REAL-VALUED LIPSCHITZ FUNCTIONS AND METRIC PROPERTIES OF FUNCTIONS

GERALD BEER AND M. ISABEL GARRIDO

Abstract. The purpose of this article is to explore the very general phenomenon that a function between metric spaces has a particular metric property if and only if whenever it is followed in a composition by an arbitrary real-valued Lipschitz function, the composition has this property. The key tools we use are the Efremovič lemma [21] and a theorem of Garrido and Jaramillo [22] that says that a function h between metric spaces is Lipschitz if and only if whenever it is followed by a Lipschitz real-valued function in a composition, the composition is Lipschitz. We also present a streamlined proof of the Garrido-Jaramillo result itself, but one that still relies on their natural continuous linear operator from the Lipschitz space for the target space to the Lipschitz space for the domain. Separately, we include a highly applicable uniform closure theorem that yields the most important uniform density theorems for Lipschitz-type functions as special cases.

1. Introduction

Analysts have little interest in topological spaces that fail to satisfy the Hausdorff separation property. They would also prefer to work in the framework of completely regular Hausdorff spaces, also called Tychonoff spaces. For one thing, in this context, such spaces have Hausdorff compactifications; in fact a topological space can be embedded in a compact Hausdorff space if and only if it is Tychonoff. Similarly, only in this framework does one have a compatible separated diagonal uniformity, so that notions such as completeness and uniform continuity can be formulated. They subsume the locally convex topological vector spaces and the locally compact Hausdorff spaces. But perhaps most importantly, it is only in this setting that the real-valued continuous functions determine the topology.

Given a family of real-valued functions $\{f_i : i \in I\}$ defined on a nonempty set Y, the *initial topology* $\tau_{\{f_i : i\in I\}}$ on Y is the topology generated by the family of subsets $\{f_i^{-1}(V) : i \in I \text{ and } V \text{ is open in the real line}\}$. This is the coarsest topology one can place on Y such that each f_i is continuous. A function h from a second topological space into Y equipped with such an initial topology is continuous if and only if for each $i \in I$, $f_i \circ h$ is continuous [40, Theorem 8.10].

As an example, if $\langle Y, ||\cdot||\rangle$ is a real normed linear space, the initial topology on Y determined by its continuous dual is called the *weak topology* on Y. This will agree with the norm topology if and only if the space is finite dimensional.

Date: November 19, 2019.

2010 *Mathematics Subject Classification.* Primary 54E40, 26A16; Secondary 46E15, 54C30.

Key words and phrases. Lipschitz spaces, uniformly continuous function, Cauchy continuous function, locally Lipschitz function, Lipschitz in the small function.

The first author thanks Universidad Complutense for its hospitality in September 2019. The second author was partially supported by DGES grant PGC2018-097286-B-I00 (Spain).

Now given a topological space $\langle Y, \tau \rangle$ and a family of continuous real-valued functions $\{f_i : i \in I\}$ on Y, the initial topology determined by the family is in general coarser than τ. The two topologies agree provided the family separates points from closed sets, which means that whenever A is a τ-closed subset of Y and $p \in Y \backslash A$, there exists some f_i such that $f_i(p)$ lies outside of the closure of $f(A)$ [40, Corollary 8.15]. The standard definition of complete regularity [40, pp. 94-95] implies that the continuous functions from $\langle Y, \tau \rangle$ to $[0,1]$ separate points from closed sets, so any larger class of continuous functions will do so as well. In particular, a function h into a completely regular space $\langle Y, \tau \rangle$ will be continuous if and only if whenever f is continuous and real-valued, $f \circ h$ is continuous.

Now if $\langle Y, \rho \rangle$ is a metric space with induced metric topology τ_ρ, then the family of all Lipschitz functions on Y separates points from closed sets because the family of all distance functionals $y \mapsto \rho(y, p)$ where p runs over Y already does. In particular, if $h : \langle X, d \rangle \to \langle Y, \rho \rangle$ is a function, then h is continuous if and only if whenever f is a real-valued Lipschitz function on Y, $f \circ h$ is continuous. The purpose of this article is show that most of the important classes of functions between metric spaces behave analogously, including the uniformly continuous functions, the Lipschitz functions, the locally Lipschitz functions, the functions that map Cauchy sequences to Cauchy sequences, and the class of (uniformly continuous) functions that are *Lipschitz in the small* as introduced by Luukkainen [33] : there exist $\delta > 0$ and $\lambda > 0$ such that whenever $d(x_1, x_2) < \delta$ in X, we have $\rho(h(x_1), h(x_2)) \leq \lambda \cdot d(x_1, x_2)$.

Evidently, a bounded Lipschitz in the small function $h : X \to Y$ with parameters δ and λ is already Lipschitz, for if the diameter of $h(X)$ is M and $d(x_1, x_2) \geq \delta$, then $\rho(h(x_1), h(x_2)) \leq \frac{M}{\delta} d(x_1, x_2)$. If we replace the metric ρ on the target space Y by $\tilde{\rho} = \min\{\rho, 1\}$, then h is Lipschitz in the small if and only if h regarded as a function from X to $\langle Y, \tilde{\rho} \rangle$ is Lipschitz [23, p. 283]. For real-valued functions, the Lipschitz in the small functions are particularly important in that such functions are uniformly dense in the real-valued uniformly continuous functions [10, 23]. At the end of the article, we give a uniform closure result for the real-valued continuous functions that are Lipschitz in the small when restricted to a particular family of subsets, provided the family is shielded from closed sets. From this single result, we can easily identify uniformly dense subclasses of Lipschitz-type functions within the most important classes of continuous functions. Our results argue for the primacy of the Lipschitz in the small functions in the study of real-valued continuous functions on metric spaces.

Along the way, we introduce and study the class of (locally Lipschitz) functions that are Lipschitz when restricted to Bourbaki bounded subsets.

2. Preliminaries

All metric spaces are assumed to contain at least two points. We denote the set of limit points of the space $\langle X, d \rangle$ by X'. We write $S_d(x, \varepsilon)$ for the open ball of center x and radius $\varepsilon > 0$ in X. We write $S_d(A, \varepsilon)$ for $\cup_{a \in A} S_d(a, \varepsilon)$; this set is often called the *ε-enlargement* of A [5]. For $x \in X$ and A a nonempty subset of $\langle X, d \rangle$, we put $d(x, A) := \inf\{d(x, a) : a \in A\}$; clearly, $S_d(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}$ for A nonempty.

We denote the usual d-diameter of a nonempty subset A of X by $\mathrm{diam}_d(A)$. We call nonempty subsets A and B of X *near* provided $\inf\{d(a, b) : a \in A,\ b \in B\} = 0$;

intuitively, nearness for two nonempty subsets means they either intersect or are asymptotic.

A subset A of X is called *bounded* if it is either empty or has finite diameter; equivalently, A is a subset of some open ball in X. A subset A of X is called *totally bounded* if for each $\varepsilon > 0$ there exists a finite subset F of X with $A \subseteq S_d(F, \varepsilon)$. It is called *relatively compact* if it has compact closure. Each relatively compact subset of X is totally bounded, and the converse is true if and only if the metric d is complete. If A is a nonempty subset of X and $\varepsilon > 0$, we define $S_d^n(A, \varepsilon)$ recursively by $S_d^1(A, \varepsilon) = S_d(A, \varepsilon)$ and $S_d^{n+1}(A, \varepsilon) = S_d(S_d^n(A, \varepsilon), \varepsilon)$. A subset A of X is called *Bourbaki bounded* [9, 24] if for each $\varepsilon > 0$, there exists a nonempty finite subset F of X and $n \in \mathbb{N}$ with $A \subseteq S_d^n(F, \varepsilon)$. Each totally bounded subset is Bourbaki bounded and each Bourbaki subset is metrically bounded. Unlike boundedness and total boundedness, Bourbaki boundedness is not an intrinsic property of the subset. For example, in ℓ_2, the standard orthonormal base $\{e_n : n \in \mathbb{N}\}$ is a Bourbaki bounded subset, but the set fails to be a Bourbaki bounded subset of itself, considered as a metric subspace of ℓ_2.

As a subset of a metric space is (a) relatively compact if and only if each sequence in it has a convergent subsequence, and (b) totally bounded if and only if each sequence in it has a Cauchy subsequence, a subset is Bourbaki bounded if and only if each sequence in it has a Bourbaki-Cauchy subsequence [24, Theorem 4], as we now define, in a way paralleling the standard definition of Cauchy sequence.

Definition 2.1. A sequence $\langle x_n \rangle$ in a metric space $\langle X, d \rangle$ is called *Bourbaki-Cauchy* provided for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ and $m \in \mathbb{N}$ such that whenever n and j exceed k, we have $x_n \in S_d^m(\{x_j\}, \varepsilon)$.

A metric space is called *Bourbaki complete* provided each Bourbaki-Cauchy sequence in it clusters [24]. A metric space is Bourbaki complete if and only if each Bourbaki bounded subset is relatively compact [24, Theorem 9]; that each relatively compact set is Bourbaki bounded is always true as each relatively compact set is totally bounded.

