

Distances on the tropical line determined by two points

M.J. de la Puente

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Abstract

Let $p', q' \in \mathbb{R}^n$. Write $p' \sim q'$ if $p' - q'$ is a multiple of $(1, \dots, 1)$. Two different points p and q in \mathbb{R}^n / \sim uniquely determine a tropical line $L(p, q)$, passing through them, and stable under small perturbations. This line is a balanced unrooted semi-labeled tree on n leaves. It is also a metric graph.

If some representatives p' and q' of p and q are the first and second columns of some real normal idempotent order n matrix A , we prove that the tree $L(p, q)$ is described by a matrix F , easily obtained from A . We also prove that $L(p, q)$ is caterpillar. We prove that every vertex in $L(p, q)$ belongs to the tropical linear segment joining p and q . A vertex, denoted pq , closest (w.r.t tropical distance) to p exists in $L(p, q)$. Same for q . The distances between pairs of adjacent vertices in $L(p, q)$ and the distances $d(p, pq)$, $d(qq, q)$ and $d(p, q)$ are certain entries of the matrix $|F|$. In addition, if p and q are generic, then the tree $L(p, q)$ is trivalent. The entries of F are differences (i.e., sum of principal diagonal minus sum of secondary diagonal) of order 2 minors of the first two columns of A .

1 Introduction

Tropical algebra, geometry and analysis are novelties in mathematics. As for algebra (also called extremal algebra, max-algebra, etc.) it is just algebra performed with unusual operations: \max (for addition) and $+$ (for multiplication). As for geometry, it can be understood as a degeneration (or shadow) of classical algebraic geometry.

Tropical mathematics is an exciting fast growing field of research; see the collective works [20, 27, 28], some general references for tropical algebra [1, 2, 8, 12, 18], some general references for tropical geometry [5, 6, 16, 17, 21, 22, 29, 30, 32,

34, 38, 39] and some pioneer works [11, 19, 40, 41, 42] among others. In [3, 9] tropical curves are presented as metric graphs.

In classical projective geometry, it is easy to determine the line passing through two different given points p and q . If $[p_1, p_2, \dots, p_n]$ and $[q_1, q_2, \dots, q_n]$ are projective coordinates over a field, then the points $x = [x_1, x_2, \dots, x_n]$ in such a line are described by the rank condition

$$\text{rk} \begin{bmatrix} p_1 & q_1 & x_1 \\ p_2 & q_2 & x_2 \\ \vdots & \vdots & \vdots \\ p_n & q_n & x_n \end{bmatrix} = 2.$$

A basic question in tropical mathematics is to establish the properties of the unique tropical line $L(p, q)$, stable under small perturbations, determined by two given points p and q (to be precise, $L(p, q)$ is the limit, as ϵ tends to zero, of the tropical lines going through perturbed points $p^{v_\epsilon}, q^{v_\epsilon}$. Here, p^{v_ϵ} denotes a translation of p by a vector v_ϵ whose length is ϵ). The aim of this paper is to answer this question in a particular instance, namely, when coordinates of p and q are columns of some normal idempotent square real matrix A .

Tropical algebraic varieties can be defined algebraically (by means of ideals) or geometrically (by means of amoebas). Tropical curves can also be defined combinatorially (by means of balanced weighted graphs). For tropical lines, weights can be disregarded, since they all are equal to one. This paper is about the *combinatorial description* of the line $L(p, q)$. Moreover, we obtain $L(p, q)$ as a metric graph, with additional information. Indeed, in metric graphs, leaves have infinite length, while edges have finite length. The point p (which, in general, is not a vertex of $L(p, q)$) sits on a certain leaf of $L(p, q)$, and we determine the length from p to the closest inner vertex of $L(p, q)$ (same for q). These two lengths are extra information for the metric graph $L(p, q)$.

In this paper we never use $-\infty$. Write $\oplus = \max$ and $\odot = +$. These are the tropical operations addition and multiplication in \mathbb{R}^n . Let (e_1, e_2, \dots, e_n) denote the canonical basis in \mathbb{R}^n . We work in the quotient space $Q^{n-1} := \mathbb{R}^n / \sim$; see (3). There is a bijection between Q^{n-1} and \mathbb{R}^{n-1} .

Given different $p, q \in Q^{n-1}$, there may exist many tropical lines passing through p and q , but there is only one such line which is stable under small perturbations; see [22, 17, 32, 37]. It is denoted $L(p, q)$.

What do we know about tropical lines in Q^{n-1} ? The cases $n = 2$ or 3 are easy. Set $n = 4$. In the generic case, a *tropical line* in Q^3 is a *balanced polyhedral*

complex consisting of four rays r_1, r_2, r_3, r_4 and an edge r , so that

$$L(p, q) = r \cup \bigcup_{j=1}^4 r_j.$$

The ray r_4 extends infinitely in the direction of $e_1 + e_2 + e_3$ and positive sense, and the rays r_j do so in the negative e_j direction, for $j = 1, 2, 3$.

For arbitrary n , a generic line L in Q^{n-1} is a *balanced unrooted trivalent semi-labeled tree T on leaves marked $1, 2, \dots, n$* . Leaf marked j in T corresponds to ray r_j in L . This tree is *semi-labeled* because its inner vertices are left unlabeled. This is all well-known; see [17, 21, 22, 30, 32, 35].

What do we prove about $L(p, q)$? Let $\text{tconv}(p, q)$ denote the tropical segment joining p and q in Q^{n-1} . We have $\text{tconv}(p, q) \subset L(p, q)$, following [14]. Suppose that p, q have representatives in \mathbb{R}^n whose coordinates are the first and second columns of some normal idempotent square real matrix A of order n . In this paper we prove that every vertex of $L(p, q)$ belongs to $\text{tconv}(p, q)$; see theorem 13. This is not true in less restrictive conditions. Since $\text{tconv}(p, q)$ is compact, then there is a vertex in $L(p, q)$ closest to p (same for q), with respect to tropical distance (see (5) for the definition and properties of tropical distance). Moreover, the tree $L(p, q)$ is caterpillar. If p and q are generic, then $L(p, q)$ is trivalent; see also theorem 13.

The paper goes as follows. First, we define the *difference* of an order 2 matrix; see definition 1. We define the *matrix of differences* $F = (f_{kl})$ relative to two columns of A . Then, for $n = 4$ we prove that the combinatorics of the tree $L(p, q)$ are determined by the sign of f_{34} ; see remark in p. 13. Moreover, the tropical distances $d(p, pq)$, $d(pq, qp)$, $d(q, qp)$ and $d(p, q)$ are certain entries of the matrix of absolute values $|F|$. Here pq (resp. qp) denotes the vertex of $L(p, q)$ closest to p (resp. to q), with respect to tropical distance. Notice that pq and qp are the only vertices of the line $L(p, q)$, for $n = 4$. This is theorem 7. Then, theorem 13 is an extension of theorem 7 to arbitrary n .

The key to theorem 13 is additivity of matrix F , as stated in (12). To prove that $d(p, q) = |f_{12}|$ is straightforward; see lemma 4. The proof of theorem 13 is recursive. It goes as follows. The combinatorics of the tree $L(p, q)$ and the distances between consecutive vertices in it are determined in $n - 3$ steps. For each step, we deal with an old tree T' and a new tree T . The tree T has one more leaf than T' . More precisely, T is a *tropical modification* of T' (see [5, 6, 29] for the meaning of modification in tropical geometry). All the distances in T are kept the same as in T' with one exception: a distance in T' breaks up into two, due to the tropical modification that has happened. We make this breaking precise by defining *fractures*; see definition 11. For the understanding of the whole process, example 14 is provided in full detail, step by step, with accompanying figures 5 to 9.

We work with only two columns of a normal idempotent matrix (NI, for short). These matrices $A = (a_{ij})$ are defined by extremely simple linear equalities and inequalities; see (1). These inequalities are crucial for us to carry computations through! Normal matrices were first studied by Yoeli (under another name) in [41]. Normal idempotent matrices are related to metrics in [23, 36]. See [31] for applications of NI matrices to alcoved polytopes, and [26] for applications of normal and NI matrices to tropical commutativity.

Our results and definitions are gathered in sections 3, 4 and 5. Lemma 5 and theorem 7 were obtained with A. Jiménez and appeared before in [24]. Strictly speaking, the contents of section 4 are included in section 5. However, we prefer to keep section 4 as it stands, because it is helpful for the grasping of the rest of the paper.

2 Background

For $n \in \mathbb{N}$, set $[n] := \{1, 2, \dots, n\}$. Let $\mathbb{R}^{n \times m}$ denote the set of real matrices having n rows and m columns. Define tropical sum and product of matrices following the same rules of classical linear algebra, but replacing addition (multiplication) by tropical addition (multiplication). We will never use classical multiplication of matrices, in this note.

We will always write the coordinates of points in columns.

By definition, a square real matrix $A = (a_{ij})$ is *normal* if $a_{ii} = 0$ and $a_{ij} \leq 0$, all $i, j \in [n]$. Any real matrix can be normalized, not uniquely; see [7, 8] for details. A matrix is *idempotent* if $A = A \odot A$. If each diagonal entry of $A = (a_{ij})$ vanishes, then $A \leq A \odot A$, because for each $i, j \in [n]$, we have

$$a_{ij} \leq \max_{k \in [n]} a_{ik} + a_{kj} = (A \odot A)_{ij}.$$

We will work with *normal idempotent matrices* (NI, for short). Being NI is characterized by the following linear equalities and inequalities:

$$a_{ii} = 0, \quad a_{ij} \leq 0, \quad a_{ik} + a_{kj} \leq a_{ij}, \quad i, j, k \in [n], \quad \text{card}\{i, j, k\} \geq 2. \quad (1)$$

In particular, $a_{ik} + a_{ki} \leq 0$, for $i, k \in [n]$.

