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New models for extreme gravitational systems

Nuevos modelos asociados a sistemas gravitacionales extremos

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To all my family.

A toda mi familia.

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Contents

Abbreviations

BH Black Hole

BK Bartnik McKinnon

BR Belinfante Rosenfeld

EC Einstein Cartan

EH Einstein Hilbert

EYM Einstein Yang Mills

GR General Relativity

LC Levi Civita

PG Poincaré Gauge

RC Riemann Cartan

RN Reissner Nordström

YM Yang Mills

Abstract

A large number of classes of modified theories of gravity have been developed for a long time. They have attracted much attention from physicists, since they show different aspects concerning gravitational interaction. In fact, these aspects may extend the role of gravity not only at large scales but at microscopic regimes, so that they have been systematically related to fundamental issues such as the occurrence of space-time singularities, the loss of renormalizability or the origin of the accelerated expansion of the universe, among others. Despite the successful predictions and the highly tested accuracy of General Relativity (GR) in describing the gravitational phenomena, the absence of an appropriate explanation for these issues has stimulated the investigation of new alternative models of gravitation.

The extension of the conventional approach can be addressed by the introduction into the gravitational action of higher order corrections depending on the metric tensor alone. Such a procedure preserves the geometric structure of the space-time and potentially yields new propagating degrees of freedom related to metric, which means that not only the phenomenological compatibility with GR must be considered by the new framework but also the viability of its stability conditions. On the other hand, it is also possible to define a more complex geometry by the modification of the affine connection. Namely, the Levi-Civita connection of GR is subject to the fulfillment of two independent constraints: the conservation of the metric tensor under parallel transport and the vanishing of its antisymmetric component. Hence, in this case there is an increase in the number of degrees of freedom contained in the connection, which can involve new geometrical and dynamical effects in the space-time. Furthermore, from a theoretical point of view, the resulting post-Riemannian geometry can be related to the existence of a new fundamental symmetry in nature by applying the gauge principles. This scheme leads to the appearance of new theories of gravitation, such as the Metric-Affine or the Poincaré Gauge theory.

In the present thesis, we use these notions to investigate the nature and the implications of the space-time torsion in the framework of the Poincaré Gauge theory. Thereby, we deal with a metric-compatible asymmetric connection and analyse the foundations and viability of different models within this framework. First, in Chapter 1, we present an introduction of the specific motivations to consider a post-

Riemannian regime, by emphasizing the most relevant consequences and differences from the standard case of GR. The intrinsic relation between torsion and the spin density tensor of matter is especially remarkable. It is also worthwhile to stress the potential effects of the torsion tensor on the propagation and motion of Dirac particles, as well as its dynamical contribution to the geometry of space-time. In this regard, we describe in Chapter 2 the most relevant configurations provided by a dynamical torsion in a vacuum space-time. These types of scenarios allow an assessment to be made of the possible roles assumed by torsion and furthermore of the characteristic effects involved in its interaction with matter fields. We present new black hole solutions for both the cases with massless and massive torsion, which introduce significant corrections to the Schwarzschild solution of GR. The existence of a dynamical axial mode related to torsion highlights the relevance of these solutions, since this is the unique component implicated in the interaction with Dirac fields, according to the minimal coupling principle. On the other hand, the new geometry can modify additional fundamental constraints, such as the appearance of space-time singularities or instabilities. Therefore, in Chapter 3, we revise the singularity theorems of pseudo-Riemannian geometry and study this issue within the new framework, in order to extend their general applicability and address the appropriate changes in the presence of torsion. By focusing on a particular set of assumptions, we also perform an intensive analysis to find new ghost and tachyon-free conditions related to torsion, which must be satisfied by the Lagrangian coefficients to avoid unsuitable instabilities. Finally, in Chapter 4, we extrapolate the external symmetries provided by post-Riemannian geometry to construct new models within the Einstein-Yang-Mills theory of internal symmetry groups, which focuses on the interaction between gravity and non-Abelian gauge fields. Indeed, the search of a correspondence between both approaches allows the simplification of the complexity involved by the highly nonlinear character of these elements, which in turn facilitates the obtention of different non-Abelian exact solutions to the field equations. Appendix A is devoted to the expressions of the general field equations induced by curvature and torsion in the gauge formalism, which associate both geometrical quantities with the energy-momentum and spin density tensors of matter. In addition, the space-time symmetries applied to simplify the extreme difficulty of the field equations and to categorize the resulting new black hole solutions are present in Appendix B, whereas a detailed derivation of the $SU(2)$ gauge connection in static and spherically symmetric space-times is shown in Appendix C.

The results achieved in this thesis provide new bases and methodologies to describe and measure the possible existence of a space-time torsion in the universe. Since this quantity appears to be directly connected to the intrinsic angular momentum of elementary particles, it is expected to generate negligible effects at macroscopic scales. Therefore, the focusing on extreme gravitational systems that may intensify such effects is especially relevant to overcome these observational issues.

Resumen

Un gran número de teorías de gravedad modificada se ha venido desarrollando desde hace décadas. Debido a las múltiples propiedades teóricas que proporcionan al campo gravitatorio, éstas han atraído la atención de muchos investigadores desde sus inicios. Dichas propiedades pueden modificar el papel de la gravedad y extenderlo, no sólo a gran escala, sino también a un régimen microscópico, por lo que se han venido relacionando sistemáticamente con cuestiones fundamentales como la ocurrencia de singularidades en el espacio-tiempo, la pérdida de renormalizabilidad o el origen de la expansión acelerada del universo. A pesar de las exitosas predicciones de la Teoría de la Relatividad General (GR) y de su carácter predictivo altamente probado, la ausencia de una solución adecuada a estas cuestiones ha estimulado la investigación de nuevos modelos alternativos de la gravedad.

La extensión del marco teórico convencional puede realizarse mediante la introducción en la acción gravitacional de correcciones geométricas de orden superior dependientes del tensor métrico. Este procedimiento preserva la estructura geométrica del espacio-tiempo y agrega nuevos grados de libertad a la teoría, lo que significa que no sólo es importante asegurar la compatibilidad con GR desde un punto de vista fenomenológico, sino también su propia estabilidad dinámica. Por otro lado, también es posible definir una geometría más compleja introduciendo nuevos grados de libertad en la conexión afín. En concreto, la conexión afín de Levi-Civita presente en GR satisface dos ligaduras, al implicar la conservación de la métrica bajo el transporte paralelo y omitir la inclusión de una componente antisimétrica. Los grados de libertad geométricos resultantes al liberar el cumplimiento de cualesquiera de estas dos condiciones, sumados a los ya existentes en el marco teórico estándar, pueden dar lugar a nuevos efectos dinámicos en el espacio-tiempo. Desde un punto de vista teórico, esta nueva geometría postRiemanniana puede relacionarse con una nueva simetría fundamental aplicando los principios de invariancia gauge. Este enfoque ha dado lugar al nacimiento de nuevas teorías de la gravitación, como la Teoría Métrica Afín o la Teoría Gauge de Poincaré.

En la presente tesis, usamos estas nociones para investigar la naturaleza y las posibles implicaciones derivadas de una torsión espacio-temporal en el marco de la Teoría Gauge de Poincaré. De esta forma, consideraremos una conexión afín

asimétrica que preserve la métrica y analizaremos los fundamentos y la viabilidad de diferentes modelos sujetos a estas directrices. En primer lugar, en el Capítulo 1, introducimos detalladamente las motivaciones para considerar un nuevo régimen postRiemanniano, destacando sus consecuencias más relevantes y sus diferencias con respecto al caso estándar de GR. En este sentido, la relación existente entre el tensor momento angular de espín de la materia y la torsión del espacio-tiempo es especialmente destacable dentro de este nuevo marco teórico. Asimismo, señalamos los posibles efectos dinámicos producidos por la torsión en la propagación de partículas de Dirac y en la propia geometría del espacio-tiempo. A este respecto, en el Capítulo 2, describimos las configuraciones geométricas más relevantes originadas por la existencia de una torsión dinámica en el vacío. Estos tipos de escenarios permiten evaluar las propiedades físicas de dicha magnitud geométrica y sus efectos en la interacción con la materia. El hallazgo de nuevas soluciones de tipo agujero negro, asociadas a los casos con torsión no masiva y masiva, se incluye también en este capítulo. Estos resultados muestran correcciones significativas a la solución de vacío de Schwarzschild de GR proporcionadas por la torsión. La existencia de un modo de torsión axial dinámico aumenta la relevancia de estas soluciones, al tratarse de la única componente de la torsión capaz de interaccionar con campos de Dirac, de acuerdo al principio de acoplamiento mínimo. Por otro lado, en el régimen postRiemanniano, otras condiciones fundamentales pueden verse alteradas, como la ocurrencia de singularidades o de inestabilidades físicas. Por tanto, en el Capítulo 3, revisamos los teoremas de singularidades presentes en la geometría pseudoRiemanniana y estudiamos su generalización al caso con torsión. Del mismo modo, imponiendo una serie de restricciones sobre la torsión y la métrica, llevamos a cabo un exhaustivo análisis para determinar nuevas condiciones de estabilidad de la teoría, las cuales pueden describirse mediante sencillas ligaduras entre los coeficientes del lagrangiano. Por último, en el Capítulo 4, hacemos uso de todas estas nociones teóricas de invariancia gauge asociadas a simetrías externas para construir nuevos modelos de campos de Einstein-Yang-Mills asociados a simetrías internas, los cuales describen la dinámica de campos gauge no abelianos en espacio-tiempo curvo bajo el marco de la GR. La búsqueda de una correspondencia entre ambos enfoques permite simplificar de manera notable su complejidad matemática, provista por el carácter altamente no lineal de sus elementos, lo que facilita la obtención de diferentes soluciones exactas a las ecuaciones de Einstein-Yang-Mills.

El Apéndice A contiene las expresiones generales de las ecuaciones de campos inducidas por los tensores de curvatura y torsión en el formalismo gauge, las cuales asocian estas magnitudes geométricas con los tensores de energía-impulso y densidad de espín de la materia. Las simetrías espacio-temporales aplicadas para simplificar la complejidad de estas ecuaciones y para categorizar las nuevas soluciones de tipo agujero negro se presentan en el Apéndice B, mientras que en el Apéndice C se muestra en detalle el análisis para la obtención de una conexión gauge de SU(2) simplificada, en presencia de un espacio-tiempo curvo estático y esféricamente simétrico.

Los resultados alcanzados en esta tesis proporcionan nuevas bases y metodologías para describir y medir la posible existencia de una torsión espacio-temporal en el universo. Al tratarse de una magnitud directamente conectada con el momento angular intrínseco de las partículas elementales, se espera que en general produzca efectos despreciables a gran escala. Por lo tanto, el estudio de sistemas gravitacionales extremos que puedan intensificar sus efectos es especialmente relevante a la hora de intentar superar estas limitaciones observacionales.

Chapter 1

Introduction to post-Riemannian geometries

1.1 Motivation and generalities

Since the early twentieth century, General Relativity (GR) has been established as the theory that best and most deeply describes, from a phenomenological point of view, the gravitational field and its interaction with matter. Since its inceptions, the theory formulated by Albert Einstein completely modified the general understanding of the universe. The most fascinating postulate assumed by Einstein's approach was the fact that the universe itself acquires a non-vanishing curvature due to the presence of gravity and matter fields. Furthermore, its theoretical bases led to the conclusion that this effect is naturally modulated by the energy-momentum properties of matter, in a form that it is preserved in all reference frames, according to the principle of general covariance [1].

From a mathematical point of view, the model was developed in terms of Riemannian geometry by establishing a correspondence between space-time and a differentiable manifold endowed with a curvature tensor, which is associated with the gravitational field. Such a description involves the existence of a metric tensor and a metric-compatible affine connection in a form that all the geometrical quantities defined on the manifold can be expressed in terms of them. These elements enable the definition of the distance and parallel transport concepts within the manifold. Thereby, one of the assumptions of the theory is the vanishing of the antisymmetric part of the affine connection, so that it can be written in terms of the metric tensor:

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\mu\rho} - \partial_{\rho} g_{\mu\nu} \right) \,, \tag{1.1}$$

where latin and greek indices refer to anholonomic and coordinate basis, respectively.

This type of connection is called the Levi-Civita (LC) connection and it is straightforward to verify the metric-compatible property because of the vanishing of the covariant derivative of the metric tensor [1]:

$$\nabla_\lambda \, g_{\mu\nu} = 0 \,. \tag{1.3}$$

In addition, this structure involves the existence of a curvature tensor depending on the metric tensor alone:

$$\left[\nabla_\mu, \nabla_\nu\right] v^\lambda = R^\lambda{}_{\rho\mu\nu} v^\rho \,, \tag{1.4}$$

where:

$$R_{\lambda\rho\mu\nu} = \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\rho \partial x^\mu} + \frac{\partial^2 g_{\rho\mu}}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 g_{\lambda\mu}}{\partial x^\rho \partial x^\nu} - \frac{\partial^2 g_{\rho\nu}}{\partial x^\lambda \partial x^\mu}\right) + g_{\sigma\omega}\left(\Gamma^\omega{}_{\rho\mu}\Gamma^\sigma{}_{\lambda\nu} - \Gamma^\omega{}_{\rho\nu}\Gamma^\sigma{}_{\lambda\mu}\right) \,. \tag{1.5}$$

These geometrical foundations are enclosed with an action principle to describe the dynamic properties of the gravitational field and the energy-momentum of matter. Namely, the Einstein-Hilbert (EH) action was formulated as an invariant functional of first order in the curvature tensor which, together with the action of matter, give rise to general field equations by performing variations with respect to the metric tensor [2]:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\mathcal{L}_m - R\right) \,, \tag{1.6}$$

$$\delta S = -\frac{1}{16\pi}\int \left(G_{\mu\nu} - 8\pi \, T_{\mu\nu}\right)\delta g^{\mu\nu}\sqrt{-g} \, d^4x \,, \tag{1.7}$$

where, additionally, the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R constitute the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2}g_{\mu\nu}$ and $T_{\mu\nu} = \frac{1}{8\pi\sqrt{-g}}\frac{\delta(\mathcal{L}_m\sqrt{-g})}{\delta g^{\mu\nu}}$ defines the energy-momentum tensor.

This construction encompasses the appropriate Newtonian limit and conservation laws in virtue of the divergenceless of the Einstein tensor. Furthermore, it establishes

[1] The covariant derivative of a general world tensor is defined as follows:

$$\begin{aligned} \nabla_\lambda T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} &= \partial_\lambda T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} + \Gamma^{\mu_1}{}_{\rho\lambda} T^{\rho \dots \mu_m}{}_{\nu_1 \dots \nu_n} + \dots + \Gamma^{\mu_m}{}_{\rho\lambda} T^{\mu_1 \dots \rho}{}_{\nu_1 \dots \nu_n} \\ &- \Gamma^\rho{}_{\nu_1\lambda} T^{\mu_1 \dots \mu_m}{}_{\rho \dots \nu_n} - \dots - \Gamma^\rho{}_{\nu_n\lambda} T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \rho} \,. \end{aligned} \tag{1.2}$$

[2] Notice that we will use Planck units throughout this work ($G = c = \hbar = 1$).

a complete correspondence between gravitation and the geometry of space-time by assigning the physical trajectories to a geodesic motion in absence of external forces [2].

A large number of further implications were also originally studied and predicted by scientists by means of the theory, like for example the equivalence principle, orbital precession of macroscopic bodies, deflection of light, gravitational redshift and lensing, time dilation or the existence of black holes (BHs) and gravitational waves, among others [3]. All these events have been systematically tested even nowadays, providing a strong supporting evidence of its accuracy and precision [4, 5].

Nevertheless, from a theoretical point of view, there exist additional issues that presumably require going beyond GR towards a more complete theory of gravity. Some of these fundamental problems are the impossibility of renormalizing the EH action unlike that given by other quantum field theories and the existence of unavoidable space-time singularities [6, 7].

Numerous attempts have been accomplished to address these questions and formulate an improved modified theory of gravity, even in the framework of Riemannian geometry [8–10]. Many of these new schemes, in fact, modify the gravity action by aggregating higher order corrections, which are at least quadratic in the curvature tensor. But additional modifications can be introduced in the realm of post-Riemannian geometry, which incorporates new degrees of freedom into the geometric structure of the manifold. Specifically, as mentioned previously, the antisymmetric and non-metricity components of the affine connection are assumed to vanish in the standard case, but this situation changes in the presence of an affinely connected metric space-time. In such a case, the components of the affine connection are expressed in the following way [3]:

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu}\,, \tag{1.8}$$

where $K^{\lambda}{}_{\mu\nu}$ represents a metric-compatible component depending on the antisymmetric part of the connection and $L^{\lambda}{}_{\mu\nu}$ is related to non-metricity. By defining $T^{\lambda}{}_{\mu\nu} = 2\tilde{\Gamma}^{\lambda}{}_{[\mu\nu]}$ as the stressed antisymmetric component and $Q_{\lambda\mu\nu} = \tilde{\nabla}_{\lambda} g_{\mu\nu}$ as the non-metricity part of the affine connection, then the previous quantities are written as follows:

$$K^{\lambda}{}_{\mu\nu} = \frac{1}{2}(T^{\lambda}{}_{\mu\nu} - T_{\mu}{}^{\lambda}{}_{\nu} - T_{\nu}{}^{\lambda}{}_{\mu})\,, \tag{1.9}$$

[3] We use notation with tilde to denote quantities depending on torsion and without tilde for the torsion-free components of such quantities.

$$L^{\lambda}{}_{\mu\nu} = \frac{1}{2}(Q^{\lambda}{}_{\mu\nu} - Q_{\mu}{}^{\lambda}{}_{\nu} - Q_{\nu}{}^{\lambda}{}_{\mu}) \,. \tag{1.10}$$

Note that these post-Riemannian components possess a tensorial character, whereas the Riemannian part of the connection still changes inhomogeneously under an infinitesimal coordinate transformation $x^{\mu} \to x'^{\mu} = x^{\mu} + \xi^{\mu}$:

$$\Gamma^{\lambda}{}_{\mu\nu} \to \Gamma'^{\lambda}{}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial x'^{\rho}} \frac{\partial x'^{\sigma}}{\partial x^{\mu}} \frac{\partial x'^{\omega}}{\partial x^{\nu}} \Gamma^{\rho}{}_{\sigma\omega} + \frac{\partial^2 x'^{\rho}}{\partial x^{\mu} \partial x^{\nu}} \frac{\partial x^{\lambda}}{\partial x'^{\rho}} \,. \tag{1.11}$$

Then, the antisymmetric part $T^{\lambda}{}_{\mu\nu}$ of the affine connection always transforms as a tensor and it is called torsion tensor, whereas the resulting piece $K^{\lambda}{}_{\mu\nu}$ on the connection is called contortion tensor. On the other hand, the tensorial nature of the metric and the covariant derivative is appropriately induced on the non-metricity component.

In analogy to the rest of the extended theories of gravity, these geometrical characteristics define additional scalar invariants into the gravitational action and hence modify the dynamical aspects provided by the gravitational field. Nevertheless, it is expected that these higher order corrections produce neglected effects at low energy scales and thus they are remarkable only around extreme gravitational systems.

1.2 Riemann-Cartan manifolds: the space-time torsion

The particular case of an affinely connected metric manifold with a metric-compatible connection is named Riemann-Cartan (RC) manifold. Hence, these types of topological spaces are characterized by a vanishing non-metricity tensor:

$$Q^{\lambda}{}_{\mu\nu} = 0 \,. \tag{1.12}$$

The resulting geometric structure is then provided by a metric tensor and an asymmetric affine connection that preserves lengths and angles under parallel transport. Since the affine connection is directly connected to the definition of the covariant derivative, the presence of an antisymmetric component within such a connection introduces deep geometrical consequences. First, it is straightforward to notice the change on the commutation relations of the covariant derivatives:

$$[\tilde{\nabla}_{\mu}, \tilde{\nabla}_{\nu}] \, v^{\lambda} = \tilde{R}^{\lambda}{}_{\rho\mu\nu} \, v^{\rho} + T^{\rho}{}_{\mu\nu} \, \tilde{\nabla}_{\rho} v^{\lambda} \,, \tag{1.13}$$

where:

$$\tilde{R}^{\lambda}{}_{\rho\mu\nu} = \partial_{\mu}\tilde{\Gamma}^{\lambda}{}_{\rho\nu} - \partial_{\nu}\tilde{\Gamma}^{\lambda}{}_{\rho\mu} + \tilde{\Gamma}^{\lambda}{}_{\sigma\mu}\tilde{\Gamma}^{\sigma}{}_{\rho\nu} - \tilde{\Gamma}^{\lambda}{}_{\sigma\nu}\tilde{\Gamma}^{\sigma}{}_{\rho\mu}\,. \tag{1.14}$$

Thereby, it is important to distinguish between the Riemann curvature and the RC curvature. The latter also satisfies its proper Bianchi identities in the RC space-time [4]:

$$\tilde{R}^{\lambda}{}_{[\mu\nu\rho]} + \tilde{\nabla}_{[\mu}T^{\lambda}{}_{\nu\rho]} + T^{\sigma}{}_{[\mu\nu}T^{\lambda}{}_{\rho]\sigma} = 0\,, \tag{1.16}$$

$$\tilde{\nabla}_{[\sigma|}\tilde{R}^{\lambda}{}_{\rho|\mu\nu]} - T^{\omega}{}_{[\sigma\mu|}\tilde{R}^{\lambda}{}_{\rho\omega|\nu]} = 0\,, \tag{1.17}$$

and allows the existence of a non-vanishing antisymmetric component of the Ricci tensor:

$$\tilde{R}_{[\mu\nu]} = \frac{1}{2}\nabla_{\lambda}T^{\lambda}{}_{\mu\nu} + \frac{1}{2}\left(\nabla_{\mu}T^{\lambda}{}_{\nu\lambda} - \nabla_{\nu}T^{\lambda}{}_{\mu\lambda}\right) + \frac{1}{2}T^{\lambda}{}_{\rho\lambda}T^{\rho}{}_{\mu\nu} + \frac{1}{4}\left(T_{\mu\lambda\rho}T^{\rho\lambda}{}_{\nu} - T_{\nu\lambda\rho}T^{\rho\lambda}{}_{\mu}\right)\,. \tag{1.18}$$

Furthermore, the torsion tensor provides a sort of displacement of vectors along infinitesimal paths that generally involves the breaking of standard parallelograms, in a way that their translational closure failure proportionally depends on the mentioned tensor [11, 12]. Suppose two vectors ϵ_1^{λ} and ϵ_2^{λ} at a given point x^{λ}, then the following identity describes the open contour of the infinitesimal parallelogram constructed by them in the presence of torsion:

$$\left(x^{\lambda} + \epsilon_2^{\lambda} + \epsilon_1'^{\lambda}\right) - \left(x^{\lambda} + \epsilon_1^{\lambda} + \epsilon_2'^{\lambda}\right) = T^{\lambda}{}_{\mu\nu}\,\epsilon_1^{\mu}\epsilon_2^{\nu}\,, \tag{1.19}$$

with $\epsilon_1'^{\lambda}$ and $\epsilon_2'^{\lambda}$ the resulting vectors obtained by the parallel transport of ϵ_1^{λ} and ϵ_2^{λ}, at the point of coordinates $x^{\lambda} + \epsilon_2^{\lambda}$ and $x^{\lambda} + \epsilon_1^{\lambda}$, in the direction of ϵ_2^{λ} and ϵ_1^{λ}, respectively. This quality represents an important and singular geometrical effect, since it cannot be yielded by any other quantity, but only by the torsion tensor.

In addition, these features allow the establishment of an equivalence between torsion and dislocation defects of three-dimensional crystal lattices [13–15]. The RC manifold then may be considered as an effective geometrical construction arising

[4]Note that the torsion tensor also implies a non-trivial relation under the following exchange of indices of the RC curvature:

$$\tilde{R}_{\lambda\rho\mu\nu} - \tilde{R}_{\mu\nu\lambda\rho} = \frac{3}{2}\left(\tilde{R}_{\lambda[\rho\mu\nu]} + \tilde{R}_{\rho[\mu\lambda\nu]} + \tilde{R}_{\mu[\rho\lambda\nu]} + \tilde{R}_{\nu[\rho\mu\lambda]}\right)\,. \tag{1.15}$$

from a microscopic structure endowed with dislocation defects, which are described by torsion in the limit where they form a continuous distribution.

In order to establish a general classification of torsion, it can be decomposed into its respective irreducible parts under the Lorentz group [16, 17]. Namely, a trace vector T_μ, an axial vector S_μ and a traceless and also pseudotraceless tensor $q^\lambda{}_{\mu\nu}$:

$$T^\lambda{}_{\mu\nu} = \frac{1}{3}\left(\delta^\lambda{}_\nu T_\mu - \delta^\lambda{}_\mu T_\nu\right) + \frac{1}{6}\, g^{\lambda\rho}\varepsilon_{\rho\sigma\mu\nu}S^\sigma + q^\lambda{}_{\mu\nu}\,, \tag{1.20}$$

where $\varepsilon_{\rho\sigma\mu\nu}$ is the four-dimensional LC symbol. From a phenomenological point of view, this sort of geometrical classification can be associated with a large number of physically relevant configurations, such as the minimal coupling between the Dirac fields and the axial vector or the vanishing of its tensorial modes in a spatially homogeneous and isotropic universe, as is assumed by the cosmological principle [18, 19].

By taking into account these notions, it is possible to construct a large class of scalar invariants from the RC curvature and the torsion tensor and define a modified gravitational action in the framework of the RC geometry. It means that the RC space-time constitutes the kinematical arena of every type of extended theory of gravity with torsion. On the other hand, the dynamical aspects of torsion also depend on the order of such geometrical invariants included in the Lagrangian. Specifically, the full linear case describes a non-propagating torsion tied to material sources, whereas higher order corrections describe a Lagrangian with propagating torsion, which generally involves dynamical effects in vacuum.

