change of variables } L=L(p)\\
& && \text{and } M_LL_p=M_p\\
&=\int_\gamma\frac{\mathrm{d}p}{2\pi\mathrm{i}}L^{m-1+r}\Omega_{n,p} && \text{in virtue of (18)}\\
&=(-m+1-r)\int_\gamma\frac{\mathrm{d}p}{2\pi\mathrm{i}}L^{m-1+r-1}L_p\Omega_n && \text{integration by parts}\\
&=(-m+1-r)\int_\gamma\frac{\mathrm{d}L}{2\pi\mathrm{i}}L^{m-1+r-1}\omega_n && \text{change of variables } p=p(L)
\end{aligned}$$

and therefore

$$\frac{\partial S}{\partial t_n}=L^{n+1-r}+\sum_{m<1-r}\Big(\int_\gamma\frac{\mathrm{d}L}{2\pi\mathrm{i}}L^{-m-1}\omega_n\Big)L^m.$$

Now,

$$\int_\gamma\frac{\mathrm{d}L}{2\pi\mathrm{i}}L^{-m-1}\omega_n=\int_\gamma\frac{\mathrm{d}L}{2\pi\mathrm{i}}L^{-m-1}(L^{n+1-r}-P_<L^{n+1-r})=-\int_\gamma\frac{\mathrm{d}L}{2\pi\mathrm{i}}L^{-m-1}P_<L^{n+1-r}$$

as $m\neq n+1-r$ for $m<1-r$. So that

$$\frac{\partial S}{\partial t_n}=\omega_n+P_<(L^{n+1-r})-\sum_{m<1-r}\Big(\int_\gamma\frac{\mathrm{d}L}{2\pi\mathrm{i}}L^{-m-1}P_<(L^{n+1-r})\Big)L^m,$$

but

$$P_<(L^{n+1-r})=\sum_{m\in\mathbb{Z}}\Big(\int_\gamma\frac{\mathrm{d}L}{2\pi\mathrm{i}}L^{-m-1}P_<(L^{n+1-r})\Big)L^m=\sum_{m<1-r}\Big(\int_\gamma\frac{\mathrm{d}L}{2\pi\mathrm{i}}L^{-m-1}P_<(L^{n+1-r})\Big)L^m$$

and the result follows. For $r\neq 1$, let us prove that $\dfrac{\partial\tilde S}{\partial x}=\dfrac{p^{1-r}}{1-r}$; equations (17) imply

$$\frac{\partial\tilde M}{\partial x}=\frac{\partial\tilde M}{\partial\tilde L}\frac{\partial\tilde L}{\partial x}+\sum_{n=0}^{\infty}\frac{\partial\tilde w_n}{\partial x}\tilde L^{-n}.$$

and hence

$$\begin{aligned}
\frac{\partial\tilde w_m}{\partial x}&=\int_\gamma\frac{\mathrm{d}L}{2\pi\mathrm{i}}\Big(\tilde L^{m-1}\frac{\partial\tilde M}{\partial x}-\tilde L^{m-1}\frac{\partial\tilde M}{\partial\tilde L}\frac{\partial\tilde L}{\partial x}\Big),\\
&=\int_\gamma\frac{\mathrm{d}p}{2\pi\mathrm{i}}\tilde L^{m-1}(\tilde M_x\tilde L_p-\tilde L_x\tilde M_p) && \text{change of variables } \tilde L=\tilde L(p)\\
& && \text{and } \tilde M_{\tilde L}\tilde L_p=\tilde M_p\\
&=\int_\gamma\frac{\mathrm{d}p}{2\pi\mathrm{i}}\tilde L^{m-1+r}p^{-r} && \text{in virtue of (18)}\\
&=(-m+1-r)\int_\gamma\frac{\mathrm{d}p}{2\pi\mathrm{i}}\tilde L^{m-1+r-1}\tilde L_p\frac{p^{1-r}}{1-r} && \text{integration by parts}\\
&=(-m+1-r)\int_\gamma\frac{\mathrm{d}\tilde L}{2\pi\mathrm{i}}\tilde L^{m-1+r-1}\frac{p^{1-r}}{1-r} && \text{change of variables } p=p(\tilde L).
\end{aligned}$$

We now proceed computing

$$\frac{\partial\tilde S(\tilde L,x,\boldsymbol{t})}{\partial x}=\sum_{n=0}^{\infty}\frac{1}{-n+1-r}\tilde w_{n,x}(x,\boldsymbol{t})\tilde L^{-n+1-r}=\sum_{m\leq 1-r}\Big(\int_\gamma\frac{\mathrm{d}\tilde L}{2\pi\mathrm{i}}\tilde L^{-m-1}\frac{p^{1-r}}{1-r}\Big)\tilde L^m,$$

noting that $\dfrac{p^{1-r}}{1-r}\in\mathfrak{g}_{\leq}$ we get the desired result. For $r=1$ we have

$$\frac{\partial\tilde w_m}{\partial x}=\int_\gamma\frac{\mathrm{d}\log p}{2\pi\mathrm{i}}\tilde L^m$$

assuming, as suggested by (10), that

$$\log p=\log\tilde L-\tilde\Lambda,\quad \tilde\Lambda:=\log v+\frac{v_0}{v}\tilde L^{-1}+\Big(v_1+\frac{v_0^2}{2v^2}\Big)\tilde L^{-2}+\cdots$$

we have

$$\begin{aligned}
\frac{\partial\tilde w_m}{\partial x}&=\int_\gamma\frac{\mathrm{d}\tilde L}{2\pi\mathrm{i}}\tilde L^{m-1}-\int_\gamma\frac{\mathrm{d}\tilde L}{2\pi\mathrm{i}}\tilde\Lambda_{\tilde L}\tilde L^m\\
&=m\int_\gamma\frac{\mathrm{d}\tilde L}{2\pi\mathrm{i}}\tilde\Lambda L^{m-1},\quad m\geq 1.
\end{aligned}$$

Thus, we calculate

$$\begin{aligned}
\frac{\partial\tilde S(\tilde L,x,\boldsymbol{t})}{\partial x}&=\log\tilde L-\sum_{m=1}^{\infty}\frac{1}{m}\tilde w_{m,x}(x,\boldsymbol{t})\tilde L^{-m}=\log\tilde L-\sum_{m<0}\Big(\int_\gamma\frac{\mathrm{d}\tilde L}{2\pi\mathrm{i}}\tilde L^{-m-1}\tilde\Lambda(\tilde L)\Big)\tilde L^m\\
&=\log\tilde L-\tilde\Lambda=\log p.
\end{aligned}$$

To check the formula $\frac{\partial \tilde{S}}{\partial t_n} = \tilde{\omega}_n$, we procede in a similar way. Equation (21) implies that

$$\frac{\partial \tilde{S}}{\partial t_n} = \tilde{L}^{n+1-r} + \sum_{m=0}^{\infty} \frac{1}{-m+1-r} \frac{\partial \tilde{w}_m}{\partial t_n} \tilde{L}^{-m+1-r},$$

now, from (17) we deduce

$$\frac{\partial \tilde{M}}{\partial t_n} = \frac{\partial \tilde{M}}{\partial \tilde{L}} \frac{\partial \tilde{L}}{\partial t_n} + (n+1-r)\tilde{L}^n + \sum_{m=0}^{\infty} \frac{\partial \tilde{w}_m}{\partial t_n} \tilde{L}^{-m}. \tag{24}$$

As for the r-dmKP case we have the following chain of observations

$$\begin{aligned}
\frac{\partial \tilde{w}_m}{\partial t_n} =& \int_\gamma \frac{\mathrm{d}\tilde{L}}{2\pi \mathrm{i}} \Big(\tilde{L}^{m-1} \frac{\partial \tilde{M}}{\partial t_n} - \tilde{L}^{m-1} \frac{\partial \tilde{M}}{\partial \tilde{L}} \frac{\partial \tilde{L}}{\partial t_n} \Big), && \text{from (24)} \\
=& \int_\gamma \frac{\mathrm{d}\tilde{L}}{2\pi \mathrm{i}} \Big(\tilde{L}^{m-1} \{\tilde{\Omega}_n, \tilde{M}\} - \tilde{L}^{m-1} \frac{\partial \tilde{M}}{\partial \tilde{L}} \{\tilde{\Omega}_n, \tilde{L}\} \Big), && \text{derived from (6) and (19)} \\
=& \int_\gamma \frac{\mathrm{d}p}{2\pi \mathrm{i}} \tilde{L}^{m-1} p^r \tilde{\Omega}_{n,p} (\tilde{M}_x \tilde{L}_p - \tilde{L}_x \tilde{M}_p) && \text{change of variables } \tilde{L} = \tilde{L}(p) \\
=& (-m+1-r) \int_\gamma \frac{\mathrm{d}p}{2\pi \mathrm{i}} \tilde{L}^{m-1+r-1} \tilde{L}_p \tilde{\Omega}_n && \text{integration by parts} \\
=& (-m+1-r) \int_\gamma \frac{\mathrm{d}\tilde{L}}{2\pi \mathrm{i}} \tilde{L}^{m-1+r-1} \tilde{\Omega}_n && \text{change of variables } p = p(\tilde{L})
\end{aligned}$$

that lead to

$$\frac{\partial \tilde{S}}{\partial t_n} = \tilde{L}^{n+1-r} + \sum_{m \le 1-r} \Big(\int_\gamma \frac{\mathrm{d}\tilde{L}}{2\pi \mathrm{i}} \tilde{L}^{-m-1} \tilde{\Omega}_n \Big) \tilde{L}^m.$$

Also we have

$$\int_\gamma \frac{\mathrm{d}\tilde{L}}{2\pi \mathrm{i}} \tilde{L}^{-m-1} \tilde{\Omega}_n = \int_\gamma \frac{\mathrm{d}\tilde{L}}{2\pi \mathrm{i}} \tilde{L}^{-m-1} (\tilde{L}^{n+1-r} - P_\le \tilde{L}^{n+1-r}) = -\int_\gamma \frac{\mathrm{d}\tilde{L}}{2\pi \mathrm{i}} L^{-m-1} P_\le \tilde{L}^{n+1-r}$$

as $m \neq n+1-r$ for $m \le 1-r$. Therefore,

$$\frac{\partial \tilde{S}}{\partial t_n} = \tilde{\Omega}_n + P_\le(\tilde{L}^{n+1-r}) - \sum_{m \le 1-r} \Big(\int_\gamma \frac{\mathrm{d}\tilde{L}}{2\pi \mathrm{i}} \tilde{L}^{-m-1} P_\le(\tilde{L}^{n+1-r}) \Big) \tilde{L}^m,$$

and from

$$P_\le(\tilde{L}^{n+1-r}) = \sum_{m \in \mathbb{Z}} \Big(\int_\gamma \frac{\mathrm{d}\tilde{L}}{2\pi \mathrm{i}} \tilde{L}^{-m-1} P_\le(\tilde{L}^{n+1-r}) \Big) \tilde{L}^m = \sum_{m \le 1-r} \Big(\int_\gamma \frac{\mathrm{d}\tilde{L}}{2\pi \mathrm{i}} \tilde{L}^{-m-1} P_\le(\tilde{L}^{n+1-r}) \Big) \tilde{L}^m$$

we arrive to the claimed result. □

3.3 The S-function formulation of the integrable hierarchies

Here we show that the integrable hierarchies can be formulated in terms of S-functions. This is a key observation for the reduction procedure that we shall present in the next section.