The *tropical determinant* (also called *tropical permanent*, *max-algebraic permanent*, etc.; see [8, 32]) of $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is defined as

$$|A|_{\text{trop}} = \max_{\sigma \in S_n} a_{1\sigma(1)} + a_{2\sigma(2)} + \dots + a_{n\sigma(n)},$$

where S_n denotes the permutation group in n symbols. The matrix A is *tropically singular* if this maximum is attained twice, at least. Otherwise, A is *tropically*

regular. We will never use classical determinants in this note. See [15] for tropical rank issues.

Two different points p', q' in \mathbb{R}^n determine the following set of tropical linear combinations:

$$\{\lambda \odot p' \oplus \mu \odot q' \in \mathbb{R}^n : \lambda, \mu \in \mathbb{R}\}. \quad (2)$$

This set is closed under tropical multiplication by any real number ν i.e., it is closed under classical addition of vectors νu , for $u = (1, \dots, 1)$. Therefore, it is useful to work in the quotient space

$$Q^{n-1} := \mathbb{R}^n / \sim \quad (3)$$

where $(a_1, a_2, \dots, a_n) \sim (b_1, b_2, \dots, b_n)$ if

$$(a_1, a_2, \dots, a_n) = \lambda \odot (b_1, b_2, \dots, b_n) = (\lambda + b_1, \lambda + b_2, \dots, \lambda + b_n),$$

for some $\lambda \in \mathbb{R}$. The class of $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ will be denoted $[a_1, \dots, a_n]$ or \bar{a} . The operations \oplus and \odot carry over to Q^{n-1} . Each element in Q^{n-1} has a *unique representative whose last coordinate is null*; in particular, Q^{n-1} can be identified with the classical hyperplane

$$H_n := \{x \in \mathbb{R}^n : x_n = 0\} \quad (4)$$

inside \mathbb{R}^n . As vector spaces, H_n is isomorphic to \mathbb{R}^{n-1} . We will often identify Q^{n-1} with H_n in the sequel. By this identification, the topology induced by the tropical distance corresponds to the usual topology.

Given different points $p, q \in Q^{n-1}$, consider representatives p', q' in \mathbb{R}^n . The image of (2) in Q^{n-1} is denoted $\text{tconv}(p, q)$ and called the *tropical line segment* determined by p and q . By [14], the set $\text{tconv}(p, q)$, viewed in H_n , is the concatenation of, at most, $n - 1$ ordinary line segments, and the slope of each such line segment is a zero-one vector. For negative λ , very large in absolute value, we get $\lambda \odot p' \oplus \mu \odot q' = \mu \odot q'$, whence $\lambda \odot p \oplus \mu \odot q = q$ is an endpoint of $\text{tconv}(p, q)$. (Here we have a difference between classical and tropical mathematics. In classical mathematics, expression (2) corresponds to a line, not a segment!) The tropical segment $\text{tconv}(p, q)$ is compact and connected, classically.

For $p \in \mathbb{R}^n$, set

$$||p|| := \max_{i,j \in [n]} \{|p_i|, |p_i - p_j|\}.$$

For $p, q \in Q^{n-1}$, choose (unique) representatives $p', q' \in \mathbb{R}^n$ with null last coordinate and set

$$d(p, q) := ||p' - q'|| = \max_{i,j \in [n]} \{|p_i - q_i|, |p_i - q_i - p_j + q_j|\}. \quad (5)$$

This defines a distance (or a metric) in (Q^{n-1}, \oplus, \odot) , called *tropical distance*; see [10, 13, 14, 31]. We will not use any other distance in this paper.

Recall that the *integer length* (also called lattice length) of a classical segment ab in \mathbb{R}^n joining points a and b is the ratio between the Euclidean length of ab and the minimal Euclidean length of integer vectors parallel to ab . If $a, b \in \mathbb{Z}^2$, then the integer length of ab is one less the number on integer points on the segment ab .

Recall that the tropical segment $\text{tconv}(p, q)$ is a concatenation of classical bounded segments. Thus, the integer length of $\text{tconv}(p, q)$ is the sum of the integer lengths of those segments (see p. 5). It turns out that $d(p, q)$ equals the integer length of $\text{tconv}(p, q)$.

Notice that d is additive for tropically collinear points. For example, given p, q, r and $s \in Q^2$ (represented in figure 1 by points in $H_3 \simeq \mathbb{R}^2$), with $p' = (-2, -2, 0)^t$, $q' = (0, 0, 0)^t$, $r' = (-5, -2, 0)^t$ and $s' = (-2, -5, 0)^t$, we have $d(p, q) = 2$ (not $2\sqrt{2}$!), $d(r, s) = \max\{3, 6\} = 6 = 3 + 3$ and $d(r, q) = \max\{5, 2, 3\} = 5 = 3 + 2 = d(s, q)$.

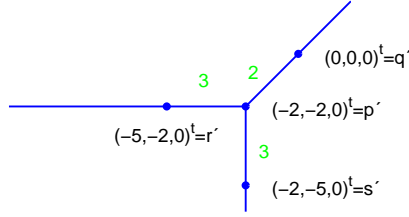


Figure 1: Tropical line in Q^2 with vertex at the point $p = [-2, -2, 0]^t$. It looks like a tripod. Distances are indicated in green.

For any $S \subseteq [n]$, write $e_S := \sum_{j \in S} e_j$ and notice that

$$\overline{e_S} = -\overline{e_{S^c}} \quad \text{in } Q^{n-1}, \quad (6)$$

where S^c is the complementary to S in $[n]$. In particular, $\overline{e_{12\dots n}} = \overline{0}$.

Any unbounded closed segment in \mathbb{R}^{n-1} in the direction of some canonical basis vector and negative sense is called a *ray*. Write r_j for a ray in the e_j direction, for $j \in [n-1]$. Any unbounded closed segment in the direction of $e_{12\dots n-1}$ and positive sense is also called a *ray*. By abuse of notation, we denote such a ray by r_n . A ray r_j is *maximal* inside a line L if the endpoint of r_j is a vertex of L . An *edge* is a bounded closed segment.

We have $\oplus = \max$ and $\odot = +$. Then, a *tropical monomial* is a classical linear form $\sum_i a_i x_i$, and a *tropical polynomial* is a maximum

$$P(x_1, x_2, \dots, x_n) = \max_{a \in \mathcal{A}} c_a + a_1 x_1 + a_2 x_2 + \dots + a_n x_n, \quad c_a \in \mathbb{R},$$

and $\mathcal{A} \subset \mathbb{N}^n$ finite. The corresponding function $P : \mathbb{R}^n \rightarrow \mathbb{R}$ is piecewise linear and concave. The *tropical hypersurface* determined by P in \mathbb{R}^n is the set of points where the *maximum is attained twice, at least*. Equivalently, it is the set of points where P is not differentiable; see [5, 6, 17, 22, 30, 32, 33]. In particular, we have tropical lines, planes and hyperplanes in \mathbb{R}^n . Then we mod out by \sim , to get tropical lines, planes and hyperplanes in Q^{n-1} .

We work in (Q^{n-1}, \oplus, \odot) . Algebraically, a tropical line in codimension one (i.e., in Q^2) is determined by one tropical polynomial of degree one. A tropical line in higher codimension is determined by an ideal generated by degree-one tropical polynomials. Tropical lines have been thoroughly studied in [33]. The paper [32] contains a detailed description of tropical lines in Q^3 ; see below p. 8.

A generic line L in Q^2 looks like a tripod in $H_3 \simeq \mathbb{R}^2$; see figure 1. It consists of three rays r_1, r_2, r_3 meeting at vertex. If $L = L(p, q)$, then the vertex is computed by the *tropical Cramer's rule*; see [32, 35, 37]. It goes as follows: given coordinates $[p_1, p_2, p_3]^t, [q_1, q_2, q_3]^t$ for p and q , consider the 2×2 tropical minors:

$$m_{ij} := \left| \begin{array}{cc} p_i & q_i \\ p_j & q_j \end{array} \right|_{\text{trop}} = \max\{p_i + q_j, p_j + q_i\}. \quad (7)$$

Then the vertex of $L(p, q)$ is

$$[-m_{23}, -m_{13}, -m_{12}]^t. \quad (8)$$

Fix $n = 4$. Let us identify Q^3 with $H_4 \simeq \mathbb{R}^3$. Set theoretically, a *tropical line* L in \mathbb{R}^3 consists of four rays r_1, r_2, r_3, r_4 and, in the generic case, an edge r :

$$L = r \cup \bigcup_{j=1}^4 r_j.$$

We have $r_j \cap r \neq \emptyset$, for all $j \in [4]$. If r collapses to a point (in the non-generic case), then $r_j \cap r_k \neq \emptyset$, for all $j, k \in [4]$. A line L in Q^3 belongs to one of the following *combinatorial types*:

$$\{12, 34\}, \quad \{13, 24\}, \quad \{14, 23\}, \quad \{1234\}.$$

Indeed, the line L is of type $\{ij, kl\}$ if and only if L has two vertices, denoted v^{ij} and v^{kl} , and the segments r, r_i, r_j meet at v^{ij} and r, r_k, r_l meet at v^{kl} , where

$\{i, j, k, l\} = [4]$. In particular, types can be written in various ways: for example, $\{12, 34\} = \{21, 34\} = \{21, 43\} = \{34, 12\} = \{43, 12\}$, etc. Moreover, the line L is a trivalent tree if its type is $\{12, 34\}$, $\{13, 24\}$ or $\{14, 23\}$, and this is the generic case; see figure 2. Let $\{i, j, k, l\} = [4]$. We can assume that $i \neq 4 \neq j$, without loss of generality. Notice that the direction of the segment r of a line L of type $\{ij, kl\}$ is e_{ij} , by the balancing condition. On the other hand, if the type of L is $\{1234\}$, then the edge r has collapsed to a point, and the four rays r_1, r_2, r_3, r_4 meet at a point, called vertex of L and denoted v^{1234} .

