

GENERATORS OF C_0 -SEMIGROUPS OF WEIGHTED COMPOSITION OPERATORS

EVA A. GALLARDO-GUTIÉRREZ, ARISTOMENIS G. SISKAKIS, AND DMITRY YAKUBOVICH

ABSTRACT. We prove that in a large class of Banach spaces of analytic functions in the unit disc \mathbb{D} an (unbounded) operator $Af = G \cdot f' + g \cdot f$ with G, g analytic in \mathbb{D} generates a C_0 -semigroup of weighted composition operators if and only if it generates a C_0 -semigroup. Particular instances of such spaces are the classical Hardy spaces. Our result generalizes previous results in this context and it is related to cocycles of flows of analytic functions on Banach spaces. Likewise, for a large class of non-separable Banach spaces X of analytic functions in \mathbb{D} contained in the Bloch space, we prove that no non-trivial holomorphic flow induces a C_0 -semigroup of weighted composition operators on X . This generalizes previous results in [6] and [1] regarding C_0 -semigroup of (unweighted) composition operators.

1. INTRODUCTION

A natural question in the study of semigroups of operators is to characterize their generators, whenever they are defined. Recently, in [2] and [14] generators of semigroups of composition operators were characterized in the context of a large variety of Banach spaces of analytic functions in the unit disc \mathbb{D} . In particular, if an (unbounded) operator $Af = G \cdot f'$ on the classical Hardy space H^p , $1 \leq p < \infty$ generates a C_0 -semigroup, this is a semigroup of composition operators. See also the previous works [3], [4] or [9].

The study of semigroups of composition operators on various function spaces of analytic functions traces back to the pioneering work of Berkson and Porta [8] in the late seventies. As a particular instance, they gave a characterization of the generators of semigroups of composition operators acting on the classical Hardy spaces H^p induced by *holomorphic flows* of analytic self-maps of the unit disc \mathbb{D} showing, in turns, that these semigroups are always strongly continuous. Later on, Cowen [11] and Siskakis [26] found applications of this theory to the study of Cesàro and other averaging operators. Semigroups of composition operators acting on other holomorphic function spaces as

Date: October 12, 2021.

2010 Mathematics Subject Classification. 47B33 (primary), 30C45 (secondary).

Key words and phrases. Weighted composition operators, C_0 -semigroups, Cocycles of Holomorphic Flows.

First author is partially supported by Plan Nacional I+D grant no. PID2019-105979GB-I00, Spain.

Third author is partially supported by Spanish Ministry of Science, Innovation and Universities (grant no. PGC2018-099124-B-I00).

The first and the third author also acknowledge support from “Severo Ochoa Programme for Centres of Excellence in R&D” (CEX2019-000904-S) of the Ministry of Economy and Competitiveness of Spain and by the European Regional Development Fund and from the Spanish National Research Council, through the “Ayuda extraordinaria a Centros de Excelencia Severo Ochoa” (20205CEX001).

well as the inducing holomorphic flows have been extensively studied in the last three decades (we refer to the recent monograph [7] for more on flows).

Nevertheless, in what refers to semigroups of weighted composition operators the panorama differs completely. Seminal works in the context of Hardy spaces were [28] and [22] by the end of the eighties. More recently, in 2012, the authors of [20] studied which strongly continuous semigroups of operators on Banach spaces of analytic functions arise from holomorphic flows, where semigroups of weighted composition operators play an important role.

Our work has two main results, which in some sense, complement each other. The first one characterizes generators of strongly continuous semigroup (i.e. C_0 -semigroups) of weighted composition operators acting on a large class of Banach spaces X of analytic functions in the unit disc \mathbb{D} . More precisely, in Section 3 we will show that whenever the polynomials are dense in the Banach space X , an (unbounded) operator A acting on X like

$$Af = G \cdot f' + g \cdot f,$$

with G and g analytic in \mathbb{D} generates a C_0 -semigroup of weighted composition operators if and only if it generates a C_0 -semigroup. In order to prove it, we will make use of some of the ideas in [14], along with the concept of cocycles of flows of analytic functions on Banach spaces.

Our second result deals with classes of Banach spaces X where polynomials are no longer dense, namely, with spaces satisfying

$$H^\infty \subseteq X \subseteq \mathcal{B},$$

where \mathcal{B} denotes the Bloch space and H^∞ is the algebra of bounded analytic functions on \mathbb{D} .

In particular, in Section 4 we will prove that in such a case, a nontrivial holomorphic flow cannot give rise to a non-trivial weighted composition C_0 -semigroup. This extends previous results in [6] and [1] regarding composition operators C_0 -semigroups, that is, the unweighted case.

2. PRELIMINARIES

Recall that a one-parameter family $\Phi = \{\varphi_t\}_{t \geq 0}$ of analytic self-maps of \mathbb{D} is called a *holomorphic flow* (or *holomorphic semiflow* by some authors) provided that it is a continuous family that has a semigroup property with respect to composition, namely

- 1) $\varphi_0(z) = z, \forall z \in \mathbb{D}$;
- 2) $\varphi_{t+s}(z) = \varphi_t \circ \varphi_s(z), \forall t, s \geq 0, \forall z \in \mathbb{D}$;
- 3) For any $s \geq 0$ and any $z \in \mathbb{D}$, $\lim_{t \rightarrow s} \varphi_t(z) = \varphi_s(z)$,

(see [25], for instance). In the trivial case, $\varphi_t(z) = z$ for all $t \geq 0$. Otherwise, we say that Φ is nontrivial.

The infinitesimal generator of $\Phi = \{\varphi_t\}_{t \geq 0}$ is the function G , defined by

$$G(z) = \lim_{t \rightarrow 0^+} \frac{\varphi_t(z) - z}{t} = \frac{\partial \varphi_t(z)}{\partial t} \Big|_{t=0}.$$

This limit exists uniformly on compact subsets of \mathbb{D} and for each $z \in \mathbb{D}$, the mapping $w : t \rightarrow \varphi_t(z)$ satisfies the differential equation

$$\frac{d}{dt}w(t) = G(w(t))$$

with $w(0) = z$ (see [8]).

