UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

Bell violations in quantum information: non-signalling, communication and multipartite scenarios

Violaciones de Bell en información cuántica: escenarios de no señalización, con comunicación y multipartitos

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Abderramán Amr Rey

DIRECTORES

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UNIVERSIDAD COMPLUTENSE DE MADRID

Facultad de Ciencias Matemáticas

Departamento de Análisis y Matemática Aplicada

A thesis submitted in fulfillment of the requirements
for the degree of Doctor of Philosophy by

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Under the supervision of

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Resumen

En fundamentos de información cuántica se estudian las posibles ventajas de utilizar recursos cuánticos frente a recursos clásicos. Su relevancia se debe al gran desarrollo actual de las tecnologías cuánticas.

Una de las líneas sobre las que se basa este estudio fue iniciada por Bell en el año 1964. Estableció que ciertos resultados obtenidos en mediciones separadas en sistemas cuánticos compuestos son incompatibles con una descripción de la naturaleza local y realista, incluso en el supuesto en el que permitamos la existencia de variables ocultas. Este fenómeno se conoce como no localidad cuántica.

Además del interés teórico que esto tiene, se ha demostrado, desde la década de los 90, que este fenómeno cuántico se puede aprovechar para obtener ventajas en criptografía, complejidad de comunicación y amplificación de aleatoriedad.

Desde un punto de vista matemático, el trabajo de Bell se reduce a estudiar funcionales actuando sobre distribuciones de probabilidad condicionada. Esto dirige nuestra investigación hacia la teoría de violaciones cuánticas de desigualdades de Bell. Si dado un funcional lineal llamamos su valor cuántico al máximo que este funcional alcanza sobre las correlaciones cuánticas y su valor clásico al que alcanza sobre las correlaciones clásicas, una cantidad frecuentemente utilizada para entender la diferencia entre distribuciones de probabilidad locales y cuánticas es precisamente este ratio de violación relativo de los distintos funcionales de Bell.

Expresando este escenario en el lenguaje moderno de juegos, Tsirelson estudió el modelo físico de dos jugadores separados espacialmente, que reciben un número arbitrario de preguntas, tienen dos contestaciones posibles, y pueden usar o bien recursos cuánticos o bien recursos clásicos. Caracterizó este escenario mediante el uso de normas definidas sobre el espacio de las matrices de tamaño determinado por el número de preguntas. Así, cada norma define un conjunto de matrices por medio de su bola unidad que se corresponde con el conjunto de correlaciones que pueden generar los jugadores cuando tienen acceso a un cierto recurso físico. Además demostró un resultado crucial: el ratio de violación cuántico-clásico de cualquier funcional está acotado por la llamada constante de Grothendieck.

Posteriormente, una serie de trabajos continuaron el estudio de este ratio en escenarios más generales de forma exitosa, obteniendo situaciones en las que el ratio anterior puede ser arbitrariamente grande. Se puso así de manifiesto no solo cómo de bueno es un recurso en comparación con el otro desde un punto de vista fundamental, sino que también se ha podido dar cotas a la dimensión del espacio de Hilbert que tenemos que utilizar, o, incluso, se ha conocido cómo de resistente al ruido es el estado que estamos considerando.

El objetivo de esta tesis es seguir con esta línea de investigación. Estudiaremos el ratio que hemos mencionado anteriormente en diferentes contextos de especial relevancia en información cuántica, consiguiendo, en la medida de lo posible, cotas óptimas. Siguiendo con la idea de Tsirelson, asociaremos conjuntos de distribuciones asociadas a cada recurso físico con normas tensoriales.

La no localidad es el punto de partida de nuestra tesis. Este fenómeno puede desligarse de la teoría cuántica y estudiarse en sí mismo. Requiriendo a nuestro marco de referencia que sea compatible con la teoría de relatividad, es decir, prohibiendo la comunicación instantánea entre las partes espacialmente separadas, se puede formular elegantemente la teoría de no señalización. Cualquier teoría, clásica, cuántica o postcuántica queda englobada en este marco al basarse en principios tan fundamentales. Así pues, el primer escenario que consideramos es el de las distribuciones de probabilidad de no señalización dentro de los escenarios bipartitos. Utilizaremos las técnicas antes mencionadas para poder computar el ratio de violación entre este tipo de recurso y la teoría clásica, estudiando así las última limitaciones de cualquier teoría física.

Otro escenario importante en teoría de la información es el de complejidad de comunicación. En este se estudia el número mínimo de bits o de cúbits que dos partes, físicamente separadas y habiendo recibido dos entradas cualesquiera, han de intercambiar para computar correctamente una cierta función booleana. En esta tesis trataremos en cambio un escenario ligeramente diferente siguiendo con la idea de que los funcionales de Bell son capaces de describir el escenario que estamos considerando. Estudiaremos las distintas distribuciones de probabilidad que pueden generar dos partes mediante un envío limitado de información. En nuestro trabajo veremos un juego en el cual el envío unidireccional de $\log n$ cúbits es tan bueno como el envío bidireccional de n bits clásicos.

Por último trataremos el escenario multipartito, en el que aumentamos el número de partes interactuantes a un número arbitrario, en general mayor que dos. En este caso surgen diferentes nociones de no localidad, haciendo la clasificación de este escenario más compleja. La extensión más natural de la noción de localidad es la de totalmente no local, en las cuales las medidas realizadas por cada una de las partes son locales y realistas. El problema con esta definición es que el hecho de que una distribución sea totalmente no local no significa que todas las partes contribuyan en la no localidad, pues el recurso no local podría ser compartido por un subconjunto estricto de las partes. Fue por ello que Svetlichny introdujo un nuevo tipo de no localidad, la bilocalidad. Demostró también que existen correlaciones cuánticas que muestran una no localidad mayor que esta. En nuestro trabajo compararemos asintóticamente el ratio de violación cuántico-bilocal y mostraremos que en el caso tripartito, y por tanto en cualquier otro caso con más partes, la utilización de recursos cuánticos puede ser arbitrariamente mejor que cualquier recurso bilocal.

Summary

In foundations of quantum information, the possible advantages of using quantum resources compared to classical resources are studied. Its relevance is due to the great current development of quantum technologies.

One of the lines on which this study is based was started by Bell in 1964. He established that certain results obtained by measuring composite quantum systems separately are incompatible with a local and realistic description of nature, even in the assumption in which we allow the existence of hidden variables. This phenomenon is known as quantum non-locality.

In addition to the theoretical interest that this has, it has been shown, since the 1990s, that this quantum phenomenon can be used to obtain advantages in cryptography, communication complexity and randomness amplification.

From a mathematical point of view, Bell's work is reduced to studying functionals acting on conditional probability distributions. This directs our research towards the theory of quantum violations of Bell inequalities. If, given a linear functional, we call its quantum value the maximum that this functional reaches over quantum correlations and its classical value the one it reaches over classical correlations, a quantity which is frequently used to understand the difference between local and quantum probability distributions is precisely this relative ratio of violation of the different Bell functionals.

Expressing this scenario in the modern language of games, Tsirelson studied the physical model of two spatially separated players, who receive an arbitrary number of questions, have two possible answers, and can use either quantum or classical resources. He characterized this scenario using tensor norms on the matrix space of dimension equal to the number of questions. Thus, each norm defines a set of matrices by means of its unit ball that corresponds to the set of correlations that players can generate when they have access to a certain physical resource. He also proved a crucial result: the quantum-classical ratio of violation of any functional is bounded by Grothendieck's constant.

Subsequently, a series of studies successfully continued the study of this ratio in more general scenarios, obtaining situations in which the previous ratio could be arbitrarily large. It was thus revealed not only how good one resource is compared to the other from a fundamental point of view, but also that it has also been possible to give bounds to the dimension of the Hilbert space that we have to use, or even, to know how resistant to noise the state we are considering is.

The objective of this thesis is to continue with this line of research. We will study the aforementioned ratio in different contexts of special relevance in quantum information, achieving, as far as possible, optimal bounds. Continuing with Tsirelson's idea, we will associate tensor norms with the set of probability distributions generated with a certain physical resource.

Non-locality is the starting point of our thesis. This phenomenon can be detached from quantum theory and studied on its own. By requiring our frame of reference to be compatible with the theory of relativity, that is, by prohibiting instantaneous communication between spatially separated parties, the non-signalling theory can be elegantly formulated. Any theory, classical,

quantum or post-quantum, is included in this framework as it is based on such fundamental principles. Thus, the first scenario that we consider is that of the non-signalling probability distributions within the bipartite scenarios. We will use the aforementioned techniques to compute the ratio of violation between this type of resource and the classical theory, thus studying the ultimate limitations of any physical theory.

Another important scenario in information theory is that of communication complexity. In this scenario one studies the minimum number of bits or qubits that two physically separated parties have to exchange in order to correctly compute a certain Boolean function. Instead, in this thesis we will deal with a slightly different scenario, following the idea that Bell functionals are capable of describing the scenario we are considering. We will study the different probability distributions that two parties can generate using a limited amount of information. In our work we will look at a game in which the one-way communication of $\log n$ qubits is as good as the two-way communication of n classical bits.

Finally we will deal with the multipartite scenario, in which we increase the number of interacting parties to an arbitrary number, generally greater than two. In this case, different notions of non-locality can be established, making the classification of this scenario more complex. The most natural extension of the notion of locality is that of totally non-local, in which the measurements made by each of the parties are local and realistic. The problem with this definition is that the fact that a distribution is totally non-local does not mean that all parties contribute in the non-locality, since the non-local resource could be shared by a strict subset of the parties. It was for this reason that Svetlichny introduced a new type of non-locality, the bi-locality. He also showed the existence of quantum correlations that show a non-locality stronger than this. In our work, we will asymptotically compare the quantum-bilocal ratio of violation and show that in the tripartite case, and therefore in any other case with more parties, the use of quantum resources can be arbitrarily better than any bilocal resource.

Contents

Chapter 1

Introduction

The relevance of quantum information nowadays comes from the development of quantum technologies. Potentially quantum resources can be much more powerful than classical ones. Thus, it is more important than ever to have an extensive and well-founded theory about the possible advantages that quantum information can offer us.

Quantum mechanics began to develop in the mid-1920s. Its four postulates have provided a framework capable of describing reality with astonishing precision. However, the probabilistic behaviour of quantum mechanics has caused many scientists to question this theory and to remain skeptical about the non-deterministic behaviour of nature.

In 1935, Einstein, Podolsky and Rosen [35] proposed a theoretical experiment that could lead to possible incompleteness or invalidity of quantum mechanics. Two interacting particles move off in opposite directions and cease to interact. Then, the measurement of either the position or the momentum of one of the particles allows you to deduce the position or the momentum of the other particle without directly measuring it. They argued that both physical quantities should be then "elements of reality". Since, according to Heisenberg's uncertainty principle, for two non-commuting operators, the knowledge of one quantity precludes the knowledge of the other, they concluded that there should exist more "elements of reality" beyond our control that would determine these physical quantities. This motivated Bohm's work [15] to consider hidden variables to underlie quantum phenomena. Another possible explanation to this paradox, although it was not well accepted initially, would consider some non-local property of nature: one part of the system would be able to modify instantly the other separated part of the system. This was known as "spooky action at distance", and it was believed to violate causality. The very same year, Schrödinger used, motivated by this, the term "verschränkter Zustand" (entangled state) for the first time in history.

We had to wait almost 30 years until an explanation of this paradox was given. In 1964, Bell [12] proposed an easy experiment to disprove the hidden variable model. It established that the results obtained by spatially separated measurements on composite quantum systems are incompatible with a local variable model. The first experiment in this direction was realized by Aspect in 1982 [3], but it was not until recently that a loophole-free experiment was finally performed [42]. This experiment can be considered then as an irrefutable piece of evidence against a local and realistic model to describe Nature.

These types of theoretical experiments arose from the human being's interest in the description of natural phenomena. But aside from its fundamental relevance, this physical behaviour, known as *quantum non-locality*, is behind many important applications in quantum information theory. Quantum cryptography [36, 1, 75], communication complexity [25], randomness expansion [65] and randomness amplification [28, 39] are just a few examples of how the non-local properties of entangled states can be used.

After three decades of great achievements (see e.g. the review [20]), quantum non-locality is still an ongoing topic of research. In this thesis we will try to obtain a better understanding inside the foundations of quantum information, in particular in the topics of the upper non-communication limit, known as non-signalling, the comparison between quantum and classical communication and the multipartite scenario. For this reason, and in order to make this work self-contained, we will start by explaining the simplest scenario that enables quantum non-locality: two isolated parties in a Bell scenario.

1.1 Physical scenario and basic definitions

In the Bell scenario there are two spatially separated parties, called Alice and Bob. They have different measurement devices available that they will use according to the input they receive in order to generate an outcome. The number of inputs and outputs are both assumed to be finite and the outputs do not need to be deterministic.

More precisely, given $N \in \mathbb{N}$, denote $[N] = \{1, \dots, N\}$. In our physical scenario, Alice and Bob will receive x and y in $[N]$ and they will output a and b in $[K]$ with certain probability $P(a, b|x, y)$. The main object of the study is the tensor

$$P = \{P(a, b|x, y)\}_{x,y;a,b=1}^{N,K},$$

which has the properties $P(a, b|x, y) \geq 0$ for all x, y, a, b and

$$\sum_{a,b} P(a, b|x, y) = 1,$$

for every x and y. Moreover, from an algebraic point of view, P is an element in $\mathbb{R}^{N^2K^2}$. Let us denote by $\mathcal{P}$ the subset of $\mathbb{R}^{N^2K^2}$ given by such elements and we will refer to them as *bipartite probability distributions* or simply *probability distributions*[1].

It was noticed by Bell that the assumption of a physical theory to explain the previous experiment leads to a subset of $\mathcal{P}$ formed by those probability distributions compatible with such a theory.

The fact that Alice and Bob are spatially separated imposes some restrictions on the probability distributions that they can generate, especially since they do not know the other agent's input. Hence, Alice's outcomes should not depend on the input of Bob and viceversa. This is known as the non-signalling principle. Mathematically speaking the *non-signalling conditions* are:

$$\sum_a P(a, b|x, y) = \sum_a P(a, b|x', y) \text{ for all } x, x', y, b, \tag{1.1.1}$$

$$\sum_b P(a, b|x, y) = \sum_b P(a, b|x, y') \text{ for all } y, y', x, a. \tag{1.1.2}$$

The set formed by the probability distributions that follow these conditions is called the *non-signalling* set. We will denote it by $\mathcal{NS}$.

[1]Notice that, although we call P a probability distribution, only $(P(a, b|x, y))_{a,b}$ is a probability distribution for all x and y.

Notice that if we have a probability distribution such that, for instance, the first condition does not hold, then, Bob would be able to infer, by doing statistics, some information about the input that Alice has used. This would imply instant communication from Alice to Bob violating the principles of relativity. Such theory would be then not valid to describe the world.

Coming back to the work of EPR [35], the authors Einstein, Podolsky and Rosen tried to give an explanation to the paradox by the use of "elements of reality". The study of theories based on hidden variable models is partially motivated by the attempt to avoid the intrinsic uncertainty in Nature, assumed by quantum mechanics. A local hidden variable model (LHVM) assumes that the possible outcomes that Alice and Bob obtain on a given measurement depend on its respective inputs and it also may depend on some unknown parameters, called hidden variables. We model them by a probability space (Λ, λ) where the λ's in Λ are the hidden variables.

Definition 1.1.1. *Given a probability distribution $P \in \mathcal{P}$, we will say that P is* Classical *or LHVM if we can write it as:*

$$P(a,b|x,y) = \int_\Lambda P_\lambda(a|x) Q_\lambda(b|y) d\lambda \quad \text{for every } x, y, a, b,$$

where (Λ, λ) is a probability space, $P_\lambda(a|x) \geq 0$ and $\sum_a P_\lambda(a|x) = 1$ for all a, x, ω; and analogous conditions hold for $Q_\lambda(b|y)$. The set of classical probability distributions will be denoted by $\mathcal{L}$.

The set $\mathcal{L}$ is a polytope. In particular, it is convex.

In Appendix A we recall the postulates of quantum mechanics and explain why the following definition describes those probability distributions if we accept quantum mechanics as a model of nature. We will use the bra-ket notation, introduced by Dirac [33], where the vectors v of a Hilbert space $\mathcal{H}$ are denoted by $|v\rangle$ and the scalar product between $|u\rangle$ and $|v\rangle$ is written as $\langle u|v\rangle$.

Definition 1.1.2. *We say that a probability distribution $P \in \mathcal{P}$ is* Quantum *if there exist two Hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_B$ such that*

$$P(a,b|x,y) = \langle\psi| E_x^a \otimes F_y^b |\psi\rangle \quad \text{for every } x, y, a, b,$$

where $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is a vector of norm one and $(E_x^a)_{x,a} \subset B(\mathcal{H}_A)$, $(F_y^b)_{y,b} \subset B(\mathcal{H}_B)$ are two sets of operators representing POVM measurements on Alice's and Bob's system respectively. That is, E_x^a is semidefinite positive and $\sum_a E_x^a = \mathrm{id}_{\mathcal{H}_A}$ for every a, x; and analogous conditions hold for $(F_y^b)_{y,b}$. We will denote by $\mathcal{Q}$ the set of quantum probability distributions.

Note that in Definition 1.1.2, $E_x^a \otimes F_y^b$ is an operator in $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $E_x^a \otimes F_y^b|\psi\rangle$ is an element in $\mathcal{H}_A \otimes \mathcal{H}_B$, so it makes sense to consider the inner product with $|\psi\rangle$.

It is well known that quantum probability distributions form also a convex set, which is not closed in general [71].

Recall that these sets, $\mathcal{L}$, $\mathcal{Q}$, $\mathcal{NS}$ and $\mathcal{P}$, change according to the number of inputs and the number of outputs that we are considering. Therefore, in general, for a set of probability distributions $\mathcal{A}$ we should write $\mathcal{A}(N, K)$. We can even go further considering the number of inputs and outputs for Alice different to those for Bob. Nevertheless, in order to simplify the notation, we will write simply $\mathcal{A}$ when we assume N inputs and K outputs for both Alice and Bob.

It is easy to see that $\mathcal{L} \subset \mathcal{Q}$, which means that any classical probability distribution can be produced using quantum resources. But according to Bell's theorem [12], the inclusion of $\mathcal{L}$ into $\mathcal{Q}$ is strict, i.e., these sets are different. Hence, we can separate the disjoint points by using the hyperplane separation theorem. An hyperplane can be seen as the kernel of a certain non-zero linear functional, which in our case will act on the set of bipartite probability distributions. More precisely,

Definition 1.1.3. *Given $M = \{M_{xy}^{ab}\}_{x,y,a,b} \in \mathbb{R}^{N^2K^2}$, the action of M over $P \in \mathcal{P}$ is given by:*

$$\langle M, P\rangle = \sum_{x,y,a,b} M_{xy}^{ab} P(a,b|x,y).$$

These functionals receive the name of Bell functionals.

In literature the situation in which there exists $\alpha \in \mathbb{R}$ such that $\langle M, P\rangle \leq \alpha$ for all $P \in \mathcal{L}$, is called a *Bell inequality*. And if there exists $P_0 \in \mathcal{Q}$ such that $\langle M, P_0\rangle > \alpha$, then we have a *violation of a Bell inequality*.

There is an interesting mathematical construction associated to the two-output scenario which is particularly simple and has been widely studied. In this case, with $K = 2$, each probability distribution is a point in the space $\mathbb{R}^{4N^2}$. Without loosing too much information, we can shrink it into an object in $\mathbb{R}^{N^2}$. To construct this, first name the two possible outputs a and b as $\{-1, +1\}$. Then, given a bipartite probability distribution P, consider the expectation:

$$\gamma_{xy} = \mathbb{E}_P[a \cdot b|x, y] = \sum_{a,b} abP(a,b|x,y).$$

The object, called *correlation*, is a matrix $(\gamma_{xy})_{xy}$ in $\mathbb{R}^{N^2}$. We say $\gamma \in \mathcal{L}_{cor}$ (resp. in $\mathcal{Q}_{cor}$, in $\mathcal{NS}_{cor}$) if and only if there exists P in $\mathcal{L}(N,2)$ (resp. in $\mathcal{Q}(N,2)$, in $\mathcal{NS}(N,2)$) such that $\gamma_{xy} = \mathbb{E}_P[a \cdot b|x,y]$ for all x and y.

Analogously to the probability distributions case, we can define Bell functionals acting on correlations. They will be matrices $T = (T_{xy})_{xy}$, whose action on correlations is given by:

$$\langle T, \gamma\rangle = \sum_{x,y} T_{xy}\gamma_{xy}. \tag{1.1.3}$$

Clauser, Horne, Shimony and Holt [24] proposed an easy experiment to test local hidden variable theories considering 2 inputs and 2 outputs for Alice and Bob using Bell inequalities. They defined a Bell functional T_{CHSH} for which $|\langle T_{\text{CHSH}}, \gamma\rangle| \leq 2$ for all $\gamma \in \mathcal{L}_{cor}$, while there exist a quantum correlation γ_0 such that $|\langle T_{\text{CHSH}}, \gamma_0\rangle| = 2\sqrt{2}$.

The fact that the quantum state $|\psi\rangle$ is lying in the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ by postulate 4, can make the measurements of Alice and Bob somehow correlated with each other's. Hence, they are able to obtain joint probability distributions that are otherwise impossible to obtain in a local universe. This property is called quantum non-locality.

Besides its theoretical interest, Bell inequalities are a key object in Quantum Information Theory. They have many interesting applications of which we will list a few: they can be used to obtain unconditionally secure quantum key distribution in quantum cryptography ([1, 2, 60, 59]), they have been also applied in the theory of multipartite interactive proof systems inside complexity theory ([13, 27, 26, 44, 34, 51, 53]), in entangled games ([53, 52, 64]) or in communication complexity (see e.g. the review [22]), and they have been also applied to estimate the dimension of the underlying Hilbert space ([18, 19, 76, 77]).

In this thesis we will work with Bell inequalities in a more abstract sense as we will use other sets of probability distributions apart from $\mathcal{Q}$ or $\mathcal{L}$. Hence, we need the following notation.

Definition 1.1.4. *Given a Bell functional M and certain set of probability distributions $\mathcal{A}$, denote the value of M as:*

$$\omega_{\mathcal{A}}(M) = \sup_{P \in \mathcal{A}} |\langle M, P\rangle|.$$

There are two different cases of Bell functionals that we are going to consider in this work: General Bell functionals, where its coefficients can have any positive or negative value, and two-prover one-round games, or simply games. In them, the two players are collaborating and should answer some outputs after being asked certain question by the referee. They have a great relevance in computer science, since many interesting problems can be rephrased in terms of them.

Definition 1.1.5. *A Bell functional* $G = \{G_{xy}^{ab}\}_{x,y,a,b} \in \mathbb{R}^{N^2K^2}$ *is a game if the coefficients* G_{xy}^{ab} *can be written as*

$$G_{xy}^{ab} = \pi(x,y)V_{xy}^{ab},$$

for all x, y, a *and* b. *Here,* $\pi : [N] \times [N] \to [0,1]$ *is a probability distribution over the questions and it fulfills* $\sum_{x,y} \pi(x,y) = 1$, *and* V *is the verifier function such that* V_{xy}^{ab} *takes values in* $\{0,1\}$ *for all* x, y, a, b.

The value 0 for the verifier function means that the players have lost the game, while the value 1 means that they have won. The quantity $\langle G, P\rangle$ for a certain probability distribution P means the expectation of winning the game using a strategy defined by P. And the quantity $\omega_{\mathcal{A}}(G)$ denotes the highest probability of winning when the players are restricted to the use of, for example, classical resources ($\mathcal{A} = \mathcal{L}$) or quantum resources ($\mathcal{A} = \mathcal{Q}$). All the coefficients of G are non-negative and then we will always have $0 \leq \omega_{\mathcal{A}}(G) \leq 1$.

Analogously, given a Bell functional T acting on the set of correlations $\mathcal{A}_{cor}$ by Equation (1.1.3), we denote:

$$\omega_{\mathcal{A}_{cor}}(T) = \sup_{\gamma \in \mathcal{A}_{cor}} |\langle T, \gamma\rangle|.$$

There is a correspondence of a certain type of games, the *XOR games*, with the correlation scenario.

Definition 1.1.6. *An XOR game* G *is a two output game where the verifier function depends only on the parity of the outputs, i.e.,* $V_{x,y}^{a,b} = 1$ *if* $a \cdot b = f(x,y)$ *and* 0 *otherwise, where* $f : [N] \times [N] \to \{-1,+1\}$ *is a Boolean function. In this case, the coefficients of the XOR game* T *are defined as* $T_{x,y} = \pi(x,y)f(x,y)$ *for all* x *and* y.

With this definition, two quantities can be considered: $\omega_{\mathcal{A}}(G)$ and $\omega_{\mathcal{A}_{cor}}(T)$. The relation between them can be done with the following formula:

$$\omega_{\mathcal{A}_{cor}}(T) = 2\omega_{\mathcal{A}}(G) - 1.$$

Note that any XOR game played with a random classical strategy P will give you the value of $\langle G, P\rangle = 1/2$.

Nevertheless, and from a physical point of view, studying the particular value of a functional M for the sets $\mathcal{L}$ and $\mathcal{Q}$ is not enough to have a good knowledge about the relative power as resources of two such sets. It is better to consider the relative ratio of violation. More precisely, we consider the following definition:

Definition 1.1.7. *Given* $\mathcal{A}$ *and* $\mathcal{B}$ *certain subsets of probability distributions, we define*

$$LV(\mathcal{A},\mathcal{B}) = \sup_M \frac{\omega_{\mathcal{A}}(M)}{\omega_{\mathcal{B}}(M)}, \tag{1.1.4}$$

where the supremum is over all possible linear functionals M *acting on probability distributions. We call this quantity the largest violation between* $\mathcal{A}$ *and* $\mathcal{B}$.

The reader may wonder whether this quantity is well defined, since there could be an M such that $\omega_{\mathcal{B}}(M) = 0$ and $\omega_{\mathcal{A}}(M) \neq 0$. A sufficient condition to avoid this problem is that $\mathcal{A}$ has to be contained in the affine hull of $\mathcal{B}$. In this case we would have that $\omega_{\mathcal{B}}(M) = 0$ for some M implies $\omega_{\mathcal{A}}(M) = 0$. Then, we can consider general linear functionals by just defining $0/0 = 0$.

Moreover, the fact that $\omega_{\mathcal{A}}(M) = \sup_{P\in\mathcal{A}} |\langle M, P\rangle|$ uses an absolute value on the definition is important for general Bell functionals, although it does not play a role for Bell functionals with positive coefficients. Thus, in the general case, $\omega_{\mathcal{A}}(M) = 0$ if and only if $\langle M, P\rangle = 0$ for every $P \in \mathcal{A}$.

Typically we consider the cases where $\mathcal{B}$ is contained in $\mathcal{A}$. Either way, contained or not contained, if $\mathcal{B}$ is closed and bounded, the quantity $LV(\mathcal{A}, \mathcal{B})$ is useful to identify when two convex sets are not equal. If $LV(\mathcal{A}, \mathcal{B}) > 1$ for some convex sets of probability distributions, then there exists a Bell functional M, a real number α and a probability distribution $P_0 \in \mathcal{A}$ such that $|\langle M, P_0\rangle| > \alpha$ while $\omega_{\mathcal{B}}(M) \leq \alpha$. And this situation obviously implies $P_0 \notin \mathcal{B}$. The converse is also true, i.e., having P_0 in $\mathcal{A}$ but not in $\mathcal{B}$ implies $LV(\mathcal{A}, \mathcal{B}) > 1$ by the existence of a Bell functional M and a real number α such that $|\langle M, P_0\rangle| > \alpha$ and $\omega_{\mathcal{B}}(M) \leq \alpha$.

Definition 1.1.7 is the most important quantity that we are going to use in the thesis and will be the central object of our study. It gives a quantitative notion of the relative power as a resource of the probability distributions in $\mathcal{A}$ compared to those in $\mathcal{B}$. It is thus a comparison from a fundamental point of view.

The quantity $LV(\mathcal{A}, \mathcal{B})$ restricted to games is a measure of how much better the strategies of $\mathcal{A}$ are compared to those of $\mathcal{B}$. Moreover, there is a *geometrical interpretation* that was thoroughly analyzed in [47]. If we define $\widetilde{\mathcal{A}} = co\big(\mathcal{A} \cup -\mathcal{A}\big)$, and similarly for $\widetilde{\mathcal{B}}$, then, the quantity $LV(\mathcal{A}, \mathcal{B})$ is the smallest positive number λ such that $\widetilde{\mathcal{A}} \subseteq \lambda\widetilde{\mathcal{B}}$.

We will distinguish between general linear functionals and pointwise non-negative linear functionals, such as games. Hence, we denote as $LV^+(\mathcal{A}, \mathcal{B})$ the largest Bell violation such that the supremum considers only pointwise non-negative Bell functionals.

Besides $LV(\mathcal{A}, \mathcal{B})$, other quantities can be defined.

$$LV^P(\mathcal{B}) = \sup_M \frac{|\langle M, P\rangle|}{\sup_{P'\in\mathcal{B}} |\langle M, P'\rangle|} \quad \text{and} \quad LV_M(\mathcal{A}, \mathcal{B}) = \frac{\sup_{P\in\mathcal{A}} |\langle M, P\rangle|}{\sup_{P\in\mathcal{B}} |\langle M, P\rangle|}.$$

It can be seen that $\sup_{P\in\mathcal{A}} LV^P(\mathcal{B}) = \sup_M LV_M(\mathcal{A}, \mathcal{B}) = LV(\mathcal{A}, \mathcal{B})$.

Recall that taking the ratio in Equation (1.1.4) is crucial for a meaningful definition of the amount of violation. Other quantities, such as the difference between them $(\omega_{\mathcal{A}}(M) - \omega_{\mathcal{B}}(M))$, are susceptible of changes under transformations of the Bell functional such as $M \to \lambda M$.

To emphasize the type of results we are looking for, we go back to the classical-quantum case. The quantity $LV(\mathcal{Q}, \mathcal{L})$ has been widely studied. Since $\mathcal{Q}$ is contained in the affine hull of $\mathcal{L}$, then $LV(\mathcal{Q}, \mathcal{L})$ is well defined. As $\mathcal{L} \subset \mathcal{Q}$, then $\omega_{\mathcal{L}}(M) \leq \omega_{\mathcal{Q}}(M)$ and $LV(\mathcal{Q}, \mathcal{L}) \geq 1$. The violation of a Bell inequality stated by Bell's work [12] implies that $\mathcal{Q}$ is different to $\mathcal{L}$ by noticing that $LV(\mathcal{Q}, \mathcal{L}) > 1$.

The classical result of Tsirelson [74], states that the quantity $LV(\mathcal{Q}_{cor}, \mathcal{L}_{cor})$ is upper bounded by a universal constant. This happens independently of the number of inputs and the Hilbert space dimension. This constant receives the name of Grothendieck's constant and it can be understood as a limitation of the advantages of quantum mechanics.

Interestingly, it has been shown that, in a scenario with an arbitrary number of outputs, this quantity can be unbounded:

$$\lim_{N,K\to\infty} LV(\mathcal{Q}(N, K), \mathcal{L}(N, K)) = \infty.$$

The first unbounded largest violation is an application of the Raz parallel repetition theorem [67]. Unfortunately, this estimate is far from being optimal. The best upper bounds known for $LV(\mathcal{Q},\mathcal{L})$, at least in the order, are a consequence of the work of [47, 63] and they are stated here:

$$LV^+(\mathcal{Q},\mathcal{L}) \leq \min\{N,K\} \quad \text{and} \quad LV(\mathcal{Q},\mathcal{L}) \leq C\min\{N,K\} \tag{1.1.5}$$

where C is a universal constant. Note that, throughout this work, C will always denote a general constant, not necessarily the same one each time that it appears. See the works [6, 58], where some improvements of the previos upper bounds are obtained (although the order of the bounds are the same).

Currently, the best lower bounds for $LV(\mathcal{Q},\mathcal{L})$ are the two following estimates, which almost close the gap to the known upper bounds in terms of the number of inputs, outputs and in the dimension of the Hilbert space used to define $\mathcal{Q}$. In [23] the authors obtained a Bell functional M_0 with $2^n/n$ inputs and n outputs such that $LV_{M_0}(\mathcal{Q},\mathcal{L}) \geq n/(\log n)^2$. While in [47] the authors showed the existence of a general bipartite Bell functional M_1 with N inputs, N outputs such that $LV_{M_1}(\mathcal{Q},\mathcal{L}) \geq \sqrt{N}/\log N$ using an N-dimensional Hilbert space.

In the article [49] the authors analyze the multiple physical interpretations of the quantity $LV^P(\mathcal{Q},\mathcal{L})$. They showed that it quantifies the amount of noise that a quantum distribution P can endure before becoming local. It can also be used as a "dimension witness" [21], estimating the dimension of the corresponding Hilbert space needed to achieve such a value. Or even it can be applied to communication complexity giving lower bounds to the number of bits needed to be sent between Alice and Bob [32].

The main results of this thesis are related with finding upper and lower bounds to the quantity $LV(\mathcal{A},\mathcal{B})$ when different sets of probability distributions in different scenarios are used. We will continue the study of the bipartite scenario with the non-signalling probability distributions. They correspond to the ultimate upper bound that Alice and Bob can generate without communication and englobe the classical, quantum and any hypothetical post-quantum theory.

1.2 Non-Signalling

Quantum non-locality is the result of the four postulates of quantum mechanics. The fact that a joint quantum state lays in the tensor product of Alice's and Bob's Hilbert spaces, gives rise to probability distributions impossible to obtain with local measurements in a classical universe. As we have previously commented, the applications of quantum non-locality have been very numerous. But there are some applications of non-locality, such as device-independent quantum cryptography (see for instance [1]), where one is particularly interested in avoiding hypotheses about the adversaries (such as being quantum). And there exists a especially useful framework related to non-locality that goes beyond the formalism of quantum mechanics. This is the non-signalling formalism.

This study traces its roots back to the work of Popescu and Rorhlich in 1994. They tried to explore whether quantum correlations could be explained through relativistic causality. According to Bell's theorem, the probability distributions that two parties are able to compute depend on the physical model they are assuming. Popescu and Rorhlich formulated the question of which theories can give rise to correlations that preserve causality. They hypothesized about a box being able to produce outcomes to Alice and Bob and rephrased Einstein's causality to the requirement that Alice's outcomes should not depend on the input choice of Bob and viceversa.

Actually, these can be seen as a principle of nature, something that any physical theory should accomplish because they correspond to the maximal limits of non-locality before the theory itself becomes signalling and loses then its physical meaning. Both classical and quantum probability distributions naturally fulfill these two conditions, and it is believed that any post-quantum theory should fulfill them too. This situation gives the following chain of inclusions:

$$\mathcal{L} \subsetneq \mathcal{Q} \subsetneq \mathcal{NS} \subsetneq \mathcal{P}.$$

Since the non-signalling set is defined by the non-signalling conditions (Equations (1.1.1) and (1.1.2)), and these are a finite number of linear inequalities, then the set of non-signalling probability distributions is a polytope. The geometry of $\mathcal{NS}$ for the case of two inputs and two outputs is particularly simple and it has been used to violate the CHSH inequality with the maximum possible value. Some work has been done to compute all the vertices of $\mathcal{NS}$ in the case of two outputs and an arbitrary number of inputs [10, 46] and in the case of two inputs and an arbitrary number of outputs [9].

Then, in this work, when we study the quantity $LV(\mathcal{NS}, \mathcal{L})$, apart from a proper understanding about the non-signalling set, we are also studying the most extreme possible scenario, the ultimate limitations of any *meaningful* physical theory.

In order to obtain upper bounds we characterize the non-signalling probability distributions by a norm. As local distributions have been already characterized in terms of tensor norms, then, by standard Banach space techniques, we have been able to obtain the following theorem:

Theorem 1.2.1. *The following holds:*

$$LV^+(\mathcal{NS}, \mathcal{L}) \leq \min\{N, K\} \quad \textit{and} \quad LV(\mathcal{NS}, \mathcal{L}) \leq C \min\{N, \sqrt{NK}\},$$

where C is a universal constant.

Notice that the bounds for $LV^+(\mathcal{N}S, \mathcal{L})$ and the ones for $LV^+(\mathcal{Q}, \mathcal{L})$ given in Equation (1.1.5) are the same. However, one cannot expect both bounds to coincide for general Bell functionals because no upper bound given on $LV(\mathcal{N}S, \mathcal{L})$ can depend only on the number of outputs, contrary to what it happens in $LV(\mathcal{Q}, \mathcal{L})$ (we will explain this point in more detail in Chapter 3). In this last case it has been proven that K as the number of outputs and $\sqrt{N}$ as the number of inputs are the best lower bounds. But it is not known yet whether the upper bound in N is attained. Hence, we cannot conclude that the largest non-signalling Bell violation is comparable to the largest quantum Bell violation, but this result could be very likely. The upper bound we obtain is “morally” comparable to the one for the quantum value of Bell inequalities. This emphasizes the idea that, in some sense, quantum mechanical resources can be as good as any other physical post-quantum theory.

Moreover, we have been able to give a Bell functional which is near optimal in all parameters (the number of inputs and the number of outputs), as the next theorem shows.

Theorem 1.2.2. *There exists a bipartite game M_0 with N inputs and N outputs for which*

$$\frac{\omega_{\mathcal{NS}}(M_0)}{\omega_{\mathcal{L}}(M_0)} \geq C \frac{N}{\log N},$$

where C is a universal constant.

With this bound we close our study of $LV(\mathcal{NS}, \mathcal{L})$. All these results can be found in Chapter 3.

After studying the scenario in which the communication scenario is forbidden, we will study the scenario in which they will be allowed to communicate.

1.3 Games with communication

Beyond quantum non-locality, other scenario in which difference in performance between quantum and classical resources has been studied is the communication complexity scenario [55]. In this context, Alice and Bob receive inputs x and y and they have to compute a certain Boolean function. For this purpose they are allowed to exchange information and the object of study is the minimum number of bits or qubits that they have to exchange in order to compute correctly (or up to a bounded probability error) this function. This quantity is called the communication complexity of a function, and it is usually calculated in the worst case scenario of inputs, although a distribution over the inputs can also be considered. In this scenario, partial Boolean functions have been found for which quantum and classical communication complexities are exponentially separated [68].

Note the similarity of the communication complexity scenario with the XOR games, in which the product of the answers of Alice and Bob should match a function that depends on the inputs.

In [50] the authors introduced a new setting, which, from a conceptual point of view, can be seen as a mixture of the two previously defined scenarios. In this new setting, Alice and Bob try to win an XOR game using shared classical randomness together with the communication of a limited number of bits, either classical or quantum. The two spatially separated parties, Alice and Bob, after receiving the inputs x and y, communicate with each other, using either classical or quantum bits. The amount of information that they use to communicate will be bounded. Finally, they will answer a and b. Hence, both questions and answers will be classical.

The probability distributions that they can generate will be different according to the allowed communication. It depends on whether the communication is quantum or classical, whether it is in only one direction, lets say from Alice to Bob, or two directions with several rounds of communication. Hence, we denote by $\mathcal{L}^{c\rightarrow}$ those probability distributions in which Alice sends c bits to Bob. By $\mathcal{Q}^{c\rightarrow}$ those in which Alice sends c qubits to Bob, and by $\mathcal{L}^{c\leftrightarrow}$ those in which there are several rounds of two-way communication, with a total amount of c bits interchanged. Recall that in this last case one should fix the number of rounds of communication, who gives the first message, and the size of each message. We will consider all those sets of probability distributions.