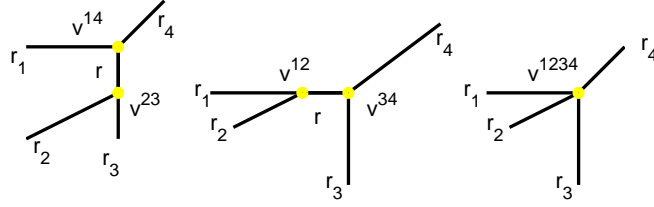


Figure 2: Some tropical lines in 3-space: type $\{14, 23\}$ on the left, type $\{12, 34\}$ center and type $\{1234\}$, on the right. These are non-planar balanced polyhedral complexes in $H_4 \simeq \mathbb{R}^3$, where the ray r_4 points in the direction e_{123} , positive sense. The segment r separates rays r_1, r_4 from r_2, r_3 in the $\{14, 23\}$ case.

It is well-known that two different points $p, q \in Q^3$ determine a unique tropical line $L(p, q)$ passing through them and stable under small perturbations; see [14, 32, 33]. If $L = L(p, q)$ and we want to compute the vertices of this line, first we must find out the combinatorial type of L . Here we follow [32]. For $1 \leq i < j \leq 4$, consider the 2×2 tropical minors m_{ij} defined in (7). These minors can be arranged into an upper triangular matrix

$$M = \begin{pmatrix} m_{12} & m_{13} & m_{14} \\ & m_{23} & m_{24} \\ & & m_{34} \end{pmatrix}. \quad (9)$$

The m_{ij} are not independent: they satisfy the *tropical Plücker relation*, i.e., the

following maximum is attained twice, at least:

$$m := \max\{m_{12} + m_{34}, m_{13} + m_{24}, m_{14} + m_{23}\}. \quad (10)$$

Then, by [32],

- the type of $L(p, q)$ is $\{12, 34\}$ when $m_{12} + m_{34} < m$,
- the type of $L(p, q)$ is $\{13, 24\}$ when $m_{13} + m_{24} < m$,
- the type of $L(p, q)$ is $\{14, 23\}$ when $m_{14} + m_{23} < m$,
- the type of $L(p, q)$ is $\{1234\}$ when the maximum m is attained three times.

A point x belongs to $L(p, q)$ if and only if

$$\text{rk} \begin{bmatrix} p_1 & q_1 & x_1 \\ p_2 & q_2 & x_2 \\ p_3 & q_3 & x_3 \\ p_4 & q_4 & x_4 \end{bmatrix}_{\text{trop}} = 2;$$

This *tropical rank* condition means that the value of each of the following 3×3 tropical minors is attained twice, at least:

$$m_1(x) := \begin{vmatrix} p_2 & q_2 & x_2 \\ p_3 & q_3 & x_3 \\ p_4 & q_4 & x_4 \end{vmatrix}_{\text{trop}} = \max\{x_2 + m_{34}, x_3 + m_{24}, x_4 + m_{23}\}$$

$$m_2(x) := \begin{vmatrix} p_1 & q_1 & x_1 \\ p_3 & q_3 & x_3 \\ p_4 & q_4 & x_4 \end{vmatrix}_{\text{trop}} = \max\{x_1 + m_{34}, x_3 + m_{14}, x_4 + m_{13}\}$$

$$m_3(x) := \begin{vmatrix} p_1 & q_1 & x_1 \\ p_2 & q_2 & x_2 \\ p_4 & q_4 & x_4 \end{vmatrix}_{\text{trop}} = \max\{x_1 + m_{24}, x_2 + m_{14}, x_4 + m_{12}\}$$

$$m_4(x) := \begin{vmatrix} p_1 & q_1 & x_1 \\ p_2 & q_2 & x_2 \\ p_3 & q_3 & x_3 \end{vmatrix}_{\text{trop}} = \max\{x_1 + m_{23}, x_2 + m_{13}, x_3 + m_{12}\}.$$

Each tropical determinant above has been expanded by the last column, by the tropical Laplace's rule. Now, for any positive, large enough $u \in \mathbb{R}$, the points

$$y_1(u) = \begin{bmatrix} -u \\ -m_{34} \\ -m_{24} \\ -m_{23} \end{bmatrix}, y_2(u) = \begin{bmatrix} -m_{34} \\ -u \\ -m_{14} \\ -m_{13} \end{bmatrix}, y_3(u) = \begin{bmatrix} -m_{24} \\ -m_{14} \\ -u \\ -m_{12} \end{bmatrix}, y_4(u) = \begin{bmatrix} -m_{23} \\ -m_{13} \\ -m_{12} \\ -u \end{bmatrix}$$

satisfy that the maximum $m_j(y_j(u))$ is attained three times, for each $j \in [4]$. Moreover, the point $y_j(u)$ moves along a ray r_j , as u tends to $+\infty$.

Say the type of $L(p, q)$ is $\{12, 34\}$. Then values $u_1, u_2, u_3, u_4 \in \mathbb{R}$ can be determined so that $y_1(u_1) = y_2(u_2) := v^{12}$ and $y_3(u_3) = y_4(u_4) := v^{34}$, obtaining the following vertices for $L(p, q)$ in Q^3 :

$$v^{12} = \begin{bmatrix} m_{13} - m_{23} - m_{34} \\ -m_{34} \\ -m_{24} \\ -m_{23} \end{bmatrix}, \quad v^{34} = \begin{bmatrix} -m_{24} \\ -m_{14} \\ m_{13} - m_{12} - m_{14} \\ -m_{12} \end{bmatrix}.$$

Say the type of $L(p, q)$ is $\{13, 24\}$. Similar calculations yield the following vertices for the line $L(p, q)$, in this case:

$$v^{13} = \begin{bmatrix} -m_{24} \\ -m_{14} \\ -m_{24} - m_{14} + m_{34} \\ -m_{12} \end{bmatrix}, \quad v^{24} = \begin{bmatrix} -m_{23} \\ -m_{13} \\ -m_{12} \\ -m_{13} - m_{12} + m_{14} \end{bmatrix}. \quad (11)$$

Say the type of $L(p, q)$ is $\{1234\}$. Then we get

$$v^{1234} = \begin{bmatrix} m_{13} + m_{14} - m_{34} \\ m_{12} \\ m_{13} \\ m_{14} \end{bmatrix}.$$

Computations are similar for type $\{14, 23\}$.

Suppose now n that is arbitrary. A generic line L in Q^{n-1} is (identified with) a *balanced unrooted trivalent semi-labeled tree* T on leaves marked $1, 2, \dots, n$ inside $H_n \simeq \mathbb{R}^{n-1}$. Leaf j of T corresponds to ray r_j of the line L , while the inner vertices of T are left unlabeled. In particular, generic tropical lines sitting in Q^{n-1} and Q^{m-1} cannot be homeomorphic, if $n \neq m$.

We consider the space \mathcal{T}_n of phylogenetic trees, studied in detail in [4, 33] (although this space is denoted $\mathcal{G}_{2,n}'''$ in [33]). Then \mathcal{T}_n is a simplicial complex of pure dimension equal to $n - 4$. The number of facets of \mathcal{T}_n is

$$(2n - 5)!!$$

(i.e., the product of all odd numbers between $2n - 5$ and 1, called *Schröder number*). Each facet of \mathcal{T}_n corresponds to a combinatorial type of unrooted trivalent semi-labeled trees on n leaves, i.e., to a combinatorial type of generic line in Q^{n-1} . In particular, for $n = 4$, there are 3 types (we have seen these types above; they were

denoted $\{12, 34\}$, $\{13, 24\}$ and $\{14, 23\}$); for $n = 5$, there are 15 types; for $n = 6$, there are 105 types, and so on.

It is known (see [35]) that \mathcal{T}_5 is the Petersen graph: it has 15 edges (these correspond to the 15 types of generic tropical lines in Q^4) and 10 vertices (these correspond to types of non-generic tropical lines, where the degree of some vertex of the line is 4). Every generic tropical line in Q^4 is a trivalent caterpillar tree on 5 leaves; see [33, 35].

Recall that a tree is *caterpillar* if it contains a path passing through every vertex of degree ≥ 2 . For instance, every tree on four leaves is caterpillar. See figure 3 for trivalent caterpillar and snowflake trees on six leaves.

It is known that \mathcal{T}_6 has 25 vertices, 105 edges and 105 triangles (i.e., there are 105 types of generic tropical lines in Q^5): 90 triangles correspond to trivalent caterpillar trees on 6 leaves, and 15 triangles to trivalent snowflake trees on 6 leaves; see [32, 33].

Any trivalent semi-labeled tree T on n leaves can be described by a finite family of *bipartitions* of $[n]$: a bipartition for each inner edge of T .

Given points $p, q \in Q^{n-1}$, we will have to describe $L(p, q)$ as a tree, combinatorially. If $L(p, q)$ is trivalent, this will be achieved by giving a family of bipartitions of $[n]$:

$$\{S_1, S_1^c\}, \{S_2, S_2^c\}, \dots, \{S_t, S_t^c\},$$

for some $t \in \mathbb{N}$ and $S_j \subset [n]$, $j \in [t]$.

3 Differences and tropical distances

Definition 1. Given numbers $a, b, c, d \in \mathbb{R}$, the difference of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $a + d - b - c$ (principal diagonal minus secondary diagonal).