Associated to the holomorphic flow $\Phi = \{\varphi_t\}_{t \geq 0}$ is the family of composition operators $\{C_t\}_{t \geq 0}$, defined on the space of analytic functions on \mathbb{D} by

$$C_t f = f \circ \varphi_t.$$

Clearly, $\{C_t\}_{t \geq 0}$ has the semigroup property:

- 1) $C_0 = I$ (the identity operator);
- 2) $C_t C_s = C_{t+s}$ for all $t, s \geq 0$.

Moreover, recall that if an operator semigroup $\{T_t\}_{t \geq 0}$ acts on an (abstract) Banach space X , then it is called *strongly continuous* or C_0 -semigroup, if it satisfies

$$\lim_{t \rightarrow 0^+} T_t f = f$$

for any $f \in X$. As mentioned in the introduction, Berkson and Porta [8] proved that every holomorphic flow always induces a strongly continuous semigroup on the classical Hardy spaces H^p and extensions to other spaces of analytic functions have been extensively studied in the last years.

Given a C_0 -semigroup $\{T_t\}_{t \geq 0}$ on a Banach space X , recall that its generator is the closed and densely defined linear operator A , defined by

$$Af = \lim_{t \rightarrow 0^+} \frac{T_t f - f}{t}$$

with domain $\mathcal{D}(A) = \{f \in X : \lim_{t \rightarrow 0^+} \frac{T_t f - f}{t} \text{ exists}\}$. The semigroup is determined uniquely by its generator.

On the other hand, given a holomorphic flow $\Phi = \{\varphi_t\}_{t \geq 0}$ in \mathbb{D} , a multiplicative *cocycle* for Φ is a continuous complex-valued function $m : [0, +\infty) \times \mathbb{D} \rightarrow \mathbb{C}$ such that

- 1) $m(t, \cdot)$ is analytic on \mathbb{D} for all $t \geq 0$;
- 2) $m(0, z) = 1, \forall z \in \mathbb{D}$;
- 3) $m(t + s, z) = m(s, z) m(t, \varphi_s(z)), \forall t, s \geq 0, \forall z \in \mathbb{D}$.

For the sake of simplicity, we will denote $m_t(z) = m(t, z)$ for all $t \geq 0$ and $z \in \mathbb{D}$.

The third equation above is often called the *cocycle identity* and it implies that $m_0(z)$ is either 1 or 0, so the second equation is simply a non-triviality condition. König [22] investigated weighted holomorphic flows on the unit disc and provided a characterization of smooth cocycles in the Hardy space. In addition, he proved that the cocycle identity implies that m is not vanishing (see [22, Lemma 2.1]).

A particular instance of a cocycle for a holomorphic flow $\Phi = \{\varphi_t\}_{t \geq 0}$ is a *coboundary*, that is, a continuous complex-valued function m on $[0, +\infty) \times \mathbb{D}$ such that there exists holomorphic function α on \mathbb{D} , non-vanishing except possibly at the common fixed point of φ_t , satisfying

$$(2.1) \quad m(t, z) = \frac{\alpha(\varphi_t(z))}{\alpha(z)} \text{ for all } (t, z) \in [0, +\infty) \times \mathbb{D}.$$

Cocycles and, in particular, coboundaries arise naturally in the theory of weighted composition operators semigroups. Observe that given a holomorphic flow $\Phi = \{\varphi_t\}_{t \geq 0}$ and a cocycle m for Φ , the weighted composition operators $\{W_t : t \geq 0\}$ defined on the space of analytic functions on \mathbb{D} by

$$(2.2) \quad W_t f(z) = m_t(z)(C_t f)(z) = m_t(z)f(\varphi_t(z)), \quad (z \in \mathbb{D})$$

form a semigroup. Indeed the converse also holds. Namely, (2.2) defines a semigroup if and only if m is a cocycle for Φ . In addition, if m is a coboundary then (2.1) easily yields that

$$W_t = M_{\frac{1}{\alpha}} C_t M_\alpha$$

for every $t \geq 0$, where M_α is the operator of multiplication by $\alpha(z)$. This says that for every $t \geq 0$, W_t and C_t are similar as operators on the space of all analytic functions on \mathbb{D} .

Throughout the rest of the manuscript, X will be a Banach space of holomorphic functions on \mathbb{D} and $\mathcal{L}(X)$ will denote the space of bounded linear operators acting on X . The space consisting of all analytic functions on \mathbb{D} is denoted by $\text{Hol}(\mathbb{D})$ while $\mathcal{O}(\overline{\mathbb{D}})$ stands for the set of all functions, holomorphic on (a neighborhood of) the closed unit disc $\overline{\mathbb{D}}$ endowed with the usual topologies (regarding $\mathcal{O}(\overline{\mathbb{D}})$ we refer to [5, pp. 81], for instance). We will always assume the following natural condition:

$$(\star) \quad \mathcal{O}(\overline{\mathbb{D}}) \hookrightarrow X \hookrightarrow \text{Hol}(\mathbb{D}),$$

being both embeddings continuous.

In order to complete this preliminary section, we reproduce a theorem essentially contained in [22] for H^p spaces (see also [28] for the case of coboundaries). The proof works as well for Banach spaces X satisfying (\star) .

Theorem 2.1. *Let $\{W_t\}_{t \geq 0}$ be a weighted composition C_0 -semigroup acting on X defined by (2.2), and let G be the infinitesimal generator of $\{\varphi_t\}$. Then the following holds.*

(i) *The equation*

$$(2.3) \quad g(z) = \frac{\partial m_t(z)}{\partial t} \Big|_{t=0}, \quad \text{for } z \in \mathbb{D}$$

defines an analytic function in \mathbb{D} , and m_t is a cocycle for Φ . Moreover, m_t is expressed by

$$(2.4) \quad m_t(z) = \exp \left(\int_0^t g(\varphi_s(z)) ds \right), \quad t \geq 0, z \in \mathbb{D}.$$

(ii) *The generator A of the semigroup $\{W_t\}_{t \geq 0}$ and its domain are given by*

$$(2.5) \quad Af = G \cdot f' + g \cdot f, \quad \mathcal{D}(A) = \{f \in X : G \cdot f' + g \cdot f \in X\}.$$

Roughly speaking, our main result in Section 3 can be seen as the converse of this theorem.