In [50], the one-way communication case for correlations was studied. That is, the case when communication is only allowed from one of the parties to the other and there are only two outputs per party. Among other results, the authors showed an example of a game for which $O(\sqrt{n})$ bits of one-way classical communication are needed in order to achieve the same value as the one that can be attained with $\log n$ qubits of one-way communication. They left open the question of the existence of a game for which one can obtain the same order of exponential separation between the one-way quantum communication and the general (two-way) classical communication.

In our work we answer this last question in the positive. We show that actually the same game appearing in [50] achieves exponential separation between one-way quantum communication and two-way classical communication. As in the one-way case, the separation obtained is the maximum possible up to a logarithmic factor. More precisely:

Theorem 1.3.1. *For every $n \in \mathbb{N}$, there exist an XOR game T with 2^{2n} inputs for Alice and 2^{n^2} inputs for Bob such that, for every $k \in \mathbb{N}$,*

$$\frac{\omega_{\mathcal{Q}_{cor}^{\log n\rightarrow}}(T)}{\omega_{\mathcal{L}_{cor}^{\log k\leftrightarrow}}(T)} \geq C\frac{\sqrt{n}}{\log k},$$

where C is a universal constant. Moreover, this bound is essentially optimal.

The technical part of our proof is to characterize the two-way classical communication value of an XOR game in order to give an upper bound for this game in particular. A general two-way classical communication protocol consists on t-rounds of communication. In the round i, Alice will send a message of c_i bits to Bob, and Bob will send back to Alice a message of d_i bits. After that, they will output a and b. Recall that each message will depend on the initial input, and also on the previous messages already received. Thus, careful reasoning will be necessary to handle all the dependencies in order to relate this value with a tensor norm.

This content can be found on Chapter 4.

1.4 Multipartite

Most of the results obtained about quantum non-locality refer to the bipartite scenario, while the multipartite scenario is still much less understood. This last scenario has a greater complexity and offers a much richer source of correlations. Consequently, the potential applications of these phenomena to quantum networks or many body physics has aroused in the last years considerable interest in the study of quantum non-locality [5].

First we have to revisit some definitions in order to expand the notions of local, quantum and non-signalling probability distributions to the multipartite setting. We will consider here the Bell scenario with k parties, in which the party i receives input x_i and produces output a_i according to the probability distribution,

$$(P(a_1, \cdots, a_k|x_1, \cdots, x_k))_{x_1,\ldots,x_k}^{a_1,\ldots,a_k}. \tag{1.4.1}$$

For simplicity, we will assume for all parties the same number of inputs, N, and the same number of outputs, K. We will denote the set of k-party probability distributions as $\mathcal{P}^k(N, K)$, or simply $\mathcal{P}^k$.

The quantum extension to the multipartite setting is very natural. A probability distribution (1.4.1) is quantum if

$$P(a_1, \cdots, a_k|x_1, \cdots, x_k) = \langle\psi|E^1_{a_1,x_1} \otimes \cdots \otimes E^k_{a_k,x_k}|\psi\rangle, \tag{1.4.2}$$

where $(E^i_{a_i,x_i})_{x_i,a_i}$ is a family of measurements for the i^{th}-party (that is, for each party i, $E^i_{a_i,x_i} \geq 0\ \forall a_i, x_i$ and $\sum_{a_i} E^i_{a_i,x_i} = \mathbb{1}\ \forall x_i$) and $|\psi\rangle$ is a k-partite pure quantum state. We will denote the set of probability distributions of this form by $\mathcal{Q}^k$.

Denote by $\mathcal{NS}^k$ the set of k-partite non-signalling probability distributions with N inputs and K outputs per party. This definition follows from a generalization of the non-signalling conditions (see [57, Definition 1]). Given S a subset of $\{1, \cdots, k\}$, define $\bar{S} = \{1, \cdots, k\} \setminus S$ and denote by $|S|$ the cardinality of S. Then, $P \in \mathcal{P}^k$ is in $\mathcal{NS}^k$ if and only if for all non-empty subsets S of $\{1, \cdots, k\}$ there exists $Q \in \mathcal{P}^{k-|S|}$ such that:

$$\sum_{a_i : i \in S} P(a_1, \cdots, a_i, \cdots, a_k|x_1, \cdots, x_i, \cdots, x_k) = Q((a_i)_{i\in\bar{S}}|(x_i)_{i\in\bar{S}})$$

for all x_i and a_i such that $i \in \bar{S}$.

However we can extend the notion of locality given in Definition 1.1.1 to the multipartite case in different ways, giving rise to a more complex structure. Historically, the first extension was the fully local. In it, the only source of correlations among the parties is a local common cause. We say that a k-party probability distribution is *fully local* if

$$P(a_1, \cdots, a_k|x_1, \cdots, x_k) = \int P_1(a_1|x_1, \lambda) \cdots P_k(a_k|x_k, \lambda)\, d\lambda, \tag{1.4.3}$$

where x_i and a_i denote the inputs and outputs in each party, $d\lambda$ denotes a probability distribution and $(P^i_\lambda(a_i|x_i))_{x_i,a_i}$ is a probability distribution in the i^{th} party. We will denote the set of probability distributions of this form by $\mathcal{L}^k$.

There are several known relations among the sets that we have already commented. For instance, $\mathcal{L}^k \subsetneq \mathcal{Q}^k \subsetneq \mathcal{NS}^k$. The first strict inclusion is the content of Bell's theorem while the second is due to Tsirelson [74] and Popescu and Rohrlich [66]. They are already different for $K = 2$, as we have explained before.

However, the verification of non-fully-local correlations does not necessarily imply non-locality shared among all parties since a non-local resource distributed among only two parties can be enough to falsify these models. Thus, different notions of non-locality can be established in order to capture the idea that the non-local correlations must be truly shared among all parties.

The second extension of the notion of locality is the so called bilocality. In it, the non-local resource is shared among a strict subset of the parties. Bilocal probability distributions admit the following decomposition:

$$\begin{aligned} &P(a_1, \cdots, a_k|x_1, \cdots, x_k) \\ &= \sum_S p_S \int_{\Lambda_S} P_S((a_i)_{i\in S}|(x_i)_{i\in S}, \lambda) P_{\bar{S}}((a_i)_{i\in \bar{S}}|(x_i)_{i\in \bar{S}}, \lambda) d\lambda, \end{aligned} \tag{1.4.4}$$

where S runs over all strict non-empty subsets of $\{1, \cdots, k\}$, $\bar{S} = \{1, \cdots, k\} \setminus S$, $(p_S)_S$ is a probability distribution, (Λ_S, λ) is a probability space for all S and $P_S((a_i)_{i\in S}|(x_i)_{i\in S}, \lambda)$ and $P_{\bar{S}}((a_i)_{i\in \bar{S}}|(x_i)_{i\in \bar{S}}, \lambda)$ are probability distributions on the parties in S and in $\bar{S}$ respectively for all values of the local variable λ. Notice that the notion of bilocality is stronger than the notion of full locality, since the former can be strictly included in the latter.

Svetlichny gave the definition of bilocality for the tripartite case and proved, in his celebrated paper [72], that there exist tripartite quantum correlations that cannot be reproduced by bilocal models, implying that the non-locality attained through quantum mechanics can be very strong. The idea of bilocality was extended later to an arbitrary number of parties in [29, 70] and now we call genuine multipartite non-locality (GMNL) if the non-locality resource is shared among all parties. The impossibility of a bilocal model to reproduce certain correlations enables the verification of GMNL.

Equation (1.4.4) introduces a rich theoretical structure due to the imposition of different conditions on the different probability distributions for the subsets of parties $P_S, P_{\bar{S}}$ [45, 8]. If no restriction is added we have Svetlichny's notion of bilocality and we will denote the set of such probability distributions by $\mathcal{BL}^k_{\mathcal{G}}$, general bilocal. As observed in [38, 7], in certain scenarios allowing for signalling correlations in the definition of bilocality leads to ill-defined resource theories and grandfather-style paradoxes. A natural way to get rid of these problems is to consider bilocal models in which correlations among subsets of parties are required to be non-signalling (see e.g. [4, 30]). We will refer to these models as non-signalling bilocal and we will denote the corresponding set by $\mathcal{BL}^k_{\mathcal{NS}}$. It readily follows from the definitions that $\mathcal{L}^k \subset \mathcal{BL}^k_{\mathcal{NS}} \subset \mathcal{BL}^k_{\mathcal{G}}$. Svetlichny's result states that $\mathcal{Q}^k \nsubseteq \mathcal{BL}^k_{\mathcal{G}}$. Notice that it also holds that $\mathcal{BL}^k_{\mathcal{NS}} \nsubseteq \mathcal{Q}^k$.

The way in which Svetlichny falsified the inclusion of $\mathcal{Q}^k$ into $\mathcal{BL}^k$ was also with Bell functionals, since the set of bilocal probability distributions happens to be convex, just as classical or quantum probability distributions. In the multipartite scenario, a linear functional M is characterized by real numbers $\{M^{a_1\cdots a_k}_{x_1\ldots x_k}\}$ acting on the set of k-partite joint probability distributions by:

$$\langle M, P\rangle = \sum_{a_1,\ldots,a_k} \sum_{x_1,\ldots,x_k} M^{a_1\ldots a_k}_{x_1\ldots x_k} P(a_1, \cdots, a_k|x_1, \cdots, x_k).$$

In this work we will consider the different notions of bilocal probability distributions and we will analyze its asymptotic behaviour with respect to the tripartite quantum probability distributions. We will answer very natural questions in this topic based on the use of some known Bell inequalities.

The classical result of Tsirelson [74] stated that the quantity $LV(\mathcal{Q}^2_{cor}, \mathcal{L}^2_{cor})$ is upper bounded by a universal constant. And Tsirelson himself posed the question of whether a similar result was true for tripartite correlations. This question was answered in [64] in the negative. That is, tripartite quantum correlations can be used to get unbounded Bell violations, which can be understood as "unlimited advantages" over local correlations.

In the setting of correlations we consider binary outputs and we take the expectation over the product of the outputs. More precisely, in the k-partite scenario, we can consider the probability distribution

$$(P(a_1, \cdots, a_k|x_1, \cdots, x_k))_{x_1,\dots,x_k}^{a_1,\dots,a_k} \in \mathbb{R}^{N^k K^k},$$

such that $a_i \in \{-1, 1\}$ for every i. Then, we define the correlation associated to P,

$$\gamma = (\gamma_{x_1\dots x_k})_{x_1,\dots,x_k} \in \mathbb{R}^{N^k},$$

as

$$\begin{aligned}\gamma_{x_1\dots x_k} = \mathbb{E}[a_1 \dots a_k|x_1, \dots, x_k] &= \sum_{a_1,\dots,a_k} a_1 \cdots a_k P(a_1, \dots, a_k|x_1, \dots, x_k) \\ &= P(a_1 \cdots a_k{=}1|x_1, \dots, x_k) - P(a_1 \cdots a_k{=}{-1}|x_1, \dots, x_k).\end{aligned}$$

We will say that a certain correlation is local (resp. quantum, non-signalling bi-local and general bilocal) if there exists a local (resp. quantum, non-signalling bi-local and general bilocal) probability distribution such that γ is the correlation associated to it. In this way, we will denote $\mathcal{L}^k_{cor}$ (resp. $\mathcal{Q}^k_{cor}$, $\mathcal{BL}^k_{cor,\mathcal{NS}}$, $\mathcal{BL}^k_{cor,\mathcal{G}}$) the set of local (resp. quantum, non-signalling bilocal and general bilocal) correlations with N inputs per party. We will notice that the set of general bilocal correlations and the set of non-signalling bilocal correlations are equal.

According to the more general definitions of non-locality given in (1.4.4), one can wonder whether the main result in [64] is still true in this context. That is, replacing the fully local tripartite correlations by the bilocal ones.

Our first result is that in the two-output correlation scenario, and contrary to the full-locality case, there is a universal constant which prevents from having unbounded violations, as it happens in Tsirelson's result for bipartite correlations.

Theorem 1.4.1. *Given N a natural number, the following holds:*

$$LV\big(\mathcal{Q}^3_{cor}, \mathcal{BL}^3_{cor}\big) \leq K_G.$$

That is, tripartite quantum correlations cannot be significantly better than those correlations where two of the three parties can apply any non-local resource between them, but they are local with respect to the third one.

This result motivates the study of the same question for general probability distributions. Our main result is that in this case quantum GMNL systems lead to unbounded Bell violations with respect to general bilocal models. Hence, the quantity $LV(\mathcal{Q}^3, \mathcal{BL}^3_{\mathcal{G}})$ can be arbitrarily large. Notice that we are using here the strongest notion of bilocality and therefore the result holds with respect to any other bilocal model. More precisely,

Theorem 1.4.2. *There exist a game G with N^2 outputs and $2^{N^2}/(N^2)$ inputs for which*

$$\frac{\omega_{\mathcal{Q}^3}(G)}{\omega_{\mathcal{BL}^3_G}(G)} \geq C\frac{N^2}{\log N},$$

where C is a universal constant. Moreover, this bound is essentially optimal in the number of outputs.

The game used in Theorem 1.4.2 is defined as an extension of a bipartite game in the multipartite scenario, using notions of tensor products and parallel repetition. Finally, it is already enough to consider tripartite systems and, hence, the result extends trivially to an arbitrary number of parties.

The content of this section refers to Chapter 5.

Publications and Preprints

The results of this thesis have appeared in the following articles.

- Abderramán Amr, Carlos Palazuelos, Ignacio Villanueva, *Optimal non-signalling violations via tensor norms*, Revista Matemática Complutense, 33, 661-694 (2020).

  (Chapter 3)

- Abderramán Amr, Carlos Palazuelos, Julio I de Vicente, *Unbounded Bell violations for quantum genuine multipartite non-locality*, Journal of Physics A: Mathematical and Theoretical, 53 (27) 275301 (2020).

  (Chapter 5)

Moreover, some results have also appeared in the following preprint.

- Abderramán Amr, Ignacio Villanueva, *Quantum one-way vs. classical two-way communication in XOR games* arXiv:2003.09747 (2020).

  (Chapter 4)

Chapter 2

Abstract approach to the problem

2.1 Introduction

As we said before, the main object of study in this work is the quantity $LV(\mathcal{A}, \mathcal{B})$ introduced in Definition 1.1.7, where the sets $\mathcal{A}$ and $\mathcal{B}$ correspond to some probability distributions or correlations. The sets $\mathcal{A}$ and $\mathcal{B}$ are convex, and then, finding upper and lower bounds to $LV(\mathcal{A}, \mathcal{B})$ is related with separating convex sets from each other. We will fulfill this task by means of Banach space techniques. This approach was firstly used by Tsirelson in the setting of correlations. Because of the intrinsic symmetry, in the correlation setting, the problem reduces to compare norms between Banach spaces. However, the case of general probability distributions becomes increasingly complex, specially when we consider the use of general linear functionals.

The sets $\mathcal{A}$ and $\mathcal{B}$ are in a space that depends on several parameters, such as the number of inputs, number of outputs, the dimension of the used Hilbert space or the amount of information sent. And that is the reason why the upper and lower bounds will be given in terms of these parameters. In general we will be only interested in giving these results in the asymptotic limit and thus, up to a constant.

In this chapter we will introduce the necessary basic mathematical notions related to Banach spaces and tensor products. Then we will use those notions to show a general approach to our problem.

2.2 Preliminary notions and basic definitions on Banach spaces

Given a normed space X, denote by $\|\cdot\|_{\mathsf{X}}$ its norm and by $\mathsf{B}_{\mathsf{X}} = \{x \in \mathsf{X} \text{ such that } \|x\|_{\mathsf{X}} \leq 1\}$ its closed unit ball. The dual space consisting of linear and continuous maps from X to the scalar field $\mathbb{R}$ will be denoted by X^* and its norm has the natural expression $\|x^*\|_{\mathsf{X}^*} = \sup_{x \in \mathsf{B}_{\mathsf{X}}} |\langle x^*, x\rangle|$. In this work we will restrict to finite dimensional real Banach spaces.

Recall that B_{X} is a convex and compact. In fact, given a symmetric convex set S we can define a norm on the linear span of S by using the Minkowski functional $q_S(x) = \inf\{\lambda \geq 0 : x \in \lambda S\}$.

Given two finite dimensional normed spaces X and Y, a bounded operator is a linear transformation $M : \mathsf{X} \to \mathsf{Y}$. The smallest $L \geq 0$ verifying $\|M(x)\|_{\mathsf{Y}} \leq L\|x\|_{\mathsf{X}}$ for all $x \in \mathsf{X}$ is called the operator norm of M, or just $\|M\|$. The set of bounded operators from X to Y is denoted by $\mathcal{B}(\mathsf{X}, \mathsf{Y})$. We use the notation $\mathcal{B}(\mathsf{X}, \mathsf{X}) = \mathcal{B}(\mathsf{X})$.

Two normed spaces X and Y are isomorphic if there exists a linear and bijective map $M : \mathsf{X} \to \mathsf{Y}$ such that both M and M^{-1} are bounded. Since all the Banach spaces that we will consider here are finite dimensional, they are all isomorphic if they have the same dimension. For two isomorphic Banach spaces X and Y we can define the Banach-Mazur distance as $d(X,Y) = \inf\{\|T\|\|T^{-1}\|$ such that T is an isomorphism from X to $\mathsf{Y}\}$[73]. We will be specially interested in those Banach-Mazur distances which can be bounded by a constant independent from the dimension of the Banach spaces that we are considering.

During this work we will be particularly interested in the spaces ℓ_1^N and ℓ_∞^N, which correspond to the norm sum and the norm max. Denoting by $\{e_i\}_{i=1}^N$ to the elements of the canonical basis of $\mathbb{R}^N$, we have that the extreme points of $\mathsf{B}_{\ell_\infty^N}$ are exactly the elements of the form $\sum_{i=1}^N a_i e_i$, where $a_i = \pm 1$ for every i, while the extreme points of $\mathsf{B}_{\ell_1^N}$ are exactly the elements of the form $a_i e_i$, where $a_i = \pm 1$. Moreover, we will enhance these two spaces, ℓ_1^N and ℓ_∞^N, with other Banach spaces, lets say X, to create $\ell_1^N(\mathsf{X})$ and $\ell_\infty^N(\mathsf{X})$. A typical element of these spaces is a sequence of N elements in X, $u = \{x_i\}_{i=1}^N$, whose norm is:

$$\|u\|_{\ell_1^N(\mathsf{X})} = \sum_{i=1}^N \|x_i\|_{\mathsf{X}},$$
$$\|u\|_{\ell_\infty^N(\mathsf{X})} = \max_{1\le i\le N} \|x_i\|_{\mathsf{X}}.$$

It is well known that

$$(\ell_\infty^N(\mathsf{X}))^* = \ell_1^N(\mathsf{X}^*) \quad \text{and} \quad (\ell_1^N(\mathsf{X}))^* = \ell_\infty^N(\mathsf{X}^*),$$

isometrically.

Recall that a sequence of N elements in X, $u = \{x_i\}_{i=1}^N$ can be naturally seen as an element in the tensor product $\mathbb{R}^N \otimes \mathsf{X}$, with the identification being $u = \sum_{i=1}^N e_i \otimes x_i$, where e_i are the vectors of the canonical basis of $\mathbb{R}^N$.

Whenever we have two finite dimensional normed spaces X and Y, we can consider the tensor product of them $\mathsf{X} \otimes \mathsf{Y}$ and endow it with different norms compatible with the tensor product structure (see [31, Section 2] or [69] for the following definitions and relations). For a given $u \in \mathsf{X} \otimes \mathsf{Y}$ the α-norm is denoted by $\|u\|_{\mathsf{X}\otimes_\alpha \mathsf{Y}}$ and we use the notation $\mathsf{X} \otimes_\alpha \mathsf{Y}$ to refer to the space $\mathsf{X} \otimes \mathsf{Y}$ endowed with the previous norms. The ϵ-norm and the π-norm are defined by:

$$\|u\|_{\mathsf{X}\otimes_\epsilon \mathsf{Y}} = \sup\Big\{ |\langle u, x^* \otimes y^*\rangle| : x^* \in \mathsf{B}_{\mathsf{X}^*}, y^* \in \mathsf{B}_{\mathsf{Y}^*} \Big\}, \tag{2.2.1}$$
$$\|u\|_{\mathsf{X}\otimes_\pi \mathsf{Y}} = \inf\Big\{ \sum_{i=1}^N \|x_i\|_{\mathsf{X}}\|y_i\|_{\mathsf{Y}} : N \in \mathbb{N} \text{ and } u = \sum_{i=1}^N x_i \otimes y_i \Big\}.$$

Given a norm α, we can define its dual norm α^* by $\|u\|_{\mathsf{X}^*\otimes_{\alpha^*}\mathsf{Y}^*} = \|u\|_{(\mathsf{X}\otimes_\alpha \mathsf{Y})^*}$ for $u \in \mathsf{X}^* \otimes \mathsf{Y}^*$. The ϵ and the π norm are dual to each other (in finite dimensional spaces):

$$(\mathsf{X} \otimes_\epsilon \mathsf{Y})^* = \mathsf{X}^* \otimes_\pi \mathsf{Y}^* \text{ and } (\mathsf{X} \otimes_\pi \mathsf{Y})^* = \mathsf{X}^* \otimes_\epsilon \mathsf{Y}^* \quad (\textit{isometrically}).$$

It can be proved that $\mathsf{B}_{\mathsf{X}\otimes_\pi \mathsf{Y}} = co(\mathsf{B}_\mathsf{X} \otimes \mathsf{B}_\mathsf{Y})$, where $co(\mathsf{A} \otimes \mathsf{B})$ denotes the convex hull of the set $\{a \otimes b : a \in \mathsf{A}, b \in \mathsf{B}\}$. And it follows from the definitions that $\ell_1^N(\mathsf{X}) = \ell_1^N \otimes_\pi \mathsf{X}$ and also $\ell_\infty^N(\mathsf{X}) = \ell_\infty^N \otimes_\epsilon \mathsf{X}$ isometrically.

It is easy to see that given $u \in \mathsf{X} \otimes \mathsf{Y}$, then $\|u\|_{\mathsf{X}\otimes_\epsilon \mathsf{Y}} \le \|u\|_{\mathsf{X}\otimes_\pi \mathsf{Y}}$. But there are other norms compatible with the tensor product structure. We say that a norm α in $\mathsf{X} \otimes \mathsf{Y}$ is a crossnorm if:

- $\|x \otimes y\|_{\mathsf{X} \otimes_\alpha \mathsf{Y}} \leq \|x\|_{\mathsf{X}} \|y\|_{\mathsf{Y}}$ for every $x \in \mathsf{X}$ and $y \in \mathsf{Y}$.
- For every $\psi \in \mathsf{X}^*$ and $\phi \in \mathsf{Y}^*$, the linear functional $\psi \otimes \phi$ on $\mathsf{X} \otimes \mathsf{Y}$ is bounded and $\|\psi \otimes \phi\| \leq \|\psi\|_{\mathsf{X}^*} \|\phi\|_{\mathsf{Y}^*}$.

Moreover, α is a tensor norm if it also fulfills the metric mapping property. This property says that for all normed spaces W and Z and for all linear maps $T : \mathsf{X} \to \mathsf{W}$, $S : \mathsf{Y} \to \mathsf{Z}$, we have

$$\|T \otimes S : \mathsf{X} \otimes_\alpha \mathsf{Y} \to \mathsf{W} \otimes_\alpha \mathsf{Z}\| = \|T\| \|S\|.$$

It can be seen in the mentioned references that the ϵ-norm and the π-norm fulfill the metric mapping property.

Tensor norms are relevant to us because of the following reason: we can relate the different sets of correlations or probability distributions with different tensor norms. Then, we can develop and use techniques of the latter to deduce properties of the former. In the next section we will develop this situation in order to see which norms and under which circumstances.

2.3 Correlations

The set of correlations is formed by taking the expectation over probability distributions that are defined on binary outputs $\{-1, +1\}$. Given $P \in \mathcal{P}^k$, the set of k-party probability distributions, then:

$$\gamma_{x_1 \dots x_k} = \mathbb{E}[a_1 \dots a_k | x_1, \dots, x_k] = \sum_{a_1, \dots, a_k} a_1 \cdots a_k P(a_1, \dots, a_k | x_1, \dots, x_k).$$

This set is naturally contained in $\mathbb{R}^{N^k}$ and it can be proved to be convex, using linearity of the expectation, and also symmetric with respect to the origin, using the transformation $a \leftrightarrow -a$. Hence, we can define a norm in its linear span using the Minkowski functional.

If we are able to relate a certain set of correlations $\mathcal{A}_{cor}$ with the unit ball of a certain Banach space X, we will also be able to associate a norm to the value of a Bell functional $T = (T_{x,y})_{x,y}$ by:

$$\omega_{\mathcal{A}_{cor}}(T) = \sup_{\gamma \in \mathcal{A}_{cor}} |\langle T, \gamma \rangle| = \sup_{\gamma \in \mathsf{B}_{\mathsf{X}}} |\langle T, \gamma \rangle| = \|T\|_{\mathsf{X}^*}.$$

If we can proceed analogously for another set of correlations $\mathcal{B}_{cor}$ and another Banach space Y in such a way that $\sup_{\gamma \in \mathcal{B}_{cor}} |\langle T, \gamma \rangle| = \|T\|_{\mathsf{Y}^*}$, then, we can compute the largest violation attained between correlations $\mathcal{B}_{cor}$ and $\mathcal{A}_{cor}$ by:

$$LV(\mathcal{A}_{cor}, \mathcal{B}_{cor}) = \sup_T \frac{\sup_{\gamma \in \mathcal{A}_{cor}} |\langle T, \gamma \rangle|}{\sup_{\gamma \in \mathcal{B}_{cor}} |\langle T, \gamma \rangle|} = \sup_T \frac{\|T\|_{\mathsf{X}^*}}{\|T\|_{\mathsf{Y}^*}}. \tag{2.3.1}$$

In this case we can say that:

$$LV(\mathcal{A}_{cor}, \mathcal{B}_{cor}) = \|id : \mathsf{Y}^* \longrightarrow \mathsf{X}^*\| = \|id : \mathsf{X} \longrightarrow \mathsf{Y}\|.$$

This association was done implicitly by Tsirelson [74] in the case of bipartite classical and quantum correlations. In the following we will state the relation of unit balls of certain Banach spaces with these two physical sets. The natural Banach space for them is $\ell_\infty^N \otimes \ell_\infty^N$. This makes reference to the situation where Alice and Bob are answering ± 1. And depending on the chosen physical model, we associate to it a different tensor norm.

Non-signalling correlations can be also described using a tensor norm on the space $\ell_\infty^N \otimes \ell_\infty^N$, but this situation will be analyzed in Section 3.2. We warn the reader that the quantum case is announced here for completeness, but its characterization by means of tensor norms will not be used in the rest of the thesis.

Example 2.3.1. *It is easy to see from Definition 1.1.1 that classical correlations can be defined as*

$$\mathcal{L} = co\{(a_x b_y)_{x,y} \text{ such that } a_x = \pm 1 \text{ and } b_y = \pm 1 \text{ for all } x, y\}.$$

Moreover, using the definition of the π *norm, it is easy to see [74, 63] that*

$$\gamma \in \mathcal{L}_{cor} \text{ if and only if } \|\gamma\|_{\ell_\infty^N \otimes_\pi \ell_\infty^N} \leq 1.$$

Hence, the classical value for a certain correlation Bell functional is:

$$\omega_{\mathcal{L}_{cor}}(T) = \sup_{\gamma \in \mathcal{L}_{cor}} |\langle T, \gamma \rangle| = \sup_{\|\gamma\|_{\ell_\infty^N \otimes_\pi \ell_\infty^N} \leq 1} |\langle T, \gamma \rangle| = \|T\|_{\ell_1^N \otimes_\epsilon \ell_1^N}.$$

Example 2.3.2. *Quantum correlations are those* $(\gamma_{xy})_{x,y}$ *that can be written as* $\gamma_{xy} = \langle \psi | A_x \otimes B_y | \psi \rangle$*, where* ψ *is a state on a Hilbert space* $\mathcal{H}_A \otimes \mathcal{H}_B$ *and* A_x *and* B_y *are self-adjoint operators acting on* $\mathcal{H}_A$ *and* $\mathcal{H}_B$ *such that* $\max_{x,y}\{\|A_x\|, \|B_y\|\} \leq 1$.

It was analyzed by Tsirelson [74] that we can calculate the quantum value by setting:

$$\omega_{\mathcal{Q}_{cor}}(T) = \sup \Big\{ |\sum_{x,y} T_{xy} \langle u_x, v_y \rangle| : \|u_x\|, \|v_y\| \leq 1 \text{ for all } x, y \Big\}.$$

Considering the tensor norm γ_2 *[31] for a given* $\gamma = (\gamma_{xy})_{x,y}$ *on the space* $\ell_\infty^N \otimes \ell_\infty^N$ *as*

$$\|\gamma\|_{\ell_\infty^N \otimes_{\gamma_2} \ell_\infty^N} = \inf \Big\{ \max_{1 \leq x,y \leq N} \|u_x\|_{\ell_2^N} \|v_y\|_{\ell_2^N} : \gamma_{xy} = \langle u_x, v_y \rangle \text{ for every } x, y \Big\},$$

where u_x *and* v_y *are unit vectors in a real Hilbert space for all* x *and* y*, we have that* $\mathcal{Q}_{cor} = \mathsf{B}_{\ell_\infty^N \otimes_{\gamma_2} \ell_\infty^N}$*. Hence:*

$$\omega_{\mathcal{Q}_{cor}}(T) = \|T\|_{\ell_1^N \otimes_{\gamma_2^*} \ell_1^N}.$$

Given any Bell functional for correlations, an upper bound between the ratio of these two bias was also given by Tsirelson [74] as an application of the Grothendieck theorem [41]. It states that:

$$LV(\mathcal{Q}_{cor}, \mathcal{L}_{cor}) \leq K_G, \tag{2.3.2}$$

where, K_G is the real Grothendieck's constant. This constant verifies that [31, 16]:

$$1,67696 \leq K_G < \frac{\pi}{2\log(1+\sqrt{2})} = 1,7822139781.$$

This tells us that the quantum value and the classical value are close from each other since there could not be an unbounded violation between them. Yet, they are different, as it can be seen with the CHSH example, where

$$LV_{T_{CHSH}}(\mathcal{Q}_{cor}, \mathcal{L}_{cor}) = \frac{\omega_{\mathcal{Q}_{cor}}(T_{CHSH})}{\omega_{\mathcal{L}_{cor}}(T_{CHSH})} = \sqrt{2}.$$

2.4 General approach for probability distributions

In this section we continue the study of $LV(\mathcal{A},\mathcal{B})$ with the difference that now the sets $\mathcal{A}$ and $\mathcal{B}$ correspond to different sets of probability distributions. Contrary to correlations, the set of probability distributions $\mathcal{P}^k$, and therefore any subset of it, is contained in a proper affine subspace of $\mathbb{R}^{N^k K^k}$, because $\sum_{a_1 \dots a_k} P(a_1,\dots,a_k|x_1,\dots,x_k) = 1$ for all $x_1,\dots,x_k$.

The origin is thus far from this set and it is impossible to associate the unit ball of a Banach space to any $\mathcal{A} \subset \mathcal{P}^k$, as happened in the case of correlations.

Despite this fact, the general procedure that we will follow in order to find upper bounds to the largest violation consists on:

1. Finding Banach spaces X and Y such that

$$k_1\|M\|_{\mathsf{X}^*} \leq \omega_{\mathcal{A}}(M) \leq k_2\|M\|_{\mathsf{X}^*} \quad \text{and} \quad k_3\|M\|_{\mathsf{Y}^*} \leq \omega_{\mathcal{B}}(M) \leq k_4\|M\|_{\mathsf{Y}^*},$$

   for some constants $k_1,k_2,k_3,k_4 \in \mathbb{R}$ and for every M.

2. Finding upper bounds to the identity norm $\|id : \mathsf{X} \longrightarrow \mathsf{Y}\|$ since

$$\frac{k_1}{k_4}\sup_M \frac{\|M\|_{\mathsf{X}^*}}{\|M\|_{\mathsf{Y}^*}} \leq LV(\mathcal{A},\mathcal{B}) = \sup_M \frac{\omega_{\mathcal{A}}(M)}{\omega_{\mathcal{B}}(M)} \leq \frac{k_2}{k_3}\sup_M \frac{\|M\|_{\mathsf{X}^*}}{\|M\|_{\mathsf{Y}^*}},$$

   and

$$\sup_M \frac{\|M\|_{\mathsf{X}^*}}{\|M\|_{\mathsf{Y}^*}} = \|id : \mathsf{X} \longrightarrow \mathsf{Y}\| = \|id : \mathsf{Y}^* \longrightarrow \mathsf{X}^*\|.$$

In general, we will proceed differently depending whether we are considering the case of non-negative coefficients, such as games, or general Bell functionals.

On the contrary, finding lower bounds corresponds to finding the right Bell functional. The fact that the sets of probability distributions will be somehow associated with Banach spaces will help us on this task, since we will be able to use Banach space techniques. In particular, there are two types of Bell functionals that we are going to consider in this thesis. One type uses combinatoric constructions and requires a large number of inputs, i.e. exponential with respect to the violation. The second one is the random construction of Bell functionals (see [62, 63] for some surveys on this topic). With this method it was proved in [47, 48], using operator spaces techniques, the existence of Bell functionals in which the number of inputs and outputs grow polynomially with the amount of violation. However, this procedure is non-constructive and it will come at the expenses of not identifying an explicit Bell functional for such task.

2.4.1 Games

First we are going to consider the linear functionals G such that all its coefficients are non-negative. The one-round two-player games would fit into this group. The method consists in finding a Banach space X such that it fulfills the following two conditions:

Condition 2.4.1. $\mathcal{A} \subset \mathsf{B}_{\mathsf{X}}$

Condition 2.4.2. *For all $P \in \mathsf{B}_{\mathsf{X}}$ there exists $P' \in \mathcal{A}$ such that $|P| \leq P'$ pointwise. Here, $|P|$ denotes the element resulting when taking the absolute value of the entries of P.*

In this case we will be able to state the following:

Proposition 2.4.3. *Given a set of probability distributions $\mathcal{A} \subset \mathcal{P}^k$ and a Banach space X such that it fulfills Conditions 2.4.1 and 2.4.2, then*

$$\mathsf{B}_\mathsf{X} \cap \mathcal{P}^k = \mathcal{A}.$$

Proof. It is obvious that Condition 2.4.1 implies $\mathcal{A} \subset \mathsf{B}_\mathsf{X} \cap \mathcal{P}^k$. For the reverse inequality consider $P \in \mathsf{B}_\mathsf{X} \cap \mathcal{P}^k$. Then, using Condition 2.4.2 we can find P' in $\mathcal{A} \subset \mathcal{P}^k$ such that $P \leq P'$. The fact that

$$1 = \sum_{a_1,\ldots,a_k} P(a_1,\ldots,a_k|x_1,\ldots,x_k) \leq \sum_{a_1,\ldots,a_k} P'(a_1,\ldots,a_k|x_1,\ldots,x_k) = 1$$

for all $x_1,\ldots,x_k$ makes $P = P'$. □

Lemma 2.4.4. *Given $\mathcal{A}$ a subset of probability distributions $\mathcal{P}^k$ and given a Banach space X that fulfills Conditions 2.4.1 and 2.4.2, then, for every G with non-negative entries we have:*

$$\omega_\mathcal{A}(G) = \|G\|_{\mathsf{X}^*}$$

Proof. Using Condition 2.4.1,

$$\omega_\mathcal{A}(G) = \sup_{P\in\mathcal{A}} |\langle G, P\rangle| \leq \sup_{P\in\mathsf{B}_\mathsf{X}} |\langle G, P\rangle| = \|G\|_{\mathsf{X}^*}.$$

On the other hand, given any $P \in \mathsf{B}_\mathsf{X}$, by Condition 2.4.2 there exists $P' \in \mathcal{A}$ such that $|P| \leq P'$ pointwise. Using moreover that the components of G are non-negative,

$$\langle G, P\rangle \leq \langle G, |P|\rangle \leq \langle G, P'\rangle \leq \sup_{P\in\mathcal{A}} |\langle G, P\rangle| = \omega_\mathcal{A}(G).$$

Then, $\|G\|_{\mathsf{X}^*} = \sup_{P\in\mathsf{B}_\mathsf{X}} |\langle G, P\rangle| \leq \omega_\mathcal{A}(G)$ and the result follows. □

In this case we are also able to say that:

$$LV(\mathcal{A},\mathcal{B}) = \|id : \mathsf{X}^* \longrightarrow \mathsf{Y}^*\|_+,$$

where $\|\cdot\|_+$ indicates the norm of a map when it is restricted to positive elements. Sometimes this may imply an advantage over the general value of the norm.

We will show three different cases where this method has been used before: generic probability distributions, classical and quantum probability distributions.

Example 2.4.5. *In this first example we are going to set $\mathcal{A} = \mathcal{P}^1$, which was defined in (1.4.1) and corresponds to the one-party probability distributions. That is,*

$$\mathcal{P}^1 = \{(P(a|x))_{x,a} : P(a|x) \geq 0 \textit{ for all } x, a \textit{ and } \sum_a P(a|x) = 1 \textit{ for all } x\}.$$

The Banach space to consider is simply $\ell_\infty^N(\ell_1^K)$. Condition 2.4.1 follows from considering $\mathcal{P}^1 \subset \mathsf{B}_{\ell_\infty^N(\ell_1^K)}$, because for any $P \in \mathcal{P}^1$, $\|P\|_{\ell_\infty^N(\ell_1^K)} = 1$. To prove condition 2.4.2 consider any $P \in \mathsf{B}_{\ell_\infty^N(\ell_1^K)}$, then we can define the element P' by setting $P'(a|x) = |P(a|x)|$ for all x and $1 \leq a \leq K-1$. And $P'(K,x) = 1 - \sum_{a=1}^{K-1} |P(a|x)|$ for all x. It trivially fulfills that $|P'| \leq P$ and $P \in \mathcal{P}^1$.

Therefore, if $G = (G_x^a)_{x,a}$ is a functional with non-negative coefficients, then,

$$\omega_{\mathcal{P}^1}(G) = \|G\|_{\ell_1^N(\ell_\infty^K)}.$$

A bipartite probability distribution P is a point in $\mathbb{R}^{N^2K^2}$ that can be written algebraically as:

$$P = (P(a,b|x,y))_{x,y,a,b} = \sum_{x,y;a,b=1}^{N,K} P(a,b|x,y) e_x \otimes e_a \otimes e_y \otimes e_b.$$

If we make use of the identification $\mathbb{R}^{NK} \otimes \mathbb{R}^{NK} = \mathbb{R}^{N^2K^2}$ and use Example 2.4.5 to recall that Alice and Bob are giving answers separately depending on the inputs received, then we can naturally consider P to be contained in the space $\ell_\infty^N(\ell_1^K) \otimes \ell_\infty^N(\ell_1^K)$. We will illustrate this situation with bipartite classical and quantum probability distributions, where the additional restrictions imposed by the considered physical framework will have an impact on the tensor norm. However, we will not always be able to find a tensor norm to describe the physical set. This is the case of non-signalling probability distributions, which will make the problem more difficult to handle.