Proposition 3. *Let L and $\tilde{L}$ be functions with expansions as given in* (4) *and* (10)*, respectively, and Π_r as defined in* (16)*. Let $S(L,x,\boldsymbol{t})$ and $\tilde{S}(\tilde{L},x,\boldsymbol{t})$ be S-functions; i.e.*

$$\begin{aligned}\frac{\partial S}{\partial x}&=\Pi_r, & \frac{\partial \tilde{S}}{\partial x}&=\Pi_r,\\ \frac{\partial S}{\partial t_n}&=P_{\geq}(L^{n+1-r}), & \frac{\partial \tilde{S}}{\partial t_n}&=P_{>}(\tilde{L}^{n+1-r}),\quad n=1,2,\dots.\end{aligned}$$

Then, L and $\tilde{L}$ do satisfy the r-dmKP and r-dDym hierarchies (6) *and* (12)*, respectively; i.e.,*

$$\frac{\partial L}{\partial t_n}=\{P_{\geq}(L^{n+1-r}),L\},\quad \frac{\partial \tilde{L}}{\partial t_n}=\{P_{>}(\tilde{L}^{n+1-r}),\tilde{L}\}.$$

Proof. From $S_x=\Pi_r$ we deduce

$$\frac{\partial p}{\partial t_n}=p^r\frac{\partial^2 S}{\partial t_n\partial x}=p^r(\Omega_n(p(L,x,\boldsymbol{t}),x,\boldsymbol{t}))_x=p^r(\Omega_{n,p}p_x+\Omega_x).$$

Thus,

$$\frac{\partial L}{\partial t_n}=-\frac{\partial L}{\partial p}\frac{\partial p}{\partial t_n}=-L_p p^r(\Omega_{n,p}p_x+\Omega_x)=p^r(-\Omega_{n,p}L_x+\Omega_x L_p)=\{\Omega_n,L\},$$

as claimed. The statement for the r-dDym hierarchy follows analogously. □

4 Reductions

The reductions we consider here are motivated by the reductions we studied in [18] of the dispersionless KP hierarchy. We assume that the $\boldsymbol{t}$-dependence appears always in terms of $\boldsymbol{U}=(U_1,\dots,U_N)$, a set of N functions of $\boldsymbol{t}$ and x. This dependence is defined trough the following equations for the function $p=p(L,\boldsymbol{U})$ or $p=p(\tilde{L},\boldsymbol{U})$

$$\frac{\partial p}{\partial U_i}=R_i(p,\boldsymbol{U}),\quad i=1,\dots,N, \tag{25}$$

which in terms of the Lax functions are

$$\frac{\partial L}{\partial U_i}+R_i(p,\boldsymbol{U})\frac{\partial L}{\partial p}=0,\quad i=1,\dots,N, \tag{26}$$

$$\frac{\partial \tilde{L}}{\partial U_i}+\tilde{R}_i(p,\boldsymbol{U})\frac{\partial \tilde{L}}{\partial p}=0,\quad i=1,\dots,N. \tag{27}$$

We shall assume that the compatibility conditions for (25) are fulfilled; i.e. both sets of functions $\{R_i\}_{i=1}^N$ and $\{\tilde{R}_i\}_{i=1}^N$ fulfill

$$\frac{\partial R_i}{\partial U_j}-\frac{\partial R_j}{\partial U_i}+R_j\frac{\partial R_i}{\partial p}-R_i\frac{\partial R_j}{\partial p}=0. \tag{28}$$

We will also suppose that R_i and $\tilde{R}_i$, $i = 1, \ldots, N$, are rational functions with N simple poles, $\pi_i = \pi_i(\boldsymbol{U})$ and $\tilde{\pi}_i = \tilde{\pi}_i(\boldsymbol{U})$ $i = 1, \ldots, N$, respectively. Recalling the expansions (4) and (10) and taking into account formulae (26) and (27), we request to R_i to be of order $O(1)$ when $p \to \infty$ and $\tilde{R}_i$ of order $O(p)$ when $p \to \infty$. Hence, our functions R will be of the form

$$R_i(\boldsymbol{U}) = p \sum_{j=1}^{N} \frac{\rho_{ij}(\boldsymbol{U})}{p - \pi_j(\boldsymbol{U})}, \tag{29}$$

$$\tilde{R}_i(\boldsymbol{U}) = p^2 \sum_{j=1}^{N} \frac{\tilde{\rho}_{ij}(\boldsymbol{U})}{p - \tilde{\pi}_j(\boldsymbol{U})}. \tag{30}$$

The asymptotic behaviors for $p \to \infty$ are

$$\begin{aligned} R_i &= R_{i,0} + R_{i,1}p^{-1} + R_{i,2}p^{-2} + \cdots, \\ \tilde{R}_i &= \tilde{R}_{i,0}p + \tilde{R}_{i,1} + \tilde{R}_{i,2}p^{-1} + \cdots, \end{aligned}$$

were we have used the following notation

$$R_{i,n} := \sum_{j=1}^{N} \rho_{ij}\pi_j^n, \quad \tilde{R}_{i,n} := \sum_{j=1}^{N} \tilde{\rho}_{ij}\tilde{\pi}_j^n.$$

The equation (26) together with (29), in the r-dmKP hierarchy case, imply

$$\begin{aligned} \frac{\partial u_0}{\partial U_i} &= -R_{i,0}, \\ \frac{\partial u_1}{\partial U_i} &= -R_{i,1}, \\ \frac{\partial u_2}{\partial U_i} &= -R_{i,2} + R_{i,0}u_1, \\ &\vdots \end{aligned} \tag{31}$$

Observe that all the coefficients u_n are expressed recursively in terms of the functions $\{\pi_k, \rho_{ik}\}$ defining R_i. The equation (27) together with (30), in the r-dDym hierarchy case, leads to

$$\begin{aligned} \frac{\partial v}{\partial U_i} &= -\tilde{R}_{i,0}v, \\ \frac{\partial v_0}{\partial U_i} &= -\tilde{R}_{i,1}v, \\ \frac{\partial v_1}{\partial U_i} &= -\tilde{R}_{i,2}v + \tilde{R}_{i,0}v_1, \\ &\vdots \end{aligned} \tag{32}$$

Observe that all the coefficients v and v_n are expressed recursively in terms of the functions $\tilde{\pi}_k, \tilde{\rho}_{ik}\}$ defining R_i.

4.1 On the compatibility conditions

We now discuss the compatibility conditions for (28).

r**-dDym** The compatibility equations for (28) with the choice (30) are

$$\tilde{\rho}_{il}\frac{\partial\tilde{\pi}_l}{\partial U_j}-\tilde{\rho}_{jl}\frac{\partial\tilde{\pi}_l}{\partial U_i}=\sum_{k\neq l}\frac{\tilde{\rho}_{ik}\tilde{\rho}_{jl}-\tilde{\rho}_{il}\tilde{\rho}_{jk}}{\tilde{\pi}_k-\tilde{\pi}_l}\tilde{\pi}_l^2, \tag{33a}$$

$$\frac{\partial\tilde{\rho}_{il}}{\partial U_j}-\frac{\partial\tilde{\rho}_{jl}}{\partial U_i}=2\sum_{k\neq l}\frac{\tilde{\rho}_{ik}\tilde{\rho}_{jl}-\tilde{\rho}_{il}\tilde{\rho}_{jk}}{(\tilde{\pi}_l-\tilde{\pi}_k)^2}\tilde{\pi}_k\tilde{\pi}_l. \tag{33b}$$

From equations (33a) and (33b) we deduce

Proposition 4. *There exist a pair of potentials $\tilde{\sigma}$ and $\tilde{\rho}$ such that*

$$\tilde{R}_{i,0}=-\frac{\partial\tilde{\sigma}}{\partial U_i}, \tag{34}$$

$$\tilde{R}_{i,1}=-\frac{\partial\tilde{\rho}}{\partial U_i}-\tilde{\rho}\frac{\partial\tilde{\sigma}}{\partial U_i}. \tag{35}$$

Proof. Firstly, we observe that

$$\frac{\partial R_{i,0}}{\partial U_j}-\frac{\partial R_{j,0}}{\partial U_i}=\sum_{l=1}^{N}\Big(\frac{\partial\tilde{\rho}_{il}}{\partial U_j}-\frac{\partial\tilde{\rho}_{jl}}{\partial U_i}\Big)=2\sum_{\substack{l=1,\dots,N\\k\neq l}}\frac{\tilde{\rho}_{ik}\tilde{\rho}_{jl}-\tilde{\rho}_{il}\tilde{\rho}_{jk}}{(\tilde{\pi}_l-\tilde{\pi}_k)^2}\tilde{\pi}_k\tilde{\pi}_l=0$$

Secondly, we evaluate

$$\begin{aligned}\frac{\partial R_{i,1}}{\partial U_j}-R_{j,0}R_{i,1}-\Big(\frac{\partial R_{j,1}}{\partial U_i}-R_{i,0}R_{j,1}\Big)&=\sum_{\substack{l=1,\dots,N\\k\neq l}}\Big[(\tilde{\rho}_{ik}\tilde{\rho}_{jl}-\tilde{\rho}_{il}\tilde{\rho}_{jk})\Big(\frac{2\tilde{\pi}_l^2\tilde{\pi}_k}{(\tilde{\pi}_l-\tilde{\pi}_k)^2}-\frac{\tilde{\pi}_l^2}{\tilde{\pi}_l-\tilde{\pi}_k}+\tilde{\pi}_l\Big)\Big]\\&=\sum_{\substack{l=1,\dots,N\\k\neq l}}(\tilde{\rho}_{ik}\tilde{\rho}_{jl}-\tilde{\rho}_{il}\tilde{\rho}_{jk})\tilde{\pi}_k\tilde{\pi}_l(\tilde{\pi}_k+\tilde{\pi}_l)=0.\end{aligned}$$

The stated result is a direct consequence of these two equations. □

r**-dmKP** The compatibility equations (28) for (29) are

$$\rho_{il}\frac{\partial\pi_l}{\partial U_j}-\rho_{jl}\frac{\partial\pi_l}{\partial U_i}=\sum_{k\neq l}\frac{\rho_{ik}\rho_{jl}-\rho_{il}\rho_{jk}}{\pi_k-\pi_l}\pi_l, \tag{36a}$$

$$\frac{\partial\rho_{il}}{\partial U_j}-\frac{\partial\rho_{jl}}{\partial U_i}=\sum_{k\neq l}\frac{\rho_{ik}\rho_{jl}-\rho_{il}\rho_{jk}}{(\pi_l-\pi_k)^2}(\pi_k+\pi_l). \tag{36b}$$

As in Proposition 4 we can show that

Proposition 5. *There exist a potential ρ such that*

$$R_{i,0}=-\frac{\partial\rho}{\partial U_i}. \tag{37}$$

The following Proposition shows a connection between the compatibility conditions for the r-dDym and r-dmKP equations. Further, it also shows a connection among these and the compatibility conditions

$$r_{il}\frac{\partial p_l}{\partial U_j}-r_{jl}\frac{\partial p_l}{\partial U_i}=\sum_{k\neq l}\frac{r_{jl}r_{ik}-r_{il}r_{jk}}{p_k-p_l}, \tag{38a}$$

$$\frac{\partial r_{il}}{\partial U_j}-\frac{\partial r_{jl}}{\partial U_i}=2\sum_{k\neq l}\frac{r_{jl}r_{ik}-r_{il}r_{jk}}{(p_l-p_k)^2}. \tag{38b}$$

for similar reductions of the dispersionless KP hierarchy, that we discussed in some length in [18].