Finally, we remark that whenever the cocycle m for Φ is *bounded*, that is, for every $t \geq 0$ the analytic function m_t is bounded on \mathbb{D} , the weighted composition operator W_t is a bounded operator on the Hardy spaces H^p for every $1 \leq p \leq \infty$. In the opposite direction, if g is analytic with $\operatorname{Re} g(z) < M < \infty$ on \mathbb{D} , then the above formulas induce a strongly continuous semigroup $\{W_t\}_{t \geq 0}$.

3. C_0 -SEMIGROUPS OF WEIGHTED COMPOSITION OPERATORS: GENERATORS

In this section we will prove that in a large class of Banach spaces X of analytic functions in the unit disc \mathbb{D} , including Hardy spaces H^p ($1 \leq p < \infty$), an (unbounded) operator A defined by (2.5) generates a C_0 -semigroup of weighted composition operators if and only if it generates a C_0 -semigroup of operators.

Theorem 3.1. *Suppose X is a Banach space of analytic functions on \mathbb{D} , satisfying (\star) .*

Let $\{S_t\}_{t \geq 0}$ be a C_0 -semigroup on X , whose generator A is defined by (2.5), where G, g are analytic functions in \mathbb{D} . Then there exists a holomorphic flow $\Phi = \{\varphi_t\}_{t \geq 0}$, whose infinitesimal generator is G and a cocycle m for Φ , satisfying (2.3) and (2.4), such that S_t is the weighted composition operator

$$S_t f(z) = m_t(z) f(\varphi_t(z))$$

for every $t \geq 0, z \in \mathbb{D}$ and $f \in X$.

In order to prove the result, as it was mentioned in the Introduction, we will make use of some ideas in [14]. Note that the natural assumption (\star) on X was also made in [14].

Proof. Assume the generator A of the C_0 -semigroup $\{S_t\}_{t \geq 0}$ is given by $Af = G \cdot f' + g \cdot f$ where $G, g \in \operatorname{Hol}(\mathbb{D})$. We begin by posing the Cauchy Problem as in [14]

$$(CP) \quad \begin{cases} \frac{\partial \varphi_t(z)}{\partial t} = G(\varphi_t(z)) \\ \varphi_0(z) = z \end{cases} \quad (z \in D(0, r) = \{z \in \mathbb{D} : |z| < r\}),$$

where $r \in (0, 1)$ is a fixed radius. Hence, upon applying the theory of ordinary differential equations in complex domain, there exists $t_0 > 0$ and an analytic solution $\{\varphi_t(z)\}$ of (CP), defined for $z \in D(0, r)$ and all complex t , $|t| < t_0$. In particular, for these values of z and t , $|\varphi_t(z)| < 1$.

Moreover, this solution is unique in the class of smooth functions and has the semigroup property:

$$(3.1) \quad \varphi_{t+s}(z) = \varphi_t \circ \varphi_s(z)$$

whenever $t, s, t+s \in (-t_0, t_0)$ and $z, \varphi_s(z) \in D(0, r)$.

Choose $r' \in (0, r)$ such that $\varphi_t(z) \in D(0, r)$ for $t \in (-t_0, t_0)$ and $z \in D(0, r')$ (here we may need to replace t_0 with a smaller positive number). Then

$$(3.2) \quad \varphi_{-t} \circ \varphi_t(z) = z \quad \text{if } t \in (-t_0, t_0) \text{ and } z \in D(0, r').$$

First, we prove the following:

Claim 1: For any $f \in \mathcal{D}(A)$

$$(3.3) \quad S_t f(z) = \exp \left(\int_0^t g(\varphi_s(z)) ds \right) (f \circ \varphi_t)(z), \quad z \in D(0, r), 0 \leq t < t_0.$$

Proof of Claim 1. By (CP) we have

$$\frac{\partial \varphi_{-t}(z)}{\partial t} = -G(\varphi_{-t}(z)),$$

for any $z \in D(0, r)$ and any $t \in (0, t_0)$. Let us fix $f \in \mathcal{D}(A)$ and denote by

$$f_t(z) = S_t f(z), \quad z \in \mathbb{D}, t \geq 0.$$

Calculating the derivative of $f_t \circ \varphi_{-t}(z)$ with respect to t we obtain

$$\begin{aligned} \frac{\partial (f_t \circ \varphi_{-t})(z)}{\partial t} &= \frac{\partial f_t}{\partial t}(\varphi_{-t}(z)) + \frac{\partial f_t}{\partial z}(\varphi_{-t}(z)) \frac{\partial \varphi_{-t}(z)}{\partial t} \\ &= (A f_t)(\varphi_{-t}(z)) - \frac{\partial f_t}{\partial z}(\varphi_{-t}(z)) G(\varphi_{-t}(z)) \\ &= G(\varphi_{-t}(z)) \frac{\partial f_t}{\partial z}(\varphi_{-t}(z)) + g(\varphi_{-t}(z)) f_t(\varphi_{-t}(z)) - \frac{\partial f_t}{\partial z}(\varphi_{-t}(z)) G(\varphi_{-t}(z)), \end{aligned}$$

where the second line follows since A is the generator of the semigroup, and hence $\frac{\partial f_t}{\partial t} = A f_t$. Accordingly,

$$(3.4) \quad \frac{\partial (f_t \circ \varphi_{-t})(z)}{\partial t} = g(\varphi_{-t}(z)) f_t(\varphi_{-t}(z)), \quad z \in D(0, r), 0 \leq t < t_0.$$