We denote the continuous functions from $\langle X, d \rangle$ to $\langle Y, \rho \rangle$ by $C(X, Y)$. We now formally introduce some subclasses of $C(X, Y)$ in order of increasing size. Suppose f is a function from X to Y.

- f is called *Lipschitz* if for some $\lambda > 0$, whenever $\{x_1, x_2\} \subseteq X$, we have $\rho(f(x_1), f(x_2)) \leq \lambda \cdot d(x_1, x_2)$. For additional specificity, we can say that f is λ-*Lipschitz*.
- f is called *uniformly continuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x_1, x_2) < \delta \Rightarrow \rho(f(x_1), f(x_2)) < \varepsilon$.
- f is called *Cauchy continuous* [11, 16, 39] if whenever $\langle x_n \rangle$ is a Cauchy sequence in X, $\langle f(x_n) \rangle$ is a Cauchy sequence in Y.

We denote these classes by $\text{Lip}(X, Y)$, $UC(X, Y)$ and $CC(X, Y)$ respectively. The classes $C(X, Y)$ and $CC(X, Y)$ coincide for all target spaces Y if and only if the domain space is complete [39], while the well-studied class of domain spaces for which $C(X, Y) = UC(X, Y)$ properly contains the class of compact metric spaces (see, e.g., [3, 5, 30, 36]). This class is usually called the *UC-spaces* or the *Atsuji spaces* in the literature. The most visual way to describe a UC-space is as follows:

X' is compact (although possibly empty), and for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x \notin S_d(X', \varepsilon) \Rightarrow S_d(x, \delta) = \{x\}$. We have $CC(X, Y) = UC(X, Y)$ for all target spaces Y if and only if the metric completion of $\langle X, d\rangle$ is a UC-space [4, 31]. We mention this additional fact about Cauchy continuity connecting it to uniform continuity that we will use in the sequel: $f : X \to Y$ is Cauchy continuous if and only if its restriction to each totally bounded subset is uniformly continuous [14, 39].

If $f \in \text{Lip}(X, Y)$, we put $L(f) := \inf\{\lambda > 0 : f \textit{ is } \lambda - \text{Lipschitz}\}$. Clearly, $\text{Lip}(X, \mathbb{R})$ is a vector space. However, $f \mapsto L(f)$ on $\text{Lip}(X, \mathbb{R})$ is only a seminorm as $L(f) = 0$ provided f is any constant function. We describe one way to fix this problem. Let $x_0 \in X$ be arbitrary; the *Lipschitz norm* on $\text{Lip}(X, \mathbb{R})$ with respect to the base point x_0 is defined by

$$||f||_{\text{Lip}} := \max\{|f(x_0)|, L(f)\} \qquad (f \in \text{Lip}(X, \mathbb{R})).$$

Replacing x_0 by a different point yields an equivalent norm as does replacing the maximum by a sum. It is well-know that this norm makes $\text{Lip}(X, \mathbb{R})$ a Banach space. Convergence with respect to the Lipschitz norm implies uniform convergence on bounded subsets of X; the converse holds if and only if X is bounded and there exists $\delta > 0$ such that for each $x \in X$, $S_d(x, \delta) = \{x\}$ [13, Theorem 3.7].

The next result due to Michael [34] alone makes $\langle \text{Lip}(X, \mathbb{R}), ||\cdot||_{\text{Lip}}\rangle$ worth studying. It, of course, gives another path to the construction of the completion of a metric space. Michael used this result to give an attractive proof of the Arens-Eells theorem [2]: each metric space can be isometrically embedded as a closed subspace of a normed linear space.

Theorem 2.2. *Let $\langle X, d\rangle$ be a metric space and equip $Lip(X, \mathbb{R})$ with a Lipschitz norm $||\cdot||_{Lip}$ with base point $x_0 \in X$. For each $x \in X$, let $\hat{x} : Lip(X, \mathbb{R}) \to \mathbb{R}$ be the evaluation map, i.e., $\hat{x}(f) = f(x)$. Then $x \mapsto \hat{x}$ maps X isometrically into the continuous dual of $\langle Lip(X, \mathbb{R}), ||\cdot||_{Lip}\rangle$ equipped the usual operator norm.*

Recall that a function f between metric spaces is called *locally Lipschitz* if for each $x \in X, \exists \delta_x > 0$ such that f restricted to $S_d(x, \delta_x)$ is Lipschitz. Obviously each locally Lipschitz function is continuous. The real-valued locally Lipschitz functions are uniformly dense in $C(X, \mathbb{R})$ [11, 22, 35].

Between the class of locally Lipschitz functions and the class of Lipschitz in the small functions lies the class of *Cauchy-Lipschitz functions* which are the functions that are Lipschitz when restricted to the range of each Cauchy sequence in X. The real-valued Cauchy-Lipschitz functions are a uniformly dense subclass of $CC(X, \mathbb{R})$ [11, Theorem 4.5]. Pairwise coincidence of our three classes of locally Lipschitz functions has been determined (see [10, Theorem 3.3] and [11, Theorem 3.5 and Theorem 3.6]):

- the Cauchy-Lipschitz functions on $\langle X, d\rangle$ agree with the locally Lipschitz functions if and only if $\langle X, d\rangle$ is a complete metric space;
- the locally Lipschitz functions on $\langle X, d\rangle$ agree with Lipschitz in the small functions if and only if $\langle X, d\rangle$ is a UC-space;
- the Lipschitz in the small functions on $\langle X, d\rangle$ agree with the Cauchy-Lipschitz functions if and only if the completion of $\langle X, d\rangle$ is a UC-space.

We emphasize the following parallel facts by stating them in a proposition.

Proposition 2.3. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces and let $f : X \to Y$.*

(1) f is locally Lipschitz if and only if its restriction to each relatively compact subset of X is Lipschitz.
(2) f is Cauchy-Lipschitz if and only if its restriction to each totally bounded subset of X is Lipschitz.

The first fact is a folk-theorem; a proof in the case of real-valued functions is given in [38, Theorem 2.1]. The second fact was established by the authors in [11, Proposition 3.4]. Furthermore, a nonempty subset A of a metric space is relatively compact (resp. totally bounded) if and only if each locally Lipschitz function (resp. Cauchy-Lipschitz function) on X is Lipschitz when restricted to A (see [10, Theorem 4.2 and Theorem 4.3]). To draw a further parallel, it is easy to verify that a function between metric spaces is locally Lipschitz if and only if it is Lipschitz when restricted to the range of each convergent sequence [11, Theorem 3.1].

The class of subsets of a metric space on which each Lipschitz in the small function is actually Lipschitz - called the class of *small-determined subsets* - has been recently internally characterized by Leung and Tang [32]. A small-determined subset of a metric space need not be metrically bounded; for example, in $X = \{0,1\} \times \mathbb{R}$ as a metric subspace of the Euclidean plane, both $\{0\} \times \mathbb{R}$ and $\{1\} \times \mathbb{R}$ are small-determined subsets. As the entire space is not small-determined, the small-determined subsets are not in general stable under finite union. To see this, consider $f : X \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \alpha & \text{if } x = (0, \alpha) \text{ for some } \alpha \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases};$$

This function is Lipschitz in the small where $\delta = \frac{1}{2}$ and $\lambda = 1$ but it fails to be globally Lipschitz.

Metric spaces that are small-determined in themselves are particularly notable, for it is in exactly in these spaces that $\mathrm{Lip}(X, \mathbb{R})$ is uniformly dense in $UC(X, \mathbb{R})$ [23, Theorem 7]. Each Bourbaki bounded subset is small-determined, and for bounded subsets of a metric space, the two notions coincide [10].

One of the disappointing aspects of the class of Lipschitz in the small functions is that it cannot be characterized as a family of functions that is Lipschitz when restricted to a particular family of subsets, for if it could, then the small-determined subsets would also be an adequate family of subsets for this purpose. The following example shows that this is not so.

Example 2.4. A real-valued function on a metric space $\langle X, d\rangle$ that is Lipschitz restricted to each small-determined subset may not be Lipschitz in the small. In ℓ_2 with standard orthonormal base $\{e_n : n \in \mathbb{N}\}$, let X be this metric subspace:

$$X = \Big\{x \in \ell_2 : \exists n \in \mathbb{N} \text{ with } ||x - e_n||_2 \leq \frac{1}{2}\Big\}.$$

A subset A of this bounded metric space is small-determined if and only if it is Bourbaki bounded, that is, A intersects at most finitely many of the balls of radius 1/2 that comprise the space. With these facts in mind, if f is the function

$x \mapsto n||x - e_n||_2$ where e_n is the unique member of the orthonormal base with $||x - e_n||_2 \leq 1/2$, then f restricted to each Bourbaki bounded subset is evidently Lipschitz. However, f is not Lipschitz in the small.

Each of the seven function class we have discussed is stable under function composition. In the case of real-valued functions, each function class forms a vector lattice with respect to the usual pointwise defined operations on functions.

3. Results

Our first result, included for completeness, lacks any depth.