Example 2.4.6. *Now we are going to set $\mathcal{A} = \mathcal{L}$, the bipartite classical probability distributions*[1]*. The Banach space to consider in this case is $\ell_\infty^N(\ell_1^K) \otimes_\pi \ell_\infty^N(\ell_1^K)$.*

To prove condition 2.4.1 consider any extreme point $P = (P(a,b|x,y))_{x,y,a,b}$ from $\mathcal{L}$. It has the form $P(a,b|x,y) = \delta_{a,\tilde{a}(x)}\delta_{b,\tilde{b}(y)}$ where $\tilde{a} : [N] \to [K]$ and $\tilde{b} : [N] \to [K]$ are some functions. Then:

$$\|P\|_{\ell_\infty^N(\ell_1^K)\otimes_\pi \ell_\infty^N(\ell_1^K)} = \|\sum_{x,a} \delta_{a,\tilde{a}(x)} e_x \otimes e_a\|_{\ell_\infty^N(\ell_1^K)} \|\|\sum_{b,y} \delta_{b,\tilde{b}(y)} e_y \otimes e_b\|_{\ell_\infty^N(\ell_1^K)} = 1.$$

To prove condition 2.4.2 consider $P \in \mathsf{B}_{\ell_\infty^N(\ell_1^K)\otimes_\pi \ell_\infty^N(\ell_1^K)} = co(\mathsf{B}_{\ell_\infty^N(\ell_1^K)} \otimes \mathsf{B}_{\ell_\infty^N(\ell_1^K)})$. Then we can write for a certain probability space (Λ, λ)

$$P(a,b,x,y) = \int_\Lambda p_\lambda Q_\lambda(a,x) R_\lambda(b,y) d\lambda,$$

with $Q = (Q_\lambda(a,x))_{x,a}$ and $R = (R_\lambda(b,y))_{y,b}$ are in $\mathsf{B}_{\ell_\infty^N(\ell_1^K)}$ for all λ. Using Example 2.4.5 we can find $\widetilde{Q}_\lambda$ and $\widetilde{R}_\lambda$ in $\mathcal{P}^1$ for each λ such that $|Q_\lambda| \leq \widetilde{Q}_\lambda$ and $|R_\lambda| \leq \widetilde{R}_\lambda$. Define $\widetilde{P} = (\widetilde{P}(a,b|x,y))_{x,y,a,b}$ by setting

$$\widetilde{P}(a,b|x,y) = \int_\Lambda p_\lambda \widetilde{Q}_\lambda(a,x) \widetilde{R}_\lambda(b,y) d\lambda.$$

Then, it is easy to see that $|P| \leq \widetilde{P}$.

Finally, we can say:

$$\omega_{\mathcal{L}}(G) = \|G\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}.$$

The quantum case is far more complicated. In the works [47, 48, 49, 63, 64] the authors showed that, in order to use a norm formulation of the Bell functionals, one should consider *operator spaces*. They are a natural framework for the study of quantum Bell inequalities, but this study is beyond the purpose of this thesis. Hence, we will only enunciate here the main result.

Example 2.4.7. *[49] Given a non-negative Bell functional G, then*

$$\omega_{\mathcal{Q}}(G) = \|G\|_{\ell_1^N(\ell_\infty^K)\otimes_{min} \ell_1^N(\ell_\infty^K)},$$

where the min norm is a norm defined in the category of operator spaces.

[1] During this work we will use $\mathcal{A}$ to refer the bipartite probability distributions and $\mathcal{A}^k$ with $k \neq 2$ to refer the k-partite probability distributions.

Remark 2.4.8. *Finally, upper bounding the quantity $LV^+(\mathcal{Q},\mathcal{L})$ consists on bounding the following norm:*

$$\|id : \ell_1^N(\ell_\infty^K) \otimes_\epsilon \ell_1^N(\ell_\infty^K) \to \ell_1^N(\ell_\infty^K) \otimes_{min} \ell_1^N(\ell_\infty^K)\|_+$$

Then, for this case we have [63]:

$$LV^+(\mathcal{Q},\mathcal{L}) \leq C \min\{N_1, N_2, K_1, K_2\}, \tag{2.4.1}$$

where C is a universal constant. Here we have distinguished the inputs (N_1 and N_2) and outputs (K_1 and K_2) for Alice and Bob (respectively).

The main advantage of this first method for non-negative functionals is that we can consider simpler Banach spaces. As we have seen, those Banach spaces are just simply combinations of tensor products of $\ell_\infty^N(\ell_1^K)$. The inconvenient here is that it is not useful for general Bell functionals.

2.4.2 General Bell functionals

The study of general Bell functionals presents adicional problems since the dimension of $\mathcal{P}^k$ (that is, the smallest dimension of the affine subspace containing that set) and the dimension of $\mathbb{R}^{N^k K^k}$ are different for all k. Thus, it is easy to show that there exist some Bell functional M different from zero such that $\langle M, P\rangle = 0$ for all P in $\mathcal{A} \subset \mathcal{P}^k$, making the value $\omega_{\mathcal{A}}(M)$ inequivalent to a norm. To circumvent this problem we will have to consider a more complex Banach space whose unit ball is $\mathsf{B}_\mathsf{X} = \overline{co(\mathcal{A} \cup -\mathcal{A})}^2$. If we are able to find such a Banach space then we can state the following theorem:

Lemma 2.4.9. *Given M a Bell functional, suppose there exists a Banach space X such that $\mathsf{B}_\mathsf{X} = \overline{co(\mathcal{A} \cup -\mathcal{A})}$ for certain set $\mathcal{A} \subset \mathcal{P}^k$. Then,*

$$\sup_{P\in\mathcal{A}} |\langle M, P\rangle| = \|M\|_{\mathsf{X}^*}.$$

Proof. The proof of $\sup_{P\in\mathcal{A}} |\langle M, P\rangle| \leq \|M\|_{\mathsf{X}^*}$ is trivial. For the reverse inequality we can consider that any $P \in \mathsf{B}_\mathsf{X}$ can be written as $P = \mu_1 P_1 + \mu_2 P_2$ such that P_1 and P_2 are in $\mathcal{A}$ and $|\mu_1| + |\mu_2| \leq 1$[3]. Then, it is clear that

$$\begin{aligned} |\langle M, P\rangle| &= |\langle M, \mu_1 P_1 + \mu_2 P_2\rangle| \leq |\mu_1 \langle M, P_1\rangle| + |\mu_2 \langle M, P_2\rangle| \\ &\leq (|\mu_1| + |\mu_2|) \sup_{P\in\mathcal{A}} |\langle M, P\rangle| \leq \sup_{P\in\mathcal{A}} |\langle M, P\rangle|. \end{aligned}$$

□

In the following we want to show examples of these Banach spaces.

One-partite probability distributions

We will continue with the one party probability distributions, $\mathcal{P}^1$. In order to understand this situation, we follow an approach similar to what was done in [47]. In that paper, in order to study quantum violation of general Bell inequalities, the authors introduced an auxiliary Banach space, the $\mathsf{NSG}(N, K)$, or simply NSG. In addition, it was shown that the space NSG^* is a "twisted version" of the space $\ell_1^N(\ell_\infty^K)$, with dimension $NK - N + 1$.

[2]Given a set $D \in \mathbb{R}^n$ with $n \in \mathbb{N}$, then the closure of D is denoted by $\overline{D}$.

[3]Notice that given a bounded set $D \subset \mathbb{R}^n$ with $n \in \mathbb{N}$, the supremum of a linear functional over D and the supremum over $\overline{D}$ are equal.

Definition 2.4.10. *Define the space* NSG *as the Banach space formed by*

$$\mathsf{NSG} = \{\{R(x|a)\}_{x,a=1}^{N,K} \in \mathbb{R}^{NK} : \sum_{a=1}^{K} R(x|a) = \text{constant} \in \mathbb{R} \textit{ for every } x\},$$

endowed with the norm

$$\|R\|_{\mathsf{NSG}} = \max_x \sum_a |R(x|a)| = \|R\|_{\ell_\infty^N(\ell_1^K)}.$$

The following theorem can be found in [47], but here we present an alternative proof.

Theorem 2.4.11. *The following holds:*

$$\mathsf{B}_{\mathsf{NSG}} = co(\mathcal{P}^1 \cup -\mathcal{P}^1).$$

Proof. Taking and element $u \in co(\mathcal{P}^1 \cup -\mathcal{P}^1)$, then $u = \mu P_1 + (1-\mu) P_2$. Since it is clear that for every $P \in \mathcal{P}^1$, we have $\max_{x=1,\cdots,N} \sum_{a=1}^K |P(x|a)| = 1$, then it is obvious, by triangle inequality, that $\max_{x=1,\cdots,N} \sum_{a=1}^K |u(x,a)| \leq 1$.

In order to show the converse inclusion, let us consider an element in $R \in \mathsf{NSG}$ such that $\max_{x=1,\cdots,N} \sum_{a=1}^K |R(x|a)| \leq 1$ and we will show that $R \in co(\mathcal{P}^1 \cup -\mathcal{P}^1)$.

Let us denote, for a fixed x,

$$A_x^+ = \{a : R(x|a) \geq 0\} \quad \text{and} \quad A_x^- = \{a : R(x|a) < 0\},$$

and

$$M = \max_x \sum_{a \in A_x^+} R(x|a) \quad \text{and} \quad m = \max_x \Big| \sum_{a \in A_x^-} R(x|a) \Big|.$$

The fact that $\sum_a R(x|a) = K$ for every x guarantees that the previous max and min are attained in the same x. In particular, note that $M - m = K$ and $M + m = \max_x \sum_a |R(x|a)|$.

Therefore, we can write $R = MP_1 - mP_2$, where we define for each x:

$$P_1(x|a) = \begin{cases} \frac{R(x|a)}{M} & \text{if } R(x|a) \geq 0 \text{ and } 1 \leq a \leq K-1, \\ 0 & \text{if } R(x|a) < 0 \text{ and } 1 \leq a \leq K-1, \\ 1 - \sum_{a=1}^{K-1} P_1(x|a) & \text{if } a = K; \end{cases}$$

$$P_2(x|a) = \begin{cases} -\frac{R(x|a)}{m} & \text{if } R(x|a) < 0 \text{ and } 1 \leq a \leq K-1, \\ 0 & \text{if } R(x|a) \geq 0 \text{ and } 1 \leq a \leq K-1, \\ 1 - \sum_{a=1}^{K-1} P_2(x|a) & \text{if } a = K. \end{cases}$$

Since $P_1, P_2 \in \mathcal{P}^1$ we conclude that $R \in co(\mathcal{P}^1 \cup -\mathcal{P}^1)$ and we finish the proof. □

Now we can apply Lemma 2.4.9 to establish the equivalence between the value of certain general Bell functional $M \in \mathbb{R}^{NK}$ played with strategies in $\mathcal{P}^1$ and the norm of this functional computed in the space NSG^*:

$$\omega_{\mathcal{P}^1}(M) = M_{\mathsf{NSG}^*}.$$

This space NSG is a twisted version of $\ell_\infty^N(\ell_1^K)$ and this can be seen using the isomorphism $\widetilde{T}$

$$\begin{aligned} \widetilde{T} : \mathsf{NSG} &\to \ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \mathbb{R} \\ \{R(x,a)\} &\to \left(\{R(x,a)\}_{x=1,a=1}^{N,K-1}, \sum_{a=1}^{K} R(x,y,a,b) \right), \end{aligned}$$

and, consequently, $\widetilde{T}^{-1}$

$$\widetilde{T}^{-1} : \ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \mathbb{R} \to \mathsf{NSG}$$
$$\left(\{R(x,a)\}_{x=1,a=1}^{N,K-1}, R \right) \to \left\{ \{R(x,a)\}_{a=1}^{K-1}, R - \sum_{a=1}^{K-1} R(x,a) \right\}_{x=1}^{N}.$$

It was proven in [47] the following theorem:

Theorem 2.4.12. *The Banach-Mazur distance between* NSG *and* $\ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \mathbb{R}$ *is upper bounded by 3.*

Bipartite classical probability distributions

The second example that we are going to mention is the classical bipartite probability distributions, $\mathcal{L}$. In this case, the Banach space to consider is $\mathsf{NSG} \otimes_\pi \mathsf{NSG}$, which results to be a twisted version of $\ell_\infty^N(\ell_1^K) \otimes_\pi \ell_\infty^N(\ell_1^K)$ with dimension $(N(K-1)+1)^2$. The following result was proved in [47].

Theorem 2.4.13. *The following holds:*

$$co(\mathcal{L} \cup -\mathcal{L}) = \mathsf{B}_{\mathsf{NSG}\otimes_\pi \mathsf{NSG}}.$$

Moreover, $\mathsf{NSG} \otimes_\pi \mathsf{NSG}$ *is isomorphic to* $(\ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \mathbb{R}) \otimes_\pi (\ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \mathbb{R})$ *with Banach-Mazur distance less or equal than 9, independently of the dimension.*

The first part of the proof follows from using Theorem 2.4.11 and the property of the π-norm:

$$\mathsf{B}_{\mathsf{NSG}\otimes_\pi \mathsf{NSG}} = co(\mathsf{B}_{\mathsf{NSG}} \otimes \mathsf{B}_{\mathsf{NSG}}) = co((\mathcal{P}^1 \cup -\mathcal{P}^1) \otimes (\mathcal{P}^1 \cup -\mathcal{P}^1)),$$

altogether with a characterization $\mathcal{L} = co(\mathcal{P}^1 \otimes \mathcal{P}^1)$. The part related with the Banach-Mazur distance comes from considering the metric mapping property and Theorem 2.4.12.

Bipartite quantum probability distributions

Once again, a precise formulation of the quantum case requires notions on operator spaces that are beyond the scope of this thesis. In any case we show here for completeness the characterization of the quantum probability distributions by a Banach space. It requires a complex version of NSG. In this case, the Banach-Mazur distance from NSG and $\ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \mathbb{C}$ is upper bounded by 9. It was proven in [47] that:

$$co(\mathcal{Q} \cup -\mathcal{Q}) = \mathsf{B}_{\mathsf{NSG}\otimes_\wedge \mathsf{NSG}},$$

where $\otimes_\wedge$ is the dual of the *min* norm in the operator space category. Moreover, the Banach-Mazur distance between $\mathsf{NSG} \otimes_\wedge \mathsf{NSG}$ and $\ell_\infty^N(\ell_1^K) \otimes_\wedge \ell_\infty^N(\ell_1^K)$ is upper bounded by 81.

Remark 2.4.14. *Finally, upper bounding the quantity* $LV(\mathcal{Q},\mathcal{L})$ *consists on bounding up to a constant the quantity:*

$$\|id : \mathsf{NSG}(N_1,K_1) \otimes_\wedge \mathsf{NSG}(N_2,K_2) \longrightarrow \mathsf{NSG}(N_1,K_1) \otimes_\pi \mathsf{NSG}(N_2,K_2)\|.$$

This was done in [47, 63] with the following result:

$$LV(\mathcal{Q}(N_1,N_2,K_1,K_2), \mathcal{L}(N_1,N_2,K_1,K_2)) \leq C \min\{N_1, N_2, \sqrt{K_1K_2}\}, \tag{2.4.2}$$

where C *is a universal constant.*

Chapter 3

Non-Signalling

3.1 Introduction

Non-signalling probability distributions are those bipartite probability distributions $P \in \mathcal{P}$ that satisfy the following two conditions.

$$\sum_a P(a,b|x,y) = \sum_a P(a,b|x',y) \quad \text{for all } x,x',y,b, \tag{3.1.1}$$

$$\sum_b P(a,b|x,y) = \sum_b P(a,b|x,y') \quad \text{for all } y,y',x,a. \tag{3.1.2}$$

These conditions are known as the *non-signalling contions*. And the set is denoted by $\mathcal{NS}$. If these two equations hold, then the marginals are well defined, i.e. there exist probability distributions Q and R in $\mathcal{P}^1$ (see Equation (1.4.1) such that:

$$\sum_b P(a,b|x,y) = Q(a|x) \quad \text{for all } a,x,y,$$

$$\sum_a P(a,b|x,y) = R(b|y) \quad \text{for all } b,x,y.$$

Recall that if the first condition is not fulfilled, then Bob would be able to obtain information about the inputs that Alice has received. This would imply instant communication and it is prohibited by the theory of relativity.

In this chapter we analyze the set of bipartite non-signalling probability distributions. We find a characterization in terms of a norm that will be useful to give upper bounds of the Bell violation ratio of the non-signalling over the locals probability distributions. Moreover, we prove that these bounds are optimal.

3.2 Non-signalling correlations

The non-signalling set for correlations is denoted by $\mathcal{NS}_{cor}$. Recall that in this case the possible outputs a and b take the values $\{-1,+1\}$ and, by definition, $\gamma \in \mathcal{NS}_{cor}$ if and only if there exists $P \in \mathcal{NS}$ such that $\gamma_{xy} = \mathbb{E}_P[a \cdot b|x,y]$.

The set $\mathcal{NS}_{cor}$ has already been studied and characterized [74]. It is known that a non-signalling distribution is uniquely determined by the expected correlations and the expected marginals, defined as $M_A(x) = \mathbb{E}[a|x]$ and $M_B(y) = \mathbb{E}[b|y]$ [32, Proposition 1].

However here we will interested in another characterization of the non-signalling set of correlations which does not take into account the marginals. We will make use of tensor norms, but first, we need the following mathematical notion:

Remark 3.2.1. *Given $u = \sum_{i,j} u_{ij} e_i \otimes e_j \in \mathbb{R}^N \otimes \mathbb{R}^N$, according to the definition of the ϵ-norm in Equation (2.2.1):*

$$\|u\|_{\ell_\infty^N \otimes_\epsilon \ell_\infty^N} = \sup_{z_1, z_2 \in \mathsf{B}_{\ell_1^N}} |\langle z_1 \otimes z_2, u\rangle| = \sup_{x,y} |\langle \pm e_x \otimes \pm e_y, \sum_{i,j} u_{ij} e_i \otimes e_j\rangle| = \sup_{x,y} |u_{xy}| = \|u\|_{\ell_\infty^{N^2}}.$$

In the second inequality, in order to compute the supremum, we take into account only the extreme points (which is equivalent, by convexity), and, in the case of $\mathsf{B}_{\ell_1^N}$, they have the form $\pm e_x$ for all x.

Now we can state the characterization for non-signalling correlations (see [32, Proposition 1] for $M_A(x) = M_B(y) = 0$ for all x and y):

Lemma 3.2.2. *Given a correlation $\gamma = \sum_{x,y} \gamma_{xy} e_x \otimes e_y \in \mathbb{R}^N \otimes \mathbb{R}^N$, the following holds:*

$$\gamma \in \mathcal{NS}_{cor} \text{ if and only if } |\gamma_{xy}| \leq 1 \text{ for all } x, y, \text{ if and only if } \|\gamma\|_{\ell_\infty^N \otimes_\epsilon \ell_\infty^N} \leq 1.$$

Proof. The equivalence between $|\gamma_{xy}| \leq 1$ for all x, y and $\|\gamma\|_{\ell_\infty^N \otimes_\epsilon \ell_\infty^N} \leq 1$ folllows from Remark 3.2.1.

Moreover, given any $P \in \mathcal{NS}$, we clearly have that:

$$|\sum_{a,b} ab P(a,b|x,y)| \leq \sum_{a,b} |ab||P(a,b|x,y)| \leq \sum_{a,b} P(a,b|x,y) = 1.$$

For the other implication, given a correlation γ_{xy} satisfying $|\gamma_{xy}| \leq 1$ for every x, y, consider the probability distribution defined as:

$$P(a,b|x,y) = \begin{cases} \frac{1+\gamma_{xy}}{4} & \text{if} \quad ab = 1, \\ \frac{1-\gamma_{xy}}{4} & \text{if} \quad ab = -1. \end{cases}$$

Then, it is clear that P is in $\mathcal{NS}$, with $\sum_a P(a,b|x,y) = \sum_b P(a,b|x,y) = 1/2$. □

This completes the studied done in the last chapter about the tensor norm characterization of bipartite correlations, which are naturally in the space $\ell_\infty^N \otimes \ell_\infty^N$. Thus, we have described the classical correlations using the π-norm, the quantum correlations using the dual of the γ_2-norm and the non-signalling correlations using the ϵ-norm. Notice that it follows easily that the correlations that arise from non-signalling probability distributions and from general probability distributions are the same.

Now that we have a tensor norm characterization for non-signalling correlations, we can also use the characterization for classical correlations given in Example 2.3.1 to study the largest violation that can be attained by correlation Bell functionals. Those linear Bell functionals act on correlations and have the form $T = (T_{xy})_{x,y=1}^N = \sum_{x,y} T_{xy} e_x \otimes e_y \in \mathbb{R}^N \otimes \mathbb{R}^N$. Then, we can say the following:

$$LV(\mathcal{NS}_{cor},\mathcal{L}_{cor}) = \sup_T \frac{\sup_{\gamma\in\mathcal{NS}_{cor}}|\langle T|\gamma\rangle|}{\sup_{\gamma\in\mathcal{L}_{cor}}|\langle T|\gamma\rangle|} = \sup_T \frac{\sup_{\|\gamma\|_{\ell_\infty^N\otimes_\epsilon\ell_\infty^N}\le 1}|\langle T|\gamma\rangle|}{\sup_{\|\gamma\|_{\ell_\infty^N\otimes_\pi\ell_\infty^N}\le 1}|\langle T|\gamma\rangle|} = \sup_T \frac{\|T\|_{\ell_1^N\otimes_\pi\ell_1^N}}{\|T\|_{\ell_1^N\otimes_\epsilon\ell_1^N}}. \tag{3.2.1}$$

It is well known that $\|\cdot\|_{\ell_1^N\otimes_\pi\ell_1^N} \le \sqrt{2N}\|\cdot\|_{\ell_1^N\otimes_\epsilon\ell_1^N}$ (see for instance [56, Ex. 29]). Therefore, the largest non-signalling violation attainable in the correlation situation cannot be larger than $O(\sqrt{N})$, and this order is attained.

To show the optimality of the order $\sqrt{N}$ we will need the following well known remark:

Remark 3.2.3. *Given $n\in\mathbb{N}$, the Hadamard matrix $H_{2^n} = (h_{xy})_{x,y=1}^{2^n}$ has the property $H_{2^n}H_{2^n}^T = 2^n\mathbb{1}$. The restriction on the dimension to be a power of 2 guarantees that a Hadamard matrix exists (see e. g. [78]). A construction was given by Sylvester in 1867 in a recursive way:*

$$H_{2^0} = 1 \qquad H_{2^1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad H_{2^k} = \begin{bmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{bmatrix}.$$

Moreover, the upper bound

$$\|H_{2^n}\|_{\ell_1^N\otimes_\epsilon\ell_1^N} \le (2^n)^{3/2}$$

is proved in [56, Ex. 29].

Given a general $N\in\mathbb{N}$, consider now n such that $N/2 < 2^n \le N$ and let $H_{2^n} = (h_{xy})_{x,1=1}^{2^n}$ be a Hadamard matrix from Remark 3.2.3. Define the Bell functional $T = (T_{xy})_{x,y=1}^N$ as $T_{xy} = h_{xy}$ for $1\le x,y\le 2^n$ and $T_{xy}=0$ otherwise. Then:

$$\|T\|_{\ell_1^N\otimes_\pi\ell_1^N} = \sum_{x,y=1}^{N}|T_{xy}| = \sum_{x,y=1}^{2^n}|h_{xy}| \ge \frac{N^2}{4}.$$

At the same time,

$$\|T\|_{\ell_1^N\otimes_\epsilon\ell_1^N} = \sup_{a_x,b_y=\pm1}\Big|\sum_{x,y=1}^{2^n} h_{xy}a_xb_y\Big| \le (2^n)^{3/2} \le N^{3/2}.$$

This shows the optimality of the order mentioned before.

Observe that, while the largest Bell violation between non-signalling and classical correlations $LV(\mathcal{NS}_{cor},\mathcal{L}_{cor})$ has a value of $\mathcal{O}(\sqrt{N})$, $LV(\mathcal{Q}_{cor},\mathcal{L}_{cor})$, the largest Bel violation between quantum and classical correlations, has a value of $\mathcal{O}(1)$, independently of N. Actually, by Section 2.3, it can be seen that this value is upper bounded by Grothendieck's constant. This shows us that the non-signalling theory can be much more powerful that the quantum theory, at least for the case of correlations.

Now the question is to study what happens when we consider any number of outputs, i.e., general probability distributions. The problem that we will find here is, contrary to the case of correlations, the absence of a tensor norm structure. Indeed, it can be seen that no tensor norm in $\ell_1^N(\ell_\infty^K)\otimes\ell_1^N(\ell_\infty^K)$ can describe the non-signalling value of a game, even up to a constant (see Appendix B).

3.3 The non-signalling norm

As we have seen, the relation between classical and non-signalling correlations is well understood, and this relation can be expressed and analyzed using the tensor norm language. We start now to follow this approach for the study of full probability distributions.

We begin by defining a suitable norm for non-signalling probability distributions. Any given $P \in \mathcal{P} \subset \mathbb{R}^{N^2K^2}$ can be seen as a 4-tensor in $\mathbb{R}^N \otimes \mathbb{R}^K \otimes \mathbb{R}^N \otimes \mathbb{R}^K$ with indexes x, a, y and b.

For a given $z = \{z(x,a,y,b)\}_{x,a,y,b} \in \mathbb{R}^N \otimes \mathbb{R}^K \otimes \mathbb{R}^N \otimes \mathbb{R}^K$, we consider:

$$\|z\|_{\ell_\infty^N(\ell_1^K(\ell_\infty^N(\ell_1^K)))} = \max_x \sum_a \max_y \sum_b |z(x,a,y,b)|,$$
$$\|z\|_{\ell_1^N(\ell_\infty^K(\ell_1^N(\ell_\infty^K)))} = \sum_x \max_a \sum_y \max_b |z(x,a,y,b)|.$$

Definition 3.3.1. *Define the Banach spaces,* NS, $\mathsf{NS1}$ *and* $\mathsf{NS2}$ *by using the vector space* $\mathbb{R}^{N^2K^2}$ *with the following norms for any given* $P \in \mathbb{R}^{N^2K^2}$*,*

$$\|P\|_{\mathsf{NS1}} = \max_x \sum_a \max_y \sum_b |P(a,b|x,y)|,$$
$$\|P\|_{\mathsf{NS2}} = \max_y \sum_b \max_x \sum_a |P(a,b|x,y)|,$$
$$\|P\|_{\mathsf{NS}} = \max\{\|P\|_{\mathsf{NS1}}, \|P\|_{\mathsf{NS2}}\}.$$

Notation 1. *The Banach space* NS *depends on* N *and* K*, the number of inputs and outputs, and should be written as* $\mathsf{NS}(N,K)$*. But, in order to simplify the notation we will generically write* NS *referring to the* N,K *situation and it will be specifically denoted otherwise. This also applies to* $\mathsf{NS1}$ *and* $\mathsf{NS2}$*.*

Given $P \in \mathbb{R}^{N^2K^2}$, we define $P^T = \mathsf{flip}(P)$ where

$$\mathsf{flip} : \mathbb{R}^N \otimes \mathbb{R}^K \otimes \mathbb{R}^N \otimes \mathbb{R}^K \longrightarrow \mathbb{R}^N \otimes \mathbb{R}^K \otimes \mathbb{R}^N \otimes \mathbb{R}^K$$
$$e_x \otimes e_a \otimes e_y \otimes e_b \longrightarrow e_y \otimes e_b \otimes e_x \otimes e_a.$$

With this notation, $\|P^T\|_{\mathsf{NS1}} = \|P\|_{\mathsf{NS2}}$. Note that both $\mathsf{NS1}$ and $\mathsf{NS2}$ are isomorphic to $\ell_\infty^N(\ell_1^K(\ell_\infty^N(\ell_1^K)))$.

The space NS is going to play the role associated to X in Chapter 2. From the following result it can be deduced that $\mathcal{NS} = \mathsf{B}_{\mathsf{NS}} \cap \mathcal{P}$.

Proposition 3.3.2. *Let* $P \in \mathbb{R}^{N^2K^2}$*. Then* $P \in \mathcal{NS}$ *if and only if* $P \in \mathcal{P}$ *and* $\|P\|_{\mathsf{NS}} = 1$*.*

Proof. Suppose $P \in \mathcal{NS}$. Since $\mathcal{NS} \subset \mathcal{P}$, P belongs to $\mathcal{P}$ and, moreover, there exist Q and R in $\mathcal{P}^1$ such that $\sum_a P(a,b|x,y) = Q(b|y)$ for all b and y and $\sum_b P(a,b|x,y) = R(a|x)$ for all x and a. Then,

$$\max_x \sum_a \max_y \sum_b |P(a,b|x,y)| = \max_x \sum_a \max_y R(a|x) = \max_x \sum_a R(a|x) = 1.$$

And similarly for $\max_y \sum_b \max_x \sum_a |P(a,b|x,y)|$. Therefore, $\|P\|_{\mathsf{NS}} = 1$.

For the other implication, suppose $P \in \mathcal{P}$ and $P \notin \mathcal{NS}$. Then, we can assume that P does not fulfill Equation (3.1.1) (the other case being analogous) and therefore we can assume that there exist b_0, y_0, x_0 and x_1 such that $\sum_a P(a,b_0|x_0,y_0) > \sum_a P(a,b_0|x_1,y_0)$. Hence,

$$\|P\|_{\mathsf{NS}} \geq \max_{y=1,\cdots,N} \sum_{b=1}^{K} \max_{x=1,\cdots,N} \sum_{a=1}^{K} P(a,b|x,y) \geq \sum_{b=1}^{K} \max_{x=1,\cdots,N} \sum_{a=1}^{K} P(a,b|x,y_0)$$
$$= \sum_{b\neq b_0} \max_{x=1,\cdots,N} \sum_{a=1}^{K} P(a,b|x,y_0) + \max_{x=1,\cdots,N} \sum_{a=1}^{K} P(a,b_0|x,y_0)$$
$$\geq \sum_{b\neq b_0} \sum_{a=1}^{K} P(a,b|x_1,y_0) + \sum_{a=1}^{K} P(a,b_0|x_0,y_0)$$
$$> \sum_{b\neq b_0} \sum_{a=1}^{K} P(a,b|x_1,y_0) + \sum_{a=1}^{K} P(a,b_0|x_1,y_0)$$
$$= \sum_{b=1}^{K} \sum_{a=1}^{K} P(a,b|x_1,y_0) = 1.$$

Therefore, $\|P\|_{\mathsf{NS}} > 1$, which is a contradiction. Hence, we conclude that $P \in \mathcal{NS}$. □

The following set, closely related to the non-signalling probability distributions, will be very useful for our reasonings. It was introduced in [57].

Definition 3.3.3. *The set SNOS $\subset \mathbb{R}^{N^2K^2}$ consists of the non-negative elements $P(a,b|x,y)$ in $\mathbb{R}^{N^2K^2}$ such that, for every $1 \leq x,y \leq N$, there exist $(Q_1(a|x))_{a=1}^K$ and $(Q_2(b|y))_{b=1}^K$ probability distributions verifying that, for every x,y,a,b, $\sum_a P(a,b|x,y) \leq Q_2(b|y)$ and $\sum_b P(a,b|x,y) \leq Q_1(a|x)$.*

Proposition 3.3.4. *Given $P \in \mathbb{R}^{N^2K^2}$ with non-negative entries, the condition of P in SNOS is equivalent to the existence of $\tilde{P}$ in $\mathcal{NS}$ such that $P(a,b|x,y) \leq \tilde{P}(a,b|x,y)$ for all x,y,a,b. In this case we use the notation $P \leq \tilde{P}$.*

Proof. There are different proofs showing the non-trivial implication. One can be found in [43, Claim 1] and another one in Appendix C of this thesis. □

The next result is necessary to see that the Banach space NS fulfills Condition 2.4.2.

Proposition 3.3.5. *Consider $P \in \mathbb{R}^{N^2K^2}$ with non-negative entries. Then, $P \in$ SNOS if and only if $\|P\|_{\mathsf{NS}} \leq 1$.*

Proof. Let $P \in$ SNOS. For every $1 \leq x,y \leq N$, let $(Q_1(a|x))_{a=1}^K$, $(Q_2(b|y))_{b=1}^K$ be as in Definition 3.3.3. Then

$$\max_x \sum_a \max_y \sum_b P(a,b|x,y) \leq \max_x \sum_a \max_y Q_1(a|x) = \max_x \sum_a Q_1(a|x) = 1,$$
$$\max_y \sum_b \max_x \sum_a P(a,b|x,y) \leq \max_y \sum_b \max_x Q_2(b|y) = \max_y \sum_b Q_2(b|y) = 1.$$

Consequently, $\|P\|_{\mathsf{NS}} \leq 1$.

Conversely, if $\|P\|_{\mathsf{NS}} \leq 1$, define, for all y and b, $\max_x \sum_a P(a,b|x,y) = \tilde{Q}_2(b|y)$ and for all x and a, $\max_y \sum_b P(a,b|x,y) = \tilde{Q}_1(a|x)$. Note that $\tilde{Q}_1$ and $\tilde{Q}_2$ need not be probability distributions (although they are non-negative). For this reason, we define:

$$Q_1(a|x) = \begin{cases} 1 - \sum_{s\neq 1} \tilde{Q}_1(s|x) & \text{if } a = 1 \\ \tilde{Q}_1(a|x) & \text{if } a \neq 1 \end{cases} \qquad Q_2(b|y) = \begin{cases} 1 - \sum_{t\neq 1} \tilde{Q}_2(t|y) & \text{if } b = 1 \\ \tilde{Q}_2(b|y) & \text{if } b \neq 1 \end{cases}$$

It is easy to see that Q_1 and Q_2 guarantee that P belongs to SNOS. □

Remark 3.3.6. *It follows from Proposition 3.3.5 that the set SNOS is convex and it has the same dimension as* $\mathbb{R}^{N^2K^2}$

Condition 2.4.1 is fulfilled by Proposition 3.3.2 while Condition 2.4.2 is fulfilled by Proposition 3.3.5 with the fact that for any given $P \in \mathsf{B}_{\mathsf{NS}}$ we have also that $|P| \in \mathsf{B}_{\mathsf{NS}}$. Then, Lemma 2.4.4 can be applied to the non-signalling norm to provide the next corollary:

Corollary 3.3.7. *Given a tensor* $G \in \mathbb{R}^{N^2K^2}$ *with non-negative coefficients, we have*

$$\omega_{\mathcal{NS}}(G) = \|G\|_{\mathsf{NS}^*}.$$

Since the value of winning a game using non-signalling strategies can be calculated using this norm, we can say that it corresponds to "something physical". We leave here the explicit expression after introducing some necessary mathematical notions.

Given two Banach spaces X and Y that are linear and topological subspaces of some other Banach space Z, we can consider two new spaces $\mathsf{X} \cap \mathsf{Y}$ and $\mathsf{X} + \mathsf{Y}$ with the following norms:

$$\begin{aligned} \|x\|_{\mathsf{X}\cap\mathsf{Y}} &= \max\{\|x\|_{\mathsf{X}}, \|x\|_{\mathsf{Y}}\}, \\ \|x\|_{\mathsf{X}+\mathsf{Y}} &= \inf\{\|x_1\|_{\mathsf{X}} + \|x_2\|_{\mathsf{Y}} \text{ such that } x = x_1 + x_2\}. \end{aligned} \tag{3.3.1}$$

One can check [14, Chapter 2] that if $\mathsf{X}\cap\mathsf{Y}$ is dense in both X and Y, then $(\mathsf{X}\cap\mathsf{Y})^* = \mathsf{X}^* + \mathsf{Y}^*$, isometrically. In the case we will be interested, X and Y will be finite dimensional with the same dimension. Since all norms on a finite dimensional space are equivalent, we can consider Z to be either X or Y and $\mathsf{X}\cap\mathsf{Y}$ will be not only dense in X and Y, but actually will coincide (as a vector space) with both of them.

Using Equation (3.3.1) with the non-signalling norm we can identify $\mathsf{NS} = \mathsf{NS1} \cap \mathsf{NS2}$. Then, by (3.3.1) we can say that $\mathsf{NS}^* = \mathsf{NS1}^* + \mathsf{NS2}^*$. Note also that $\|M\|_{\mathsf{NS1}^*} = \sum_x \max_a \sum_y \max_b |M(a,b|x,y)|$ and $\|M\|_{\mathsf{NS2}^*} = \sum_y \max_b \sum_x \max_a |M(a,b|x,y)|$. It allows us to write that:

$$\begin{aligned} \|M\|_{\mathsf{NS}^*} &= \inf\{\|M_1\|_{\mathsf{NS1}^*} + \|M_2\|_{\mathsf{NS2}^*} : M = M_1 + M_2\} \\ &= \inf\left\{\|M_1\|_{\ell_1^N(\ell_\infty^K(\ell_1^N(\ell_\infty^K)))} + \|M_2^T\|_{\ell_1^N(\ell_\infty^K(\ell_1^N(\ell_\infty^K)))} : M = M_1 + M_2\right\}. \end{aligned} \tag{3.3.2}$$

Corollary 3.3.8. *If* G *is two-prover one-round game, then the non-signalling value of the game can be written as:*

$$\omega_{\mathcal{NS}}(G) = \inf\left\{\|G_1\|_{\ell_1^N(\ell_\infty^K(\ell_1^N(\ell_\infty^K)))} + \|G_2^T\|_{\ell_1^N(\ell_\infty^K(\ell_1^N(\ell_\infty^K)))} : G = G_1 + G_2\right\}.$$

3.4 Upper bounds

In this section we want to give upper bounds to the quantity $LV(\mathcal{NS}, \mathcal{L})$, the largest violation between non-signalling and classical probability distributions. We will consider separately two different types of Bell functionals, one in which all components are non-negative, which are related to games, and the other in which the components can be positive or negative indistinctly. We will proceed as it was specified in section 2.4.1.

3.4.1 Upper bounds for games

We have seen in Corollary 3.3.7 how the non-signalling value of a game (more generally, of any functional G with non-negative entries) verifies $\omega_{\mathcal{NS}}(G) = \|G\|_{\mathsf{NS}^*}$. It is also known ([63, Section 4]) that the classical value of a game G verifies $\omega_{\mathcal{L}}(G) = \|G\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}$.

So, the next step is to upper bound the distance between the norms associated to the classical and non-signalling probability distributions in terms of the number of inputs and outputs.

$$\|id : \ell_1^N(\ell_\infty^K) \otimes_\epsilon \ell_1^N(\ell_\infty^K) \longrightarrow \mathsf{NS}^*\|.$$

To this aim we will use that the Banach space NS is related to $\ell_\infty^N(\ell_1^K(\ell_\infty^N(\ell_1^K)))$ via the Banach spaces $\mathsf{NS1}$ and $\mathsf{NS2}$.