Proposition 6. *1. If $\tilde\pi_i$ and $\tilde\rho_{ij}$ solves the compatibility conditions* (33a) *and* (33b) *then*

$$\pi_i=\mathrm{e}^{\tilde\sigma}\,\tilde\pi_i,\quad \rho_{ij}=\mathrm{e}^{\tilde\sigma}\,\tilde\rho_{ij}\tilde\pi_j \tag{39}$$

solves the compatibility conditions (36a) *and* (36b). *Moreover, we may take as the potential ρ the following function*

$$\rho=\mathrm{e}^{\tilde\sigma}\,\tilde\rho.$$

2. If π_i and ρ_{ij} solves the compatibility conditions (36a) *and* (36b) *then*

$$p_i=\pi_i+\rho,\quad r_{ij}=\rho_{ij}\pi_j \tag{40}$$

solves the compatibility conditions (38a) *and* (38b).

Proof. 1. With the expressions (39) and the formulae (33a) and (33b) we evaluate

$$\begin{aligned}
\rho_{il}\frac{\partial \pi_l}{\partial U_j}-\rho_{jl}\frac{\partial \pi_l}{\partial U_i}&=\mathrm{e}^{2\tilde\sigma}\left(\Big(\tilde\rho_{il}\frac{\partial\tilde\pi_l}{\partial U_j}-\tilde\rho_{jl}\frac{\partial\tilde\pi_l}{\partial U_i}\Big)\tilde\pi_l-(\tilde\rho_{il}\tilde R_{j,0}-\tilde\rho_{jl}\tilde R_{i,0})\tilde\pi_l^2\right)\\
&=\mathrm{e}^{2\tilde\sigma}\sum_{k\neq l}(\tilde\rho_{ik}\tilde\rho_{jl}-\tilde\rho_{il}\tilde\rho_{jk})\tilde\pi_l^2\Big(\frac{\tilde\pi_l}{\tilde\pi_k-\tilde\pi_l}+1\Big)\\
&=\mathrm{e}^{2\tilde\sigma}\sum_{k\neq l}(\tilde\rho_{ik}\tilde\rho_{jl}-\tilde\rho_{il}\tilde\rho_{jk})\tilde\pi_k\tilde\pi_l\frac{\tilde\pi_l}{\tilde\pi_k-\tilde\pi_l}\\
&=\sum_{k\neq l}\frac{\rho_{ik}\rho_{jl}-\rho_{il}\rho_{jk}}{\pi_k-\pi_l}\pi_l\\
\frac{\partial\rho_{il}}{\partial U_j}-\frac{\partial\rho_{jl}}{\partial U_i}&=\mathrm{e}^{\tilde\sigma}\left(\frac{\partial\tilde\rho_{il}}{\partial U_j}-\frac{\partial\tilde\rho_{jl}}{\partial U_i}-(\tilde\rho_{il}\tilde R_{j,0}-\tilde\rho_{jl}\tilde R_{i,0})\tilde\pi_l+\tilde\rho_{il}\frac{\partial\tilde\pi_l}{\partial U_j}-\tilde\rho_{jl}\frac{\partial\tilde\pi_l}{\partial U_i}\right)\\
&=\mathrm{e}^{\tilde\sigma}\sum_{k\neq l}(\tilde\rho_{ik}\tilde\rho_{jl}-\tilde\rho_{il}\tilde\rho_{jk})\Big(\frac{2\tilde\pi_k\tilde\pi_l^2}{(\tilde\pi_k-\tilde\pi_l)^2}+\frac{\tilde\pi_k\tilde\pi_l}{\tilde\pi_k-\tilde\pi_l}\Big)\\
&=\mathrm{e}^{\tilde\sigma}\sum_{k\neq l}(\tilde\rho_{ik}\tilde\rho_{jl}-\tilde\rho_{il}\tilde\rho_{jk})\tilde\pi_k\tilde\pi_l\frac{\tilde\pi_k+\tilde\pi_l}{(\tilde\pi_k-\tilde\pi_l)^2}\\
&=\sum_{k\neq l}\frac{\rho_{ik}\rho_{jl}-\rho_{il}\rho_{jk}}{(\pi_l-\pi_k)^2}(\pi_k+\pi_l)
\end{aligned}$$

and as claimed we have gotten (36a) and (36b).

From (39) we get

$$R_{i,0} = \mathrm{e}^{\tilde{\sigma}}\,\tilde{R}_{i,1};$$

i.e.,

$$\frac{\partial \rho}{\partial U_i} = \mathrm{e}^{\tilde{\sigma}}\left(\frac{\partial \rho}{\partial U_i} + \tilde{\rho}\frac{\partial \tilde{\sigma}}{\partial U_i}\right) = \frac{\partial(\mathrm{e}^{\tilde{\sigma}}\,\tilde{\rho})}{\partial U_i}.$$

2. We now use (40) together with (36a) and (36b) to get

$$\begin{aligned}
r_{il}\frac{\partial p_l}{\partial U_j} - r_{jl}\frac{\partial p_l}{\partial U_i} &= \Big(\rho_{il}\frac{\partial \pi_l}{\partial U_j} - \rho_{jl}\frac{\partial \pi_l}{\partial U_i}\Big)\pi_l - \rho_{il}R_{j,0} + \rho_{jl}R_{i,0} \\
&= \sum_{k\neq l}(\rho_{ik}\rho_{jl} - \rho_{il}\rho_{jk})\pi_l\Big(\frac{\pi_l}{\pi_k - \pi_l} + 1\Big) \\
&= \sum_{k\neq l}(\rho_{ik}\rho_{jl} - \rho_{il}\rho_{jk})\pi_k\pi_l\frac{1}{\pi_k - \pi_l} \\
&= \sum_{k\neq l}\frac{r_{ik}r_{jl} - r_{il}r_{jk}}{p_k - p_l} \\
\frac{\partial r_{il}}{\partial U_j} - \frac{\partial r_{jl}}{\partial U_i} &= \Big(\frac{\partial \rho_{il}}{\partial U_j} - \frac{\partial \rho_{jl}}{\partial U_i}\Big)\pi_l + \rho_{il}\frac{\partial \pi_l}{\partial U_j} - \rho_{jl}\frac{\partial \pi_l}{\partial U_i} \\
&= \sum_{k\neq l}(\rho_{ik}\rho_{jl} - \rho_{il}\rho_{jk})\Big(\frac{(\pi_k + \pi_l)\pi_l}{(\pi_k - \pi_l)^2} + \frac{\pi_l}{\pi_k - \pi_l}\Big) \\
&= 2\sum_{k\neq l}(\rho_{ik}\rho_{jl} - \rho_{il}\rho_{jk})\pi_k\pi_l\frac{1}{(\pi_k - \pi_l)^2} \\
&= 2\sum_{k\neq l}\frac{r_{ik}r_{jl} - r_{il}r_{jk}}{(p_l - p_k)^2}.
\end{aligned}$$

□

The inverse statement also holds.

Proposition 7. *1. Let π_i, ρ_{ij} be solutions of* (36a) *and* (36b)*, then there exists a potential function $\tilde{\sigma}$ such that*

$$\frac{\partial \tilde{\sigma}}{\partial U_i} = -\sum_{l=1}^{N}\frac{\rho_{il}}{\pi_l}.$$

Moreover,

$$\tilde{\pi}_i = \mathrm{e}^{-\tilde{\sigma}}\,\pi_i, \quad \tilde{\rho}_{ij} = \frac{\rho_{ij}}{\pi_j},$$

provides us with solutions to (33a) *and* (33b)*.*

2. Let p_i, r_{ij} solutions (38a) *and* (38b) *then there exists a potential function ρ such that*

$$\frac{\partial \rho}{\partial U_i} = \sum_{l=1}^{N}\frac{r_{il}}{\rho - p_l}.$$

Moreover,

$$\pi_i := p_j - \rho, \quad \rho_{ij} := \frac{r_{ij}}{p_j - \rho}$$

solves the equations (36a) *and* (36b).

Proof. 1. Let us evaluate

$$\begin{aligned}
\frac{\partial}{\partial U_j}\Big(\sum_{l=1}^{N}\frac{\rho_{il}}{\pi_l}\Big) - \frac{\partial}{\partial U_i}\Big(\sum_{l=1}^{N}\frac{\rho_{jl}}{\pi_l}\Big) &= \sum_{l=1}^{N}\Big(\frac{\rho_{il,j}-\rho_{jl,i}}{\pi_l} - \frac{\rho_{il}\pi_{l,j}-\rho_{jl}\pi_{l,i}}{\pi_l^2}\Big) \\
&= \sum_{\substack{l=1,\dots,N\\ k\neq l}}\Big((\rho_{ik}\rho_{jl}-\rho_{il}\rho_{jk})\Big(\frac{\pi_k+\pi_l}{\pi_l(\pi_l-\pi_k)^2} - \frac{\pi_k-\pi_l}{\pi_l(\pi_l-\pi_k)^2}\Big)\Big) \\
&= 2\sum_{\substack{l=1,\dots,N\\ k\neq l}}\frac{\rho_{ik}\rho_{jl}-\rho_{il}\rho_{jk}}{(\pi_l-\pi_k)^2} = 0.
\end{aligned}$$

So that, locally, the existence of the mentioned potential holds, and the relation (39) allows us to identify it with $\tilde{\sigma}$. Moreover, the identities

$$\begin{aligned}
\rho_{il}\frac{\partial\pi_l}{\partial U_j} - \rho_{jl}\frac{\partial\pi_l}{\partial U_i} &= \mathrm{e}^{2\tilde{\sigma}}\left(\Big(\tilde{\rho}_{il}\frac{\partial\tilde{\pi}_l}{\partial U_j} - \tilde{\rho}_{jl}\frac{\partial\tilde{\pi}_l}{\partial U_i}\Big)\tilde{\pi}_l - (\tilde{\rho}_{il}\tilde{R}_{j,0} - \tilde{\rho}_{jl}\tilde{R}_{i,0})\tilde{\pi}_l^2\right) \\
\sum_{k\neq l}\frac{\rho_{ik}\rho_{jl}-\rho_{il}\rho_{jk}}{\pi_k-\pi_l}\pi_l &= \mathrm{e}^{2\tilde{\sigma}}\sum_{k\neq l}(\tilde{\rho}_{ik}\tilde{\rho}_{jl}-\tilde{\rho}_{il}\tilde{\rho}_{jk})\tilde{\pi}_k\tilde{\pi}_l\frac{\tilde{\pi}_l}{\tilde{\pi}_k-\tilde{\pi}_l} \\
\frac{\partial\rho_{il}}{\partial U_j} - \frac{\partial\rho_{jl}}{\partial U_i} &= \mathrm{e}^{\tilde{\sigma}}\left(\frac{\partial\tilde{\rho}_{il}}{\partial U_j} - \frac{\partial\tilde{\rho}_{jl}}{\partial U_i} - (\tilde{\rho}_{il}\tilde{R}_{j,0} - \tilde{\rho}_{jl}\tilde{R}_{i,0})\tilde{\pi}_l + \tilde{\rho}_{il}\frac{\partial\tilde{\pi}_l}{\partial U_j} - \tilde{\rho}_{jl}\frac{\partial\tilde{\pi}_l}{\partial U_i}\right) \\
\sum_{k\neq l}\frac{\rho_{ik}\rho_{jl}-\rho_{il}\rho_{jk}}{(\pi_l-\pi_k)^2}(\pi_k+\pi_l) &= \mathrm{e}^{\tilde{\sigma}}\sum_{k\neq l}(\tilde{\rho}_{ik}\tilde{\rho}_{jl}-\tilde{\rho}_{il}\tilde{\rho}_{jk})\tilde{\pi}_k\tilde{\pi}_l\frac{\tilde{\pi}_k+\tilde{\pi}_l}{(\tilde{\pi}_k-\tilde{\pi}_l)^2},
\end{aligned}$$

imply our statements.