Theorem 3.1. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces and let $h : X \to Y$. Then h is bounded if and only if whenever $f \in Lip(Y, \mathbb{R})$, the composition $f \circ h$ is bounded.*

Proof. If h bounded and is followed by a Lipschitz function in composition, then we get a bounded function, as a Lipschitz function on Y is bounded when restricted to any bounded subset of Y. Conversely, if the range of h is not bounded, fixing y_0 in Y, let $f = \rho(\cdot, y_0) \in \mathrm{Lip}(Y, \mathbb{R})$. As f restricted to $h(X)$ is unbounded, $f \circ h$ is unbounded. □

We next look at uniform continuity. We start with a statement of the Efremovič lemma for metric spaces given in [5, p. 82]; for a more abstract formulation, we refer the reader to [21, 37].

Theorem 3.2. *Let $\langle y_n\rangle$ and $\langle w_n\rangle$ be sequences in a metric space $\langle Y, \rho\rangle$ such that $\forall n \in \mathbb{N},\ \rho(y_n, w_n) > \varepsilon$. Then there is an infinite subset $\mathbb{N}_\varepsilon$ of $\mathbb{N}$ such that*

$$inf\ \{\rho(y_j, w_k) : (j, k) \in \mathbb{N}_\varepsilon \times \mathbb{N}_\varepsilon\} \geq \frac{\varepsilon}{4}.$$

Theorem 3.3. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces and let $h : X \to Y$. Then $h \in UC(X, Y)$ if and only if whenever $f \in Lip(Y, \mathbb{R})$, we have $f \circ h \in UC(X, \mathbb{R})$.*

Proof. Necessity is obvious as each Lipschitz function is uniformly continuous, and the uniformly continuous functions are stable under composition. For sufficiency, we prove the contrapositive. Suppose $h \notin UC(X, Y)$; by definition, for some $\varepsilon > 0$, we can find for each positive integer k points x_k, w_k in X such that $0 < d(x_k, w_k) < 1/k$ while $\rho(h(x_k), h(w_k)) > \varepsilon$. In view of the Efremovič lemma, by passing to an infinite subset of $\mathbb{N}$, we may assume without loss of generality that

$$(\spadesuit) \qquad \inf\ \{\rho(h(x_j), h(w_k)) : (j, k) \in \mathbb{N} \times \mathbb{N}\} \geq \tfrac{\varepsilon}{4}.$$

Now put $f := \rho(\cdot, \{h(w_k) : k \in \mathbb{N}\}) \in \mathrm{Lip}(Y, \mathbb{R})$. Then $f \circ h$ fails to be uniformly continuous as by ($\spadesuit$), for each positive integer k, $\rho((f \circ h)(x_k), (f \circ h)(w_k)) \geq \frac{\varepsilon}{4}$. □

As an alternative proof of sufficiency in Theorem 3.3, we could have used the fact that a function h between metric spaces is uniformly continuous if and only if h preserves nearness for nonempty subsets (see, e.g. [37, p. 35]). Thus, if $h \notin UC(X, Y)$ we can find nonempty near subsets A and B of X such that $h(A)$ and $h(B)$ are not near with respect to ρ. Letting $f = \rho(\cdot, h(A))$ we see that $f \circ h$ does not preserve nearness either. But this characterization of uniform continuity in terms of nearness is itself a consequence of the Efremovič lemma!

Proposition 3.4. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces. Let $\mathcal{B}$ be a family of nonempty subsets of $\langle X, d\rangle$ and put*

$$\Omega_{\mathcal{B}} := \{h \in C(X,Y) : \forall B \in \mathcal{B},\ h|_B \text{ is uniformly continuous}\}.$$

Then $h \in C(X,Y)$ belongs to $\Omega_{\mathcal{B}}$ if and only if for each $f \in Lip(Y,\mathbb{R})$ and for each $B \in \mathcal{B}$ we have $(f \circ h)|_B \in UC(B,\mathbb{R})$.

Proof. For sufficiency, by Theorem 3.3, for each $B \in \mathcal{B}$ we see that $h|_B$ is uniformly continuous. But by definition, this means that $h \in \Omega_{\mathcal{B}}$. □

Corollary 3.5. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces and let $h : X \to Y$. Then $h \in CC(X,Y)$ if and only if whenever $f \in Lip(Y,\mathbb{R})$, we have $f \circ h \in CC(X,\mathbb{R})$.*

Proof. By [14, Proposition 4.11], a (continuous) function on X maps Cauchy sequences to Cauchy sequences if and only if it is uniformly continuous restricted to each totally bounded subset of X. This means that $CC(X,Y) = \Omega_{\mathcal{B}}$ where $\mathcal{B}$ is the family of nonempty totally bounded subsets of X. Apply the last proposition. □

Let $h : \langle X, d\rangle \to \langle Y, \rho\rangle$ and let A be a nonempty subset of X. We say h is *strongly uniformly continuous* on A if $\forall \varepsilon > 0\ \exists \delta > 0$ such that whenever $a \in A$ and $x \in X$ with $d(a,x) < \delta$, we have $\rho(h(a),h(x)) < \varepsilon$ [14, 15]. This variational notion is stronger than the uniform continuity of the restriction of h to A even if $h \in C(X,Y)$. Continuity itself can be explained in terms of strong uniform continuity: h is continuous at $a \in X$ if and only if h is strongly uniformly continuous on $\{a\}$. The standard sequential proof by contradiction that if $h \in C(X,Y)$ and A is nonempty and compact, then $h|_A$ is uniformly continuous shows that strong uniform continuity on A ensues. For that matter, our proof of Theorem 3.3 goes through to establish the next result.

Theorem 3.6. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces and let A be a nonempty subset of X. Then $h : X \to Y$ is strongly uniformly continuous on A if and only if whenever $f \in Lip(Y,\mathbb{R})$, the composition $f \circ h$ is strongly uniformly continuous on A.*

Note that Theorem 3.6 subsumes Theorem 3.3; just take $A = X$. The class of subsets of X on which each continuous function on X is strongly uniformly continuous can be characterized in compelling ways (see, e.g., [14, Theorem 3.1 and Theorem 5.2]). We return to strong uniform continuity in the final section of this article.

As promised in the abstract, we now give a simplified proof of the theorem of Garrido and Jaramillo [22, pp. 140-141] which says that a function h between metric spaces is Lipschitz if and only if for each real-valued Lipschitz function f on the target space, the composition $f \circ h$ is Lipschitz.

Theorem 3.7. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces and let $h : X \to Y$. Then $h \in Lip(X,Y)$ if and only if whenever $f \in Lip(Y,\mathbb{R})$, we have $f \circ h \in Lip(X,\mathbb{R})$.*

Proof. Only sufficiency requires proof. Let $x_0 \in X$ be fixed, put $y_0 = h(x_0)$, and equip $\mathrm{Lip}(X,\mathbb{R})$ (resp. $\mathrm{Lip}(Y,\mathbb{R})$) with Lipschitz norms with base point x_0 (resp. y_0). Define the linear transformation $T : \mathrm{Lip}(Y,\mathbb{R}) \to \mathrm{Lip}(X,\mathbb{R})$ by $T(f) = f \circ h$. It is easy to see that T is continuous from the classical closed graph theorem of functional analysis [27, p. 158], which we may apply since both the domain and target space are Banach spaces. Suppose $f, f_1, f_2, f_3, \ldots$ is a sequence in $\text{Lip}(Y, \mathbb{R})$ such that $||f_n - f||_{\text{Lip}} \to 0$ while $||T(f_n) - g||_{\text{Lip}} \to 0$ where $g \in \text{Lip}(X, \mathbb{R})$. Since convergence in a Lipschitz norm forces pointwise convergence, $\langle f_n \rangle$ converges pointwise to f and $\langle T(f_n) \rangle$ converges pointwise to g. This means that for each $x \in X$,

$$\lim_{n\to\infty} f_n(h(x)) = f(h(x)) = T(f)(x)$$

while

$$\lim_{n\to\infty} f_n(h(x)) = \lim_{n\to\infty} T(f_n)(x) = g(x),$$

from which $T(f) = g$ as required.

With $K := ||T||_{\text{op}}$, we intend to show that h is K-Lipschitz. Fix $x_1 \neq x_2$ in X and put $y_1 = h(x_1)$ and $y_2 = h(x_2)$. Let $f \in \text{Lip}(Y, \mathbb{R})$ be defined by

$$f(y) := \rho(y, y_2) - \rho(y_0, y_2).$$

As $L(f) = 1$ and $f(y_0) = T(f)(x_0) = 0$, we conclude $||f||_{\text{Lip}} = \max\{|f(y_0)|, L(f)\}$ $= 1$ and $||T(f)||_{\text{Lip}} = L(T(f))$. The upcoming inequality string shows that h is K-Lipschitz.

$$\rho(h(x_1), h(x_2)) = \rho(y_1, y_2) = |f(y_1) - f(y_2)| = |f(h(x_1)) - f(h(x_2))|$$

$$\begin{aligned} &= |(T(f))(x_1) - (T(f))(x_2)| \leq L(T(f)) \cdot d(x_1, x_2) \\ &= ||T(f)||_{\text{Lip}} \cdot d(x_1, x_2) \leq K \cdot ||f||_{\text{Lip}} \cdot d(x_1, x_2) \\ &= K \cdot d(x_1, x_2). \end{aligned}$$ □

The next proposition serves as a companion to Proposition 3.4

Proposition 3.8. *Let $\langle X, d \rangle$ and $\langle Y, \rho \rangle$ be metric spaces. Let $\mathcal{B}$ be a family of nonempty subsets of $\langle X, d \rangle$ and put*

$$\Delta_{\mathcal{B}} := \{h \in C(X, Y) : \forall B \in \mathcal{B},\ h|_B \text{ is Lipschitz}\}.$$

Then $h \in C(X, Y)$ belongs to $\Delta_{\mathcal{B}}$ if and only if for each $f \in Lip(Y, \mathbb{R})$ and for each $B \in \mathcal{B}$ we have $(f \circ h)|_B \in Lip(B, \mathbb{R})$.