1.3 Poincaré gauge theory of gravity

From a theoretical point of view, the most consistent and successful description of torsion is formulated in the framework of the Poincaré Gauge (PG) theory of gravity [20–22]. Just as its name indicates, this theory represents a gauge approach to gravity based on the semidirect product of the Lorentz group and the space-time translations, in analogy to the unitary irreducible representations of relativistic particles labeled by their spin and mass, respectively. Then not only an energy-momentum tensor of matter arises from this approach, but also a non-trivial spin density tensor that operates as source of torsion, providing an appropriate correspondence between the respective gauge potentials and their corresponding field strength tensors.

Hence, the model requires gauging the external degrees of freedom consisting of rotations and translations, which are represented by the Poincaré group $ISO(1,3)$. This means that a gauge connection containing two principal independent variables is introduced in order to describe the gravitational field as a gauge field. These quantities constitute the gauge potentials related to the generators of translations and local Lorentz rotations, respectively:

$$A_\mu = e^a{}_\mu P_a + \omega^{ab}{}_\mu J_{ab}\,, \tag{1.21}$$

where $e^a{}_\mu$ is the vierbein field and $\omega^{ab}{}_\mu$ is the spin connection, which satisfy the following relations with the metric and the affine connection [23]:

$$g_{\mu\nu} = e^a{}_\mu\, e^b{}_\nu\, \eta_{ab}\,, \tag{1.22}$$

$$\omega^{ab}{}_\mu = e^a{}_\lambda\, e^{b\rho}\, \tilde{\Gamma}^\lambda{}_{\rho\mu} + e^a{}_\lambda\, \partial_\mu\, e^{b\lambda}\,. \tag{1.23}$$

Thus, the vierbein field and the affine connection act as translational and rotational type potentials, respectively. Moreover, the mentioned gauge connection associated with the group $ISO(1,3)$ defines a 2-form curvature, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$, which can be expressed in the following way:

$$F_{\mu\nu} = F^a{}_{\mu\nu} P_a + F^{ab}{}_{\mu\nu} J_{ab}\,, \tag{1.24}$$

with $F^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^{ab}{}_\mu e_{b\nu} - \omega^{ab}{}_\nu\, e_{b\mu}$ and $F^{ab}{}_{\mu\nu} = \partial_\mu \omega^{ab}{}_\nu - \partial_\nu \omega^{ab}{}_\mu + \omega^{ac}{}_\nu\, \omega^b{}_{c\mu} - \omega^{ac}{}_\mu\, \omega^b{}_{c\nu}$.

As with other well-known gauge theories, the field strength tensor characterizes the properties of the gravitational interaction, which in the PG framework are potentially modified by the presence of torsion. In particular, it is related to the torsion and the curvature tensor as follows:

$$F^a{}_{\mu\nu} = e^a{}_\lambda\, T^\lambda{}_{\nu\mu}\,, \tag{1.25}$$

$$F^{ab}{}_{\mu\nu} = e^a{}_\lambda e^b{}_\rho\, \tilde{R}^{\lambda\rho}{}_{\mu\nu}\,. \tag{1.26}$$

Hence, whereas curvature is related to the rotation of a vector along an infinitesimal path over the space-time, torsion is related to the translation and they appropriately constitute the field strengths of the rotation and the translation group, respectively.

In contrast with the regular Yang-Mills (YM) theories of internal symmetry groups, the complexity provided by the external symmetry group $ISO(1,3)$ allows the definition of a greater number of scalar invariants from the curvature and torsion tensors. From a theoretical point of view, these types of geometrical quantities are essential since they yield kinetic and interaction terms into the gravitational action. In general, by excluding parity violating terms, it is possible to construct six independent quadratic scalar invariants of curvature and three of torsion, besides the linear one given by the Ricci scalar. Therefore, the most general parity preserving action quadratic in the field strength tensors can be written as [5]:

$$\begin{aligned} S \;=\; \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[&\mathcal{L}_m - \tilde{R} - a_1\tilde{R}^2 + (a_3 - a_1)\,\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} + a_2\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu} \\ &+a_4\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\mu\rho\nu} + a_5\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + (a_6 + 4a_1)\,\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} \\ &+\alpha\, T_{\lambda\mu\nu}T^{\lambda\mu\nu} + \beta\, T_{\lambda\mu\nu}T^{\mu\lambda\nu} + \gamma\, T^{\lambda}{}_{\lambda\nu}T^{\mu}{}_{\mu}{}^{\nu}\Big]\,, \end{aligned} \tag{1.27}$$

where $a_1, a_2, a_3, a_4, a_5, a_6, \alpha, \beta$ and γ are constant parameters. Although the theoretical and experimental research for restrictions on the values of these coefficients still persists, they are in principle subject to the requirement of a viable set of stability conditions and to the constraints given by the experimental evidence. In any case, for deriving the field equations, it is possible to dismiss one of the coefficients associated with the scalars of curvature and reduce the set of parameters by applying the Gauss-Bonnet theorem in four-dimensional RC space-times, without loss of generality [25, 26]. Specifically, the following combination quadratic in the curvature tensor acts as a total derivative of a certain vector V^{μ} in the gravitational action:

$$\sqrt{-g}\,\left(\tilde{R}^2 + \tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} - 4\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu}\right) = \partial_{\mu}V^{\mu}\,. \tag{1.28}$$

Then, in order to derive the general field equations of the quadratic PG theory it is sufficient to perform variations with respect to the gauge potentials, resulting the following outcome:

$$X1_{\mu}{}^{\nu} + 16\pi\theta_{\mu}{}^{\nu} = 0\,, \tag{1.29}$$

$$X2_{[\mu\lambda]}{}^{\nu} + 16\pi S_{\lambda\mu}{}^{\nu} = 0\,, \tag{1.30}$$

where $X1_{\mu}{}^{\nu}$ and $X2_{[\mu\lambda]}{}^{\nu}$ are tensorial functions depending on the RC curvature and the torsion tensor, which are defined in Appendix A, whereas $\theta_{\mu}{}^{\nu}$ and $S_{\lambda\mu}{}^{\nu}$ are the canonical energy-momentum tensor and the spin density tensor, respectively, which are defined as follows:

$$\theta_{\mu}{}^{\nu} = \frac{e^{a}{}_{\mu}}{16\pi\sqrt{-g}}\frac{\delta\left(\mathcal{L}_m\sqrt{-g}\right)}{\delta e^{a}{}_{\nu}}\,, \tag{1.31}$$

[5] For an exhaustive study on the class of quadratic PG Lagrangians including parity violating terms, see reference [24].

$$S_{\lambda\mu}{}^{\nu} = \frac{e^a{}_\lambda e^b{}_\mu}{16\pi\sqrt{-g}} \frac{\delta\left(\mathcal{L}_m\sqrt{-g}\right)}{\delta\omega^{ab}{}_\nu}\,. \tag{1.32}$$

This variational procedure is a direct consequence of the gauge invariance of the Poincaré group, whose non-Abelian nature is also present in the physical model in virtue of the highly nonlinear character shown by the field equations (1.29) and (1.30). In addition, the canonical energy-momentum tensor derived from this approach is not generally symmetric in the presence of torsion and the spin density tensor is antisymmetric in its first pair of indices. Moreover, it is straightforward to obtain from the field equations the following conservation laws for these tensors:

$$\nabla_\nu \theta_\mu{}^\nu + K_{\lambda\rho\mu}\theta^{\rho\lambda} + \tilde{R}_{\lambda\rho\nu\mu} S^{\lambda\rho\nu} = 0\,, \tag{1.33}$$

$$\nabla_\mu S_{\lambda\rho}{}^\mu + 2K^\sigma{}_{[\lambda|\mu} S_{|\rho]\sigma}{}^\mu - \theta_{[\lambda\rho]} = 0\,. \tag{1.34}$$

Thereby, both quantities act as sources of gravity and represent the translational and rotational currents, respectively. They constitute the natural generalization of the conserved Noether currents associated with the external translations and rotations of the Poincaré group in a Minkowski space-time, as expected [27]:

$$\partial_\nu \theta_\mu{}^\nu = 0\,, \tag{1.35}$$

$$\partial_\mu J_{\lambda\rho}{}^\mu = 0\,, \tag{1.36}$$

where $J_{\lambda\rho}{}^\mu$ is the total angular momentum density, which is decomposed into an orbital part and an intrinsic part (i.e. the spin density tensor):

$$J_{\lambda\rho}{}^\mu = M_{\lambda\rho}{}^\mu + S_{\lambda\rho}{}^\mu\,, \tag{1.37}$$

with $M_{\lambda\rho}{}^\mu = x_{[\lambda}\,\theta_{\rho]}{}^\mu$ the resulting orbital angular momentum density, whose divergence is trivially proportional to the antisymmetric part of the canonical energy-momentum tensor:

$$\partial_\mu M_{\lambda\rho}{}^\mu = \theta_{[\rho\lambda]}\,. \tag{1.38}$$

Since the addition of total derivatives into the total Lagrangian preserves the invariance of the mentioned conservation laws, it turns out that the canonical currents are not uniquely defined and it is possible to establish fundamental relations between them. In particular, as is shown, the canonical energy-momentum tensor generally contains an antisymmetric component even when the notions of curvature and torsion are neglected (i.e. in the framework of Special Relativity), but it is possible to relocalize it by applying a symmetrization procedure [28]:

$$T_{\mu\nu} = \theta_{\mu\nu} - \partial_\lambda S_{\mu\nu}{}^{\lambda} - 2\partial_\lambda S^{\lambda}{}_{(\mu\nu)}\,. \tag{1.39}$$

In fact, we denote such a symmetric quantity as $T_{\mu\nu}$ because it was also shown that, by replacing the ordinary derivatives by torsion-free covariant derivatives, it actually coincides with the energy-momentum tensor defined from GR [29]. In virtue of this procedure, it is also common to designate this tensor as the Belinfante-Rosenfeld (BR) energy-momentum tensor. The generalization to RC space-times gives rise to the following expression [30]:

$$T_{\mu\nu} = \theta_{\mu\nu} - \overset{\star}{\nabla}_\lambda S_{\mu\nu}{}^{\lambda} - 2\overset{\star}{\nabla}_\lambda S^{\lambda}{}_{(\mu\nu)}\,, \tag{1.40}$$

with $\overset{\star}{\nabla}_\lambda = \tilde{\nabla}_\lambda - T^{\rho}{}_{\lambda\rho}$.

Thus, whereas the symmetric BR tensor represents the energy-momentum distribution of matter in GR, this situation does not hold in the PG theory of gravity due to the dynamical character of the spin density tensor in the presence of torsion. On the contrary, such a role falls on the canonical energy-momentum tensor, so the symmetric energy-momentum tensor of GR must be replaced by this quantity.

1.4 Motion of test particles in the Poincaré gauge theory

As previously stressed, the presence of a space-time torsion generalizes the conservation laws associated with the energy-momentum and spin density tensors of matter, in such a form that these currents completely coincide with the ones derived from GR when the latter vanishes:

$$\nabla_\nu \theta_\mu{}^{\nu} = 0\,, \tag{1.41}$$

$$\theta_{[\mu\nu]} = 0\,. \tag{1.42}$$

This result is a direct consequence of the deep relation existing between the torsion field and the intrinsic angular momentum of matter in the realm of the PG theory, where it operates as a source of torsion. It means that it is crucial to distinguish between the motion of spinning and spinless particles when considering the physical trajectories of test bodies from this approach. Indeed, neither the curves of extremal length given by the geodesic equations:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\lambda\rho}\frac{dx^\lambda}{ds}\frac{dx^\rho}{ds} = 0\,, \tag{1.43}$$

nor the straightest lines defined by the parallel transport of a vector to itself in terms of the autoparallel equations:

$$\frac{d^2x^\mu}{ds^2} + \tilde{\Gamma}^\mu{}_{\lambda\rho}\frac{dx^\lambda}{ds}\frac{dx^\rho}{ds} = 0\,, \tag{1.44}$$

can represent the general motion of matter in the presence of a space-time torsion. Conversely, whereas the former can only be related to spinless particles, the latter does not distinguish between particles with a different spin and then the torsion tensor affects the motion of particles with and without spin in the same way.

An appropriate expression, however, can be obtained by the conservation laws (1.33) and (1.34) by integrating over a three-dimensional spacelike section of the world tube involving the particle and employing the semiclassical approximation [2, 31]:

$$\begin{aligned} & \int \partial_\nu\left(\sqrt{-g}\,\theta^{\mu\nu}\right) d^3x' + \int \Gamma^\mu{}_{\lambda\rho}\theta^{\lambda\rho}\sqrt{-g}\,d^3x' \\ = & -\int K_{\lambda\rho}{}^\mu\theta^{\rho\lambda}\sqrt{-g}\,d^3x' - \int \tilde{R}_{\lambda\rho\sigma}{}^\mu\, S^{\lambda\rho\sigma}\sqrt{-g}\,d^3x'\,, \end{aligned} \tag{1.45}$$

with:

$$\int \partial_\nu\left(\sqrt{-g}\,\theta^{\mu\nu}\right) d^3x' = \frac{d}{dt}\int \theta^{\mu t}\sqrt{-g}\,d^3x'\,, \tag{1.46}$$

by the Gauss theorem. Then, by defining the four-momentum p^μ and the net spin angular momentum $S^{\lambda\rho}$, of the particle with four-velocity u^μ, in terms of the proper time s along its world line:

$$p^\lambda u^\rho = \frac{dt}{ds}\int \theta^{\lambda\rho}\sqrt{-g}\,d^3x'\,, \tag{1.47}$$

$$S^{\lambda\rho}u^\sigma = \frac{dt}{ds}\int S^{\lambda\rho\sigma}\sqrt{-g}\,d^3x'\,, \tag{1.48}$$

the expression (1.45) involves the following equations of motion:

$$\frac{dp^\mu}{ds} + \Gamma^\mu{}_{\lambda\rho}\, p^\lambda u^\rho + K_{\lambda\rho}{}^\mu p^\rho u^\lambda + \tilde{R}_{\lambda\rho\sigma}{}^\mu S^{\lambda\rho} u^\sigma = 0\,. \tag{1.49}$$

As can be seen, an additional generalized Lorentz force emerges depending on the intrinsic angular momentum of matter and the torsion tensor, which is contained in the RC curvature and the contortion component. This force potentially yields deviations from the geodesic trajectories and it represents another fundamental difference with the standard approach of GR. In this sense, it is straightforward to check that, for spinless matter (i.e. $S^{\lambda\rho} = 0$), the equations of motion reduce to the same geodesic equations of GR.

Nevertheless, since the spin of elementary particles is of the order of the Planck constant, it is expected that the strength of this force yields effects too tiny to be measured, as occurs in the context of other well-known gravitational theories framed beyond GR. From an experimental point of view, this means the difficulty in proving the possible existence of a non-vanishing dynamical torsion in the space-time. Moreover, it has been argued the possibility of measuring torsion effects by making use of a macroscopic rotating gyroscope (i.e. a gyroscope with vanishing net spin) [32]. Even so, the insufficiency of these types of arguments has been systematically pointed out because of the uncoupling between torsion and the orbital angular momentum of such gyroscopes [33, 34]. This situation changes when a polarized system with a net elementary particle spin is considered, although this possibility still requires more research and development, in order to generate an appreciable effect on its trajectories [35–37].

On the other hand, torsion is induced on the vierbein field by the field equations and thereby it can also operate on the geodesic motion of ordinary matter via the LC connection. This fact may involve additional effects to detect the possible existence of this geometric field.

1.5 The Dirac equation in the presence of torsion

The Dirac equation describes the wave function of spin 1/2 particles. It represents a crucial tool to analyse the influence of gravity on these sorts of particles. From a mathematical point of view, although general coordinate transformations do not have spinor representations, these fields can be described by the representations $(0, 1/2) \bigoplus (1/2, 0)$ associated with the Lorentz group [38]. Therefore, a Lorentz spin connection ω_μ is introduced in order to establish a well-defined covariant Dirac equation and to provide the dynamics of the spinor fields on a general space-time:

$$\omega_\mu = i\, \omega^{ab}{}_\mu \left[\gamma_a, \gamma_b\right]\,, \tag{1.50}$$

where $\omega^{ab}{}_{\mu}$ coincides with the Expression (1.23) when the coupling with torsion is considered and γ_a are the four constant Dirac matrices.

Then, it is possible to perform the following covariant derivative of a Dirac spinor:

$$\tilde{\nabla}_{\mu}\Psi = \partial_{\mu}\Psi - \omega^{ab}{}_{\mu}\left[\gamma_a, \gamma_b\right]\Psi\,. \tag{1.51}$$

In the minimal coupling, the ordinary derivative is simply replaced by this sort of covariant derivative, which includes the torsion tensor and therefore it can operate on Dirac spinors. Thereby, the generalized Dirac Lagrangian of a spinor with mass m minimally coupled to torsion is written in the following way [18]:

$$\mathcal{L}_{Dirac} = \frac{i}{2}\left(\bar{\Psi}\,\gamma^{\mu}\tilde{\nabla}_{\mu}\Psi - \tilde{\nabla}_{\mu}\bar{\Psi}\,\gamma^{\mu}\Psi - 2im\bar{\Psi}\Psi\right)\,, \tag{1.52}$$

where $\bar{\Psi} = \Psi^{\dagger}\gamma^0$ is the Dirac adjoint. By performing the hermitian conjugation of the Expression (1.51) and multiplying by γ^0 from the right, the identity $(\gamma^a)^{\dagger} = \gamma^0\gamma^a\gamma^0$ implies the covariant derivative of a Dirac adjoint:

$$\tilde{\nabla}_{\mu}\bar{\Psi} = \partial_{\mu}\bar{\Psi} + \omega^{ab}{}_{\mu}\bar{\Psi}\left[\gamma_a, \gamma_b\right]\,, \tag{1.53}$$

and separates the metric and torsion contributions into the Dirac Lagrangian as follows:

$$\mathcal{L}_{Dirac} = \frac{i}{2}\left(\bar{\Psi}\,\gamma^{\mu}\nabla_{\mu}\Psi - \nabla_{\mu}\bar{\Psi}\,\gamma^{\mu}\Psi - e^{a\mu}e^{b}{}_{\lambda}e^{c\rho}K^{\lambda}{}_{\rho\mu}\bar{\Psi}\left\{\gamma_a, \left[\gamma_b, \gamma_c\right]\right\}\Psi - 2im\bar{\Psi}\Psi\right)\,. \tag{1.54}$$

Therefore, in the minimal coupling, the interaction term between torsion and the Dirac spinor depends on the anticommutator $\{\gamma_a, [\gamma_b, \gamma_c]\}$. It is possible to compute this factor by considering the properties of the product of three gamma matrices:

$$\gamma_a\gamma_b\gamma_c = \eta_{ab}\gamma_c + \eta_{bc}\gamma_a - \eta_{ac}\gamma_b + i\epsilon_{abc}{}^{d}\gamma_d\gamma^5\,, \tag{1.55}$$

where $\gamma^5 = \frac{i}{4!}\epsilon^{abcd}\gamma_a\gamma_b\gamma_c\gamma_d$ is the fifth gamma matrix, that additionally satisfies the following properties:

$$\left(\gamma^5\right)^{\dagger} = \gamma^5\,, \tag{1.56}$$

$$\left(\gamma^5\right)^{2} = I_4\,, \tag{1.57}$$

$$\left\{\gamma^5, \gamma^a\right\} = 0\,. \tag{1.58}$$

By taking into account these conditions, one obtains the following outcome:

$$\{\gamma_a, [\gamma_b, \gamma_c]\} = 4i\epsilon_{abc}{}^{d}\gamma_d\gamma^5\,, \tag{1.59}$$

which means that the mentioned interaction term constitutes a totally antisymmetric quantity coupled to the component of the Lorentz spin connection depending on torsion:

$$\mathcal{L}_{Dirac} = \frac{i}{2}\left(\bar{\Psi}\,\gamma^\mu\nabla_\mu\Psi - \nabla_\mu\bar{\Psi}\,\gamma^\mu\Psi + 2i\epsilon^{\lambda\rho\mu\nu}T_{\lambda\rho\mu}\bar{\Psi}\gamma^5\gamma_\nu\Psi - 2im\bar{\Psi}\Psi\right)\,. \tag{1.60}$$

Indeed, the resulting Dirac Lagrangian can be expressed in a more compact form in terms of the axial component of torsion:

$$\mathcal{L}_{Dirac} = \frac{i}{2}\left(\bar{\Psi}\,\gamma^\mu\nabla_\mu\Psi - \nabla_\mu\bar{\Psi}\,\gamma^\mu\Psi + 2i\bar{\Psi}\gamma^5\gamma^\mu S_\mu\Psi - 2im\bar{\Psi}\Psi\right)\,. \tag{1.61}$$

This result yields an explicit interaction between torsion and Dirac spinors depending on the axial vector alone, so that the presence of the rest of the irreducible parts of the torsion tensor does not alter itself. Such components may only enter implicitly in the interaction if they are induced on the vierbein field present in the Lagrangian. On the other hand, since there is still no experimental evidence on the existence of the torsion field, the formulation of other Lagrangians non-minimally coupled to torsion may be viable [39, 40]. These types of configurations introduce corrections into the interaction scheme and enable an active role of the additional modes of torsion in the presence of fermions.

1.6 Teleparallelism

As was indicated previously, a general PG model of gravity is commonly characterized by the presence of both curvature and torsion by means of RC geometry. Nevertheless, certain degenerate cases arise when the restriction of vanishing some of these quantities is applied. For example, the linear PG Lagrangian reduces to the conventional EH Lagrangian if the condition of a vanishing torsion tensor is imposed, which means that the resulting approach is completely determined in terms of Riemannian geometry (i.e. in terms of the LC connection). On the other hand, it is also possible to construct alternative gravity theories with torsion by imposing the vanishing of the curvature tensor alone. This condition is fulfilled for a non-trivial set of values of torsion that cancels the RC curvature. Indeed, the RC curvature (1.14) can be expressed as the sum of the Riemannian torsion-free curvature and a post-Riemannian component depending on the torsion tensor:

$$\tilde{R}^{\lambda}{}_{\rho\mu\nu} = R^{\lambda}{}_{\rho\mu\nu} + \nabla_{\mu} K^{\lambda}{}_{\rho\nu} - \nabla_{\nu} K^{\lambda}{}_{\rho\mu} + K^{\lambda}{}_{\sigma\mu} K^{\sigma}{}_{\rho\nu} - K^{\lambda}{}_{\sigma\nu} K^{\sigma}{}_{\rho\mu} \,, \tag{1.62}$$

which means that it vanishes identically when the following constraint is satisfied:

$$R^{\lambda}{}_{\rho\mu\nu} = \nabla_{\nu} K^{\lambda}{}_{\rho\mu} - \nabla_{\mu} K^{\lambda}{}_{\rho\nu} + K^{\lambda}{}_{\sigma\nu} K^{\sigma}{}_{\rho\mu} - K^{\lambda}{}_{\sigma\mu} K^{\sigma}{}_{\rho\nu} \,. \tag{1.63}$$

In terms of the affine connection, it is straightforward to find a solution for this equation by imposing the vanishing of the Lorentz spin connection. This choice cancels the Lorentz gauge curvature $F^{ab}{}_{\mu\nu}$ and hence the RC curvature tensor, since both are related by the Expression (1.26). The resulting connection is called the Weitzenböck connection and thereby it provides a gauge theory of gravitation for the translation group [41–44]:

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = e_{a}{}^{\lambda} \partial_{\nu} e^{a}{}_{\mu} \,. \tag{1.64}$$

The absence of curvature enables the definition of a path-independent parallel transport within the manifold, which involves the notion of parallelism of vectors at different points. In addition, since the relation (1.63) allows the torsion-free curvature tensor to be expressed in terms of torsion, it is possible to construct a gravitational action equivalent to the EH action of GR up to a divergence term, which does not contribute to the field equations:

$$\begin{aligned} S = -\frac{1}{16\pi} \int R \sqrt{-g}\, d^4x \;\; &= \;\; \frac{1}{64\pi} \int \Big[T_{\lambda\mu\nu} T^{\lambda\mu\nu} + 2 T_{\lambda\mu\nu} T^{\mu\lambda\nu} - 4 T^{\mu}{}_{\mu\lambda} T^{\nu}{}_{\nu}{}^{\lambda} \\ &\qquad - \frac{8}{\sqrt{-g}}\, \partial_{\mu} \Big(T^{\lambda\mu}{}_{\lambda} \sqrt{-g} \Big) \Big] \sqrt{-g}\, d^4x \,, \end{aligned} \tag{1.65}$$

with:

$$T^{\lambda}{}_{\mu\nu} = e_{a}{}^{\lambda} \left(\partial_{\nu} e^{a}{}_{\mu} - \partial_{\mu} e^{a}{}_{\nu} \right) \,. \tag{1.66}$$

The resulting model is then completely expressed in terms of the torsion tensor of a Weitzenböck space-time, which means that such a quantity replaces curvature in order to describe the gravitational field. Likewise, the corresponding energy-momentum tensor derived from this approach does not act as a source of curvature, but as a source of torsion.