Consider $A \in \mathbb{R}^{n \times n}$ and write \underline{i} to denote the i -th column of A . Let $i, j, k, l \in [n]$ with $i < j$ and $k < l$. By $A(kl; ij)$ we denote the minor $\begin{bmatrix} a_{ki} & a_{kj} \\ a_{li} & a_{lj} \end{bmatrix}$.

Definition 2. Fix the i -th and j -th columns of a matrix $A \in \mathbb{R}^{n \times n}$, with $1 \leq i < j \leq n$. For $1 \leq k < l \leq n$, set $F = (f_{kl})$ with

$$f_{kl} = a_{ki} + a_{lj} - a_{kj} - a_{li}$$

i.e., f_{kl} is the difference of the minor $A(kl; ij)$. (Obviously, the matrix F depends on i and j).

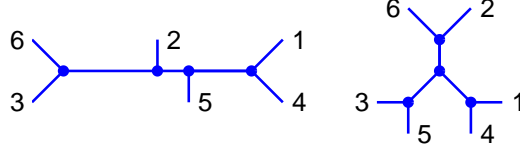


Figure 3: Two trivalent semi-labeled trees on six leaves. The inner vertices are not labeled. On the left, caterpillar having three inner edges. This tree is described the bipartitions $\{36, 1245\}$, $\{236, 145\}$, $\{2356, 14\}$. There is one inner edge separating leaves marked 3 and 6, from leaves marked 1, 2, 4 and 5. On the right, a snowflake tree having three inner edges. This tree is described by the bipartitions $\{26, 1345\}$, $\{14, 2356\}$, $\{35, 1246\}$.

Clearly,

$$f_{kl} + f_{lr} = f_{kr} \quad (12)$$

for $k < l < r$. This *additivity* (similar to that of Pascal triangle) tells us that F can be recovered from entries $f_{k-1,k}$. Compare with subadditivity of A shown in (1).

Lemma 3. *If $A \in \mathbb{R}^{n \times n}$ is NI and F is defined above, then $f_{il} \geq 0$, for $i < l$ and $f_{jl} \leq 0$, for $j < l$.*

Proof. $f_{il} = a_{lj} - a_{li} - a_{ij} \geq 0$ and $f_{jl} = a_{ji} + a_{lj} - a_{li} \leq 0$, by (1). \square

Examples of F can be found in p. 16 and 23.

For $1 \leq i < j \leq n$, let L_{ij}^A denote the line determined by columns i -th and j -th of A . Write L_{ij} , if A is understood. We will see that *some entries of the absolute value matrix $|F|$ are equal to some tropical distances between certain points of L_{ij} , the distance being defined in (5).*

To begin with, we have an easy lemma.

Lemma 4. *Assume $A \in \mathbb{R}^{n \times n}$ is NI and fix $1 \leq i < j \leq n$ with F as in definition 2. Then $d(\underline{i}, \underline{j}) = |f_{ij}|$.*

Proof. We can assume $i = 1$ and $j = 2$, by a change of coordinates. Then, by equivalence in Q^{n-1} ,

$$\underline{1} - \underline{2} = \begin{bmatrix} -a_{12} \\ a_{21} \\ a_{31} - a_{32} \\ \vdots \\ a_{n1} - a_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ a_{21} + a_{12} \\ a_{31} - a_{32} + a_{12} \\ \vdots \\ a_{n1} - a_{n2} + a_{12} \end{bmatrix}.$$

Entries in the last column are non positive, by (1), the smallest being $a_{21} + a_{12} \leq 0$, again by (1). Thus, $d(\underline{1}, \underline{2}) = |a_{21} + a_{12}| = |f_{12}|$. \square

4 Case $n = 4$

Assume that $i \neq 4 \neq j$. A generic line L is a semi-labeled trivalent tree on four leaves. It has just one inner edge r . Recall that L is of type $\{ij, kl\}$ if and only if e_{ij} is the direction of the edge r . Leaves i, j (resp. k, l) lie to one endpoint of r (resp. to the other endpoint).

Recall that L_{ij}^A denotes the line determined by columns i -th and j -th of A .

Lemma 5. Assume $A \in \mathbb{R}^{4 \times 4}$ is a NI matrix. Let $\{i, j, k, l\} = [4]$ with $i < j$. Then the type of L_{ij}^A is not $\{ij, kl\}$; it is $\{ik, jl\}$, $\{il, jk\}$ or $\{1234\}$; (easy to remember: i and j must be separated by the comma, unless the type is $\{1234\}$).

Proof. Without loss of generality, assume that $i = 1, j = 2$. Write $p = \underline{1}, q = \underline{2}$ and $L(p, q) = L_{12}^A$. Compute M in (9) and m in (10), using (1), to obtain

$$M = \begin{pmatrix} 0 & a_{32} & a_{42} \\ & a_{31} & a_{41} \\ & & \alpha \end{pmatrix}, \quad m = \max\{\alpha, a_{32} + a_{41}, a_{31} + a_{42}\}, \quad \alpha = |A(34; 12)|_{\text{trop}}. \quad (13)$$

Then, the value α is attained at the main (resp. secondary) (resp. both) diagonal(s) of $A(34; 12) = \begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}$ if and only if $\alpha = a_{31} + a_{42}$ (resp. $\alpha = a_{32} + a_{41}$) (resp. $a_{31} + a_{42} = a_{32} + a_{41}$) if and only if the type of L_{12} is $\{13, 24\}$ (resp. $\{14, 23\}$) (resp. $\{1234\}$). Thus, L_{12} is not $\{12, 34\}$. \square

Remark: looking at the former proof and definition 2, notice that the type of L_{12}^A is $\{13, 24\}$ if and only if $f_{34} > 0$. If the type were $\{14, 23\}$, then $f_{34} < 0$ and if the type were $\{1234\}$, then $f_{34} = 0$.

Recall that maximal rays inside a line were defined in p. 6.

Lemma 6. Assume $A \in \mathbb{R}^{4 \times 4}$ is NI and let $\{i, j, k, l\} = [4]$ with $i < j$. Then the vertices of the line L_{ij}^A belong to the tropical segment $\text{tconv}(\underline{i}, \underline{j})$. Moreover, $\underline{i} \in r_j$ and $\underline{j} \in r_i$, where r_i, r_j are maximal rays in L_{ij}^A .

Proof. Without loss of generality, assume that $i = 1, j = 2$. The points $\underline{1}$ and $\underline{2}$ have coordinates

$$\begin{bmatrix} 0 \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix}, \quad \begin{bmatrix} a_{12} \\ 0 \\ a_{32} \\ a_{42} \end{bmatrix},$$

respectively and we know that the coordinates of the vertices of L_{12} depend on the type of L_{12} . This type is not $\{12, 34\}$, by lemma 5.

Say the type of L_{12} is $\{13, 24\}$. Then M, m and α are shown in (13), with

$$a_{32} + a_{41} < a_{31} + a_{42} = \alpha. \quad (14)$$

Using (11), the vertices of L_{12} are

$$v^{13} = \begin{bmatrix} -a_{41} \\ -a_{42} \\ a_{31} - a_{41} \\ 0 \end{bmatrix}, \quad v^{24} = \begin{bmatrix} -a_{31} \\ -a_{32} \\ 0 \\ a_{42} - a_{32} \end{bmatrix}. \quad (15)$$

We have

$$v^{13} = \begin{pmatrix} -a_{41} \\ a_{21} - a_{41} \\ a_{31} - a_{41} \\ 0 \end{pmatrix} \oplus \begin{pmatrix} a_{12} - a_{42} \\ -a_{42} \\ a_{32} - a_{42} \\ 0 \end{pmatrix} = \overline{\underline{1} \odot (-a_{41}) \oplus \underline{2} \odot (-a_{42})}$$

and

$$v^{24} = \begin{pmatrix} -a_{31} \\ a_{21} - a_{31} \\ 0 \\ a_{41} - a_{31} \end{pmatrix} \oplus \begin{pmatrix} a_{12} - a_{32} \\ -a_{32} \\ 0 \\ a_{42} - a_{32} \end{pmatrix} = \overline{\underline{1} \odot (-a_{31}) \oplus \underline{2} \odot (-a_{32})},$$

using inequalities (1) and (14). This shows that v^{13} and v^{24} belong to $\text{tconv}(\underline{1}, \underline{2})$. Moreover

$$\underline{1} - v^{13} = \begin{bmatrix} a_{41} \\ a_{21} + a_{42} \\ a_{41} \\ a_{41} \end{bmatrix} = \begin{bmatrix} 0 \\ a_{21} + a_{42} - a_{41} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ f_{24} \\ 0 \\ 0 \end{bmatrix}, \quad (16)$$

whence $\underline{1} \in r_2$. Similarly, $\underline{2} - v^{24} = [-f_{13}, 0, 0, 0]^t$, whence $\underline{2} \in r_1$.

Computations are analogous if the type of line L_{12} is $\{14, 23\}$. \square

Recall that the tropical distance induces the usual topology. By compactness of $\text{tconv}(\underline{i}, \underline{j})$, there is a vertex in L_{ij}^A closest to \underline{i} , denoted \underline{ij} , and a vertex in L_{ij}^A closest to \underline{j} , denoted \underline{ji} , distances considered tropically. Of course, $\underline{ji} = \underline{ij}$ if and only if L_{ij} is $\{1234\}$.

In the following theorem, notice that distances depend on the type of L_{ij}^A .

Theorem 7. Assume $A \in \mathbb{R}^{4 \times 4}$ is NI and let $\{i, j, k, l\} = [4]$ with $i < j$. If the type of the line L_{ij}^A is $\{ik, jl\}$, then

1. $d(\underline{i}, \underline{ij}) = |f_{jl}|$,
2. $d(\underline{j}, \underline{ji}) = |f_{ik}|$,
3. $d(\underline{ij}, \underline{ji}) = |f_{kl}|$ (this case is easy to remember).