Proposition 3.4.1. *For every $M \in \mathbb{R}^{N^2K^2}$,*

$$\|M\|_{\ell_1^N(\ell_\infty^K(\ell_1^N(\ell_\infty^K)))} \leq N\|M\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}.$$

Proof. To prove this bound consider the following:

$$\begin{aligned}\|M\|_{\ell_1^N(\ell_\infty^K(\ell_1^N(\ell_\infty^K)))} &= \sum_x \max_a \sum_y \max_b |M_{xy}^{ab}| \leq N \max_{x,a} \sum_y \max_b |M_{xy}^{ab}| \\ &= N\|M\|_{\ell_\infty^{NK}(\ell_1^N(\ell_\infty^K))} = N\|M\|_{\ell_\infty^{NK}\otimes_\epsilon \ell_1^N(\ell_\infty^K)} \\ &= N \sup_{u\in \mathsf{B}_{\ell_1^{NK}}, v\in \mathsf{B}_{\ell_\infty^N(\ell_1^K)}} |\langle u\otimes v, M\rangle| \\ &\leq N \sup_{u\in \mathsf{B}_{\ell_\infty^N(\ell_1^K)}, v\in \mathsf{B}_{\ell_\infty^N(\ell_1^K)}} |\langle u\otimes v, M\rangle| \\ &= N\|M\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)},\end{aligned}$$

where in the third equality we have used the identification $\ell_\infty(\mathsf{X}) = \ell_\infty^N \otimes_\epsilon \mathsf{X}$, in the fourth equality we have used the definition of the ϵ norm (2.2.1) and in the second inequality we have used the inclusion $\mathsf{B}_{\ell_1^{NK}} \subset \mathsf{B}_{\ell_\infty^N(\ell_1^K)}$. □

Now we bound, in the positive case, the distance between those two same norms in terms of the number of outputs.

Proposition 3.4.2. *Let $G \in \mathbb{R}^{N^2K^2}$ have non-negative entries, then*

$$\|G\|_{\ell_1^N(\ell_\infty^K(\ell_1^N(\ell_\infty^K)))} \leq K\|G\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}.$$

Proof. Consider an element $G \in \mathbb{R}^{N^2K^2}$ with non-negative coefficients, then,

$$\begin{aligned}\|G\|_{\ell_1^N(\ell_\infty^K(\ell_1^N(\ell_\infty^K)))} &= \sum_x \max_a \sum_y \max_b |G_{xy}^{ab}| \leq \sum_x \max_a \Big(\sum_{y,b} G_{xy}^{ab}\Big) \\ &= \sup_{u\in \mathsf{B}_{\ell_\infty^N(\ell_1^K)}} \Big\langle u, \sum_{y,b} G_{xy}^{ab} e_x \otimes e_a \Big\rangle.\end{aligned}$$

By the non-negativity of G, we may assume that u is also pointwise non-negative, and we have

$$\sup_{u\in \mathsf{B}_{\ell_\infty^N(\ell_1^K)}} \Big\langle u, \sum_{y,b} G_{xy}^{ab} e_x \otimes e_a \Big\rangle = \sup_{u\in \mathsf{B}_{\ell_\infty^N(\ell_1^K)}, v\in \mathsf{B}_{\ell_\infty^{NK}}} \langle u\otimes v, G\rangle,$$

and this last supremun is attained when $v = \sum_{y,b} e_y \otimes e_b$.

It can be checked that $\mathsf{B}_{\ell_\infty^{NK}} \subset K\mathsf{B}_{\ell_\infty^N(\ell_1^K)}$. Then, we have

$$\begin{aligned}\|G\|_{\ell_1^N(\ell_\infty^K(\ell_1^N(\ell_\infty^K)))} &= \sup_{u\in \mathsf{B}_{\ell_\infty^N(\ell_1^K)}, v\in \mathsf{B}_{\ell_\infty^N(\ell_1^K)}} \langle u\otimes v, G\rangle \leq \\ &\leq \sup_{v\in \mathsf{B}_{\ell_\infty^N(\ell_1^K)}, u\in \mathsf{B}_{\ell_\infty^N(\ell_1^K)}} K|\langle u\otimes v, G\rangle| \leq \\ &\leq K\|G\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}.\end{aligned}$$

□

Remark 3.4.3. *Note that Proposition 3.4.1 is stated for general functionals M. However, Proposition 3.4.2 requires that the element M is (pointwise) non-negative. It can be seen that Proposition 3.4.2 fails for general elements. In fact, Remark 3.2.3 shows a Bell functional T in which, for $K = 2$, the quotient $\omega_{\mathcal{NS}}(T)/\omega_{\mathcal{L}}(T)$ can be arbitrarily large.*

Theorem 3.4.4. *Given a tensor $G \in \mathbb{R}^{N^2K^2}$ with non-negative coefficients,*

$$\frac{\omega_{\mathcal{NS}}(G)}{\omega_{\mathcal{L}}(G)} \leq min\{N, K\}.$$

Proof. Since we are considering a Bell inequality with non-negative coefficients, Corollary 3.3.7 states that $\sup_{P\in\mathcal{NS}}\langle G, P\rangle = \|G\|_{\mathsf{NS}^*}$. At the same time, we have

$$\sup_{P\in\mathcal{L}} |\langle M, P\rangle| = \|G\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}.$$

In addition, we have that $\|G\|_{\mathsf{NS}^*} \leq \min\{\|G\|_{\mathsf{NS1}^*}, \|G\|_{\mathsf{NS2}^*}\}$.

Now, Applying Proposition 3.4.1 and Proposition 3.4.2, we obtain

$$\min\{\|G\|_{\mathsf{NS1}^*}, \|G\|_{\mathsf{NS2}^*}\} \leq \min\{N, K\}\|G\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}.$$

Putting both inequalities together we get:

$$\begin{aligned}LV^+(\mathcal{NS}, \mathcal{L}) &= \sup_G \frac{\sup_{P\in\mathcal{NS}}\langle G, P\rangle}{\sup_{P\in\mathcal{L}}\langle G, P\rangle} = \sup_G \frac{\|G\|_{\mathsf{NS}^*}}{\|G\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}} \\ &\leq \sup_G \frac{\min\{\|G\|_{\mathsf{NS1}^*}, \|G\|_{\mathsf{NS2}^*}\}}{\|G\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}} \leq \min\{N, K\}.\end{aligned}$$

Here the supremum considers tensors with only non-negative elements. □

Remark 3.4.5. *Obvious modifications of these proofs show that if we distinguish the inputs and outputs for Alice and Bob as N_1, N_2, K_1 and K_2, then one has the following bound for pointwise non-negative elements:*

$$LV^+(\mathcal{NS}, \mathcal{L}) \leq \min\{N_1, N_2, K_1, K_2\}.$$

Recall that by Equation (2.4.1) for non-negative Bell functionals we have $LV^+(\mathcal{Q}, \mathcal{L}) \leq \min\{N, K\}$. Somehow surprisingly, we see that, although a priori non-signalling strategies can be much better than quantum strategies, the same upper bound applies in both cases. In fact, it is known that this upper bound is essentially optimal in the number of outputs K ([23]). Hence, in terms of this number, non-signalling strategies do not provide a huge advantage with respect to quantum strategies.

3.4.2 Upper bounds for general Bell functionals

The problem of considering general Bell functionals, with coefficients not necessarily non-negative, is that the relation between $\omega_{\mathcal{NS}}(M)$ and $\|M\|_{\mathsf{NS}^*}$ is not so clear anymore. In fact, it can be easily checked that $\omega_{\mathcal{NS}}(\cdot)$ is not a norm in $\mathbb{R}^{N^2K^2}$.

In the following we make a construction of a normed space based on the non-signalling distributions. This space, called ANS, has the property that its elements fulfill Equations (3.1.1) and (3.1.2) and will allow us to completely characterize the set of non-signalling probability distributions by means of its unit ball (see Theorem 3.4.10).

Definition 3.4.6. *Let* ANS *be the set of elements* $R \in \mathbb{R}^{N^2K^2}$ *for which there exist* $\{Q(y,b)\}_{y,b} \in \mathbb{R}^{NK}$, $\{P(x,a)\}_{x,a} \in \mathbb{R}^{NK}$ *and a constant* $z \in \mathbb{R}$ *such that* $\sum_a R(x,y,a,b) = Q(y,b)$ *for all* x, b, y, $\sum_b R(x,y,a,b) = P(x,a)$ *for all* y, b, x *and* $\sum_{a,b} R(x,y,a,b) = z$ *for all* x, y.

We endow the linear space ANS *with the norm*

$$\|R\|_{\mathsf{ANS}} = \max\left\{\|R\|_{\ell_\infty^N(\ell_1^K(\ell_\infty^N(\ell_1^K)))}, \|R^T\|_{\ell_\infty^N(\ell_1^K(\ell_\infty^N(\ell_1^K)))}\right\}. \tag{3.4.1}$$

For every element $P \in \mathsf{ANS}$, we have that $\|P\|_{\mathsf{ANS}} = \|P\|_{\mathsf{NS}}$. Hence, we will use the notation $\|P\|_{\mathsf{NS}}$ is this section.

We are going to show that the space ANS is a twisted version of NS with dimension $(N(K-1)+1)^2$.

We will need some notation. Given $R = (R(x,y,a,b))_{x,y=1,a,b=1}^{N,K} \in \mathbb{R}^{N^2K^2}$, we define

$$R^+(x,y,a,b) = \begin{cases} R(x,y,a,b) & \text{if } R(x,y,a,b) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$R^-(x,y,a,b) = \begin{cases} R(x,y,a,b) & \text{if } R(x,y,a,b) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $R = R^+ + R^-$

We will use the following notation for fixed x, a and y, b respectively:

$$\begin{aligned} c_{xa} &= \max_y \sum_b |R(x,y,a,b)| = \sum_b |R(x,y_{xa},a,b)|, \\ c_{xa}^{\pm} &= \sum_b |R^{\pm}(x,y_{xa},a,b)|, \\ d_{yb} &= \max_x \sum_a |R(x,y,a,b)| = \sum_b |R(x_{yb},y,a,b)|, \\ d_{yb}^{\pm} &= \sum_b |R^{\pm}(x_{yb},y,a,b)|. \end{aligned}$$

It is straightforward to check that for every x, y, a, b one has the following equalities:

$$\begin{aligned} c_{xa} &= c_{xa}^+ + c_{xa}^-, \\ P(x,a) &= c_{xa}^+ - c_{xa}^-, \\ d_{yb} &= d_{yb}^+ + d_{yb}^-, \\ Q(y,b) &= d_{yb}^+ - d_{yb}^-, \\ z &= \sum_a c_{xa}^+ - \sum_a c_{xa}^- = \sum_b d_{yb}^+ - \sum_b d_{yb}^-. \end{aligned}$$

We will need the following two lemmas.

Lemma 3.4.7. *If $R \in \mathsf{ANS}$, then*

$$\|R\|_{\mathsf{NS}} = \|R^+\|_{\mathsf{NS}} + \|R^-\|_{\mathsf{NS}}.$$

Moreover, if $\|R\|_{\mathsf{NS}} = \sum_{a,b}|R(x_0, y_a, a, b)|$, then

$$\|R^+\|_{\mathsf{NS}} = \sum_{a,b}|R^+(x_0, y_a, a, b)| \quad \text{and} \quad \|R^-\|_{\mathsf{NS}} = \sum_{a,b}|R^-(x_0, y_a, a, b)|.$$

An analogous statement holds if $\|R\|_{\mathsf{NS}} = \sum_{a,b}|R(x_b, y_0, a, b)|$.

Proof. Consider an element $R \in \mathsf{ANS}$ from Definition 3.4.6 with its notation. Suppose that $\|R\|_{\mathsf{NS1}} = \sum_{a,b} R(x_0, y_a, a, b)$, $\|R\|_{\mathsf{NS2}} = \sum_{a,b} R(x_b, y_0, a, b)$ and also assume, without loss of generality, that $\|R\|_{\mathsf{NS1}} \geq \|R\|_{\mathsf{NS2}}$. Using the notation introduce above, we have in addition,

$$\|R\|_{\mathsf{NS}} = \sum_a c^+_{x_0 a} + \sum_a c^-_{x_0 a}.$$

We are going to divide the proof in three steps. First, we note that

$$\max_y \sum_b |R^+(x, y, a, b)| = c^+_{xa} \quad \text{and} \quad \max_y \sum_b |R^-(x, y, a, b)| = c^-_{xa}.$$

To see this, recall that it follows from adding or subtracting the next equality and inequality, which hold for every $1 \leq y \leq N$:

$$\left.\begin{array}{l} c^+_{xa} - c^-_{xa} = P(x, a) = \sum_b |R^+(x, y, a, b)| - \sum_b |R^-(x, y, a, b)| \\ c^+_{xa} + c^-_{xa} = c_{xa} \geq \sum_b |R^+(x, y, a, b)| + \sum_b |R^-(x, y, a, b)| \end{array}\right\}.$$

Similarly, the same result is obtained for d^+_{yb} and d^-_{yb}.

In the second step we prove that

$$\max_x \sum_a c^+_{xa} = \sum_a c^+_{x_0 a} \quad \text{and} \quad \max_x \sum_a c^-_{xa} = \sum_a c^-_{x_0 a}.$$

Again, this follows from adding and subtracting the next equality and inequality, both of which clearly hold for every $1 \leq x \leq N$.

$$\left.\begin{array}{l} \sum_a c^+_{x_0 a} - \sum_a c^-_{x_0 a} = z = \sum_a c^+_{xa} - \sum_a c^-_{xa} \\ \sum_a c^+_{x_0 a} + \sum_a c^-_{x_0 a} = \|R\|_{\mathsf{NS1}} \geq \sum_a c^+_{xa} + \sum_a c^-_{xa} \end{array}\right\}.$$

Similarly one proves that $\max_y \sum_b d^+_{yb} = \sum_b d_{y_0 b}$ and $\max_y \sum_b d^-_{yb} = \sum_b d_{y_0 b}$ using $\|R\|_{\mathsf{NS2}}$ instead of $\|R\|_{\mathsf{NS1}}$.

The third step consists on showing that actually

$$\|R^+\|_{\mathsf{NS}} = \sum_a c^+_{x_0 a} \quad \text{and} \quad \|R^-\|_{\mathsf{NS}} = \sum_a c^-_{x_0 a}.$$

To do this, note that the next equalities and inequalities clearly hold for every $1 \leq y \leq N$:

$$\left.\begin{array}{l} \sum_a c^+_{x_0 a} - \sum_a c^-_{x_0 a} = z = \sum_b d^+_{yb} - \sum_b d^-_{yb} \\ \sum_a c^+_{x_0 a} + \sum_a c^-_{x_0 a} = \|R\|_{\mathsf{NS}} = \|R\|_{\mathsf{NS1}} \geq \|R\|_{\mathsf{NS2}} = \sum_a d^+_{yb} + \sum_a d^-_{yb} \end{array}\right\}.$$

This shows that $\sum_a c^+_{x_0 a} \geq \sum_b d^+_{yb}$ and $\sum_a c^-_{x_0 a} \geq \sum_b d^-_{yb}$ for all y, which finishes the proof. $\square$

Remark 3.4.8. *Using Lemma 3.4.7 and its notation, if $R \in \mathsf{ANS}$, then it follows that*

$$\|R^+\|_{\mathsf{NS}} - \|R^-\|_{\mathsf{NS}} = \sum_{ab} |R^+(x_0, y_a, a, b)| - \sum_{ab} |R^-(x_0, y_a, a, b)| = \sum_{ab} R(x_0, y_a, a, b) = z.$$

The following lemma is an adapted version of [43, Claim 1]. The proof is analogous and for completeness it will be given in full detail. Recall that given a set $\mathcal{A} \subset \mathbb{R}^M$ with $M \in \mathbb{N}$ and $r \in \mathbb{R}^+$ we can define $r\mathcal{A} = \{ra \text{ such that } a \in \mathcal{A}\}$ for $r \in \mathbb{R}^+$.

Lemma 3.4.9. *Given $P = (P(a, b, x, y))_{a,b,x,y} \in \mathbb{R}^{N^2K^2}$ with non-negative entries, suppose that there exist $(Q_1(x, a))_{x,a}$ and $(Q_2(y, b))_{y,b}$ such that $\sum_a P(a, b, x, y) \leq Q_2(y, b)$ for all x, y, b, $\sum_b P(a, b, x, y) \leq Q_1(x, a)$ for all x, y, a and $\sum_a Q_1(x, a) = \sum_b Q_2(y, b) = \|P\|_{\mathsf{NS}}$ for all x, y, then there exists $\tilde{P} \in \|P\|_{\mathsf{NS}}\mathcal{NS}$ such that $P(a, b, x, y) \leq \tilde{P}(a, b, x, y)$ for all x, y, a, b.*

Proof. Defining $u_{xy} = \|P\|_{\mathsf{NS}} - \sum_{a,b} P(a, b, x, y)$, $t_{xy}(b) = Q_2(y, b) - \sum_a P(a, b, x, y)$ and $s_{xy}(a) = Q_1(x, a) - \sum_b P(a, b, x, y)$ we can construct $\tilde{P} \in \|P\|_{\mathsf{NS}}\mathcal{NS}$ using:

$$\tilde{P}(a, b, x, y) = \begin{cases} P(a, b, x, y) + \frac{s_{xy}(a)t_{xy}(b)}{u_{xy}} & \text{if } u_{xy} > 0, \\ P(a, b, x, y) & \text{if } u_{xy} = 0. \end{cases}$$

To show that $\sum_a \tilde{P}(x, y, a, b) = Q_2(y, b)$, consider first the case $u_{xy} \neq 0$:

$$\begin{aligned} \sum_a \tilde{P}(a, b, x, y) &= \sum_a P(a, b, x, y) + \frac{(\sum_a s_{xy}(a))t_{xy}(b)}{u_{xy}} \\ &= \sum_a P(a, b, x, y) + \frac{(\|P\|_{\mathsf{NS}} - \sum_{ab} P(x, y, a, b))t_{xy}(b)}{\|P\|_{\mathsf{NS}} - \sum_{ab} P(x, y, a, b)} \\ &= \sum_a P(a, b, x, y) + t_{xy}(b) \\ &= \sum_a P(a, b, x, y) + Q_2(y, b) - \sum_a P(a, b, x, y) = Q_2(y, b). \end{aligned}$$

On the other side, the case $u_{xy} = 0$ (which implies $\sum_{ab} P(a, b, x, y) = \|P\|_{\mathsf{NS}}$) is incompatible with having $\sum_a P(a, b, x, y) < Q_2(y, b)$, because this last inequality implies $\sum_{ab} P(a, b, x, y) < \sum_b Q_2(y, b) = \|P\|_{\mathsf{NS}}$. Hence, in this case we also have $\sum_a \tilde{P}(x, y, a, b) = Q_2(y, b)$.

It can be seen analogously that $\sum_b \tilde{P}(a, b, x, y) = Q_1(x, a)$. Moreover $\tilde{P}$ has the property $\sum_{ab} \tilde{P}(a, b, x, y) = \sum_b Q_2(y, b) = \|P\|_{\mathsf{NS}}$ for all x, y. □

The next theorem is central in this section. It is analogous to Theorem 2.4.11 and 2.4.13 and it proves the relation between ANS and $\mathcal{NS}$, completely characterizing the non-signalling probabilty distributions by means of a Banach space. Moreover, it is a necessary condition to apply Lemma 2.4.9 in our study.

Theorem 3.4.10. *Let ANS be the linear space above endowed with the norm $\|\cdot\|_{\mathsf{NS}}$ and let us denote by $\mathsf{B}_{\mathsf{ANS}}$ its unit ball. Then,*

$$\mathsf{B}_{\mathsf{ANS}} = co(\mathcal{NS} \cup -\mathcal{NS}).$$

Proof. Take $R \in B_{\mathsf{ANS}}$. We aim to obtain $\tilde{R}^+$ from R^+ and $\tilde{R}^-$ from R^- in such a way that $\tilde{R}^{\pm} \in \|R^{\pm}\|_{\mathsf{NS}}\mathcal{NS}$ and $R^+ + R^- = \tilde{R}^+ - \tilde{R}^-$.

In that case we will have:

$$R = R^+ + R^- = \tilde{R}^+ - \tilde{R}^- = \|R\|_{\mathsf{NS}} \cdot \frac{\tilde{R}^+}{\|R^+\|_{\mathsf{NS}}} - \|R^-\|_{\mathsf{NS}} \cdot \frac{\tilde{R}^-}{\|R^-\|_{\mathsf{NS}}}.$$

Since $\tilde{R}^+/\|R^+\|_{\mathsf{NS}} \in \mathcal{NS}$ and $\tilde{R}^-/\|R^-\|_{\mathsf{NS}} \in \mathcal{NS}$, and also by Lemma 3.4.7, $\|R^+\|_{\mathsf{NS}} + \|R^-\|_{\mathsf{NS}} = \|R\|_{\mathsf{NS}} \leq 1$, we will conclude that $R \in co(\mathcal{NS} \cup -\mathcal{NS})$.

Using the definitions of $c^{\pm}_{xa}$ and $d^{\pm}_{yb}$ from Lemma 3.4.7, let for all x:

$$Q^{\pm}_1(x,a) = \begin{cases} c^{\pm}_{xa} & \text{if } a = 1, \dots, K-1, \\ \|R^{\pm}\|_{\mathsf{NS}} - \sum_{a=1}^{K-1} c^{\pm}_{xa} & \text{if } a = K. \end{cases}$$

$$Q^{\pm}_2(y,b) = \begin{cases} d^{\pm}_{yb} & \text{if } b = 1, \dots, K-1, \\ \|R^{\pm}\|_{\mathsf{NS}} - \sum_{a=1}^{K-1} d^{\pm}_{yb} & \text{if } b = K. \end{cases}$$

Let us show that $Q^{\pm}_1$ and $Q^{\pm}_2$ fulfill conditions of Lemma 3.4.9. For Q^+_1 the justification is the following (and for the rest it can be proven similarly): On the one hand, it is clear that $Q^{\pm}_1(x,a) \geq 1$ for every x, a, and

$$\sum_{a=1}^{K} Q^{\pm}_1(x,a) = \|R^{\pm}\|_{\mathsf{NS}}.$$

On the other hand, for a fixed x,

$$\sum_b R^+(x,y,a,b) \leq \sup_y \sum_b R^+(x,y,a,b) = c^+_{xa} = Q^+_1(x,a) \quad \text{for all} \quad a = 1, \cdots, K-1,$$
$$\sum_b R^+(x,y,K,b) \leq c^+_{xK} \leq \|R^+\|_{\mathsf{NS}} - \sum_{a=1}^{K-1} c^+_{xa} = Q^+_1(x,K).$$

Since we can see analogously that $\sum_a R^+(x,y,a,b) \leq Q^+_2(y,b)$ for every x, y, b, we can apply Lemma 3.4.9 to R^+ using Q^+_1 and Q^+_2 to obtain $\tilde{R}^+ \in \|R^+\|_{\mathsf{NS}}\mathcal{NS}$ and such that $R^+(x,y,a,b) \leq \tilde{R}^+(x,y,a,b)$ for every x, y, a, b. Moreover, we can show analogously that Lemma 3.4.9 can be applied to $|R^-|$ using Q^-_1 and Q^-_2 to obtain $\tilde{R}^- \in \|R^-\|_{\mathsf{NS}}\mathcal{NS}$ and such that $-R^-(x,y,a,b) = |R^-(x,y,a,b)| \leq \tilde{R}^-(x,y,a,b)$ for every x, y, a, b[1].

We still have to prove that $R = \tilde{R}^+ - \tilde{R}^-$. Note that in the construction of $\tilde{R}^+$ and $\tilde{R}^-$ using Lemma 3.4.9 one defines:

[1]Note that Lemma 3.4.9 applies on non-negative tensors, so we must use it on $-R^- = |R^-|$.

$$\begin{aligned}
s^{\pm}_{xy}(a) &= c^{\pm}_{xa} - \sum_{b=1}^{K} |R^{\pm}(x,y,a,b)| \text{ for } a = 1,\ldots,K-1,\\
s^{\pm}_{xy}(K) &= \|R^{\pm}\|_{\mathsf{NS}} - \sum_{a=1}^{K-1} c^{\pm}_{xa} - \sum_{b=1}^{K} |R^{\pm}(x,y,K,b)|,\\
t^{\pm}_{xy}(b) &= d^{\pm}_{yb} - \sum_{a=1}^{K} |R^{\pm}(x,y,a,b)| \text{ for } b = 1,\ldots,K-1,\\
t^{\pm}_{xy}(K) &= \|R^{\pm}\|_{\mathsf{NS}} - \sum_{b=1}^{K-1} d^{\pm}_{yb} - \sum_{a=1}^{K} |R^{\pm(x,y,a,K)|},\\
u^{\pm}_{xy} &= \|R^{\pm}\|_{\mathsf{NS}} - \sum_{a,b=1}^{K} |R^{\pm}(x,y,a,b)|.
\end{aligned}$$

In order to obtain:

$$\tilde{R}^{+}(x,y,a,b) = \begin{cases} R^{+}(x,y,a,b) + \frac{s^{+}_{xy}(a)t^{+}_{xy}(b)}{u^{+}_{xy}} & \text{if } u^{+}_{xy} > 0,\\ R^{+}(x,y,a,b) & \text{if } u^{+}_{xy} = 0, \end{cases}$$

$$\tilde{R}^{-}(x,y,a,b) = \begin{cases} |R^{-}(x,y,a,b)| + \frac{s^{-}_{xy}(a)t^{-}_{xy}(b)}{u^{-}_{xy}} & \text{if } u^{-}_{xy} > 0,\\ |R^{-}(x,y,a,b)| & \text{if } u^{-}_{xy} = 0. \end{cases}$$

In order to show $R = \tilde{R}^{+} - \tilde{R}^{-}$ we will prove that $s^{+}_{xy}(a) = s^{-}_{xy}(a)$, $t^{+}_{xy}(b) = t^{-}_{xy}(b)$ and $u^{+}_{xy} = u^{-}_{xy}$ for all x, y, a, b, from where the result follows straightforwardly.

On the one hand, Remark 3.4.8 guarantees that

$$\begin{aligned}
&u^{+}_{xy} = u^{-}_{xy} \Leftrightarrow \|R^{+}\|_{\mathsf{NS}} - \sum_{a,b=1}^{K} |R^{+}(x,y,a,b)| = \|R^{-}\|_{\mathsf{NS}} - \sum_{a,b=1}^{K} |R^{-}(x,y,a,b)|\\
&\Leftrightarrow \|R^{+}\|_{\mathsf{NS}} - \|R^{-}\|_{\mathsf{NS}} = \sum_{a,b=1}^{K} R^{+}(x,y,a,b) + \sum_{a,b=1}^{K} |R^{-}(x,y,a,b)| = \sum_{a,b=1}^{K} R(x,y,a,b) = z.
\end{aligned}$$

On the other hand, for all $a = 1, \cdots, K-1$,

$$\begin{aligned}
&s^{+}_{xy}(a) = s^{-}_{xy}(a)\\
&\Leftrightarrow c^{+}_{xa} - \sum_{b=1}^{K} |R^{+}(x,y,a,b)| = c^{-}_{xa} - \sum_{b=1}^{K} |R^{-}(x,y,a,b)|\\
&\Leftrightarrow c^{+}_{xa} - c^{-}_{xa} = \sum_{b=1}^{K} R^{+}(x,y,a,b) + \sum_{b=1}^{K} R^{-}(x,y,a,b) = \sum_{b=1}^{K} R(x,y,a,b) = P(x,a),
\end{aligned}$$

which follows from the comments right before Lemma 3.4.7.

For the case $a = K$, we can write

$$
\begin{aligned}
& s^+_{xy}(K) = s^-_{xy}(K) \\
& \Leftrightarrow \|R^+\|_{\mathsf{NS}} - \sum_{a=1}^{K-1} c^+_{xa} - \sum_{b=1}^{K} |R^+(x,y,K,b)| = \|R^-\|_{\mathsf{NS}} - \sum_{a=1}^{K-1} c^-_{xa} - \sum_{b=1}^{K} |R^-(x,y,K,b)| \\
& \Leftrightarrow \|R^+\|_{\mathsf{NS}} - \|R^-\|_{\mathsf{NS}} = \sum_{a=1}^{K-1} \left(c^+_{xa} - c^-_{xa}\right) + \sum_{b=1}^{K} R(x,y,K,b) = \sum_{a,b=1}^{K} R(x,y,a,b) = z.
\end{aligned}
$$

Finally, using the same arguments, replacing a with b, x with y and $c^{\pm}_{xa}$ with $d^{\pm}_{yb}$, one can show that $t^+_{xy}(b) = t^-_{xy}(b)$ for all $b = 1, \cdots, K$. □

In the following theorem we prove that the space ANS is a twisted version of the space NS.

Theorem 3.4.11. *The Banach-Mazur distance between* ANS *and the space*

$$\mathsf{NS}(N, K-1) \oplus_\infty \ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \mathbb{R}$$

is upper bounded by 9.

Proof. Define the map T as:

$$
\begin{aligned}
T : \mathsf{ANS} &\to \mathsf{NS}(N, K-1) \oplus_\infty \ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \mathbb{R} \\
R = \{R(x,y,a,b)\}_{a,b=1}^{K}{}_{x,y=1}^{N} & \\
&\to \Bigg(\{R(x,y,a,b)\}_{x,y=1,a,b=1}^{N,K-1}, \left\{ \sum_{b=1}^{K} R(x,y,a,b) \right\}_{x=1,a=1}^{N,K-1}, \\
&\qquad \left\{ \sum_{a=1}^{K} R(x,y,a,b) \right\}_{y=1,b=1}^{N,K-1}, \sum_{a,b=1}^{K} R(x,y,a,b) \Bigg).
\end{aligned}
$$

Recall that $\sum_a R(x,y,a,b)$ and also $\sum_b R(x,y,a,b)$ are well defined because $R \in \mathsf{ANS}$ and they do not depend on x or y, respectively. Moreover, $\sum_{a,b} R(x,y,a,b)$ is constant for all x, y. Using these observations, one can easily check that the map T is well defined and it is a linear map. In addition, it is easy to verify that $\|T\| \leq 1$. Indeed, this can be seen by noting that the map T can be written as $T = T_1 + T_2 + T_3 + T_4$, where T_i is a linear map and $\|T_i\| \leq 1$ for every $i = 1, \cdots, 4$.

The inverse $T^{-1} : \mathsf{NS}(N, K-1) \oplus_\infty \ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \mathbb{R} \to \mathsf{ANS}$ of the map T is defined as

$$
\begin{aligned}
& T^{-1}\Big(\{R(x,y,a,b)\}_{x,y=1,a,b=1}^{N,K-1}, \{P(x,a)\}_{x=1,a=1}^{N,K-1}, \{Q(y,b)\}_{y=1,b=1}^{N,K-1}, S \Big) \\
& = \begin{cases} R(x,y,a,b) & \text{if } 1 \leq a,b \leq K-1 \\ P(x,a) - \sum_{b'=1}^{K-1} R(x,y,a,b') & \text{if } 1 \leq a \leq K-1, b = K \\ Q(y,b) - \sum_{a'=1}^{K-1} R(x,y,a',b) & \text{if } 1 \leq b \leq K-1, a = K \\ S + \sum_{a',b'=1}^{K-1} R(x,y,a',b') - \sum_{b'=1}^{K-1} Q(y,b') - \sum_{a'=1}^{K-1} P(x,a') & \text{if } a = b = K \end{cases}
\end{aligned}
$$

Basic linear algebra can be used to show that T^{-1} is well defined; that is, $T^{-1}(R,P,Q,S) = \{O(x,y,a,b)\}_{xyab}$ is in ANS, by showing that

$$\sum_{a=1}^{K} O(x,y,a,b) = Q(y,b) \quad \text{and} \quad \sum_{a=1}^{K} O(x,y,a,K) = S - \sum_{b=1}^{K-1} Q(y,b) \quad \text{for all } x, y, b,$$

and similar equalities for $\sum_{b=1}^{K} O(x,y,a,b)$ and also that $\sum_{ab} O(x,y,a,b) = S$ for all x,y.

The fact that T is linear is obvious. Finally, to see that $T^{-1} \circ T = id$, call $T^{-1}(T(R)) = Z$ and write:

$$Z_{xyab} = \begin{cases} R(x,y,a,b) & \text{if } 1 \leq a,b \leq K-1 \\ \sum_{b'=1}^{K} R(x,y,a,b') - \sum_{b'=1}^{K-1} R(x,y,a,b') = R(x,y,a,K) & \text{if } 1 \leq a \leq K-1, b=K \\ \sum_{a'=1}^{K} R(x,y,a',b) - \sum_{a'=1}^{K-1} R(x,y,a',b) = R(x,y,K,b) & \text{if } 1 \leq b \leq K-1, a=K \\ \sum_{a',b'=1}^{K} R(x,y,a',b') + \sum_{a',b'=1}^{K-1} R(x,y,a',b') - \sum_{b'=1}^{K-1} \sum_{a'=1}^{K} R(x,y,a',b') & \\ -\sum_{a'=1}^{K-1} \sum_{b'=1}^{K} R(x,y,a',b') = R(x,y,K,K) & \text{if } a=b=K \end{cases}$$

In order to calculate the norm of T^{-1}, we can consider four different applications:

$\alpha_1 : \mathsf{NS}(N,K-1) \to \mathsf{ANS}$

$\alpha_2 : \ell_\infty^N(\ell_1^{K-1}) \to \mathsf{ANS}$

$\alpha_3 : \ell_\infty^N(\ell_1^{K-1}) \to \mathsf{ANS}$

$\alpha_3 : \mathbb{R} \to \mathsf{ANS}$

defined, respectively, by

$$\alpha_1(R)(x,y,a,b) = \begin{cases} R(x,y,a,b) & \text{if } 1 \leq a,b \leq K-1 \\ -\sum_{b'=1}^{K-1} R(x,y,a,b') & \text{if } 1 \leq a \leq K-1, b=K \\ -\sum_{a'=1}^{K-1} R(x,y,a',b) & \text{if } 1 \leq b \leq K-1, a=K \\ \sum_{a',b'=1}^{K-1} R(x,y,a',b') & \text{if } a=b=K \end{cases}$$

$$\alpha_2(P)(x,y,a,b) = \begin{cases} 0 & \text{if } 1 \leq b \leq K-1 \\ P(x,a) & \text{if } 1 \leq a \leq K-1, b=K \\ -\sum_{a'=1}^{K-1} P(x,a') & \text{if } a=K, b=K \end{cases}$$

$$\alpha_3(Q)(x,y,a,b) = \begin{cases} 0 & \text{if } 1 \leq a \leq K-1 \\ Q(y,b) & \text{if } 1 \leq b \leq K-1, a=K \\ -\sum_{b'=1}^{K-1} Q(y,b') & \text{if } a=K, b=K \end{cases}$$

$$\alpha_4(S)(x,y,a,b) = \begin{cases} S & \text{if } b=K, a=K \\ 0 & \text{otherwise }. \end{cases}$$

One can check that these are well defined linear maps. Moreover, one can write:

$$T^{-1}(R,P,Q,S) = \alpha_1(R) + \alpha_2(P) + \alpha_3(Q) + \alpha_4(S).$$

Since $\|\alpha_1(R)\|_{\mathsf{NS}} = \max\{\|\alpha_1(R)\|_{\mathsf{NS1}}, \|\alpha_1(R)\|_{\mathsf{NS2}}\}$, then,

$$\begin{aligned} \|\alpha_1(R)\|_{\mathsf{NS1}} &= \max_x \sum_a \max_y \sum_b |\alpha_1(R)(x,y,a,b)| = \sum_{a,b=1}^{K} |\alpha_1(R)(x_0,y_a,a,b)| \\ &= \sum_{a,b=1}^{K-1} |R(x_0,y_a,a,b)| + \sum_{a=1}^{K-1} |\sum_{b'=1}^{K-1} R(x_0,y_a,a,b')| + \sum_{b=1}^{K-1} |\sum_{a'=1}^{K-1} R(x_0,y_k,a',b)| \\ &+ |\sum_{a',b'=1}^{K-1} R(x_0,y_k,a',b')| \leq 4\|R\|_{\mathsf{NS1}}. \end{aligned}$$

Similarly, one can show that $\|\alpha_1(R)\|_{\mathsf{NS1}} \leq 4\|R\|_{\mathsf{NS1}}$ making $\|\alpha_1\| \leq 4$.

Using the same techniques as before the estimates $\|\alpha_2\| \leq 2$, $\|\alpha_3\| \leq 2$ and $\|\alpha_4\| \leq 1$ can be proven, concluding that

$$\|T^{-1}\| \leq \|\alpha_1\| + \|\alpha_2\| + \|\alpha_3\| + \|\alpha_4\| \leq 9.$$

□

The bound given in Proposition 3.4.2 (in terms of the number of outputs) was only valid for non-negative elements. We prove now a bound for the general case. First, we need a lemma that allows to bound the difference in norm when changing from $\ell_1^N(\ell_\infty^K(\ell_1^N) = \ell_1^N \otimes_\pi (\ell_\infty^K \otimes_\epsilon \ell_1^N)$ to $\ell_1^N(\ell_\infty^K) \otimes_\epsilon \ell_1^N = (\ell_1^N \otimes_\pi \ell_\infty^K) \otimes_\epsilon \ell_1^N$. Since the result could be of independent interest, we state it and prove it in a more general context. The proof remains essentially the same.

Lemma 3.4.12. *Let X be a Banach space and $M \in \ell_1^N \otimes \mathsf{X} \otimes \ell_1^L$. Then,*

$$\|M\|_{\ell_1^N(\mathsf{X}\otimes_\epsilon \ell_1^L)} \leq \sqrt{2N}\|M\|_{\ell_1^N(\mathsf{X})\otimes_\epsilon \ell_1^L}.$$

To prove this lemma we will need to make use of the Khintchine inequality ([31, pag. 96]). We state this theorem below and we notice that in the case of $p = 1$ it is known that $a_1 = \sqrt{2}$.

Theorem 3.4.13 (Khintchine inequality)**.** *For $1 \leq p < \infty$ there exist constants $a_p, b_p \geq 1$ such that*

$$a_p^{-1}\left(\sum_{i=1}^N |\alpha_i|^2\right)^{\frac{1}{2}} \leq \left(\int_0^1 \Big|\sum_{i=1}^N r_i(t)\alpha_i\Big|^p dt\right)^{\frac{1}{p}} \leq b_p\left(\sum_{i=1}^N |\alpha_i|^2\right)^{\frac{1}{2}}$$

for every N and all $\alpha_1, \cdots, \alpha_N \in \mathbb{R}$, where here $(r_i)_{i=1}^N$ denote the Rademacher functions.