For $z \in D(0, r)$, let us write $Y(t) = (f_t \circ \varphi_{-t})(z)$ and $\psi(t) = g(\varphi_{-t}(z))$. Then (3.4) is the differential equation

$$Y'(t) = \psi(t) Y(t),$$

so

$$Y(t) \exp \left(- \int_0^t \psi(u) du \right) = C_z,$$

where C_z is a constant (depending on z). Thus, for each $z \in D(0, r)$

$$(f_t \circ \varphi_{-t})(z) = C_z \exp \left(\int_0^t g(\varphi_{-u}(z)) du \right).$$

For $t = 0$ we have $C_z = f_0(\varphi_0(z)) = f(z)$ and therefore

$$(3.5) \quad (f_t \circ \varphi_{-t})(z) = \exp \left(\int_0^t g(\varphi_{-u}(z)) du \right) f(z)$$

for $z \in D(0, r)$. Replacing z by $\varphi_t(z)$ in (3.5) we deduce by means of analyticity that

$$\begin{aligned} f_t(z) &= \exp \left(\int_0^t g(\varphi_{t-u}(z)) du \right) f(\varphi_t(z)) \\ &= \exp \left(\int_0^t g(\varphi_s(z)) ds \right) f(\varphi_t(z)), \end{aligned}$$

for all $z \in D(0, r)$. This proves Claim 1. \square

Now, let $f \in X$. There exists a sequence $\{f_n\} \subset \mathcal{D}(A)$ such that $f_n \rightarrow f$ in X as $n \rightarrow \infty$. Hence, $f_n \rightarrow f$ uniformly on compact subsets of \mathbb{D} as $n \rightarrow \infty$, and therefore

$$\lim_{n \rightarrow \infty} S_t f_n(z) = S_t f(z)$$

for each $z \in \mathbb{D}$ and $t \geq 0$. Accordingly, by Claim 1, for any $f \in X$

$$(3.6) \quad S_t f(z) = \exp \left(\int_0^t g(\varphi_s(z)) ds \right) (f \circ \varphi_t)(z), \quad z \in D(0, r), 0 \leq t < t_0.$$

Now, upon applying (3.6) to the constant function $\mathbb{1}(z) = 1$, we deduce

$$(3.7) \quad S_t(\mathbb{1})(z) = \exp \left(\int_0^t g(\varphi_s(z)) ds \right)$$

for $z \in D(0, r)$, $0 \leq t < t_0$. Define

$$(3.8) \quad m_t(z) := S_t(\mathbb{1})(z) \quad z \in \mathbb{D}, t \geq 0.$$

Clearly, this definition agrees with (3.7) if $0 \leq t < t_0$ and $|z| < r$. Therefore $m_t(z) \neq 0$ for $0 \leq t < t_0$ and $|z| < r$. Moreover, $m_t \in X$ and in particular, m_t is analytic on \mathbb{D} , for any $t \geq 0$.

Now apply (3.6) to the identity function id , $\text{id}(z) \equiv z$. We get

$$S_t(\text{id})(z) = \exp \left(\int_0^t g(\varphi_s(z)) ds \right) \varphi_t(z) \in X$$

for $z \in D(0, r)$ and $0 \leq t < t_0$. Therefore

$$(3.9) \quad \varphi_t(z) = \frac{1}{m_t(z)} S_t(\text{id})(z).$$

for every z , $|z| < r$ and $0 \leq t < t_0$. Hence for any such t , φ_t continues to a meromorphic function in \mathbb{D} .

Analogously, applying (3.6) to the functions z^n for $n \geq 2$, it follows for every $a \in \mathbb{D}$

$$(3.10) \quad (\varphi_t(a))^n = \frac{1}{m_t(z)} S_t(z^n)(a),$$

for $0 \leq t < t_0$. If for some $0 \leq t < t_0$, φ_t had a pole in a point $a_0 \in \mathbb{D}$, then φ_t^n would have at least n th order pole at a_0 for any $n \geq 1$, which contradicts (3.10). Therefore φ_t is analytic in \mathbb{D} for any t , $0 \leq t < t_0$.

Now, a similar argument to that in [14, Claim 2] along with (3.10) yields that there exists a positive $t_1 \leq t_0$ such that $|\varphi_t(z)| < 1$ for all $z \in \mathbb{D}$ and all $0 \leq t < t_1$.

Finally, observe that from (3.1) we get

$$\varphi_s \circ \varphi_t(z) = \varphi_{s+t}(z)$$

for $s, t \geq 0$, $s+t < t_1$ and all $z \in \mathbb{D}$. Since the family $\{\varphi_t\}$ satisfies (CP), it follows that it can be continued to a holomorphic flow, defined on $[0, +\infty) \times \mathbb{D}$ (see [25, Proposition 3.3.1]). Now we can assert that the right hand side of (3.6) defines a semigroup on the space $\text{Hol}(\mathbb{D})$. Since this semigroup coincides with the semigroup $\{S_t\}$ for $0 \leq t < t_0$, it follows that (3.6) holds for all $t > 0$ and all $z \in \mathbb{D}$. In particular, by applying (3.6) to the function $\mathbb{1}$, we get (2.4).

The rest of the proof runs as in [14], which yields the statement of the Main Theorem. \square

Remark 3.2. Note that, by hypotheses, the weighted composition operators act boundedly on X . Nevertheless, we point out that even in the Hardy spaces H^p it is unknown how to characterize bounded weighted composition operators in terms of the inducing symbols (see [15] for related discussions).