Proof. Without loss of generality, assume that $i = 1, j = 2$. We know that the type of L_{12} is not $\{12, 34\}$, by lemma 5.

Say the type of L_{12} is $\{13, 24\}$, so that $k = 3, l = 4$. By definition of F and (14), we have $f_{34} > 0$. Go back to (15), where coordinates for v^{13} and v^{24} were computed, to get

$$v^{13} - v^{24} = \begin{bmatrix} a_{31} - a_{41} \\ a_{32} - a_{42} \\ a_{31} - a_{41} \\ a_{32} - a_{42} \end{bmatrix} = \begin{bmatrix} a_{31} - a_{41} - a_{32} + a_{42} \\ 0 \\ a_{31} - a_{41} - a_{32} + a_{42} \\ 0 \end{bmatrix} = \begin{bmatrix} f_{34} \\ 0 \\ f_{34} \\ 0 \end{bmatrix}$$

and we obtain

$$d(v^{13}, v^{24}) = f_{34}.$$

Moreover, from (16) and lemma 3 (for $j = 2$), we get

$$d(\underline{1}, v^{13}) = -f_{24} = |f_{24}|,$$

Similarly,

$$d(\underline{2}, v^{24}) = f_{13} = |f_{13}|.$$

Now

$$\underline{2} - v^{13} = \begin{bmatrix} a_{12} + a_{41} \\ a_{42} \\ a_{32} + a_{41} - a_{31} \\ a_{42} \end{bmatrix} = \begin{bmatrix} a_{12} + a_{41} - a_{42} \\ 0 \\ a_{32} + a_{41} - a_{31} - a_{42} \\ 0 \end{bmatrix} = \begin{bmatrix} -f_{14} \\ 0 \\ -f_{34} \\ 0 \end{bmatrix}.$$

By additivity (12), we have $f_{13} + f_{34} = f_{14}$, with $f_{13} \geq 0, f_{14} \geq 0$ and $f_{34} > 0$. Thus, by the definition of tropical distance, we get

$$d(\underline{2}, v^{13}) = \max\{f_{14}, f_{34}, f_{13}\} = f_{14}.$$

We have $d(\underline{2}, v^{24}) = f_{13} < f_{14} = d(\underline{2}, v^{13})$, showing that v^{24} is closer to $\underline{2}$ than v^{13} . Thus we can relabel as follows

$$v^{24} = \underline{21}, \quad v^{13} = \underline{12}.$$

This proves the three statements for type $\{13, 24\}$. Computations are similar if the type of L_{12} is $\{14, 23\}$. \square

Example 8. Assume that $* \in \mathbb{R}$ are such that A is NI, with

$$A = \begin{bmatrix} 0 & -12 & * & * \\ -10 & 0 & * & * \\ -11 & -14 & 0 & * \\ -15 & -13 & * & 0 \end{bmatrix},$$

(this can be achieved, for instance, taking $-20 \leq a_{kl} \leq -10$, for $k, l = 3, 4$ and $k \neq l$). We have

$$F = \begin{pmatrix} 22 & 9 & 14 \\ & -13 & -8 \\ & & 5 \end{pmatrix}$$

and $d(\underline{1}, \underline{2}) = 22$, by lemma 4. By the last part in theorem 7, we get

$$d(\underline{12}, \underline{21}) = |f_{34}| = 5 \neq 0,$$

whence the type of L_{12} is not $\{1234\}$. It can be either $\{13, 24\}$ or $\{14, 23\}$, since 1 and 2 must be separated by the comma, by lemma 5. We have

$$\underline{1} - \underline{2} = \begin{bmatrix} 12 \\ -10 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 14 \\ -8 \\ 5 \\ 0 \end{bmatrix}, \quad d(\underline{1}, \underline{2}) = 22.$$

If the type were $\{14, 23\}$, by theorem 7 we would have

$$d(\underline{1}, \underline{12}) = |f_{23}| = 13, \quad d(\underline{2}, \underline{21}) = |f_{14}| = 14, \quad 22 \neq 13 + 5 + 14,$$

contradicting that the tropical distance is additive for three tropically collinear points. Thus the type is $\{13, 24\}$ and then

$$d(\underline{1}, \underline{12}) = |f_{24}| = 8, \quad d(\underline{2}, \underline{21}) = |f_{13}| = 9, \quad 22 = 8 + 5 + 9.$$

A longer way to obtain the same result is computing M, m and α in (13). We get that the type of L_{12}^A is $\{13, 24\}$, and then formulae (11) provide the coordinates of $\underline{12}$ and $\underline{21}$.

Corollary 9. Assume $A \in \mathbb{R}^{4 \times 4}$ is NI and let $1 \leq i < j \leq 4$. If the type of the line L_{ij}^A is $\{1234\}$, then for $k \in [4] \setminus \{i, j\}$ we have

1. $d(\underline{i}, \underline{ij}) = |f_{jk}|$,
2. $d(\underline{j}, \underline{ji}) = |f_{ik}|$.

\square

5 General case

Our aim for this section is to describe the tree $L(p, q)$ through the matrix F . Let pq (resp. qp) denote the vertex closest to p (resp q) in $L(p, q)$, if such a vertex exists. These two are the only inner vertices of the line $L(p, q)$ that we will *consistently label*. Vertices of $L(p, q)$ may receive temporary labels, such as v, w, x, y, z etc.

Let $A \in \mathbb{R}^{n \times n}$ be a NI matrix. For the rest of the paper, we assume that $p = \underline{1}$ and $q = \underline{2}$, so that $L(p, q) = L(\underline{1}, \underline{2}) = L_{12}^A$. This is no loss of generality. If F is as in definition 2, then

$$f_{1k} \geq 0, \quad f_{2k} \leq 0, \quad \forall k \quad (17)$$

$$f_{12} = \max_{1 \leq k < l \leq n} |f_{kl}| \quad (18)$$

by lemma 3 and the NI condition (1).

Notation: For $3 \leq s \leq n$, let A^s (resp. F^s) denote the principal minor of A (resp. of F) of order s ; in particular, $A^n = A$. The first two columns of A^s are denoted $\underline{1}^s$ and $\underline{2}^s$. The line $L(\underline{1}^s, \underline{2}^s)$ is denoted L^s . It sits inside Q^{s-1} , which can be identified with $H_s \simeq \mathbb{R}^{s-1}$. In particular, $L^n = L(p, q)$. Let $\underline{12}^s$ (resp. $\underline{21}^s$) denote the vertex of L^s closest to $\underline{1}^s$ (resp. to $\underline{2}^s$), if such a vertex exists. Let r_j^s denote any ray in the e_j negative sense inside \mathbb{R}^{s-1} , for $j \in [s-1]$, and r_s^s any ray in the $e_{12\dots s-1}$ positive sense. We know that L^s is the finite union of s rays r_1^s, \dots, r_s^s and some edges h_1, \dots, h_t , for certain $t \in \mathbb{N} \cup \{0\}$.

Definition 10. Fix s with $3 \leq s \leq n$. If, for some $1 \leq k < l \leq s$, $|f_{kl}|$ equals either the distance between two adjacent vertices in L^s or it equals $d(\underline{1}^s, \underline{12}^s)$ or $d(\underline{21}^s, \underline{2}^s)$, then we will say that f_{kl} is s -active.

Definition 11. If $|a| = |b| + |c|$ with non-zero $a, b, c \in \mathbb{R}$, we say that a fractures by means of b . We also say that a was formerly active and that b, c are newly active.

Consider the matrix F^s and assume that f_{kl} is $(s-1)$ -active, with $1 \leq k < l \leq s-1$. Then, f_{kl} fractures by means of some entry of the s -th column, if and only if

$$|f_{kl}| > |f_{ks}|. \quad (19)$$

Indeed, we will have $|f_{kl}| = |f_{ks}| + |f_{ls}|$, following from additivity (12). In practice, to find out if a fracture occurs by means of some entry of the s -th column, we can minimize the absolute value of the entries of the s -th column of F^s .

Lemma 12. Let $A \in \mathbb{R}^{n \times n}$ be NI and $3 \leq s \leq n$. Then point $\underline{2}^s$ lies to the northwest of $\underline{1}^s$ inside $H_n \subset \mathbb{R}^n$.

Proof. By equivalence in Q^{n-1} , the coordinates of $\underline{1}^s$ and $\underline{2}^s$ in H_n are

$$\begin{bmatrix} -a_{s1} \\ a_{21} - a_{s1} \\ a_{31} - a_{s1} \\ \vdots \\ a_{s-1,1} - a_{s1} \\ 0 \end{bmatrix}, \begin{bmatrix} a_{12} - a_{s2} \\ -a_{s2} \\ a_{32} - a_{s2} \\ \vdots \\ a_{s-1,2} - a_{s2} \\ 0 \end{bmatrix}$$

where the first and second coordinates compare as follows:

$$-a_{s1} \geq a_{12} - a_{s2},$$

$$a_{21} - a_{s1} \leq -a_{s2},$$

by (1). This implies the result. \square

Theorem 13. *Let $n \geq 3$ and assume p, q are different points in Q^{n-1} having representatives p', q' in \mathbb{R}^n whose coordinates are the first and second columns of a NI matrix $A \in \mathbb{R}^{n \times n}$. Then the matrix F , as in definition 2, describes the line $L(p, q)$ as a balanced unrooted semi-labeled tree on n leaves, which is caterpillar. Every vertex in $L(p, q)$ belongs to $\text{tconv}(p, q)$. The vertices pq and qp exist in $L(p, q)$. The distances between pairs of adjacent vertices in $L(p, q)$ and the distances $d(p, pq)$, $d(qp, q)$ and $d(p, q)$ are certain entries of the matrix $|F|$. In addition, if p and q are generic, then $L(p, q)$ is trivalent.*

Proof. We have $p = \underline{1}$ and $q = \underline{2}$ and $d(\underline{1}, \underline{2}) = f_{12}$, by lemma 4 and (17). Write $L = L(\underline{1}, \underline{2}) = L_{12}^A$.