From a phenomenological point of view, teleparallelism provides an equivalent description of gravity to GR in terms of the mentioned translational field strength tensor, which is shown to be completely determined by the vierbein field. This fact reveals that curvature and torsion are simply alternative ways of describing the conventional gravitational field. In this sense, teleparallelism does not involve new physics related to torsion. Nevertheless, both approaches are conceptually different, since the geometrical correspondence existing in GR between curvature and gravitation does not hold in a teleparallel model based on torsion. Indeed, following the geometric structure of GR, the trajectories of free-falling particles present in a curved space-time results in a geodesic motion depending on the LC connection:

$$\frac{dp^{\mu}}{ds} + \Gamma^{\mu}{}_{\lambda\rho}\, p^{\lambda} u^{\rho} = 0\,, \tag{1.67}$$

whereas the introduction of a Weitzenböck connection $\tilde{\Gamma}^{\mu}{}_{\lambda\rho}$ with vanishing curvature derives straightforwardly in the following expression [45, 46]:

$$u^{\lambda}\tilde{\nabla}_{\lambda} p^{\mu} = T_{\lambda}{}^{\mu}{}_{\rho}\, p^{\lambda} u^{\rho}\,, \tag{1.68}$$

where $u^{\lambda}\tilde{\nabla}_{\lambda} p^{\mu} = \frac{dp^{\mu}}{ds} + \tilde{\Gamma}^{\mu}{}_{\lambda\rho}\, p^{\lambda} u^{\rho}$ is the four-acceleration of the particle in the consequent Weitzenböck space-time. Then, the equations of motion are modified in a form where the torsion tensor plays the role of a gravitational force operating on the particle, instead of a purely geometrical effect such as the one given by curvature in the regular case.

1.7 Gravitation with non-propagating torsion: the Einstein-Cartan theory

Another singular case of the PG theory arises when the higher order corrections present in the Lagrangian are excluded from the final scheme. Indeed, in the same way that the EH action is related to the Ricci scalar depending on the metric tensor alone, in a first-order approximation it is possible to generalize this action by means of the Ricci scalar defined on a RC space-time, providing the so called Einstein-Cartan (EC) theory [21]:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\mathcal{L}_m - \tilde{R}\right)\,, \tag{1.69}$$

where:

$$\tilde{R} = R + \frac{1}{4} T_{\lambda\mu\nu} T^{\lambda\mu\nu} + \frac{1}{2} T_{\lambda\mu\nu} T^{\mu\lambda\nu} - T^{\mu}{}_{\mu\lambda} T^{\nu}{}_{\nu}{}^{\lambda} - 2\nabla_{\lambda} T^{\rho\lambda}{}_{\rho} \,. \tag{1.70}$$

In this case, the Lagrangian contains, besides the torsion-free Ricci scalar and a total derivative, a particular combination of the three independent quadratic scalar invariants of torsion, that are computed into the field equations by performing variations with respect to the gauge potentials, as usual. Accordingly, this analysis lead to the following field equations:

$$\tilde{G}_{\mu\nu} = 8\pi \, \theta_{\nu\mu} \,, \tag{1.71}$$

$$\delta_{\mu}{}^{\nu} g^{\lambda\sigma} T^{\rho}{}_{\rho\sigma} - g^{\lambda\nu} T^{\rho}{}_{\rho\mu} - g^{\lambda\sigma} T^{\nu}{}_{\mu\sigma} = 16\pi \, S_{\mu}{}^{\lambda\nu} \,. \tag{1.72}$$

The first equation provides higher order corrections in the torsion tensor to the Riemannian component of the Einstein tensor. Consequently, it generally involves the existence of a non-vanishing antisymmetric component of the canonical energy-momentum tensor. In addition, the second equation associates directly the torsion and the spin density tensor of matter sources by an algebraic relation, rather than by a differential expression for the torsion field. This leads to a non-dynamical character for torsion under the EC theory, which prevents this quantity to propagate in a vacuum configuration and forces it to vanish when the spin density tensor is zero. Therefore, it can only generate physical effects inside spinning matter and influence directly on other sources through a spin-spin contact interaction.

Furthermore, the standard decomposition (1.40) of the canonical energy-momentum tensor into the totally symmetrized energy-momentum tensor and the spin density tensor allows the vierbein equation (1.71) to be rewritten as the standard Einstein equation of GR with an additional geometric correction quadratic in the torsion tensor (i.e. in the spin density tensor since both torsion and spin are directly related by Equation (1.72)). Indeed, within this model, it is straightforward to express torsion as a tensorial function of the spin density tensor as follows:

$$T^{\lambda}{}_{\mu\nu} = 8\pi \left(2 S_{\nu\mu}{}^{\lambda} + \delta^{\lambda}{}_{\mu} S^{\rho}{}_{\nu\rho} - \delta^{\lambda}{}_{\nu} S^{\rho}{}_{\mu\rho} \right) \,, \tag{1.73}$$

whereas the Einstein tensor in the presence of torsion is split into its torsion-free component and an extended piece depending on torsion in the following way:

$$
\begin{aligned}
\tilde{G}_{\mu\nu} &= G_{\mu\nu} + \frac{1}{2}\left(\nabla_\lambda T^\lambda{}_{\mu\nu} - \nabla_\lambda T_\mu{}^\lambda{}_\nu - \nabla_\lambda T_\nu{}^\lambda{}_\mu - 2\nabla_\nu T^\lambda{}_{\mu\lambda}\right) \\
&+ \frac{1}{2}\left(T^\lambda{}_{\rho\lambda} T^\rho{}_{\mu\nu} - T^\lambda{}_{\rho\lambda} T_\mu{}^\rho{}_\nu - T^\lambda{}_{\rho\lambda} T_\nu{}^\rho{}_\mu + T^\lambda{}_{\rho\nu} T_\mu{}^\rho{}_\lambda - \frac{1}{2} T_\nu{}^\lambda{}_\rho T_\mu{}^\rho{}_\lambda\right) \\
&+ g_{\mu\nu}\left(\nabla_\lambda T^{\rho\lambda}{}_\rho - \frac{1}{8} T_{\lambda\rho\sigma} T^{\lambda\rho\sigma} - \frac{1}{4} T_{\lambda\rho\sigma} T^{\rho\lambda\sigma} + \frac{1}{2} T^\lambda{}_{\lambda\sigma} T^\rho{}_\rho{}^\sigma\right) . \qquad (1.74)
\end{aligned}
$$

Then, the relation existing between the torsion-free Einstein tensor and the BR energy-momentum tensor is linked to a higher order correction quadratic in the spin density tensor itself:

$$
G_{\mu\nu} = 8\pi\, T_{\mu\nu} + \mathcal{O}\left(S^2\right) . \qquad (1.75)
$$

In virtue of the general construction of the EC theory, such a correction is also proportional to the square of Einstein's gravitational constant, which implies that the possible effects derived from the EC torsion may only be measured under the most extreme macroscopic conditions.

Chapter 2

Vacuum solutions of the Poincaré gauge theory

2.1 The Baekler solution: torsion and confinement type of potential

On account of the general PG field equations (1.29) and (1.30), the propagating character of torsion demands the presence of higher order curvature terms in the gravitational action. Indeed, the variational procedure derived from these terms gives rise to a set of differential expressions for the torsion field. This means the possible existence of a propagating torsion even in absence of matter sources (i.e. in physical configurations with vanishing energy-momentum and spin density tensors). From a fundamental point of view, this feature represents a deep aspect in the nature of torsion, which may also produce significant effects under these conditions in the geometry of the space-time.

In particular, Birkhoff's theorem of GR establishes that the only vacuum solution to the Einstein field equations with spherical symmetry is the Schwarzschild solution [47]. However, in the realm of PG gravity, this theorem is satisfied only in certain cases [48, 49]. Then, by considering the general PG Lagrangian with dynamical torsion, the approach leads to a large class of gravitational models endowed with a vacuum structure where an extensive number of particular and fundamental differences may arise. This fact evinces that the search and analysis of exact solutions are essential in order to improve the understanding and physical consequences of this field.

One of the most primary and remarkable solutions is the so called Baekler solution [50]. It constitutes an exact vacuum solution with propagating torsion, which refers to a PG Lagrangian whose limit to the regular gravitational model takes place in the framework of teleparallel geometry to a first approximation [51, 52]:

$$S = -\frac{1}{32\pi} \int \left(2T^{\mu}{}_{\mu\lambda} T^{\nu}{}_{\nu}{}^{\lambda} - T_{\lambda\mu\nu} T^{\lambda\mu\nu} - \frac{1}{4\kappa} \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\lambda\rho\mu\nu} \right) \sqrt{-g}\, d^4x \,, \tag{2.1}$$

where κ is a coupling constant provided by the supplementary and presumably very weak gravitational interaction. Thereby, the action is divided into a first term connected with the long-range Einstein type of gravity that comprises the Schwarzschild solution and a YM-like factor depending on the curvature tensor that introduces slight corrections to this approach, which means a richer structure than the one present in Einstein's theory.

The corresponding field equations associated with this model are then described by the following system:

$$\begin{aligned} &\frac{1}{4}\delta_{\mu}{}^{\nu} \left(2T^{\lambda}{}_{\lambda\sigma} T^{\rho}{}_{\rho}{}^{\sigma} - 4\nabla_{\lambda} T^{\rho}{}_{\rho}{}^{\lambda} - T_{\lambda\rho\sigma} T^{\lambda\rho\sigma} \right) + \nabla_{\mu} T^{\lambda}{}_{\lambda}{}^{\nu} + \nabla_{\lambda} T_{\mu}{}^{\nu\lambda} \\ = \; &\frac{1}{4\kappa} \left(\frac{1}{4}\delta_{\mu}{}^{\nu} \tilde{R}_{\lambda\rho\tau\sigma} \tilde{R}^{\lambda\rho\tau\sigma} - \tilde{R}_{\lambda\rho\mu\sigma} \tilde{R}^{\lambda\rho\nu\sigma} \right) - K^{\nu}{}_{\lambda\mu} T^{\rho}{}_{\rho}{}^{\lambda} - K_{\lambda\rho\mu} T^{\lambda\rho\nu} \,, \end{aligned} \tag{2.2}$$

$$2\kappa \left(\delta_{\mu}{}^{\nu} T^{\rho}{}_{\rho}{}^{\lambda} - g^{\lambda\nu} T^{\rho}{}_{\rho}{}^{\mu} + T^{\lambda\nu}{}_{\mu} - T_{\mu}{}^{\nu\lambda} \right) = \nabla_{\rho} \tilde{R}_{\mu}{}^{\lambda\rho\nu} + K^{\lambda}{}_{\sigma\rho} \tilde{R}_{\mu}{}^{\sigma\rho\nu} - K^{\sigma}{}_{\mu\rho} \tilde{R}_{\sigma}{}^{\lambda\rho\nu} \,. \tag{2.3}$$

As can be seen, in the limit of teleparallelism, the curvature tensor disappears from the variational equations and torsion operates as the unique geometrical quantity describing the gravitational field. Furthermore, the resulting Lagrangian with vanishing curvature encompasses the Schwarzschild metric as a solution and it presents an agreement with the standard tests of GR up to the fourth order in the post-Newtonian approximation [53]. In fact, although the expression of such a Lagrangian does not coincide exactly with the one given by the equivalent version of GR in teleparallelism, its deviations do not yield any difference for the case of static and isotropic space-times. Accurately, these deviations can be computed by the subtraction of the mentioned Lagrangians:

$$S = \frac{1}{64\pi} \int \left(T_{\lambda\mu\nu} T^{\lambda\mu\nu} - 2T_{\lambda\mu\nu} T^{\mu\lambda\nu} \right) \sqrt{-g}\, d^4x \,, \tag{2.4}$$

or, equivalently:

$$S = \frac{1}{128\pi} \int S_{\mu} S^{\mu} \sqrt{-g}\, d^4x \,, \tag{2.5}$$

where the axial mode S_μ of torsion vanishes for such static and spherically symmetric Weitzenböck space-times [54]. Consider the line element and the tetrad basis of these types of geometrical systems:

$$ds^2 = \Psi_1(r)\,dt^2 - \frac{dr^2}{\Psi_2(r)} - r^2\left(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2\right)\,, \tag{2.6}$$

$$e^{\hat{t}} = \sqrt{\Psi_1(r)}\,dt\,,\quad e^{\hat{r}} = \frac{dr}{\sqrt{\Psi_2(r)}}\,,\quad e^{\hat{\theta_1}} = r\,d\theta_1\,,\quad e^{\hat{\theta_2}} = r\sin\theta_1\,d\theta_2\,; \tag{2.7}$$

with $0 \leq \theta_1 \leq \pi$ and $0 \leq \theta_2 \leq 2\pi$. In that case, as is shown in Appendix B, the intrinsic relations between curvature and torsion involve further symmetries on this tensor, which must also satisfy the following condition:

$$\mathcal{L}_\xi T^\lambda{}_{\mu\nu} = 0\,, \tag{2.8}$$

in order to ensure that the covariant derivative commutes with the Lie derivative and preserve the invariance of the curvature tensor under isometries.

By following these remarks, the static and isotropic torsion acquires the following structure [49, 55]:

$$\begin{aligned}
T^t{}_{tr} &= a(r) \quad ,\\
T^r{}_{tr} &= b(r) \quad ,\\
T^{\theta_k}{}_{t\theta_k} &= c(r) \quad ,\\
T^{\theta_k}{}_{r\theta_k} &= g(r) \quad ,\\
T^{\theta_k}{}_{t\theta_l} &= e^{\tilde{a}\theta_k}\, e^{\tilde{b}}{}_{\theta_l}\, \epsilon_{\tilde{a}\tilde{b}}\, d(r) \quad ,\\
T^{\theta_k}{}_{r\theta_l} &= e^{\tilde{a}\theta_k}\, e^{\tilde{b}}{}_{\theta_l}\, \epsilon_{\tilde{a}\tilde{b}}\, h(r) \quad ,\\
T^t{}_{\theta_k\theta_l} &= \epsilon_{kl}\, k(r)\, \sin\theta_1 \quad ,\\
T^r{}_{\theta_k\theta_l} &= \epsilon_{kl}\, l(r)\, \sin\theta_1 \quad ;
\end{aligned} \tag{2.9}$$

where a, b, c, d, g, h, k and l are eight arbitrary functions depending only on r; $k, l = 1, 2$; $\tilde{a}, \tilde{b} = 3, 4$ and $\epsilon_{\tilde{a}\tilde{b}}$ is the two-dimensional LC symbol. Thus, in the framework of teleparallelism, the additional requirement given by the presence of a Weitzenböck connection fixes the supplementary condition (1.66) on the torsion tensor, which reduces the number of degrees of freedom mentioned above and involves the vanishing of the axial vector.

On the other hand, the additional gravitational interaction given by a non-vanishing curvature tensor provides a confinement type of potential in the weak-field limit proportional to κr, besides the Newtonian one yielded by the conventional gravitational field. Hence, this confining contribution arises in the linearized approximation resulting from the traces of the variational system of equations (2.2) and (2.3) by including the energy-momentum and spin density tensors of matter sources:

$$\partial_\mu T^{\nu}{}_{\nu}{}^{\mu} = 4\pi T^{\mu}{}_{\mu} \, , \tag{2.10}$$

$$T^{\nu}{}_{\nu}{}^{\mu} + \frac{1}{4\kappa}\partial_\nu \tilde{R}^{\mu\nu} = 8\pi S^{\mu}{}_{\nu}{}^{\nu} \, . \tag{2.11}$$

Differentiation of Equation (2.11) leads this expression to the following equation:

$$\partial_\mu \partial_\nu \tilde{R}^{\mu\nu} = 16\pi\kappa \left(2\partial_\mu S^{\mu}{}_{\nu}{}^{\nu} - T^{\mu}{}_{\mu}\right) \, . \tag{2.12}$$

Thereby, the usual decomposition of the vierbein field into the background field related to the Minkowski metric and a linear perturbation:

$$e^{a}{}_{\mu} = \delta^{a}{}_{\mu} - \frac{1}{2}h^{a}{}_{\mu} \, , \tag{2.13}$$

allows the previous equations to be rewritten in terms of perturbative fields of the gauge potentials:

$$\partial_\mu \partial^\mu h^{\nu}{}_{\nu} - \partial_\mu \partial_\nu h^{\mu\nu} - 2\partial_\mu \omega^{\mu\nu}{}_{\nu} = 8\pi T^{\mu}{}_{\mu} \, , \tag{2.14}$$

$$\partial_\lambda \partial^\lambda \partial_\mu \omega^{\mu\nu}{}_{\nu} = 16\pi\kappa \left(2\partial_\mu S^{\mu}{}_{\nu}{}^{\nu} - T^{\mu}{}_{\mu}\right) \, . \tag{2.15}$$

Finally, by applying the d'Alembert operator on Equation (2.14) and the harmonic coordinate condition $2\partial_\nu h^{(\mu\nu)} = \partial^\mu h^{\nu}{}_{\nu}$, it is straightforward to obtain the following differential equation of fourth order, besides the supplementary Equation (2.15) for the spin connection:

$$\partial_\mu \partial^\mu \partial_\nu \partial^\nu h^{\lambda}{}_{\lambda} = 16\pi \left(\partial_\nu \partial^\nu T^{\mu}{}_{\mu} - 4\kappa \left(T^{\mu}{}_{\mu} - 2\partial_\mu S^{\mu}{}_{\nu}{}^{\nu}\right)\right) \, , \tag{2.16}$$

which allows the computation of the perturbative gauge potentials in terms of the energy-momentum and spin density tensors by elementary integration.

The semiclassical perfect fluid with intrinsic spin angular momentum (1.48) is associated with the following traces of the material tensors [56]:

$$T^{\mu}{}_{\mu} = -\,\rho\,, \tag{2.17}$$

$$S^{\mu}{}_{\nu}{}^{\nu} = 0\,, \tag{2.18}$$

where ρ is the matter density, which in the weak-field approximation describes a mass m concentrated in an arbitrary point of coordinates $\mathbf{r}$. Then, by substituting the expression of the traces of the material tensors into the previous system of equations:

$$\Delta\Delta h^{\lambda}{}_{\lambda} = 64\pi\kappa m\left(1 - \frac{\Delta}{4\kappa}\right)\delta(\mathbf{r})\,, \tag{2.19}$$

$$\Delta\partial_{\mu}\omega^{\mu\nu}{}_{\nu} = 16\pi\kappa m\,\delta(\mathbf{r})\,. \tag{2.20}$$

By standard integration it is straightforward to find the following weak-field solutions:

$$h^{\lambda}{}_{\lambda} = -\frac{4m}{r} + c_1 + 8m\kappa r + c_2 r^2\,, \tag{2.21}$$

$$\partial_{\mu}\omega^{\mu\nu}{}_{\nu} = \frac{4m\kappa}{r} + c_3\,, \tag{2.22}$$

with $r_{min} \leq r \leq r_{max}$, whereas c_1, c_2 and c_3 are integration constants determined by boundary conditions in this domain. Apart from the Newtonian potential associated with torsion in $h^{\lambda}{}_{\lambda}$, the additional pieces depending on r allude to a confinement type of potential related to the curvature tensor, which points out the existence of new type of exact vacuum solutions distinct from the Schwarzschild solution of the standard case.

These solutions must then fulfill the general field equations (2.2) and (2.3) associated with Lagrangian (2.1). In virtue of the highly nonlinear character of these equations, additional symmetry constraints are particularly imposed, as the presence of a static and spherically symmetric space-time. In such a case, the metric and torsion tensors acquire the form (2.6) and (2.9), respectively. Thereby, besides the two functions associated with the metric, the SO(3)-symmetrical torsion contains eight degrees of freedom, which means that the problem of solving the variational equations turns out to be still very complicated and additional restrictions are required.

Specifically, two principal constraints are also applied in order to simplify the problem. First, a reflection invariance is imposed on the torsion tensor (i.e. torsion is invariant under the group $O(3)$), which involves the vanishing of the functions $d(r), h(r), k(r)$ and $l(r)$. In addition, the so called double duality ansatz allows the cancellation of the derivative of the curvature tensor in Expression (2.3) and the simplification of this equation [57]:

$$\tilde{R}_{\lambda\rho\mu\nu} = \frac{1}{4}\,\epsilon_{\lambda\rho\alpha\beta}\epsilon_{\mu\nu\gamma\sigma}\tilde{R}^{\alpha\beta\gamma\sigma} + 4\kappa\, g_{\mu[\lambda}g_{\rho]\nu}\,. \tag{2.23}$$

By contracting indices, this restriction also implies the constancy of the Ricci scalar:

$$\tilde{R} = 12\kappa\,, \tag{2.24}$$

which means that all the possible solutions derived from this ansatz share this geometrical constraint. In particular, the Baekler solution can be easily found with the following components of the metric and torsion tensors:

$$a(r) = \frac{m}{r^2\Psi(r)}\,, \quad b(r) = \frac{m}{r^2}\,, \quad c(r) = -\,\frac{m}{r^2}\,, \quad g(r) = \frac{m}{r^2\Psi(r)}\,; \tag{2.25}$$

$$\Psi_1(r) = \Psi_2(r) \equiv \Psi(r) = 1 - \frac{2m}{r} + \kappa r^2 \quad . \tag{2.26}$$

Therefore, the metric is a Schwarzschild-de Sitter type and carries both torsion and curvature. Indeed, the components of the latter can be represented by the function $\Phi(r) = \frac{m}{r\Psi(r)}$ and the following matrix:

$$F^{ab}{}_{cd} \;=\; \kappa \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+\Phi(r) & 0 & 0 & 0 & \Phi(r) \\ 0 & 0 & 1+\Phi(r) & 0 & -\,\Phi(r) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \Phi(r) & 0 & 1-\Phi(r) & 0 \\ 0 & -\,\Phi(r) & 0 & 0 & 0 & 1-\Phi(r) \end{pmatrix}\,, \tag{2.27}$$

where the components of the six rows and columns of the matrix are labeled in the order (01, 02, 03, 23, 31, 12).

As can be seen, the correction given by the new parameter κ to the conventional gravitational field acts as a cosmological constant in the field equations and the system reduces to the Schwarzschild solution of teleparallelism in the limit where

$\kappa \rightarrow 0$. Hence, it shows the expected behaviour of the gravitational potentials presented previously and fulfills the standard tests of GR.

In addition, it can be generalized in the presence of external Coulomb electromagnetic fields generated by both electric and magnetic charges q_e and q_m, respectively, by replacing the metric function $\Psi(r)$ in the following way [58]:

$$\Psi(r) = 1 - \frac{2m}{r} + \frac{q_e^2 + q_m^2}{r^2} + \kappa r^2 \,. \tag{2.28}$$

Furthermore, the same class of solution with double duality properties and confinement type of potential can be extended to the axisymmetric case by considering a $SO(3)$-symmetrical torsion [59–62]. These results improve the understanding on the new gravitational interaction considered by the action (2.1) and confer the role of an effective cosmological constant to curvature, even when a pure constant parameter is not present in the Lagrangian.

On the other hand, other exact vacuum solutions related to different PG models uncovered by Birkhoff's theorem have been additionally found [63–66]. Some of them are not totally determined by the respective variational equations, giving rise to solutions with a high geometrical freedom and depending on arbitrary functions. This fact notably reduces the appropriate physical consistency of these models, in contrast with the particular quadratic PG theory studied above.

Finally, a large number of works on cosmology and gravitational radiation have also been accomplished in the framework of the PG theory. They implement the dynamical aspects of the torsion field into the gravitational arena and show in general interesting differences with respect to the standard regime, such as the transfer of the metric singularities to the torsion tensor or the acceleration pattern for the expansion of the universe analogous to the one given by a cosmological constant, among others (see [22, 63, 67, 68] and references therein).

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New torsion black hole solutions in Poincaré gauge theory

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Abstract. We derive a new exact static and spherically symmetric vacuum solution in the framework of the Poincaré gauge field theory with dynamical massless torsion. This theory is built in such a form that allows to recover General Relativity when the first Bianchi identity of the model is fulfilled by the total curvature. The solution shows a Reissner-Nordström type geometry with a Coulomb-like curvature provided by the torsion field. It is also shown the existence of a generalized Reissner-Nordström-de Sitter solution when additional electromagnetic fields and/or a cosmological constant are coupled to gravity.

Keywords: gravity, modified gravity, astrophysical black holes, neutron stars

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1 Introduction

General Relativity (GR) is the most successful and accurate theory of classical gravity from the last century. Its outstanding description of the gravitational interaction as a purely geometrical effect of the space-time together with a large number of experimental evidences has exalted it as the fundamental theoretical basis for modern astrophysics and cosmology [1]. Even nowadays, its elemental foundations and further implications are continually being reviewed and tested, as in the case of the recent discovery of gravitational waves from a binary black hole system [2]. Nevertheless, extensions of GR have always attracted much attention due to the deep related fundamental concepts and open questions still unsolved by the theory, as the formulation of a consistent quantum field approach to gravity, the understanding of space-time singularities or the nature of dark energy, dark matter or inflation in the very early Universe [3–6].

Another open issue consists in providing correctly the foundations of the angular momentum of gravitating sources and its suitable conservation laws in presence of a dynamical space-time within the same framework. Specifically, the intrinsic angular momentum of matter must be represented by a spin density tensor and therefore it may be expected to have it associated with a fundamental geometrical quantity. However, in standard GR, it does not couple to any distinctive geometrical property, so it is analysed possible modifications of the theory according to these lines.

In this sense, Poincaré Gauge (PG) theory provides the most elegant and promising extension of GR, in the framework of a Riemann-Cartan (RC) manifold (i.e. a manifold endowed with curvature and torsion), in order to couple the spin of particles to the torsion of the space-time [7, 8]. Indeed, within this model, both energy-momentum and spin tensors of gravitating matter act as sources of the interaction. In addition, the role of torsion depends on the order of the field strength tensors included in the Lagrangian: whereas the full linear case involves a non-propagating torsion (i.e. tied to spinning material sources), higher order corrections describe a Lagrangian with dynamical torsion [9, 10].