Remark 3.3. It is to be noticed that our results have consequences for spaces of analytic functions on domains other than the disc. More precisely let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and suppose X is a Banach space consisting of functions analytic on Ω , which satisfies the analogous condition (\star) for Ω . Suppose $\{S_t\}_{t \geq 0}$ is a C_0 -semigroup acting on X , with infinitesimal generator of the form

$$Af = G \cdot f' + g \cdot f, \quad \mathcal{D}(A) = \{f \in X : G \cdot f' + g \cdot f \in X\}$$

where G and g are analytic on Ω . Let $h : \mathbb{D} \rightarrow \Omega$ is a fixed conformal map onto Ω . Consider the Banach space $X(\mathbb{D}) = \{F = f \circ h : f \in X\}$ with the transferred norm $\|F\|_{X(\mathbb{D})} = \|f\|_X$ and assume that $X(\mathbb{D})$ satisfies (\star) . The composition operator $C_h(f) = f \circ h$ is an onto isometry between X and $X(\mathbb{D})$, with $C_h^{-1} = C_{h^{-1}}$. It is then clear that the family of operators $\{T_t\}_{t \geq 0}$,

$$T_t = C_h \circ S_t \circ C_{h^{-1}}$$

is a C_0 -semigroup on $X(\mathbb{D})$. Its infinitesimal generator Γ , for $F \in \mathcal{D}(\Gamma)$, is

$$\begin{aligned} \Gamma F &= \lim_{t \rightarrow 0^+} \frac{T_t F - F}{t} = \lim_{t \rightarrow 0^+} \frac{C_h \circ S_t \circ C_{h^{-1}}(F) - F}{t} \\ &= \lim_{t \rightarrow 0^+} \frac{S_t f \circ h - f \circ h}{t} = (Af) \circ h, \quad \text{where } f = F \circ h^{-1} \in X, \\ &= G \circ h \cdot f' \circ h + g \circ h \cdot f \circ h \\ &= \frac{1}{h'} G \circ h \cdot F' + g \circ h \cdot F \\ &= G_1 \cdot F' + g_1 \cdot F, \end{aligned}$$

where $G_1(z) = \frac{1}{h'(z)} G(h(z))$ and $g_1(z) = g(h(z))$ are analytic in \mathbb{D} . Thus by Theorem 3.1 there is a holomorphic flow $\{\varphi_t\}$ and a cocycle m for $\{\varphi_t\}$ such that

$$T_t F(z) = m_t(z) F(\varphi_t(z)), \quad t \geq 0, z \in \mathbb{D}, F \in X(\mathbb{D}).$$

So we deduce that

$$\begin{aligned} S_t(f)(z) &= (C_{h^{-1}} \circ T_t \circ C_h)(f)(z) \\ &= m_t(h^{-1}(z))(f \circ h \circ \varphi_t \circ h^{-1})(z) \\ &= \mu_t(z) f(\psi_t(z)) \end{aligned}$$

for $f \in X$ and $z \in \Omega$ where $\psi_t = h \circ \varphi_t \circ h^{-1}$ is a holomorphic flow in Ω and $\mu_t(z) = m_t(h^{-1}(z))$ is a cocycle in Ω for $\{\psi_t\}$.

More generally, let h is as before and let τ be a zero-free holomorphic function on \mathbb{D} . Then the same observation applies if $W_{h,\tau}$ is the weighted composition operator $W_{h,\tau}(f) = \tau \cdot (f \circ h)$, and the space $X(\mathbb{D})$, defined as $X(\mathbb{D}) = \{W_{h,\tau} f : f \in X\}$,

satisfies (\star) . This remark can be applied, for instance, to Hardy and to Smirnov spaces of bounded and unbounded domains (see [12] for the corresponding definitions).

4. ABSENCE OF NON-TRIVIAL C_0 -SEMIGROUPS OF WEIGHTED COMPOSITION OPERATORS IN NON-SEPARABLE BANACH FUNCTION SPACES

As mentioned in the introduction, in this section we deal with Banach spaces X of analytic functions on \mathbb{D} such that

$$(4.1) \quad H^\infty \subseteq X \subseteq \mathcal{B}.$$

Recall that the Bloch space \mathcal{B} is a Banach space endowed with the norm

$$(4.2) \quad \|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2), \quad (f \in \mathcal{B}).$$

It will be showed below that such space X is always non-separable (see Remark 4.4).

In [6], the authors showed that no non-trivial holomorphic flow $\Phi = \{\varphi_t\}_{t \geq 0}$ induces a C_0 -semigroup of composition operators on H^∞ or \mathcal{B} by considering an argument involving the Dunford-Pettis property. More recently, Anderson, Jovovic and Smith [1] proved that the same holds for the space BMOA as well as for any Banach space X satisfying (4.1) by means of a geometric function theory argument. Based on it, we will prove the following

Theorem 4.1. *Let X be a Banach space of analytic functions on \mathbb{D} such that $H^\infty \subseteq X \subseteq \mathcal{B}$. Assume that $\Phi = \{\varphi_t\}_{t \geq 0}$ is a nontrivial holomorphic flow. Let $\{W_t\}_{t \geq 0}$ be any weighted composition semigroup induced by Φ , namely, suppose there exists a cocycle m for Φ such that*

$$(4.3) \quad W_t f(z) = m_t(z) f(\varphi_t(z)), \quad (z \in \mathbb{D}),$$

for any $f \in X$. Then $\{W_t\}_{t \geq 0}$ is no longer a strongly continuous semigroup in X .

We remark that for any reasonable Banach space X of analytic functions in \mathbb{D} , the Banach space $[\varphi_t, X] \subset X$ is defined; it is the maximal subspace of X , on which the unweighted composition semigroup $\{C_t\}$ is strongly continuous. The recent work [10] is devoted to a description of flows $\{\varphi_t\}$ such that the spaces $[\varphi_t, \mathcal{B}]$ and $[\varphi_t, BMOA]$ are the minimal ones, that is, $[\varphi_t, \mathcal{B}] = \mathcal{B}_0$ and $[\varphi_t, BMOA] = VMOA$.

In the proof of Theorem 4.1, we will make use the following lemma, which can be extracted from [1], see also [13].