First, let us assume that the couple p, q is *generic*. Then, L and F are also generic.

With notation from p. 17, let us begin with the line L^2 , joining the points $\begin{bmatrix} 0 \\ a_{21} \end{bmatrix} = \begin{bmatrix} -a_{21} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a_{12} \\ 0 \end{bmatrix}$. Then

$$d(\underline{1}^2, \underline{2}^2) = |a_{12} + a_{21}| = f_{12},$$

by lemma 4. We have $f_{12} \neq 0$, by genericity of F and f_{12} is 2-active. This is the initial step.

The proof proceeds by recursion, for $3 \leq s \leq n$. In the s -th step, the line L^s is obtained from the line L^{s-1} , by tropical modification. This precisely means that exactly one $(s-1)$ -active entry of F^{s-1} fractures. Moreover, after the s -th step is completed, we have the following properties:

1. in each row of F^s , there is some s -active entry,

2. there are exactly two s -active entries in the last column of F^s ; these are newly active,
 3. there are some negative and some positive s -active entries in F^s ,
 4. the sum of the absolute values of all s -active entries in F^s is equal to f_{12} .
- if $s = 3$, then $f_{12} + f_{23} = f_{13}$, by additivity (12). By (17) and (18),

$$|f_{12}| = |f_{13}| + |f_{23}|$$

is a fracture of $d(\underline{1}^2, \underline{2}^2) = f_{12}$. The line L^3 has a vertex, which we denote w^3 , whose coordinates are given in (8)

$$w^3 = \begin{bmatrix} -m_{23} \\ -m_{13} \\ -m_{12} \end{bmatrix} = \begin{bmatrix} -a_{31} \\ -a_{32} \\ 0 \end{bmatrix} = \overline{\underline{1} \odot (-a_{31}) \oplus \underline{2} \odot (-a_{32})},$$

equalities holding by the NI hypothesis. Then

$$\underline{1}^3 - w^3 = \begin{bmatrix} a_{31} \\ a_{32} + a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ f_{23} \\ 0 \end{bmatrix}, \quad \underline{2}^3 - w^3 = \begin{bmatrix} a_{31} + a_{12} \\ a_{32} \\ a_{32} \end{bmatrix} = \begin{bmatrix} -f_{13} \\ 0 \\ 0 \end{bmatrix} \quad (20)$$

whence

$$d(\underline{1}^3, w^3) = |f_{23}| = -f_{23}, \quad d(\underline{2}^3, w^3) = |f_{13}| = f_{13}.$$

Now f_{13}, f_{23} become 3-active, while f_{12} stops being active.

Equalities (20) tell us that walking northbound from point $\underline{1}^3$ for $|f_{23}|$ units, we reach w^3 , and walking eastbound from point $\underline{2}^3$ for f_{13} units, we also reach w^3 ; see figure 5, left. The line L^3 satisfies the statement of the theorem and it is trivalent.

- if $s = 4$, there are two cases: either $f_{34} < 0$ or $f_{34} > 0$ (by genericity of F , we have $f_{34} \neq 0$). Both cases were studied in theorem 7. Being generic, the tree L^4 is of type $\{13, 24\}$ or $\{14, 23\}$, by lemma 5. This means that leaves 1 and 2 are separated already at step $s = 4$. They will remain separated ever after. In particular, we will have

$$\underline{1}^s \in r_2^s, \quad \underline{2}^s \in r_1^s, \quad \forall s \geq 4. \quad (21)$$

The fracture is

$$d(\underline{1}^3, w^3) = |f_{23}| = |f_{24}| + |f_{34}|, \quad \text{if } f_{34} < 0 \quad (22)$$

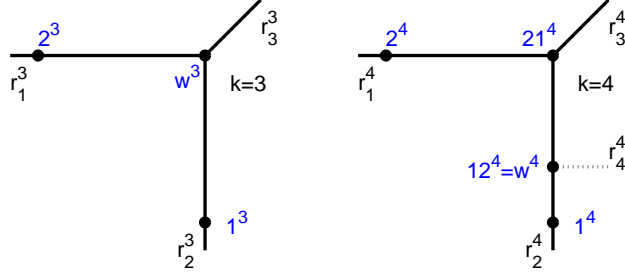


Figure 4: Modification and fracture occurring at step $s = 4$, when $f_{34} < 0$.

or

$$d(\underline{2}^3, w^3) = |f_{13}| = |f_{14}| + |f_{34}|, \quad \text{if } f_{34} > 0. \quad (23)$$

In the previous two steps ($s = 3$ or $s = 4$) two entries in the last column of F^s became s -active, while one entry of F^{s-1} stopped being active, due to the fracture. Moreover, properties 1 to 4 in p. 18 hold true.

Assuming that properties 1 to 4 hold at step $(s - 1)$, notice that exactly one fracture of one $(s - 1)$ -active entry of F^{s-1} occurs at step s , for each $5 \leq s \leq n$. Indeed, recall (19) and consider $i \in [s - 1]$ (i depending on s) such that

$$|f_{is}| = \min_{k \in [s-1]} |f_{ks}|. \quad (24)$$

By genericity of F , such an index i is unique and thus, some $(s - 1)$ -active entry on the i -th row of F^s fractures. We have only one fracture at step s , due to properties 1 to 4 and the fact that equalities (12) are not independent, for a fixed s .

Now we proceed to describe L as a tree, based on data in F . Assume, by recursion, that we have described the tree L^{n-1} and that L^{n-1} is trivalent. Write L' instead of L^{n-1} , for simplicity (similar meaning for p' , q' , F' , etc.). Being trivalent, L' is described by a finite family of bipartitions of $[n - 1]$:

$$\{S_1, S_1^c\}, \{S_2, S_2^c\}, \dots, \{S_t, S_t^c\},$$

where $t = n - 4$ is the number of inner edges of L' (by recursion), $S_j \subset [n - 1]$, with $\text{card } S_j \geq 2$ and $\text{card } S_j^c \geq 2$ (by trivalency). Moreover, the distances between pairs of adjacent vertices in L' and the distances $d(p', p'q')$, $d(q'p', q')$ and $d(p', q')$ are certain entries of $|F'|$. Now, the tree L is a *tropical modification* of L' . That means that a ray r_n^n sprouts up from L' at some point of L' , labeled w temporarily, with the *balancing condition* holding at w inside L . The point w becomes a vertex of L (although, it is not a vertex in L'). By genericity, we face two cases:

1. If w belongs to the relative interior of some inner edge r of L' . Say this segment corresponds to the bipartition $\{S_t, S_t^c\}$. We know that the leaves 1 and 2 are separated since step $s = 4$, so that

$$\{1, 2\} \cap S_t \neq \emptyset \text{ and } \{1, 2\} \cap S_t^c \neq \emptyset.$$

Say $1 \in S_t$ and $2 \in S_t^c$. Removal of the relative interior of r splits the tree L' into two subtrees, L'_1 and L'_2 , named so that 1 is a leaf in L'_1 . Then, the tree L is described by

$$\{\widehat{S_1}, \widehat{S_1^c}\}, \dots, \{\widehat{S_{t-1}}, \widehat{S_{t-1}^c}\}, \{S_t \cup \{n\}, S_t^c\}, \{S_t, S_t^c \cup \{n\}\},$$

where

$$\widehat{S} = \begin{cases} S \cup \{n\}, & \text{if } S^c \text{ is a subset of leaves of } L'_1 \text{ or of } L'_2, \\ S, & \text{otherwise.} \end{cases}$$

Moreover, we know that the endpoints of r are vertices of L' : let us label them v_1, v_2 temporarily, so that $v_1 \in L'_1$. Then

$$d(v_1, v_2) = |f_{kl}|,$$

for some $1 \leq k < l \leq n - 1$ and so f_{kl} is $(n - 1)$ -active. Due to tropical modification, this entry fractures, yielding

$$|f_{kl}| = |f_{kn}| + |f_{ln}|$$

and so

$$d(v_1, w) = |f_{ln}|, \quad d(v_2, w) = |f_{kn}|, \tag{25}$$

or

$$d(v_1, w) = |f_{kn}|, \quad d(v_2, w) = |f_{ln}|. \tag{26}$$

We decide between (25) and (26) by computing the coordinates of w in two different ways: beginning from $\underline{1}$ and beginning from $\underline{2}$.

2. If w belongs to the relative interior of a ray r'_j , some $j \in [n-1]$. Then L is given by

$$\{\{j, n\}, \{1, \dots, j-1, j+1, \dots, n-1\}\}, \{\widehat{S}_1, \widehat{S}_1^c\}, \{\widehat{S}_2, \widehat{S}_2^c\}, \dots, \{\widehat{S}_t, \widehat{S}_t^c\},$$

where

$$\widehat{S} = \begin{cases} S \cup \{n\}, & \text{if } j \in S, \\ S, & \text{otherwise.} \end{cases}$$

Due to tropical modification, one fracture of one $(n-1)$ -active f_{kl} occurs:

$$|f_{kl}| = |f_{kn}| + |f_{ln}|.$$

By recursion, we have $|f_{kl}| = d(\underline{1}', \underline{12}')$ or $|f_{sl}| = d(\underline{2}', \underline{21}')$, and recalling that $\underline{1}' \in r'_2$ and $\underline{2}' \in r'_1$ (this holds true since step $s = 4$), we get

$$j = 2 \text{ or } j = 1. \quad (27)$$

- If $|f_{kl}| = d(\underline{1}', \underline{12}')$, then $j = 2$. We relabel w as $\underline{12}$, relabel $\underline{12}'$ as v and obtain

$$d(\underline{1}, \underline{12}) = |f_{ln}|, \quad d(\underline{12}, v) = |f_{kn}|, \quad (28)$$

or

$$d(\underline{1}, \underline{12}) = |f_{kn}|, \quad d(\underline{12}, v) = |f_{ln}|. \quad (29)$$

We decide between (28) and (29) by computing the coordinates of w in two different ways: beginning from $\underline{1}$ and beginning from $\underline{2}$.