Furthermore, the vacuum structure of the space-time also differs depending on this critical role, especially when a certain class of PG models provides the existence of propagating torsion modes in vacuum. Specifically, Birkhoff's theorem establishing that the only vacuum solution with spherical symmetry is the Schwarzschild solution, is satisfied only in certain cases of the PG theory [11, 12]. In this work, we consider a particular PG theory described by a Lagrangian of first and second order in the curvature terms, which reduces to ordinary GR when torsion satisfies a general condition connected to the first Bianchi identity. Only in such a case, it loses its physical relevance. It is shown that within this framework, the Birkhoff's theorem is not satisfied and a new analytical SO(3) spherically symmetric and static vacuum solution with dynamical torsion emerges. This solution describes a Reissner-Nordström type configuration characterized exclusively for its mass and the torsion field contribution, in analogy to the electric charge in Maxwell's theory. Thus, by this contribution of the torsion field to the space-time geometry, neither other physical sources nor electromagnetic fields are necessary to generate this type of solutions. On the other hand, we also stress that it is always possible to find a generalized Reissner-Nordström-de Sitter solution endowed with both electric and magnetic charges, as well as with a cosmological constant within this construction. Finally, the equations of motion for a general test particle in such a space-time are obtained from the respective conservation law of the energy-momentum tensor of matter.

This work is organized as follows. First, in section II, we briefly present the general mathematical foundations of PG theory paying spetial attention to our model. Field equations and analyses of general solutions beyond the Birkhoff's theorem for GR and different classes of PG theories are shown in section III. Our new analytical solution within this framework, as well as its natural generalization to include external Coulomb electric and magnetic fields with a non-vanishing cosmological constant are presented and analysed in section IV. In section V, we obtain from the general conservation law of the energy-momentum tensor, the equations of motion for a test particle belonging to a RC manifold connected to our model. Finally, we present the conclusions of our work in section VI. A general demonstration for the conservation law of the energy-momentum tensor is also presented in appendix A.

Before proceeding to the main discussion and general results, we briefly introduce the notation and physical units to be used throughout this article. Latin a, b and greek μ, ν indices refer to anholonomic and coordinate basis, respectively. We use notation with tilde for magnitudes including torsion (i.e. defined within a RC manifold) and without tilde for torsionless objects. Finally, we will use Planck units ($G = c = \hbar = 1$).

2 Quadratic Poincaré gauge gravity model

A model of PG gravity requires gauging the external degrees of freedom consisting of rotations and translations, which are represented by the Poincaré group $ISO(1,3)$. Therefore, a gauge connection containing two principal independent variables is introduced in order to describe the gravitational field. These quantities constitute the gauge potentials related to the generators of translations and local Lorentz rotations, respectively:

$$A_\mu = e^a{}_\mu P_a + \omega^{ab}{}_\mu J_{ab}\,, \tag{2.1}$$

where $e^a{}_\mu$ is the vierbein field and $\omega^{ab}{}_\mu$ the spin connection, which satisfy the following relations with the metric g and the affine connection $\tilde{\Gamma}$ within the RC manifold [13]:

$$g_{\mu\nu} = e^a{}_\mu \, e^b{}_\nu \, \eta_{ab}\,, \tag{2.2}$$

$$\omega^{ab}{}_\mu = e^a{}_\lambda \, e^{b\rho} \, \tilde{\Gamma}^\lambda{}_{\rho\mu} + e^a{}_\lambda \, \partial_\mu \, e^{b\lambda}\,. \tag{2.3}$$

Note that in a RC manifold the affine connection constitutes a metric-compatible connection (i.e. $\tilde{\nabla}_\lambda \, g_{\mu\nu} = 0$). Moreover, it can split into the Levi-Civita connection and the so called contortion tensor in the following way:

$$\tilde{\Gamma}^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} + K^\lambda{}_{\mu\nu} \,. \tag{2.4}$$

Additionally, P_a are the generators of the space-time translations and J_{ab} the generators of the space-time rotations, which satisfy the following commutative relations:

$$[P_a, P_b] = 0 \,, \tag{2.5}$$
$$[P_a, J_{bc}] = i \, \eta_{a[b} \, P_{c]} \,, \tag{2.6}$$
$$[J_{ab}, J_{cd}] = \frac{i}{2} \left(\eta_{ad} \, J_{bc} + \eta_{cb} \, J_{ad} - \eta_{db} \, J_{ac} - \eta_{ac} \, J_{bd} \right) \,. \tag{2.7}$$

Then, the corresponding $ISO(1,3)$ gauge field strength tensor defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ takes the form:

$$F_{\mu\nu} = F^a{}_{\mu\nu} P_a + F^{ab}{}_{\mu\nu} J_{ab} \,, \tag{2.8}$$

with $F^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^{ab}{}_\mu e_{b\nu} - \omega^{ab}{}_\nu \, e_{b\mu}$, and $F^{ab}{}_{\mu\nu} = \partial_\mu \omega^{ab}{}_\nu - \partial_\nu \omega^{ab}{}_\mu + \omega^{ac}{}_\nu \, \omega^b{}_{c\mu} - \omega^{ac}{}_\mu \, \omega^b{}_{c\nu}$.

As in the case of other known gauge theories, the field strength tensor characterizes the properties of the gravitational interaction, that in the PG framework are potentially modified by the presence of torsion. In particular, it is related to the torsion and the curvature of the space-time as follows:

$$F^a{}_{\mu\nu} = e^a{}_\lambda \, T^\lambda{}_{\nu\mu} \,, \tag{2.9}$$
$$F^{ab}{}_{\mu\nu} = e^a{}_\lambda e^b{}_\rho \, \tilde{R}^{\lambda\rho}{}_{\mu\nu} \,, \tag{2.10}$$

where $T^\lambda{}_{\mu\nu}$ and $\tilde{R}^{\lambda\rho}{}_{\mu\nu}$ are the components of the torsion and the curvature tensor respectively:

$$T^\lambda{}_{\mu\nu} = 2\tilde{\Gamma}^\lambda{}_{[\mu\nu]} \,, \tag{2.11}$$
$$\tilde{R}^\lambda{}_{\rho\mu\nu} = \partial_\mu \tilde{\Gamma}^\lambda{}_{\rho\nu} - \partial_\nu \tilde{\Gamma}^\lambda{}_{\rho\mu} + \tilde{\Gamma}^\lambda{}_{\sigma\mu} \tilde{\Gamma}^\sigma{}_{\rho\nu} - \tilde{\Gamma}^\lambda{}_{\sigma\nu} \tilde{\Gamma}^\sigma{}_{\rho\mu} \,. \tag{2.12}$$

These components modify the commutative relations of the covariant derivatives for a general vector field v^λ over a RC manifold in the following way:

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] \, v^\lambda = \tilde{R}^\lambda{}_{\rho\mu\nu} \, v^\rho + T^\rho{}_{\mu\nu} \, \tilde{\nabla}_\rho v^\lambda \,, \tag{2.13}$$

with $\tilde{\nabla}_\mu \, v^\lambda = \partial_\mu \, v^\lambda + \tilde{\Gamma}^\lambda{}_{\rho\mu} \, v^\rho$.

Hence, whereas curvature is related to the rotation of a vector along an infinitesimal path over the space-time, torsion is related to the translation and it has deep geometrical implications, such as breaking infinitesimal parallelograms on the manifold [14]. Furthermore, the RC manifold may be regarded as an effective geometrical construction arising from a microscopic structure endowed with dislocation defects, which are described by torsion in the limit where they form a continuous distribution [15, 16]. In this sense, it is expected that the field strength tensor defined within this RC manifold gives rise to the pattern of dislocations density in terms of a dynamical torsion (i.e. even in the absence of matter fields).

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In addition, both curvature and torsion tensors can also be classified by the decomposition into their irreducible parts under the Lorentz group [17, 18]. Especially, torsion can be divided into three irreducible components given by distinct contributions: a trace vector, an axial vector and a traceless and also pseudotraceless tensor. From a phenomenological point of view, this sort of geometrical classification can be associated with a large number of physically relevant situations, such as the coupling between the Dirac fields and the totally antisymmetric part of the torsion or the vanishing of its tensorial modes in a spatially homogeneous and isotropic universe, as it is assumed by the cosmological principle (see [19] for a more detailed account and alternative classifications). However, there exist more complex systems that require the non-vanishing of the rest of the modes, such as the given by a general static and spherically symmetric space-time, which is deeply considered in this work.

In the basic version of the PG theory, the presence of torsion is sourced by the spin of matter, so that it introduces new independent characteristics from the standard theory and it achieves a dynamical role defining an invariant Lagrangian quadratic in the field strength tensors. In this work, we focus on a PG model whose second order contributions are only due to the existence of this kind of non-vanishing and also massless torsion:

$$
\begin{aligned}
S = \; & \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[\mathcal{L}_m - R - \frac{1}{4}\left(d_1+d_2\right)\tilde{R}^2 - \frac{1}{4}\left(d_1+d_2+4c_1+2c_2\right)\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} \\
& + c_1\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu} + c_2\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\mu\rho\nu} + d_1\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + d_2\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu}\Big]\,, \qquad (2.14)
\end{aligned}
$$

where c_1, c_2, d_1 and d_2 are four constant parameters. Note that in order to construct the Expression (2.14), we can use the identity $\tilde{R} = R - 2\nabla_\lambda T^{\rho\lambda}{}_\rho + \frac{1}{4}T_{\lambda\mu\nu}T^{\lambda\mu\nu} + \frac{1}{2}T_{\lambda\mu\nu}T^{\mu\lambda\nu} - T^{\mu}{}_{\mu\lambda}T^{\nu}{}_{\nu}{}^{\lambda}$, which allows to rewrite the general PG Lagrangian with massless torsion in terms of the torsionless Einstein-Hilbert Lagrangian.

In the elementary case where torsion does not propagate, all these constants vanish and the action leads to the standard Einstein theory. However, as it is remarked above, we are interested in the presence of higher order curvature terms in the action because in such a case, torsion becomes dynamical. Furthermore, in the limit where the first Bianchi identity of GR still holds for the total curvature (i.e. $\tilde{R}^{\lambda}{}_{[\mu\nu\rho]} = 0$[1]), then the Lagrangian leads to the sum of the Einstein-Hilbert Lagrangian and the Gauss-Bonnet term. As it is well known, the latter is a topological invariant in the four dimensional case, so it does not contribute to the field equations and the theory coincides locally with GR.

According to the first Bianchi identity in a RC space-time [20]:

$$
\tilde{R}^{\lambda}{}_{[\mu\nu\rho]} + \tilde{\nabla}_{[\mu}T^{\lambda}{}_{\nu\rho]} + T^{\sigma}{}_{[\mu\nu}\,T^{\lambda}{}_{\rho]\sigma} = 0\,, \qquad (2.18)
$$

[1] The symmetric and antisymmetric parts of a generic covariant tensor $A_{a_1\ldots a_q}$ are denoted by parenthesis and brackets, respectively:

$$
A_{a_1\ldots a_q} = A_{(a_1\ldots a_q)} + A_{[a_1\ldots a_q]}\,, \qquad (2.15)
$$

with:

$$
A_{(a_1\ldots a_q)} = \frac{1}{q!}\sum_{\pi} A_{a_{\pi(1)}\ldots a_{\pi(q)}}\,, \qquad (2.16)
$$

and

$$
A_{[a_1\ldots a_q]} = \frac{1}{q!}\sum_{\pi} \delta_\pi A_{a_{\pi(1)}\ldots a_{\pi(q)}}\,, \qquad (2.17)
$$

where the sum is taken over all permutations π of $1,\ldots,q$ and δ_π is $+1$ for even permutations and -1 for odd permutations.

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the Expression (2.14) reduces to the regular gravity action when $\tilde{\nabla}_{[\mu}T^{\lambda}{}_{\nu\rho]} + T^{\sigma}{}_{[\mu\nu}T^{\lambda}{}_{\rho]\sigma} = 0$. Note that this expression does not imply the vanishing of the torsion tensor, but a less constraining condition fulfilled by this quantity for recovering GR.

3 Field equations

In order to derive the field equations, we may simplify the expression above without loss of generality applying the Gauss-Bonnet theorem in RC spaces [21, 22]. Indeed, the following term is a total derivative of a certain vector V^{μ}:

$$\sqrt{-g}\left(\tilde{R}^2 + \tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} - 4\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu}\right) = \partial_{\mu}V^{\mu}\,. \tag{3.1}$$

Then (2.14) is locally equivalent to the following action:

$$\begin{aligned} S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[\mathcal{L}_m - R - \frac{1}{2}\left(2c_1 + c_2\right)\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} + c_1\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu} \\ + c_2\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\mu\rho\nu} + d_1\tilde{R}_{\mu\nu}\left(\tilde{R}^{\mu\nu} - \tilde{R}^{\nu\mu}\right)\Big]\,. \end{aligned} \tag{3.2}$$

In the absence of matter, i.e. $\mathcal{L}_m = 0$, Birkhoff's theorem is satisfied only in certain cases of the PG theory [11, 12]. We observe that our particular PG model does not generally satisfy this theorem, so the analysis of new static and spherically symmetric vacuum solutions to the field equations is necessary.

Before computing the vacuum equations, we define the following geometric quantities:

$$G_{\mu}{}^{\nu} = R_{\mu}{}^{\nu} - \frac{R}{2}\delta_{\mu}{}^{\nu}\,, \tag{3.3}$$

$$T1_{\mu}{}^{\nu} = \tilde{R}_{\lambda\rho\mu\sigma}\tilde{R}^{\lambda\rho\nu\sigma} - \frac{1}{4}\delta_{\mu}{}^{\nu}\tilde{R}_{\lambda\rho\tau\sigma}\tilde{R}^{\lambda\rho\tau\sigma}\,, \tag{3.4}$$

$$T2_{\mu}{}^{\nu} = \tilde{R}_{\lambda\rho\mu\sigma}\tilde{R}^{\lambda\nu\rho\sigma} + \tilde{R}_{\lambda\rho\sigma\mu}\tilde{R}^{\lambda\sigma\rho\nu} - \frac{1}{2}\delta_{\mu}{}^{\nu}\tilde{R}_{\lambda\rho\tau\sigma}\tilde{R}^{\lambda\tau\rho\sigma}\,, \tag{3.5}$$

$$T3_{\mu}{}^{\nu} = \tilde{R}_{\lambda\rho\mu\sigma}\tilde{R}^{\nu\sigma\lambda\rho} - \frac{1}{4}\delta_{\mu}{}^{\nu}\tilde{R}_{\lambda\rho\tau\sigma}\tilde{R}^{\tau\sigma\lambda\rho}\,, \tag{3.6}$$

$$H1_{\mu}{}^{\nu} = \tilde{R}^{\nu}{}_{\lambda\mu\rho}\tilde{R}^{\lambda\rho} + \tilde{R}_{\lambda\mu}\tilde{R}^{\lambda\nu} - \frac{1}{2}\delta_{\mu}{}^{\nu}\tilde{R}_{\lambda\rho}\tilde{R}^{\lambda\rho}\,, \tag{3.7}$$

$$H2_{\mu}{}^{\nu} = \tilde{R}^{\nu}{}_{\lambda\mu\rho}\tilde{R}^{\rho\lambda} + \tilde{R}_{\lambda\mu}\tilde{R}^{\nu\lambda} - \frac{1}{2}\delta_{\mu}{}^{\nu}\tilde{R}_{\lambda\rho}\tilde{R}^{\rho\lambda}\,, \tag{3.8}$$

$$C1_{\mu}{}^{\lambda\nu} = \nabla_{\rho}\tilde{R}_{\mu}{}^{\lambda\rho\nu} + K^{\lambda}{}_{\sigma\rho}\tilde{R}_{\mu}{}^{\sigma\rho\nu} - K^{\sigma}{}_{\mu\rho}\tilde{R}_{\sigma}{}^{\lambda\rho\nu}\,, \tag{3.9}$$

$$C2_{\mu}{}^{\lambda\nu} = \nabla_{\rho}\left(\tilde{R}_{\mu}{}^{\nu\lambda\rho} - \tilde{R}_{\mu}{}^{\rho\lambda\nu}\right) + K^{\lambda}{}_{\sigma\rho}\left(\tilde{R}_{\mu}{}^{\nu\sigma\rho} - \tilde{R}_{\mu}{}^{\rho\sigma\nu}\right) - K^{\sigma}{}_{\mu\rho}\left(\tilde{R}_{\sigma}{}^{\nu\lambda\rho} - \tilde{R}_{\sigma}{}^{\rho\lambda\nu}\right)\,, \tag{3.10}$$

$$C3_{\mu}{}^{\lambda\nu} = \nabla_{\rho}\tilde{R}^{\rho\nu\lambda}{}_{\mu} + K^{\lambda}{}_{\sigma\rho}\tilde{R}^{\rho\nu\sigma}{}_{\mu} - K^{\sigma}{}_{\mu\rho}\tilde{R}^{\rho\nu\lambda}{}_{\sigma}\,, \tag{3.11}$$

$$Y1_{\mu}{}^{\lambda\nu} = \delta_{\mu}{}^{\nu}\nabla_{\rho}\tilde{R}^{\lambda\rho} - \nabla_{\mu}\tilde{R}^{\lambda\nu} + \delta_{\mu}{}^{\nu}K^{\lambda}{}_{\sigma\rho}\tilde{R}^{\sigma\rho} + K^{\rho}{}_{\mu\rho}\tilde{R}^{\lambda\nu} - K^{\nu}{}_{\mu\rho}\tilde{R}^{\lambda\rho} - K^{\lambda}{}_{\rho\mu}\tilde{R}^{\rho\nu}\,, \tag{3.12}$$

$$Y2_{\mu}{}^{\lambda\nu} = \delta_{\mu}{}^{\nu}\nabla_{\rho}\tilde{R}^{\rho\lambda} - \nabla_{\mu}\tilde{R}^{\nu\lambda} + \delta_{\mu}{}^{\nu}K^{\lambda}{}_{\sigma\rho}\tilde{R}^{\rho\sigma} + K^{\rho}{}_{\mu\rho}\tilde{R}^{\nu\lambda} - K^{\nu}{}_{\mu\rho}\tilde{R}^{\rho\lambda} - K^{\lambda}{}_{\rho\mu}\tilde{R}^{\nu\rho}\,. \tag{3.13}$$

It is worthwhile to stress that all these quantities have a tensor character induced by the nature of the curvature and the torsion tensors, so that the physics equations depending on them retain the same form independently of the choice of coordinates on the manifold, according to the principle of general covariance.

Then, the field equations are derived from the PG action by performing variations with respect to the gauge potentials:

$$\delta S = \frac{1}{16\pi} \int \left(e_a{}^{\mu} X1_{\mu}{}^{\nu} \delta e^a{}_{\nu} + e_a{}^{\mu} e_{b\lambda} X2_{\mu}{}^{\lambda\nu} \delta\omega^{ab}{}_{\nu} \right) \sqrt{-g}\, d^4x\,, \tag{3.14}$$

so that they constitute the following system of equations:

$$X1_{\mu}{}^{\nu} = 0\,, \tag{3.15}$$

$$X2_{[\mu\lambda]}{}^{\nu} = 0\,, \tag{3.16}$$

where:

$$X1_{\mu}{}^{\nu} = -2G_{\mu}{}^{\nu} + 4c_1 T1_{\mu}{}^{\nu} + 2c_2 T2_{\mu}{}^{\nu} - 2\left(2c_1 + c_2\right) T3_{\mu}{}^{\nu} + 2d_1 \left(H1_{\mu}{}^{\nu} - H2_{\mu}{}^{\nu}\right)\,, \tag{3.17}$$

$$X2_{\mu}{}^{\lambda\nu} = 4c_1 C1_{\mu}{}^{\lambda\nu} - 2c_2 C2_{\mu}{}^{\lambda\nu} + 2\left(2c_1 + c_2\right) C3_{\mu}{}^{\lambda\nu} - 2d_1 \left(Y1_{\mu}{}^{\lambda\nu} - Y2_{\mu}{}^{\lambda\nu}\right)\,. \tag{3.18}$$

On the other hand, the static spherically symmetric line element and the respective tetrad basis are chosen as:

$$ds^2 = \Psi_1(r)\, dt^2 - \frac{dr^2}{\Psi_2(r)} - r^2 \left(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2\right)\,, \tag{3.19}$$

$$e^{\hat{t}} = \sqrt{\Psi_1(r)}\, dt\,, \quad e^{\hat{r}} = \frac{dr}{\sqrt{\Psi_2(r)}}\,, \quad e^{\hat{\theta}_1} = r\, d\theta_1\,, \quad e^{\hat{\theta}_2} = r \sin\theta_1\, d\theta_2\,; \tag{3.20}$$

with $0 \leq \theta_1 \leq \pi$ and $0 \leq \theta_2 \leq 2\pi$.

In addition, torsion must satisfy the condition $\mathcal{L}_{\xi} T^{\lambda}{}_{\mu\nu} = 0$ (i.e. the Lie derivative in the direction of the Killing vector ξ on $T^{\lambda}{}_{\mu\nu}$ vanishes), in order to preserve the symmetry properties of the system. Then, the only non-vanishing components of $T^{\lambda}{}_{\mu\nu}$ are [12, 23]:

$$\begin{aligned}
T^{t}{}_{tr} &= a(r)\,, \\
T^{r}{}_{tr} &= b(r)\,, \\
T^{\theta_k}{}_{t\theta_l} &= \delta^{\theta_k}{}_{\theta_l}\, c(r)\,, \\
T^{\theta_k}{}_{r\theta_l} &= \delta^{\theta_k}{}_{\theta_l}\, g(r)\,, \\
T^{\theta_k}{}_{t\theta_l} &= e^{a\theta_k}\, e^{b}{}_{\theta_l}\, \epsilon_{ab}\, d(r)\,, \\
T^{\theta_k}{}_{r\theta_l} &= e^{a\theta_k}\, e^{b}{}_{\theta_l}\, \epsilon_{ab}\, h(r)\,, \\
T^{t}{}_{\theta_k\theta_l} &= \epsilon_{kl}\, k(r)\, \sin\theta_1\,, \\
T^{r}{}_{\theta_k\theta_l} &= \epsilon_{kl}\, l(r)\, \sin\theta_1\,;
\end{aligned} \tag{3.21}$$

where a, b, c, d, g, h, k and l are arbitrary functions depending only on r; $k, l = 1, 2$, and ϵ_{ab} is the totally antisymmetric Levi-Civita symbol, given by:

$$\epsilon_{ab} = \begin{cases} +1\,, & \text{for} \quad a\,b = 1\,2. \\ -1\,, & \text{for} \quad a\,b = 2\,1. \\ \;\;0\,, & \text{for all other combinations.} \end{cases} \tag{3.22}$$

As can be seen, the SO(3)-symmetrical torsion exhibits eight degrees of freedom and it allows us to consider the most general expression for the torsion tensor. It means the possible existence of more complex solutions than the O(3)-symmetrical torsion case, where only four degrees of freedom survive.

Nevertheless, it is possible to impose an additional restriction involving these torsion components by taking the trace of eq. (3.16) in the weak-field approximation:

$$(4c_1+c_2+d_1)\,\nabla_\rho \tilde{R}^{[\lambda\rho]} = 2c_1 K_{\mu\nu\rho}\left(\tilde{R}^{\nu\lambda\rho\mu} - \tilde{R}^{\rho\mu\nu\lambda}\right) + \frac{3}{2}c_2 K_{\mu\nu\rho}\left(\tilde{R}^{\mu[\rho\nu\lambda]} + \tilde{R}^{\rho[\mu\lambda\nu]}\right) + d_1\left(K_{\mu\rho}{}^{\lambda}\tilde{R}^{[\mu\rho]} + T^{\rho}{}_{\mu\rho}\tilde{R}^{[\lambda\mu]}\right) - (4c_1+c_2+d_1)K^{\lambda}{}_{\nu\rho}\tilde{R}^{[\nu\rho]}\,. \tag{3.23}$$

Then, by neglecting torsion terms of second order, only the first term of the equation contributes. The equations of motion for the torsion tensor in linear approximation read

$$\nabla_\mu \nabla^\mu T^{\nu}{}_{\lambda\nu} + \nabla_\mu \nabla_\nu T^{\nu\mu}{}_{\lambda} - \nabla_\mu \nabla_\lambda T^{\nu\mu}{}_{\nu} = 0\,, \tag{3.24}$$

for theories with $4c_1 + c_2 + d_1 \neq 0$.

In terms of the torsion components, this constraint is equivalent to the relation:

$$b(r) = rc'(r) + c(r) + \frac{p}{r}\sqrt{\frac{\Psi_1(r)}{\Psi_2(r)}}\,, \tag{3.25}$$

where p is an integration constant.