Proof of Lemma 3.4.12. Let $M = \sum_{i=1}^N \sum_{j=1}^L e_i \otimes M_{ij} \otimes e_j$, with $M_{ij} \in \mathsf{X}$. Then, we have

$$\begin{aligned}
\|M\|_{\ell_1^N(\mathsf{X}\otimes_\epsilon \ell_1^L)} &= \sum_{i=1}^N \Big\|\sum_{j=1}^L M_{ij}\otimes e_j\Big\|_{\mathsf{X}\otimes_\epsilon \ell_1^L} \\
&\overset{(1)}{=} \sum_{i=1}^N \sup_{x_i^*\in \mathsf{B}_{\mathsf{X}^*}} \sum_{j=1}^L |x_i^*(M_{ij})| \\
&= \sup_{(x_1^*,\cdots,x_N^*)\in \mathsf{B}_{\ell_\infty^N(\mathsf{X}^*)}} \sum_{i=1}^N\sum_{j=1}^L |x_i^*(M_{ij})| \\
&\overset{(2)}{\leq} \sqrt{N} \sup_{(x_1^*,\cdots,x_N^*)\in \mathsf{B}_{\ell_\infty^N(\mathsf{X}^*)}} \sum_{j=1}^L \Big(\sum_{i=1}^N |x_i^*(M_{ij})|^2\Big)^{\frac{1}{2}} \\
&\overset{(3)}{\leq} \sqrt{2N} \sup_{(x_1^*,\cdots,x_N^*)\in \mathsf{B}_{\ell_\infty^N(\mathsf{X}^*)}} \sum_{j=1}^L \int_0^1 \Big|\sum_{i=1}^N r_i(t)x_i^*(M_{ij})\Big|dt \\
&= \sqrt{2N} \sup_{(x_1^*,\cdots,x_N^*)\in \mathsf{B}_{\ell_\infty^N(\mathsf{X}^*)}} \int_0^1 \sum_{j=1}^L \Big|\sum_{i=1}^N r_i(t)x_i^*(M_{ij})\Big|dt \\
&\leq \sqrt{2N} \sup_{(x_1^*,\cdots,x_N^*)\in \mathsf{B}_{\ell_\infty^N(\mathsf{X}^*)}} \sup_{t\in[0,1]} \sum_{j=1}^L \Big|\sum_{i=1}^N r_i(t)x_i^*(M_{ij})\Big|
\end{aligned}$$

$$\leq \sqrt{2N} \sup_{(x_1^*,\cdots,x_N^*)\in \mathsf{B}_{\ell_\infty^N(\mathsf{X}^*)}} \sup_{(t_1,\cdots,t_N)\in\{-1,1\}^N} \sum_{j=1}^{L} \Big| \sum_{i=1}^{N} t_i x_i^*(M_{ij}) \Big|$$
$$\stackrel{(4)}{=} \sqrt{2N} \sup_{(x_1^*,\cdots,x_N^*)\in \mathsf{B}_{\ell_\infty^N(\mathsf{X}^*)}} \sum_{j=1}^{L} \Big| \sum_{i=1}^{N} x_i^*(M_{ij}) \Big|$$
$$\stackrel{(5)}{=} \sqrt{2N} \|M\|_{\ell_1^N(\mathsf{X})\otimes_\epsilon \ell_1^L}.$$

Here, $\stackrel{(1)}{=}$ follows from the definition of the ϵ norm, $\stackrel{(2)}{\leq}$ follows from the fact that $\|id:\ell_2^N \to \ell_1^N\| \leq \sqrt{N}$, $\stackrel{(3)}{\leq}$ follows from Khintchine inequality, $\stackrel{(4)}{=}$ is clear since $\|t_i x_i^*\|_{\mathsf{X}} = \|x_i^*\|_{\mathsf{X}}$ for $t_i = \pm 1$ and $\stackrel{(5)}{=}$ follows again from the definition of the ϵ norm. □

Using this result, we can bound the difference between the norms in $\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)$ and $\ell_1^N(\ell_\infty^K(\ell_1^N(\ell_\infty^K)))$.

Proposition 3.4.14. *There exists a universal constant C independent of N, K such that, given $M \in \mathbb{R}^{N^2K^2}$, one has*

$$\|M\|_{\ell_1^N(\ell_\infty^K(\ell_1^N(\ell_\infty^K)))} \leq C\sqrt{NK}\|M\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}.$$

Proof. The Banach-Mazur distance between ℓ_∞^K and ℓ_1^K is $d(\ell_\infty^K, \ell_1^K) \leq C\sqrt{K}$, with C certain constant independent of the dimension [73, Proposition 37.6]. This means that there exists an isomorphism $T : \ell_\infty^K \to \ell_1^K$ such that $\|T\|\|T^{-1}\| \leq C\sqrt{K}$. We will use the metric mapping property of the π [31, pag. 27] and the ϵ [31, pag. 46] norm, which says that for all linear maps $T : \mathsf{X} \to \mathsf{W}$, $S : \mathsf{Y} \to \mathsf{Z}$, we have

$$\|T \otimes S : \mathsf{X}\otimes_\alpha \mathsf{Y} \to \mathsf{W}\otimes_\alpha \mathsf{Z}\| = \|T\|\|S\| \quad \text{for} \quad \alpha = \pi, \epsilon.$$

In particular, if we consider a normed space X and the mapping $id \otimes T : \mathsf{X}\otimes_\pi \ell_\infty^K \longrightarrow \mathsf{X}\otimes_\pi \ell_1^K$, then, for every $M \in \mathsf{X}\otimes \ell_\infty^K$ one has $\|(id\otimes T)(M)\|_{\mathsf{X}\otimes_\pi \ell_1^K} \leq \|T\|\|M\|_{\mathsf{X}\otimes_\pi \ell_\infty^K}$. Similar statements hold if we replace T by T^{-1}.

Let $M \in \mathbb{R}^{N^2K^2}$. The reasonings above, together with Lemma 3.4.12 replacing the space X in the lemma by ℓ_∞^K yield the following:

$$\begin{aligned}\|M\|_{\ell_1^N(\ell_\infty^K(\ell_1^N(\ell_\infty^K)))} &\leq \|T^{-1}\|\|M\|_{\ell_1^N(\ell_\infty^K(\ell_1^{NK})} \leq \sqrt{2N}\|T^{-1}\|\|M\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^{NK}} \\ &\leq \sqrt{2N}\|T\|\|T^{-1}\|\|M\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)} \\ &\leq C'\sqrt{NK}\|\|M\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}.\end{aligned}$$

□

Remark 3.4.15. *A dual statement of Proposition 3.4.14 is that*

$$\|id : \ell_\infty^N(\ell_1^K(\ell_\infty^N(\ell_1^K))) \to \ell_\infty^N(\ell_1^K)\otimes_\pi \ell_\infty^N(\ell_1^K)\| \leq C\sqrt{NK}.$$

Moreover, a dual statement of Proposition 3.4.1 is

$$\|id : \ell_\infty^N(\ell_1^K(\ell_\infty^N(\ell_1^K))) \to \ell_\infty^N(\ell_1^K)\otimes_\pi \ell_\infty^N(\ell_1^K)\| \leq N.$$

In particular, this trivially implies that

$$\|id : \mathsf{BNS}(N,(K-1)) \to \ell_\infty^N(\ell_1^{K-1})\otimes_\pi \ell_\infty^N(\ell_1^{K-1})\| \leq C\min\{N, \sqrt{NK}\},$$

where the space $\mathsf{BNS}(NK)$ *was defined right after Definition 3.3.1.*

At this point we have found two Banach spaces for $\mathcal{NS}$ and for $\mathcal{L}$ that fulfill the conditions of Lemma 2.4.9. Therefore, we just have to upper bound the following norm:

$$\|id : \mathsf{ANS} \to \mathsf{NSG} \otimes_\pi \mathsf{NSG}\|.$$

And that is precisely what we do in the following theorem.

Theorem 3.4.16. *Let N and K be two natural numbers. Then,*

$$LV(\mathcal{NS}, \mathcal{L}) \leq C min\{N, \sqrt{NK}\},$$

where C is a universal constant.

Proof. As we have said in the introduction, $LV(\mathcal{NS}, \mathcal{L})$ is the smallest constant such that

$$\widetilde{\mathcal{NS}} \subseteq LV(\mathcal{NS}, \mathcal{L}) \cdot \widetilde{\mathcal{L}},$$

where $\widetilde{\mathcal{A}} = co(\mathcal{A} \cup -\mathcal{A})$.

Since the equalities $\mathsf{B}_{\mathsf{ANS}} = \widetilde{\mathcal{NS}}$ and $\mathsf{B}_{\mathsf{NSG}\otimes_\pi\mathsf{NSG}} = \widetilde{\mathcal{L}}$ are known from Theorem 3.4.10 and Theorem 2.4.13, the statement of the theorem is equivalent to prove that

$$\|id : \mathsf{ANS} \to \mathsf{NSG} \otimes_\pi \mathsf{NSG}\| \leq C\min\{N, \sqrt{NK}\},$$

where C is a universal constant.

Let us define the Banach spaces:

$$\begin{aligned} \mathsf{X} &= \mathsf{NS}(N, K-1) \oplus_\infty \ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \mathbb{R}, \\ \mathsf{Y} &= \big(\ell_\infty^N(\ell_1^{K-1}) \otimes_\pi \ell_\infty^N(\ell_1^{K-1})\big) \oplus_\infty \ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \ell_\infty^N(\ell_1^{K-1}) \oplus_\infty \mathbb{R}. \end{aligned}$$

We will decompose the identity map between ANS and $\mathsf{NSG} \otimes_\pi \mathsf{NSG}$ as

$$T^{-1} \circ id \circ T : \mathsf{ANS} \to \mathsf{X} \to \mathsf{Y} \to \mathsf{NSG} \otimes_\pi \mathsf{NSG},$$

where T is the map used in the proof of Theorem 3.4.11. Now, in that theorem we showed that

$$\|T : \mathsf{ANS} \to \mathsf{X}\| \leq 1.$$

Moreover, a direct consequence of Remark 3.4.15 is that

$$\|id : \mathsf{X} \to \mathsf{Y}\| \leq C \min\{N, \sqrt{NK}\}.$$

Hence, we have that

$$\|id : \mathsf{ANS} \to \mathsf{NSG} \otimes_\pi \mathsf{NSG}\| \leq C\min\{N, \sqrt{NK}\} \|T^{-1} : Y \to \mathsf{NSG} \otimes_\pi \mathsf{NSG}\|$$

and the theorem will follow from the estimate

$$\|T^{-1} : \mathsf{Y} \to \mathsf{NSG} \otimes_\pi \mathsf{NSG}\| \leq 9.$$

To see this last bound, we proceed as in the proof of Theorem 3.4.11 by decomposing the map $T^{-1} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and upper bounding each of the norms. Let us first consider

$$\alpha_1 : \ell_\infty^N(\ell_1^{K-1}) \otimes_\pi \ell_\infty^N(\ell_1^{K-1}) \to \mathsf{NSG} \otimes_\pi \mathsf{NSG}.$$

Now, in order to upper bound the norm of this map, it suffices to consider elements of the form $R = (P_1(a,x)P_2(b,y))_{x,y;a,b=1}^{N,K-1}$ such that $\|P_1\|_{\ell_\infty^N(\ell_1^{K-1})} \leq 1$ and $\|P_2\|_{\ell_\infty^N(\ell_1^{K-1})} \leq 1$. It is easy to see that

$$\alpha_1(R) = \big(Q_1(a,x)Q_2(b,y)\big)_{x,y,a,b=1}^{N,K},$$

where for every x, y,

$$Q_1(a|x) = \begin{cases} P_1(a,x) & \text{if } 1 \leq a \leq K-1 \\ -\sum_{a'=1}^{K-1} P_1(a,x) & \text{if } a = K. \end{cases},$$

$$Q_2(b|y) = \begin{cases} P_2(b,y) & \text{if } 1 \leq b \leq K-1 \\ -\sum_{b'=1}^{K-1} P_2(b,y) & \text{if } b = K. \end{cases}.$$

Hence, for these particular elements, it is clear that

$$\|\alpha_1(R)\|_{\mathsf{NSG}\otimes_\pi \mathsf{NSG}} = \|Q_1\|_{\mathsf{NSG}}\|Q_2\|_{\mathsf{NSG}} = \|Q_1\|_{\ell_\infty^N(\ell_1^K)}\|Q_2\|_{\ell_\infty^N(\ell_1^K)},$$

where in the last equality we have used Definition 2.4.10. Now, it is very easy to check that $\|Q_1\|_{\ell_\infty^N(\ell_1^K)} \leq 2$ and $\|Q_2\|_{\ell_\infty^N(\ell_1^K)} \leq 2$, from where we conclude that $\|\alpha_1\| \leq 4$.

Let us consider now

$$\alpha_2 : \ell_\infty^N(\ell_1^{K-1}) \to \mathsf{NSG} \otimes_\pi \mathsf{NSG}.$$

Given $P \in \ell_\infty^N(\ell_1^{K-1})$ with $\|P\|_{\ell_\infty^N(\ell_1^{K-1})} \leq 1$, it can be easily checked that

$$\alpha_2(P) = \big(Q_1(a,x)Q_2(b,y)\big)_{x,y,a,b=1}^{N,K},$$

where for every x, y,

$$Q_1(a|x) = \begin{cases} P(a,x) & \text{if } 1 \leq a \leq K-1 \\ -\sum_{a'=1}^{K-1} P_1(a',x) & \text{if } a = K \end{cases},$$

$$Q_2(b|y) = \begin{cases} 0 & \text{if } 1 \leq b \leq K-1 \\ 1 & \text{if } b = K \end{cases}.$$

As in the case if α_1, we can deduce that $\|\alpha_2(P)\|_{\mathsf{NSG}\otimes_\pi \mathsf{NSG}} = \|Q_1\|_{\ell_\infty^N(\ell_1^K)}\|Q_2\|_{\ell_\infty^N(\ell_1^K)} \leq 2$, so that $\|\alpha_2\| \leq 2$. Moreover, the case of α_3 can be analyzed exactly in the same way to deduce $\|\alpha_3\| \leq 2$.

Finally, for the case of $\alpha_4 : \mathbb{R} \to \mathsf{NSG} \otimes_\pi \mathsf{NSG}$, one can check that for a given $|s| \leq 1$, we have $\alpha_4(s) = \big(Q_1(a,x)Q_2(b,y)\big)_{x,y,a,b=1}^{N,K}$, where, for every x, y,

$$Q_1(a|x) = \begin{cases} 0 & \text{if } 1 \leq a \leq K-1 \\ s & \text{if } a = K. \end{cases},$$

$$Q_2(b|y) = \begin{cases} 0 & \text{if } 1 \leq b \leq K-1 \\ 1 & \text{if } b = K. \end{cases},$$

and one trivially deduces that $\|\alpha_4\| \leq 1$.

Since $\|T^{-1}\| \leq \|\alpha_1\| + \|\alpha_2\| + \|\alpha_3\| + \|\alpha_4\| \leq 9$, we conlude the proof. □

Remark 3.4.17. *For this case it can also be seen that using the same techniques, when we distinguish the inputs and outputs for Alice and Bob as N_1, N_2, K_1 and K_2, the following bounds can be obtained:*

$$LV(\mathcal{NS}, \mathcal{L}) \leq \mathcal{O}(\min\{N_1, N_2, \sqrt{CN_1K_2}, \sqrt{CN_2K_1}\})$$

3.5 Lower bounds

As the space NS is related to $\mathsf{NS1}$ and $\mathsf{NS2}$, we will first analyze whether the upper bounds in Proposition 3.4.1 and Proposition 3.4.2 are sharp. We answer this question in the positive using the element

$$M_0(x,y,a,b) = \begin{cases} 1 & \text{if } x = b \quad \text{and} \quad y = a = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.5.1}$$

where $x, y, a, b = 1, \cdots, N$. Indeed, in this case we have that $N = K$, $\|M_0\|_{\ell_1^N(\ell_\infty^N(\ell_1^N(\ell_\infty^N)))} = N$ and $\|M_0\|_{\ell_1^N(\ell_\infty^N)\otimes_\epsilon \ell_1^N(\ell_\infty^N)} = 1$.

One could wonder whether this element M_0 can be used to give an optimal ratio

$$LV(\mathcal{NS},\mathcal{L}) = \sup_M \frac{\|M\|_{\mathsf{NS}^*}}{\|M\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}}.$$

However, it is easy to see that $\|M_0\|_{\mathsf{NS2}^*} = \|\mathsf{flip}(M_0)\|_{\ell_1^N(\ell_\infty^N(\ell_1^N(\ell_\infty^N)))} = 1$. Hence, $\|M_0\|_{\mathsf{NS}^*} \leq 1$ and the ratio in this case would not be greater than one.

In order to prove that the upper bounds of $LV(\mathcal{NS},\mathcal{L})$ given in Theorem 3.4.16 are optimal up to a logarithmic factor we will need a more complex element. This will be seen in the next theorem.

Theorem 3.5.1. *For every natural number n there exists a pointwise non-negative tensor $G_n \in \mathbb{R}^{N^2K^2}$, with $N = K = n$, such that*

$$\frac{\omega_{\mathcal{NS}}(G)}{\omega_{\mathcal{L}}(G)} \geq D\frac{n}{\log n},$$

where D is a universal constant.

Consider a family of elements $\{\sigma_{xy}\}_{x,y=1}^N$ where, for all inputs x and y, σ_{xy} is a permutation of the outputs. Let S_K be the symmetric group over $[K]$. Thus, $\sigma_{xy} \in S_K$ for all x and y.

For every such family we define the linear functional with non negative entries:

$$G = \sum_{x,y=1}^N \sum_{j=1}^K e_x \otimes e_j \otimes e_y \otimes e_{\sigma_{xy}(j)}.$$

For the interested reader, we remark that, properly normalized, M can be seen as a unique game [37], with the uniform distribution on the inputs (x, y) and the verifier function defined as 1 if and only if $b = \sigma_{xy}(a)$, and 0 otherwise. We will not explicitly use this fact, though.

NS value of M:
We prove next that $\|G\|_{\mathsf{NS}^*} = N^2$. We consider the following strategy:

$$P = \frac{1}{K}\sum_{x,y=1}^N \sum_{j=1}^K e_x \otimes e_j \otimes e_y \otimes e_{\sigma_{xy}(j)}.$$

It can be easily seen that it is a non-signalling probability distribution. Then if we consider the value of G acting on P we obtain:

$$\langle G, P\rangle = \Big\langle \sum_{x,y}\sum_{j} e_x \otimes e_j \otimes e_y \otimes e_{\sigma_{xy}(j)}, \frac{1}{K}\sum_{x',y'}\sum_{j'} e_{x'} \otimes e_{j'} \otimes e_{y'} \otimes e_{\sigma_{x'y'}(j')} \Big\rangle$$
$$= \sum_{x,x',y,y',j,j'} \frac{1}{K}\langle e_x, e_{x'}\rangle\langle e_j, e_{j'}\rangle\langle e_y, e_{y'}\rangle\langle e_{\sigma_{xy}(j)}, e_{\sigma_{x'y'}(j')}\rangle = \sum_{x,y,j}\frac{1}{K} = N^2.$$

Therefore we have that $\|G\|_{\mathsf{NS}^*} \geq N^2$. At the same time, it is easy to see that $\langle G, P\rangle \leq N^2$ for every $P \in \mathcal{P}$ (even a signalling one). Hence $\|G\|_{\mathsf{NS}^*} = N^2$.

Classical value of G:

We study now the classical value of G. As $\mathcal{L}$ is a convex polytope [74] and G is a convex (in fact, linear) function acting on $\mathcal{L}$, applying convexity arguments it is clear that we only need to consider classical extremal strategies. A classical extremal strategy P is uniquely determined by two functions $a, b : \{1, \dots, N\} \longrightarrow \{1, \dots, K\}$ in such a way that:

$$P = \sum_{x,y} e_x \otimes e_{a(x)} \otimes e_y \otimes e_{b(y)}.$$

Then, G acting on P verifies:

$$\langle G, P\rangle = \Big\langle \sum_{x,y}\sum_{j} e_x \otimes e_j \otimes e_y \otimes e_{\sigma_{xy}(j)}, \sum_{x',y'} e_{x'} \otimes e_{a(x')} \otimes e_{y'} \otimes e_{b(y')} \Big\rangle$$
$$= \sum_{x,x',y,y',j} \langle e_x, e_{x'}\rangle\langle e_j, e_{a(x')}\rangle\langle e_y, e_{y'}\rangle\langle e_{\sigma_{xy}(j)}, e_{b(y')}\rangle = \sum_{x,y,j}\langle e_j, e_{a(x)}\rangle\langle e_{\sigma_{xy}(j)}, e_{b(y)}\rangle$$
$$= \sum_{x,y}\langle e_{\sigma_{xy}(a(x))}, e_{b(y)}\rangle.$$

We apply now probabilistic reasonings. For every $1 \leq x, y \leq N$, we consider the permutation σ_{xy} to be a random variable uniformly distributed in S_K. For $(x, y) \neq (x', y')$ we consider σ_{xy} and $\sigma_{x'y'}$ to be independent random variables. That is, G is a random variable in the probability space $\Omega := (S_K)^{\otimes N^2}$, considered with the uniform probability.

We fix a classical extremal strategy P characterized by functions $a, b : \{1, \dots, N\} \longrightarrow \{1, \dots, K\}$ as above. For one such P and for every pair of inputs x and y, we can define a random variable $Z^P_{xy} : S_K \to \{0, 1\}$ by

$$Z^P_{xy} = \langle e_{\sigma_{xy}(a(x))}, e_{b(y)}\rangle.$$

Recall that the superindex P makes reference to the extremal probability distribution, which uniquely determines the functions a and b. This random variable takes the following values:

$$Z^P_{xy} = \begin{cases} 1 & \text{if } \sigma_{xy}(a(x)) = b(y), \\ 0 & \text{if } \sigma_{xy}(a(x)) \neq b(y). \end{cases}$$

Clearly, the probability of $\sigma_{xy}(a(x)) = b(y)$ is $1/K$. Therefore Z^P_{xy} is a Bernoulli variable of parameter $1/K$.

We recall the following Chernoff-type bound [40]:

Theorem 3.5.2. *Let $X_1, X_2, \ldots, X_n$ be independent* 0-1 *random variables with $\mathbb{P}[X_i = 1] = p_i$. Denote $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$. Then for all $\delta > 1$:*

$$\mathbb{P}[X \geq (1+\delta)\mu] \leq \exp\Big(\frac{-\delta^2\mu}{2+\delta}\Big).$$

We define a new random variable

$$Z^P = \sum_{x,y=1}^N Z_{xy}^P = \sum_{x,y=1}^N \langle e_{\sigma_{xy}(a(x))}, e_{b(y)}\rangle.$$

Clearly, if $(x,y) \neq (x',y')$, then Z_{xy}^P and $Z_{x'y'}^P$ are independent. It is easy to see that $\mathbb{E}\left[Z^P\right] = N^2/K$; we can choose then $N = K$ and apply Theorem 3.5.2 to obtain:

$$\mathbb{P}\left[Z^P \geq (1+\delta)N\right] \leq \exp\Big(\frac{-\delta^2 N}{2+\delta}\Big).$$

There are N^N different possibilities for the a function and also for the b function. That means, there are in total N^{2N} different classical extremal strategies, which we label as P_i for $i = 1, \ldots, N^{2N}$.

Now we apply the union bound and we obtain

$$\begin{aligned}\mathbb{P}\left[\bigcup_{i=1}^{N^{2N}} \big(Z^{P_i} \geq (1+\delta)N\big)\right] &\leq \sum_{i=1}^{N^{2N}} \mathbb{P}\left[Z^{P_i} \geq (1+\delta)N\right] \leq N^{2N} \exp\Big(\frac{-\delta^2 N}{2+\delta}\Big) \\ &= \exp\Big(\log N^{2N}\Big)\exp\Big(\frac{-\delta^2 N}{2+\delta}\Big) = \exp\Big(2N\log N - \frac{\delta^2 N}{2+\delta}\Big).\end{aligned}$$

Choosing $\delta = 3\log N - 2$, we have

$$\begin{aligned}\exp\Big(2N\log N - \frac{\delta^2 N}{2+\delta}\Big) &= \exp\Big(2N\log N - \frac{(3\log N)^2 - 12\log N + 4}{3\log N}N\Big) \\ &= \exp\Big(-N(\log N - 4) - \frac{4N}{3\log N}\Big) < 1,\end{aligned}$$

for $N \geq 5$. Therefore,

$$\mathbb{P}\left[\left(\bigcup_{i=1}^{N^{2N}} \big(Z^{P_i} \geq (3\log N - 1)N\big)\right)^c\right] = \mathbb{P}\left[\bigcap_{i=1}^{N^{2N}} \big(Z^{P_i} < (3\log N - 1)N\big)\right] > 0$$

for $N \geq 5$.

Hence, we know the existence of a family of N^2 permutations, $(\sigma_{xy})_{x,y=1}^N$ defining a linear functional G, such that $\|G\|_{\mathsf{NS}^*} = N^2$ and $\|G\|_{\ell_1^N(\ell_\infty^N)\otimes_\epsilon \ell_1^N(\ell_\infty^N)} \leq (3\log N - 1)N$. This concludes the analysis of the classical bound.

Remark 3.5.3. *The same result can be obtained with a more restrictive type of games, the XOR-d games considered in [11]. Alice and Bob receive questions (x,y) from $X \times Y$ and reply with answers $a, b \in (\mathbb{Z}_N, +)$ where $\mathbb{Z}_N$ is the cyclic group of N elements with inner operation $+$. The winning constraint is now $a + b = \sigma_{xy}$ for some function $\sigma : X \times Y \to \mathbb{Z}_N$, $\sigma(x,y) = \sigma_{xy}$. Choosing σ_{xy} to be uniformly random and independent in (x,y), we obtain the same bounds.*

3.6 Conclusions

In this chapter we have studied the non-signalling probability distributions, which can be understood as the ultimate limitation of any reasonable physical theory. We have extended the study of relative Bell violations to non-signalling over local resources by means of tensor norms.

First, we have analyzed the correlation scenario. In contrast to the quantum case, in which the ratio of bipartite Bell violations is upper bounded by the Grothendieck's constant, not-signalling Bell violations can be unbounded. In fact, it was already known that $LV(\mathcal{NS}_{cor}, \mathcal{L}_{cor}) = \mathcal{O}(\sqrt{N})$. Moreover, the characterization in terms of tensor norms turns out to be satisfactory since the unit ball $\ell_\infty^N \otimes_\epsilon \ell_\infty^N$ coincides with the set of non-signalling correlations.

Next, we have considered the case of general probability distributions. The non-signalling value of a game can also be described by a certain natural norm in $\mathbb{R}^{N^2K^2}$, giving raise to the Banach space NS. However, a difficulty that we have encountered here is that this norm, and contrary to the classical case, is not compatible with the tensor product structure $\ell_1^N(\ell_\infty^K) \otimes \ell_1^N(\ell_\infty^K)$. This situation is shown in Appendix C.

Nevertheless, although the space NS can be tricky, we manage to give some upper bounds in Theorem 3.4.4. Interestingly, we see that these upper bounds coincide with the upper bounds already known for quantum Bell violations (see Equation (1.1.5)).

The procedure of embedding $\mathcal{NS}$ into $\mathbb{R}^{N^2K^2}$ and consider the non-signalling norm in this space suits perfectly to relate norms and values (local, non-signalling) of Bell inequalities with non-negative coefficients. But if we consider general Bell functionals, with coefficients not necessarily non-negative, then the relation between $\omega_{\mathcal{NS}}(M)$ and $\|M\|_{\mathsf{NS}^*}$ is not so clear anymore (it can be easily checked that $\omega_{\mathcal{NS}}(\cdot)$ does not define a norm in $\mathbb{R}^{N^2K^2}$).

This is the reason why we define the ANS space, which is a twisted version of NS with the particularity that $\mathsf{B}_{\mathsf{ANS}} = co(\mathcal{NS} \cup -\mathcal{NS})$. As a consequence of this, we can use techniques from Banach space theory to obtain the upper bound of Theorem 3.4.16.

Finally, we have shown the existence of a game for which the ratio of violation is near optimal with respect to the number of inputs and outputs. The use of a non-constructive procedure to obtain the Bell functional guarantees its existence with high probability, but it comes at the expense of not identifying an explicit Bell functional for this task. Nevertheless, this closes the gap between the upper bound and the lower bound, leaving no open questions at this matter.

The initial motivation to define non-signalling probability distributions was to reconcile quantum mechanics with the theory of relativity. Forbidding instant communication establishes a limit on how non-local can be any theory before loosing physical meaning. But, since for correlations we have that $LV(\mathcal{NS}_{cor}, \mathcal{Q}_{cor}) = \mathcal{O}(\sqrt{N})$, they seemed to be very different. With our results we recover the idea that quantum Bell violations can be as large as any other post-quantum theory. They show that $LV(\mathcal{NS}, \mathcal{L})$ is "morally" comparable to $LV(\mathcal{Q}, \mathcal{L})$. We finally guess that quantum mechanical resources could give the same advantage that any other post-quantum theory.

As possible future lines of work, the most natural extension is to try to apply these techniques to the multipartite setting. It would be also interesting to use our results in other settings, such as, for instance, parallel repetition or cryptography.

Chapter 4

Classical and quantum communication

4.1 Introduction

Quantum communication can be much more efficient than classical communication to perform certain tasks. In this chapter we study the probability distributions that arise when classical and quantum communication are allowed in order to study games assisted with communication. Both questions and answers are classical and the two parties, Alice and Bob, will have unlimited power of calculation and shared randomness.

This line was already introduced in [50], where the authors characterized the correlations generated with one-way classical and quantum communication. They found the existence of an XOR game in which Alice and Bob need to communicate $k = \mathcal{O}(\sqrt{n})$ classical bits to obtain the same value as the one obtained with $\log n$ qubits. And they left open the question whether the same bound could also apply in the case of allowing the communication to be in both directions. We will answer this question in the positive. That is, the same exponential separation between quantum and classical resources for the XOR game given in [50], can be attained between quantum one-way communication and classical two-way communication protocols. To this purpose, we will study thoroughly the scenario where two-way classical communication is permitted. Any general protocol would consist on an arbitrary number of rounds where each message depends on the input and the messages previously received.

This chapter is organized as follows: First we will introduce the correlations that can be generated with one-way classical and quantum communication. Then, we will present the form of a general two-way classical communication protocol explicitly. We will use this expression to calculate the value of an XOR game played with them in terms of tensor norms. And finally, we will use this expression to upper bound the classical value of the same game given in [50].

4.2 One-way classical and quantum communication

4.2.1 Classical

In this section we are going to describe the correlations that Alice and Bob are able to generate when they share an unlimited amount of randomness and, additionally, Alice is allowed to send c classical bits to Bob. We call this set as $\mathcal{L}_{cor}^{c\rightarrow}$ and we notice that the reverse case, in which

Bob sends c bits to Alice would be symmetric and denoted by $\mathcal{L}_{cor}^{c\leftarrow}$. This scenario was already studied in [50] and it is included here for completeness.

The randomness can be defined via a probability space (Λ, λ) in such a way that Alice's answer is a function $a : [N] \times \Lambda \to \{-1, +1\}$, the message of c-bits is a function $m : [N] \times \Lambda \to [2^c]$ and Bob's answer is a function $b : [N] \times \Lambda \times [2^c] \to \{-1, +1\}$. Therefore, the joint correlation can be described by

$$\gamma_{xy} = \int_\Lambda a(x, \lambda) b(y, \lambda, m(x, \lambda)) d\lambda.$$

For a fixed λ the correlation can be written as $\gamma_{xy}^\lambda = a(x)b(y, m(x))$. This allows us to deduce the extremal points and to give the following definition for $\mathcal{L}_{cor}^{c\rightarrow}$ as:

$$\mathcal{L}_{cor}^{c\rightarrow} = co\Big\{ \big(a(x)b(y, m(x))\big)_{x,y=1}^N \text{ such that } m : [N] \to [2^c] \text{ is a function,} \\ a(x) = \pm 1, b(y, t) = \pm 1 \text{ for all } x, y \text{ in } [N] \text{ and } t \text{ in } [2^c] \Big\}.$$

Given an XOR game T, then we can define the value played with a one-way classical communication by

$$\omega_{\mathcal{L}_{cor}^{c\rightarrow}}(T) = \sup_{\gamma \in \mathcal{L}_{cor}^{c\rightarrow}} |\langle T, \gamma \rangle|.$$

To characterize this value with a norm we are going to use the element $T \otimes id$. T is the XOR game, and then $T = \sum_{x,y} T_{xy} e_x \otimes e_y$ and id is the identity, which can be written algebraically as $id = \sum_{i=1}^{2^c} e_i \otimes e_i$. Then:

$$T \otimes id = \sum_{x,y} \sum_{k,j=1}^{2^c} T_{xy} \delta_{k,j} (e_x \otimes e_k) \otimes (e_y \otimes e_j) = \sum_{x,y,k} T_{xy} (e_x \otimes e_k) \otimes (e_y \otimes e_k).$$

This element can be seen as an tensor in

$$\ell_1^N(\ell_\infty^{2^c}) \otimes_\epsilon \ell_1^N(\ell_1^{2^c}).$$

Proposition 4.2.1. *[50, Lemma 6.1] Let $T = (T_{xy})_{x,y}$ be an XOR game. Then,*

$$\omega_{\mathcal{L}_{cor}^{c\rightarrow}}(T) = \|T \otimes id\|_{\ell_1^N(\ell_\infty^{2^c}) \otimes_\epsilon \ell_1^N(\ell_1^{2^c})}.$$

We can trivially embed $\mathcal{L}_{cor}$ into $\mathcal{L}_{cor}^{c\rightarrow}$ for any given $c \in \mathbb{N}$ and therefore we will always have $\omega_{\mathcal{L}_{cor}}(T) \leq \omega_{\mathcal{L}_{cor}^{c\rightarrow}}(T)$ for any XOR game T.

4.2.2 Quantum

In this section we are going to explain the correlations that Alice and Bob can generate when Alice sends a message to Bob consisting in c qubits. The message will consist on a state living in a 2^c-dimensional Hilbert space. Alice will produce an answer and a state ρ_x according to the input x received and Bob will produce an answer according to the input y and the state ρ_x received. Both of them are allowed to use shared randomness that can be modeled via a probability space (Λ, λ). Finally, the correlation that they generate can be written as:

$$\gamma_{x,y} = \int_\Lambda a(x, \lambda) \operatorname{tr}(B(y, \lambda)) \rho(x, \lambda) d\lambda,$$

where $B(y,\lambda)$ is a self adjoint matrix related to a dichotomic POVM for all y and λ. Moreover, $a(x,\lambda) = \pm 1$ for all x and λ. We denote the set of correlation that can be written in this form as $\mathcal{Q}^{c\rightarrow}_{cor}$.

Moreover, given a correlation Bell functional T, we denote the correlation value of the XOR game by:

$$\omega_{\mathcal{Q}^{c\rightarrow}_{cor}}(T) = \sup_{\gamma \in \mathcal{Q}^{c\rightarrow}_{cor}} |\langle T, \gamma \rangle|. \tag{4.2.1}$$

To characterize this value with a norm we are going to use the element $T \otimes id_{M^{2^c}}$. T is the XOR game, and then it can be written as $T = \sum_{x,y} T_{xy} e_x \otimes e_y$, while $id_{M^{2^c}}$ is the identity on the matrix space, which can be written algebraically as $id = \sum_{i=1}^{2^c} e_{ij} \otimes e_{ij}$. Here e_{ij} is a matrix that is zero everywhere except in the position (i,j) where it has the value 1.

$$G \otimes id_{S_1^{2^c}} := \sum_{xy} \sum_{ij} T_{xy} (e_x \otimes e_{ij}) \otimes (e_y \otimes e_{ji}) \in \ell_1^N(S_\infty^d) \otimes \ell_1^N(S_1^{2^c})).$$

Here $S_1^{2^c}$ is the linear space of matrices of dimension 2^c with the trace norm and $S_\infty^{2^c}$ is the linear space of matrices with the operator norm.

It was proven in [50] that we can relate Equation (4.2.1) with a tensor norm using the following lemma:

Lemma 4.2.2. *Let T be an XOR game with coefficients $(T_{xy})_{x,y}$. Then,*

$$\omega_{\mathcal{Q}^{c\rightarrow}_{cor}}(T) = \sup \Big| \sum_{x,y} T_{xy} \operatorname{tr}(B_y R_x) \Big|.$$

The supremum runs over all families of self-adjoint operators $(B_y)_y$, $(R_x)_x$ in M_{2^c} verifying $\|R_x\|_{S_1^{2^c}} \leq 1$ for every x and $\|B_y\|_{M_{2^c}} \leq 1$.

And moreover, it can be related to some tensor norm:

Lemma 4.2.3. *Let T be an XOR game with coefficients $(T_{x,y})_{x,y}$. Then,*

$$\frac{1}{4} \|T \otimes id_{S_1^{2^c}}\|_{\ell_1^N(S_\infty^{2^c}) \otimes_\epsilon \ell_1^N(S_1^{2^c})} \leq \omega_{\mathcal{Q}^{c\rightarrow}_{cor}}(T) \leq \|T \otimes id_{S_1^{2^c}}\|_{\ell_1^N(S_\infty^{2^c}) \otimes_\epsilon \ell_1^N(S_1^{2^c})}.$$

4.2.3 Bell functional

The example that we are going to consider was given in [50] and corresponds to the setting of correlations.

Definition 4.2.4. *Define $T = (T_{\tilde{x}y})_{\tilde{x},y} = (T_{(xz)y})_{(x,z),y}$ as the XOR game for which the input of Alice is an element $\tilde{x} = (x,z) \in \{\pm 1\}^n \times \{\pm 1\}^n$ and the input of Bob is an element $y \in \{\pm 1\}^{n^2}$, and the coefficients can be written as:*

$$T_{(xz)y} = \frac{1}{L} \sum_{i,j=1}^{n} x_i z_j y_{ij},$$

Here, L is a normalization factor in order to fulfill $\sum_{xzy} |T_{(xz)y}| = 1$, This means that $L = \sum_{xyz} |\sum_{ij} x_i z_j y_{ij}|$.

That is, the probability of question $(\tilde{x}, y)$ is $\frac{1}{L} |\sum_{ij} x_i z_j y_{ij}|$ and the condition that the players must fulfill with their answers in that case is $ab = \operatorname{sign} \sum_{i,j} x_i z_j y_{ij}$.

Remark 4.2.5. *The following estimate for the value of L follows as an application of Khintchine inequality for $p = 1$ (Theorem 3.4.13) and is given in [50, Lemma 5.3]:*

$$\frac{1}{\sqrt{2}} n 2^{n^2+2n} \leq L \leq n 2^{n^2+2n}.$$

The next result shows an exponential separation on the communication needed when playing the XOR game. The players need to communicate $\mathcal{O}(k)$ classical bits to obtain the same value as with $\mathcal{O}(\log k)$ quantum bits.

Theorem 4.2.6. *[50, Theorem 1.3] For every $n \in \mathbb{N}$, consider the XOR game T from Definition 4.2.4 with 2^{2n} inputs for Alice and 2^{n^2} inputs for Bob. Then for every $k \in \mathbb{N}$ with $k \geq e^2$,*

$$\frac{\omega_{\mathcal{Q}_{cor}^{\log n \rightarrow}}(T)}{\omega_{\mathcal{L}_{cor}^{\log k \rightarrow}}(T)} \geq C \frac{\sqrt{n}}{\log k},$$

where C is a constant independent of n, k. Moreover, this bound is essentially optimal since for any XOR game T:

$$\frac{\omega_{\mathcal{Q}_{cor}^{\log n \rightarrow}}(T)}{\omega_{\mathcal{L}_{cor}^{\log k \rightarrow}}(T)} \leq K_G \sqrt{n}.$$

The authors left open the question whether the same bound holds in the case one considers the communication to be in both directions. This motivates the next section.

4.3 Two-way classical communication

4.3.1 Protocol and first definitions

We consider a general deterministic protocol with t-rounds of two-way classical communication between Alice and Bob for the context of correlations. In round i, first Alice will send c_i bits to Bob and, after receiving them, Bob will send d_i bits to Alice. After that, the round $i + 1$ can begin. At the end, Alice would have sent $\widetilde{m}_1, \widetilde{m}_2 \dots, \widetilde{m}_t$ and Bob, $\widetilde{n}_1, \widetilde{n}_2, \dots, \widetilde{n}_t$. When they finish the t-rounds of messages, Alice and Bob will output ± 1.