2. The compatibility conditions for

$$\frac{\partial\rho}{\partial U_i} = \sum_{l=1}^{N}\frac{r_{il}}{\rho - p_l}.$$

ares precisely the equations (38a) and (38b), see [18].

Now, the remaining results follow from the equations

$$\begin{aligned}
r_{il}\frac{\partial p_l}{\partial U_j} - r_{jl}\frac{\partial p_l}{\partial U_i} &= \Big(\rho_{il}\frac{\partial\pi_l}{\partial U_j} - \rho_{jl}\frac{\partial\pi_l}{\partial U_i}\Big)\pi_l - \rho_{il}R_{j,0} + \rho_{jl}R_{i,0} \\
\sum_{k\neq l}\frac{r_{ik}r_{jl}-r_{il}r_{jk}}{p_k-p_l} &= \sum_{k\neq l}(\rho_{ik}\rho_{jl}-\rho_{il}\rho_{jk})\pi_l\Big(\frac{\pi_l}{\pi_k-\pi_l}+1\Big) \\
\frac{\partial r_{il}}{\partial U_j} - \frac{\partial r_{jl}}{\partial U_i} &= \Big(\frac{\partial\rho_{il}}{\partial U_j} - \frac{\partial\rho_{jl}}{\partial U_i}\Big)\pi_l + \rho_{il}\frac{\partial\pi_l}{\partial U_j} - \rho_{jl}\frac{\partial\pi_l}{\partial U_i} \\
2\sum_{k\neq l}\frac{r_{ik}r_{jl}-r_{il}r_{jk}}{(p_l-p_k)^2} &= \sum_{k\neq l}(\rho_{ik}\rho_{jl}-\rho_{il}\rho_{jk})\Big(\frac{(\pi_k+\pi_l)\pi_l}{(\pi_k-\pi_l)^2} + \frac{\pi_l}{\pi_k-\pi_l}\Big),
\end{aligned}$$

□

The relations(31), (32), (34) and (35) allow us to take

$$u_0 = \rho.$$

and

$$v = \exp(\tilde{\sigma}),$$
$$v_0 = \tilde{\rho}\exp(\tilde{\sigma}).$$

Diagonal reductions appear when $\rho_{ij} = \rho_i\delta_{ij}$ and $\tilde{\rho}_{ij} = \tilde{\rho}_i\delta_{ij}$ then

$$R_i = \frac{p\rho_i}{p-\pi_i}, \quad \tilde{R}_i = \frac{p^2\tilde{\rho}_i}{p-\tilde{\pi}_i},$$

and (36a) and (36a) become

$$\frac{\partial \pi_i}{\partial U_j} - = \frac{\rho_j}{\pi_i - \pi_j}\pi_i, \tag{41a}$$
$$\frac{\partial \rho_i}{\partial U_j} = -\frac{\rho_i\rho_j}{(\pi_i-\pi_j)^2}(\pi_j+\pi_i). \tag{41b}$$

while (33a) and (33b) read

$$\frac{\partial \tilde{\pi}_i}{\partial U_j} - = \frac{\tilde{\rho}_j}{\tilde{\pi}_i - \tilde{\pi}_j}\tilde{\pi}_i^2, \tag{42a}$$
$$\frac{\partial \tilde{\rho}_i}{\partial U_j} = -2\frac{\tilde{\rho}_i\tilde{\rho}_j}{(\tilde{\pi}_i-\tilde{\pi}_j)^2}\tilde{\pi}_j\tilde{\pi}_i. \tag{42b}$$

4.2 Reductions for the r-dmKP hierarchy

Here we shall consider the reduction (26) for R_i as in (29). We also consider functions $s_< \in \mathfrak{g}_<$ satisfying

$$\frac{\partial s_<}{\partial p}R_i + \frac{\partial s_<}{\partial U_i} = p^{1-r}\sum_{j=1}^{N}\frac{\rho_{ij}f_j}{p-p_j} \tag{43}$$

where we suppose that the compatibility conditions for (43)

$$\rho_{il}\frac{\partial f_l}{\partial U_j} - \rho_{jl}\frac{\partial f_l}{\partial U_i} = -\sum_{k\neq l}(\rho_{ik}\rho_{jl} - \rho_{il}\rho_{jk})\Big(r + \frac{\pi_k}{\pi_l - \pi_k}\Big)\frac{f_l - f_k}{\pi_l - \pi_k}$$

hold. If we deal with a diagonal reduction of the type $\rho_{ij} = \rho_i\delta_{ij}$ the above compatibility conditions become

$$\frac{\partial f_i}{\partial U_j} = \frac{\rho_j}{\pi_i - \pi_j}\Big(r + \frac{\pi_j}{\pi_i - \pi_j}\Big)(f_i - f_j). \tag{44}$$

Then, we have

Proposition 8. *Let $s_<(p,\boldsymbol{U})\in\mathfrak{g}_<$ be a function satisfying (43) and define*

$$s_\geq(p,\boldsymbol{U},\boldsymbol{t}):=\sum_{n=1}^\infty t_n\Omega_n(p,\boldsymbol{U})\in\mathfrak{g}_\geq,$$
$$s(p,\boldsymbol{U},x,\boldsymbol{t}):=s_\geq(p,\boldsymbol{U},\boldsymbol{t})+\Pi_r(p)x+s_<(p,\boldsymbol{U}).$$

Suppose that $\boldsymbol{U}=\boldsymbol{U}(x,\boldsymbol{t})$ is determined by the following s hodograph system

$$\sum_{n=1}^\infty t_n\frac{\partial\Omega_n}{\partial p}(\pi_i(\boldsymbol{U}),\boldsymbol{U})+x\pi_i(\boldsymbol{U})^{-r}+\pi_i(\boldsymbol{U})^{-r}f_i(\boldsymbol{U})=0,\quad i=1,\dots,N. \tag{45}$$

Then,

$$S(L,x,\boldsymbol{t}):=s(p(L,\boldsymbol{U}),\boldsymbol{U},x,\boldsymbol{t})$$

is an S-function.

Proof. As we have

$$\frac{\partial S}{\partial t_n}=\omega_n(L,\boldsymbol{U})+\sum_{i=1}^N\frac{\partial s(p(L,\boldsymbol{U}),\boldsymbol{U},x,\boldsymbol{t})}{\partial U_i}\Big|_{\boldsymbol{U}=\boldsymbol{U}(\boldsymbol{t})}\frac{\partial U_i}{\partial t_n},$$
$$\frac{\partial S}{\partial x}=\Pi_r+\sum_{i=1}^N\frac{\partial s(p(L,\boldsymbol{U}),\boldsymbol{U},x,\boldsymbol{t})}{\partial U_i}\Big|_{\boldsymbol{U}=\boldsymbol{U}(\boldsymbol{t})}\frac{\partial U_i}{\partial x},$$

the stated result follows from the identity

$$\frac{\partial s(p(L,\boldsymbol{U}),\boldsymbol{U},x,\boldsymbol{t})}{\partial U_i}\Big|_{\boldsymbol{U}=\boldsymbol{U}(\boldsymbol{t})}=0,$$

that we shall show to hold. For this aim we first observe that

$$\frac{\partial s(p(L,\boldsymbol{U}),\boldsymbol{U},x,\boldsymbol{t})}{\partial U_i}=\frac{\partial s}{\partial p}R_i+\frac{\partial s}{\partial U_i}. \tag{46}$$

Then, multiplying (43) by $\prod_{l=1}^N(p-p_l)$, recalling that $s_<$ is regular at $p=\pi_i$, and taking the limit $p\to\pi_i$ we get

$$\frac{\partial s_<}{\partial p}\Big|_{p=\pi_i}=\pi_i^{-r}f_i,$$

which together with (45) implies

$$\frac{\partial s}{\partial p}\Big|_{p=\pi_i}=0. \tag{47}$$

Now, observing

$$\frac{\partial s(p(L,\boldsymbol{U}),\boldsymbol{U},x,\boldsymbol{t})}{\partial U_i}=\sum_{n=1}^\infty t_n\frac{\partial\omega_n(L,\boldsymbol{U})}{\partial U_i}+xp^{-r}R_i+\frac{\partial s_<}{\partial U_i}$$

and recalling

$$\omega_n(L,\boldsymbol{U}) := \Omega_n(p(L,\boldsymbol{U}),\boldsymbol{U}) = L^{n+1-r} - P_<(L^{n+1-r}) \Rightarrow \frac{\partial\omega_n(L,\boldsymbol{U})}{\partial U_i} \in \mathfrak{g}_<,$$
$$R_i = O(1), \quad p\to\infty \Rightarrow p^{-r}R_i \in \mathfrak{g}_<,$$

we conclude

$$\frac{\partial s(p(L,\boldsymbol{U}),\boldsymbol{U},\boldsymbol{t})}{\partial U_i} \in \mathfrak{g}_<,$$

that, when applied to equation (46), gives

$$\frac{\partial s(p(L,\boldsymbol{U}),\boldsymbol{U},x,\boldsymbol{t})}{\partial U_i} = P_<\Big(\frac{\partial s}{\partial p}R_i\Big) + \frac{\partial s_<}{\partial U_i}. \tag{48}$$

Let us introduce a function $E = E(p,\boldsymbol{U}) \in \mathfrak{g}_\geq$ such that

$$\frac{\partial E}{\partial p}(\pi_i,\boldsymbol{U}) = \pi_i^{-r} f_i(\boldsymbol{U}), \tag{49}$$

for example we may take

$$\frac{\partial E}{\partial p} = p^{-r}\sum_{i=1}^{N} f_i \prod_{j\neq i}\frac{p-\pi_j}{\pi_i-\pi_j}.$$

Then, denoting $\hat{s}_\geq := s_\geq + \Pi_r x$,

$$P_<\Big(\frac{\partial s}{\partial p}R_i\Big) = P_<\Big(\frac{\partial(\hat{s}_\geq + E)}{\partial p}R_i\Big) + P_<\Big(\frac{\partial(s_< - E)}{\partial p}R_i\Big).$$

In the one hand we notice that from (47) and (45) we have

$$\frac{\partial(\hat{s}_\geq + E)}{\partial p} = p^{-r}\Big(\prod_{i=1}^{N}(p-p_i)\Big)(\alpha_0+\alpha_1 p+\cdots)$$

and hence

$$\frac{\partial(\hat{s}_\geq + E)}{\partial p}R_i = p^{1-r}(\alpha_0+\alpha_1 p+\cdots)\sum_{j=1}^{N}\rho_{ij}\Big(\prod_{\substack{l=1,\dots,N\\ l\neq j}}(p-p_l)\Big) \in \mathfrak{g}_\geq,$$

so that

$$P_<\Big(\frac{\partial(\hat{s}_\geq + E)}{\partial p}R_i\Big) = 0.$$

In the other hand, we formula

$$P_<\Big(\frac{\partial s_<}{\partial p}R_i\Big) = \frac{\partial s_<}{\partial p}R_i$$

which follows from $R_i = O(1)$ when $p\to\infty$ and $\dfrac{\partial s_<}{\partial p} \in \mathfrak{g}_<$, and we have the relation

$$P_<\Big(\frac{\partial E}{\partial p}R_i\Big) = p^{1-r}\sum_{j=1}^{N}\frac{\rho_{ij}f_j}{p-p_j}.$$