- If $|f_{kl}| = d(\underline{2}', \underline{21}')$, then the result is similar.

If the couple p, q is not generic, a sufficiently small perturbation \tilde{p}, \tilde{q} of them is generic. We apply the previous paragraphs to \tilde{p}, \tilde{q} and we obtain a line \tilde{L} . Then, the line L can be viewed as the result of the collapsing of some adjacent vertices on \tilde{L} , or the points p and pq may coincide. Same for q and qp . Passing from \tilde{L} to L amounts to vanishing of some s -active \tilde{f}_{kl} , with $1 \leq k < l \leq s \leq n$. The tree L is caterpillar, though it might not be trivalent. \square

Example 14. For $n = 7$,

$$\begin{bmatrix} 0 & -19 \\ -15 & 0 \\ -17 & -14 \\ -16 & -14 \\ -20 & -21 \\ -18 & -17 \\ -27 & -15 \end{bmatrix}$$

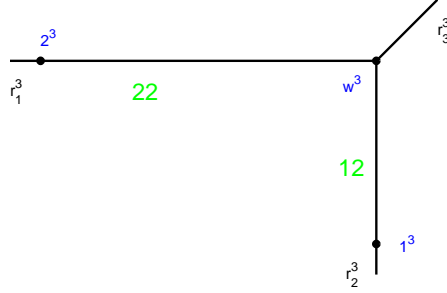


Figure 5: Construction of the tree L in example 14: step $s = 3$.

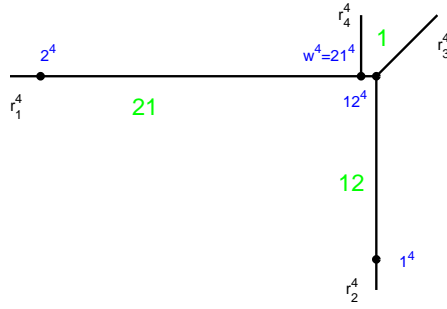


Figure 6: Construction of the tree L in example 14: step $s = 4$.

are the first two columns of a NI matrix $A = (a_{ij})$ (take, for instance $-28 \leq a_{st} \leq -14$, if $s \neq t$ and $3 \leq t \leq 7$). Then $d(\underline{1}, \underline{2}) = d(\underline{1}^2, \underline{2}^2) = |f_{12}| = f_{12} = 34$, by lemma 4 and

$$F = \begin{pmatrix} 34 & 22 & 21 & 18 & 20 & 31 \\ & -12 & -13 & -16 & -14 & -3 \\ & & -1 & -4 & -2 & 9 \\ & & & -3 & -1 & 10 \\ & & & & 2 & 13 \\ & & & & & 11 \end{pmatrix}.$$

For $3 \leq s \leq 7$, active entries of F^s will be boxed.

- $s = 3$ (see figure 5). The vertex of the line L^3 , denoted w^3 , is $[-a_{31}, -a_{32}, 0]^t =$

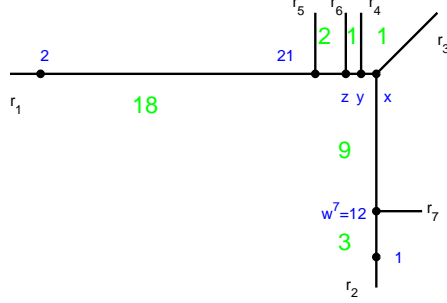


Figure 9: Construction of the tree L in example 14: final step.

The 3-active f_{sl} are boxed below:

$$F = \begin{pmatrix} 34 & \boxed{22} & 21 & 18 & 20 & 31 \\ & \boxed{-12} & -13 & -16 & -14 & -3 \\ & & -1 & -4 & -2 & 9 \\ & & & -3 & -1 & 10 \\ & & & & 2 & 13 \\ & & & & & 11 \end{pmatrix}.$$

- $s = 4$ (see figure 6). We have $-1 = f_{34} < 0$ so, by the remark after lemma 5, the type of L^4 is $\{14, 23\}$. This means that r_4^4 and r_1^4 meet, and r_2^4 and r_3^4 meet too inside L^4 . This is case 2 of the previous proof, with $j = 2$. Since $\underline{2}^4 \in r_1^4$, then the point where r_4^4 and r_1^4 meet must be $\underline{21}^4$. The entry f_{13} is 3-active and we have the fracture

$$\begin{aligned} 22 &= 21 + 1 \\ |f_{13}| &= |f_{14}| + |f_{34}|. \end{aligned}$$

of $d(\underline{2}^3, w^3) = |f_{13}|$. In fact, w^4 can be relabeled $\underline{21}^4$ and

$$\begin{aligned} \underline{12}^4 &= \underline{1}^4 + 12\overline{e_2} = \begin{bmatrix} 0 \\ -3 \\ -17 \\ -16 \end{bmatrix}, & d(\underline{1}^4, \underline{12}^4) &= |f_{23}| = 12, \\ \underline{21}^4 &= \underline{2}^4 + 21\overline{e_1} = \begin{bmatrix} 2 \\ 0 \\ -14 \\ -14 \end{bmatrix}, & d(\underline{2}^4, \underline{21}^4) &= |f_{14}| = 21, \end{aligned}$$

$$\underline{21}^4 - \underline{12}^4 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad d(\underline{12}^4, \underline{21}^4) = 1 = |f_{34}|,$$

$$34 = 21 + 1 + 12$$

and

$$F = \begin{pmatrix} 34 & & 22 & \boxed{21} & 18 & 20 & 31 \\ & \boxed{-12} & & -13 & -16 & -14 & -3 \\ & & \boxed{-1} & & -4 & -2 & 9 \\ & & & & -3 & -1 & 10 \\ & & & & & 2 & 13 \\ & & & & & & 11 \end{pmatrix}.$$

- $s = 5$ (see figure 7). Then $|f_{14}| > |f_{15}|$ and f_{14} is 4-active, so that

$$21 = 18 + 3$$

$$|f_{14}| = |f_{15}| + |f_{45}|$$

is a fracture of $d(\underline{2}^4, \underline{21}^4) = |f_{14}|$. This is case 2 of previous proof with $j = 1$. Thus, the tree L^5 is given by

$$\{15, 234\}, \{145, 23\}$$

and it is caterpillar. We have

$$\underline{1}^5 + 12\overline{e_2} + 1\overline{e_{23}} + 3\overline{e_{234}} = \begin{bmatrix} 0 \\ 1 \\ -13 \\ -13 \\ -20 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -14 \\ -14 \\ -21 \end{bmatrix} = \underline{2}^5 + 18\overline{e_1}, \quad (31)$$

so that this point is w^5 . Then,

$$d(\underline{1}^5, \underline{12}^5) = |f_{23}| = 12, \quad d(\underline{2}^5, \underline{21}^5) = |f_{15}| = 18.$$

In addition to $\underline{1}^5, \underline{12}^5, \underline{21}^5$ and $\underline{2}^5$, there is one more vertex in L^5 , denoted v temporarily, and we have

$$d(\underline{12}^5, v) = |f_{34}| = 1, \quad d(\underline{21}^5, v) = |f_{45}| = 3,$$

$$34 = 18 + 3 + 1 + 12,$$

$$F = \begin{pmatrix} 34 & 22 & 21 & \boxed{18} & 20 & 31 \\ & \boxed{-12} & -13 & -16 & -14 & -3 \\ & & \boxed{-1} & -4 & -2 & 9 \\ & & & \boxed{-3} & -1 & 10 \\ & & & & 2 & 13 \\ & & & & & 11 \end{pmatrix}.$$

- $s = 6$ (see figure 8). Then $|f_{45}| > |f_{46}|$ and f_{45} is 5-active, so that

$$\begin{aligned} 3 &= 1 + 2 \\ |f_{45}| &= |f_{46}| + |f_{56}| \end{aligned}$$

is a fracture of $d(\underline{21}^5, v) = |f_{45}|$. A ray r_6^6 sprouts up from the segment of L^5 joining $\underline{21}^5$ and v . This is case 1 of the previous proof. This happens at a point denoted w^6 temporarily and, therefore, tree L^6 is given by

$$\{15, 2346\}, \{156, 234\}, \{1456, 23\}.$$

Thus, L^6 is caterpillar and we have

$$\underline{1}^6 + 12\overline{e_2} + 1\overline{e_{23}} + 1\overline{e_{234}} = \begin{bmatrix} 0 \\ -1 \\ -15 \\ -15 \\ -20 \\ -18 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -14 \\ -14 \\ -19 \\ -17 \end{bmatrix} = \underline{2}^6 + 18\overline{e_1} + 2\overline{e_{15}},$$

and this point is w^6 . Thus,

$$\begin{aligned} d(\underline{21}^6, w^6) &= |f_{56}| = 2, & d(w^6, v) &= |f_{46}| = 1, \text{ (this information is new)} \\ d(\underline{2}^6, \underline{21}^6) &= |f_{15}| = 18, & d(v, \underline{12}^6) &= |f_{34}| = 1, & d(\underline{12}^6, \underline{1}^6) &= |f_{23}| = 12, \end{aligned}$$

$$34 = 18 + 2 + 1 + 1 + 12,$$

$$F = \begin{pmatrix} 34 & 22 & 21 & \boxed{18} & 20 & 31 \\ & \boxed{-12} & -13 & -16 & -14 & -3 \\ & & \boxed{-1} & -4 & -2 & 9 \\ & & & -3 & \boxed{-1} & 10 \\ & & & & \boxed{2} & 13 \\ & & & & & 11 \end{pmatrix}.$$