These messages can be seen as functions depending on their inputs x and y and the previous messages they have received, into the space of all possible messages. We will distinguish the inputs for Alice and Bob with N_1 and N_2, respectively. Then, we can view the first message m_1 of Alice as an application

$$\begin{aligned} \widetilde{m}_1 : [N_1] &\longrightarrow [2^{c_1}] \\ x &\longrightarrow \widetilde{m}_1(x) \coloneqq \widetilde{m}_1. \end{aligned}$$

Bob's first message is a mapping

$$\begin{aligned} \widetilde{n}_1 : [N_2] \times [2^{c_1}] &\longrightarrow [2^{d_1}] \\ (y, \widetilde{m}_1) &\longrightarrow \widetilde{n}_1(y) \coloneqq \widetilde{n}_1. \end{aligned}$$

Similarly, Alice's and Bob's last messages are mappings

$$\begin{aligned} \widetilde{m}_t : [N_1] \times [2^{d_1}] \times \cdots \times [2^{d_{t-1}}] &\longrightarrow [2^{c_t}] \\ (x, \widetilde{n}_1, \dots, \widetilde{n}_{t-1}) &\longrightarrow \widetilde{m}_t(x, \widetilde{n}_1, \dots, \widetilde{n}_{t-1}) \coloneqq \widetilde{m}_t. \end{aligned}$$

and

$$\begin{aligned} \widetilde{n}_t : [N_2] \times [2^{c_1}] \times \cdots \times [2^{c_t}] &\longrightarrow [2^{d_t}] \\ (y, \widetilde{m}_1, \dots, \widetilde{m}_t) &\longrightarrow \widetilde{n}_t(y, \widetilde{m}_1, \dots, \widetilde{m}_t) := \widetilde{n}_t. \end{aligned}$$

After they interchange messages, Alice and Bob produce ±1-valued outputs $a(x, \widetilde{n}_1, \dots, \widetilde{n}_t)$, $b(y, \widetilde{m}_1, \dots, \widetilde{m}_t)$. Therefore, Alice's strategy is a function

$$\begin{aligned} a : [N_1] \times [2^{d_1}] \times \cdots \times [2^{d_t}] &\longrightarrow \{\pm1\} \\ (x, \widetilde{n}_1, \dots, \widetilde{n}_t) &\longrightarrow a(x, \widetilde{n}_1, \dots, \widetilde{n}_t) := a_{x\widetilde{n}_1 \dots \widetilde{n}_t}. \end{aligned}$$

Similarly, Bob's strategy is given by a function

$$\begin{aligned} b : [N_2] \times [2^{c_1}] \times \cdots \times [2^{c_t}] &\longrightarrow \{\pm1\} \\ (y, \widetilde{m}_1, \dots, \widetilde{m}_t) &\longrightarrow b(y, \widetilde{m}_1, \dots, \widetilde{m}_t) := b_{y\widetilde{m}_1 \dots \widetilde{m}_t}. \end{aligned}$$

Recall that the answer for Alice depends on the input x and the messages given by Bob, which, at the same time, depend on the input y and the messages given by Alice. And vice versa for the answer of Bob.

Definition 4.3.1. *Define the two-way local correlations with t-rounds of communication as the following convex set:*

$$\begin{aligned} \mathcal{L}_{cor}^{c\leftrightarrow} = co\{ (a_{x\widetilde{n}_1 \dots \widetilde{n}_t} b_{y\widetilde{m}_1 \dots \widetilde{m}_t})_{x,y} \text{ such that } a_{x\widetilde{n}_1 \dots \widetilde{n}_t}, b_{y\widetilde{m}_1 \dots \widetilde{m}_t} = \pm1 \\ \text{for all } x, y \text{ and the protocol of messages } \widetilde{m}_1, \widetilde{n}_1, \dots, \widetilde{n}_t \}. \end{aligned}$$

Definition 4.3.2. *Define the value of winning an XOR game T with t-rounds of two-way communication as the supremum over all possible correlations that consider all possible protocols for the messages:*

$$\omega_{\mathcal{L}_{cor}^{c\leftrightarrow}}(T) = \sup_{\gamma \in \mathcal{L}_{cor}^{\leftrightarrow}} |\langle T, \gamma \rangle| = \sup \Big| \sum_{x,y=1}^{n} T_{xy} a_{x\widetilde{n}_1 \dots, \widetilde{n}_t} b_{y\widetilde{m}_1 \dots \widetilde{n}_t} \Big|. \tag{4.3.1}$$

4.3.2 The value of a game of two-way communication with a norm

The purpose of this section is to show that the value of any XOR game assisted with a general two-way classical communication protocol can be described by a tensor norm. We will need some mathematical notions.

Remark 4.3.3. *The element*

$$z = \{z(m_1, n_1, m_2, n_2, \dots, m_t, n_t)\}_{m_1, n_1, \dots, m_t, n_t} \in \mathbb{R}^{c_1 d_1 \dots c_t d_t} = \mathbb{R}^{c_1} \otimes \mathbb{R}^{d_1} \otimes \dots \otimes \mathbb{R}^{d_t} \tag{4.3.2}$$

can be seen as an element in the space $\ell_\infty^{c_1}(\ell_1^{d_1}(\dots \ell_\infty^{c_t}(\ell_1^{d_t})))$. Considered in that space, the norm of z is

$$\|z\|_{\ell_\infty^{c_1}(\ell_1^{d_1}(\dots \ell_\infty^{c_t}(\ell_1^{d_t})))} = \max_{m_1} \sum_{n_1} \dots \max_{m_t} \sum_{n_t} |z(m_1, n_1, \dots, m_t, n_t)|.$$

The next lemma will be useful later and its proof is very easy.

Lemma 4.3.4. *The following holds:*

1. *Given a Banach space X, the extreme points of $\mathsf{B}_{\ell_\infty^N(\mathsf{X})}$ are exactly the elements of the form $\sum_{i=1}^N e_i \otimes x_i$, where x_i is an extreme point of B_X for every i, and we use the tensor notation to identify $\ell_\infty^N(\mathsf{X})$ and $\ell_\infty^N \otimes \mathsf{X}$*

2. *Given a Banach space X, the extreme points of $\mathsf{B}_{\ell_1^N(\mathsf{X})}$ are exactly the elements of the form $e_i \otimes x_i$, where x_i is an extreme point of B_X and we use the tensor notation as above.*

Lemma 4.3.5. *The extreme points of the unit ball of $\ell_\infty^{N_1}(\ell_1^{2^{c_1}}(\ell_\infty^{2^{d_1}}(\dots(\ell_1^{2^{c_t}}(\ell_\infty^{2^{d_t}})))))$ are exactly the elements of the form:*

$$\sum_{x,m_1,n_1,\dots,m_t,n_t} z_{xn_1\dots n_t}\delta_{m_1,m_1(x)}\delta_{m_2,m_2(x,n_1)}\cdots\delta_{m_t,m_t(x,n_1,\dots,n_{t-1})}e_x\otimes e_{m_1}\otimes e_{n_1}\otimes\dots\otimes e_{m_t}\otimes e_{n_t},$$

where $z_{xn_1\dots n_t}=\pm1$ for all x, $n_1,\dots,n_t$ and $m_1:[N_1]\to[2^{c_1}]$, $m_2:[N_1]\times[2^{d_1}]\to[2^{c_2}]$ and so on, are functions.

Similarly, the extreme points of the unit ball of $\ell_\infty^{N_2 2^{c_1}}(\ell_1^{2^{d_1}}(\ell_\infty^{2^{c_2}}(\ell_1^{2^{d_2}}(\dots(\ell_\infty^{2^{c_t}}(\ell_1^{2^{d_t}}))))))$ are exactly the elements of the form:

$$\sum_{y,n_1,m_1,\dots,n_t,m_t} z_{ym_1\dots m_t}\delta_{n_1,n_1(y,m_1)}\cdots\delta_{n_t,n_t(y,m_1,m_2,\dots,m_t)}e_y\otimes e_{m_1}\otimes e_{n_1}\otimes\dots\otimes e_{m_t}\otimes e_{n_t},$$

where, $z_{xm_1\dots m_t}=\pm1$ for all x, $m_1,\dots,m_t$ and $n_1:[N_2]\times[2^{c_1}]\to[2^{d_1}]$, $n_2:[N_2]\times[2^{c_1}]\times[2^{c_2}]\to[2^{d_2}]$ and so on, are functions.

Proof. The proof follows easily from Lemma 4.3.4 and induction. For the sake of clarity we write out the proof for the case of $\ell_\infty^{N_2 2^{c_1}}(\ell_1^{2^{d_1}}(\ell_\infty^{2^{c_2}}(\ell_1^{2^{d_2}})))$, which corresponds to $t=2$ in the second statement of the lemma.

First note that following Lemma 4.3.4, the extreme elements of the unit ball of $\ell_\infty^{2^{c_2}}(\ell_1^{2^{d_2}})$ are of the form

$$\sum_{m_2,n_2=1}^{2^{c_2},2^{d_2}} z_{m_2}\delta_{n_2,n_2(m_2)}e_{m_2}\otimes e_{n_2},$$

where $n_2:[2^{c_2}]\to[2^{d_2}]$ runs over all possible functions and $z_{m_2}=\pm1$ for all m_2.

Then, with the aid of the δ notation, the extreme points of the of the unit ball of $\ell_1^{2^{d_1}}(\ell_\infty^{2^{c_2}}(\ell_1^{2^{d_2}}))$ can be written as

$$\sum_{n_1=1}^{2^{d_1}}\sum_{n_2,m_2} z_{m_2}\delta_{n_2,n_2(m_2)}\delta_{n_1,n_0}e_{n_1}\otimes e_{m_2}\otimes e_{n_2}, \tag{4.3.3}$$

where $n_0\in[2^{d_1}]$.

Finally, to describe the extreme points of the unit ball of $\ell_\infty^{N_2 2^{c_1}}(\ell_1^{2^{d_1}}(\ell_\infty^{2^{c_2}}(\ell_1^{2^{d_2}})))$, first note that $\mathbb{R}^{N_2 2^{c_1}}=\mathbb{R}^{N_2}\otimes\mathbb{R}^{2^{c_1}}$. Then, applying again Lemma 4.3.4, for every y and m_1, we obtain that the extreme points of the unit ball of $\ell_\infty^{N_2 2^{c_1}}(\ell_1^{2^{d_1}}(\ell_\infty^{2^{c_2}}(\ell_1^{2^{d_2}})))$ are exactly those of the form

$$\sum_{y,m_1=1}^{N_2,2^{c_1}} e_y \otimes e_{m_1} \otimes \left(\sum_{n_1=1}^{2^{d_1}} \sum_{n_2,m_2} z_{m_2} \delta_{n_2,n_2(m_2)} \delta_{n_1,n_0} e_{n_1} \otimes e_{m_2} \otimes e_{n_2} \right).$$

In that expression, the functions z_{m_2}, $\delta_{n_2,n_2(m_2)}$ and δ_{n_1,n_0} depend also on y and m_1, and therefore we can rewrite the formula above as:

$$\sum_{y,m_1,n_1,m_2,n_2} z_{ym_1m_2} \delta_{n_1,n_1(m_1,y)} \delta_{n_2,n_2(y,m_1,m_2)} e_y \otimes e_{m_1} \otimes e_{n_1} \otimes e_{m_2} \otimes e_{n_2},$$

where $n_2 : [N_2] \times [2^{c_1}] \times [2^{c_2}] \to [2^{d_2}]$ and $n_1 : [N_2] \times [2^{c_1}] \to [2^{d_1}]$ are functions and $z_{y,m_1,m_2} = \pm 1$ for all y, m_1, m_2.

□

Using the lemmas stated above, we can now find an expression for the value of an XOR game $T = (T_{xy})_{x,y=1}^{N_1,N_2}$ with the protocol defined in Section 4.3.1. Analogously to Section 4.2.1, we can also define the element

$$T \otimes id \otimes \ldots \otimes id = \sum_{\substack{x,y \\ n_1,m_1,\ldots,n_t,m_t, \\ n_1',m_1',\ldots,n_t',m_t'}} T_{xy} \delta_{n_1,n_1'} \ldots \delta_{n_t,n_t'} \delta_{m_1,m_1'} \ldots \delta_{m_t,m_t'} \tag{4.3.4}$$

$$\cdot (e_x \otimes e_{m_1} \otimes e_{n_1} \otimes \ldots \otimes e_{m_t} \otimes e_{n_t}) \otimes (e_y \otimes e_{m_1'} \otimes e_{n_1'} \otimes \ldots \otimes e_{m_t'} \otimes e_{n_t'})$$

$$= \sum_{\substack{x,y \\ n_1,m_1,\ldots,n_t,m_t}} T_{xy} (e_x \otimes e_{m_1} \otimes e_{n_1} \otimes \ldots \otimes e_{m_t} \otimes e_{n_t}) \otimes (e_y \otimes e_{m_1} \otimes e_{n_1} \otimes \ldots \otimes e_{m_t} \otimes e_{n_t}),$$

that can be seen as a tensor in

$$\ell_1^{N_1}(\ell_\infty^{2^{c_1}}(\ell_1^{2^{d_1}}(\ldots(\ell_\infty^{2^{c_t}}(\ell_1^{2^{d_t}}))))) \otimes_\epsilon \ell_1^{N_2 2^{c_1}}(\ell_\infty^{2^{d_1}}(\ldots(\ell_1^{2^{c_t}}(\ell_\infty^{2^{d_t}})))).$$

Theorem 4.3.6. *Consider a two-way protocol with t-rounds of communication in which a total amount of c-bits are exchanged. In each round i of the protocol Alice sends c_i bits to Bob and, after that, Bob sends d_i bits to Alice. Hence, $\sum_{i=1}^t c_i + \sum_{i=1}^t d_i = c$. The value of an XOR game T restricted to the aforementioned protocols is:*

$$\omega_{\mathcal{L}_{cor}^{c\leftrightarrow}}(T) = \|T \otimes id \otimes id \otimes \ldots \otimes id\|_{\ell_1^N(\ell_\infty^{2^{c_1}}(\ell_1^{2^{d_1}}(\ldots(\ell_\infty^{2^{c_t}}(\ell_1^{2^{d_t}})))))\otimes_\epsilon \ell_1^{N 2^{c_1}}(\ell_\infty^{2^{d_1}}(\ldots(\ell_1^{2^{c_t}}(\ell_\infty^{2^{d_t}}))))}.$$

Proof. Let $T \otimes id \otimes \ldots \otimes id$ be the element from Equation 4.3.4. To compute its norm, consider A and B extreme points of the following sets:

$$A \in \mathsf{B}_{\ell_\infty^{N_1}(\ell_1^{2^{c_1}}(\ell_\infty^{2^{d_1}}(\ldots(\ell_1^{2^{c_t}}(\ell_\infty^{2^{d_t}})))))},$$

$$B \in \mathsf{B}_{\ell_\infty^{N_2 2^{c_1}}(\ell_1^{2^{d_1}}(\ell_\infty^{2^{c_2}}(\ldots(\ell_\infty^{2^{c_t}}(\ell_1^{2^{d_t}})))))}.$$

Their expression is given by Lemma 4.3.5. Then,

$$\langle T \otimes id \otimes \ldots \otimes id, A \otimes B \rangle =$$

$$= \sum_{\substack{x,y \\ m_1,\ldots,m_t \\ n_1,\ldots,n_t}} T_{xy} z_{xn_1\ldots n_t} \delta_{m_1,\widetilde{m}_1(x)} \ldots \delta_{m_t,\widetilde{m}_t(x,n_1,\ldots,n_{t-1})} s_{ym_1\ldots m_t} \delta_{n_1,\widetilde{n}_1(y,m_1)} \ldots \delta_{n_t,\widetilde{n}_t(y,m_1,\ldots,m_t)}$$

$$= \sum_{x,y} T_{xy} z_{x\widetilde{n}_1(y,\widetilde{m}_1)\ldots\widetilde{n}_t(y,\widetilde{m}_1,\ldots,\widetilde{m}_t)} s_{y\widetilde{m}_1(x)\ldots\widetilde{m}_t(x,\widetilde{n}_1,\ldots,\widetilde{n}_{t-1})}.$$

In the last equality we have used the deltas to replace m_1 by $m_1(x)$, n_1 by $\widetilde{n}_1(y, \widetilde{m}_1(x))$ and so on. Finally, proving that the element $(z_{x\widetilde{n}_1(y,\widetilde{m}_1)\dots\widetilde{n}_t(y,\widetilde{m}_1,\dots,\widetilde{m}_t)} s_{y\widetilde{m}_1(x)\dots\widetilde{m}_t(x,\widetilde{n}_1,\dots,\widetilde{n}_{t-1})})_{x,y}$ is in $\mathcal{L}_{cor}^{\leftrightarrow}$ is trivial because it corresponds to the protocol of messages to be $\widetilde{m}_1, \widetilde{n}_1, \dots, \widetilde{m}_t, \widetilde{n}_t$. As we see, each extreme point from the unit ball corresponds for a two-way communication protocol for Alice and Bob, and the result follows.

□

4.3.3 Correlation Bell functional

To show the difference between sets $\mathcal{Q}_{cor}^{c\rightarrow}$ and $\mathcal{L}_{cor}^{c\leftrightarrow}$ we are going to use the bell functional from Definition 4.2.4. This game appeared in [50] to prove a similar bound to show the difference between $\mathcal{Q}_{cor}^{c\rightarrow}$ and $\mathcal{L}_{cor}^{c\rightarrow}$.

In order to prove a version of Theorem 4.2.6 we need to show a lower bound for the value with quantum communication and an upper bound for the value with classical communication. The quantum value was already proven in [50] and it is stated in Proposition 4.3.7. Our main contribution is the upper bound for the value with two-way classical communication. To make the proof easier to follow, we state first some lemmas. Some of them were already used in [50].

Proposition 4.3.7. *[50, Proposition 5.6] Let T be the XOR game defined in Definition 4.2.4. Then,*

$$\omega_{\mathcal{Q}_{cor}^{\log n \rightarrow}}(T) \geq \frac{C}{\sqrt{n}},$$

where C is a constant independent of n.

We show here the main idea of the proof. For every $\tilde{x} = (x, z) \in \{\pm 1\}^n \times \{\pm 1\}^n$, define the n-dimensional states

$$|\varphi_x\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n x_i |i\rangle \quad \text{and} \quad |\varphi_z\rangle = \frac{1}{\sqrt{n}} \sum_{j=1}^n z_j |j\rangle.$$

And also consider for every $y \in \{\pm 1\}^{n^2}$ the matrix $A_y = (y_{ij})_{i,j=1}^n \in M_n$.

Alice will create a n-dimensional state with one of the positive components of the not self-adjoint in general operator $\rho_{\tilde{x}} = |\varphi_x\rangle\langle\varphi_z|$, properly normalized. She will send it to Bob and he will measure it with one of the self-adjoint components of A_y, again properly normalized.

First we state Khintchine and Double Khintchine inequalities in the precise form we will use. A proof of the double Khintchine inequality can be found in [31, pag. 455].

Theorem 4.3.8 (Khintchine inequalities)**.** *For $1 \leq p < \infty$ there exist constants $a_p, b_p \geq 1$ such that*

$$a_p^{-1} \left(\sum_{i=1}^n |\alpha_i|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{y \in \{\pm 1\}^n} \frac{1}{2^n} \Big| \sum_{i=1}^n \alpha_i y_i \Big|^p \right)^{\frac{1}{p}} \leq b_p \left(\sum_{i=1}^n |\alpha_i|^2 \right)^{\frac{1}{2}} \tag{4.3.5}$$

for every $n \in \mathbb{N}$ and all $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$.

Moreover,

$$a_p^{-2} \left(\sum_{i,j=1}^n |\alpha_{ij}|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{x,z \in \{\pm 1\}^n} \frac{1}{2^{2n}} \Big| \sum_{i,j=1}^n \alpha_{ij} x_i z_j \Big|^p \right)^{\frac{1}{p}} \leq b_p^2 \left(\sum_{i,j=1}^n |\alpha_{ij}|^2 \right)^{\frac{1}{2}} \tag{4.3.6}$$

for every $n \in \mathbb{N}$ and all $\alpha_{11}, \alpha_{12}, \cdots, \alpha_{nn} \in \mathbb{C}$.

In our reasonings we actually need the trasposed version of both Khintchine inequalities. We state the precise result.

Lemma 4.3.9. *Let $1 < p < \infty$ and let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then, for every $n \in \mathbb{N}$ and for every sequence of numbers $(\alpha(y))_{y\in\{-1,1\}^n}$,*

$$\left(\sum_{i=1}^{n}\left(\sum_{y\in\{-1,1\}^n} y_i\alpha(y)\right)^2\right)^{\frac{1}{2}} \leq b_{p'}^2\,(2^n)^{\frac{1}{p'}}\left(\sum_{y\in\{-1,1\}^n}|\alpha(y)|^p\right)^{\frac{1}{p}},$$

where $b_{p'}$ is the constant appearing in Lemma 4.3.8 for p'.

Moreover, for every $n \in \mathbb{N}$ and for every finite sequence of numbers $(\alpha(x,z))_{(x,z)\in\{-1,1\}^n\times\{\pm1\}^n}$,

$$\left(\sum_{i,j=1}^{n}\left(\sum_{(x,z)} x_i z_j\alpha(x,z)\right)^2\right)^{\frac{1}{2}} \leq b_{p'}^2\,\left(2^{2n}\right)^{\frac{1}{p'}}\left(\sum_{(x,z)}|\alpha(x,z)|^p\right)^{\frac{1}{p}},$$

where the sums in (x,z) are over $\{\pm1\}^n \times \{\pm1\}^n$ and $b_{p'}$ is again the constant appearing in Lemma 4.3.8 for p'.

Proof. The second statement follows from (4.3.6). The proof can be seen in [50, Lemma 5.4]. The proof of the first statement is done similarly, using (4.3.5) rather than (4.3.6). □

We will also need the following simple consequence of Holder's inequality.

Lemma 4.3.10. *For every $1 < p < \infty$ and for every finite sequence of real numbers $(\alpha_i)_{i=1}^d$,*

$$\sum_{i=1}^{d}|\alpha_i| \leq d^{1/p'}\Big(\sum_{i=1}^{d}|\alpha_i|^p\Big)^{1/p},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$

Remark 4.3.11. *Consider two sequences of numbers $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$, with n a natural number. It is easy to see that:*

$$\sum_{i=1}^{n}|\alpha_i\beta_i| \leq \Big(\max_{i=1}^{n}|\alpha_i|\Big)\Big(\sum_{i=1}^{n}|\beta_i|\Big)$$

We state and prove one more technical simple result.

Lemma 4.3.12. *Take $a \in \mathsf{B}_{\ell_1^{c_1}(\ell_\infty^{c_2}(\ldots(\ell_1^{c_{t-1}}(\ell_\infty^{c_t}))\ldots))}$ and $b \in \mathsf{B}_{\ell_\infty^{c_1}(\ell_1^{c_1}(\ldots(\ell_\infty^{c_{t-1}}(\ell_1^{c_t}))\ldots))}$. Then:*

$$\sum_{m_1\ldots m_t}|a_{m_1\ldots m_t}b_{m_1\ldots m_t}| \leq 1.$$

Proof. Using Remark 4.3.11 a number t of times we can compute:

$$\begin{aligned}
&\sum_{m_1,\dots,m_t} |a_{m_1\dots m_t} b_{m_1\dots m_t}| = \sum_{m_1,\dots,m_{t-1}} \sum_{m_t} |a_{m_1\dots m_t} b_{m_1\dots m_t}| \\
&\leq \sum_{m_1,\dots,m_{t-2}} \sum_{m_{t-1}} \Big(\Big(\max_{m_t} |a_{m_1\dots m_t}| \Big) \Big(\sum_{m_t} |b_{m_1\dots m_t}| \Big) \Big) \\
&\leq \sum_{m_1,\dots,m_{t-2}} \Big(\Big(\sum_{m_{t-1}} \max_{m_t} |a_{m_1\dots m_t}| \Big) \Big(\max_{m_{t-1}} \sum_{m_t} |b_{m_1\dots m_t}| \Big) \Big) \\
&\leq \dots \leq \Big(\sum_{m_1} \max_{m_1} \dots \sum_{m_{t-1}} \max_{m_t} |a_{m_1\dots m_t}| \Big) \Big(\max_{m_1} \sum_{m_2} \dots \max_{m_{t-1}} \sum_{m_t} |b_{m_1\dots m_t}| \Big) \leq 1.
\end{aligned}$$

The last inequality follows from considering a and b in their respective unit balls.

To see the last inequality, it is enough to keep on summing in the same order, that is, in n_{t-1}, then in m_{t-1}, then in n_{t-2}, etc.

□

We will use the notation $\overline{m}, \overline{n}$ for the multiindices $(m_1, \dots, m_t)$, $(n_1 \dots, n_t)$.

Now we can upper bound the value of the XOR game T using any two-way classical communication protocol.

Proposition 4.3.13. *Let T be the XOR game from Definition 4.2.4. Then*

$$\omega_{\mathcal{L}_{cor}^{\log k \leftrightarrow}}(T) \leq \frac{4\sqrt{2} e^{5/2} (\log k)^{3/2}}{n}.$$

Proof. We have to bound the following norm:

$$\|T \otimes id \otimes id \otimes \dots \otimes id\|_{\ell_1^{2^{2n}}(\ell_\infty^{2^{c_1}}(\ell_1^{2^{d_1}}(\dots(\ell_\infty^{2^{c_t}}(\ell_1^{2^{d_t}})))))\otimes_\epsilon \ell_1^{2^{n^2 c_1}}(\ell_\infty^{2^{d_1}}(\dots(\ell_1^{2^{c_t}}(\ell_\infty^{2^{d_t}}))))}.$$

This corresponds to generic two-way protocols with t-rounds of communication with $\log k$ as the total amount of bits exchanged, $\log k = \sum_{i=1}^t c_i + d_i$. Assume that Alice starts the communication, being the other case similar. Consider $a = (a(\tilde{x}, \overline{m}, \overline{n}))_{x,\overline{m},\overline{n}} \in B_{\ell_\infty^{2^{2n}}(\ell_1^{2^{c_1}}(\ell_\infty^{2^{d_1}}(\dots(\ell_1^{2^{c_t}}(\ell_\infty^{2^{d_t}})))))}$ and $b = (b(y, \overline{m}, \overline{n}))_{y,\overline{m},\overline{n}} \in B_{\ell_\infty^{2^{n^2}}(\ell_\infty^{2^{c_1}}(\ell_1^{2^{d_1}}(\dots(\ell_\infty^{2^{c_t}}(\ell_1^{2^{d_t}})))))}$. Then,

$$\langle T \otimes id \otimes \dots \otimes id, a \otimes b \rangle = \sum_{\substack{\tilde{x},y \\ \overline{m},\overline{n}}} T_{\tilde{x}y} a(\tilde{x}, \overline{m}, \overline{n}) b(y, \overline{m}, \overline{n}).$$

We have

$$
\begin{aligned}
&\sum_{\substack{\tilde{x},y\\ \overline{m},\overline{n}}} T_{\tilde{x}y} a(\tilde{x},\overline{m},\overline{n}) b(y,\overline{m},\overline{n}) \leq \sum_{\overline{m},\overline{n}} \left| \sum_{\tilde{x},y} T_{\tilde{x}y} a(\tilde{x},\overline{m},\overline{n}) b(y,\overline{m},\overline{n}) \right| \\
&\leq k^{\frac{1}{p'}} \left(\sum_{\overline{m},\overline{n}} \left| \sum_{\tilde{x},y} T_{\tilde{x}y} a(\tilde{x},\overline{m},\overline{n}) b(y,\overline{m},\overline{n}) \right|^p \right)^{\frac{1}{p}} \\
&= \frac{k^{\frac{1}{p'}}}{L} \left(\sum_{\overline{m},\overline{n}} \left| \sum_{(x,z),y} \sum_{i,j} x_i z_j y_{ij} a(\tilde{x},\overline{m},\overline{n}) b(y,\overline{m},\overline{n}) \right|^p \right)^{\frac{1}{p}} \\
&= \frac{k^{\frac{1}{p'}}}{L} \left(\sum_{\overline{m},\overline{n}} \left| \sum_{i,j} \left(\sum_{(x,z)} x_i z_j a(\tilde{x},\overline{m},\overline{n}) \right) \left(\sum_{y} y_{ij} b(y,\overline{m},\overline{n}) \right) \right|^p \right)^{\frac{1}{p}},
\end{aligned}
$$

where the second inequality follows from Lemma 4.3.10.

We note now that, for every choice of $\overline{m},\overline{n}$,

$$
\begin{aligned}
&\left| \sum_{i,j} \left(\sum_{(x,z)} x_i z_j a(x,z,\overline{m},\overline{n}) \right) \left(\sum_{y} y_{ij} b(y,\overline{m},\overline{n}) \right) \right| \\
&\leq \left(\sum_{i,j} \left(\sum_{(x,z)} x_i z_j a(x,z,\overline{m},\overline{n}) \right)^2 \right)^{\frac{1}{2}} \left(\sum_{i,j} \quad \sum_{y} y_{ij} b(y,\overline{m},\overline{n}) \Bigg)^2 \right)^{\frac{1}{2}} \\
&\leq b_{p'}^3 \left(2^{2n+n^2} \right)^{\frac{1}{p'}} \quad \sum_{x,z} |a(x,z,\overline{m},\overline{n})|^p \Bigg)^{\frac{1}{p}} \quad \sum_{y} |b(y,\overline{m},\overline{n})|^p \Bigg)^{\frac{1}{p}},
\end{aligned}
$$

where the first inequality follows from Cauchy-Schwartz inequality and the second one follows from Lemma 4.3.9.

Using this, we have that

$$
\begin{aligned}
&\left| \sum_{\substack{\tilde{x},y\\ \overline{m},\overline{n}}} T_{\tilde{x}y} a(\tilde{x},\overline{m},\overline{n}) b(y,\overline{m},\overline{n}) \right| \\
&\leq \frac{k^{\frac{1}{p'}}}{L} b_{p'}^3 \left(2^{2n+n^2} \right)^{\frac{1}{p'}} \left(\sum_{\overline{m},\overline{n}} \quad \sum_{x,z} |a(x,z,\overline{m},\overline{n})|^p \right) \left(\sum_{y} |b(y,\overline{m},\overline{n})|^p \right) \Bigg)^{\frac{1}{p}} \\
&= \frac{k^{\frac{1}{p'}}}{L} b_{p'}^3 \left(2^{2n+n^2} \right)^{\frac{1}{p'}} \left(\sum_{x,z,y} \sum_{\overline{m},\overline{n}} |a(x,z,\overline{m},\overline{n}) b(y,\overline{m},\overline{n})|^p \right)^{\frac{1}{p}} \\
&\leq \frac{k^{\frac{1}{p'}}}{L} b_{p'}^3 \left(2^{2n+n^2} \right)^{\frac{1}{p'}} \left(2^{2n+n^2} \right)^{\frac{1}{p}},
\end{aligned}
$$

where in the last inequality we have used Lemma 4.3.12 and the simple fact that, for every

$1 < p < \infty$, if

$$\sum_{\overline{m},\overline{n}} |a(x,z,\overline{m},\overline{n})b(y,\overline{m},\overline{n})| \leq 1,$$

then also

$$\sum_{\overline{m},\overline{n}} |a(x,z,\overline{m},\overline{n})b(y,\overline{m},\overline{n})|^p \leq 1.$$

To finish, we use that $L \geq \frac{1}{\sqrt{2}} n 2^{n^2+2n}$ by Remark 4.2.5. We also use that $b_{p'} \leq \sqrt{2ep'}$ (see [31, Section 8.5]) and we make the choice $p' = \log k$. Then, we have:

$$\sum_{\substack{\tilde{x},y \\ \overline{m},\overline{n}}} T_{\tilde{x}y} a(\tilde{x},\overline{m},\overline{n}) b(y,\overline{m},\overline{n}) \leq \frac{4e^{\frac{5}{2}}(\log k)^{\frac{3}{2}}}{n}.$$

□

Theorem 1.3.1 follows from Proposition 4.3.13 and Proposition 4.3.7.

4.3.4 General two-way communication

The techniques of Theorem 4.3.6 can be easily applied to general games, that is, games with a general number of outputs.

A general deterministic protocol with t-rounds of two-way classical communication between Alice and Bob will consist on the messages $\widetilde{m}_1, \widetilde{n}_1$ to $\widetilde{m}_t, \widetilde{n}_t$. After they interchange messages, Alice and Bob will produce the outputs $a(x, \widetilde{n}_1, \ldots, \widetilde{n}_t)$ and $b(y, \widetilde{m}_1, \ldots, \widetilde{m}_t)$, respectively. Therefore, Alice's strategy is a function

$$\begin{aligned} \widetilde{a} : [N_1] \times [2^{d_1}] \times \cdots \times [2^{d_t}] &\longrightarrow [K_1] \\ (x, \widetilde{n}_1, \ldots, \widetilde{n}_t) &\longrightarrow \widetilde{a}(x, \widetilde{n}_1, \ldots, \widetilde{n}_t) := \widetilde{a}_{x\widetilde{n}_1 \ldots \widetilde{n}_t}, \end{aligned}$$

and, similarly, Bob's strategy is given by a function

$$\begin{aligned} \widetilde{b} : [N_2] \times [2^{c_1}] \times \cdots \times [2^{c_t}] &\longrightarrow [K_2] \\ (y, \widetilde{m}_1, \ldots, \widetilde{m}_t) &\longrightarrow \widetilde{b}(y, \widetilde{m}_1, \ldots, \widetilde{m}_t) := \widetilde{b}_{y\widetilde{m}_1 \ldots \widetilde{m}_t}. \end{aligned}$$

Therefore we can define the value of a game when playing with the above protocols:

Definition 4.3.14. *Define the two-way local probability distributions with t-rounds of communication as the following set:*

$$\mathcal{L}^{c\leftrightarrow} = co\{(\delta_{a,\widetilde{a}(x\widetilde{n}_1 \ldots \widetilde{n}_t)} \delta_{b,\widetilde{b}(y\widetilde{m}_1 \ldots \widetilde{m}_t)})_{x,y,a,b} \text{ such that } \widetilde{a} \text{ and } \widetilde{b} \text{ are functions } \}$$

Given a Bell functional M, its classical two-way communication value for protocols with t-rounds of communication, c_i and d_i bits exchanged in round i, where $c = \sum_{i=1}^t c_i + d_i$, is:

$$\omega_{\mathcal{L}^{c\leftrightarrow}}(M) = \sup \Big| \sum_{x,a,y,b} M_{xy}^{ab} \delta_{a,\widetilde{a}(x,\widetilde{n}_1,\widetilde{n}_2,\ldots,\widetilde{n}_t)} \delta_{b,\widetilde{b}(y,\widetilde{m}_1,\widetilde{m}_2\ldots,\widetilde{m}_t)} \Big|. \tag{4.3.7}$$

Recall that the supremum runs over all possible extremal strategies, thus, with the delta function, both the messages and the final answer are fixed.

In order to find an expression for the value of the game, $G = (G_{xy}^{ab})_{x,y=1\,a,b=1}^{N_1,N_2\,K_1,K_2}$ consider the following element:

$$G \otimes id \otimes \ldots \otimes id = \sum_{\substack{x,y,a,b \\ \overline{n},\overline{m} \\ \overline{n}',\overline{m}'}} G_{xy}^{ab} \delta_{n_1,n_1'} \ldots \delta_{n_t,n_t'} \delta_{m_1,m_1'} \ldots \delta_{m_t,m_t'} \tag{4.3.8}$$

$$\cdot (e_x \otimes e_a \otimes e_{m_1} \otimes e_{n_1} \otimes \ldots \otimes e_{m_t} \otimes e_{n_t}) \otimes (e_y \otimes e_a \otimes e_{m_1'} \otimes e_{n_1'} \otimes \ldots \otimes e_{m_t'} \otimes e_{n_t'})$$
$$= \sum_{\substack{x,y \\ \overline{n},\overline{m}}} G_{xy}^{ab} (e_x \otimes e_a \otimes e_{m_1} \otimes e_{n_1} \otimes \ldots \otimes e_{m_t} \otimes e_{n_t}) \otimes (e_y \otimes e_b \otimes e_{m_1} \otimes e_{n_1} \otimes \ldots \otimes e_{m_t} \otimes e_{n_t}),$$

that can be seen as a tensor in

$$\ell_1^{N_1}(\ell_\infty^{2^{c_1}}(\ell_1^{2^{d_1}}(\ldots(\ell_\infty^{2^{c_t}}(\ell_1^{2^{d_t}}(\ell_\infty^{K_1})))))) \otimes_\epsilon \ell_1^{N_2 2^{c_1}}(\ell_\infty^{2^{d_1}}(\ldots(\ell_1^{2^{c_t}}(\ell_\infty^{2^{d_t}K_2})))).$$

Theorem 4.3.15. *Given a game $G = \sum_{xyab} G_{xy}^{ab} e_x \otimes e_a \otimes e_y \otimes e_b$, the following holds:*

$$\omega_{\mathcal{L}^{c\leftrightarrow}}(G) = \|G \otimes id \otimes id \otimes \ldots \otimes id\|_{\ell_1^N(\ell_\infty^{2^{c_1}}(\ell_1^{2^{d_1}}(\ldots(\ell_\infty^{2^{c_t}}(\ell_1^{2^{d_t}}(\ell_\infty^K))))))\otimes_\epsilon \ell_1^{N2^{c_1}}(\ell_\infty^{2^{d_1}}(\ldots(\ell_1^{2^{c_t}}(\ell_\infty^{K2^{d_t}}))))}.$$

Proof. Let $G \otimes id \otimes \ldots \otimes id$ be the element from Equation 4.3.8. Since it is pointwise positive, we can consider only pointwise positive elements to compute its norm. Take A an extreme point of $B_{\ell_\infty^{N_1}(\ell_1^{2^{c_1}}(\ell_\infty^{2^{d_1}}(\ldots(\ell_1^{2^{c_t}}(\ell_\infty^{2^{d_t}}(\ell_1^{K_1})))))}$. Then it can be written using Lemma 4.3.5

$$A = \sum_{\substack{x,y \\ \overline{m},\overline{n}}} \delta_{m_1,\widetilde{m}_1(x)} \ldots \delta_{m_t,\widetilde{m}_t(x,n_1,\ldots,n_{t-1})} \delta_{a,\widetilde{a}(x,n_1,n_2,\ldots,n_t)} e_x \otimes e_a \otimes e_{m_1} \otimes e_{n_1} \otimes \ldots \otimes e_{m_t} \otimes e_{n_t}$$

And also take B an extreme point of $\mathsf{B}_{\ell_\infty^{N_2 2^{c_1}}(\ell_1^{2^{d_1}}(\ell_\infty^{2^{c_2}}(\ldots(\ell_\infty^{2^{c_t}}(\ell_1^{2^{d_t}K_2})))))}$, which, according to Lemma 4.3.5, is

$$\sum_{\substack{x,y \\ \overline{m},\overline{n}}} \delta_{n_1,\widetilde{n}_1(y)} \ldots \delta_{n_t,n_t(y,m_1,\ldots,m_t)} \delta_{b,\widetilde{b}(y,m_1,m_2\ldots,m_t)} e_y \otimes e_b \otimes e_{m_1} \otimes e_{n_1} \otimes \ldots \otimes e_{m_t} \otimes e_{n_t}$$

Then,

$$\langle G \otimes id \otimes \ldots \otimes id, A \otimes B \rangle =$$
$$= \sum_{\substack{x,y \\ m_1,\ldots,m_t \\ n_1,\ldots,n_t}} G_{xy}^{ab} \delta_{m_1,\widetilde{m}_1(x)} \ldots \delta_{m_t,\widetilde{m}_t(x,n_1,\ldots,n_{t-1})} \delta_{n_1,\widetilde{n}_1(y)} \ldots \delta_{n_t,\widetilde{n}_t(y,m_1,\ldots,m_t)}$$
$$= \sum_{x,y} G_{xy}^{ab} \delta_{a,\widetilde{a}(x,\widetilde{n}_1,\widetilde{n}_2,\ldots,\widetilde{n}_t)} \delta_{b,\widetilde{b}(y,\widetilde{m}_1,\widetilde{m}_2\ldots,\widetilde{m}_t)}.$$

Which coincides with a deterministic protocol. □

4.4 Conclusions

We have started this chapter by defining the correlations that arise when either one-way classical or one-way quantum communication is allowed. This setting was studied in [50] and the authors defined an XOR game in which Alice and Bob need to communicate $k = \mathcal{O}(\sqrt{n})$ classical bits to obtain the same value as the one obtained with $\log n$ qubits.