Therefore,

$$P_<\Big(\frac{\partial(s_< - E)}{\partial p}R_i\Big) = \frac{\partial s_<}{\partial p}R_i - p^{1-r}\sum_{j=1}^N \frac{\rho_{ij} f_j}{p-p_j}.$$

Coming back to (48) we get

$$\frac{\partial s(p(L,\boldsymbol{U}),\boldsymbol{U},x,\boldsymbol{t})}{\partial U_i} = \frac{\partial s_<}{\partial p}R_i + \frac{\partial s_<}{\partial U_i} - p^{1-r}\sum_{j=1}^N \frac{\rho_{ij} f_j}{p-p_j}$$

which vanishes in virtue of (43). □

4.3 Hydrodynamic type systems and the r-dmKP hierarchy

Here we will briefly discuss how the reduction scheme derived from (26) is associated with hydrodynamic type systems. First, we remark that, assuming L to be regular at the points $p=\pi_i$, $i=1,\dots,N$, we have

$$\frac{\partial L}{\partial p}\Big|_{p=\pi_i} = 0,$$

so that (6) implies

$$\sum_{j=1}^N \ell_{ij}\frac{\partial U_j}{\partial t_n} = D_{in}\sum_{j=1}^N \ell_{ij}\frac{\partial U_j}{\partial x}$$

where

$$L_i := L\big|_{p=\pi_i}, \quad D_{in} := \pi_i^r\frac{\partial \Omega_n}{\partial p}\Big|_{p=\pi_i}, \quad \ell_{ij} := \frac{\partial L_i}{\partial U_j}.$$

Thus, if we define the following matrices

$$D_n := \mathrm{diag}(D_{1n},\dots,D_{Nn}), \quad \ell := (\ell_{ij}), \quad A_n := \ell^{-1}D_n\ell,$$

we have the following hydrodynamic type system

$$\frac{\partial \boldsymbol{U}}{\partial t_n} = A_n(\boldsymbol{U})\frac{\partial \boldsymbol{U}}{\partial x}.$$

Let us study in more detail the t_1-flow, as we know

$$\Omega_1 = p^{2-r} + (2-r)u_0 p^{1-r}$$

so that

$$\frac{\partial L}{\partial t_1} = (2-r)\Big(\big(p+(1-r)u_0\big)\frac{\partial L}{\partial x} - p\frac{\partial u_0}{\partial x}\frac{\partial L}{\partial p}\Big) \tag{50}$$

that implies

$$\frac{\partial u_n}{\partial t_1} = (2-r)\Big(\frac{\partial u_{n+1}}{\partial x} + (1-r)u_0\frac{\partial u_n}{\partial x} + nu_n\frac{\partial u_0}{\partial x}\Big), \quad n\geq 0,$$

which we may think of as an r-modified Benney moment equations —recall that in the dispersionless KP hierarchy the first non-trivial flow comprimiese precisely the Benney moment equations—.

If we define

$$\Delta_n := p^r \frac{\partial \Omega_n}{\partial p} \Rightarrow \Delta_1 = (2-r)(p + (1-r)u_0)$$

we may write

$$A_n = \Delta_n(\hat{A}_1), \quad \hat{A}_1 := \frac{A_1}{2-r} - (1-r)u_0$$

which is equivalent to the Kodama–Gibbons formula for the dispersionless KP equation [12]. The relevance of the r-modified Benney equations also appears in relation with the reduction (26). If we introduce the reduction in (50) we get

$$\sum_{j=1}^{N} \Big(\sum_{i=1}^{N} \frac{\partial L}{\partial U_i} (\hat{A}_{1,ij} - p\delta_{ij}) + p \frac{\partial u_0}{\partial U_j} \frac{\partial L}{\partial p} \Big) \frac{\partial U_j}{\partial x} = 0,$$

and assuming the linear independence of $\dfrac{\partial U_j}{\partial x}$, $j = 1, \dots, N$, we have

$$R_i = p \sum_{j=1}^{N} (\hat{A}_1 - p)^{-1}_{ji} \frac{\partial u_0}{\partial U_j}.$$

4.4 Examples of hodograph solutions of the r-dmKP hierarchy

We analyze here some solutions derived, from Proposition 8, of the r-dmKP system (7) — which implies the r-dmKP equation (8)—. As we are interested in (7) we shall set $t_n = 0$ for $n = 3, 4, \dots$. For the Ω's we have

$$\begin{aligned}
\Omega_1 &= p^{2-r} + (2-r)u_0 p^{1-r}, \\
\Omega_2 &= p^{3-r} + (3-r)u_0 p^{2-r} + (3-r)\Big(u_1 + \frac{2-r}{2} u_0^2\Big) p^{1-r}.
\end{aligned}$$

4.4.1 1-component reduction

For $N = 1$ there are no compatibility conditions to fulfill and therefore we may take

$$R_1 = -\frac{p}{p - \pi_1(U)}, \quad U = U_1.$$

Equations (31) are

$$\frac{\partial u_0}{\partial U} = 1, \quad \frac{\partial u_1}{\partial U} = \pi_1, \quad \frac{\partial u_2}{\partial U} = \pi_1^2 - u_1,$$

so that we may take

$$u_0 = U, \quad u_1 = \int^U \pi_1(s) \mathrm{d}s, \quad u_2 = \int^U \pi_1(s)^2 \mathrm{d}s - \int^U (\int^s \pi_1(s') \mathrm{d}s') \mathrm{d}s. \tag{51}$$

In this situation $\hat{s}_{\geq}$ is

$$\hat{s}_{\geq} = \Big(p^{3-r} + (3-r)u_0 p^{2-r} + (3-r)\Big(u_1 + \frac{2-r}{2} u_0^2\Big) p^{1-r}\Big) t_2 + (p^{2-r} + (2-r)u_0 p^{1-r}) t_1 + \Pi_r x.$$

Therefore, the hodograph condition (45) is

$$(3-r)\Big(\pi_1^2+(2-r)u_0\pi_1+(1-r)\Big(u_1+\frac{2-r}{2}u_0^2\Big)\Big)t_2+(2-r)(\pi_1+(1-r)u_0)t_1+x+f(U)=0$$

where f is an arbitrary function. If we now introduce formulae (51) we get the following

Proposition 9. *Given two arbitrary functions $\pi_1(U), f(U)$, and a function $U(x,t_1,t_2)$ determined by the hodograph equation*

$$(3-r)\Big(\pi_1(U)^2+(2-r)U\pi_1(U)+(1-r)\Big(\int^U\pi_1(s)\mathrm{d}s+\frac{2-r}{2}U^2\Big)\Big)t_2 \\ +(2-r)(\pi_1(U)+(1-r)U)t_1+x+f(U)=0. \quad (52)$$

then

$$u_0=U,\quad u_1=\int^U\pi_1(s)\mathrm{d}s,\quad u_2=\int^U\pi_1(s)^2\mathrm{d}s-\int^U\int^s\pi_1(s')\mathrm{d}s'\mathrm{d}s. \quad (53)$$

solves the r-dmKP system (7).

While this is a consequence of Proposition 8 the following direct proof is available.

Proof. If we request to u_0,u_1 and u_2 as in (51) to solve the system of PDE's (7) then $U(x,t_1,t_2)$ satisfy

$$\begin{aligned}\frac{U_{t_1}}{U_x}&=(2-r)(\pi_1(U)+(1-r)U),\\ \frac{U_{t_2}}{U_x}&=(3-r)\Big(\pi_1(U)^2+(2-r)U\pi_1(U)+(1-r)\Big(\int^U\pi_1(s)\mathrm{d}s+\frac{2-r}{2}U^2\Big)\Big).\end{aligned} \quad (54)$$

But, taking x, t_1 and t_2 derivatives in (52) we get

$$\begin{aligned}(f'(U)+B'(U)t_2+A'(U)t_1)U_x+1=0,\\ (f'(U)+B'(U)t_2+A'(U)t_1)U_{t_1}+A(U)=0,\\ (f'(U)+B'(U)t_2+A'(U)t_1)U_{t_2}+B(U)=0,\end{aligned}$$

where

$$\begin{aligned}A&:=(2-r)(\pi_1(U)+(1-r)U),\\ B&:=(3-r)\bigg(\pi_1(U)^2+(2-r)U\pi_1(U)+(1-r)\Big(\int^U\pi_1(s)\mathrm{d}s+\frac{2-r}{2}U^2\Big)\bigg),\end{aligned}$$

which imply (54). Thus, solutions $U(x,t_1,t_2)$ of (52) provide us with solutions to (7). □

A simple solution appears with the choice $\pi_1=kU$, $f:=0$. Observe that we are dealing, as follows from (26), with the following reduction

$$L=\begin{cases}p\,\mathrm{e}^{Up^{-1}} & k=1,\\ p\,(1+(1-k)Up^{-1})^{\frac{1}{1-k}}, & k\neq 1.\end{cases}$$

If this is the case we get

$$A = \alpha U \quad B := \beta U^2$$

with

$$\alpha := (2-r)(k+1-r), \quad \beta := (3-r)\Big(k^2 + (5-3r)k + \frac{(1-r)(2-r)}{2}\Big)$$

and the corresponding solution, for $\beta \neq 0$, is

$$u_0 = U = -\frac{\alpha t_1}{2\beta t_2} \pm \sqrt{\frac{\alpha^2 t_1^2}{4\beta^2 t_2^2} - \frac{x}{\beta t_2}}. \tag{55}$$

There are two particular values for k

$$k = -(2-r), -\frac{1-r}{2}$$

such that

$$\beta = 0 \text{ and } \alpha = -(2-r) \text{ and } \frac{(2-r)(1-r)}{2}, \text{ respectively.}$$

In this case we get two simple t_2-invariant solutions

$$u_0 = \begin{cases} -\dfrac{x}{(2-r)t_1} & \text{for } k = r-2, \\ \dfrac{2x}{(1-r)(2-r)t_1} & \text{for } k = (r-1)/2. \end{cases}$$

For example, if in instead of $f = 0$ we set $f = U^3$ we get the solution

$$u_0 = U = g - \frac{3\alpha t_1 - \beta^2 t_2^2}{9g} - \frac{\beta t_2}{3}$$

where g is defined by

$$g := \frac{1}{6}\sqrt[3]{-108x + 36\alpha\beta t_1 t_2 - 8\beta^2 t_2^2 + 12\sqrt{81x^2 + 12\alpha^3 t_1^3 - 54\alpha\beta x t_1 t_2 + 12\beta^3 x t_2^3 - 3\alpha^2\beta^2 t_1^2 t_2^2}}$$

For $\beta = 0$ we get the following t_2-independent solution

$$u_0 = U = g - \frac{3\alpha t_1}{9g}, \text{ where } g := \frac{1}{6}\sqrt[3]{-108x + 12\sqrt{81x^2 + 12\alpha^3 t_1^3}} \text{ for } \alpha = -(2-r), \frac{(2-r)(1-r)}{2}.$$