Proof. For sufficiency, by Theorem 3.7, for each $B \in \mathcal{B}$ we see that $h|_B$ is Lipschitz. But by definition, this means that $h \in \Delta_{\mathcal{B}}$. □

As a consequence of Proposition 2.3 we get these two corollaries of Proposition 3.8. In the first, the family of subsets $\mathcal{B}$ is the family of relatively compact sets, and in the second, $\mathcal{B}$ is the family of totally bounded sets.

Corollary 3.9. *Let $\langle X, d \rangle$ and $\langle Y, \rho \rangle$ be metric spaces and let $h : X \to Y$. Then h is locally Lipschitz if and only if whenever $f \in Lip(Y, \mathbb{R})$, the function $f \circ h$ is locally Lipschitz.*

Corollary 3.10. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces and let $h : X \to Y$. Then h is Cauchy-Lipschitz if and only if whenever $f \in Lip(Y, \mathbb{R})$, the function $f \circ h$ is Cauchy-Lipschitz.*

Corollary 3.11. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces and let $h : X \to Y$. Then h is Lipschitz restricted to Bourbaki bounded subsets of X if and only if this is so for each composition $f \circ h$ where $f \in Lip(Y, \mathbb{R})$.*

We note that if a function between metric spaces is Lipschitz in the small, then it is bounded when restricted to each Bourbaki bounded subset [9, Theorem 3.3], and is thus Lipschitz on each Bourbaki bounded subset. But what does Lipschitzian comportment on Bourbaki bounded sets mean sequentially? We need an idea that parallels the notion of a Cauchy-Lipschitz sequence.

Definition 3.12. Let $h : \langle X, d\rangle \to \langle Y, \rho\rangle$. We say that h is *Bourbaki-Cauchy-Lipschitz* if it is Lipschitz restricted to the range of each Bourbaki-Cauchy sequence in X

Proposition 3.13. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces. Then $h : X \to Y$ is Bourbaki-Cauchy-Lipschitz if and only if for each nonempty Bourbaki bounded subset B of X, the restriction of h to B is Lipschitz.*

Proof. As the range of each Bourbaki-Cauchy sequence is a Bourbaki bounded subset, sufficiency is trivial. For necessity, suppose h is Bourbaki-Cauchy-Lipschitz and let $B \subseteq X$ be a nonempty Bourbaki bounded subset. First, we are going to see that $h|_B$ is bounded. Otherwise, there exists a sequence $\langle b_n\rangle$ in B for which $\langle h(b_n)\rangle$ stays outside each metrically bounded subset of Y eventually. By passing to a subsequence, we may assume $\langle b_n\rangle$ is Bourbaki-Cauchy [24, Theorem 4]. Then h can't be Lipschitz on the metrically bounded set $\{b_n : n \in \mathbb{N}\}$, and so h is not Bourbaki-Cauchy-Lipschitz. The remainder of the proof duplicates the second half of the proof of [11, Proposition 3.4] *verbatim.* □

With Proposition 3.13 in mind, we can restate Corollary 3.11 in a way that parallels the previous two corollaries: $h : X \to Y$ is Bourbaki-Cauchy-Lipschitz if and only if for each $f \in \text{Lip}(Y, \mathbb{R})$, $f \circ h$ is Bourbaki-Cauchy-Lipschitz.

Consistent with past lines of inquiry of the authors, we present necessary and sufficient conditions on a metric space $\langle X, d\rangle$ so that the Bourbaki-Cauchy-Lipschitz functions on X coincide with each of the other three classes of locally Lipschitz functions under consideration here. We start with the largest class.

Theorem 3.14. *Let $\langle X, d\rangle$ be a metric space. The following conditions are equivalent:*

(1) the metric space is Bourbaki complete;
(2) each locally Lipschitz function from $\langle X, d\rangle$ to an arbitrary metric space $\langle Y, \rho\rangle$ is Bourbaki-Cauchy-Lipschitz;
(3) each real-valued locally Lipschitz function on $\langle X, d\rangle$ is Bourbaki-Cauchy-Lipschitz.

Proof. (1) $\Rightarrow$ (2). As we mentioned in the preliminaries, each locally Lipschitz function is Lipschitz when restricted to relatively compact subsets. Now since every Bourbaki bounded subset in a Bourbaki complete space is relatively compact [24, Theorem 9], condition (2) follows from Proposition 3.13.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). We prove the contrapositive. Suppose $\langle X, d\rangle$ fails to be Bourbaki complete; again by [24, Theorem 9], we can find a Bourbaki bounded subset B that is not relatively compact. Let $\langle b_n\rangle$ be a sequence in B with distinct terms that fails to cluster in X. As $\{b_n : n \in \mathbb{N}\}$ is a closed discrete subset of X, by the Tietze extension theorem we can find $g \in C(X, \mathbb{R})$ such that for each $n \in \mathbb{N}$, $g(b_n) = n$. Now from the uniform density of the locally Lipschitz functions in $C(X, \mathbb{R})$ [11, 22, 35], we can choose a locally Lipschitz real-valued function g_0 on X such that $\sup_{x\in X} |g(x) - g_0(x)| < 1$. From this, g_0 is unbounded on B and so $g_0|_B$ cannot be Lipschitz because B is metrically bounded. Apply Proposition 3.13 to conclude that g_0, while locally Lipschitz, is not Bourbaki-Cauchy-Lipschitz. □

Theorem 3.15. *Let $\langle X, d\rangle$ be a metric space. The following conditions are equivalent:*

1. *the completion $\langle \hat{X}, \hat{d}\rangle$ of $\langle X, d\rangle$ is Bourbaki complete;*
2. *each Bourbaki bounded subset of X is totally bounded;*
3. *each Cauchy-Lipschitz function from $\langle X, d\rangle$ to an arbitrary metric space $\langle Y, \rho\rangle$ is Bourbaki-Cauchy-Lipschitz;*
4. *each real-valued Cauchy-Lipschitz function on $\langle X, d\rangle$ is Bourbaki-Cauchy-Lipschitz.*

Proof. The equivalence of conditions (1) and (2) was established concurrently and independently by Aggarwal and Kundu [1, Theorem 2.7] and Garrido and Meroño [25, Proposition 3.6], and the implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious. Our proof will be finished if we can establish (4) $\Rightarrow$ (2).

If condition (2) fails, let $B \subseteq X$ be Bourbaki bounded but not totally bounded. This means that for some $\delta > 0$, B cannot be covered by a finite number of open balls of radius δ whose centers lie in B; equivalently, we can find a sequence $\langle b_n\rangle$ in B such that $n \neq j$ implies $d(b_n, b_j) \geq \delta$. Denote the range of this sequence by B_0, and define $f : B_0 \to \mathbb{R}$ by $f(b_n) = n$. As B_0 is a closed subset of the completion $\langle \hat{X}, \hat{d}\rangle$ and any (real-valued) function on B_0 is locally Lipschitz, by the Czipszer-Gehér extension theorem [19, 20], there exists a locally Lipschitz real-valued function $\hat{f}$ on $\hat{X}$ that extends f. By the completeness of $\hat{d}$, $\hat{f}$ is Cauchy-Lipschitz [11], and so $\hat{f}$ restricted to X is Cauchy-Lipschitz. But the restriction is not Bourbaki-Cauchy-Lipschitz because it is not bounded on B_0, a Bourbaki bounded subset of X. □

Aggarwal and Kundu [1] stated condition (2) above in this sequential form: each Bourbaki-Cauchy sequence in X has a Cauchy subsequence. We refer the interested reader to their article for additional equivalences. For metrizable spaces that admit a compatible metric of this type, see [29, Corollary 4.6].

The Lipschitz in the small functions are contained in the Bourbaki-Cauchy-Lipschitz functions because each Bourbaki bounded subset is small-determined. We delay our discussion as to when these function classes coincide until the end of the next section when additional machinery is available to us.

4. The case of Lipschitz in the small functions

In view of Example 2.4, there is no hope of using Proposition 3.8 to provide a parallel theorem for the class of Lipschitz in the small functions. Nevertheless, we can prove that this property is determined by compositions with real-valued

Lipschitz functions using an argument that directly invokes Theorem 3.7 upon replacing the metric on the target space by a bounded metric.