This example motivated us to study whether the same bound is obtained against a general two-way classical communication protocol. And this is our main result of this chapter: In Theorem 1.3.1 we show that this same bound can be obtained between one-way quantum communication and two-way classical communication.

To fulfill this task we characterize the value of an XOR game with a tensor norm with a clumsy statement due to the intrinsic difficulties of describing two-way communication. The idea of Theorem 4.3.6 is to relate the deterministic communication protocols with the extreme points of the unit ball of $\ell_\infty^{N_2 2^{c_1}}(\ell_1^{2^{d_1}}(\dots(\ell_\infty^{2^{c_t}}(\ell_1^{2^{d_t}}))))$ and $\ell_\infty^{N_1}(\ell_1^{2^{c_1}}(\ell_\infty^{2^{d_1}}(\dots(\ell_1^{2^{c_t}}(\ell_\infty^{2^{d_t}})))))$, and then to use these extreme points to compute the ϵ norm of the operator $T \otimes id \otimes id \otimes \dots \otimes id$.

Then, in Proposition 4.3.13 we prove the upper bound for any classical two-way protocol relying on techniques from the local theory of Banach spaces, in particular on a careful use of the Khintchine and double Khintchine inequalities. Also, careful reasoning is needed when handling the dependencies appearing between a message and the previous and following messages.

Finally, we want to make the following remark. Contrary to the standard setting of communication complexity, in which one looks on how much communication do we need to obtain certain result, in our case we fix the communication and we look at the value we can obtain with that amount of communication.

One could wonder how this task can fit into the standard setting of communication complexity. At the end of the protocol Alice can send her output to Bob and then Bob can output the value of the function. But, in our case, the value of the game depends on a certain distribution on the inputs instead of the worst case scenario. And moreover, we do not consider the standard "bounded-error" setting, but a setting in which the classical or quantum success probability is allowed to be arbitrarily low.

We notice that Klartag and Regev [68] already gave an exponential separation between one-way quantum communication and two-way classical communication. However, in the result of Klartag and Regev, "only" $\mathcal{O}(n^{1/3})$ classical bits to obtain the same value as $\log n$ qubits, while in our result a tight bound of $\mathcal{O}(n^{1/2})$ is necessary for this purpose. And not only that, an alternative proof using tensor norms can be very valuable in order to find new interesting examples in communication complexity.

Chapter 5

Multipartite setting

5.1 Introduction

The different notions of locality that can be stablished in the multipartite setting provide an evidence of the complexity of the classification of non-local resources. Genuine multipartite non-local distributions apprehend truly multipartite effects, given that these probability distributions cannot be reproduced by bilocal models.

We show here that, while in the correlation scenario the relative violation of bilocal Bell inequalities by quantum resources is bounded, i.e. it does not grow arbitrarily with the number of inputs, it turns out to be unbounded in the general case. We identify Bell functionals that take the form of non-local games for which the ratio of the quantum and bilocal values grows unboundedly as a function of the number of inputs and outputs.

5.2 General definitions

As the general multipartite definitions have been already introduced in Chapter 1, we will revisit all those definitions about the multipartite system, but only for the tripartite case.

Consider the standard Bell scenario in which Alice, Bob and Charlie produce outputs a, b and c upon receiving inputs x, y, and z, respectively, according to the joint probability distribution,

$$(P(a,b,c|x,y,z))_{x,y,z}^{a,b,c}. \tag{5.2.1}$$

We will denote by N the number of possible inputs and by K the number of possible outputs. A distribution (5.2.1) is said to be *fully local* if[1]

$$P(a,b,c|x,y,z) = \sum_{\lambda} p_\lambda P_\lambda^1(a|x) P_\lambda^2(b|y) P_\lambda^3(c|z), \tag{5.2.2}$$

where $(p_\lambda)_\lambda$ denotes a probability distribution and P_λ^i are in $\mathcal{P}^1$ for all i and λ. We will denote by $\mathcal{L}^3$, the set of tripartite fully local probability distributions. On the other hand, bilocal probability distributions admit a more general model of the form

[1] As there is a finite number of extremal points, we can consider a sum instead of an integral for the definition of $\mathcal{L}^3$.

$$P(a,b,c|x,y,z) \\ = \sum_\lambda p_\lambda P^1_\lambda(a,b|x,y)Q^1_\lambda(c|z) + \sum_\lambda q_\lambda P^2_\lambda(b,c|y,z)Q^2_\lambda(a|x) + \sum_\lambda r_\lambda P^3_\lambda(c,a|z,x)Q^3_\lambda(b|y), \quad (5.2.3)$$

where $\sum_\lambda p_\lambda + q_\lambda + r_\lambda = 1$, $P^i_\lambda \in \mathcal{P}^2$ and $Q^i_\lambda \in \mathcal{P}^1$ for all λ and i.

If no restriction is added to these probability distributions for the subsets of parties P^i_λ for all i, we have Svetlichny's notion of *bilocality*, denoted simply by $\mathcal{BL}_\mathcal{G}$. If, on the other hand, the probability distributions for the subsets P^i_λ are required to be non-signalling, we will refer to these probability distributions as *non-signalling bilocal* and we will denote the corresponding set by $\mathcal{BL}_\mathcal{NS}$.

We recall that non-signalling probability distributions are such that each party's marginal probability distribution is independent of the other parties' inputs. We will denote it by $\mathcal{NS}^3$.

Finally, a probability distribution (5.2.1) is quantum if

$$P(a,b,c|x,y,z) = \langle\psi|E^1_{a,x} \otimes E^2_{b,y} \otimes E^3_{c,z}|\psi\rangle, \quad (5.2.4)$$

where $(E^i_{\alpha,\beta})_{\alpha,\beta}$ is a family of measurements for the ith-party and $|\psi\rangle$ is a tripartite pure quantum state. We will denote the set of probability distributions of this form by $\mathcal{Q}^3$.

There are several known relations among these sets. For instance,

$$\mathcal{L}^3 \subsetneq \mathcal{Q}^3 \subsetneq \mathcal{NS}^3 \quad \text{and} \quad \mathcal{L}^3 \subset \mathcal{BL}^3_\mathcal{NS} \subset \mathcal{BL}^3_\mathcal{G}.$$

Svetlichny's result states that $\mathcal{Q}^3 \not\subseteq \mathcal{BL}$. Notice that it also holds that $\mathcal{BL}^3_\mathcal{NS} \not\subseteq \mathcal{Q}^3$.

In the multipartite scenario, a linear functional M is characterized by real numbers $\{M^{abc}_{xyz}\}$ acting on the set of tripartite joint probability distributions by:

$$\langle M, P\rangle = \sum_{a,b,c}\sum_{x,y,z} M^{abc}_{xyz} P(a,b,c|x,y,z).$$

The 3-prover one-round games are particular Bell functionals, whose coefficients are of the form:

$$G^{abc}_{xyz} = \pi(x,y,z)V^{abc}_{xyz}, \quad (5.2.5)$$

where $(\pi(x,y,z))_{x,y,z}$ is a probability distribution and V is a predicate function taking values one or zero. Note that, in particular, G has non-negative coefficients. Games describe a setting where each of the players is asked a certain question according to the probability distribution π and must answer a certain output, being the condition of winning the game that the questions and answers verify $V^{abc}_{xyz} = 1$. In this context, the quantity $\omega_\mathcal{A}(G) = \sup_{P\in\mathcal{A}}\langle G, P\rangle$ represents the winning probability of the game if the players are restricted to the use of strategies defined by the set $\mathcal{A}$.

Nevertheless, for the purpose of this work, we will consider a slightly more general definition for games and we will treat them as functionals G with non negative coefficients such that

$$\sum_{x,y,z} \max_{a,b,c} G^{abc}_{xyz} \leq 1 \quad \text{(normalization condition)}. \quad (5.2.6)$$

5.3 Correlation scenario

The correlation scenario arises when all outputs are binary, i.e. $\{-1,1\}$ and, instead of considering the full joint probability distribution (5.2.1), only expectations over the product of the outputs are considered. Then, we define the correlation associated to a probability distribution (5.2.1),

$$\gamma = (\gamma_{xyz})_{x,y,z} \in \mathbb{R}^{N^3},$$

by

$$\begin{aligned}\gamma_{xyz} &= \mathbb{E}[abc|x,y,z] = \sum_{a,b,c} abcP(a,b,c|x,y,z) \\ &= P(a\cdot b\cdot c{=}1|x,y,z) - P(a\cdot b\cdot c{=}{-1}|x,y,z).\end{aligned}$$

It is well known that a given correlation γ is in $\mathcal{L}^3_{cor}$ if and only if

$$\gamma \in co\{(a_x b_y c_z)_{x,y,z} : a_x, b_y, c_z = \pm 1, x,y,z = 1,\cdots,N\},$$

or, equivalently,

$$\|\gamma\|_{\ell_\infty^N \otimes_\pi \ell_\infty^N \otimes_\pi \ell_\infty^N} \leq 1.$$

On the other hand, γ is in $\mathcal{Q}^3_{cor}$ if and only if

$$\gamma_{xyz} = \langle\psi|A_x \otimes B_y \otimes C_z|\psi\rangle \quad \text{for every } x,y,z,$$

where A_x, B_y and C_z are norm-one selfadjoint operators for every x, y and z and $|\psi\rangle$ is a tripartite pure quantum state. Contrary to the bipartite case, a Banach space norm is no longer useful to characterize tripartite quantum correlations. Nonetheless, they can be related with a norm on $\ell_\infty^N \otimes \ell_\infty^N \otimes \ell_\infty^N$ which must be defined in the category of operator spaces.

Returning briefly to the multipartite case, a characterization of the set $\mathcal{NS}^k_{cor}$ can be done by the following lemma (see Lemma 3.2.2 for the proof in the case $k=2$):

Lemma 5.3.1. *Given a correlation γ, it is in $\mathcal{NS}^k_{cor}$ if and only if $\|\gamma\|_{\ell_\infty^N \otimes_\epsilon \cdots \otimes_\epsilon \ell_\infty^N} \leq 1$ if and only if $|\gamma_{x_1\ldots x_k}| \leq 1$ for every $x_1,\ldots,x_k$.*

Proof. The equivalence between $\|\gamma\|_{\ell_\infty^N \otimes_\epsilon \cdots \otimes_\epsilon \ell_\infty^N} \leq 1$ and $|\gamma_{x_1\ldots x_k}| \leq 1$ for every $x_1,\ldots,x_k$ follows from the multipartite extension of Remark 3.2.1.

Proving that $|\gamma_{x_1\ldots x_k}|$ is less than or equal to 1 if it is in $\mathcal{NS}^k_{cor}$ follows from the definition. For the other implication, given a correlation $\gamma_{x_1\ldots x_k}$ satisfying $|\gamma_{x_1\ldots x_k}| \leq 1$ for every $x_1,\ldots,x_k$, consider the probability distribution defined as:

$$P(a_1,\ldots,a_k|x_1,\ldots,x_k) = \begin{cases} \frac{1+\gamma_{x_1\ldots x_k}}{2^k} & \text{if} \quad a_1a_2\ldots a_k = 1, \\ \frac{1-\gamma_{x_1\ldots x_k}}{2^k} & \text{if} \quad a_1a_2\ldots a_k = -1. \end{cases}$$

This probability distribution can be easily seen to be in $\mathcal{NS}^k(N,2)$ as a consequence that for all i we have

$$\begin{aligned}&\sum_{a_i} P(a_1,\ldots,a_k|x_1,\ldots,x_k) \\ &= P(a_1,\ldots,\underbrace{1}_{i},\ldots,a_k|x_1,\ldots,x_k) + P(a_1,\ldots,\underbrace{-1}_{i},\ldots,a_k|x_1,\ldots,x_k) \\ &= \frac{1\pm\gamma_{x_1\ldots x_k}}{2^k} + \frac{1\mp\gamma_{x_1\ldots x_k}}{2^k} = \frac{1}{2^{k-1}}.\end{aligned}$$

□

In particular, notice that Lemma 5.3.1 implies that correlations associated to non-signalling probability distributions are the same as correlations associated to general probability distributions. So we have

$$\mathcal{BL}^k_{cor,\mathcal{NS}} = \mathcal{BL}^k_{cor,\mathcal{G}}.$$

Let us then just denote $\mathcal{BL}^k_{cor}$ in this case.

Now we are going to characterize correlations associated to bilocal probability distributions.

Proposition 5.3.2. *A correlation $(\gamma_{xyz})_{x,y,z}$ is in $\mathcal{BL}^3_{cor}$ if and only if*

$$\gamma \in co\big\{(\alpha_{xy}c_z)_{x,y,z},\ (\beta_{yz}a_x)_{x,y,z},\ (\gamma_{xz}b_y)_{x,y,z}\big\},$$

where $(\alpha_{xy})_{x,y}$, $(\beta_{yz})_{y,z}$, $(\gamma_{xz})_{x,z}$ are elements in $\mathcal{NS}^2_{cor}$ and $a_x, b_y, c_z = \pm 1$ for every x, y, z.

Proof. An extremal probability distribution P of the set $\mathcal{BL}^3_{\mathcal{G}}(N, 2)$ will have one of the following forms:

$$(Q(a,b|x,y)R(c|z))^{a,b,c}_{x,y,z},\ (Q(b,c|y,z)R(a|x)))^{a,b,c}_{x,y,z},\ (Q(a,c|x,z)R(b|y)))^{a,b,c}_{x,y,z}, \tag{5.3.1}$$

where in all cases Q and R are general probability distributions. Let us assume that this extreme point has the form $P = (Q(a,b|x,y)R(c|z))^{a,b,c}_{x,y,z}$ and denote β and α the corresponding correlations from the probability distributions $(Q(a,b|x,y))^{a,b}_{x,y}$ and $(R(c|z))^{c}_{z}$, respectively. Then, given x, y, z, we have

$$\begin{aligned}\gamma_{xyz} &= \mathbb{E}[a\cdot b\cdot c|x,y,z] = \sum_{a,b,c} abcP(a,b,c|x,y,z) = \sum_{a,b,c} abcQ(a,b|x,y)R(c|z)\\ &= \Big(\sum_{a,b} abQ(a,b|x,y)\Big)\Big(\sum_{c} cR(c|z)\Big) = \mathbb{E}[a\cdot b|x,y]\mathbb{E}[c|z] = \beta_{xy}\alpha_z.\end{aligned}$$

By definition, $(\beta_{xy})_{xy}$ is in $\mathcal{NS}^2_{cor}$ whenever Q is in $\mathcal{NS}^2$ and, clearly, $|\mathbb{E}[c|z]| \leq 1$. Since the other two cases in (5.3.1) are completely analogous, the result follows by convexity. □

In order to characterize the bilocal correlations, $\mathcal{BL}^3_{cor}$, in terms of a norm we define the following sets:

$$\begin{aligned}\mathcal{L}^2_{1,cor} &= co\{(\alpha_{xy}c_z)_{x,y,z} : \alpha_{xy} = \pm 1, c_z = \pm 1, \text{ for all } x, y, z\},\\ \mathcal{L}^2_{2,cor} &= co\{(\alpha_{yz}c_x)_{x,y,z} : \alpha_{yz} = \pm 1, c_x = \pm 1, \text{ for all } x, y, z\},\\ \mathcal{L}^2_{3,cor} &= co\{(\alpha_{zx}c_y)_{x,y,z} : \alpha_{zx} = \pm 1, c_y = \pm 1, \text{ for all } x, y, z\}.\end{aligned}$$

With the above definitions and Proposition 5.3.2 it easily follows that:

$$\mathcal{BL}^3_{cor} = co(\mathcal{L}^2_{1,cor} \cup \mathcal{L}^2_{2,cor} \cup \mathcal{L}^2_{3,cor}).$$

The relation of each $\mathcal{L}^2_{i,cor}$ with a the unit ball of a Banach space follows from Example 2.3.1. Thus, given $\gamma \in \mathbb{R}^{N^3}$, we have:

$$\begin{aligned}\gamma \in \mathcal{L}^2_{1,cor} &\text{ if and only if } \|\gamma\|_{\ell^{N^2}_\infty \otimes_\pi \ell^N_\infty} \leq 1,\\ \gamma \in \mathcal{L}^2_{2,cor} &\text{ if and only if } \|\gamma'\|_{\ell^{N^2}_\infty \otimes_\pi \ell^N_\infty} \leq 1,\\ \gamma \in \mathcal{L}^2_{3,cor} &\text{ if and only if } \|\gamma''\|_{\ell^{N^2}_\infty \otimes_\pi \ell^N_\infty} \leq 1.\end{aligned}$$

Here we have used that $\gamma' = \mathsf{swap}(\gamma)$ and $\gamma'' = \mathsf{swap}(\mathsf{swap}(\gamma))$ and swap is the linear map defined as:

$$\begin{aligned} \mathsf{swap} : \mathbb{R}^N \otimes \mathbb{R}^N \otimes \mathbb{R}^N &\longrightarrow \mathbb{R}^N \otimes \mathbb{R}^N \otimes \mathbb{R}^N \\ e_x \otimes e_y \otimes e_z &\longrightarrow e_y \otimes e_z \otimes e_x \end{aligned}$$

We name $\mathsf{L}_{i,cor}$ to the Banach space such that the unit ball is $\mathcal{L}_i^2$, for $i = 1, 2, 3$. Before giving the relation of $\mathcal{BL}_{cor}$ with a norm we need the next remark:

Remark 5.3.3. *Given* X *and* Y *Banach spaces from Equation (3.3.1), it is easy to see that:*

$$\mathsf{B}_{\mathsf{X}\cap\mathsf{Y}} = \mathsf{B}_\mathsf{X} \cap \mathsf{B}_\mathsf{Y} \quad \textit{and} \quad \mathsf{B}_{\mathsf{X}+\mathsf{Y}} = co(\mathsf{B}_\mathsf{X} \cup \mathsf{B}_\mathsf{Y}).$$

Hence, we can conclude that $\mathsf{BL}_{cor} = \mathsf{L}_{1,cor} + \mathsf{L}_{2,cor} + \mathsf{L}_{3,cor}$ and $\mathsf{BL}^*_{cor} = \mathsf{L}^*_{1,cor} \cap \mathsf{L}^*_{2,cor} \cap \mathsf{L}^*_{3,cor}$. It trivially follows the next theorem:

Theorem 5.3.4. *Given* T *a tripartite Bell functional for correlations, then:*

$$\omega_{\mathcal{BL}_{cor}}(T) = \|T\|_{\mathsf{BL}^*_{cor}} = \max\{\|T\|_{\mathsf{L}^*_{1,cor}}, \|T\|_{\mathsf{L}^*_{2,cor}}, \|T\|_{\mathsf{L}^*_{3,cor}}\}.$$

We are now in position to address the main aim of this section: characterize the relative Bell violations for the quantum and bilocal sets in the correlation scenario. Before presenting these results, let us first briefly recall several known results in this direction. The result of Tsirelson ([74]) states that $LV(\mathcal{Q}^2_{cor}, \mathcal{L}^2_{cor}) \leq K_G$, for every number of inputs, where K_G denotes the real Grothendieck's constant. On the other hand, we have seen right after Equation (3.2.1) that $LV(\mathcal{NS}^2_{cor}, \mathcal{L}^2_{cor}) \leq \sqrt{2N}$ and that this upper bound is essentially optimal.

Regarding the tripartite scenario, in [48] the authors showed that the relative violation between the quantum and the fully local set turned out to be unbounded, answering an old question stated by Tsirelson ([74]). Later, the estimates proved therein were improved in [17], by showing that

$$LV\big(\mathcal{Q}^3_{cor}, \mathcal{L}^3_{cor}\big) \geq \mathcal{O}(N^{\frac{1}{4}}),$$

up to logarithmic factors. Moreover, it is known that this estimate is not far from being optimal, since the following inequality holds for every N (see [17]):

$$LV\big(\mathcal{Q}^3_{cor}, \mathcal{L}^3_{cor}\big) \leq C\sqrt{N},$$

where C is a universal constant.

In the following proposition, we show that Tsirelson's result already prevents $LV(\mathcal{Q}^3_{cor}, \mathcal{BL}^3_{cor})$ from being unbounded. We also analyze the ratio between the set of bilocal correlations and the sets of fully local and quantum correlations.

Proposition 5.3.5. *Given* N*, the following inequalities hold:*

1. $LV\big(\mathcal{Q}^3_{cor}(N), \mathcal{BL}^3_{cor}(N)\big) \leq K_G$.
2. $LV\big(\mathcal{BL}^3_{cor}(N), \mathcal{L}^3_{cor}(N)\big) \leq \sqrt{2N}$. *This implies* $LV\big(\mathcal{BL}^3_{cor}(N), \mathcal{Q}^3_{cor}(N)\big) \leq \sqrt{2N}$. *Moreover, the order* $\sqrt{N}$ *is optimal in these inequalities, since, in particular,*

$$LV\big(\mathcal{BL}^3_{cor}(N), \mathcal{Q}^3_{cor}(N)\big) \geq \sqrt{N}/(4K_G).$$

Proof. To prove (1) consider a general Bell functional $M = (M_{xyz})_{x,y,z=1}^N$. Then, for a given tripartite quantum correlation of the form

$$\gamma = \Big(\langle \psi | A_x \otimes B_y \otimes C_z | \psi \rangle \Big)_{x,y,z=1}^N ,$$

we have

$$\begin{aligned} \langle M, \gamma \rangle &= \Big| \sum_{x,y,z} M_{xyz} \langle \psi | A_x \otimes B_y \otimes C_z | \psi \rangle \Big| = \Big| \sum_{x,y,z} M_{xyz} \langle \psi | A_x \otimes D_{yz} | \psi \rangle \Big| \\ &\leq K_G \sup_{a_x = \pm 1, b_{yz} = \pm 1} \Big| \sum_{x,y,z} M_{xyz} a_x b_{yz} \Big| \leq K_G \sup_{\delta \in \mathcal{BL}_{cor}^3(N)} |\langle M, \delta \rangle|, \end{aligned}$$

where in the second equality we have denoted $D_{yz} = B_y \otimes C_z$, which is a self adjoint norm one operator, in the first inequality we have used Equation (2.3.2) and in the last inequality we have used Proposition 5.3.2.

To prove (2) consider a generic extremal strategy for the set $\mathcal{BL}_{cor}^3(N)$, which we will assume, without loss of generality, that is of the form

$$\gamma = (\gamma_{xy}^{\mathcal{G}} \gamma_z)_{x,y,z=1}^N ,$$

where $\gamma^{\mathcal{G}} = (\gamma_{xy}^{\mathcal{G}})_{xy} \in \mathcal{NS}_{cor}^2(N)$. Then,

$$\begin{aligned} \langle M, \gamma \rangle &= \Big| \sum_{x,y,z} M_{xyz} \gamma_{xy}^{\mathcal{G}} \gamma_z \Big| = \sqrt{2N} \Big| \sum_{x,y,z} M_{xyz} \Big(\frac{\gamma_{xy}^{\mathcal{G}}}{\sqrt{2N}} \Big) \gamma_z \Big| \\ &= \sqrt{2N} \Big| \sum_{x,y,z} M_{xyz} \widetilde{\gamma}_{xy}^{\mathcal{C}} \gamma_z \Big| \leq \sqrt{2N} \sup_{\mathcal{L}_{cor}^3(N)} |\langle M, \gamma \rangle|. \end{aligned}$$

Here, we have that $(\widetilde{\gamma}_{xy}^{\mathcal{C}})_{x,y} := \Big(\frac{\gamma_{xy}^{\mathcal{G}}}{\sqrt{2N}} \Big)_{x,y} \in \mathcal{L}_{cor}^2$ and so $\Big(\widetilde{\gamma}_{xy}^{\mathcal{C}} \gamma_z \Big)_{x,y,z} \in \mathcal{L}_{cor}^3$. This follows from Equation (3.2.1) and proves the first inequality in (2).

An easy form to prove (2), considering norms, follows by:

$$\| id : \ell_1^{N^2} \otimes_\epsilon \ell_1^{N^2} \longrightarrow \ell_1^N \otimes_\epsilon \ell_1^N \otimes_\epsilon \ell_1^N \| \leq \| id : \ell_1^{N^2} \longrightarrow \ell_1^N \otimes_\epsilon \ell_1^N \| \| id : \ell_1^N \longrightarrow \ell_1^N \| \leq \sqrt{2N},$$

where we have used the metric mapping property for the ϵ norm, $\| id_{\ell_1^N} \| = 1$, and

$$\| \mathsf{swap}(T) \|_{\ell_1^N \otimes_\epsilon \ell_1^N \otimes_\epsilon \ell_1^N} = \| T \|_{\ell_1^N \otimes_\epsilon \ell_1^N \otimes_\epsilon \ell_1^N}$$

because of the commutativity of the ϵ-norm.

The second inequality in (2) is straighfordward from the first one and the fact that $\mathcal{L}_{cor}^3(N) \subset \mathcal{Q}_{cor}^3(N)$.

Finally, let us show the optimality of the order $\sqrt{N}$. To this end consider n such that $N/2 < 2^n \leq N$ and let $H_{2^n} = (h_{xy})_{x,1=1}^{2^n}$ be a Hadamard matrix (see Remark 3.2.3). Then, define the Bell functional $M = (M_{xyz})$ as $M_{xyz} = h_{xy}$ for $1 \leq x, y \leq 2^n$, $z = 1$ and $M_{xyz} = 0$ otherwise. We will study the values $\omega_{\mathcal{BL}_{cor}^3(N)}(M)$ and $\omega_{\mathcal{Q}_{cor}^3(N)}(M)$.

The element $\gamma = (\gamma_{xyz})_{xyz}$ defined by $\gamma_{xyz} = M_{xyz}$ for all x, y, z is clearly in $\mathcal{BL}_{cor}^3(N)$ (since $|\gamma_{xyz}| \leq 1$ for every x, y, z). Then,

$$\omega_{\mathcal{BL}_{cor}^3(N)}(M) \geq \Big| \sum_{x,y,z} M_{xyz} \gamma_{xyz} \Big| = \Big| \sum_{x,y=1}^{2^n} h_{xy}^2 \Big| = 2^{2n} > \frac{N^2}{4}.$$

On the other hand, for every quantum correlation $\gamma = \Big(\langle\psi|A_x \otimes B_y \otimes C_z|\psi\rangle\Big)_{x,y,z=1}^N$, we have

$$\Big|\sum_{x,y,z} M_{xyz}\gamma_{xyz}\Big| = \Big|\sum_{x,y} h_{xy}\langle\psi|A_x \otimes B_y \otimes C_1|\psi\rangle\Big| = \Big|\sum_{x,y} h_{xy}\langle u_x|v_y\rangle\Big|,$$

where we have defined $|u_x\rangle = A_x \otimes \mathbb{1} \otimes C_1|\psi\rangle$ and $|u_y\rangle = \mathbb{1} \otimes B_y \otimes \mathbb{1}|\psi\rangle$. We can now apply Equation (2.3.2) to upper bound the previous expression by

$$K_G \sup_{a_x,b_y=\pm 1} \Big|\sum_{x,y=1}^{2^n} h_{xy}a_x b_y\Big| \leq K_G(2^n)^{3/2} \leq K_G N^{3/2},$$

where the last inequality is proved in [56, Ex. 29].

Since the previous estimate holds for all quantum correlations, the upper bound $\omega_{\mathcal{Q}^3_{cor}(N)}(M) \leq K_G N^{3/2}$ follows. Hence, we deduce

$$LV\big(\mathcal{BL}^3_{cor}(N), \mathcal{Q}^3_{cor}(N)\big) \geq \frac{\sqrt{N}}{4K_G}.$$

□

5.4 General probability distributions

The previous section motivates the question of whether we can obtain unbounded violations of tripartite quantum probability distributions over bilocal probability distributions. As we have seen, this is impossible in the setting of correlations and we want to investigate here if

$$\lim_{\substack{N\to\infty \\ K\to\infty}} LV\big(\mathcal{Q}^3(N,K), \mathcal{BL}_{\mathcal{A}}(N,K)\big) = \infty, \text{ for } \mathcal{A} = \mathcal{NS} \text{ or } \mathcal{G},$$

holds. Note that the previous question is well posed since it is well known that the set of tripartite quantum probability distributions is contained in the affine hull of the set of tripartite fully local probability distributions. We will show that this is in fact true in the strongest case, $\mathcal{A} = \mathcal{G}$, using 3-prover one-round games, which are particular Bell functionals.

5.4.1 Upper bounds

In the tripartite scenario that we are considering we can establish three sets of bipartite classical probability distributions:

Definition 5.4.1. *Define the sets $\mathcal{L}^2_1$, $\mathcal{L}^2_2$ and $\mathcal{L}^2_3$ as:*

$$\begin{aligned}
\mathcal{L}^2_1 &= co\{(P(a,b|x,y)Q(c|z))_{x,a,y,b,z,c} : P \in \mathcal{P}^2, Q \in \mathcal{P}^1\},\\
\mathcal{L}^2_2 &= co\{(P(b,c|y,z)Q(a|x))_{x,a,y,b,z,c} : P \in \mathcal{P}^2, Q \in \mathcal{P}^1\},\\
\mathcal{L}^2_3 &= co\{(P(c,a|z,x)Q(b|y))_{x,a,y,b,z,c} : P \in \mathcal{P}^2, Q \in \mathcal{P}^1\}.
\end{aligned}$$

With these last definitions and Definition 5.2.3 it is easy to see that:

$$\mathcal{BL}^3_{\mathcal{G}} = co(\mathcal{L}^2_1 \cup \mathcal{L}^2_2 \cup \mathcal{L}^2_3).$$

Note that those 3 sets, $\mathcal{L}^2_1$, $\mathcal{L}^2_2$ and $\mathcal{L}^2_3$, are bilocal classical probability distributions defined over the three-partite probability distributions. Therefore, using Example 2.4.6 we can relate them with a norm. Let $P^\dagger = \mathsf{switch}(P)$ and $P^{\dagger\dagger} = \mathsf{switch}(\mathsf{switch}(P))$ where we define the switch operator as:

$$\begin{aligned}\mathsf{switch}: \mathbb{R}^N \otimes \mathbb{R}^K \otimes \mathbb{R}^N \otimes \mathbb{R}^K \otimes \mathbb{R}^N \otimes \mathbb{R}^K &\longrightarrow \mathbb{R}^N \otimes \mathbb{R}^K \otimes \mathbb{R}^N \otimes \mathbb{R}^K \otimes \mathbb{R}^N \otimes \mathbb{R}^K \\ e_x \otimes e_a \otimes e_y \otimes e_b \otimes e_z \otimes e_c &\longrightarrow e_y \otimes e_b \otimes e_z \otimes e_c \otimes e_x \otimes e_a.\end{aligned}$$

And also define the Banach spaces L_1, L_2 and L_3 as $\mathbb{R}^{N^3K^3}$ joint with the following norms:

$$\begin{aligned}\|P\|_{\mathsf{L}_1} &= \|P\|_{\ell_\infty^{N^2}(\ell_1^{K^2})\otimes_\pi \ell_\infty^N(\ell_1^K)},\\ \|P\|_{\mathsf{L}_2} &= \|P^\dagger\|_{\ell_\infty^{N^2}(\ell_1^{K^2})\otimes_\pi \ell_\infty^N(\ell_1^K)},\\ \|P\|_{\mathsf{L}_3} &= \|P^{\dagger\dagger}\|_{\ell_\infty^{N^2}(\ell_1^{K^2})\otimes_\pi \ell_\infty^N(\ell_1^K)}.\end{aligned}$$

It is then immediate to see that the relation between $\mathcal{L}_i^2$ and L_i for $i = 1, 2, 3$ is:

$$\omega_{\mathcal{L}_i^2}(M) = \|M\|_{(\mathsf{L}_i^2)^*},$$

for any given Bell functional M.

Define $\mathsf{BL} = \mathsf{L}_1 + \mathsf{L}_2 + \mathsf{L}_3$. We want to see that indeed BL is the Banach space associated with the set of bilocal probability distributions $\mathcal{BL}_\mathcal{G}^3$.

Theorem 5.4.2. *The norm related with the Banach space* BL *fulfills Conditions 2.4.1 and 2.4.2.*

Proof. To prove the first condition notice that, using Example 2.4.6, we know that $\mathcal{L}_i^2 \subset \mathsf{B}_{\mathsf{L}_i}$ for $i = 1, 2, 3$. Then,

$$\mathcal{BL}_\mathcal{G}^3 = co(\mathcal{L}_1^2 \cup \mathcal{L}_2^2 \cup \mathcal{L}_3^2) \subset co(\mathsf{B}_{\mathsf{L}_1} \cup \mathsf{B}_{\mathsf{L}_2} \cup \mathsf{B}_{\mathsf{L}_3}) = \mathsf{B}_{\mathsf{BL}}$$

For the second condition, take $P \in \mathsf{B}_{\mathsf{BL}}$. Then there exist $Q_i \in \mathsf{L}_i$ for $i = 1, 2, 3$ such that $P = \sum_i Q_i$ and $\|P\|_{\mathsf{BL}} = \sum_i \|Q_i\|_{\mathsf{L}_i} \le 1$. We apply Condition 2.4.2 to each $Q_i/\|Q_i\|_{\mathsf{L}_i}$ to obtain $\widetilde{Q}_i \in \mathcal{L}_i^2$, such that $|Q_i| \le \|Q_i\|_{\mathsf{L}_i}\widetilde{Q}_i$. Setting,

$$P' = \frac{1}{\|P\|_{\mathsf{BL}}}\Big(\sum_i \|Q_i\|_{\mathsf{L}_i}\widetilde{Q}_i\Big),$$

the result follows by noticing that $P' \in \mathcal{BL}_\mathcal{G}^3$ and

$$|P| = |\sum_i Q_i| \le \sum_i |Q_i| \le \sum_i \|Q_i\|_{\mathsf{L}_i}\widetilde{Q}_i \le \frac{1}{\|P\|_{\mathsf{BL}}}\Big(\sum_i \|Q_i\|_{\mathsf{L}_i}\widetilde{Q}_i\Big).$$

□

Corollary 5.4.3. *With the definitions above, given a tripartite Bell functional M,*

$$\omega_{\mathcal{BL}_\mathcal{G}^3}(M) = \max\{\omega_{\mathcal{L}_1^2}(M), \omega_{\mathcal{L}_2^2}(M), \omega_{\mathcal{L}_3^2}(M)\}.$$

Moreover, given a tripartite Bell functional G with non-negative entries:

$$\omega_{\mathcal{BL}_\mathcal{G}^3}(G) = \|G\|_{\mathsf{BL}^*} = \max\{\|G\|_{\ell_1^{N^2}(\ell_\infty^{K^2})\otimes_\epsilon \ell_1^N(\ell_\infty^K)}, \|G^\dagger\|_{\ell_1^{N^2}(\ell_\infty^{K^2})\otimes_\epsilon \ell_1^N(\ell_\infty^K)}, \|G^{\dagger\dagger}\|_{\ell_1^{N^2}(\ell_\infty^{K^2})\otimes_\epsilon \ell_1^N(\ell_\infty^K)}\}.$$

In order to analyze the quantum case, consider the following sets inside the tripartite probability distributions.

Definition 5.4.4. *Define $\mathcal{Q}_1^2$ as those bipartite probability distributions in the tripartite scenario $(P(a,b,c|x,y,z))_{x,y,z}^{a,b,c}$ such that they can be written as:*

$$P(a,b,c|x,y,z) = \langle \psi | \Pi_{x,y}^{a,b} \otimes \Lambda_z^c | \psi \rangle,$$

where $|\psi\rangle$ is a vector in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\{\Pi_{x,y}^{a,b}\}_{a,b}$ and $\{\Lambda_z^c\}_c$ are POVM for all x, y and z. Similarly, define $\mathcal{Q}_i^2$ changing the indexes according to Definition 5.4.1.

Lemma 5.4.5. *Given a tripartite game G, if we denote by $\omega_{\mathcal{Q}_d^3}(G)$ the quantum value of G when at least one of the player is restricted to local dimension d, then*

$$\omega_{\mathcal{Q}_d^3}(G) \leq d\, \omega_{\mathcal{BL}_{\mathcal{G}}^3}(G).$$

Proof of Lemma 5.4.5. The proof can be obtained from a slight modification of the comments below [63, Proposition 5.2]. Indeed, let us fix a quantum distribution P which is defined by a quantum state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^n \otimes \mathbb{C}^m$ and some POVMs $\{\Pi_x^a\}_a$, $\{\Lambda_y^b\}_b$ and $\{\Upsilon_z^c\}_c$ acting on $\mathbb{C}^d$, $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, for every x, y, z. Then, from the Schmidt decomposition, we deduce that the state $|\psi\rangle$ can be written as

$$|\psi\rangle = \sum_{i=1}^{d} \lambda_i |f_i\rangle |g_i\rangle,$$

where $\sum_i |\lambda_i|^2 = 1$, and $|f_i\rangle$ and $|g_i\rangle$ are orthonormal systems in the unit ball of $\mathbb{C}^d$ and $\mathbb{C}^{nm}$ respectively. Then, we have

$$\begin{aligned} |\langle G, P\rangle| &= \Big| \sum_{x,y,z,a,b,c} G_{xyz}^{abc} \langle \psi | \Pi_x^a \otimes \Lambda_y^b \otimes \Upsilon_z^c |\psi\rangle \Big| \\ &\leq \sum_{i,j} |\lambda_i||\lambda_j| \sum_{x,y,z,a,b,c} G_{xyz}^{abc} |\langle f_i|\Pi_x^a|f_j\rangle| |\langle g_i|\Lambda_y^b \otimes \Upsilon_z^c|g_j\rangle| \\ &\leq d \max_{i,j} \sum_{x,y,z,a,b,c} G_{xyz}^{abc} |\langle f_i|\Pi_x^a|f_j\rangle| |\langle g_i|\Lambda_y^b \otimes \Upsilon_z^c|g_j\rangle|, \end{aligned}$$

where we have used the well known fact $\sum_{i=1}^d |\lambda_i| \leq \sqrt{d}\Big(\sum_{i=1}^d |\lambda_i|^2\Big)^{\frac{1}{2}}$.

Now, as it is shown in the comments below [63, Proposition 5.2], Cauchy-Schwarz inequality implies that for every i and j, and for every x, y and z, we have

$$\sum_a |\langle f_i|\Pi_x^a|f_j\rangle| \leq 1 \quad \text{and} \quad \sum_{b,c} |\langle g_i|\Lambda_y^b \otimes \Upsilon_z^c|g_j\rangle| \leq 1.$$

Hence, we deduce that $\langle G, P\rangle \leq d\, \omega_{\mathcal{BL}_{\mathcal{G}}^3}(G)$, which concludes the proof. □

Remark 5.4.6. *An easy modification of the last proof, separating positive and negative terms, makes the result hold for general coefficients at the expenses of paying a constant 2.*

We will distinguish the different inputs and outputs that Alice, Bob and Charlie receive by naming them N_1, N_2, N_3 and K_1, K_2, K_3, respectively.