In addition to a cosmological constant, we only focus on suitable solutions that may exist in presence of Coulomb electric and magnetic fields, as in the standard Einstein-Maxwell framework of GR, so the solutions are restricted to verify $\Psi_1(r) = \Psi_2(r) \equiv \Psi(r)$ in order to satisfy the Maxwell's equations in the RC manifold. These restrictions substantially simplify the problem. In any case, the field equations constitute a highly nonlinear system involving a large number of degrees of freedom and it forms an underdetermined system with different classes of solutions. We will require a final additional condition: suitable solutions must take an appropriate form referred to the rotated basis $\vartheta^a = \Lambda^a{}_b e^b$, given by the following vector fields:

$$\begin{aligned}
\vartheta^{\hat{t}} &= \frac{1}{2}\left\{[\Psi(r)+1]\,dt + \left[1 - \frac{1}{\Psi(r)}\right]dr\right\}\,;\\
\vartheta^{\hat{r}} &= \frac{1}{2}\left\{[\Psi(r)-1]\,dt + \left[1 + \frac{1}{\Psi(r)}\right]dr\right\}\,;\\
\vartheta^{\hat{\theta}_1} &= r\,d\theta_1\,;\\
\vartheta^{\hat{\theta}_2} &= r\sin\theta_1\,d\theta_2\,.
\end{aligned} \tag{3.26}$$

This orthogonal coframe has already been used in previous literature to simplify the form of the Baekler solution, that belongs to a different class of PG models containing an O(3)-symmetrical torsion [24]. Especially, besides to its considerable simplification of the solution, it has the advantage of leading to a conformally flat Lorentz connection [25, 26]. In our case, we expect that the rotated Lorentz connection defined on the RC manifold recovered its Minkowski values for the vanishing of the free parameters associated with the torsion tensor and then the remaining physical configuration reduced to GR. Note that, in order to reach this limit, it is not necessary that each component of torsion vanishes identically, but only the fulfillment of the first Bianchi identity of GR for the total curvature, as remarked in the previous section.

At the same time, any solution $F^a{}_{bc}$ referring to the mentioned orthogonal coframe can be written as follows:

$$\mathcal{F}^{\hat{t}}{}_{\hat{t}\hat{r}} = \frac{1}{2}\left\{[1+\Psi(r)]\,a(r) + \left[1 - \frac{1}{\Psi(r)}\right]b(r)\right\}\,;$$

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$$
\begin{aligned}
\mathcal{F}^{\hat{r}}{}_{\hat{t}\hat{r}} &= \frac{1}{2}\left\{\left[1+\frac{1}{\Psi(r)}\right]b(r)-\left[1-\Psi(r)\right]a(r)\right\}\,;\\
\mathcal{F}^{\hat{\theta}_1}{}_{\hat{t}\hat{\theta}_1} &= \mathcal{F}^{\hat{\theta}_2}{}_{\hat{t}\hat{\theta}_2}=\frac{1}{2}\left\{\left[1+\frac{1}{\Psi(r)}\right]c(r)+\left[1-\Psi(r)\right]g(r)\right\}\,;\\
\mathcal{F}^{\hat{\theta}_1}{}_{\hat{r}\hat{\theta}_1} &= \mathcal{F}^{\hat{\theta}_2}{}_{\hat{r}\hat{\theta}_2}=\frac{1}{2}\left\{\left[1+\Psi(r)\right]g(r)-\left[1-\frac{1}{\Psi(r)}\right]c(r)\right\}\,;\\
\mathcal{F}^{\hat{\theta}_2}{}_{\hat{t}\hat{\theta}_1} &= -\mathcal{F}^{\hat{\theta}_1}{}_{\hat{t}\hat{\theta}_2}=\frac{1}{2}\left\{\left[1+\frac{1}{\Psi(r)}\right]d(r)+\left[1-\Psi(r)\right]h(r)\right\}\,;\\
\mathcal{F}^{\hat{\theta}_2}{}_{\hat{r}\hat{\theta}_1} &= -\mathcal{F}^{\hat{\theta}_1}{}_{\hat{r}\hat{\theta}_2}=\frac{1}{2}\left\{\left[1+\Psi(r)\right]h(r)-\left[1-\frac{1}{\Psi(r)}\right]d(r)\right\}\,;\\
\mathcal{F}^{\hat{t}}{}_{\hat{\theta}_1\hat{\theta}_2} &= \frac{1}{2r^2}\left\{\left[1+\Psi(r)\right]k(r)+\left[1-\frac{1}{\Psi(r)}\right]l(r)\right\}\,;\\
\mathcal{F}^{\hat{r}}{}_{\hat{\theta}_1\hat{\theta}_2} &= \frac{1}{2r^2}\left\{\left[1+\frac{1}{\Psi(r)}\right]l(r)-\left[1-\Psi(r)\right]k(r)\right\}\,.
\end{aligned}
\tag{3.27}
$$

Therefore, in order to obtain a class of suitable non-singular solutions (excluding the point $r=0$), the components of the torsion tensor must satisfy the following relations:

$$
b(r)=a(r)\,\Psi(r)\,,\quad c(r)=-\,g(r)\,\Psi(r)\,,\quad d(r)=-\,h(r)\,\Psi(r)\,,\quad l(r)=k(r)\,\Psi(r)\,. \tag{3.28}
$$

We find out that these constraints also involve the vanishing of the three independent quadratic torsion invariants (i.e. $T_{\lambda\mu\nu}\,T^{\lambda\mu\nu}=T_{\lambda\mu\nu}\,T^{\mu\lambda\nu}=T^{\mu}{}_{\mu\lambda}\,T_{\nu}{}^{\nu\lambda}=0$).

4 Solutions

By taking into account all these remarks, the following SO(3)-symmetric vacuum solution can be easily found for $c_1=-\,d_1/4$ and $c_2=-\,d_1/2$:

$$
\begin{aligned}
&a(r)=\frac{\Psi'(r)}{2\Psi(r)}\,, && b(r)=\frac{\Psi'(r)}{2}\,, && c(r)=\frac{\Psi(r)}{2r}\,, && g(r)=-\,\frac{1}{2r}\,,\\
&d(r)=\frac{\kappa}{r}\,, && h(r)=-\,\frac{\kappa}{r\Psi(r)}\,, && k(r)=l(r)=0\,; &&
\end{aligned}
\tag{4.1}
$$

with

$$
\Psi(r)=1-\frac{2m}{r}+\frac{d_1\kappa^2}{r^2}\quad. \tag{4.2}
$$

Hence, the relation (3.25) is completely fulfilled and the constant p vanishes.

This solution describes a Reissner-Nordström type geometry, supported only by the metric and torsion fields rather than an electric or magnetic source. The new contribution is proportional to the square of the new parameter κ. Indeed, this parameter determines the intensity of the strength tensor corresponding to the torsion:

$$
F^{ab}{}_{cd}=\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -\,\kappa/2r^2 & 0 & -\,\kappa/2r^2 & 0\\
0 & \kappa/2r^2 & 0 & 0 & 0 & -\,\kappa/2r^2\\
-\,\kappa/r^2 & 0 & 0 & -\,1/r^2 & 0 & 0\\
0 & \kappa/2r^2 & 0 & 0 & 0 & -\,\kappa/2r^2\\
0 & 0 & \kappa/2r^2 & 0 & \kappa/2r^2 & 0
\end{pmatrix}\,, \tag{4.3}
$$

where the six rows and columns of the matrix are labeled the components in the order (01, 02, 03, 23, 31, 12).

The values above for the Lagrangian coefficients and their respective signs define the strength and properties of the torsion field in the PG framework. In existing literature, particular results containing a certain set of viable coefficient combinations for the purely massless PG theory have been developed under the linear field approximation requiring the absence of both ghost and tachyon modes [27] or only the ghost-free condition [28, 29]. Nevertheless, it has also been shown that the Hamiltonian constraint formalism differs from these results where the highly nonlinear effects of the PG theory are included [30]. Furthermore, some other authors have pointed out several mistakes and incompleteness in various of the mentioned analyses, reaching important contradictions with the commented conclusions [31, 32]. In this sense, the stability of these models is still an open issue.

On the other hand, by following our constraints (3.25) and (3.28), we note that any other combination for the constant parameters of eq. (2.14) involves a vacuum configuration described strictly by the Schwarzschild metric. Hence, in the present case, there is a unique combination that allows a vacuum configuration different from the Schwarzschild geometry. It is the Reissner-Nordström solution above. Moreover, by solving the field equations it is possible to demonstrate this statement even for the case $\Psi_1(r) \neq \Psi_2(r)$. It is also shown that the torsion decreases at infinity and the metric is asymptotically flat. So the corresponding Newtonian limit is satisfied by the solution as demanded by different approaches [33].

It is also straightforward to notice that the condition $\tilde{\nabla}_{[\mu} T^{\lambda}{}_{\nu\rho]} + T^{\sigma}{}_{[\mu\nu} T^{\lambda}{}_{\rho]\sigma} = 0$ is fulfilled for this solution when $\kappa = 0$. In such a case, although the rest of the non-vanishing components of the torsion tensor still remain, the Action (3.2) is equivalent to the Einstein-Hilbert one and the GR approach is totally recovered. These non-vanishing components yield an inert RC spin connection and curvature, which emerge to the physical structure only when the parameter κ switches on and the torsion becomes dynamical. This fact contrasts with the alternative ways of recovering the regular gravity action given by the rest of the PG models present in previous literature, such as the mentioned Baekler solution where this limit is carried out in the framework of teleparallelism [34]. Teleparallel Gravity is the gauge theory for the translation group based on the curvature-free Weitzenböck connection and it is constructed in such a form that provides an equivalent description of gravity to GR, but in terms of torsion so that there exist conceptual differences between them (see [35] for a recent overview).

Additionally, the expression for the Lorentz connection referred to ϑ^a exhibits a similar property to its counterpart of the Baekler solution. It takes Minkowski values within the RC manifold for $\kappa = 0$ and it does not depend on any other magnitude in such a case:

$$\hat{A} = -\frac{\kappa}{r} J_{\hat{\theta}\hat{\phi}}\, dt + \frac{\kappa}{r\Psi(r)} J_{\hat{\theta}\hat{\phi}}\, dr + \frac{1}{2}\left(J_{\hat{r}\hat{\theta}} - J_{\hat{t}\hat{\theta}}\right) d\theta + \sin\theta \left[\frac{1}{2}\left(J_{\hat{r}\hat{\phi}} - J_{\hat{t}\hat{\phi}}\right) + \cot\theta J_{\hat{\theta}\hat{\phi}}\right] d\phi\,. \tag{4.4}$$

This solution can be trivially generalized to include the existence of a non-vanishing cosmological constant Λ and Coulomb electromagnetic fields produced by both electric and magnetic charges q_e and q_m, respectively. For this purpose, it is assumed that photons are decoupled from torsion as it is dictated by the minimum coupling principle. Then, it is easy to extend the solution by modifying the metric function $\Psi(r)$ by the following expression:

$$\Psi(r) = 1 - \frac{2m}{r} + \frac{d_1\kappa^2 + q_e^2 + q_m^2}{r^2} + \frac{\Lambda}{3} r^2\,. \tag{4.5}$$

As can be seen, the term derived by the dynamical torsion has the same structure than the terms provided by the electric and magnetic monopole charges and it is possible to collect these three contributions along with the cosmological constant onto a common space-time. Therefore, these factors involve geometrical effects on the PG field strength tensors, even though the electromagnetic field is not coupled directely to the torsion field. Switching off the parameter κ, the solution reduces to the Reissner-Nordström-de Sitter solution of ordinary GR as expected. Thereby, this solution shows similarities between the torsion and the electromagnetic fields, even though they are independent quantities.

It is worthwhile to stress the further relation between this type of geometry and other well known post-Riemannian approaches, such as the metric-affine gauge (MAG) theory of gravity, where the RC space-time and the PG group are both replaced by a general affinely connected metric manifold with non-metricity condition (i.e. $\tilde{\nabla}_\lambda g_{\mu\nu} \neq 0$) and its associated affine gauge group [36, 37]. Indeed, analogous results were found out in terms of the dilation and the shear charges associated with the non-metricity tensor, which can involve a vacuum Reissner-Nordström configuration in this context [38, 39]. Nevertheless, the so called gravito-electric and gravito-magnetic terms present in all these solutions fall completely on the non-metricity field, so that when the latter vanishes those terms disappear from the metric tensor, even in presence of a non-vanishing torsion component. This result differs from our PG solution since the Reissner-Nordström structure provided by the torsion field can even exist when the connection is metric-compatible and the non-metricity tensor vanishes. This fact together with the mentioned achievements of the MAG point out a richer structure of spherical and static solutions in gravitational theories characterized by a general affine connection.

5 Equations of motion

As any test particle or physical field uncoupled to torsion cannot experiment deviations from their geodesic trajectories, the respective equations of motion within the RC space-time connected to our PG model must distinguish between both classes of spinless and spinning matter. For this purpose, it is critical to deal with the principal conservation law of the total energy-momentum tensor $\theta^{\mu\nu}$ derived by the invariance of Action (3.2):

$$\nabla_\nu \theta^{\mu\nu} + K_{\lambda\rho}{}^{\mu} \theta^{\rho\lambda} + \tilde{R}_{\lambda\rho\sigma}{}^{\mu}\, S^{\lambda\rho\sigma} = 0\,, \tag{5.1}$$

where $S^{\lambda\rho\sigma}$ is the spin density tensor.

An analysis for the achievement of this result based on our particular PG model is shown in the appendix A. The mentioned conservation law allows to obtain the equations of motion for a test particle in such a RC space-time by integrating the expression above over a three dimensional space-like section of the world tube involving the particle and employing the semiclassical approximation [40, 41]:

$$\begin{aligned} &\int \partial_\nu \left(\sqrt{-g}\,\theta^{\mu\nu}\right) d^3x' + \int \Gamma^{\mu}{}_{\lambda\rho} \theta^{\lambda\rho} \sqrt{-g}\, d^3x' \\ &+ \int K_{\lambda\rho}{}^{\mu} \theta^{\rho\lambda} \sqrt{-g}\, d^3x' + \int \tilde{R}_{\lambda\rho\sigma}{}^{\mu}\, S^{\lambda\rho\sigma} \sqrt{-g}\, d^3x' = 0, \end{aligned} \tag{5.2}$$

with

$$\int \partial_\nu \left(\sqrt{-g}\,\theta^{\mu\nu}\right) d^3x' = \frac{d}{dt} \int \theta^{\mu t} \sqrt{-g}\, d^3x'\,, \tag{5.3}$$

due to the Gauss theorem and by neglecting surface terms. As eq. (5.2) must be fulfilled for any integration volume, it is equivalent to the differential equation of motion:

$$\frac{dp^\mu}{ds} + \Gamma^{\mu}{}_{\lambda\rho}\, p^\lambda u^\rho + K_{\lambda\rho}{}^{\mu} p^\rho u^\lambda + \tilde{R}_{\lambda\rho\sigma}{}^{\mu} S^{\lambda\rho} u^\sigma = 0\,, \tag{5.4}$$

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where we have used the following definitions

$$\theta^{\lambda\rho} = \frac{dt}{ds} \int p^{\lambda} u^{\rho} \sqrt{-g}\, d^3 x' \,, \tag{5.5}$$

and

$$S^{\lambda\rho\sigma} = \frac{dt}{ds} \int S^{\lambda\rho} u^{\sigma} \sqrt{-g}\, d^3 x' \,. \tag{5.6}$$

Here, s is the proper time along the particle world line, p^{μ} the four-momentum of the particle and u^{μ} its four-velocity. Therefore, the presence of a dynamical torsion in the space-time and the interaction between the curvature and the spin of matter originate in general, a generalized Lorentz force acting on this type of matter. Thus, this force potentially yields deviations from the geodesic trajectories. Of course, this generally non-geodesic motion turns out to be another essential difference with gravitational theories endowed with vanishing torsion, such as ordinary GR. Nevertheless, for spinless matter with $S^{\lambda\rho} = 0$ and $p^{\lambda} \propto u^{\lambda}$, the equations of motion reduce to the same geodesic equations of GR.

This fundamental difference might be used in order to prove experimentally the possible existence of a non-vanishing dynamical torsion in the space-time. Nevertheless, it is expected to yield too tiny effects to be measured, as occurs with the rest of the well known PG models. Additionally, torsion is induced on the vierbein field by the field equations and thereby it can also operate on the geodesic motion of ordinary matter via the Levi-Civita connection. In particular, for a standard Reissner-Nordström geometry, the respective point charges have well known consequences on the geodesic paths of test particles and light rays [42].

Presumably, the effects of this type of geometry are also very small at astrophysics or cosmological scales, because of the vanishing of the spin density tensor in the most macroscopical bodies. However, this situation may differ around extreme gravitational systems as neutron stars or black holes with intense magnetic fields and sufficiently oriented elementary spins. In such a case, it is expected that the RC space-time described by the PG theory modulates these events.

Further analyses can be performed by comparing the gravitational interaction of the spin and the orbital angular momentum of a rotating rigid test body [43, 44]. In this sense, it is especially interesting their natural extension towards the MAG theory when the motion of a rotating and deformable test body is considered [45]. All these achievements allow to systematically study the behaviour of gravitating matter with microstructure and to establish additional differences between a large extreme gravitational systems, such as the one present in our PG model and the one previously mentioned supported by MAG.

6 Conclusions

In the present work, we have investigated the PG theory with massless torsion based on a gravitational model directly connected to GR when the dynamical role of torsion is frozen via the first Bianchi identity. In the general case, this approach allows the torsion tensor to constitute a dynamical degree of freedom. We have shown that the vacuum structure of the theory may differ from the Einstein's theory and, specifically, distinct classes of solutions can exist besides the Schwarzschild solution given by the Birkhoof's theorem within the standard framework of GR. Hence, in order to improve the understanding of such a theory of gravity, the search and analysis of exact solutions are fundamental.

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The large degree of symmetry assumed and the requirement of the existence of a suitable electromagnetic-like vacuum structure analogous to the Einstein-Maxwell framework together with the use of a convenient rotated basis allow to reduce notably the difficulty of the highly nonlinear nature present in the theory. Under these requirements, we have obtained a new static and spherically symmetric vacuum solution. This solution provides a Reissner-Nordström type geometry with a SO(3)-symmetrical torsion depending on a parameter κ and it has been deduced without the use of the double duality ansatz for the RC curvature, often employed in previous literature in order to restrict the PG field equations into a very highly simplified system [46]. Its existence shows the dynamical character of the torsion field, which can even be induced on the metric tensor via the field equations generating a distinct class of solutions, beyond the Schwarzschild scheme and the Birkhoff's theorem of GR.

The corresponding generalized Reissner-Nordström-de Sitter configuration is also obtained when external electromagnetic fields and a non-vanishing cosmological constant are included, by analogy with the standard case. In this scheme, the torsion field contribution is perfectly distinguishable from the rest of physical degrees of freedom and the solution reduces to the standard case when its dynamical role is switched off. Therefore, the solution presents similarities between the torsion and the electromagnetic fields. It is expected that these similarities still remain in more general systems, such as axisymmetric space-times.

The foundations presented in this article have also been employed in previous works for the analysis and the achievement of exact solutions in extended models of gravity, such as the well known Einstein-Yang-Mills theory. The results obtained in this work show the flexibility and usefulness of the method described in [47, 48]. Furthermore, the recurrence of the fundamental schemes derived by our analyses in the extensive MAG framework is also remarked. It shows deeper relations between the solutions and the vacuum structure provided by these approaches, which improve their physical understanding and applicability. Specifically, the role of the non-metricity present in MAG has been typically categorized into earlier eppochs of the universe, whereas the one of the torsion field is expected to represent a larger number of physical scenarios, even in our current universe, such as extreme gravitational systems described by neutron stars or black holes with intense spin densities.

Finally, the equations of motion for a general test particle are derived and the differences with the geodesic trajectories of GR are stressed. These differences are also very important to understand the physical properties and further implications of our solution. Their theoretical consequences or observational effects in astrophysics and cosmology will be studied in future work.

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A Energy-momentum conservation

The conservation law for the total energy-momentum tensor associated with our model can be obtained directly from the PG Lagrangian:

$$
\begin{aligned}
S = \frac{1}{16\pi}\int d^4x\,\sqrt{-g}\Big\{&\mathcal{L}_m - R \\
&+\frac{d_1}{4}\left[2\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} - \tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu} - 2\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\mu\rho\nu} + 4\tilde{R}_{\mu\nu}\left(\tilde{R}^{\mu\nu} - \tilde{R}^{\nu\mu}\right)\right]\Big\}.
\end{aligned}
\tag{A.1}
$$

We can obtain this result by the computation of the torsion-free divergence acting on the vierbein equation:

$$
\begin{aligned}
\nabla_\nu X1_\mu{}^\nu = d_1\Big[&\tilde{R}_{\lambda\rho\mu\sigma}\left(\nabla_\nu\tilde{R}^{\lambda\sigma\rho\nu} - \nabla_\nu\tilde{R}^{\lambda\rho\nu\sigma} - \nabla_\nu\tilde{R}^{\lambda\nu\rho\sigma} + 2\nabla_\nu\tilde{R}^{\nu\sigma\lambda\rho}\right) + 2\left(\tilde{R}^{\lambda\nu} - \tilde{R}^{\nu\lambda}\right)\nabla_\nu\tilde{R}_{\lambda\mu} \\
&+\frac{1}{2}\tilde{R}_{\lambda\rho\omega\sigma}\left(\nabla_\mu\tilde{R}^{\lambda\rho\omega\sigma} + 2\nabla_\mu\tilde{R}^{\lambda\omega\rho\sigma} - 2\nabla_\mu\tilde{R}^{\omega\sigma\lambda\rho}\right) - 2\left(\tilde{R}^{\lambda\rho} - \tilde{R}^{\rho\lambda}\right)\nabla_\mu\tilde{R}_{\lambda\rho} + 2\left(\tilde{R}^{\lambda\rho} - \tilde{R}^{\rho\lambda}\right)\nabla_\nu\tilde{R}^\nu{}_{\lambda\mu\rho} \\
&+2\tilde{R}^\nu{}_{\lambda\mu\rho}\nabla_\nu\left(\tilde{R}^{\lambda\rho} - \tilde{R}^{\rho\lambda}\right) + 2\tilde{R}_{\lambda\mu}\nabla_\nu\left(\tilde{R}^{\lambda\nu} - \tilde{R}^{\nu\lambda}\right) + \nabla_\nu\tilde{R}_{\lambda\rho\mu\sigma}\left(\tilde{R}^{\lambda\sigma\rho\nu} - \tilde{R}^{\lambda\rho\nu\sigma} - \tilde{R}^{\lambda\nu\rho\sigma} + 2\tilde{R}^{\nu\sigma\lambda\rho}\right)\Big].
\end{aligned}
\tag{A.2}
$$

The information of the additional field equation $X2_{[\mu\lambda]}{}^\nu = -16\pi S_{\lambda\mu}{}^\nu$, can be introduced in the equation above with the result:

$$
\begin{aligned}
\nabla_\nu X1_\mu{}^\nu = 16\pi\tilde{R}_{\lambda\rho\sigma\mu}\,S^{\lambda\rho\sigma} + d_1\Big\{&\tilde{R}^\lambda{}_{\rho\mu\sigma}\Big[K^\rho{}_{\omega\nu}\left(\tilde{R}_\lambda{}^{\omega\nu\sigma} + 2\tilde{R}^{\nu\sigma\omega}{}_\lambda + \tilde{R}_\lambda{}^{\nu\omega\sigma} - \tilde{R}_\lambda{}^{\sigma\omega\nu}\right) \\
&-K^\omega{}_{\lambda\nu}\left(\tilde{R}_\omega{}^{\rho\nu\sigma} + 2\tilde{R}^{\nu\sigma\rho}{}_\omega + \tilde{R}_\omega{}^{\nu\rho\sigma} - \tilde{R}_\omega{}^{\sigma\rho\nu}\right) + 2\delta^\sigma_\lambda\nabla_\nu\left(\tilde{R}^{\rho\nu} - \tilde{R}^{\nu\rho}\right) - 2\nabla_\lambda\left(\tilde{R}^{\rho\sigma} - \tilde{R}^{\sigma\rho}\right) \\
&+2\delta^\sigma_\lambda K^\rho{}_{\omega\nu}\left(\tilde{R}^{\omega\nu} - \tilde{R}^{\nu\omega}\right) + 2K^\nu{}_{\lambda\nu}\left(\tilde{R}^{\rho\sigma} - \tilde{R}^{\sigma\rho}\right) - 2K^\sigma{}_{\lambda\nu}\left(\tilde{R}^{\rho\nu} - \tilde{R}^{\nu\rho}\right) - 2K^\rho{}_{\nu\lambda}\left(\tilde{R}^{\nu\sigma} - \tilde{R}^{\sigma\nu}\right)\Big] \\
&+\nabla_\nu\tilde{R}_{\lambda\rho\mu\sigma}\left(\tilde{R}^{\lambda\sigma\rho\nu} - \tilde{R}^{\lambda\rho\nu\sigma} - \tilde{R}^{\lambda\nu\rho\sigma} + 2\tilde{R}^{\nu\sigma\lambda\rho}\right) + \frac{1}{2}\nabla_\mu\tilde{R}_{\lambda\rho\nu\sigma}\left(\tilde{R}^{\lambda\rho\nu\sigma} + 2\tilde{R}^{\lambda\nu\rho\sigma} - 2\tilde{R}^{\nu\sigma\lambda\rho}\right) \\
&-2\left(\tilde{R}^{\lambda\rho} - \tilde{R}^{\rho\lambda}\right)\nabla_\mu\tilde{R}_{\lambda\rho} + 2\left(\tilde{R}^{\lambda\rho} - \tilde{R}^{\rho\lambda}\right)\nabla_\nu\tilde{R}^\nu{}_{\lambda\mu\rho} + 2\tilde{R}^\nu{}_{\lambda\mu\rho}\nabla_\nu\left(\tilde{R}^{\lambda\rho} - \tilde{R}^{\rho\lambda}\right) \\
&+2\tilde{R}_{\lambda\mu}\nabla_\nu\left(\tilde{R}^{\lambda\nu} - \tilde{R}^{\nu\lambda}\right) + 2\left(\tilde{R}^{\lambda\nu} - \tilde{R}^{\nu\lambda}\right)\nabla_\nu\tilde{R}_{\lambda\mu}\Big\},
\end{aligned}
\tag{A.3}
$$

where

$$
\begin{aligned}
&\tilde{R}^\lambda{}_{\rho\mu\sigma}\delta^\sigma_\lambda\nabla_\nu\left(\tilde{R}^{\rho\nu} - \tilde{R}^{\nu\rho}\right) + \tilde{R}_{\lambda\mu}\nabla_\nu\left(\tilde{R}^{\lambda\nu} - \tilde{R}^{\nu\lambda}\right) \\
&= \tilde{R}^\nu{}_{\lambda\mu\rho}\nabla_\nu\left(\tilde{R}^{\lambda\rho} - \tilde{R}^{\rho\lambda}\right) - \tilde{R}^\lambda{}_{\rho\mu\sigma}\nabla_\lambda\left(\tilde{R}^{\rho\sigma} - \tilde{R}^{\sigma\rho}\right) = 0\,.
\end{aligned}
\tag{A.4}
$$