Lemma 4.2. *For any holomorphic flow $\{\varphi_t\}_{t \geq 0}$, one of the two statements holds:*

- (1) *There is a point $\gamma_0 \in \partial\mathbb{D}$ such that*

$$\exists \lim_{r \rightarrow 1^-} \varphi_t(r\gamma_0) := \varphi_t(\gamma_0), \quad t > 0,$$

and $\varphi_t(\gamma_0) \in \mathbb{D}$ for any $t > 0$;

- (2) *φ_t is an automorphism of \mathbb{D} onto itself for any $t \geq 0$.*

This is implicit in the proof of Theorem 3.1 in [1], which is based on the König model of $\{\varphi_t\}_{t \geq 0}$. Namely, there is a Riemann map h of \mathbb{D} onto a simply connected domain Ω with $h(0) = 0$ and a constant c , $\operatorname{Re} c \geq 0$, such that either

$$(4.4) \quad \varphi_t(z) = h^{-1}(e^{-ct}h(z)), \quad z \in \mathbb{D}, t \geq 0$$

or

$$(4.5) \quad \varphi_t(z) = h^{-1}(h(z) + ct), \quad z \in \mathbb{D}, t \geq 0.$$

In the first case Ω has to be spirallike, $we^{-ct} \in \Omega$ for each $w \in \Omega$ and $t \geq 0$. In the second one, Ω is close to convex, $w + ct \in \Omega$ for each $w \in \Omega$ and $t \geq 0$. We refer to [24] or [25] for more on the subject.

A key ingredient in the proof will be a result which allows us to estimate the derivative of an infinite interpolating Blaschke products in pseudohyperbolic discs. Of particular help will be Lemma 3.5 from the paper [17] by Girela, Peláez and Vukotic. In order to formulate it, we recall that given $\theta \in \mathbb{R}$ and a sequence of (not necessarily distinct) points $\{z_k\}$ in \mathbb{D} satisfying the *Blaschke condition*

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty,$$

the infinite product

$$B(z) = e^{i\theta} \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z_k}z}$$

converges uniformly on compact subsets of \mathbb{D} .

For each $k \geq 1$, we will denote by b_{z_k} the factor $\frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z_k}z}$. Whenever $z_k = 0$, the above expression $|z_k|/z_k$ is taken to be equal to -1 , so that $b_{z_k}(z) = z$ in this case. The holomorphic function B is called the *Blaschke product with zero sequence* $\{z_k\}$. It satisfies

- i) B vanishes precisely at the points $\{z_k\}$, with the corresponding multiplicities (that is, $\{z_k\}$ is the zero-sequence of B), and
- ii) $|B(z)| < 1$ for every $z \in \mathbb{D}$.
- iii) $\lim_{r \rightarrow 1^-} |B(re^{i\theta})| = 1$ almost everywhere on the boundary $\partial\mathbb{D}$.

Recall that a sequence $\{z_n\}$ is said to be interpolating if for any bounded sequence of complex numbers $\{w_n\}$ there exists a function $f \in H^\infty$ such that $f(z_n) = w_n$ for all n . A Blaschke product is interpolating if its zero sequence is interpolating (in this case, all z_n are distinct). It is known that B is interpolating if and only if there is a constant $\delta > 0$ such that $|B_n(z_n)| > \delta > 0$ for all n , where $B_n = B/b_{z_n}$. Whenever $1 - |z_{n+1}| \leq \alpha(1 - |z_n|)$ for all n , where $\alpha < 1$ is a constant, the sequence $\{z_n\}$ is interpolating.

For these and other properties of Blaschke products, we refer to Garnett's book [16].

In the next statement, $\Delta(a, r)$ will denote the pseudohyperbolic disc

$$\Delta(a, r) = \{z \in \mathbb{D} : \rho(z, a) < r\},$$

where $\rho(z, w)$ stands for the pseudohyperbolic distance between two points w, z in \mathbb{D} , namely, $\rho(z, w) = \left| \frac{w - z}{1 - \overline{w}z} \right|$.

Lemma 4.3. *Suppose B is an infinite Blaschke product in \mathbb{D} . Let $\{a_n\}$ be a part of its zeros. Suppose that it is an interpolating sequence and that*

$$|(B/b_{a_n})(a_n)| \geq \delta > 0.$$

Then there are positive constants α and β , which depend only on δ , such that the pseudo-discs $\Delta(a_n, \alpha)$, $n \geq 1$ are pairwise disjoint and

$$|B'(z)| \geq \frac{\beta}{1 - |a_n|}, \quad z \in \Delta(a_n, \alpha).$$

Though the statement of Lemma 4.3 is formally more general than the one of Lemma 3.5 in [17], the proof is obtained by means of the same argument.

We are now in position to prove Theorem 4.1.

Proof of Theorem 4.1. By the Closed Graph Theorem, the inclusion $X \subseteq \mathcal{B}$ is continuous. Hence it suffices to prove that there exist a function $f \in H^\infty \subseteq X$ and a constant $\delta > 0$ such that

$$(4.6) \quad \limsup_{t \rightarrow 0^+} \|W_t f - f\|_{\mathcal{B}} \geq \delta.$$

Assume first that in Lemma 4.2, case (1) holds. Without loss of generality, we can assume that $\gamma_0 = 1$. We are going to choose an increasing sequence $r_n \nearrow 1$ and a decreasing one $t_n \searrow 0$ such that $\{r_n\} \cup \{\varphi_{t_n}(r_n)\}$ is an interpolating sequence in \mathbb{D} (in particular, the sequence $\{r_n\}$ is disjoint from $\{\varphi_{t_n}(r_n)\}$). Moreover, we require that

$$|\varphi_{t_1}(r_1)| < r_1 < \dots < |\varphi_{t_{n-1}}(r_{n-1})| < r_{n-1} < |\varphi_{t_n}(r_n)| < r_n < \dots$$

and

$$(4.7) \quad 1 - r_n < \frac{1}{2}(1 - |\varphi_{t_n}(r_n)|) < \frac{1}{4}(1 - r_{n-1})$$

for all $n \geq 1$. (For $n = 1$, we only require the first inequality.) This will be done by an inductive construction.