Theorem 4.1. *A function h from $\langle X, d\rangle$ to $\langle Y, \rho\rangle$ is Lipschitz in the small if and only if whenever $f \in Lip(Y, \mathbb{R})$, the function $f \circ h$ is Lipschitz in the small.*

Proof. One direction is clear. For the other, with $\tilde{\rho} = \min\{1, \rho\}$, we use the fact that $h : \langle X, d\rangle \to \langle Y, \rho\rangle$ is Lipschitz in the small provided h regarded as a function $\tilde{h}$ from $\langle X, d\rangle$ to $\langle Y, \tilde{\rho}\rangle$ is Lipschitz (see condition (iii) on [23, p. 283]). To this end, we apply the Garrido-Jaramillo result Theorem 3.7.

Let f be a real-valued function on Y that is Lipschitz with respect to $\tilde{\rho}$; clearly, f is bounded. As the identity map from $\langle Y, \rho\rangle$ to $\langle Y, \tilde{\rho}\rangle$ is 1-Lipschitz, f is Lipschitz with respect to ρ as well. By hypothesis, $f \circ h$ is Lipschitz in the small, and as $f \circ h = f \circ \tilde{h}$ we see that $f \circ \tilde{h}$ is Lipschitz because it is a bounded function. Applying Theorem 3.7, $\tilde{h}$ is Lipschitz as required. □

We wish to prove a more precise result, namely, that if $f \circ h$ is Lipschitz in the small for each $f \in \text{Lip}(Y, \mathbb{R})$ with a common distance parameter δ but a variable Lipschitz constant parameter λ, then δ also works for h. This is quite involved.

To formalize things, we will say that a function h between $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ is *Lipschitz in the small with distance control* δ if there exists some positive λ such that $\rho(h(x_1), h(x_2)) \leq \lambda \cdot d(x_1, x_2)$ whenever $d(x_1, x_2) < \delta$. A weaker property that we are forced to consider is given in the next definition.

Definition 4.2. We call a function h from $\langle X, d\rangle$ to $\langle Y, \rho\rangle$ *bounded in the small with distance control* δ if there exists $M > 0$ such that whenever $d(x_1, x_2) < \delta$ in X, then $\rho(h(x_1), h(x_2)) < M$.

Obviously a Lipschitz in the small function with distance control δ and Lipschitz constant parameter λ is bounded in the small with the same distance control: if $d(x_1, x_2) < \delta$, then $\rho(h(x_1), h(x_2)) < \lambda\delta$.

Our next result may be viewed as a uniform local boundedness principle for functions between metric spaces.

Proposition 4.3. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces and let $h : X \to Y$. Then h is bounded in the small with distance control δ if and only if whenever $f \in Lip(Y, \mathbb{R})$, the function $f \circ h$ is bounded in the small with distance control δ.*

Proof. For necessity, suppose whenever $d(x_1, x_2) < \delta$, we have $\rho(h(x_1), h(x_2)) < M$. Then if $f : Y \to \mathbb{R}$ is λ-Lipschitz, we get $|(f \circ h)(x_1) - (f \circ h)(x_2)| < M\lambda$ whenever $d(x_1, x_2) < \delta$. Conversely, suppose h fails to be bounded in the small with distance control δ. This means that we can find sequences $\langle x_n\rangle$ and $\langle w_n\rangle$ in X such that for each $n \in \mathbb{N}$, we have $d(x_n, w_n) < \delta$ while $\rho(h(x_n), h(w_n)) > n$. We need only consider these two mutually exclusive and exhaustive cases for our sequences:

(a) by passing to a subsequence, one of the image sequences $\langle h(x_n)\rangle$ or $\langle h(w_n)\rangle$ is bounded - say $\langle h(w_n)\rangle$;

(b) each image sequence eventually is outside of each bounded subset of Y.

In case (a) choose $y_0 \in Y$ and $\alpha > 0$ such that $\forall n \in \mathbb{N},\ h(w_n) \in S_\rho(y_0, \alpha)$. Since $\rho(h(x_n), h(w_n)) > n$ for each n, whenever $j \neq n$, the triangle inequality yields

$$\rho(h(x_n), h(w_j)) \geq \rho(h(x_n), h(w_n)) - 2\alpha.$$

Put $E = \{h(w_n) : n \in \mathbb{N}\}$ and let $f = \rho(\cdot, E) \in \mathrm{Lip}(Y, \mathbb{R})$. Then

$$|(f \circ h)(x_n) - (f \circ h)(w_n)| = \rho(h(x_n), E) \geq \rho(h(x_n), h(w_n)) - 2\alpha,$$

which becomes arbitrarily large with increasing n.

Case (b) is more complex. We define subsequences $\langle x_{n_k} \rangle$ and $\langle w_{n_k} \rangle$ recursively. Put $x_{n_1} = x_1$ and $w_{n_1} = w_1$. For each $k \geq 1$, let $A_k = \{h(w_{n_1}), h(w_{n_2}), \ldots, h(w_{n_k})\}$ and let $B_k = \{h(x_{n_1}), h(x_{n_2}), \ldots, h(x_{n_k})\}$. Since enlargements of bounded sets are bounded, we can find $n_{k+1} > n_k$ such that both (i) $h(x_{n_{k+1}}) \notin S_d(A_k, k+1)$ and (ii) $h(w_{n_{k+1}}) \notin S_d(B_k, k)$. By (i) if $j < k$ we get

$$\rho(h(x_{n_k}), h(w_{n_j})) \geq (k-1) + 1 = k,$$

and by (ii) if $j > k$ we get

$$\rho(h(x_{n_k}), h(w_{n_j})) \geq j - 1 \geq k.$$

By assumption, $\rho(h(x_{n_k}), h(w_{n_k})) > n_k \geq k$. With $E = \{h(w_{n_j}) : j \in \mathbb{N}\}$, for each $k \in \mathbb{N}$, we have $\rho(h(x_{n_k}), E) \geq k$ so that $\rho(h(x_{n_k}), E) - \rho(h(w_{n_k}), E) \geq k$. Thus, with $f = \rho(\cdot, E)$, whenever $k \in \mathbb{N}$

$$|(f \circ h)(x_{n_k}) - (f \circ h)(w_{n_k})| \geq k.$$

In either case (a) or (b), we have produced a real-valued Lipschitz function f for which $f \circ h$ fails to be bounded in the small with distance control δ. $\square$

Theorem 4.4. *Let $\langle X, d \rangle$ and $\langle Y, \rho \rangle$ be metric spaces and let $h : X \to Y$. Then h is Lipschitz in the small with distance control $\delta > 0$ if and only if for each $f \in Lip(Y, R)$ the composition $f \circ h$ is Lipschitz in the small with distance control δ.*

Proof. Necessity is obvious as usual. If we know that the composition with each real-valued Lipschitz function is Lipschitz in the small with common distance control δ, then in particular we know that each such composition is Lipschitz in the small, and then from Theorem 4.1, h is Lipschitz in the small. This means there are $r > 0$ and $\lambda > 0$ such that

$$\rho(h(x_1), h(x_2)) \leq \lambda \cdot d(x_1, x_2) \quad \text{whenever} \quad d(x_1, x_2) < r.$$

Now if $\delta \leq r$, we are done. Otherwise, Proposition 4.3 allows us to assert that the following number M is finite:

$$M := \sup \{\rho(h(x_1), h(x_2)) : \{x_1, x_2\} \subseteq X \text{ and } d(x_1, x_2) < \delta\}.$$

Thus, whenever $r \leq d(x_1, x_2) < \delta$, we have

$$\rho(h(x_1), h(x_2)) \leq M \leq M \cdot \frac{d(x_1, x_2)}{r}.$$

Summarizing h is Lipschitz in the small with distance control δ and with Lipschitz constant $\max\{\lambda, \frac{M}{r}\}$. □

Over the last four years, fundamental characterizations of when $UC(X,\mathbb{R})$ is stable under pointwise product have been discovered [6, 12, 17, 18]. It is surprising to the authors that these results were not known many years ago. Our final result of this section is another contribution in this direction. It resolves the question: when do the Bourbaki-Cauchy-Lipschitz functions on a metric space coincide with the Lipschitz in the small functions?

In our proof, we denote the real-valued Lipschitz in the small functions (resp. Bourbaki-Cauchy-Lipschitz functions) by $LS(X,\mathbb{R})$ (resp. $BCL(X,\mathbb{R})$).

Theorem 4.5. *Let $\langle X, d\rangle$ be a metric space. The following conditions are equivalent.*

(1) $UC(X,\mathbb{R})$ is a ring;
(2) each subset of X is either Bourbaki bounded or contains an infinite uniformly isolated subset - a subset B such that $\inf\{d(b, X\backslash\{b\}) : b \in B\} > 0$;
(3) each Bourbaki-Cauchy-Lipschitz function from $\langle X, d\rangle$ to an arbitrary metric space $\langle Y, \rho\rangle$ is Lipschitz in the small;
(4) each real-valued Bourbaki-Cauchy-Lipschitz function on $\langle X, d\rangle$ is Lipschitz in the small.