First, we focus on the differential form of Riemann tensors and express the torsion-free operator ∇ in terms of $\tilde{\nabla}$ and the contortion tensor:

$$
\nabla_\sigma\tilde{R}_{\lambda\rho\mu\nu} = \tilde{\nabla}_\sigma\tilde{R}_{\lambda\rho\mu\nu} + K^\omega{}_{\lambda\sigma}\tilde{R}_{\omega\rho\mu\nu} + K^\omega{}_{\rho\sigma}\tilde{R}_{\lambda\omega\mu\nu} + K^\omega{}_{\mu\sigma}\tilde{R}_{\lambda\rho\omega\nu} + K^\omega{}_{\nu\sigma}\tilde{R}_{\lambda\rho\mu\omega}\,. \tag{A.5}
$$

Thus, by simplifying the resulting expression and rearranging terms, we obtain the following equation:

$$
\begin{aligned}
\nabla_\nu X1_\mu{}^\nu = d_1\Big\{&\tilde{\nabla}_\nu\tilde{R}_{\lambda\rho\mu\sigma}\left(\tilde{R}^{\lambda\sigma\rho\nu} - \tilde{R}^{\lambda\rho\nu\sigma} - \tilde{R}^{\lambda\nu\rho\sigma} + 2\tilde{R}^{\nu\sigma\lambda\rho}\right) + \frac{1}{2}\tilde{\nabla}_\mu\tilde{R}_{\lambda\rho\nu\sigma}\left(\tilde{R}^{\lambda\rho\nu\sigma} + 2\tilde{R}^{\lambda\nu\rho\sigma} - 2\tilde{R}^{\nu\sigma\lambda\rho}\right) \\
&+2\tilde{R}^\lambda{}_{\rho\mu\sigma}\Big[\delta^\sigma_\lambda K^\rho{}_{\omega\nu}\left(\tilde{R}^{\omega\nu} - \tilde{R}^{\nu\omega}\right) + K^\nu{}_{\lambda\nu}\left(\tilde{R}^{\rho\sigma} - \tilde{R}^{\sigma\rho}\right) - K^\sigma{}_{\lambda\nu}\left(\tilde{R}^{\rho\nu} - \tilde{R}^{\nu\rho}\right) - K^\rho{}_{\nu\lambda}\left(\tilde{R}^{\nu\sigma} - \tilde{R}^{\sigma\nu}\right)\Big] \\
&+\frac{1}{2}\left(K^\omega{}_{\lambda\mu}\tilde{R}_{\omega\rho\nu\sigma} + K^\omega{}_{\rho\mu}\tilde{R}_{\lambda\omega\nu\sigma} + K^\omega{}_{\nu\mu}\tilde{R}_{\lambda\rho\omega\sigma} + K^\omega{}_{\sigma\mu}\tilde{R}_{\lambda\rho\nu\omega}\right)\left(\tilde{R}^{\lambda\rho\nu\sigma} + 2\tilde{R}^{\lambda\nu\rho\sigma} - 2\tilde{R}^{\nu\sigma\lambda\rho}\right) \\
&+\left(K^\omega{}_{\mu\nu}\tilde{R}_{\lambda\rho\omega\sigma} + K^\omega{}_{\sigma\nu}\tilde{R}_{\lambda\rho\mu\omega}\right)\left(\tilde{R}^{\lambda\sigma\rho\nu} - \tilde{R}^{\lambda\rho\nu\sigma} - \tilde{R}^{\lambda\nu\rho\sigma} + 2\tilde{R}^{\nu\sigma\lambda\rho}\right) + 2\left(\tilde{R}^{\lambda\rho} - \tilde{R}^{\rho\lambda}\right)\nabla_\nu\tilde{R}^\nu{}_{\lambda\mu\rho} \\
&-2\left(\tilde{R}^{\lambda\rho} - \tilde{R}^{\rho\lambda}\right)\nabla_\mu\tilde{R}_{\lambda\rho} + 2\left(\tilde{R}^{\lambda\rho} - \tilde{R}^{\rho\lambda}\right)\nabla_\rho\tilde{R}_{\lambda\mu}\Big\} + 16\pi\tilde{R}_{\lambda\rho\sigma\mu}\,S^{\lambda\rho\sigma}\,.
\end{aligned}
\tag{A.6}
$$

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According to the second Bianchi identity for a RC manifold, the components of the Riemann tensor satisfy [20]:

$$\tilde{\nabla}_{[\lambda|}\tilde{R}^{\sigma}{}_{\rho|\mu\nu]} - T^{\omega}{}_{[\lambda\mu|}\tilde{R}^{\sigma}{}_{\rho\omega|\nu]} = 0\,, \tag{A.7}$$

so that we can simplify even more terms and obtain the following expression:

$$\begin{aligned}\nabla_\nu X1_\mu{}^\nu = d_1\Bigg\{&\frac{1}{2}\left(K^{\omega}{}_{\rho\mu}\tilde{R}_{\lambda\omega\nu\sigma} - K^{\omega}{}_{\lambda\mu}\tilde{R}_{\rho\omega\nu\sigma}\right)\left(\tilde{R}^{\lambda\rho\nu\sigma} + 2\tilde{R}^{\lambda\nu\rho\sigma} - 2\tilde{R}^{\nu\sigma\lambda\rho}\right) - 2\left(\tilde{R}^{\lambda\rho} - \tilde{R}^{\rho\lambda}\right)\nabla_\mu\tilde{R}_{\lambda\rho}\\
&+2\left(\tilde{R}^{\lambda\rho} - \tilde{R}^{\rho\lambda}\right)\left(\nabla_\nu\tilde{R}^{\nu}{}_{\lambda\mu\rho} - K^{\nu}{}_{\lambda\sigma}\tilde{R}^{\sigma}{}_{\nu\mu\rho} + K^{\omega}{}_{\nu\lambda}\tilde{R}^{\nu}{}_{\rho\mu\omega} - K^{\nu}{}_{\sigma\nu}\tilde{R}^{\sigma}{}_{\rho\mu\lambda} + \nabla_\rho\tilde{R}_{\lambda\mu} - \tilde{R}_{\nu\mu}K^{\nu}{}_{\lambda\rho}\right)\\
&+\frac{1}{2}\left(\tilde{R}_{\lambda\rho\omega(\nu|}T_{\mu}{}^{\omega}{}_{|\sigma)} + \tilde{R}_{\lambda\rho\omega(\nu}T^{\omega}{}_{\sigma)\mu} + \tilde{R}_{\lambda\rho\omega(\nu}T_{\sigma)}{}^{\omega}{}_{\mu} - \tilde{\nabla}_{(\nu|}\tilde{R}_{\lambda\rho\mu|\sigma)}\right)\left(\tilde{R}^{\lambda\rho\nu\sigma} - 2\tilde{R}^{\nu\sigma\lambda\rho}\right)\\
&+\tilde{R}_{\lambda\rho\mu\omega}\left[T^{\omega}{}_{\nu\sigma}\tilde{R}^{\lambda(\nu\sigma)\rho} + 2T_{(\nu\sigma)}{}^{\omega}\tilde{R}^{\lambda[\nu\sigma]\rho} + T_{(\nu\sigma)}{}^{\omega}\left(2\tilde{R}^{\nu\sigma\lambda\rho} - \tilde{R}^{\lambda\rho\nu\sigma}\right)\right]\Bigg\} + 16\pi\tilde{R}_{\lambda\rho\sigma\mu}S^{\lambda\rho\sigma}\,.\end{aligned} \tag{A.8}$$

The last factors vanish because of the contraction between the symmetric and anti-symmetric parts of the tensors above. Then, by repeating the same procedure on the Ricci tensors:

$$\begin{aligned}\nabla_\nu X1_\mu{}^\nu = d_1\Bigg[&\frac{1}{2}\left(K^{\omega}{}_{\rho\mu}\tilde{R}_{\lambda\omega\nu\sigma} - K^{\omega}{}_{\lambda\mu}\tilde{R}_{\rho\omega\nu\sigma}\right)\left(\tilde{R}^{\lambda\rho\nu\sigma} + 2\tilde{R}^{\lambda\nu\rho\sigma} - 2\tilde{R}^{\nu\sigma\lambda\rho}\right) + 2K^{\omega}{}_{\lambda\mu}\tilde{R}^{\rho\lambda}\tilde{R}_{\omega\rho}\\
&+2\left(\tilde{R}^{\lambda\rho} - \tilde{R}^{\rho\lambda}\right)K^{\omega}{}_{\nu\mu}\tilde{R}^{\nu}{}_{\lambda\omega\rho} + 8\tilde{R}^{[\lambda\rho]}K^{\omega}{}_{\nu(\lambda}\tilde{R}^{\nu}{}_{\rho)\mu\omega}\Bigg] + 16\pi\tilde{R}_{\lambda\rho\sigma\mu}S^{\lambda\rho\sigma}\,,\end{aligned} \tag{A.9}$$

where, once again, the contraction $\tilde{R}^{[\lambda\rho]}K^{\omega}{}_{\nu(\lambda}\tilde{R}^{\nu}{}_{\rho)\mu\omega} = 0$. On the other hand, the anti-symmetric part of the energy-momentum tensor is related via the vierbein equation to the following quantity:

$$\begin{aligned}X1^{[\mu\nu]} = \frac{d_1}{2}\Bigg[&\tilde{R}_{\lambda\rho}{}^{\nu}{}_{\sigma}\left(\tilde{R}^{\lambda\mu\rho\sigma} - 2\tilde{R}^{\mu\sigma\lambda\rho}\right) - \tilde{R}^{\nu}{}_{\sigma\lambda\rho}\left(\tilde{R}^{\rho\sigma\lambda\mu} - 2\tilde{R}^{\lambda\rho\mu\sigma}\right) + 2\left(\tilde{R}^{\mu\lambda}\tilde{R}_{\lambda}{}^{\nu} - \tilde{R}^{\lambda\mu}\tilde{R}^{\nu}{}_{\lambda}\right)\\
&+2\left(\tilde{R}_{\lambda\rho} - \tilde{R}_{\rho\lambda}\right)\left(\tilde{R}^{\nu\lambda\mu\rho} - \tilde{R}^{\mu\lambda\nu\rho}\right)\Bigg]\,.\end{aligned} \tag{A.10}$$

Therefore, it is straightforward to express this torsion-free divergence into a very concise form:

$$\nabla_\nu X1_\mu{}^\nu = K_{\lambda\rho\mu}X1^{\lambda\rho} + 16\pi\tilde{R}_{\lambda\rho\sigma\mu}S^{\lambda\rho\sigma}\,, \tag{A.11}$$

and the general conservation law of the total energy-momentum tensor states from the equation $X1^{\mu\nu} = -16\pi\theta^{\mu\nu}$ in the following way:

$$\nabla_\nu\theta_\mu{}^\nu + K_{\lambda\rho\mu}\theta^{\rho\lambda} + \tilde{R}_{\lambda\rho\sigma\mu}S^{\lambda\rho\sigma} = 0\,. \tag{A.12}$$

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Extended Reissner–Nordström solutions sourced by dynamical torsion

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We find a new exact vacuum solution in the framework of the Poincaré Gauge field theory with massive torsion. In this model, torsion operates as an independent field and introduces corrections to the vacuum structure present in General Relativity. The new static and spherically symmetric configuration shows a Reissner–Nordström-like geometry characterized by a spin charge. It extends the known massless torsion solution to the massive case. The corresponding Reissner–Nordström–de Sitter solution is also compatible with a cosmological constant and additional $U(1)$ gauge fields.

1. Introduction

The fundamental relation of the energy and momentum of matter with the space–time geometry is one of the most important foundations of General Relativity (GR). Namely, the energy-momentum tensor acts as the source of gravity, which is appropriately described in terms of the curvature tensor. In an analogous way, it may be expected that the intrinsic angular momentum of matter may also act as an additional source of the interaction and extend such a geometrical scheme.

Poincaré Gauge (PG) theory of gravity is the most consistent extension of GR that provides a suitable correspondence between spin and the space–time geometry by assuming an asymmetric affine connection defined within a Riemann–Cartan (RC) manifold (i.e. endowed with curvature and torsion) [1,2]. It represents a gauge approach to gravity based on the semidirect product of the Lorentz group and the space–time translations, in analogy to the unitary irreducible representations of relativistic particles labeled by their spin and mass, respectively. Then not only an energy-momentum tensor of matter arises from this approach, but also a non-trivial spin density tensor that operates as source of torsion and allows the existence of a gravitating antisymmetric component of the former, which may induce changes in the geometrical structure of the space–time, as the rest of the components of the mentioned tensor. This fact contrasts with the established by GR, where all the possible geometrical effects occurred in the Universe can be only provided by a symmetric component of the energy-momentum tensor, despite the existence of dynamical configurations endowed with asymmetric energy-momentum tensors [3,4].

Accordingly, a gauge invariant Lagrangian can be constructed from the field strength tensors to introduce the extended dynamical effects of the gravitational field. In this sense, it is well-known that the role of torsion depends on the order of the mentioned field strength tensors present in the Lagrangian, in a form that only quadratic or higher order corrections in the curvature tensor involve the presence of a non-trivial dynamical torsion, whose effects can propagate even in a vacuum space–time.

Likewise, the distinct restrictions on the Lagrangian parameters lead to a large class of gravitational models where an extensive number of particular and fundamental differences may arise. For example, in analogy to the standard approach of GR, it was shown that the Birkhoff's theorem is satisfied only in certain cases of the PG theory [5,6]. Indeed, the dynamical role of the new degrees of freedom involved in such a theory can modify the space–time geometry and even predominate in their respective domains of applicability. The

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search and study of exact solutions are therefore essential in order to improve the understanding and physical interpretation of the new framework.

A large class of exact solutions have been found since the formulation of the theory, especially for the case of static and spherically symmetric vacuum space–times, where one of the most primary and remarkable solutions is the so called Baekler solution, associated with a sort of PG models that encompass a weak-field limit with an additional confinement type of potential besides the Newtonian one [7], giving rise to a Schwarzschild–de Sitter geometry in analogy to the effect caused by the presence of a cosmological constant in the regular gravity action [8]. Furthermore, additional results have also been systematically obtained for a large class of PG configurations, such as axisymmetric space–times, cosmological systems or generalized gravitational waves (see [2,9–11] and references therein).

Recently, the authors of this work found a new exact solution with massless torsion associated with a PG model containing higher order corrections quadratic in the curvature tensor, in such a way that the standard framework of GR is naturally recovered when the total curvature satisfies the first Bianchi identity of the latter. This construction ensures that all the new propagating degrees of freedom introduced by the model fall on the torsion field, so that this quantity extend the domain of applicability of the standard case. Thus, it was shown that the regular Schwarzschild geometry provided by the Birkhoff's theorem of GR can be replaced by a Reissner–Nordström (RN) space–time with RC Coulomb-like curvature when this sort of dynamical torsion is considered [12]. This result contrasts with other post-Riemannian solutions, such as the derived in the framework of the Metric-Affine Gauge (MAG) theory, where the non-metricity tensor can involve an analogous vacuum RN configuration [13,14]. In addition, it is reasonable to expect that such a configuration may be extended for the case where additional non-vanishing mass modes of the torsion tensor are present in the Lagrangian, in order to analyze the equivalent PG model with massive torsion. As we will show, we have found the associated RN solution with massive torsion and generalized the previous approach according to the scheme performed in that simpler case.

This paper is organized as follows. First, in Section 2, we introduce our PG model with massive torsion and briefly describe its general mathematical foundations. The analysis and application of the resulting field equations in the static spherically symmetric space–time is shown in Section 3, in order to find the appropriate vacuum solutions for the selected case. In section 4, we present the required new PG solution with massive torsion and extend our previous results related to the massless case. We present the conclusions of our work in Section 5. Finally, we detail in Appendix A the geometrical quantities involved in the vacuum field equations associated with this model.

Before proceeding to the main discussion and general results, we briefly introduce the notation and physical units to be used throughout this article. Latin a, b and greek μ, ν indices refer to anholonomic and coordinate basis, respectively. We use notation with tilde for magnitudes including torsion and without tilde for torsion-free quantities. On the other hand, we will denote as P_a the generators of the space–time translations as well as J_{ab} the generators of the space–time rotations and assume their following commutative relations:

$$[P_a, P_b] = 0\,, \tag{1}$$

$$[P_a, J_{bc}] = i\,\eta_{a[b}\,P_{c]}\,, \tag{2}$$

$$[J_{ab}, J_{cd}] = \frac{i}{2}\,(\eta_{ad}\,J_{bc} + \eta_{cb}\,J_{ad} - \eta_{db}\,J_{ac} - \eta_{ac}\,J_{bd})\,. \tag{3}$$

Finally, we will use Planck units ($G = c = \hbar = 1$) throughout this work.

2. Quadratic Poincaré gauge gravity model with massive torsion

We start from the general gravitational action associated with our original PG model and incorporate the three independent quadratic scalar invariants of torsion into this expression, which represent the mass terms of the mentioned quantity:

$$\begin{aligned} S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[\mathcal{L}_m - \tilde{R} - \frac{1}{4}(d_1+d_2)\,\tilde{R}^2 - \frac{1}{4}(d_1+d_2+4c_1+2c_2)\,\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} + c_1\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu} \\ +c_2\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\mu\rho\nu} + d_1\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + d_2\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} + \alpha\,T_{\lambda\mu\nu}T^{\lambda\mu\nu} + \beta\,T_{\lambda\mu\nu}T^{\mu\lambda\nu} + \gamma\,T^{\lambda}{}_{\lambda\nu}T^{\mu}{}_{\mu}{}^{\nu}\Big]\,, \end{aligned} \tag{4}$$

where $c_1, c_2, d_1, d_2, \alpha, \beta$ and γ are constant parameters.

The field strength tensors above derive from the gauge connection of the Poincaré group $ISO(1,3)$, which can be expressed in terms of the generators of translations and local Lorentz rotations in the following way:

$$A_\mu = e^a{}_\mu P_a + \omega^{ab}{}_\mu J_{ab}\,, \tag{5}$$

where $e^a{}_\mu$ is the vierbein field and $\omega^{ab}{}_\mu$ the spin connection of a RC manifold, related to the metric tensor and the metric-compatible affine connection as usual [15]:

$$g_{\mu\nu} = e^a{}_\mu\,e^b{}_\nu\,\eta_{ab}\,, \tag{6}$$

$$\omega^{ab}{}_\mu = e^a{}_\lambda\,e^{b\rho}\,\tilde{\Gamma}^\lambda{}_{\rho\mu} + e^a{}_\lambda\,\partial_\mu\,e^{b\lambda}\,. \tag{7}$$

The affine connection is decomposed into the torsion-free Levi-Civita connection and a contortion component, which transforms as a tensor due to the tensorial nature of torsion since it describes the antisymmetric part of the affine connection:

$$\tilde{\Gamma}^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} + K^\lambda{}_{\mu\nu}\,. \tag{8}$$

Thus, the presence of torsion potentially introduces changes in the properties of the gravitational interaction and it involves the following $ISO(1,3)$ gauge field strength tensors:

$$F^{a}{}_{\mu\nu} = e^{a}{}_{\lambda}\, T^{\lambda}{}_{\nu\mu}\,, \tag{9}$$

$$F^{ab}{}_{\mu\nu} = e^{a}{}_{\lambda} e^{b}{}_{\rho}\, \tilde{R}^{\lambda\rho}{}_{\mu\nu}\,, \tag{10}$$

where $T^{\lambda}{}_{\mu\nu}$ and $\tilde{R}^{\lambda\rho}{}_{\mu\nu}$ are the components of the torsion and the curvature tensor, respectively:

$$T^{\lambda}{}_{\mu\nu} = 2\tilde{\Gamma}^{\lambda}{}_{[\mu\nu]}\,, \tag{11}$$

$$\tilde{R}^{\lambda}{}_{\rho\mu\nu} = \partial_{\mu}\tilde{\Gamma}^{\lambda}{}_{\rho\nu} - \partial_{\nu}\tilde{\Gamma}^{\lambda}{}_{\rho\mu} + \tilde{\Gamma}^{\lambda}{}_{\sigma\mu}\tilde{\Gamma}^{\sigma}{}_{\rho\nu} - \tilde{\Gamma}^{\lambda}{}_{\sigma\nu}\tilde{\Gamma}^{\sigma}{}_{\rho\mu}\,. \tag{12}$$

Therefore, within this framework, torsion appears naturally related to the translations whereas curvature is related to the rotations, as expected. Furthermore, both quantities can decompose into distinct modes by computing their irreducible representations under the Lorentz group [16,17]. Specifically, torsion can be divided into three irreducible components: a trace vector T_{μ}, an axial vector S_{μ} and a traceless and also pseudotraceless tensor $q^{\lambda}{}_{\mu\nu}$:

$$T^{\lambda}{}_{\mu\nu} = \frac{1}{3}\left(\delta^{\lambda}{}_{\nu}T_{\mu} - \delta^{\lambda}{}_{\mu}T_{\nu}\right) + \frac{1}{6}\, g^{\lambda\rho}\varepsilon_{\rho\sigma\mu\nu}S^{\sigma} + q^{\lambda}{}_{\mu\nu}\,, \tag{13}$$

where $\varepsilon_{\rho\sigma\mu\nu}$ is the four-dimensional Levi-Civita symbol.

Hence, each of the cited modes can be massive or massless, what can be implemented in the general action of the theory by introducing the corresponding explicit torsion square pieces, as it is shown in the Expression (4). Then, the extended field equations can be derived by performing variations with respect to the gauge potentials, as usual. In addition, the resulting system of equations can be simplified without loss of generality by the Gauss–Bonnet theorem in RC spaces [18,19]. Namely, the following combination quadratic in the curvature tensor acts as a total derivative of a certain vector V^{μ} in the previous gravitational action:

$$\sqrt{-g}\left(\tilde{R}^{2} + \tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} - 4\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu}\right) = \partial_{\mu}V^{\mu}\,. \tag{14}$$

Thereby, this constraint allows to reduce the gravitational action and to obtain the following system of variational equations:

$$X1_{\mu}{}^{\nu} + 16\pi\,\theta_{\mu}{}^{\nu} = 0\,, \tag{15}$$

$$X2_{[\mu\lambda]}{}^{\nu} + 16\pi\, S_{\lambda\mu}{}^{\nu} = 0\,, \tag{16}$$

where $X1_{\mu}{}^{\nu}$ and $X2_{[\mu\lambda]}{}^{\nu}$ are tensorial functions depending on the RC curvature and the torsion tensor, which are defined in Appendix A, whereas $\theta_{\mu}{}^{\nu}$ and $S_{\lambda\mu}{}^{\nu}$ are the canonical energy-momentum tensor and the spin density tensor, respectively:

$$\theta_{\mu}{}^{\nu} = \frac{e^{a}{}_{\mu}}{16\pi\sqrt{-g}}\frac{\delta\left(\mathcal{L}_{m}\sqrt{-g}\right)}{\delta e^{a}{}_{\nu}}\,, \tag{17}$$

$$S_{\lambda\mu}{}^{\nu} = \frac{e^{a}{}_{\lambda}e^{b}{}_{\mu}}{16\pi\sqrt{-g}}\frac{\delta\left(\mathcal{L}_{m}\sqrt{-g}\right)}{\delta A^{ab}{}_{\nu}}\,. \tag{18}$$

These quantities act as sources of gravity and constitute the natural generalization of the conserved Noether currents associated with the external translations and rotations of the Poincaré group in a Minkowski space–time [20]. Indeed, it is straightforward to note from the field equations above the fulfillment of the following conservation laws:

$$\nabla_{\nu}\theta_{\mu}{}^{\nu} + K_{\lambda\rho\mu}\theta^{\rho\lambda} + \tilde{R}_{\lambda\rho\nu\mu}\, S^{\lambda\rho\nu} = 0\,, \tag{19}$$

$$\nabla_{\mu}S_{\lambda\rho}{}^{\mu} + 2K^{\sigma}{}_{[\lambda|\mu}S_{|\rho]\sigma}{}^{\mu} - \theta_{[\lambda\rho]} = 0\,. \tag{20}$$

Therefore, the canonical energy-momentum tensor generally contains an antisymmetric component even when the notions of curvature and torsion are neglected (i.e. in the framework of Special Relativity):

$$\partial_{\nu}\theta_{\mu}{}^{\nu} = 0\,, \tag{21}$$

$$\partial_{\mu}M_{\lambda\rho}{}^{\mu} + \partial_{\mu}S_{\lambda\rho}{}^{\mu} = 0\,, \tag{22}$$

where $M_{\lambda\rho}{}^{\mu} = x_{[\lambda}\,\theta_{\rho]}{}^{\mu}$ is the orbital angular momentum density, whose divergence is trivially proportional to the mentioned antisymmetric part of the canonical energy-momentum tensor:

$$\partial_{\mu}M_{\lambda\rho}{}^{\mu} = \theta_{[\rho\lambda]}\,. \tag{23}$$

Thus, as it is shown, there exists a complete correspondence between the main currents of matter sources and the space–time geometry in the framework of PG theory. However, the theoretical construction present in GR encodes all the possible geometrical effects, derived by the presence of the gravitational field, only into the symmetric part of the canonical energy-momentum tensor of matter. Specifically, it postulates the symmetrized Belinfante–Rosenfeld energy-momentum tensor as the unique material quantity coupled to gravity [21]:

$$T_{\mu\nu} = \theta_{\mu\nu} - \nabla_{\lambda}S_{\mu\nu}{}^{\lambda} - \nabla_{\lambda}S^{\lambda}{}_{\mu\nu} - \nabla_{\lambda}S^{\lambda}{}_{\nu\mu}\,, \tag{24}$$

and omits from the gravitational scheme all the possible dynamical contributions provided by the rest of features of matter. Some remarkable implications derived by this post-Riemannian approach involve the prevention of space–time singularities and the generation of an accelerating cosmological expansion in terms of the torsion field, among others [22–26]. In this sense, apart from its potential influence in the cosmological and astrophysical arena, the space–time torsion represents a fundamental quantity that may improve our understanding on the correspondence between geometry and physics, what it means that any kind of dynamical aspect associated with it may be crucial to identify its different roles or to detect it.