- $s = 7$ (see figure 9). Then $|f_{23}| > |f_{27}|$ and f_{23} is 6-active, whence

$$12 = 3 + 9$$

$$|f_{23}| = |f_{27}| + |f_{37}|$$

is a fracture of $d(\underline{1}^6, \underline{12}^6) = |f_{23}|$. A ray r_7^7 sprouts out of r_1^6 (this is case 2 of previous proof, with $j = 2$) at a point labeled w^7 . The tree $L = L^7$ is given by

$$\{15, 23467\}, \{156, 2347\}, \{1456, 237\}, \{13456, 27\}$$

and we have

$$\underline{1} + 3\overline{e_2} = \begin{bmatrix} 0 \\ -12 \\ -17 \\ -16 \\ -20 \\ -18 \\ -27 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ -5 \\ -4 \\ -8 \\ -6 \\ -15 \end{bmatrix} = \underline{2} + 18\overline{e_1} + 2\overline{e_{15}} + 1\overline{e_{156}} + 1\overline{e_{1456}} + 9\overline{e_{13456}} \quad (32)$$

so that this point is w^7 . Now, we relabel w^7 as $\underline{12}$. In addition to vertices $\underline{12}$ and $\underline{21}$, there are three more vertices in L , labeled x, y and z . We have

$$\begin{aligned} d(\underline{1}, \underline{12}) &= 3, & d(\underline{12}, x) &= 9, \text{ (this information is new)} \\ d(x, y) &= d(y, z) = 1, & d(z, \underline{21}) &= 2, & d(\underline{21}, \underline{2}) &= 18, \\ 34 &= 18 + 2 + 1 + 1 + 9 + 3, \end{aligned}$$

$$F = \begin{pmatrix} 34 & 22 & 21 & \boxed{18} & 20 & 31 \\ & -12 & -13 & -16 & -14 & \boxed{-3} \\ & & \boxed{-1} & -4 & -2 & \boxed{9} \\ & & & -3 & \boxed{-1} & 10 \\ & & & & \boxed{2} & 13 \\ & & & & & 11 \end{pmatrix},$$

and finally

$$|F| = \begin{pmatrix} * & * & * & d(\underline{2}, \underline{21}) & * & * \\ & * & * & * & * & d(\underline{1}, \underline{12}) \\ & & d(x, y) & * & * & d(\underline{12}, x) \\ & & & * & d(y, z) & * \\ & & & & d(z, \underline{21}) & * \\ & & & & & * \end{pmatrix}.$$

Remark: an algorithm is implicit in the the previous process; the details of it are postponed to a future paper.

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References

- [1] M. Akian, R. Bapat and S. Gaubert, *Max-plus algebra*, chapter 25 in *Handbook of linear algebra*, L. Hobgen (ed.) Chapman and Hall, 2007.
- [2] F.L. Baccelli, G. Cohen, G.J. Olsder and J.P. Quadrat, *Synchronization and linearity*, John Wiley; Chichester; New York, 1992.
- [3] M. Baker and X. Faber, *Metric properties of the tropical Abel–Jacobi map*, J. Algebr. Comb. **33**, (2011), 349–381.
- [4] L.J. Billera, S. P. Holmes and K. Vogtmann, *Geometry of the space of phylogenetic trees*, Adv. Appl. Math., **27** n.4, (2001), 733–767.
- [5] E. Brugallé, *Un peu de géométrie tropicale*, Quadrature, **74**, (2009), 10–22.
- [6] E. Brugallé, *Some aspects of tropical geometry*, Newsletter of the European Mathematical Society, **83**, (2012), 23–28.
- [7] P. Butkovič, *Simple image set of $(\max, +)$ linear mappings*, Discrete Appl. Math. **105**, (2000), 73–86.
- [8] P. Butkovič, *Max-plus linear systems: theory and algorithms*, (2010), Springer.
- [9] M. Chan, *Tropical hyperelliptic curves*, J. Algebr. Comb. **37**, (2013), 331–359.
- [10] G. Cohen, S. Gaubert and J.P. Quadrat, *Duality and separation theorems in idempotent semimodules*, Linear Algebra Appl. **379**, (2004), 395–422.
- [11] R. Cuninghame–Green, *Minimax algebra*, LNEMS, **166**, Springer, 1970.
- [12] R.A. Cuninghame–Green, *Minimax algebra and applications*, in *Adv. Imag. Electr. Phys.*, **90**, P. Hawkes, (ed.), Academic Press, 1–121, 1995.
- [13] R.A. Cuninghame–Green, P. Butkovič, *Bases in max-algebra*, Linear Algebra Appl. **389**, (2004) 107–120.
- [14] M. Develin, B. Sturmfels, *Tropical convexity*, *Doc. Math.* **9**, 1–27, (2004); Erratum in *Doc. Math.* **9** (electronic), 205–206, (2004).

- [15] M. Develin, F. Santos, B. Sturmfels, *On the rank of a tropical matrix*, in *Discrete and Computational Geometry*, E. Goodman, J. Pach and E. Welzl, eds., MSRI Publications, Cambridge Univ. Press, 2005.
- [16] M. Einsiedler, M. Kapranov and D. Lind, *Non-archimedean amoebas and tropical varieties*, J. Reine Angew. Math. **601** (2006), 139–157.
- [17] A. Gathmann, *Tropical algebraic geometry*, Jahresbericht der DMV, **108**, n.1, (2006), 3–32.
- [18] S. Gaubert and Max Plus, *Methods and applications of $(\max, +)$ linear algebra*, in R. Reischuk and M. Morvan, (eds.), STACS’97, **1200** in LNCS, 261–282, Lübeck, March 1997, Springer.
- [19] M. Gondran and M. Minoux, *Graphs, dioids and semirings. New models and algorithms*, Springer, 2008.
- [20] J. Gunawardena (ed.), *Idempotency*, Publications of the Newton Institute, Cambridge U. Press, 1998.
- [21] I. Itenberg, E. Brugallé, B. Tessier, *Géométrie tropicale*, Editions de l’École Polytechnique, 2008.
- [22] I. Itenberg, G. Mikhalkin and E. Shustin, *Tropical algebraic geometry*, Birkhäuser, 2007.
- [23] M. Johnson and M. Kambites, *Idempotent tropical matrices and finite metric spaces*, to appear in Adv. Geom.; arXiv: 1203.2480, 2012.
- [24] A. Jiménez and M.J. de la Puente, *Six combinatorial classes of maximal convex tropical polyhedra*, arXiv: 1205.4162, 2012.
- [25] D. Joyner, A. Ksir and C.G. Melles, *Automorphism group on tropical curves. Some cohomology calculations*, Beitr. Algebra Geom. **53**, n. 1, (2012), 41–56.
- [26] J. Linde and M.J. de la Puente, *Matrices commuting with a given normal tropical matrix*, arXiv: 1209.0660v2, 2014.
- [27] G.L. Litvinov, V.P. Maslov, (eds.) *Idempotent mathematics and mathematical physics*, Proceedings Vienna 2003, American Mathematical Society, Contemp. Math. **377**, (2005).
- [28] G.L. Litvinov, S.N. Sergeev, (eds.) *Tropical and idempotent mathematics*, Proceedings Moscow 2007, American Mathematical Society, Contemp. Math. **495**, (2009).

- [29] G. Mikhalkin, *Tropical geometry and its applications*, in Invited lectures, v. II, Proceedings of the ICM, Madrid, 2006, (M. Sanz–Solé et al. eds.) 827–852; arXiv:math.AG/0601041, 2006.
- [30] G. Mikhalkin, *What is a tropical curve?*, Notices AMS, April 2007, 511–513.
- [31] M. J. de la Puente, *On tropical Kleene star matrices and alcoved polytopes*, Kybernetika, **49**, n.6, (2013) 897–910.
- [32] J. Richter–Gebert, B. Sturmfels, T. Theobald, *First steps in tropical geometry*, in [27], 289–317.
- [33] D. Speyer, B. Sturmfels, *The tropical grassmannian*, Adv. Geom. **4**, 389–411, (2004).
- [34] D. Speyer, B. Sturmfels, *Tropical mathematics*, Math. Mag. **82** (2009) 163–173.
- [35] B. Sturmfels, *Solving systems of polynomial equations*, CBMS Regional Conference Series in Math. **97**, AMS, Providence, RI, 2002.
- [36] B. Sturmfels and J. Yu, *Classification of six–point metrics*, Electronic Journal of Combinatorics, **11**, 44 p., (2004).
- [37] L.F. Tabera, *Tropical constructive Pappus’s theorem*, IMRN **39**, 2373–2389 (2005).
- [38] O. Viro, *Dequantization of real algebraic geometry on logarithmic paper*, European Congress of Mathematics, Vol.I (Barcelona 2000), Prog. Math. 201, Birkhäuser, Basel, 2001, 135–146.
- [39] O. Viro, *On basic concepts of tropical geometry*, Proceedings of the Steklov Inst. Math., **273** (2011), 252–282.
- [40] E. Wagner, *Finitely generated moduloïds. The existence and unicity problem for bases*, in LNCIS **111**, Springer, 1988.
- [41] M. Yoeli, *A note on a generalization of boolean matrix theory*, Amer. Math. Monthly **68**, n.6, (1961) 552–557.
- [42] K. Zimmermann, *Extremální algebra*, Výzkumná publikace ekonomicko–matematické laboratoře při ekonomickém ústavě ČSAV, **46**, Prague, 1976, in Czech.