Proof. The equivalence of conditions (1) and (2) is a result of Cabello-Sánchez [18]. The equivalence of conditions (3) and (4) follows from Theorem 4.1. In order to see that condition (1) implies condition (4), we use proof by contradiction. Suppose there exists $f \in BCL(X,\mathbb{R})$ that is not Lipschitz in the small. For every $n \in \mathbb{N}$ there exists x_n and w_n in X such that $0 < d(x_n, w_n) < \frac{1}{n}$, yet

$$|f(x_n) - f(w_n)| > n \cdot d(x_n, w_n).$$

Clearly, $\lim_{n\to\infty} d(x_n, X\backslash\{x_n\}) = 0$ and $\lim_{n\to\infty} d(w_n, X\backslash\{w_n\}) = 0$. By condition (2) of [12, Theorem 3.9] which characterizes $UC(X,\mathbb{R})$ being a ring, both $\{x_n : n \in \mathbb{N}\}$ and $\{w_n : n \in \mathbb{N}\}$ must be Bourbaki bounded subsets. But this implies f is not Lipschitz on the Bourbaki bounded set

$$\{x_n : n \in \mathbb{N}\} \cup \{w_n : n \in \mathbb{N}\},$$

violating $f \in BCL(X,\mathbb{R})$.

We finally turn to (4) $\Rightarrow$ (2). Condition (4) says that the two families of real-valued functions coincide because each Bourbaki bounded subset is small-determined [10, Theorem 4.4]. Since each Lipschitz function on a Bourbaki bounded subset is bounded, $BCL(X,\mathbb{R})$ is a ring. Thus, $LS(X,\mathbb{R})$ is a ring as well. As recently shown by Beer, Garrido and García-Lirola [8], this condition is equivalent to condition (2). □

5. A META-THEOREM

Our penultimate section is presented on a more abstract level, in an attempt to pin down the essential aspects of a class of continuous functions between metric spaces that is determined by real-valued Lipschitz functions. Let Ω be a class of functions between metric spaces. For a given pair of metric spaces $(\langle X,d\rangle,\langle Y,\rho\rangle)$ we put $\Omega(X,Y) := \{f : X \to Y : f \in \Omega\}$.

Definition 5.1. We call a class of functions Ω between metric spaces *real Lipschitz determined* if it satisfies the following three properties:

(1) for all metric spaces X, we have $\mathrm{Lip}(X,\mathbb{R}) \subseteq \Omega(X,\mathbb{R}) \subseteq C(X,\mathbb{R})$;
(2) for all metric spaces X and Y, whenever $h \in \Omega(X,Y)$ and $f \in \Omega(Y,\mathbb{R})$, we have $f \circ h \in \Omega(X,\mathbb{R})$;
(3) for all metric spaces X and Y, and $h : X \to Y$, then $h \in \Omega(X,Y)$ provided whenever $f \in \mathrm{Lip}(Y,\mathbb{R})$, we have $f \circ h \in \Omega(X,\mathbb{R})$.

All of the function classes within the continuous functions that we have discussed are of this type.

Theorem 5.2. *Let Ω be a real Lipschitz determined class of functions. Then Ω has the following properties:*

(a) for all metric spaces X and Y we have $Lip(X,Y) \subseteq \Omega(X,Y) \subseteq C(X,Y)$;
(b) Ω is stable under composition;
(c) for all metric spaces X and Y and $h : X \to Y$, then $h \in \Omega(X,Y)$ provided whenever $f \in \Omega(Y,\mathbb{R})$, we have $f \circ h \in \Omega(X,\mathbb{R})$.

Proof. For the first inclusion in (a), let $h \in \mathrm{Lip}(X,Y)$. By property (1) of Ω, whenever $f \in \mathrm{Lip}(Y,\mathbb{R})$, we have $f \circ h \in \Omega(X,\mathbb{R})$. Invoking property (3), we get $h \in \Omega(X,Y)$. For the second inclusion of (a), let $h \in \Omega(X,Y)$. By properties (1) and (2) of Ω, whenever $f \in \mathrm{Lip}(Y,\mathbb{R})$ we get

$$f \circ h \in \Omega(X,\mathbb{R}) \subseteq C(X,\mathbb{R}).$$

Thus, h followed by each real-valued Lipschitz function is continuous, and since $\mathrm{Lip}(Y,\mathbb{R})$ separates points from closed sets, we conclude that h is continuous.

For (b) let $h \in \Omega(X,Y)$ and let $g \in \Omega(Y,W)$ where X, Y and W are metric spaces. By property (3) of Ω, it suffices to show that whenever $f \in \mathrm{Lip}(W,\mathbb{R})$, we have $f \circ (g \circ h) \in \Omega(X,\mathbb{R})$. However, by properties (1) and (2), $f \circ g$ belongs to $\Omega(Y,\mathbb{R})$ and applying property (2) again, $(f \circ g) \circ h$ belongs to $\Omega(X,\mathbb{R})$.

Condition (c) immediately follows from properties (1) and (3) of Ω. □

6. A uniform closure theorem with applications

Let $\mathcal{B}$ be a family of nonempty subsets of a metric space $\langle X,d\rangle$. In this section we will show that

$$\nabla_{\mathcal{B}} := \{f \in C(X,\mathbb{R}) : \forall B \in \mathcal{B}, f|_B \text{ Lipschitz in the small}\}$$

is uniformly dense in

$$\Omega_{\mathcal{B}} := \{f \in C(X,\mathbb{R}) : \forall B \in \mathcal{B}, f|_B \text{ is uniformly continuous}\}$$

under the well-studied assumption that $\mathcal{B}$ be shielded from closed sets [7, 15]. From this result, we can derive the most important uniform density results for Lipschitz-type functions that we know of in the literature. Taken as a whole, the results of this section show conclusively that the Lipschitz in the small functions are building blocks in the study of classes of continuous real-valued functions defined on a metric space.

Definition 6.1. Let $\mathcal{B}$ be a family of nonempty subsets of $\langle X, d\rangle$. We say that $\mathcal{B}$ is *shielded from closed sets* provided each $B \in \mathcal{B}$ has a superset $B_1 \in \mathcal{B}$ such that each neighborhood of B_1 contains an enlargement of B.

We say B_1 is a shield for B when it has the stated property. Put in another way, B_1 is a shield for B if each nonempty closed subset of $X\backslash B_1$ fails to be near B. The family of nonempty relatively compact subsets is shielded from closed sets, as for each nonempty relatively compact set $B, \mathrm{cl}(B)$ serves as a shield for B. Any family of subsets that contains X as a member is shielded from closed sets, as X is a shield for any of its subsets. If each $B \in \mathcal{B}$ has an enlargement contained in some $B_1 \in \mathcal{B}$, then $\mathcal{B}$ is shielded from closed sets. In particular, the family of nonempty metrically bounded subsets is shielded from closed sets as each enlargement of a bounded set is bounded.

The family of nonempty totally bounded subsets is not in general shielded from closed sets. For example, in the context of normed linear spaces, this is so if and only if the space is a Banach space [15, Proposition 4.6]. More generally, if $\langle X, d\rangle$ is a complete metric space, then the totally bounded subsets are shielded from closed sets as the totally bounded subsets agree with the relatively compact subsets with completeness. The next example shows that the nonempty Bourbaki bounded subsets need not be shielded from closed sets in a complete metric space.

Example 6.2. Let B be the closed unit ball in ℓ_2 and let $\{e_n : n \in \mathbb{N}\}$ be the standard orthonormal base. Let X be this closed and therefore complete metric subspace of ℓ_2:

$$X := B \cup \{e_n + \frac{1}{(n+1)} e_j : (n, j) \in \mathbb{N}^2\}.$$

It is easy to see that for each $n \in \mathbb{N},\ E_n := \{e_n + \frac{1}{(n+1)} e_j : j \in \mathbb{N}\}$ is not Bourbaki bounded. Since the family of Bourbaki bounded subsets is hereditary, if B_1 is a Bourbaki bounded superset of B in X, then for each $n \in \mathbb{N}$, there exists $x_n \in E_n$ with $x_n \notin B_1$. As $d(x_n, e_n) < \frac{1}{n}$, $\{x_n : n \in \mathbb{N}\}$ is a closed subset of $X\backslash B_1$ that is near B.

We leave the proof of the following transparent sequential characterization of shielding in the context of Bourbaki bounded subsets to the reader (cf. [15, Theorem 4.7]).

Proposition 6.3. *Let B be a nonempty Bourbaki bounded subset of $\langle X, d\rangle$ and suppose $B \subseteq B_1 \subseteq X$. Then B_1 is a shield for B if and only each Bourbaki-Cauchy sequence $\langle x_n\rangle$ in $X\backslash B_1$ with $lim_{n\to\infty}\ d(x_n, B) = 0$ clusters.*

The key fact we need about families of nonempty subsets that are shielded from closed subsets was essentially proved in [15, Theorem 4.3]. For the convenience of the reader, we state it now as a proposition.