Concerning the vacuum structure of the theory, the material tensors above vanish and it is sufficient to deal with the following system of equations:

$$X1_{\mu}{}^{\nu} = 0\,, \tag{25}$$

$$X2_{[\mu\lambda]}{}^{\nu} = 0\,. \tag{26}$$

It is straightforward to note that the standard approach of GR is completely recovered when the first Bianchi identity of such a theory is fulfilled by the total curvature (i.e. $\tilde{R}^{\lambda}{}_{[\mu\nu\rho]} = 0$) and all the mass coefficients of torsion vanish. However, in the massless torsion solution [12], it was shown that such a limit can be obtained by switching off the dynamical axial component of the torsion tensor, so that even for the case where both the trace vector and the tensorial component of torsion are massless, the same procedure may be trivially applied in presence of a massive axial component of torsion.

3. Space–time symmetries and consistency constraints

In order to solve the vacuum field equations of the theory for a static and spherically symmetric space–time, we consider the corresponding line element and tetrad basis as follows:

$$ds^2 = \Psi_1(r)\,dt^2 - \frac{dr^2}{\Psi_2(r)} - r^2\left(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2\right)\,, \tag{27}$$

$$e^{\hat{t}} = \sqrt{\Psi_1(r)}\,dt\,,\quad e^{\hat{r}} = \frac{dr}{\sqrt{\Psi_2(r)}}\,,\quad e^{\hat{\theta}_1} = r\,d\theta_1\,,\quad e^{\hat{\theta}_2} = r\sin\theta_1\,d\theta_2\,; \tag{28}$$

with $0 \leq \theta_1 \leq \pi$ and $0 \leq \theta_2 \leq 2\pi$.

The intrinsic relations between curvature and torsion involve that the latter is also influenced by the space–time symmetries and it must satisfy the condition $\mathcal{L}_{\xi} T^{\lambda}{}_{\mu\nu} = 0$ (i.e. the Lie derivative in the direction of the Killing vector ξ on $T^{\lambda}{}_{\mu\nu}$ vanishes). Indeed, this constraint ensures that the covariant derivative commutes with the Lie derivative, what in turn preserves the invariance of the curvature tensor under isometries.

Therefore, the static spherically symmetric torsion acquires the following structure [6,27]:

$$\begin{aligned}
T^{t}{}_{tr} &= a(r)\quad,\\
T^{r}{}_{tr} &= b(r)\quad,\\
T^{\theta_k}{}_{t\theta_l} &= \delta^{\theta_k}{}_{\theta_l}\,c(r)\quad,\\
T^{\theta_k}{}_{r\theta_l} &= \delta^{\theta_k}{}_{\theta_l}\,g(r)\quad,\\
T^{\theta_k}{}_{t\theta_l} &= e^{a\theta_k}\,e^{b}{}_{\theta_l}\,\epsilon_{ab}\,d(r)\quad,\\
T^{\theta_k}{}_{r\theta_l} &= e^{a\theta_k}\,e^{b}{}_{\theta_l}\,\epsilon_{ab}\,h(r)\quad,\\
T^{t}{}_{\theta_k\theta_l} &= \epsilon_{kl}\,k(r)\,\sin\theta_1\quad,\\
T^{r}{}_{\theta_k\theta_l} &= \epsilon_{kl}\,l(r)\,\sin\theta_1\quad;
\end{aligned} \tag{29}$$

where a, b, c, d, g, h, k and l are eight arbitrary functions depending only on r; $k, l = 1, 2$, and ϵ_{ab} is the two-dimensional Levi-Civita symbol:

$$\epsilon_{ab} = \begin{cases} +1\,, & \text{for } \ a\,b = 1\,2,\\ -1\,, & \text{for } \ a\,b = 2\,1,\\ \ \ 0\,, & \text{for all other combinations.}\end{cases} \tag{30}$$

These symmetry properties strongly reduce the possible classes of solutions, but even though the field equations constitute a highly nonlinear system involving a large number of degrees of freedom, so that the problem turns out to be still very complicated and furthermore underdetermined. In fact, one of the features associated with a large number of PG models is the existence of a high geometrical freedom, where it is possible to find solutions depending on arbitrary functions and thereby underdetermined by the variational equations [28–30]. It is worthwhile to stress that, for the particular case given by the presence of a dynamical massless torsion, the traceless of the tetrad field equations requires the vanishing of the torsion-free scalar curvature, which in turn represents a strong geometrical constraint involving the degrees of freedom of the metric tensor alone. Furthermore, in presence of an external Coulomb electric field, the compatibility with the Maxwell field equations in spherically symmetric space–times requires the additional constraint given by $\Psi_1 = \Psi_2$, so that in this case the geometry acquires the form of a RN space–time and such a type of arbitrariness does not emerge, in contrast with other PG models with explicit torsion square pieces. In this sense, as previously stressed, we simply extend our previous results with massless torsion to a generic PG model with these torsion square corrections, what it means an easy way to obtain solutions due to the analyses performed in that simpler case. On the other hand, it is worthwhile to emphasize that the existence and unicity of solutions within these torsion models can be established under appropriate energy conditions [31].

According to the massless torsion scheme, it is always possible to impose an additional constraint by applying the weak-field approximation for the torsion tensor through the trace of Eq. (26). This restriction ensures that our PG model appropriately encompasses such a limit. Then, by neglecting torsion terms of second order, the equations of motion for the torsion tensor in linear approximation read

$$\nabla_\mu \nabla^\mu T^\nu{}_{\lambda\nu} + \nabla_\mu \nabla_\nu T^{\nu\mu}{}_\lambda - \nabla_\mu \nabla_\lambda T^{\nu\mu}{}_\nu = \frac{2\alpha+\beta+3\gamma+2}{4c_1+c_2+2d_1} T^\nu{}_{\lambda\nu} \,. \tag{31}$$

In the special case where $2\alpha+\beta+3\gamma+2=0$, it turns out that the mass modes of torsion do not contribute to the weak-field approximation and then this constraint reduces to the following relation among the torsion and metric components:

$$b(r) = rc'(r) + c(r) + \frac{p}{r}\sqrt{\frac{\Psi_1(r)}{\Psi_2(r)}} \,, \tag{32}$$

where p is an integration constant.

In analogy to the massless torsion case [12], we demand the condition $\Psi_1 = \Psi_2 \equiv \Psi(r)$ to guarantee the compatibility requirement with external electric and magnetic fields, as in the standard Einstein–Maxwell framework of GR. Finally, we also require the avoidance of undesirable singularities from any solution $F^a{}_{bc}$ referred to the rotated basis $\vartheta^a = \Lambda^a{}_b e^b$ given by the following vector fields:

$$\begin{aligned}
\vartheta^{\hat{t}} &= \frac{1}{2}\left\{[\Psi(r)+1]\,dt + \left[1-\frac{1}{\Psi(r)}\right]dr\right\} \;; \\
\vartheta^{\hat{r}} &= \frac{1}{2}\left\{[\Psi(r)-1]\,dt + \left[1+\frac{1}{\Psi(r)}\right]dr\right\} \;; \\
\vartheta^{\hat{\theta}_1} &= r\,d\theta_1 \;; \\
\vartheta^{\hat{\theta}_2} &= r\sin\theta_1\,d\theta_2 \;.
\end{aligned} \tag{33}$$

Accordingly, in order to avoid geometrical divergences in the roots of the metric function $\Psi(r)$, the following relations among the torsion components are taken into account:

$$b(r) = a(r)\,\Psi(r)\,, \quad c(r) = -g(r)\,\Psi(r)\,, \quad d(r) = -h(r)\,\Psi(r)\,, \quad l(r) = k(r)\,\Psi(r)\,. \tag{34}$$

It is worthwhile to note that these constraints involve the vanishing of the three independent quadratic torsion invariants. Namely, in terms of its irreducible components:

$$T_\mu T^\mu = S_\mu S^\mu = q_{\lambda\mu\nu}\, q^{\lambda\mu\nu} = 0\,. \tag{35}$$

Furthermore, the additional quartic torsion invariants also vanish under these conditions:

$$T_\mu T_\nu S^\mu S^\nu = T^\lambda T_\rho\, q_{\mu\nu\lambda} q^{\mu\nu\rho} = S^\lambda S_\rho\, q_{\mu\nu\lambda} q^{\mu\nu\rho} = T^\lambda S_\rho\, q_{\mu\nu\lambda} q^{\mu\nu\rho} = 0\,, \tag{36}$$

$$T_\lambda T_\mu T_\nu\, q^{\lambda\mu\nu} = S_\lambda S_\mu S_\nu\, q^{\lambda\mu\nu} = T_\lambda T_\mu S_\nu\, q^{\lambda\mu\nu} = T_\lambda S_\mu S_\nu\, q^{\lambda\mu\nu} = 0\,, \tag{37}$$

$$T_\sigma\, q_{\mu\nu\rho}\, q^{\mu\nu\lambda}\, q_\lambda{}^{\rho\sigma} = S_\sigma\, q_{\mu\nu\rho}\, q^{\mu\nu\lambda}\, q_\lambda{}^{\rho\sigma} = q_{\mu\nu\lambda}\, q^{\mu\nu\rho}\, q_{\sigma\omega}{}^\lambda\, q^{\sigma\omega}{}_\rho = q_{\lambda\sigma\mu}\, q^{\lambda\omega\nu}\, q^\sigma{}_{\rho\nu}\, q_\omega{}^{\rho\mu} = 0\,. \tag{38}$$

4. Solutions

By taking into account the previous remarks, the following constraints among the metric and torsion components are necessarily imposed together with the field equations and the basic space–time symmetry properties, in order to establish an appropriate physical consistency to the regarded PG model:

$$\Psi_1 = \Psi_2 \equiv \Psi(r)\,, \tag{39}$$

$$b(r) = rc'(r) + c(r) + \frac{p}{r}\,, \tag{40}$$

$$b(r) = a(r)\,\Psi(r)\,, \quad c(r) = -g(r)\,\Psi(r)\,, \quad d(r) = -h(r)\,\Psi(r)\,, \quad l(r) = k(r)\,\Psi(r)\,. \tag{41}$$

Note that these requirements do not demand the additional assumption of the double duality ansatz, usually considered by many authors due to its strong simplification of the field equations into a particular easier form [32]. Indeed, from a physical point of view, there is not any compelling reason to apply such a higher restriction, but a particular mathematical reduction in the difficulty of the computations, what in certain cases usually involves a loss of accuracy and generality that are incompatible with other possible configurations.

Then, the original model is appropriately simplified, and the following SO(3)-symmetric vacuum solution can be easily found for $c_1 = -d_1/4$, $c_2 = -d_1/2$, $\alpha = \frac{1}{2}(1-\beta)$ and $\gamma = -1$:

$$\begin{aligned}
&a(r) = \frac{\Psi'(r)}{2\Psi(r)} + \frac{wr}{\Psi(r)}\,, \quad b(r) = \frac{\Psi'(r)}{2} + wr\,, \quad c(r) = \frac{\Psi(r)}{2r} + \frac{wr}{2}\,, \quad g(r) = -\frac{1}{2r} - \frac{wr}{2\Psi(r)}\,, \\
&d(r) = \frac{\kappa}{r}\,, \qquad h(r) = -\frac{\kappa}{r\Psi(r)}\,, \qquad k(r) = l(r) = 0 \;;
\end{aligned} \tag{42}$$

with

$$\Psi(r) = 1 - \frac{2m}{r} + \frac{d_1 \kappa^2}{r^2} \,, \tag{43}$$

$$w = \frac{(1-2\beta)}{d_1} \,. \tag{44}$$

It is straightforward to note that the solution belongs to the special case where the contribution of the mass modes to the weak-field approximation of the torsion field is negligible. Then the relation (32) is completely fulfilled by taking $p = 0$. In addition, the trace vector and the tensorial component of torsion remain massless whereas the axial mode becomes massless for $\beta = \frac{1}{2}$, what it means that our previous RN solution with massless torsion is recovered in such a case. This is an expected result, since it is shown that the dynamical behavior of torsion falls on the mentioned mode. Indeed, the axial component of torsion acts as a Coulomb-like potential depending on the parameter κ, which is related to the existence of a spin charge, in analogy to the relation between torsion and its spinning sources. Its geometrical effect is induced on the metric tensor by modifying the regular Schwarzschild vacuum structure of GR with the RN space–time associated with the following RC curvature tensor:

$$F^{ab}{}_{cd} = \begin{pmatrix} -w & 0 & 0 & 0 & 0 & 0 \\ 0 & -w\,\chi_-(r)/2 & -\chi_+(r)\,(\kappa/2r^2) & 0 & -\chi_+(r)\,(\kappa/2r^2) & -w\,\chi_+(r)/2 \\ 0 & \chi_+(r)\,(\kappa/2r^2) & -w\,\chi_-(r)/2 & 0 & w\,\chi_+(r)/2 & -\chi_+(r)\,(\kappa/2r^2) \\ -\kappa/r^2 & 0 & 0 & -\left(1/r^2 + w/2\right) & 0 & 0 \\ 0 & \chi_-(r)\,(\kappa/2r^2) & -w\,\chi_+(r)/2 & 0 & -3w\,\zeta(r)/2 & -\chi_-(r)\,(\kappa/2r^2) \\ 0 & w\,\chi_+(r)/2 & \chi_-(r)\,(\kappa/2r^2) & 0 & \chi_-(r)\,(\kappa/2r^2) & -3w\,\zeta(r)/2 \end{pmatrix}, \tag{45}$$

where the six rows and columns of the matrix are labeled the components in the order (01, 02, 03, 23, 31, 12) and the following functions have been defined:

$$\chi_\pm(r) = 1 \pm \frac{wr^2}{\Psi(r)} \,, \tag{46}$$

$$\zeta(r) = 1 + \frac{wr^2}{3\Psi(r)} \,. \tag{47}$$

Then, according to the first Bianchi identity in a RC space–time [33], the solution reduces to the standard Schwarzschild geometry of GR when $\tilde{\nabla}_{[\mu} T^{\lambda}{}_{\nu\rho]} + T^{\sigma}{}_{[\mu\nu}\, T^{\lambda}{}_{\rho]\sigma} = 0$, namely when the parameter κ of the axial component vanishes.

It is also straightforward to notice the absence of singularities, excluding the point $r = 0$, in the six independent quadratic scalar invariants defined from the curvature tensor, as expected from relations (34):

$$\tilde{R}^2 = \frac{4}{r^4}\left(1 + 6wr^2\right)^2 \,, \tag{48}$$

$$\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu} = \frac{4}{r^4}\left(1 - \kappa^2 + 2wr^2\left(1 + 3wr^2\right)\right) \,, \tag{49}$$

$$\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} = \frac{4}{r^4}\left(1 - 2\kappa^2 + 2wr^2\left(1 + 3wr^2\right)\right) \,, \tag{50}$$

$$\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\mu\rho\nu} = \frac{2}{r^4}\left(1 - \kappa^2 + 2wr^2\left(1 + 3wr^2\right)\right) \,, \tag{51}$$

$$\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} = \frac{2}{r^4}\left(1 + \kappa^2 + 6wr^2\left(1 + 3wr^2\right)\right) \,, \tag{52}$$

$$\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} = \frac{2}{r^4}\left(1 - \kappa^2 + 6wr^2\left(1 + 3wr^2\right)\right) \,. \tag{53}$$

On the other hand, the solution leads to a specific set of values for the Lagrangian coefficients, which should additionally define a viable and stable gravitational theory. According to the unitary and causality requirements, this consistency demands the absence of both ghosts and tachyons in the particle spectrum of the model, what has been systematically carried out by distinct approaches for the case of massive propagating torsion as well as for the case with zero-mass modes, where extra gauge symmetries can appear besides the fundamental Poincaré gauge symmetry [35,36,34,37–40]. Nevertheless it should be noted that, apart from some particular differences and disagreements in their conclusions, all these approaches are not developed as perturbative analyses around any specific curved background which may be induced by the presence of a dynamical torsion, but on a rigid flat space–time where the possible effects of the torsion field are completely neglected. In fact, as can be seen, within our PG model the presence of a non-vanishing propagating torsion modifies the vacuum structure with the above RN geometry, where the axial component of the torsion tensor emerges in the metric tensor and hence it cannot be unilaterally excluded from the background. Furthermore, it is straightforward to note from (31) that our PG model encompasses a weak-field approximation for the torsion field that cannot be separated from the background space–time (i.e. the torsion-free covariant derivatives of the Expression (31) cannot be replaced by ordinary derivatives). Therefore, there exists a strong limitation around the cited stability studies, what it means that future analyses should be performed in order to examine the stability of these types of PG models.

Additionally, the solution can be naturally generalized to include the existence of a non-vanishing cosmological constant Λ and Coulomb electromagnetic fields with electric and magnetic charges q_e and q_m respectively, which are decoupled from torsion under the assumption of the minimum coupling principle. This simple extension is obtained by modifying the metric function $\Psi(r)$ by the following expression:

$$\Psi(r) = 1 - \frac{2m}{r} + \frac{d_1\kappa^2 + q_e^2 + q_m^2}{r^2} + \frac{\Lambda}{3}r^2 \,. \tag{54}$$

Thereby, the solution shows similarities between the torsion and the electromagnetic fields, even though they are independent quantities. Note that it is referred to an extensive and regular PG theory, unlike other monopole-type solutions that can be constructed by modifying the model towards a different approach embedded within the complex Einstein–Yang–Mills theory [41].

The mass factors present in the solution may also involve corrections in the motion of spinning matter. Nevertheless, these deviations from the geodesic motion of ordinary matter are expected to be very small at astrophysics or cosmological scales, because of the vanishing of the spin density tensor in the most macroscopical bodies. This situation may differ around extreme gravitational systems as neutron stars or black holes with intense magnetic fields and sufficiently oriented elementary spins. In such a case, it is expected that the RC space–time described by the PG theory modulates these events. In addition, the influence of the mass of torsion on Dirac fields depends on the coupling considered between these and the torsion tensor. For Dirac fields minimally coupled to torsion, it turns out that only the axial vector carries out the interaction, whereas the trace vector and the tensorial mode are completely decoupled [42]. However, as can be seen from our RN solution, the parameter of mass associated with the axial mode falls on the rest of components of the torsion tensor, what it means that its effects may only be induced on Dirac fields non-minimally coupled to torsion.

5. Conclusions

In the present work, we have extended the correspondence between torsion and vacuum RN geometries in the framework of PG theory with massive torsion. This correspondence was first stressed in a previous work for the particular case given by a dynamical massless torsion alone, that can be associated with a PG model that contains quadratic order corrections in the curvature tensor [12]. Similar foundations were also introduced in [43,44] in order to find an alternative method to solve the Einstein–Yang–Mills equations in extended gravitational theories. We investigate its generalization to the case with non-vanishing torsion mass modes by including the respective explicit torsion square pieces in the gravitational action. Then, we obtain the corresponding RN solution with massive torsion by imposing the appropriate space–time symmetries on the metric and torsion tensor, as well as additional consistency constraints in order to avoid all the possible unsuitable singularities and encompass the weak-field limit associated with torsion in a framework compatible with external Coulomb electric and magnetic fields, as in the standard case of GR.

In this scheme, the dynamical role of the torsion tensor is carried out by its axial mode, in a way that this mode can be massive or massless, whereas the mass modes of the trace vector and of the tensorial component remain vanishing. The presence of such a non-vanishing mass modifies the rest of the torsion components of the solution and it may introduce deviations in the trajectories of spinning matter. Nevertheless, it is shown that for the case of Dirac fields the non-minimal coupling to torsion is necessary. Even though, it is expected that the possible consequent effects are negligible at macroscopic scales and they may become significant only at extremely high densities.

Finally, the corresponding Reissner–Nordström–de Sitter solution with cosmological constant and external electromagnetic fields is also obtained, by analogy with the standard case. The existence of these sorts of configurations reveals the dynamical role of the space–time torsion and provides new features associated with this field, what involves a richer vacuum structure of post-Riemannian gravitational theories endowed with both curvature and torsion.

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Appendix A. Expressions of the field equations

The Lagrangian (4) imposes the vanishing of the tensors $X1_\mu{}^\nu$ and $X2_\mu{}^{\lambda\nu}$ in vacuum, whose expressions can be written as:

$$X1_\mu{}^\nu = -2\tilde{G}^\nu{}_\mu + 4c_1 T1_\mu{}^\nu + 2c_2 T2_\mu{}^\nu - 2\,(2c_1 + c_2)\,T3_\mu{}^\nu + 2d_1\left(H1_\mu{}^\nu - H2_\mu{}^\nu\right) + \alpha\, I1_\mu{}^\nu + \beta\, I2_\mu{}^\nu + \gamma\, I3_\mu{}^\nu \,, \tag{A.1}$$

$$X2_\mu{}^{\lambda\nu} = \overset{\star}{T}_\mu{}^{\lambda\nu} + 4c_1 C1_\mu{}^{\lambda\nu} - 2c_2 C2_\mu{}^{\lambda\nu} + 2\,(2c_1 + c_2)\,C3_\mu{}^{\lambda\nu} - 2d_1\left(Y1_\mu{}^{\lambda\nu} - Y2_\mu{}^{\lambda\nu}\right) - \alpha\, Z1_\mu{}^{\lambda\nu} - \beta\, Z2_\mu{}^{\lambda\nu} - \gamma\, Z3_\mu{}^{\lambda\nu} \,, \tag{A.2}$$

where it is given the explicit dependence with the following geometrical quantities:

$$\tilde{G}_\mu{}^\nu = \tilde{R}_\mu{}^\nu - \frac{\tilde{R}}{2}\delta_\mu{}^\nu \,, \tag{A.3}$$

$$T1_\mu{}^\nu = \tilde{R}_{\lambda\rho\mu\sigma}\tilde{R}^{\lambda\rho\nu\sigma} - \frac{1}{4}\delta_\mu{}^\nu \tilde{R}_{\lambda\rho\tau\sigma}\tilde{R}^{\lambda\rho\tau\sigma} \,, \tag{A.4}$$

$$T2_\mu{}^\nu = \tilde{R}_{\lambda\rho\mu\sigma}\tilde{R}^{\lambda\nu\rho\sigma} + \tilde{R}_{\lambda\rho\sigma\mu}\tilde{R}^{\lambda\sigma\rho\nu} - \frac{1}{2}\delta_\mu{}^\nu \tilde{R}_{\lambda\rho\tau\sigma}\tilde{R}^{\lambda\tau\rho\sigma} \,, \tag{A.5}$$

$$T3_\mu{}^\nu = \tilde{R}_{\lambda\rho\mu\sigma}\tilde{R}^{\nu\sigma\lambda\rho} - \frac{1}{4}\delta_\mu{}^\nu \tilde{R}_{\lambda\rho\tau\sigma}\tilde{R}^{\tau\sigma\lambda\rho} \,, \tag{A.6}$$

$$H1_\mu{}^\nu = \tilde{R}^\nu{}_{\lambda\mu\rho}\tilde{R}^{\lambda\rho} + \tilde{R}_{\lambda\mu}\tilde{R}^{\lambda\nu} - \frac{1}{2}\delta_\mu{}^\nu \tilde{R}_{\lambda\rho}\tilde{R}^{\lambda\rho} \,, \tag{A.7}$$

$$H2_\mu{}^\nu = \tilde{R}^\nu{}_{\lambda\mu\rho}\tilde{R}^{\rho\lambda} + \tilde{R}_{\lambda\mu}\tilde{R}^{\nu\lambda} - \frac{1}{2}\delta_\mu{}^\nu \tilde{R}_{\lambda\rho}\tilde{R}^{\rho\lambda}\,, \tag{A.8}$$

$$I1_\mu{}^\nu = 4\left(T_{\lambda\rho\mu}T^{\lambda\rho\nu} + \nabla_\lambda T_\mu{}^{\nu\lambda} - K^\rho{}_{\mu\lambda}T_\rho{}^{\nu\lambda} - \frac{1}{4}\delta_\mu{}^\nu T_{\lambda\rho\sigma}T^{\lambda\rho\sigma}\right), \tag{A.9}$$

$$I2_\mu{}^\nu = 2\left(T_{\lambda\rho\mu}T^{\nu\rho\lambda} + T_{\lambda\rho\mu}T^{\rho\lambda\nu} + \nabla_\lambda T^{\lambda\nu}{}_\mu - \nabla_\lambda T^{\nu\lambda}{}_\mu + K^\rho{}_{\mu\lambda}\left(T^{\nu\lambda}{}_\rho - T^{\lambda\nu}{}_\rho\right) - \frac{1}{2}\delta_\mu{}^\nu T_{\lambda\rho\sigma}T^{\rho\lambda\sigma}\right), \tag{A.10}$$