Theorem 5.4.7. *The following holds:*

$$LV^+(\mathcal{Q}_d^3, \mathcal{BL}_{\mathcal{G}}^3) \leq \min\{d, N_1, N_2, N_3, K_1, K_2, K_3\},$$

and also:

$$LV(\mathcal{Q}_d^3, \mathcal{BL}_{\mathcal{G}}^3) \leq C \min\{d, N_1, N_2, N_3, \sqrt{K_1 K_2 K_3}\},$$

where C is a universal constant.

Proof. Notice that for every Bell functional M, $\omega_{\mathcal{Q}^3}(G) \leq \omega_{\mathcal{Q}_i^2}(G)$ for $i = 1, 2, 3$.

Given a tripartite Bell functional with non-negative coefficients G, denote the sum modulo 3 of the set $\{1, 2, 3\}$ by $\oplus$, thus:

$$\begin{aligned}\omega_{\mathcal{Q}^3}(G) &\leq \min_{i=1,2,3} \omega_{\mathcal{Q}_i^2}(G) \leq \min_{i=1,2,3} \min\{N_i, K_i, N_{i\oplus 1}N_{i\oplus 2}, K_{i\oplus 1}K_{i\oplus 2}\}\omega_{\mathcal{L}_i^2}(G) \\ &\leq \min\{N_1, N_2, N_3, K_1, K_2, K_3\}\omega_{\mathcal{BL}}(G).\end{aligned}$$

In the second inequality we have used Equation (2.4.1).

Given a tripartite Bell functional M, thus:

$$\begin{aligned}\omega_{\mathcal{Q}^3}(M) &\leq \min_{i=1,2,3} \omega_{\mathcal{Q}_i^2}(M) \leq C \min_{i=1,2,3} \min\{N_i, N_{i\oplus 1}N_{i\oplus 2}, \sqrt{K_1K_2K_3}\}\omega_{\mathcal{L}_i^2}(M) \\ &\leq C \min_{i=1,2,3} \min\{N_1, N_2, N_3, \sqrt{K_1K_2K_3}\}\omega_{\mathcal{BL}}(M).\end{aligned}$$

In the second inequality we have used Equation (2.4.2).

Using this with Lemma 5.4.5 and the remark below, the result follows. □

5.4.2 Lower bounds

Let us first recall that, given a bipartite game $G = (G_{xy}^{ab})_{x,y,a,b}$ with N inputs and K outputs per party, we denote by $G^{\otimes_2}$ the bipartite game with N^2 inputs and K^2 outputs per party, whose coefficients are:

$$G_{x_1x_2y_1y_2}^{a_1a_2b_1b_2} = G_{x_1y_1}^{a_1b_1}G_{x_2y_2}^{a_2b_2}.$$

That is, Alice's and Bob's inputs are (x_1, x_2) and (y_1, y_2), respectively, and Alice's and Bob's outputs are (a_1, a_2) and (b_1, b_2), respectively. This means that the parties are playing two instances of the game simultaneously. Studying the classical value $G^{\otimes_2}$ for a given game G is the core of the results about parallel repetition theorems, which are of great relevance in computer science.

To make the following result more intuitive, let us explain that our aim is to define a tripartite game $\widetilde{G}$ from a bipartite one G. The first (and somehow easiest) construction we considered was based on two instances of the bipartite game, one for Alice and Bob and the other for Bob and Charlie. In this situation, Alice receives input x and she outputs a, Bob receives (y_1, y_2) and outputs (b_1, b_2) (the first is for the game he is playing with Alice and the second, with Charlie) and, finally, Charlie receives input z and outputs c. Then the coefficients have the form:

$$\widetilde{G}_{xy_1y_2z}^{ab_1b_2c} = G_{xy_1}^{ab_1}G_{y_2z}^{b_2c}.$$

In order to find an example such that it does not only give unbounded violations between $\mathcal{Q}^3$ and $\mathcal{BL}_{\mathcal{G}}^3$, but it is also optimal in some parameters, we will present here another construction using three instead of two instances of the game. More precisely, one instance of the game will be asked to Alice and Bob, another to Bob and Charlie and another to Charlie and Alice. Hence, the coefficients of the new game will have the following form:

$$\hat{G}_{x_1x_2y_1y_2z_1z_2}^{a_1a_2b_1b_2c_1c_2} = G_{x_1y_1}^{a_1b_1}G_{x_2z_1}^{a_2c_1}G_{y_2z_2}^{b_2c_2}.$$

We are going to obtain a Bell inequality that is optimal in two of the three parameters.

Theorem 5.4.8. *Let G be a bipartite game with N inputs and K outputs per party. Then, the construction of the paragraph above leads to a tripartite game $\hat{G}$ with N^2 inputs and K^2 outputs per player, such that*

$$\frac{\omega_{\mathcal{Q}^3}(\hat{G})}{\omega_{\mathcal{BL}^3_G}(\hat{G})} \geq \frac{\omega_{\mathcal{Q}^2}(G)^3}{\omega_{\mathcal{L}^2}(G^{\otimes 2})}. \tag{5.4.1}$$

Moreover, if $\omega_{\mathcal{Q}^2}(G)$ is attained with local dimension d, then Equation (5.4.1) is attained with local dimension d^2.

Proof. To show that $\omega_{\mathcal{Q}^3}(\hat{G}) \geq \omega_{\mathcal{Q}^2}(G)^3$, first note that there must exist a quantum strategy which uses a quantum state $|\phi\rangle$ in some Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ and POVMs $\{\Pi_x^a\}_{a=1}^n$ and $\{\Lambda_y^b\}_{b=1}^n$ acting on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ in such a way that[2]:

$$\Big|\sum_{x,y,a,b} G_{xy}^{ab} \langle\phi|\Pi_x^a \otimes \Lambda_y^b|\phi\rangle\Big| = \omega_{\mathcal{Q}^2}(G).$$

Then we can consider the tripartite quantum state $|\psi\rangle = |\phi\rangle|\phi\rangle|\phi\rangle \in (\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes (\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes (\mathcal{H}_1 \otimes \mathcal{H}_2)$ and we can define the operators $E_{x_1,x_2}^{a_1,a_2} = \Pi_{x_1}^{a_1} \otimes \Pi_{x_2}^{a_2}$, $F_{y_1,y_2}^{b_1,b_2} = \Lambda_{y_1}^{b_1} \otimes \Pi_{y_2}^{b_2}$ and $G_{z_1,z_2}^{c_1,c_2} = \Lambda_{z_1}^{c_1} \otimes \Lambda_{z_2}^{c_2}$. It is clear that $\{E_{x_1,x_2}^{a_1,a_2}\}_{a_1,a_2}$, $\{F_{y_1,y_2}^{b_1,b_2}\}_{b_1,b_2}$ and $\{G_{z_1,z_2}^{c_1,c_2}\}_{c_1,c_2}$ are POVMs for all x_1, x_2, y_1, y_2, z_1, z_2. Moreover,

$$\begin{aligned}
\omega_{\mathcal{Q}^3}(\hat{G}) &\geq \sum \hat{G}_{x_1x_2y_1y_2z_1z_2}^{a_1a_2b_1b_2c_1c_2} \langle\psi|E_{x_1,x_2}^{a_1,a_2} \otimes F_{y_1,y_2}^{b_1,b_2} \otimes G_{z_1,z_2}^{c_1,c_2}|\psi\rangle \\
&= \Big(\sum_{x_1,y_1,a_1,b_1} G_{x_1y_1}^{a_1b_1} \langle\phi|\Pi_{x_1}^{a_1} \otimes \Lambda_{y_1}^{b_1}|\phi\rangle\Big) \times \Big(\sum_{x_2,z_1,a_2,c_1} G_{x_2z_1}^{a_2c_1} \langle\phi|\Pi_{x_2}^{a_2} \otimes \Lambda_{z_1}^{c_1}|\phi\rangle\Big) \\
&\times \Big(\sum_{y_2,z_2,b_2,c_2} G_{y_2z_2}^{b_2c_2} \langle\phi|\Pi_{y_2}^{b_2} \otimes \Lambda_{z_2}^{c_2}|\phi\rangle\Big) = \omega_{\mathcal{Q}^2}(G)^3,
\end{aligned}$$

where the first sum runs over all indices.

Note also that, if we assume $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = d$, then, by construction, the local dimension of the quantum state $|\psi\rangle$ is d^2.

In order to prove the corresponding upper bound for the classical value, let us consider a bilocal probability distribution P of the form

$$\big(P_1(a_1, a_2, b_1, b_2|x_1, x_2, y_1, y_2)P_2(c_1, c_2|z_1, z_2)\big)_{x_1,x_2,y_1,y_2,z_1,z_2}^{a_1,a_2,b_1,b_2,c_1,c_2}$$

and the other two cases will follow by symmetry.

First of all, notice that, given a certain positive pointwise element $(f(a_2, b_2, x_2, y_2))_{a_2,b_2,x_2,y_2}$ such that $\sum_{a_2,b_2} f(a_2, b_2, x_2, y_2) \leq 1$ for all x_2 and y_2, we can find a probability distribution $\widetilde{P}$ for which all its components are greater than or equal to those of f by defining:

$$\widetilde{P}(a_2, b_2|x_2, y_2) = \begin{cases} f(a_2, b_2, x_2, y_2) & \text{if } 1 \leq a_2, b_2 \leq K,\ (a_2, b_2) \neq (K, K), \\ 1 - \sum_{(a_2', b_2') \neq (K,K)} f(a_2', b_2', x_2, y_2) & \text{if } a_2 = b_2 = K. \end{cases}$$

[2] Although the value $\omega_{\mathcal{Q}^2}(G)$ could be not attained, we can find a quantum strategy up to arbitrarily high precision. We avoid writing inequalities up to ϵ.

Then, using the upper bound for the classical value of $G^{\otimes_2}$, we can write

$$\begin{aligned}
&\sum_{x_2,z_1,a_2,c_1,y_2,z_2,b_2,c_2} G^{a_2c_1}_{x_2z_1} G^{b_2c_2}_{y_2z_2} f(a_2,b_2,x_2,y_2) P(c_1,c_2|z_1,z_2) \\
&\leq \sum_{x_2,z_1,a_2,c_1,y_2,z_2,b_2,c_2} G^{a_2c_1}_{x_2z_1} G^{b_2c_2}_{y_2z_2} \widetilde{P}(a_2,b_2|x_2,y_2) P(c_1,c_2|z_1,z_2) \leq \omega_{\mathcal{L}^2}(G^{\otimes_2}).
\end{aligned} \tag{5.4.2}$$

Hence, we have

$$\begin{aligned}
\langle \hat{G}, P\rangle &= \sum G^{a_1b_1}_{x_1y_1} G^{a_2c_1}_{x_2z_1} G^{b_2c_2}_{y_2z_2} P(a_1,a_2,b_1,b_2|x_1,x_2,y_1,y_2) P(c_1,c_2|z_1,z_2) \\
&= \sum_{x_2,z_1,a_2,c_1,y_2,z_2,b_2,c_2} G^{a_2c_1}_{x_2z_1} G^{b_2c_2}_{y_2z_2} \Big(\sum_{x_1,y_1,a_1,b_1} G^{a_1b_1}_{x_1y_1} P(a_1,a_2,b_1,b_2|x_1,x_2,y_1,y_2) \Big) P(c_1,c_2|z_1,z_2) \\
&= \sum_{x_2,z_1,a_2,c_1,y_2,z_2,b_2,c_2} G^{a_2c_1}_{x_2z_1} G^{b_2c_2}_{y_2z_2} f(a_2,b_2,x_2,y_2) P(c_1,c_2|z_1,z_2) \leq \omega_{\mathcal{L}^2}(G^{\otimes_2}),
\end{aligned} \tag{5.4.3}$$

where the first sum runs over all indices, we have defined

$$f(a_2,b_2,x_2,y_2) = \sum_{x_1,y_1,a_1,b_1} G^{a_1b_1}_{x_1y_1} P(a_1,a_2,b_1,b_2|x_1,x_2,y_1,y_2)$$

and the last inequality in Equation (5.4.3) follows from Equation (5.4.2) and the fact that $\sum_{a_2,b_2} f(a_2,b_2,x_2,y_2) \leq 1$ for all x_2 and y_2. To show this last claim, fix x_2 and y_2, and write

$$\begin{aligned}
&\sum_{a_2,b_2} \sum_{x_1,y_1,a_1,b_1} G^{a_1b_1}_{x_1y_1} P(a_1,a_2,b_1,b_2|x_1,x_2,y_1,y_2) \\
&= \sum_{x_1,y_1,a_1,b_1} G^{a_1b_1}_{x_1y_1} \sum_{a_2,b_2} P(a_1,a_2,b_1,b_2|x_1,x_2,y_1,y_2) \\
&\leq \sum_{x_1,y_1} \max_{a_1,b_1} G^{a_1b_1}_{x_1y_1} \sum_{a_1,a_2,b_1,b_2,} P(a_1,a_2,b_1,b_2|x_1,x_2,y_1,y_2) \\
&= \sum_{x_1,y_1} \max_{a_1,b_1} G^{a_1b_1}_{x_1y_1} \leq 1,
\end{aligned}$$

because of Equation (5.2.6). □

There are two interesting applications of the previous theorem. The first one comes from the application to pseudotelepathy games. That is, those bipartite games which can be won perfectly with quantum strategies but not with classical ones (as it is, for instance, the magic square game [27]). As a consequence, our construction leads to the existence of pseudotelepathy against bilocality.

Corollary 5.4.9. *Let G be a pseudotelepathy game. Applying the construction of Theorem 5.4.8 we obtain a tripartite game $\hat{G}$ such that $\omega_{\mathcal{Q}^3}(\hat{G}) = 1$ and $\omega_{\mathcal{BL}^3_{\mathcal{G}}}(\hat{G}) < 1$.*

The second, and more important, application is to obtain an unbounded violation between tripartite quantum and bilocal probability distributions. For that purpose we will use the *Khot-Vishnoi game*, G_{KV} or KV game [54], which we briefly explain here. For any $n = 2^l$ with $l \in \mathbb{N}$ and every $\eta \in [0, \frac{1}{2}]$ we consider the group $\{0,1\}^n$ and the Hadamard subgroup H. Then, we consider the quotient group $G = \{0,1\}^n/H$ which is formed by $\frac{2^n}{n}$ cosets $[x]$ each with n elements. The questions of the games (x,y) are associated to the cosets whereas the answers a and b are indexed in $[n]$. The referee chooses a uniformly random coset $[x]$ and one element $z \in \{0,1\}^n$ according to the probability distribution $pr(z(i)=1)=\eta$, $pr((z(i)=0)=1-\eta$ independently of i. Then, the referee asks question $[x]$ to Alice and question $[x \oplus z]$ to Bob. Alice and Bob must answer one element of their corresponding cosets and they win the game if and only if $a \oplus b = z$. Although the KV game is not a two-prover one-round game in the sense of Equation (5.2.5), it is very easy to see that it verifies the normalization condition given in Equation (5.2.6).

Hence, the Khot-Vishnoi game has $N = 2^n/n$ inputs and $K = n$ outputs per player and it can be proved ([23, Theorem 7]) that

$$\omega_{\mathcal{L}^2}(G_{KV}) \leq C/n \quad \text{and} \quad \omega_{\mathcal{Q}^2}(G_{KV}) \geq D/\ln^2 n, \tag{5.4.4}$$

for some universal constants C and D.

The next lemma is necessary in order to apply the Khot-Vishnoi to Theorem 5.4.8 and it essentially shows that the classical value of the game is multiplicative.

Lemma 5.4.10. *Let G_{KV} be the Khot-Vishnoi game. Then,*

$$\omega_{\mathcal{L}^2}(G_{KV}^{\otimes_2}) \leq C\frac{1}{n^2},$$

where C is a universal constant.

Proof. The proof of this result follows exactly the same steps as in the proof of [23, Theorem 4.1]. As it is explained there, a deterministic strategy (which corresponds to an extremal classical probability distribution) can be identified with a couple of Boolean functions $A, B : \{0,1\}^n \to \{0,1\}$ such that each of them verifies that, restricted to each coset $[x]$ (see explanation of the game right before this lemma), takes value one for one of the elements and zero for the rest. Then, the winning probability of the game can be written as

$$n\mathbb{E}_u\mathbb{E}_z[A(u)B(u \oplus z)],$$

where u is sampled uniformly at random in $\{0,1\}^n$ and $z \in \{0,1\}^n$ is sampled pointwise independently according to the probability distribution $pr(z(i)=1)=\eta$, $pr((z(i)=0)=1-\eta$. We fix here $\eta = 1/2 - 1/\log n$. Then, Cauchy-Schwarz inequality followed by a use of the hypercontractive inequality lead to the classical upper bound stated in Equation (5.4.4).

In the case of $G_{KV}^{\otimes_2}$, a deterministic strategy can be identified with a couple of Boolean functions $A, B : \{0,1\}^{2n} = \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ such that each of them verifies that, restricted to each pair $[x] \times [y]$, takes value one for one of the elements and zero for the rest. Then, the key point is that sampling $u = (u_1, u_2)$ so that u_i is sampled uniformly in $\{0,1\}^n$ for $i = 1, 2$ is the same as sampling u uniformly in $\{0,1\}^{2n}$. At the same time, since z is sampled pointwise independently, we can sample in the form $z = (z_1, z_2)$ where $z_i \in \{0,1\}^n$ is sampled as in the single game for $i = 1, 2$. Then, the winning probability of the game can be written as

$$n^2\mathbb{E}_{u_1,u_2}\mathbb{E}_{z_1,z_2}[A(u_1,u_2)B((u_1,u_2) \oplus (z_1,z_2))] = n^2\mathbb{E}_u\mathbb{E}_z[A(u)B(u \oplus z)].$$

Then, doing the same computations as in the proof of [23, Theorem 4.1] we obtain the bound $n^2\left(\frac{1}{n^2}\right)^{1/(1-\eta)} \leq C/n^2$. This concludes the proof. □

Corollary 5.4.11. *The KV game leads to a tripartite game $\hat{G}$ with $(2^n/n)^2$ inputs and n^2 outputs per player, such that*

$$\frac{\omega_{\mathcal{Q}^3}(\hat{G})}{\omega_{\mathcal{BL}^3_{\mathcal{G}}}(\hat{G})} \geq C\frac{n^2}{\ln^6 n}, \tag{5.4.5}$$

and the quantum lower bound in the previous equation is attained with a quantum state of local dimension n^2.

Moreover, this estimate is essentially optimal in the number of outputs and in the local dimension of the Hilbert space.

5.5 Conclusions

In this work we have extended the study of relative Bell violations of quantum resources over local and fully local ones to the genuinely multipartite scenario by comparing the power of quantum strategies over bilocal models. We have considered first the correlation scenario. It is shown in Proposition 5.3.5 that, as in the bipartite case, the ratio of Bell violation of quantum probability distributions over bilocal ones is upper bounded by Grothendieck's constant for any number of inputs and, hence, there cannot be unbounded Bell violations. Since not all bilocal correlations are reproducible by quantum models, we have also investigated the relative power of the former over the latter. We have shown that this ratio is upper bounded by $O(\sqrt{N})$ and that this order is optimal.

Next, we have considered the case of general probability distributions. Contrary to the previous case, we have obtained here that quantum strategies lead to unbounded Bell violations over general bilocal probability distributions. In order to do so, we have proved in Theorem 5.4.8 that if one considers tensor products of bipartite games the ratio of the quantum value over the general bilocal value is related to the ratio of the quantum and local values for the bipartite game. This has allowed us to prove, in Corollary 5.4.9, the existence of pseudotelepathy again bilocality, and also it has allowed us to establish, in Corollary 5.4.11 by considering explicit games such as the Khot-Vishnoi game, that there exist games for which the ratio of the quantum and general bilocal values grows unboundedly with the number of inputs and outputs. We moreover have proven that for a particular choice of games the given estimate of the asymptotic behaviour of this ratio is essentially optimal in the number of outputs and in the dimension of the Hilbert space. The problem with this construction is that it was necessary the use of exponentially many inputs to obtain the separation. We leave as an open question to make a construction that uses only polynomially many inputs with respect to the separation.

It should be noticed that the two results about the ratio of Bell violations of quantum probability distributions over bilocal ones – boundedness in the correlation setting and unboundedness in the general case – hold irrespectively of whether we consider general bilocal or non-signalling bilocal models. In the first case, this holds because both sets of models happen to coincide in the correlation scenario, as discussed in Sec. 5.3. In the second case, unboudedness with respect to general bilocal probability distributions automatically implies the same with respect to non-signalling bilocal probability distributions due to the fact that this latter set is included in the former. Thus, our result can also be understood as showing an unlimited advantage of GMNL quantum probability distributions irrespective of the underlying definition of bilocality. As mentioned in the introduction, the correlations contained in general bilocal models might be so strong that lead to undesirable unphysical effects in certain scenarios and this has motivated to consider more constrained hybrid models. Despite this fact, not only general bilocal models are unable to simulate all quantum probability distributions as proven by Svetlichny in [72], but our results show that quantum-mechanical resources can be, in a certain sense, unboundedly better than this strongest form of bilocality.

Appendix A

Postulates of Quantum mechanics

In this section we want to explain bipartite quantum probability distributions given in Definition 1.1.2 using the four postulates of quantum mechanics, which provide a mathematical framework. For a more complete description see [61]:

1. **System and states**. Any isolated physical system is described by a complex Hilbert space $\mathcal{H}$. In this work we are going to assume this space to be finite dimensional, thus there exists d such that $\mathcal{H} = \mathbb{C}^d$. A physical state is represented by a unit vector $|\psi\rangle$ in this space.

   Quantum mechanics also allows for the description of states which are not completely known by the use of density operators. They are semidefinite positive operators with trace 1 acting on the state space of the system. A density operator of the form $\sum_i p_i \rho_i$ means that the quantum system is in state ρ_i with probability p_i.

2. **Evolution**. The evolution of a closed quantum system is described by a *unitary transformation*. That is, the state $|\psi\rangle$ of the system at time t_1 is related to the state $|\psi'\rangle$ of the system at time t_2 by a unitary operator U that depends only on t_1 and t_2 by $|\psi'\rangle = U|\psi\rangle$. Similarly, for density operators we can relate ρ at time t_1 and ρ' at time t_2 by $\rho' = U\rho U^\dagger$.

3. **Quantum measurements**. They are described by a family of *measurement operators* $\{M_n\}_n$ acting on the Hilbert space $\mathcal{H}$. They are indexed on the possible outcomes of the experiment. If the physical systems is in the state $|\psi\rangle$ (respectively ρ), then the probability of observing the result m is given by

   $$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle \quad \text{(respectively} \quad p(m) = \mathrm{tr}(M_m^\dagger M_m \rho)).$$

   While the post-measurement state is

   $$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}} \quad \text{(respectively} \quad \frac{M_m \rho M_m^\dagger}{\mathrm{tr}(M_m^\dagger M_m \rho)}).$$

   In this work we are interested in the "Positive Operator-Valued Measure" (POVM) formalism, which is especially well adapted to compute the outcome probabilities, with the counterpart that it is less clear how to find the post-measurement state. The set $\{E_m\}_m$ is known as POVM if and only if E_m is a semidefinite positive operator such that $\sum_m E_m = \mathbb{1}$ and $p(m) = \langle\psi|E_m|\psi\rangle$. The set of operators E_m are thus sufficient to determine the probabilities of the different measurement outcomes.

An example of POVM are the projective measurements P_m. These operators are orthogonal projectors such that $P_m P_{m'} = \delta_{m,m'} P_m$. In this case, all the POVM elements are the same as the measurement operators.

4. **Composite systems**. The composition of several physical system is given by the tensor product of them. Given N systems, $\mathcal{H}_1, \ldots, \mathcal{H}_N$, each of them in the state $|\psi_i\rangle \in \mathcal{H}_i$, then the global state is $|\psi_1\rangle \otimes \ldots \otimes |\psi_N\rangle \in \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N = \mathcal{H}$. Similarly, if system i is prepared in the state ρ_i, then the joint state of the total system is $\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_n$.

Now consider the scenario where Alice and Bob share a quantum state. By Postulate 1 we can say that this state is represented by a unit vector $|\psi\rangle$ on a Hilbert space $\mathcal{H}$. As Alice and Bob are physically separated and each of them has a quantum device, we can say that the Hilbert space $\mathcal{H}$ is the composition of two physical systems, $\mathcal{H}_A$ and $\mathcal{H}_B$, which corresponds to Alice and Bob, respectively. Thus, by Postulate 4, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

Alice and Bob perform measurements with their quantum device. To describe the possible outcomes of the experiment we make use of the POVM formalism described in Postulate 3. As each of them receive an input x and y, then they can perform different measurements according to the input they have received. Thus, two sets of POVM are defined, $(E_x^a)_a \subset B(\mathcal{H}_A)$ for all x and $(F_y^b)_{y,b} \subset B(\mathcal{H}_B)$ for all y.

The POVM formalism gives the probability of obtaining a certain outcome. Then, the probability of obtaining outputs a and b given inputs x and y is given by:

$$P(a,b|x,y) = \langle\psi|E_x^a \otimes F_y^b|\psi\rangle \quad \text{for every } x, y, a, b,$$

Similarly, we can describe this probability using a density operator ρ by:

$$P(a,b|x,y) = \mathrm{tr}(E_x^a \otimes F_y^b \rho) \quad \text{for every } x, y, a, b.$$

This definition can be extended to the multipartite setting.

Appendix B

Non-signalling norm is not a tensor norm

We start this section by noticing that the non-signalling norm describes the non-signalling probability distributions. The most natural space in which we can consider bipartite probability distributions is $\ell_\infty^N(\ell_1^K) \otimes \ell_\infty^N(\ell_1^K)$. We are going to check that the non-signalling norm is not a tensor in this space.

We recall here the definition of crossnorm and tensor norm.

Given X and Y Banach spaces, we say that a norm α in $\mathsf{X} \otimes \mathsf{Y}$ is a cross norm if and only if it has the following properties:

- $\|x \otimes y\|_{\mathsf{X} \otimes_\alpha \mathsf{Y}} \leq \|x\|_{\mathsf{X}} \|y\|_{\mathsf{Y}}$ for every $x \in \mathsf{X}$ and $y \in \mathsf{Y}$.
- For every $\psi \in \mathsf{X}^*$ and $\phi \in \mathsf{Y}^*$, the linear functional $\psi \otimes \phi$ on $\mathsf{X} \otimes \mathsf{Y}$ is bounded and $\|\psi \otimes \phi\| \leq \|\psi\|_{\mathsf{X}^*} \|\phi\|_{\mathsf{Y}^*}$.

Moreover, α is a tensor norm if it also fulfills the metric mapping property. This property says that for all normed spaces W and Z and for all linear maps $T : \mathsf{X} \to \mathsf{W}$, $S : \mathsf{Y} \to \mathsf{Z}$, we have

$$\|T \otimes S : \mathsf{X} \otimes_\alpha \mathsf{Y} \to \mathsf{W} \otimes_\alpha \mathsf{Z}\| = \|T\| \|S\|.$$

Proposition B.0.1. *The norm corresponding to* NS *is a cross norm in the space* $\ell_\infty^N(\ell_1^K) \otimes \ell_\infty^N(\ell_1^K)$.

Proof. In order to prove this we will check the two conditions that any crossnorm should fulfill. Given two tensors $P = (P_{xa})_{xa}$ and $Q = (Q_{yb})_{yb}$ in $\ell_\infty^N(\ell_1^K)$, suppose w.l.o.g. that $\|P \otimes Q\|_{\mathsf{NS}} = \|P \otimes Q\|_{\mathsf{NS1}}$, then

$$\begin{aligned} \|P \otimes Q\|_{\mathsf{NS}} = \|P \otimes Q\|_{\mathsf{NS1}} &= \max_x \sum_a \max_y \sum_b |P(a,x)Q(b,y)| \\ &= \max_x \sum_a |P(a,x)| \max_y \sum_b |Q(b,y)| = \|P\|_{\ell_\infty^N(\ell_1^K)} \|Q\|_{\ell_\infty^N(\ell_1^K)}. \end{aligned}$$

Similarly, we can consider two tensors $M = (M_{xa})_{xa}$ and $N = (N_{yb})_{yb}$ in $\ell_1^N(\ell_\infty^K)$. Then

$$\|M\otimes N\|_{\mathsf{NS}^*}\leq\|M\otimes N\|_{\mathsf{NS1}^*}=\sum_x\max_a\sum_y\max_b|M(a,x)N(y,b)|$$
$$=\sum_x\max_a|M(a,x)|\sum_y\max_b|N(b,y)|=\|M\|_{\ell_1^N(\ell_\infty^K)}\|N\|_{\ell_1^N(\ell_\infty^K)}.$$

□

Proposition B.0.2. *The norm corresponding to* NS *is not a tensor norm in the space* $\ell_\infty^N(\ell_1^K)\otimes\ell_\infty^N(\ell_1^K)$ *since it does not fulfill the metric mapping property.*

Proof. To prove that the norm $\|\cdot\|_{\mathsf{NS}}$ does not satisfy the metric mapping property in the space $\ell_\infty^N(\ell_1^K)\otimes\ell_\infty^N(\ell_1^K)$ we have to find some $T,L\in\mathcal{L}(\ell_\infty^N(\ell_1^K))$ and P such that:

$$\|(T\otimes L)P\|_{\mathsf{NS}}>\|T\|\|L\|\|P\|_{\mathsf{NS}}.$$

To this purpose, let $H_{2^n}=(h_{xy})_{x,y=1}^{2^n}$ be a Hadamard matrix (see Remark 3.2.3), which has the property $H_{2^n}H_{2^n}^T=2^n\mathbb{1}$. Note that we are considering here $N=K=2^n$.

On the one side, we choose L to be the identity, $L=id$, hence $\|L\|=1$. On the other side, T will be defined as:

$$T(e_x\otimes e_a)=e_a\otimes\sum_k h_{kx}e_k.$$

We can redefine it as $T=(i\otimes H_N)\circ X$ where $i:\ell_1^N\to\ell_\infty^N$ is the formal inclusion, $H_N:\ell_\infty^N\to\ell_1^N$ is given by $H_N(e_i)=\sum_j h_{ji}e_j$ and $X:\ell_\infty^N\otimes_\epsilon\ell_1^N\to\ell_1^N\otimes_\epsilon\ell_\infty^N$, by $X(e_x\otimes e_a)=e_a\otimes e_x$.

All the elements have norm 1 but $\|H_N\|$. Using [56, Ex. 29] we can say that

$$\|H_N\|=\|H_N\|_{\ell_1^N\otimes_\epsilon\ell_1^N}\leq N^{3/2}.$$

Now fix $1\leq a_0,b_0\leq N$, and consider the element:

$$P=\sum_{x,y}h_{yx}e_x\otimes e_{a_0}\otimes e_y\otimes e_{b_0}\in\ell_\infty^N(\ell_1^N)\otimes\ell_\infty^N(\ell_1^N).$$

This element has the following non-signalling norm:

$$\|P\|_{\mathsf{NS}}=\max_x\sum_a\max_y\sum_b|P(a,b|x,y)|=\max_{x,y}|h_{yx}|=1.$$

Then we can compute:

$$(T\otimes L)P=\sum_{x,y}h_{yx}T(e_x\otimes e_{a_0})\otimes e_y\otimes e_{b_0}=\sum_{y,k}\left(\sum_x h_{yx}h_{kx}\right)e_{a_0}\otimes e_k\otimes e_y\otimes e_{b_0}.$$

Putting it in a different way, $(T\otimes L)P=\tilde{P}$ with $\tilde{P}(k,b|a,y)=\sum_x h_{yx}h_{kx}$ for all k and y if and only if $a=a_0$, $b=b_0$ and 0 else.

$$\|\tilde{P}\|_{\mathsf{NS}}\geq\max_a\sum_k\max_y\sum_b|\tilde{P}(k,b|a,y)|\geq\sum_k\max_y\sum_b|\tilde{P}(k,b|a_0,y)|\geq\sum_k\sum_b|\tilde{P}(k,b|a_0,k)|$$
$$=\sum_k|\tilde{P}(k,b_0|a_0,k)|=\sum_k|\sum_x h_{kx}^2|=\sum_k N=N^2.$$

This gives us a difference of $\sqrt{N}$ between the two quantities and therefore the metric mapping property does not hold. $\square$

We conclude stating that no tensor norm in the space $\ell_\infty^N(\ell_1^K) \otimes \ell_\infty^N(\ell_1^K)$ can describe the non-signalling set $\mathcal{NS}$.

Appendix C

Alternative algorithm for SNOS

In this chapter we give an alternative algorithm that is necessary to prove Proposition 3.3.4.

Lemma C.0.1. *Given $P \in \mathbb{R}_+^{N^2K^2}$, we can find $\tilde{P}$ such that $P \leq \tilde{P}$, $\|P\|_{\mathsf{NS}} = \|\tilde{P}\|_{\mathsf{NS}}$ and $\tilde{P}$ fulfills Equations (3.1.1) and (3.1.2).*

Suppose, w.l.o.g., that $\|P\|_{\mathsf{NS}} = \sum_{a,b} P(a,b|x_0,y_a)$. Then run the following algorithm:

- $\tilde{P} = P$
- $x = x_0$
  - For $a = 1 : K$
    - For $y = 1 : N$ and $y \neq y_a$
      - For $b = 1 : K - 1$

$$\lambda = \min\left(\overbrace{\sum_{b'} \tilde{P}(a,b'|x,y_a) - \sum_{b'} \tilde{P}(a,b'|x,y)}^{\text{A}},\right.$$

$$\left.\underbrace{\|P\|_{\mathsf{NS}} - \left(\sum_{b'\neq b} \max_{x'} \sum_{a'} \tilde{P}(a',b'|x',y) + \sum_{a'} \tilde{P}(a',b|x,y)\right)}_{\text{B}}\right)$$

$$\tilde{P}(a,b|x,y) = \tilde{P}(a,b|x,y) + \lambda$$

      - $b = K$

$$\lambda = \sum_{b'} \tilde{P}(a,b'|x,y_a) - \sum_{b'} \tilde{P}(a,b'|x,y)$$

$$\tilde{P}(a,b|x,y) = \tilde{P}(a,b|x,y) + \lambda$$

- For $y = 1 : N$
  - For $b = 1 : K$
    - For $x = 1 : N$ and $x \neq x_0$

- For $a = 1 : K-1$

$$\lambda = \min\left(\sum_{a'} \tilde{P}(a',b|x_0,y) - \sum_{a'} \tilde{P}(a',b|x,y),\right.$$
$$\left.\|P\|_{\mathsf{NS}} - \left(\sum_{a'\neq a} \max_{y'} \sum_{b'} \tilde{P}(a',b'|x,y') + \sum_{b'} \tilde{P}(a,b'|x,y)\right)\right)$$
$$\tilde{P}(a,b|x,y) = \tilde{P}(a,b|x,y) + \lambda$$

- $a = K$

$$\lambda = \sum_{a'} \tilde{P}(a',b|x,y_a) - \sum_{a'} \tilde{P}(a',b|x,y)$$
$$\tilde{P}(a,b|x,y) = \tilde{P}(a,b|x,y) + \lambda$$

As we see, the algorithm is divided in two parts. The first part modifies all the elements for the case $x = x_0$ and in the second part, all the rest of the elements. We will analyze here why does it work.

In the first part of the algorithm, the fact that $\lambda \geq 0$ comes from the definition of the norm and from the assumption of the terms in which it is reached:

$$\|P\|_{\mathsf{NS}} = \max_x \sum_a \max_y \sum_b P(a,b,x,y) = \sum_{a,b} P(a,b,x_0,y_a) = \sum_{a,b} \tilde{P}(a,b,x_0,y_a)$$

That implies $\sum_b \tilde{P}(a,b,x_0,y_a) \geq \sum_b \tilde{P}(a,b,x_0,y)$ for all a and y, as x_0 stays fixed, and also, $\|P\|_{\mathsf{NS}} \geq \max_y \sum_b \max_x \sum_a \tilde{P}(a,b,x,y) \geq \sum_{b\neq b'} \max_x \sum_a \tilde{P}(a,b|x,y') + \sum_{a'} \tilde{P}(a,b'|x_0,y')$ for a given y', b', and x_0.

In addition, the aforementioned makes that after each iteration the norm remains the same, at least for the cases $b = 1 : K-1$.

The definition of λ for the case $b = K$ makes sure that $\sum_b \tilde{P}(a,b,x_0,y) = \sum_b P(a,b,x_0,y_a)$. In fact, this condition will make that, after the first part of the algorithm is over, $\tilde{P}$ will fulfill Equation (3.1.2) for x_0. To prove that after this last step the norm has not increased, consider the following. Arriving to the case $b = K$ and happening that $\lambda \neq 0$ implies that for $b = 1 : K-1$, $\lambda = B$, which in particular means:

$$\|P\|_{\mathsf{NS}} = \sum_{b'\neq b} \max_{x'} \sum_{a'} \tilde{P}(a',b'|x',y) + \sum_{a'} \tilde{P}(a',b|x_0,y)$$

As a consequence $\max_x \sum_a \tilde{P}(a,b,x,y) = \sum_a \tilde{P}(a,b,x_0,y)$ and then:

$$\sum_{b=1}^{K-1} \max_x \sum_a \tilde{P}(a,b,x,y) + \sum_a \tilde{P}(a,K,x_0,y) = \sum_{b=1}^{K-1} \sum_a \tilde{P}(a,b,x_0,y) + \sum_a \tilde{P}(a,K,x_0,y)$$
$$= \sum_{a,b} \tilde{P}(a,b,x_0,y) \leq \sum_a \max_y \sum_b \tilde{P}(a,b,x_0,y) = \sum_{a,b} P(a,b,x_0,y_a) = \|P\|_{\mathsf{NS}}$$

Finally, the last step of the algorithm $a = b = K$ makes sure that $\|P\|_{\mathsf{NS}} = \sum_{a,b} \tilde{P}(a,b,x_0,y)$. We will take advantage of this situation in the second part of the algorithm. But before going into that, consider that we had the case where: $\|P\|_{\mathsf{NS}} = \sum_{a,b} P(a,b|x_0,y_a) = \sum_{a,b} P(a,b|x_b,y_0)$, for certain x_b, y_a, x_0, y_0. After applying the first part of the algorithm, we can make sure that $\|P\|_{\mathsf{NS}} = \sum_{a,b} P(a,b|x_0,y_a) = \sum_{a,b} \tilde{P}(a,b,x_0,y_0)$ and $\sum_b \tilde{P}(a,b|x_0,y_0) = \sum_b P(a,b|x_0,y_a)$, which corresponds with Equation (3.1.2). But also $\sum_{a,b} P(a,b|x_b,y_0) = \sum_{a,b} \tilde{P}(a,b,x_0,y_0)$ with $\sum_a \tilde{P}(a,b|x_0,y_0) \leq \sum_a P(a,b|x_b,y_0)$ implies that the last expression is an equality which corresponds with Equation (3.1.1).

This argument shows how the algorithm, as it advances, finds a new element, greater in each component, respecting simultaneously the non-signalling conditions and the value of the norm.

The second part of the algorithm runs over all the other elements that haven't been modified yet. They are modified in a way which is exactly equal to the first part, but changing a for b and x for y, because now we can say that $\|P\|_{\mathsf{NS}} = \sum_{a,b} P(a,b|x_0,y)$ for all y. x_0 is doing the rol of y_a, but it is constant in b. As the behaviour of the algorithm is the same, when we finish running all the elements, we will get the element $\tilde{P}$ satisfying Equations (3.1.1) and (3.1.2) with $\|\tilde{P}\|_{\mathsf{NS}} = \|P\|_{\mathsf{NS}}$.

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