Proposition 6.4. *Let $\mathcal{B}$ be a family of nonempty subsets of $\langle X, d\rangle$ that is shielded from closed sets. Let $\langle Y, \rho\rangle$ be a second metric space and let $h \in C(X, Y)$ be uniformly continuous when restricted to each element of $\mathcal{B}$. Then h is strongly uniformly continuous on each member of $\mathcal{B}$.*

Theorem 6.5. *Let $\mathcal{B}$ be a family of nonempty subsets of $\langle X, d\rangle$ that is shielded from closed sets. Then the uniform closure of the real-valued continuous functions that are Lipschitz in the small on every member of $\mathcal{B}$ is the family of real-valued continuous functions that are uniformly continuous on every member of $\mathcal{B}$.*

Proof. It is clear that the family of continuous functions that are uniformly continuous on members of $\mathcal{B}$ is uniformly closed and contains all the continuous functions that are Lipschitz in the small on these subsets. Now let $f \in C(X, \mathbb{R})$ be uniformly continuous on each member of $\mathcal{B}$. Given $\varepsilon > 0$ we will produce a continuous $v : X \to \mathbb{R}$ that is Lipschitz in the small on members of $\mathcal{B}$ such that $\forall x \in X,\ |f(x) - v(x)| < \varepsilon$. Our argument has antecedents in those for [26, Lemma 1.2] and [23, Theorem 1].

For each $m \in \mathbb{Z}$ (the set of integers), put

$$C_m := \{x \in X : (m-1)\varepsilon < f(x) < (m+1)\varepsilon\}.$$

The open cover $\{C_m : m \in \mathbb{Z}\}$ of X has these properties:

- if $f(x) = \varepsilon m$ for some $m \in \mathbb{Z}$, then x belongs to exactly one C_m, otherwise x belongs to exactly two sets C_m;
- by continuity, each x has a neighborhood that meets at most three C_m;
- $C_m \cap C_n = \emptyset$ provided $|m - n| > 1$.

If some C_m is the entire space, the constant function $v(x) \equiv m\varepsilon$ does the job. Otherwise, each C_m is the cozero set of the 1-Lipschitz function $g_m(x) = \min\{1, d(x, X \backslash C_m)\}$. Put $s(x) = \sum_{m \in \mathbb{Z}} g_m(x)$; clearly if $x \in C_m$, then $s(x) = g_{m-1}(x) + g_m(x) + g_{m+1}(x)$ where in fact at most two of the summands are nonzero so that $s(x) \leq 2$.

Now put $p_m(x) = g_m(x)/s(x)$ for $x \in X$ and $m \in \mathbb{Z}$. Then $\{p_m : m \in \mathbb{Z}\}$ is a continuous partition of unity whose corresponding family of cozero sets is locally finite. As a result

$$v(x) := \sum_{m \in \mathbb{Z}} m\varepsilon \cdot p_m(x)$$

is a globally defined continuous function. If $f(x) = m\varepsilon$ for some integer m, then $f(x) = v(x)$. Otherwise, choosing the unique m with $m\varepsilon < f(x) < (m+1)\varepsilon$, we see that $v(x)$ is a convex combination of $m\varepsilon$ and $(m+1)\varepsilon$ and thus $|f(x) - v(x)| < \varepsilon$.

It remains to show that v is Lipschitz in the small on each $B \in \mathcal{B}$. To this end, it suffices to work with $g = \frac{1}{\varepsilon} v$, that is, $g(x) = \sum_{m \in \mathbb{Z}} m \cdot p_m(x)$. Note that $\forall x \in C_m$,

$$(\diamondsuit) \qquad g(x) = (m-1)p_{m-1}(x) + m \cdot p_m(x) + (m+1)p_{m+1}(x)$$

$$= m \textstyle\sum_{j=-1}^{1}\ p_{m+j}(x) - p_{m-1}(x) + p_{m+1}(x) = m - \frac{g_{m-1}(x)}{s(x)} + \frac{g_{m+1}(x)}{s(x)}.$$

Fix $B \in \mathcal{B}$; by Proposition 6.4, choose $\delta \in (0,1)$ such that if $b \in B$ and $x \in X$, $d(b, x) < \delta \Rightarrow |f(x) - f(b)| < \varepsilon/2$. We claim that for each $b \in B$ there exists $m \in \mathbb{Z}$ such that $S_d(b, \delta) \subseteq C_m$. To see this, choose $m \in \mathbb{Z}$ with

$$m\varepsilon - \frac{\varepsilon}{2} \leq f(b) \leq m\varepsilon + \frac{\varepsilon}{2}.$$

Then if $x \in S_d(b,\delta)$, by strong uniform continuity on B and our choice of δ, we get

$$(m-1)\varepsilon < f(x) < (m+1)\varepsilon,$$

that is, $S_d(b,\delta) \subseteq C_m$. In particular, $g_m(b) = \min\{1, d(b, X\backslash C_m)\} \geq \delta$ so that $s(b) \geq \delta$ whatever $b \in B$ may be.

We intend to show that whenever b_1 and b_2 are distinct points of B with $d(b_1,b_2) < \delta$, we have $|g(b_1) - g(b_2)| \leq \frac{10}{\delta^2} d(x,y)$. As a first step to realizing this inequality, taking m with $\{b_1,b_2\} \subseteq C_m$, we compute

$$(\clubsuit) \quad |s(b_1) - s(b_2)| = |\textstyle\sum_{j=-1}^{1} g_{m+j}(b_1) - \sum_{j=-1}^{1} g_{m+j}(b_2)| \leq 3d(b_1,b_2).$$

Now using $(\diamondsuit)$, $(\clubsuit)$, the fact that each g_k is 1-Lipschitz, and $\forall b \in B,\ \delta \leq s(b) \leq 2$, we finally get

$$|g(b_1) - g(b_2)| = \Big| - \frac{g_{m-1}(b_1)}{s(b_1)} + \frac{g_{m+1}(b_1)}{s(b_1)} + \frac{g_{m-1}(b_2)}{s(b_2)} - \frac{g_{m+1}(b_2)}{s(b_2)} \Big|$$

$$\leq \frac{1}{\delta^2}|g_{m-1}(b_1)s(b_2) - g_{m-1}(b_2)s(b_1)| + \frac{1}{\delta^2}|g_{m+1}(b_1)s(b_2) - g_{m+1}(b_2)s(b_1)|$$

$$\leq \frac{1}{\delta^2}|s(b_2)(g_{m-1}(b_1) - g_{m-1}(b_2))| + \frac{1}{\delta^2}|g_{m-1}(b_2)(s(b_1) - s(b_2))|$$

$$+ \frac{1}{\delta^2}|s(b_2)(g_{m+1}(b_1) - g_{m+1}(b_2))| + \frac{1}{\delta^2}|g_{m+1}(b_2)(s(b_1) - s(b_2))|$$

$$\leq \frac{1}{\delta^2}(2d(b_1,b_2) + 3d(b_1,b_2) + 2d(b_1,b_2) + 3d(b_1,b_2)) = \frac{10}{\delta^2}d(b_1,b_2)$$

as asserted. □

If we take $\mathcal{B} = \{X\}$ we get the following known result [10, 23].

Corollary 6.6. *Let $\langle X,d\rangle$ be a metric space. Then the uniform closure of the real-valued Lipschitz in the small functions on X is $UC(X,\mathbb{R})$.*

As we have noted, the family of nonempty bounded subsets of a metric space is shielded from closed sets. Also, a function that is Lipschitz in the small on bounded subsets is globally locally Lipschitz and thus is globally continuous. From these facts, we get this second corollary that we have not seen in the literature.

Corollary 6.7. *Let $\langle X,d\rangle$ be a metric space. Then the uniform closure of the real-valued functions that are Lipschitz in the small on bounded subsets of X is the family of real-valued functions that are uniformly continuous on bounded subsets.*

The next corollary can also be proved using the fact that each open cover of a metric space has a Lipschitz partition of unity subordinated to it [11, 23, 35].

Corollary 6.8. *Let $\langle X,d\rangle$ be a metric space. Then the uniform closure of the real-valued locally Lipschitz functions on X is $C(X,\mathbb{R})$.*

Proof. Obviously, the uniform limit of a sequence of locally Lipschitz functions is continuous (but not necessarily locally Lipschitz). For the converse, let $f \in C(X, \mathbb{R})$. Then f is uniformly continuous restricted to each relatively compact subset of X and as the family of relatively compact subsets is shielded from closed sets, for each $\varepsilon > 0$, there exists a (continuous) function v that is Lipschitz in the small on each nonempty relatively compact subset such that $\forall x \in X,\ |v(x) - f(x)| < \varepsilon$. By continuity, whenever B is nonempty and relatively compact, $v(B)$ is bounded, and so $v|_B$ is actually Lipschitz on B. But by Proposition 2.3, a function that is Lipschitz on each relatively compact subset is locally Lipschitz. □

As a common special case of each of the last two corollaries, each continuous real-valued function on Euclidean space $\mathbb{R}^n$ can be uniformly approximated by functions that are Lipschitz on bounded subsets.

As we have mentioned, the family of nonempty totally bounded subsets is not in general shielded from closed sets. Still, we can use the last corollary to show that the uniform closure of the Cauchy-Lipschitz real-valued functions is $CC(X, \mathbb{R})$ by passing to the completion of the metric space (cf. [11, Theorem 4.5]).