4.4.2 2-component reduction

We shall work with the diagonal reduction given by (41a) and (41b). We may take the following solution for $N = 2$:

$$\pi_1 = -\pi_2 = \frac{U_1 - U_2}{4}, \quad \rho_1 = \rho_2 = -\frac{1}{2}.$$

The linear system (44) for f_1, f_2 becomes

$$\frac{\partial f_1}{\partial U_2} = \frac{\partial f_2}{\partial U_1} = -\frac{2r-1}{2(U_1 - U_2)}(f_1 - f_2), \tag{56}$$

which is equivalent to

$$f_1=\frac{\partial\Phi}{\partial U_1},\quad f_2=\frac{\partial\Phi}{\partial U_2},$$

with

$$\frac{\partial^2\Phi}{\partial U_1\partial U_2}+\frac{2r-1}{2(U_1-U_2)}\Big(\frac{\partial\Phi}{\partial U_1}-\frac{\partial\Phi}{\partial U_2}\Big)=0.$$

The method of separation of variables leads to the following solutions, expressed in terms of the Bessel and Neumann functions J_{-r},N_{-r} (the Neumann function is also known as the Weber function Y_{-r}), for a similar result for the dKP hierarchy see [18]:

$$\Phi=(U_1-U_2)^r\big(AJ_{-r}(k(U_1-U_2))+BN_{-r}(k(U_1-U_2))\big)\big(C\cos(k(U_1+U_2))+D\sin(k(U_1+U_2))\big)$$

and also

$$\Phi=\begin{cases}(A+B(U_1-U_2)^{2r})(C+D(U_1+U_2)), & r\neq 0,\\ (A+B\log(U_1-U_2))(C+D(U_1+U_2)) & r=0.\end{cases}$$

From

$$\begin{aligned}\frac{\partial u_0}{\partial U_i}&=\frac{1}{2},\\ \frac{\partial u_1}{\partial U_i}&=\frac{1}{2}\pi_i,\\ \frac{\partial u_2}{\partial U_i}&=\frac{1}{2}(\pi_i^2-u_1),\end{aligned}$$

for $i=1,2$, we get

$$u_0=\frac{U_1+U_2}{2},\quad u_1=\frac{(U_1-U_2)^2}{16},\quad u_2=0.$$

From the formulae

$$\pi_i^2+(2-r)u_0\pi_i+(1-r)\Big(u_1+\frac{2-r}{2}u_0^2\Big)=\begin{cases}\frac{2-r}{2}\Big((1-r)U_+^2+\frac{1}{2}U_-^2+U_+U_-\Big), & i=1,\\ \frac{2-r}{2}\Big((1-r)U_+^2+\frac{1}{2}U_-^2-U_+U_-\Big), & i=2,\end{cases}$$

$$\pi_i+(1-r)u_0=\begin{cases}(1-r)U_++\frac{1}{2}U_-, & i=1,\\ (1-r)U_+-\frac{1}{2}U_-, & i=2,\end{cases}$$

where

$$U_\pm:=\frac{U_1\pm U_2}{2}$$

we deduce the following hodograph system

$$\begin{aligned}\frac{(3-r)(2-r)}{2}\Big((1-r)U_+^2+\frac{1}{2}U_-^2+U_+U_-\Big)t_2+(2-r)\Big((1-r)U_++\frac{1}{2}U_-\Big)t_1+x=f_1,\\ \frac{(3-r)(2-r)}{2}\Big((1-r)U_+^2+\frac{1}{2}U_-^2-U_+U_-\Big)t_2+(2-r)\Big((1-r)U_+-\frac{1}{2}U_-\Big)t_1+x=f_2.\end{aligned}\tag{57}$$

Adding and subtracting the equations of (57) we obtain the equivalent system

$$\begin{gathered}\frac{(3-r)(2-r)}{2}\Big((1-r)U_+^2+\frac{U_-^2}{2}\Big)t_2+(2-r)(1-r)U_+t_1+x=\frac{f_1+f_2}{2},\\ (3-r)(2-r)U_+U_-t_2+(2-r)U_-t_1=f_1-f_2.\end{gathered}\tag{58}$$

The simplest solution of (56) is $f_1=f_2=0$; i.e., $\Phi=0$, in this case there are two possible choices

$$U_-=0 \text{ or } U_+=-\frac{1}{3-r}\frac{t_1}{t_2}.$$

The first choice imply $u_1=U_-^2/4=0$ and $u_0=U_+$ to solve the algebraic equation

$$\frac{(3-r)(2-r)(1-r)}{2}t_2U_+^2+(2-r)(1-r)t_1U_++x=0$$

whose solutions are

$$u_0=U_+=-\frac{t_1}{(3-r)t_2}\pm\sqrt{\frac{t_1^2}{(3-r)^2t_2^2}-\frac{2x}{(3-r)(2-r)(1-r)t_2}}.$$

In fact, for $u_1=u_2=0$ the system (7) becomes

$$\begin{aligned}u_{0,t_1}&=(2-r)(1-r)u_0u_{0,x},\\ u_{0,t_2}&=\frac{1}{2}(3-r)(2-r)(1-r)u_0^2u_{0,x};\end{aligned}$$

i.e., the dispersionless KdV (t_1-flow) and dispersionless modified KdV (t_2-flow) equations, and $u_0=U_+$ is a solution of both simultaneously. This solution is a particular case of the already studied 1-component solution (55); i.e $\pi_1=kU$, for $k=0$ but this value of k is not allowed as we assume that $\pi_1\neq 0$, therefore the two-component case, with $\pi_1=-\pi_2=(U_1-U_2)/4$, provides us with a rigorous alternative track to this solution.

The second choice

$$u_0=U_+=-\frac{1}{3-r}\frac{t_1}{t_2}$$

imply

$$u_1=\frac{U_-^2}{4}=\frac{(1-r)t_1^2}{2(3-r)^2}\frac{t_1^2}{t_2^2}-\frac{1}{(3-r)(2-r)}\frac{x}{t_2}.$$

It is easy to check that this pair u_0,u_1 fulfills the $u_2=0$ reduction of (7), namely

$$\begin{aligned}u_{0,t_1}&=(2-r)u_{1,x}+(2-r)(1-r)u_0u_{0,x},\\ u_{1,t_1}&=(2-r)(1-r)u_0u_{1,x}+(2-r)u_1u_{0,x},\\ u_{0,t_2}&=(3-r)(2-r)(u_0u_{1,x}+u_{0,x}u_1)+\frac{1}{2}(3-r)(2-r)(1-r)u_0^2u_{0,x}.\end{aligned}$$

A more general choice is to take $\Phi=\Phi_+(U_+)\Phi_-(U_-)$, then

$$\begin{aligned}f_1&=\frac{1}{2}(\Phi_+'(U_+)\Phi_-(U_-)+\Phi_+(U_+)\Phi_-'(U_-)),\\ f_2&=\frac{1}{2}(\Phi_+'(U_+)\Phi_-(U_-)-\Phi_+(U_+)\Phi_-'(U_-)),\end{aligned}$$

and, if we assume $\Phi_-' \neq 0$, the hodograph system (58) reads

$$\begin{gathered}\frac{(3-r)(2-r)}{2}\Big((1-r)U_+^2 + \frac{U_-^2}{2}\Big)t_2 + (2-r)(1-r)U_+t_1 + x = \frac{\Phi_+'(U_+)\Phi_-(U_-)}{4},\\ U_-((3-r)(2-r)U_+t_2 + (2-r)t_1) = \frac{\Phi_+(U_+)\Phi_-'(U_-)}{2}.\end{gathered} \tag{59}$$

Picking, for example, $\Phi_- = U_-^{2r}$, $\Phi_+ = A + BU_+$ we have for $u_1 = U_-^2/4$

$$u_1 = \frac{1}{4}U_-^2 = \frac{1}{4}\Big(\frac{r(A+BU_+)}{(2-r)t_1 + (3-r)(2-r)t_2U_+}\Big)^{\frac{1}{1-r}}$$

and $u_0 = U_+$ is implicitly determined by

$$\frac{(3-r)(2-r)(1-r)}{2}t_2U_+^2 + (2-r)(1-r)t_1U_+ + x = \frac{B}{4}(4u_1)^r - (3-r)(2-r)t_2u_1.$$

4.5 Reductions for the r-dDym hierarchy

Here we tackle the reduction (27) with $\tilde{R}_i$ as in (30) for the r-dDym hierarchy. Let us introduce $s_{\leq} \in \mathfrak{g}_{\leq}$ satisfying

$$\frac{\partial \tilde{s}_{\leq}}{\partial p}\tilde{R}_i + \frac{\partial \tilde{s}_{\leq}}{\partial U_i} = p^{2-r}\sum_{j=1}^{N}\frac{\tilde{\rho}_{ij}\tilde{f}_j}{p - \tilde{\pi}_j}. \tag{60}$$

Here we assume the compatibility conditions for (60)

$$\tilde{\rho}_{il}\frac{\partial \tilde{f}_l}{\partial U_j} - \tilde{\rho}_{jl}\frac{\partial \tilde{f}_l}{\partial U_i} = -\sum_{k\neq l}(\tilde{\rho}_{ik}\tilde{\rho}_{jl} - \tilde{\rho}_{il}\tilde{\rho}_{jk})\Big(-(1-r) + \frac{\tilde{\pi}_k}{\tilde{\pi}_l - \tilde{\pi}_k}\Big)\tilde{\pi}_l\frac{\tilde{f}_l - \tilde{f}_k}{\tilde{\pi}_l - \tilde{\pi}_k},$$

that for the diagonal reduction $\tilde{\rho}_{ij} = \tilde{\rho}_i\delta_{ij}$ are

$$\frac{\partial \tilde{f}_i}{\partial U_j} = \Big(-(1-r) + \frac{\tilde{\pi}_j}{\tilde{\pi}_i - \tilde{\pi}_j}\Big)\frac{\tilde{\rho}_j\tilde{\pi}_i}{\tilde{\pi}_i - \tilde{\pi}_j}(\tilde{f}_i - \tilde{f}_j). \tag{61}$$

We have

Proposition 10. *Let $\tilde{s}_{\leq}(p, \boldsymbol{U}) \in \mathfrak{g}_{\leq}$ satisfying (60) and define*

$$\begin{gathered}\tilde{s}_{>}(p, \boldsymbol{U}, \boldsymbol{t}) := \sum_{n=1}^{\infty} t_n\tilde{\Omega}_n(p, \boldsymbol{U}),\\ \tilde{s}(p, \boldsymbol{U}, x, \boldsymbol{t}) := \tilde{s}_{>}(p, \boldsymbol{U}, \boldsymbol{t}) + \Pi_r(p)\,x + \tilde{s}_{\leq}(p, \boldsymbol{U}).\end{gathered}$$

Suppose that $\boldsymbol{U} = \boldsymbol{U}(x, \boldsymbol{t})$ is determined by the following hodograph system

$$\sum_{n=1}^{\infty} t_n\frac{\partial \tilde{\Omega}_n}{\partial p}(\tilde{\pi}_i(\boldsymbol{U}), \boldsymbol{U}) + x\tilde{\pi}_i(\boldsymbol{U})^{-r} + \tilde{\pi}_i(\boldsymbol{U})^{-r}\tilde{f}_i(\boldsymbol{U}) = 0, \quad i = 1, \dots, N. \tag{62}$$

Then,

$$\tilde{S}(\tilde{L}, x, \boldsymbol{t}) := \tilde{s}(p(\tilde{L}, \boldsymbol{U}), \boldsymbol{U}, x, \boldsymbol{t})$$

is an S-function for the r-dDym hierarchy.