Take $t_1 > 0$ arbitrarily. Since $|\varphi_{t_1}(1)| = \lim_{r \rightarrow 1^-} |\varphi_{t_1}(r)| < 1$, one can choose $r_1 < 1$ so that

$$1 - r_1 < \frac{1}{2}(1 - |\varphi_{t_1}(r_1)|),$$

and in particular, $|\varphi_{t_1}(r_1)| < r_1$. This is the first step of the construction.

Suppose t_{n-1} and r_{n-1} have been constructed. Choose $t_n \in (0, \frac{1}{2}t_{n-1})$ so that

$$0 < 1 - |\varphi_{t_n}(1)| < \frac{1}{2}(1 - r_{n-1}).$$

Then for any $r < 1$ sufficiently close to 1,

$$1 - |\varphi_{t_n}(r)| < \frac{1}{2}(1 - r_{n-1}).$$

Hence $r_n < 1$ can be chosen in such way that

$$1 - r_n < \frac{1}{2}(1 - |\varphi_{t_n}(r_n)|) < \frac{1}{4}(1 - r_{n-1}).$$

This defines sequences $r_n \nearrow 1$ and $t_n \searrow 0$ with all desired properties.

We put

$$f = \tilde{B}\hat{B}^2$$

where the Blaschke products \tilde{B} , \hat{B} correspond to sequences of zeros $\{r_n\}$, $\{\varphi_{t_n}(r_n)\}$, respectively. By (4.7), $\tilde{B}\hat{B}$ is an interpolating Blaschke product.

Having in mind (4.2), we note that

$$\begin{aligned} \|W_t f - f\|_{\mathcal{B}} &\geq \sup_{z \in \mathbb{D}} |(m_t(f \circ \varphi_t))'(z) - f'(z)|(1 - |z|^2) \\ &\geq \sup_{0 \leq r < 1} |m_t(r) \cdot f'(\varphi_t(r))\varphi_t'(r) + m_t'(r)f(\varphi_t(r)) - f'(r)|(1 - r). \end{aligned}$$

Since $f(\varphi_{t_n}(r_n)) = f'(\varphi_{t_n}(r_n)) = 0$, by substituting $r = r_n$, $t = t_n$ in the above inequality, we get

$$\|W_{t_n} f - f\|_{\mathcal{B}} \geq |f'(r_n)|(1 - r_n) \geq \delta > 0,$$

for all n (we applied Lemma 4.3 to the Blaschke product f and to $a_n = r_n$). This gives (4.6).

Now assume Case (2) in the structure Lemma 4.2. Then φ_t for $t > 0$ is a disc automorphism, being either elliptic, parabolic or hyperbolic, and this classification is the same for all t (see [7], for instance).

Hence, the flow $\{\varphi_t\}_{t \geq 0}$ can be continued to a holomorphic flow $\{\varphi_t : t \in \mathbb{R}\}$. In the parabolic case, we will assume (without loss of generality) that 1 is not the Denjoy-Wolff point of $\{\varphi_t : t \geq 0\}$. In the hyperbolic case, we will assume that 1 is not the Denjoy-Wolff point neither of $\{\varphi_t : t \geq 0\}$ nor of $\{\varphi_{-t} : t \geq 0\}$.

Then, in all three cases, for a small $t_0 > 0$, the curves $\eta_r : [0, t_0] \rightarrow \mathbb{D}$,

$$\eta_r(t) := \varphi_t(r),$$

tend as $r \rightarrow 1^-$ in the C^1 metric to the curve η_1 . This latter curve is a one-to-one smooth parametrization of a small subarc of $\partial\mathbb{D}$, one of whose endpoints is 1.

Put $r_n = 1 - 2^{-n}$ for $n \geq N_0$, where N_0 is large enough. If the curve η_1 is contained in $\{\operatorname{Im} z \geq 0\}$, then there exist $t_n \searrow 0$ such that $\arg(\varphi_{t_n}(r_n) - 1) = 3\pi/4$, $n \geq N_0$. If η_1 is contained in $\{\operatorname{Im} z \leq 0\}$, then one can choose $t_n \searrow 0$ so that $\arg(\varphi_{t_n}(r_n) - 1) = -3\pi/4$, $n \geq N_0$. In both cases,

$$\frac{1 - \operatorname{Re} \varphi_{t_n}(r_n)}{2^{-n}} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

It follows that the sequence $\{r_n\} \cup \{\varphi_{t_n}(r_n)\}$ is contained in a Stoltz angle with vertex at 1 and is separated in the pseudohyperbolic metric $\rho(\cdot, \cdot)$. Thus, this sequence is interpolating (see [23], Lecture VII) and the rest of the proof runs as in the Case (1). \square

Remark 4.4. Any space X satisfying (4.1) is not separable. Indeed, let B be an interpolating Blaschke product with (infinitely many) zeros on the radius $[0, 1)$, for instance, at the points $1 - 2^{-n}$. Put $B_t(z) = B(\exp(-it) * z)$. Then each Blaschke product B_t belongs to X . Using Lemma 4.3, we see that the distances $\|B_t - B_s\|_{\mathcal{B}}$, where $t, s \in [0, 2\pi)$ and $t \neq s$, are uniformly bounded from below by some constant $\varepsilon > 0$ which implies our assertion.

REFERENCES

- [1] A. Anderson, M. Jovovic, W. Smith, Composition semigroups on BMOA and H^∞ , J. Math. Anal. Appl. 449 (2017), no. 1, 843–852.
- [2] W. Arendt, I. Chalendar, Generators of semigroups on Banach spaces inducing holomorphic semi-flows, Israel J. Math. 229 (2019), no. 1, 165–179.