Corollary 6.9. *Let $\langle X, d \rangle$ be a metric space. Then the uniform closure of the real-valued Cauchy-Lipschitz functions on X is $CC(X, \mathbb{R})$.*

Proof. It is easy to check that $CC(X, \mathbb{R})$ is uniformly closed, and clearly each Cauchy-Lipschitz function maps Cauchy sequences to Cauchy sequences. Let $f \in CC(X, \mathbb{R})$ be arbitrary. Let $\langle \hat{X}, \hat{d} \rangle$ be the completion of $\langle X, d \rangle$. Since X is dense in $\langle \hat{X}, \hat{d} \rangle$, we can extend f to $\hat{f} \in CC(\hat{X}, \mathbb{R}) \subseteq C(\hat{X}, \mathbb{R})$ [39]. Let $\varepsilon > 0$; by Corollary 6.8 we can find a locally Lipschitz function $\hat{v}$ on $\hat{X}$ such that $\forall \hat{x} \in \hat{X},\ |\hat{f}(\hat{x}) - \hat{v}(\hat{x})| < \varepsilon$. Put $v := \hat{v}|_X$ and let B be a nonempty totally bounded subset of X. Since the $\hat{X}$-closure of B is compact, by Proposition 2.3, the restriction of $\hat{v}$ to the closure is Lipschitz, and so $v|_B$ is thus Lipschitz. Applying Proposition 2.3 again, v is a Cauchy-Lipschitz function that ε-approximates f. □

We state the following special case of Theorem 6.5 as a theorem rather than a corollary. It follows from Theorem 6.5 using the facts that (ii) a bounded Lipschitz in the small function is already Lipschitz, and (ii) if one of two functions that are at a finite uniform distance from one another is bounded, so is the other.

Theorem 6.10. *Let $\mathcal{B}$ be a family of nonempty subsets of $\langle X, d \rangle$ that is shielded from closed sets. Then the uniform closure of the real-valued continuous functions that are both bounded and Lipschitz on every member of $\mathcal{B}$ is the family of real-valued continuous functions that are both bounded and uniformly continuous on every member of $\mathcal{B}$.*

Letting $\mathcal{B} = \{X\}$ we obtain this classical result that appears in the monograph of Heinonen [28, Theorem 6.8].

Corollary 6.11. *The bounded uniformly continuous real-valued functions on a metric space $\langle X, d \rangle$ form the uniform closure of the bounded real-valued Lipschitz functions on the space.*

We don't know if our next corollary is the best possible result, as we don't have a counterexample showing that the hypothesis that the family of Bourbaki bounded subsets be shielded from closed sets is necessary. It follows from these facts: (i) a function is Bourbaki-Cauchy-Lipschitz if and only if it is Lipschitz restricted to each nonempty Bourbaki bounded subset, and (ii) a function that is Lipschitz restricted to a nonempty Bourbaki bounded subset B must be bounded on B.

Corollary 6.12. *Suppose the family of nonempty Bourbaki bounded subsets of* $\langle X, d\rangle$ *is shielded from closed sets. Then the uniform closure of the real-valued Bourbaki-Cauchy-Lipschitz functions is the family of functions that are bounded and uniformly continuous when restricted to Bourbaki bounded subsets.*

It is left to the reader to verify that a continuous function that is uniformly continuous on a Bourbaki bounded subset need not be bounded on that subset.

References

1. M. Aggarwal and J. Kundu, *More on variants of complete metric spaces*, Acta Math. Hungarica **151** (2017), 391-415.
2. R. Arens and J. Eells, *On embedding uniform and topological spaces*, Pacific J. Math. **6** (1956), 397-403.
3. M. Atsuji, *Uniform continuity of continuous functions of metric spaces*, Pacific J. Math. **8** (1958), 11-16.
4. G. Beer, *More about metric spaces on which continuous functions are uniformly continuous*, Bull. Austral. Math. Soc. **33** (1986), 397-406.
5. G. Beer, *Topologies on closed and closed convex sets*, Kluwer Academic Publishers, Dordrecht, Holland, 1993.
6. G. Beer, *McShane's extension theorem revisited*, to appear, Vietnam J. Math.
7. G. Beer, C. Costantini and S. Levi, *Bornological convergence and shields*, Mediterr. J. Math. **10** (2013), 529-560.
8. G. Beer, L. García-Lirola and M. I. Garrido, *Stability of Lipschitz-type functions under pointwsie product and reciprocation*, preprint.
9. G. Beer and M. I. Garrido, *Bornologies and locally Lipschitz functions*, Bull. Aust. Math. Soc. **90** (2014), 257-263.
10. G. Beer and M. I. Garrido, *Locally Lipschitz functions, cofinal completeness, and UC spaces*, J. Math. Anal. Appl. **428** (2015), 804-816.
11. G. Beer and M. I. Garrido, *On the uniform approximation of Cauchy continuous functions*, Top. Appl. **208** (2016), 1-9.
12. G. Beer, M. I. Garrido, and A. S. Meroño, *Uniform continuity and a new bornology for a metric space*, Set-Valued Var. Anal. **26** (2018), 49-65.
13. G. Beer and M. Hoffman, *The Lipschitz metric for real-valued continuous functions*, J. Math. Anal. Appl. **406** (2013), 229-236.
14. G. Beer and S. Levi, *Strong uniform continuity*, J. Math. Anal. Appl. **350** (2009), 568-589.
15. G. Beer and S. Levi, *Uniform Continuity, Uniform Convergence, and Shields*, Set-Valued and Variational Anal. **18** (2010), 251-275.
16. J. Borsik, *Mappings preserving Cauchy sequences*, Časopis pěst. Mat. **113** (1988), 280-285.
17. A. Bouziad and E. Sukhacheva, *Preservation of uniform continuity under pointwise product*, Top. Appl. **254** (2019), 132-144.
18. J. Cabello-Sánchez, $\mathcal{U}(X)$ *as a ring for metric spaces* X, Filomat **31** (2017), 1981-1984.
19. S. Cobzaş, R. Miculescu and A. Nicolae, *Lipschitz functions*, Springer, Cham, Switzerland, 2019.
20. C. Czipszer and L. Gehér, *Extensions of functions satisfying a Lipschitz condition*, Acta Math. Hungarica. **6** (1955), 213-220.
21. V. Efremovič, *The geometry of proximity I*, Math. Sbornik **31** (1952), 189-200.
22. M. I. Garrido and J. Jaramillo, *Homomorphisms on function lattices*, Monatsh. Math. **141** (2004), 127-146.
23. M. I. Garrido and J. Jaramillo, *Lipschitz-type functions on metric spaces*, J. Math. Anal. Appl. **340** (2008), 282-290.
24. M. I. Garrido and A.S. Meroño, *New types of completeness in metric spaces*, Ann. Acad. Sci. Fenn. Math. **39** (2014), 733-758.

25. M. I. Garrido and A.S. Meroño, *The Samuel realcompactification of a metric space*, J. Math. Anal. Appl. **456** (2017), 1013-1039.
26. I. Garrido and F. Montalvo, *Countable covers and uniform closure*, Rend. Istit. Math. Univ. Trieste **30** (1999), 91-102.
27. J. Giles, *Introduction to the analysis of normed linear spaces*, Cambridge University Press, Cambridge, 2000.
28. J. Heinonen, *Lectures on analysis on metric spaces*, Springer, New York, 2001.
29. A. Hohti, H. Junnila and A. Meroño, *On strongly Čech-complete spaces*, to appear, Top. Appl.
30. T. Jain and S. Kundu, *Atsuji spaces: equivalent conditions*, Top. Proc. **30** (2006), 301-325.
31. T. Jain and S. Kundu, *Atsuji completions: equivalent characterizations*, Top. Appl. **154** (2007), 28-38.
32. D. Leung and W.-K. Tang, *Functions that are Lipschitz in the small*, Rev. Mat. Complut. **30** (2017), 25-34.
33. J. Luukkainen, *Rings of functions in Lipschitz topology*, Ann. Acad. Sci. Fenn. Series A. I. Math. **4** (1978-79), 119-135.
34. E. Michael, *A short proof of the Arens-Eells embedding theorem*, Proc. Amer. Math. Soc. **15** (1964), 415-416.
35. R. Miculescu, *Approximation of continuous functions by Lipschitz functions*, Real Anal. Exhange **26** (2000-2001), 449-452.
36. A. A. Monteiro and M. M. Peixoto, *Le nombre de Lebesgue et la continuité uniform*, Portugaliae Math. **10** (1951), 105-113.
37. S. Naimpally, *Proximity approach to problems in topology and analysis*, Oldenbourg Verlag, Munich, 2009.
38. C. Scanlon, *Rings of functions with certain Lipschitz properties*, Pacific J. Math. **32** (1970), 197-201.
39. R. Snipes, *Functions that preserve Cauchy sequences*, Nieuw Archief Voor Wiskunde **25** (1977), 409-422.
40. S. Willard, *General Topology*, Addison-Wesley, Reading, MA, 1970.

Department of Mathematics, California State University Los Angeles, 5151 State University Drive, Los Angeles, California 90032, USA

Email address: gbeer@cslanet.calstatela.edu

Instituto de Matemática Interdisciplinar (IMI), Departamento de Álgebra, Geometría y Topología, Universidad Complutense de Madrid, 28040 Madrid, Spain

Email address: maigarri@mat.ucm.es