Proof. The lines of this proof are almost identical to those followed in Proposition 8 for the r-dmKP hierarchy. First we observe that

$$\frac{\partial \tilde{S}}{\partial t_n} = \tilde{\omega}_n(\tilde{L},\boldsymbol{U}) + \sum_{i=1}^{N} \frac{\partial \tilde{s}(p(\tilde{L},\boldsymbol{U}),\boldsymbol{U},x,\boldsymbol{t})}{\partial U_i}\bigg|_{\boldsymbol{U}=\boldsymbol{U}(\boldsymbol{t})} \frac{\partial U_i}{\partial t_n},$$

$$\frac{\partial \tilde{S}}{\partial x} = \Pi_r + \sum_{i=1}^{N} \frac{\partial \tilde{s}(p(\tilde{L},\boldsymbol{U}),\boldsymbol{U},x,\boldsymbol{t})}{\partial U_i}\bigg|_{\boldsymbol{U}=\boldsymbol{U}(\boldsymbol{t})} \frac{\partial U_i}{\partial x},$$

and our result will follow if the formula

$$\frac{\partial \tilde{s}(p(\tilde{L},\boldsymbol{U}),\boldsymbol{U},x,\boldsymbol{t})}{\partial U_i}\bigg|_{\boldsymbol{U}=\boldsymbol{U}(\boldsymbol{t})} = 0$$

is satisfied. We will proceed observing that

$$\frac{\partial \tilde{s}(p(\tilde{L},\boldsymbol{U}),\boldsymbol{U},x,\boldsymbol{t})}{\partial U_i} = \frac{\partial \tilde{s}}{\partial p}\tilde{R}_i + \frac{\partial \tilde{s}}{\partial U_i}. \tag{63}$$

Let us multiply (60) by $\prod_{l=1}^{N}(p-\tilde{\pi}_l)$, also recall that $\tilde{s}_{\leq}$ is regular at $p=\tilde{\pi}_i$, and then take the limit $p\to\tilde{\pi}_i$ in order to get

$$\frac{\partial \tilde{s}_{\leq}}{\partial p}\bigg|_{p=\tilde{\pi}_i} = \tilde{\pi}_i^{-r}\tilde{f}_i,$$

and from (62) we obtain

$$\frac{\partial \tilde{s}}{\partial p}\bigg|_{p=\tilde{\pi}_i} = 0. \tag{64}$$

We notice that

$$\frac{\partial \tilde{s}(p(\tilde{L},\boldsymbol{U}),\boldsymbol{U},x,\boldsymbol{t})}{\partial U_i} = \sum_{n=1}^{\infty} t_n \frac{\partial \tilde{\omega}_n(\tilde{L},\boldsymbol{U})}{\partial U_i} + xp(\tilde{L},\boldsymbol{U})^{-r}\frac{\partial \tilde{p}(\tilde{L},\boldsymbol{U})}{\partial U_i} + \frac{\partial \tilde{s}_{\leq}}{\partial U_i},$$

and also that

$$\tilde{\omega}_n(\tilde{L},\boldsymbol{U}) := \tilde{\Omega}_n(p(\tilde{L},\boldsymbol{U}),\boldsymbol{U}) = \tilde{L}^{n+1-r} - P_{\leq}(\tilde{L}^{n+1-r}) \Rightarrow \frac{\partial \tilde{\omega}_n(\tilde{L},\boldsymbol{U})}{\partial U_i} \in \mathfrak{g}_{\leq},$$

$$\text{and if } r\neq 1 \; p(\tilde{L},\boldsymbol{U})^{1-r} \in \mathfrak{g}_{\leq} \Rightarrow p(\tilde{L},\boldsymbol{U})^{-r}\frac{\partial p(\tilde{L},\boldsymbol{U})}{\partial U_i} \in \mathfrak{g}_{\leq},$$

$$\text{finally, for } r=1,\; \frac{\partial p}{\partial U_i} = O(p) \text{ when } p\to\infty \;\Rightarrow p^{-1}\frac{\partial p}{\partial U_i} \in \mathfrak{g}_{\leq}.$$

Therefore, we can write

$$\frac{\partial \tilde{s}(p(\tilde{L},\boldsymbol{U}),\boldsymbol{U},\boldsymbol{t})}{\partial U_i} \in \mathfrak{g}_{\leq}$$

and from (63) we deduce that

$$\frac{\partial \tilde{s}(p(\tilde{L},\boldsymbol{U}),\boldsymbol{U},x,\boldsymbol{t})}{\partial U_i} = P_{\leq}\Big(\frac{\partial \tilde{s}}{\partial p}\tilde{R}_i\Big) + \frac{\partial \tilde{s}_{\leq}}{\partial U_i}. \tag{65}$$

Following the r-dmKP case we now introduce a function $\tilde{E}=\tilde{E}(p,\boldsymbol{U})\in\mathfrak{g}_{\geq}$ such that

$$\frac{\partial\tilde{E}}{\partial p}(\tilde{\pi}_i,\boldsymbol{U})=\tilde{\pi}_i^{-r}f_i(\boldsymbol{U}), \tag{66}$$

for example we may take

$$\frac{\partial\tilde{E}}{\partial p}=p^{-r}\sum_{i=1}^{N}\tilde{f}_i\prod_{j\neq i}\frac{p-\tilde{\pi}_j}{\tilde{\pi}_i-\tilde{\pi}_j}.$$

Let us study the following equation

$$P_{\leq}\Big(\frac{\partial\tilde{s}}{\partial p}\tilde{R}_i\Big)=P_{\leq}\Big(\frac{\partial(\tilde{s}_{>}+x\Pi_r+\tilde{E})}{\partial p}\tilde{R}_i\Big)+P_{\leq}\Big(\frac{\partial(\tilde{s}_{\leq}-\tilde{E})}{\partial p}\tilde{R}_i\Big).$$

For the first term in the r.h.s we see that, as follows that from (64),

$$\frac{\partial(\tilde{s}_{>}+x\Pi_r+\tilde{E})}{\partial p}=p^{-r}\Big(\prod_{i=1}^{N}(p-\tilde{\pi}_i)\Big)(\alpha_0+\alpha_1p+\cdots),$$

and hence

$$\frac{\partial(\tilde{s}_{>}+x\Pi_r+\tilde{E})}{\partial p}\tilde{R}_i=p^{2-r}(\alpha_0+\alpha_1p+\cdots)\sum_{j=1}^{N}\tilde{\rho}_{ij}\Big(\prod_{\substack{l=1,\dots,N\\ l\neq j}}(p-\tilde{\pi}_l)\Big)\in\mathfrak{g}_{>},$$

so that

$$P_{\leq}\Big(\frac{\partial(\tilde{s}_{>}+x\Pi_r+E)}{\partial p}\tilde{R}_i\Big)=0.$$

We also deduce that

$$P_{\leq}\Big(\frac{\partial\tilde{s}_{\leq}}{\partial p}\tilde{R}_i\Big)=\frac{\partial\tilde{s}_{\leq}}{\partial p}\tilde{R}_i,$$

which is a direct consequence of $\tilde{R}_i=O(p)$ when $p\to\infty$ and $\dfrac{\partial\tilde{s}_{\leq}}{\partial p}\in\mathfrak{g}_{\leq}$. Another relevant formula is

$$P_{\leq}\Big(\frac{\partial\tilde{E}}{\partial p}\tilde{R}_i\Big)=p^{2-r}\sum_{j=1}^{N}\frac{\tilde{\rho}_{ij}\tilde{f}_j}{p-\tilde{\pi}_j}.$$

With all these formula at hand we deduce

$$P_{\leq}\Big(\frac{\partial(\tilde{s}_{\leq}-\tilde{E})}{\partial p}\tilde{R}_i\Big)=\frac{\partial\tilde{s}_{\leq}}{\partial p}\tilde{R}_i-p^{2-r}\sum_{j=1}^{N}\frac{\tilde{\rho}_{ij}\tilde{f}_j}{p-\tilde{\pi}_j}.$$

Coming back to (65) we get

$$\frac{\partial\tilde{s}(p(\tilde{L},\boldsymbol{U}),\boldsymbol{U},x,\boldsymbol{t})}{\partial U_i}=\frac{\partial\tilde{s}_{\leq}}{\partial p}\tilde{R}_i+\frac{\partial\tilde{s}_{\leq}}{\partial U_i}-p^{1-r}\sum_{j=1}^{N}\frac{\tilde{\rho}_{ij}\tilde{f}_j}{p-\tilde{\pi}_j}$$

which vanishes in virtue of (60). □

4.6 Hydrodynamic type systems and the r-dDym hierarchy

Here we proceed as in §4.3. With the use of (12), together with $\left.\frac{\partial \tilde{L}}{\partial p}\right|_{p=\tilde{\pi}_i} = 0$, and of the notation

$$\tilde{L}_i := \tilde{L}\big|_{p=\tilde{\pi}_i}, \quad \tilde{D}_{in} := \tilde{\pi}_i^r \left.\frac{\partial \tilde{\Omega}_n}{\partial p}\right|_{p=\tilde{\pi}_i}, \quad \tilde{\ell}_{ij} := \frac{\partial \tilde{L}_i}{\partial U_j},$$
$$\tilde{D}_n := \operatorname{diag}(\tilde{D}_{1n}, \dots, \tilde{D}_{Nn}), \quad \tilde{\ell} := (\tilde{\ell}_{ij}), \quad \tilde{A}_n := \tilde{\ell}^{-1}\tilde{D}_n\tilde{\ell}.$$

we derive the following hydrodynamic type system

$$\frac{\partial \boldsymbol{U}}{\partial t_n} = \tilde{A}_n(\boldsymbol{U})\frac{\partial \boldsymbol{U}}{\partial x}.$$

To analyze the t_1-flow, we recall that

$$\tilde{\Omega}_1 = v^{2-r}p^{2-r}$$

and therefore

$$\frac{\partial \tilde{L}}{\partial t_1} = (2-r)v^{1-r}p\Big(v\frac{\partial \tilde{L}}{\partial x} - p\frac{\partial v}{\partial x}\frac{\partial \tilde{L}}{\partial p}\Big) \tag{67}$$

that implies the following Benney moment type equations

$$\frac{\partial v_n}{\partial t_1} = (2-r)v^{1-r}\Big(v\frac{\partial v_{n+1}}{\partial x} + (n+1)\frac{\partial v}{\partial x}v_{n+1}\Big), \quad n \geq -1, \quad v_{-1} := v.$$

We now define

$$\tilde{\Delta}_n := p^r\frac{\partial \tilde{\Omega}_n}{\partial p} \Rightarrow \tilde{\Delta}_1 = (2-r)v^{2-r}p$$

so that the corresponding Kodama–Gibbons formula for the r-dDym hierarchy is

$$\tilde{A}_n = \tilde{\Delta}_n(\hat{\tilde{A}}_1), \quad \hat{\tilde{A}}_1 := \frac{\tilde{A}_1}{(2-r)v^{2-r}}.$$

If we introduce the reduction in (67) we get

$$\sum_{j=1}^N\left(\sum_{i=1}^N\frac{\partial \tilde{L}}{\partial U_i}(\hat{\tilde{A}}_{1,ij} - p\delta_{ij}) + p^2v\frac{\partial v}{\partial U_j}\frac{\partial \tilde{L}}{\partial p}\right)\frac{\partial U_j}{\partial x} = 0,$$

and assuming the linear independence of $\frac{\partial U_j}{\partial x}$, $j = 1, \dots, N$, we have

$$\tilde{R}_i = p^2\sum_{j=1}^N(\hat{\tilde{A}}_1 - p)^{-1}_{ji}v\frac{\partial v}{\partial U_j}.$$