- [3] C. Avicou, I. Chalendar, J. R. Partington, A class of quasicontractive semigroups acting on Hardy and Dirichlet space, *J. Evol. Equ.* 15 (2015), 647–665.
- [4] C. Avicou, I. Chalendar, J. R. Partington, Analyticity and compactness of semigroups of composition operators, *J. Math. Anal. Appl.* 437 (2016), 545–560.
- [5] C. Berenstein and R. Gay, *Complex Analysis and Topics in Harmonic Analysis*, Springer-Verlag, New York (1995).
- [6] O. Blasco, M. Contreras, S. Díaz-Madrigal, J. Martínez, M. Papadimitrakis, A. Siskakis, Semigroups of composition operators and integral operators in spaces of analytic functions, *Ann. Acad. Sci. Fenn. Math.* 38 (2013) 67–89.
- [7] F. Bracci, M. D. Contreras and S. Díaz-Madrigal, *Continuous Semigroups of Holomorphic Self-maps of the Unit Disc*, Springer Monographs in Mathematics (2020).
- [8] E. Berkson and H. Porta, Semigroups of analytic functions and composition operators. *Michigan Math. J.* 25 (1978), no. 1, 101–115.
- [9] I. Chalendar, J. R. Partington, A class of quasicontractive semigroups acting on Hardy and weighted Hardy spaces. *Semigroup Forum* 95 (2017), no. 2, 281–292.
- [10] N. Chalmoukis, V. Daskalogiannis, Holomorphic semigroups and Sarason’s characterization of vanishing mean oscillation (2021), arXiv:2106.01294.
- [11] C. Cowen, Subnormality of the Cesàro operator and a semigroup of composition operators. *Indiana Univ. Math. J.* 33 (1984), 305–318
- [12] P.L. Duren, *Theory of H^p spaces*. Pure and Applied Mathematics, Vol. 38. Academic Press, New York-London, 1970.
- [13] P. Galanopoulos, N. Merchán, A. G. Siskakis, Semigroups of composition operators in analytic Morrey spaces. *Integral Equations Operator Theory* 92 (2020), no. 2, Paper No. 12, 15 pp.
- [14] E. A. Gallardo-Gutiérrez, D. V. Yakubovich, On generators of C_0 -semigroups of composition operators. *Israel J. Math.* 229 (2019), no. 1, 487–500.
- [15] E. A. Gallardo-Gutiérrez, R. Kumar, J. R. Partington, Boundedness, compactness and Schatten-class membership of weighted composition operators. *Integral Equations Operator Theory* 67 (2010), no. 4, 467–479.
- [16] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, 1981.
- [17] D. Girela, J. Á. Peláez, D. Vukotić, Integrability of the derivative of a Blaschke product. *Proc. Edinb. Math. Soc.* (2) 50 (2007), no. 3, 673–687.
- [18] E. Hille, *Ordinary differential equations in the complex domain*. Wiley-Interscience (John Wiley & Sons), 1976.
- [19] F. Jafari, T. Tonev, E. Toneva, K. Yale, Holomorphic flows, cocycles, and coboundaries. *Michigan Math. J.* 44 (1997), no. 2, 239–253.
- [20] F. Jafari, Z. Slodkowski, T. Tonev, Semigroups of operators on Hardy spaces and cocycles of holomorphic flows. *Complex Anal. Oper. Theory* 6 (2012), no. 1, 113–119.
- [21] F. Jafari, T. Tonev, E. Toneva, Automatic differentiability and characterization of cocycles of holomorphic flows. *Proc. Amer. Math. Soc.* 133 (2005), no. 11, 3389–3394.
- [22] W. König, Semicocycles and weighted composition semigroups on H^p . *Michigan Math. J.* 37 (1990), no. 3, 469–476.
- [23] N. K. Nikol’skiĭ, *Treatise on the shift operator. Spectral function theory. With an appendix by S. V. Hruščev [S. V. Khrushchëv] and V. V. Peller. Translated from the Russian by Jaak Peetre. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, 273. Springer-Verlag, Berlin, 1986.
- [24] Ch. Pommerenke, *Univalent functions: with a chapter on quadratic differentials by Gerd Jensen, Vandenhoeck and Ruprecht in Göttingen*, 1975.
- [25] D. Shoikhet, *Semigroups in Geometrical Function Theory*, Springer, 2001.
- [26] A. G. Siskakis, Composition semigroups and the Cesàro operator on H^p . *J. Lond. Math. Soc.* 36, 153–164 (1987).

- [27] A. G. Siskakis, Semigroups of composition operators on spaces of analytic functions, a review. Studies on composition operators (Laramie, WY, 1996), 229–252, Contemp. Math., 213, Amer. Math. Soc., Providence, RI, 1998.
- [28] A. G. Siskakis, Weighted composition semigroups on Hardy spaces. Proceedings of the symposium on operator theory (Athens, 1985). Linear Algebra Appl. 84 (1986), 359–371.

E. A. GALLARDO-GUTIÉRREZ

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO Y
MATEMÁTICA APLICADA
FACULTAD DE CC. MATEMÁTICAS,
UNIVERSIDAD COMPLUTENSE DE MADRID,
PLAZA DE CIENCIAS Nº 3, 28040 MADRID, SPAIN
AND INSTITUTO DE CIENCIAS MATEMÁTICAS ICMAT(CSIC-UAM-UC3M-UCM),
MADRID, SPAIN

Email address: `eva.gallardo@mat.ucm.es`

A. G. SISKAKIS

DEPARTMENT OF MATHEMATICS
ARISTOTLE UNIVERSITY OF THESSALONIKI
54124 THESSALONIKI, GREECE

Email address: `siskakis@math.auth.gr`

D. V. YAKUBOVICH

DEPARTAMENTO DE MATEMÁTICAS,
UNIVERSIDAD AUTÓNOMA DE MADRID,
CANTOBLANCO, 28049 MADRID, SPAIN
AND INSTITUTO DE CIENCIAS MATEMÁTICAS ICMAT (CSIC-UAM-UC3M-UCM),
MADRID, SPAIN.

Email address: `dmitry.yakubovich@uam.es`