4.7 Examples of hodograph solutions of the r-dDym hierarchy

We now ready to present here some solutions of the r-dDym system (13), which implies the r-dDym equation (14). We set $t_n = 0$ for $n = 3, 4, \dots$ and recall that

$$\tilde{\Omega}_1 = v^{2-r} p^{2-r},$$
$$\tilde{\Omega}_2 = v^{3-r} p^{3-r} + (3-r) v_0 v^{2-r} p^{2-r}.$$

4.7.1 1-component reduction

We take

$$\tilde{R}_1 = -\frac{p^2}{p - \tilde{\pi}_1(U)}, \quad U = U_1.$$

From equations (32) we deduce the following equations

$$v_U = v, \quad v_{0,U} = \tilde{\pi}_1 v, \quad v_{1,U} = \tilde{\pi}_1^2 v - v_1, \tag{68}$$

which are solve by

$$v = \mathrm{e}^U, \quad v_0 = \int^U \tilde{\pi}_1(s)\,\mathrm{e}^s\,\mathrm{d}s, \quad v_1 = \mathrm{e}^{-U} \int^U \tilde{\pi}_1(s)^2\,\mathrm{e}^{2s}\,\mathrm{d}s. \tag{69}$$

In this situation $\tilde{s}_>$ is

$$\tilde{s}_> = \big(v^{3-r} p^{3-r} + (3-r) v_0 v^{2-r} p^{2-r}\big) t_2 + (v^{2-r} p^{2-r}) t_1.$$

Therefore, the hodograph condition (62) is

$$(3-r) v^{2-r} (v \tilde{\pi}_1^2 + (2-r) v_0 \tilde{\pi}_1) t_2 + (2-r) \tilde{\pi}_1 v^{2-r} t_1 + x + f(U) = 0$$

where f is an arbitrary function, which taking into account (69) leads to the following

Proposition 11. *Given arbitrary two functions $\tilde{\pi}_1(U), f(U)$ and a function $U(x, t_1, t_2)$ determined by*

$$(3-r)\,\mathrm{e}^{(2-r)U} \Big(\mathrm{e}^U\, \tilde{\pi}_1(U)^2 + (2-r) \tilde{\pi}_1(U) \int^U \tilde{\pi}_1(s)\,\mathrm{e}^s\,\mathrm{d}s\Big) t_2 + (2-r) \tilde{\pi}_1(U)\,\mathrm{e}^{(2-r)U}\, t_1 + x + f(U) = 0 \tag{70}$$

then

$$v = \mathrm{e}^U, \quad v_0 = \int^U \tilde{\pi}_1(s)\,\mathrm{e}^s\,\mathrm{d}s, \quad v_1 = \mathrm{e}^{-U} \int^U \tilde{\pi}_1(s)^2\,\mathrm{e}^{2s}\,\mathrm{d}s.$$

solves the r-dDym system (13).

This result is a corollary of Proposition 10, however we give a direct proof.

Proof. The imposition to v, v_0 and v_1, as in (68), to solve (13) is equivalent to request $U(x,t_1,t_2)$ to fulfill

$$\begin{aligned}\frac{U_{t_1}}{U_x} &= (2-r)v^{2-r}\tilde{\pi}_1,\\ \frac{U_{t_2}}{U_x} &= (3-r)v^{2-r}(\tilde{\pi}_1(U)^2 v + (2-r)\tilde{\pi}_1(U)v_0).\end{aligned} \tag{71}$$

But if we take x, t_1 and t_2 derivatives in (70) we get

$$\begin{aligned}(f'(U) + B'(U)t_2 + A'(U)t_1)U_x + 1 &= 0,\\ (f'(U) + B'(U)t_2 + A'(U)t_1)U_{t_1} + A(U) &= 0,\\ (f'(U) + B'(U)t_2 + A'(U)t_1)U_{t_2} + B(U) &= 0,\end{aligned}$$

where

$$\begin{aligned}A &:= (2-r)v^{2-r}\tilde{\pi}_1,\\ B &:= (3-r)v^{2-r}(\tilde{\pi}_1(U)^2 v + (2-r)\tilde{\pi}_1(U)v_0),\end{aligned}$$

which imply (71) and the desired result follows. □

Let us suppose that $\tilde{\pi}_1 = \mathrm{e}^{kU}$, then (70) is

$$\frac{(3-r)(3-r+k)}{1+k} t_2 \,\mathrm{e}^{(3-r+2k)U} + (2-r)t_1 \,\mathrm{e}^{(2-r+k)U} + x = f,$$

If we now impose

$$3 - r + 2k = n(2-r+k) \Rightarrow k = \frac{3-r-n(2-r)}{n-2}$$

we have

$$\frac{(3-r)(3-r+k)}{1+k} t_2 \alpha^n + (2-r)t_1\alpha + x = 0,$$

where

$$\alpha := \mathrm{e}^{(2-r+k)U}.$$

We now explore in more detail the following three examples

n	k	Hodograph equation	$v = v(\alpha)$
3	$-3+2r$	$-\frac{(3-r)r}{2(1-r)} t_2\alpha^3 + C\alpha^2 + (2-r)t_1\alpha + x = 0$	$\alpha^{-\frac{1}{1-r}}$
-1	$\frac{-5+2r}{3}$	$(2-r)t_1\alpha^2 + x\alpha - \frac{(3-r)(4-r)}{2(1-r)} t_2 = 0$	$\alpha^{\frac{3}{1-r}}$
-2	$\frac{-7+3r}{4}$	$(2-r)t_1\alpha^3 + x\alpha^2 + C\alpha - \frac{(3-r)(5-r)}{3(1-r)} t_2 = 0$	$\alpha^{\frac{4}{1-r}}$

where C is an arbitrary constant which appears by choosing f in an appropriate manner. The hodograph equations in this cases are explicitly solved, as we are dealing with cubic and a quadratic equations. Now, we present the corresponding solution the r-dDym equation (14) in the $n=-1$ case:

$$v=\left(-\frac{1}{2(2-r)}\frac{x}{t_1}\pm\sqrt{\frac{1}{4(2-r)^2}\frac{x^2}{t_1^2}+\frac{(3-r)(4-r)}{2(1-r)(2-r)}\frac{t_2}{t_1}}\right)^{\frac{3}{1-r}},$$

for $n=3$, $C=0$, the corresponding solution is

$$v=\left(-\frac{(1-r)g(x,t_1,t_2)}{(3-r)rt_2}-\frac{2(2-r)t_1}{g(x,t_1,t_2)}\right)^{-\frac{1}{1-r}}$$

with

$$g:=\sqrt[3]{\frac{(3-r)^2r^2}{4(1-r)^2}t_2^2\Big(-108x+12\sqrt{3}\sqrt{27x^2-8\frac{(1-r)(2-r)^3}{(3-r)r}\frac{t_1^3}{t_2}}\Big)},$$

and for $n=-2$, $C=0$

$$v=\left(\frac{g(x,t_1,t_2)}{6(2-r)t_1}+\frac{2x^2}{3(2-r)t_1g(x,t_1,t_2)}-\frac{x}{3(2-r)t_1}\right)^{\frac{4}{1-r}}$$

with

$$\begin{aligned}g:=\Bigg(&-8x^3+108\frac{(2-r)^2(3-r)(5-r)}{3(1-r)}t_1^2t_2\\&+12\sqrt{3}(2-r)t_1\sqrt{\frac{(3-r)(5-r)}{3(1-r)}t_2\Big(27\frac{(2-r)^2(3-r)(5-r)}{3(1-r)}t_1^2t_2-4x^3\Big)}\Bigg)^{\frac{1}{3}}.\end{aligned}$$

4.7.2 2-component reduction

For $N=2$ we take the solution of (41a) and (41b) given by

$$\tilde{\rho}_1=-\tilde{\rho}_2=-\frac{2}{U_1-U_2},\quad \tilde{\pi}_1=-\tilde{\pi}_2=\frac{1}{4(U_1-U_2)}.$$

We need to handle (72) which in this case reads

$$\frac{\partial\tilde{f}_1}{\partial U_2}=\frac{\partial\tilde{f}_2}{\partial U_1}=-\frac{3-2r}{2(U_1-U_2)}(\tilde{f}_1-\tilde{f}_2). \tag{72}$$

which is equivalent to

$$f_1=\frac{\partial\Phi}{\partial U_1},\quad f_2=\frac{\partial\Phi}{\partial U_2},$$

with

$$\frac{\partial^2\Phi}{\partial U_1\partial U_2}+\frac{2r-3}{2(U_1-U_2)}\Big(\frac{\partial\Phi}{\partial U_1}-\frac{\partial\Phi}{\partial U_2}\Big)=0.$$

The method of separation of variables leads to the following solutions:

$$\Phi = (U_1-U_2)^{r-1}\big(AJ_{1-r}(k(U_1-U_2))+BN_{1-r}(k(U_1-U_2))\big)\big(C\cos(k(U_1+U_2))+D\sin(k(U_1+U_2))\big)$$

and also

$$\Phi = \begin{cases} (A+B(U_1-U_2)^{2r-2})(C+D(U_1+U_2)), & r\neq 1,\\ (A+B\log(U_1-U_2))(C+D(U_1+U_2)) & r=1. \end{cases}$$

From (32) we derive

$$v=(U_1-U_2)^2,\quad v_0=\frac{U_1+U_2}{2},\quad v_1=\frac{1}{16}+\frac{1}{(U_1-U_2)^2},$$

observe that with this reduction we have $v_1=\frac{1}{16}+\frac{1}{v}$. The hodograph system (62) is

$$\begin{aligned} v^{2-r}\Big((3-r)\Big(\frac{1}{16}+\frac{2-r}{8}\frac{U_+}{U_-}\Big)t_2+\frac{2-r}{8}\frac{1}{U_-}t_1\Big)+x+f_1=0,\\ v^{2-r}\Big((3-r)\Big(\frac{1}{16}-\frac{2-r}{8}\frac{U_+}{U_-}\Big)t_2-\frac{2-r}{8}\frac{1}{U_-}t_1\Big)+x+f_2=0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \frac{3-r}{8}v^{2-r}t_2+2x+f_1+f_2=0,\\ \frac{2-r}{4}\frac{v^{2-r}}{U_-}((3-r)U_+t_2+t_1)+f_1-f_2=0. \end{aligned}$$

When $\Phi=0$ we get the solution

$$\begin{aligned} v&=\Big(-\frac{16}{3-r}\frac{x}{t_2}\Big)^{\frac{1}{2-r}},\\ v_0&=-\frac{1}{3-r}\frac{t_1}{t_2},\\ v_1&=\frac{1}{16}+\Big(-\frac{3-r}{16}\frac{t_2}{x}\Big)^{\frac{1}{2-r}}. \end{aligned}$$

Finally, if we set $\Phi=U_-^{2r-2}(A+BU_+)$ so that

$$\begin{aligned} f_1+f_2&=2BU_-^{2r-2},\\ f_1-f_2&=2(2r-2)U_-^{2r-3}(A+BU_+). \end{aligned}$$

and the hodograph system is

$$\begin{aligned} \frac{3-r}{8}v^{2-r}t_2+2x+2BU_-^{2r-2}=0,\\ v^{2-r}((3-r)U_+t_2+t_1)+\frac{8(2r-2)}{2-r}U_-^{2r-2}(A+BU_+)=0. \end{aligned}$$

Observe that the first equation determines U_- (or v), and introducing this information in the second we get U_+ (or v_0).

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