$$I3_\mu{}^\nu = 2\left(T^\nu{}_{\mu\lambda}T^\rho{}_\rho{}^\lambda - \nabla_\mu T^\lambda{}_\lambda{}^\nu - K^\nu{}_{\mu\lambda}T^\rho{}_\rho{}^\lambda - \frac{1}{2}\delta_\mu{}^\nu\left(T^\lambda{}_{\lambda\sigma}T^\rho{}_\rho{}^\sigma - 2\nabla_\lambda T^\rho{}_\rho{}^\lambda\right)\right), \tag{A.11}$$

$$\overset{\star}{T}_\mu{}^{\lambda\nu} = \delta_\mu{}^\nu g^{\lambda\sigma}T^\rho{}_{\rho\sigma} - g^{\lambda\nu}T^\rho{}_{\rho\mu} - g^{\lambda\sigma}T^\nu{}_{\mu\sigma}\,, \tag{A.12}$$

$$C1_\mu{}^{\lambda\nu} = \nabla_\rho\tilde{R}_\mu{}^{\lambda\rho\nu} + K^\lambda{}_{\sigma\rho}\tilde{R}_\mu{}^{\sigma\rho\nu} - K^\sigma{}_{\mu\rho}\tilde{R}_\sigma{}^{\lambda\rho\nu}\,, \tag{A.13}$$

$$C2_\mu{}^{\lambda\nu} = \nabla_\rho\left(\tilde{R}_\mu{}^{\nu\lambda\rho} - \tilde{R}_\mu{}^{\rho\lambda\nu}\right) + K^\lambda{}_{\sigma\rho}\left(\tilde{R}_\mu{}^{\nu\sigma\rho} - \tilde{R}_\mu{}^{\rho\sigma\nu}\right) - K^\sigma{}_{\mu\rho}\left(\tilde{R}_\sigma{}^{\nu\lambda\rho} - \tilde{R}_\sigma{}^{\rho\lambda\nu}\right), \tag{A.14}$$

$$C3_\mu{}^{\lambda\nu} = \nabla_\rho\tilde{R}^{\rho\nu\lambda}{}_\mu + K^\lambda{}_{\sigma\rho}\tilde{R}^{\rho\nu\sigma}{}_\mu - K^\sigma{}_{\mu\rho}\tilde{R}^{\rho\nu\lambda}{}_\sigma\,, \tag{A.15}$$

$$Y1_\mu{}^{\lambda\nu} = \delta_\mu{}^\nu\nabla_\rho\tilde{R}^{\lambda\rho} - \nabla_\mu\tilde{R}^{\lambda\nu} + \delta_\mu{}^\nu K^\lambda{}_{\sigma\rho}\tilde{R}^{\sigma\rho} + K^\rho{}_{\mu\rho}\tilde{R}^{\lambda\nu} - K^\nu{}_{\mu\rho}\tilde{R}^{\lambda\rho} - K^\lambda{}_{\rho\mu}\tilde{R}^{\rho\nu}\,, \tag{A.16}$$

$$Y2_\mu{}^{\lambda\nu} = \delta_\mu{}^\nu\nabla_\rho\tilde{R}^{\rho\lambda} - \nabla_\mu\tilde{R}^{\nu\lambda} + \delta_\mu{}^\nu K^\lambda{}_{\sigma\rho}\tilde{R}^{\rho\sigma} + K^\rho{}_{\mu\rho}\tilde{R}^{\nu\lambda} - K^\nu{}_{\mu\rho}\tilde{R}^{\rho\lambda} - K^\lambda{}_{\rho\mu}\tilde{R}^{\nu\rho}\,, \tag{A.17}$$

$$Z1_\mu{}^{\lambda\nu} = 4T^{\lambda\nu}{}_\mu\,, \tag{A.18}$$

$$Z2_\mu{}^{\lambda\nu} = 2\left(T^{\nu\lambda}{}_\mu - T^{\lambda\nu}{}_\mu\right), \tag{A.19}$$

$$Z3_\mu{}^{\lambda\nu} = g^{\lambda\nu}T^\rho{}_{\rho\mu} - \delta_\mu{}^\nu g^{\lambda\sigma}T^\rho{}_{\rho\sigma}\,. \tag{A.20}$$

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Fermion dynamics in torsion theories

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Abstract. In this work we have studied the non-geodesical behaviour of particles with spin $1/2$ in Poincaré gauge theories of gravity, via the WKB method and the Mathisson-Papapetrou equation. We have analysed the relation between the two approaches and we have argued the different advantages associated with the WKB approximation. Within this approach, we have calculated the trajectories in a particular Poincaré gauge theory, discussing the viability of measuring such a motion.

Contents

1 Introduction

There is no doubt that General Relativity (GR) is one of the most successful theories in Physics, with a solid mathematical structure and experimental confirmation [1, 2]. As a matter of fact, we are still measuring for the first time some phenomena that was predicted by the theory a hundred years ago, like gravitational waves [3]. Nevertheless it presents some problems that need to be addressed. For example, it cannot be formulated as a renormalizable and unitary Quantum Field Theory. Also, the introduction of spin matter in the energy-momentum tensor of GR may be cumbersome, since we have to add new formalisms, like the spin connection. These problems can be solved by introducing a gauge approach to the gravitational theories. This task was addressed by Sciama and Kibble in [4] and [5], respectively, where they started to introduce the idea of a Poincaré Gauge (PG) formalism for gravitational theories. Following this description one finds that the connection must be compatible with the metric, but not necessarily symmetric. Therefore, it appears a non-vanishing torsion field, that is consequence of the asymmetric character of the connection. For an extensive review of the theories that arise through this reasoning see [6].

Since these kinds of theories were established, there has been a lot of discussion on how would particles behave in a spacetime with a torsion background. In the case of scalar particles, it is clear to see that they should follow geodesics, since the covariant derivative of a scalar field does not depend on the affine connection. In addition, by assuming the minimum coupling principle, we have that light keeps moving along null geodesics, as in the standard framework of GR. This is because it is impossible to perform the minimally coupling prescription for the Maxwell's field while maintaining the $U(1)$ gauge invariance [7]. Therefore the Maxwell equations remain in the same form. The most differential part occurs when we try to predict how particles with spin $1/2$ should move within this background. This question deserves a deeper analysis, mainly because these kinds of physical trajectories differ from the ones predicted by GR, and if we are able to measure such differences, we will be devising a method to determine the possible existence of a torsion field in our universe. Furthermore, if we know the corresponding equations of motion we can also calculate the strength of this field, although we already have some constraints thanks to torsion pendulums and cosmography observations [8, 9]. In [10] we find a comprehensive review of all the proposals that have been made to explain this behaviour. Nevertheless, even nowadays there is no consensus about which one explains it more properly. Here, we will outline the most important suggestions:

- In 1971, Ponomariev [11] proposed that the test particles move along autoparallels (curves in which the velocity is parallel transported along itself with the total connection). There was no reason given, but surprisingly this has been a recurrent proposal in the posterior literature [12, 13].
- Hehl [14], also in 1971, obtained the equation of motion via the energy-momentum conservation law, in the single-point approximation, i.e. only using first order terms in the expansion used to solve the energy-momentum equation. He also pointed out that torsion could be measured by using spin $1/2$ particles.
- In 1981, Audretsch [15] analysed the movement of a Dirac electron in a spacetime with torsion. He employed the WKB approximation, and obtained the same results that Rumpf had obtained two years earlier via an unconventional quantum mechanical approach [16]. It was with this article that the coupling between spin and torsion was understood.
- In 1991, Nomura, Shirafuji and Hayashi [17] computed the equations of motion by the application of the Mathisson-Papapetrou (MP) method to expand the energy-momentum conservation law. They obtained the equations at first order, which are the ones that Hehl had already calculated, but also made the second order approximation, finding the same spin precession as Audretsch.

In order to clarify these ideas we organise the article as follows. First, in section 2 we introduce the mathematical structure of PG theories, and establish the conventions. Then, in the two following sections we review the WKB approximation by Audretsch and the MP approach by Nomura et al., comparing them and presenting the reasons to consider the former for our principal calculations. In the fifth section we present the Raychaudhuri equation in the WKB approximation, and use one of its parameters as an indicator of the strength of the spin-torsion coupling. In section 6 we compute the acceleration and the respective trajectories of an electron in a particular solution, and compare it with the geodesical behaviour predicted by GR. The final section is devoted to conclusions and future applications.

2 Mathematical structure of Poincaré gauge theories

In this section, we give an introduction to the gravitational theories endowed with a non-symmetric connection that still fulfills the metricity condition. The most interesting fact about these theories is that they appear naturally as a gauge theory of the Poincaré Group [6, 18], making their formalism closer to that of the Standard Model of Particles, therefore postulating it as a suitable candidate to explore the quantization of gravity. We will use the same convention as [15] in order to simplify the discussion.

Since the connection is not necessarily symmetric, the torsion may be different from zero

$$T_{\mu\nu}{}^{\rho} = \Gamma_{[\mu\nu]}{}^{\rho}. \tag{2.1}$$

For an arbitrary connection, that meets the metricity condition, there exists a relation with the Levi-Civita connection

$$\mathring{\Gamma}_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho} + K_{\mu\nu}{}^{\rho}, \tag{2.2}$$

where

$$K_{\mu\nu}{}^{\rho} = T^{\rho}{}_{\nu\mu} + T^{\rho}{}_{\mu\nu} - T_{\mu\nu}{}^{\rho} \tag{2.3}$$

is the *contortion* tensor. Here, the upper index $^\circ$ denotes the quantities associated with the Levi-Civita connection.

Since the curvature tensors depend on the connection, there is a relation between the ones defined throughout the Levi-Civita connection and the general ones. For the Riemann tensor we have

$$\mathring{R}_{\mu\nu\rho}{}^{\sigma} = R_{\mu\nu\rho}{}^{\sigma} + \mathring{\nabla}_{\nu} K_{\mu\rho}{}^{\sigma} - \mathring{\nabla}_{\rho} K_{\mu\nu}{}^{\sigma} - K_{\alpha\nu}{}^{\sigma} K_{\mu\rho}{}^{\alpha} + K_{\alpha\rho}{}^{\sigma} K_{\mu\nu}{}^{\alpha}. \tag{2.4}$$

By contraction we can obtain the expression for the Ricci tensor

$$\mathring{R}_{\mu\rho} = R_{\mu\rho} + \mathring{\nabla}_{\sigma} K_{\mu\rho}{}^{\sigma} - \mathring{\nabla}_{\rho} K_{\mu\sigma}{}^{\sigma} - K_{\alpha\sigma}{}^{\sigma} K_{\mu\rho}{}^{\alpha} + K_{\alpha\rho}{}^{\sigma} K_{\mu\sigma}{}^{\alpha}, \tag{2.5}$$

and the scalar curvature

$$\mathring{R} = g^{\mu\rho} \mathring{R}_{\mu\rho} = R + \mathring{\nabla}^{\rho} K_{\sigma\rho}{}^{\sigma} - K_{\alpha\sigma}{}^{\sigma} K^{\rho}{}_{\rho}{}^{\alpha} + K_{\sigma\rho}{}^{\alpha} K_{\mu\alpha}{}^{\sigma}. \tag{2.6}$$

Here we have just exposed all of these concepts in the usual spacetime coordinates. Nevertheless, it is customary in PG theories to make calculations in the tangent space, that we assume in terms of the Minkowski metric η_{ab}. At each point of the spacetime we will have a different tangent space, that it is defined through a set of orthonormal tetrads (or *vierbein*) e^{α}_{a}, that follow the relations

$$e^{\mu}{}_{a} e_{\mu b} = \eta_{ab}, \quad e^{\mu}{}_{a} e^{\nu a} = g^{\mu\nu}, \quad e_{\mu}{}^{a} e_{\nu a} = g_{\mu\nu}, \quad e_{\mu}{}^{a} e^{\mu b} = \eta^{ab}, \tag{2.7}$$

where the latin letters refer to the tangent space and the greek ones to the spacetime coordinates. It is clear that if these properties hold, then

$$g_{\mu\nu} = e_{\mu}{}^{a} e_{\nu}{}^{b} \eta_{ab}. \tag{2.8}$$

All the calculations from now on will be considered in gravitational theories characterized by this geometrical background.

3 WKB method

In this section we summarize the results obtained by Audtresch in [15], where he calculated the precession of spin and the trajectories of Dirac particles in torsion theories. The starting point is the Dirac equation of a spinor field minimally coupled to torsion

$$i\hbar \left(\gamma^{\mu} \mathring{\nabla}_{\mu} \Psi + \frac{1}{4} K_{[\alpha\beta\delta]} \gamma^{\alpha} \gamma^{\beta} \gamma^{\delta} \Psi \right) - m\Psi = 0, \tag{3.1}$$

where the γ^{α} are the modified gamma matrices, related to the standard ones by the vierbein

$$\gamma^{\alpha} = e^{\alpha}_{a} \gamma^{a}, \tag{3.2}$$

and Ψ is a general spinor state.

It is worthwhile to note that the contribution of torsion to the Dirac equation is proportional to the antisymmetric part of the torsion tensor, therefore, a torsion field with vanishing antisymmetric component will not couple to the Dirac field. This is usually known as *inert torsion*. Since there is no analytical solution to Equation (3.1), we need to make approximations in order to solve it. As it is usual in Quantum Mechanics, we can use the WKB expansion to obtain simpler versions of this equation.

So, we can expand the general spinor in the following way

$$\Psi\left(x\right)=e^{i\frac{S(x)}{\hbar}}(-i\hbar)^{n}a_{n}\left(x\right), \tag{3.3}$$

where we have used the Einstein sum convention (with n going from zero to infinity). We have also assumed that $S\left(x\right)$ is real and $a_{n}\left(x\right)$ are spinors. As every approximation, it has a limited range of validity. In this case, we can use it as long as $\mathring{R}^{-1}\gg\lambda_{B}$, where λ_{B} is the de Broglie wavelength of the particle. This constraint expresses the fact that we cannot applied the mentioned approximation in presence of strong gravitational fields and that we cannot consider highly relativistic particles.

If we insert the expansion into the Dirac equation we obtain the following expressions for the zero and first order in $\hbar$:

$$\left(\gamma^{\mu}\mathring{\nabla}_{\mu}S+m\right)a_{0}\left(x\right)=0, \tag{3.4}$$

and

$$\left(\gamma^{\mu}\mathring{\nabla}_{\mu}S+m\right)a_{1}\left(x\right)=-\gamma^{\mu}\mathring{\nabla}_{\mu}a_{0}-\frac{1}{4}K_{[\alpha\beta\delta]}\gamma^{\alpha}\gamma^{\beta}\gamma^{\delta}a_{0}. \tag{3.5}$$

We then assume that the four-momentum of the particles is orthogonal to the surfaces of constant $S\left(x\right)$, and introduce it as

$$p_{\mu}=-\partial_{\mu}S. \tag{3.6}$$

Then, if we stick to the lowest order, as a consequence of Equation (3.4), the particles will follow geodesics, as one might expect. But, what happens if we consider the first order in $\hbar$? For the explicit calculations we refer the reader to [15], we will just state the definitions and give the main results.

To obtain the equation for spin precession we have considered the spin density tensor as

$$S^{\mu\nu}=\frac{\overline{\Psi}\sigma^{\mu\nu}\Psi}{\overline{\Psi}\Psi}, \tag{3.7}$$

where the $\sigma^{\mu\nu}$ are the modified spin matrices, given by

$$\sigma^{\alpha\beta}=\frac{i}{2}\left[\gamma^{\alpha},\,\gamma^{\beta}\right]. \tag{3.8}$$

Then, we can obtain the spin vector from this density

$$s^{\mu}=\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}u_{\nu}S_{\alpha\beta}, \tag{3.9}$$

where $\varepsilon^{\mu\nu\alpha\beta}$ is the modified Levi-Civita tensor, related to the usual one by the vierbein

$$\varepsilon^{\mu\nu\alpha\beta}=e^{\mu}{}_{a}e^{\nu}{}_{b}e^{\alpha}{}_{c}e^{\beta}{}_{d}\varepsilon^{abcd}, \tag{3.10}$$

and u^μ represents the velocity of the particle

$$u^\mu = \frac{dx^\mu}{dt} = x'^\mu. \tag{3.11}$$

Via the WKB expansion, we find that we can write the lowest order of the spin vector as

$$s_0^\mu = \bar{b}_0 \gamma^5 \gamma^\mu b_0, \tag{3.12}$$

where b_0 is the a_0 spinor but normalised.

With these definitions, we can compute the evolution of the spin vector

$$u^\alpha \mathring{\nabla}_\alpha s_0^\mu = 3K^{[\mu\beta\delta]} s_{0\,\delta} u_\beta. \tag{3.13}$$

On the other hand, the calculation of the acceleration of the particle comes from the splitting of the Dirac current via the Gordon decomposition and from the identification of the velocity with the normalised convection current. Then it can be shown that the non-geodesical behaviour is governed by the following expression

$$a_\mu = v^\varepsilon \mathring{\nabla}_\varepsilon v_\mu = \frac{\hbar}{4m_{esp}} \widetilde{R}_{\mu\nu\alpha\beta} \bar{b}_0 \sigma^{\alpha\beta} b_0 v^\nu, \tag{3.14}$$

where $\widetilde{R}_{\mu\nu\alpha\beta}$ refers to the intrinsic part of the Riemann tensor associated with the totally antisymmetric component of the torsion tensor:

$$\widetilde{\Gamma}_{\mu\nu}{}^{\lambda} = \mathring{\Gamma}_{\mu\nu}{}^{\lambda} + 3T_{[\mu\nu\alpha]} g^{\alpha\lambda}. \tag{3.15}$$

Unlike most of the literature exposed in the introduction, the expression (3.14) does not have an explicit contortion term coupled to the spin density tensor, hence all the torsion information is encrypted into the mentioned part of the Riemann tensor. Finally, it is worthwhile to note that the standard case of GR is naturally recovered for inert torsion, as expected.

4 Mathisson-Papapetrou method

In this section we will study another way to obtain the evolution of the spin vector and the acceleration of a test body. It was first explored by Mathisson [19], and later formalised by Papapetrou [20], while studying the motion of extended bodies. Normally, the equations of motion are calculated using the energy-momentum conservation law. Nevertheless, in an extended body we need to integrate this tensor over the spacelike surface orthogonal to its movement. We can simplify that by applying a multipole expansion and regarding only the lower-order terms. This approach was considered in the single-point approximation by Hehl in his well-known article [14]. In addition, Nomura, Shirafuji and Hayashi developed the pole-dipole approximation, also known as the Fock-Papapetrou method in GR, in [17].

In order to develop this method we consider an extended body, whose center of mass describes a timelike trajectory defined by $X^\mu(s)$, with velocity $u^\mu(s)$, where s is the proper time. For the vector describing a general point of the body we will use the notation y^μ. Then, the vector that goes from the center of mass to any point of the body will be denoted as $\delta x^\mu = y^\mu - X^\mu$, having $\delta x^0 = 0$.

With these remarks, we can define the following integrals over the spatial hypersurface orthogonal to the trajectory:

$$M^{\mu\nu} = u^0 \int T^{\mu\nu} dx^3, \tag{4.1}$$

and

$$M^{\rho\mu\nu} = -u^0 \int \delta x^\rho T^{\mu\nu} dx^3, \tag{4.2}$$

where $T^{\mu\nu}$ denotes the energy-momentum tensor. Indeed, these quantities are known as the monopole and dipole moments. The rest of the multipole moments can be defined just by adding another δx^μ to the (4.2) integral each time.

If we assume that our extended body is small, then the integral in the multipole moments will be very small. In this sense we introduce the single point approximation

$$M^{\mu\nu} \neq 0, \;\; M^{\rho\mu\nu} = 0, \;\; \dots \;, \tag{4.3}$$

and the pole-dipole approximation

$$M^{\mu\nu} \neq 0, \;\; M^{\rho\mu\nu} \neq 0, \;\; M^{\lambda\rho\mu\nu} = 0, \;\; \dots \;. \tag{4.4}$$

For the first approximation, one obtains the following equation after integrating the energy-momentum conservation law

$$\frac{dp^\mu}{ds} + \mathring{\Gamma}_{\nu\rho}{}^{\mu} M^{\nu\rho} - K^{\rho\mu\nu} M_{[\nu\rho]} - \frac{1}{2} R^{\rho\mu\sigma\nu} N_{\nu\rho\sigma} = 0, \tag{4.5}$$

where p^μ is the momentum and $N^{\nu\rho\sigma}$ is known as the spin current, that is defined as

$$N^{\rho\mu\nu} = -u^0 \int S^{\mu\nu\rho} dx^3, \tag{4.6}$$

with $S^{\mu\nu\rho}$ being the variation of the matter Lagrangian with respect to the spin connection. Through integration of $\partial_\nu \left(x^\rho T^{\mu\nu}\right)$ and $\partial_\rho \left(x^\sigma S^{\mu\nu\rho}\right)$ it can be calculated that

$$M^{\mu\nu} = p^\mu u^\nu. \tag{4.7}$$

On the other hand, we define the intrinsic spin as

$$S^{\mu\nu} = N^{\mu\nu\rho} u_\rho, \tag{4.8}$$

and consider that the momentum is proportional to the velocity, as in the WKB approximation, hence

$$M_{[\mu\nu]} = 0. \tag{4.9}$$

Thus, we can obtain the single-point approximation equations, that we have adapted to the convention used in the WKB method

$$u^\nu \nabla_\nu s^\mu = 0, \tag{4.10}$$

$$a_\mu = u^\rho \mathring{\nabla}_\rho u_\mu = \frac{1}{2m_{esp}} R_{\mu\lambda\rho\sigma} S^{\rho\sigma} u^\lambda, \tag{4.11}$$

$$S^{\mu\nu} u_\nu = 0. \tag{4.12}$$

The first equation provides the evolution of the spin vector, the second one shows the acceleration term and the last one constitutes a consistency constraint, known as the Pirani condition [21]. This condition is usually imposed in order to solve the propagating equations, and assures the conservation of mass along the trajectory. Nevertheless it is not a consequence of a conservation law, since although it is a sufficient condition for mass conservation, it is not a necessary one. Furthermore it cannot be derived from any other general equation involved by the theory, only by assuming the appropriate estimations such as the WKB method, in which this condition can be naturally derived from Equation (3.4).

Nevertheless, this approach provides some remarkable consequences, as already pointed out by Nomura et al. First of all, the equation of the evolution of the spin vector does not coincide with the resulting one from the WKB approximation. Secondly, and more important, in the single-point approximation the spin density tensor vanishes for Dirac particles, due to the antisymmetric character of the mentioned tensor. Therefore, under these conditions, the Dirac particles would just behave as spinless particles. Such a result is an implication of the introduction of the Pirani condition, and it is often used as an argument to analyse its implementation [6]. That is why we will explore the next order in the multiple expansion, known as the pole-dipole approximation. In this case we have the following equation, obtained by integration on the spacelike surface of the energy-momentum conservation law

$$\frac{dp^\mu}{ds} + \mathring{\Gamma}_{\rho\sigma}{}^{\mu} M^{\rho\sigma} - \partial_\nu \mathring{\Gamma}_{\rho\sigma}{}^{\mu} M^{\nu(\rho\sigma)} - K_{\rho\sigma}{}^{\mu} M^{[\rho\sigma]} + \partial_\nu K_{\rho\sigma}{}^{\mu} M^{\nu[\rho\sigma]} - \frac{1}{2} R^{\sigma\mu\nu\rho} N_{\rho\sigma\nu} = 0. \tag{4.13}$$

In a similar way as in the previous approximation, the values of $M^{\mu\nu}$ and $M^{\mu\nu\rho}$ can be obtained by integrating $\partial_\nu \left(x^\rho T^{\mu\nu}\right)$, $\partial_\rho \left(x^\sigma S^{\mu\nu\rho}\right)$ and $\partial_\nu \left(x^\rho x^\sigma T^{\mu\nu}\right)$ over the spacelike surface. Now the equations can be modified by the criteria previously explained, in order to reach the WKB assumptions. Nevertheless, in this case, the fact that the momentum is proportional to the velocity does not imply the vanishing for the evolution of the spin density tensor. After applying the mentioned conditions one obtains

$$u^\alpha \mathring{\nabla}_\alpha s^\mu = 3K^{[\mu\beta\delta]} s_\delta u_\beta, \tag{4.14}$$

$$m_{esp} u^\varepsilon \mathring{\nabla}_\varepsilon u^\mu + \frac{1}{2} K^{\mu\rho\sigma} u^\varepsilon \nabla_\varepsilon S_{\rho\sigma} - \left(\mathring{\nabla}^\mu K^{\nu\rho\sigma}\right) S_{\nu\rho} u_\sigma - \frac{1}{2} R_{\mu\lambda\rho\sigma} S^{\rho\sigma} u^\lambda = 0, \tag{4.15}$$

$$S^{\mu\nu} u_\nu = 0. \tag{4.16}$$

As we can see, the equation of the spin vector has the same form than the one obtained via the WKB approximation, therefore the first problem with the single-point approximation is solved. Also, in this case, the antisymmetry of the spin current tensor does not imply the vanishing of the spin density tensor, so that the resulting trajectory will be non-geodesic, as expected.

On the other hand, we can observe that all the differences with the single-point approximation vanish when we set the axial component of torsion to zero. This occurs because in the third term of Equation (4.15) the two non-antisymmetric indexes are contracted with an antisymmetric tensor, therefore

$$\left(\nabla^\mu K^{\nu\rho\sigma}\right) S_{\nu\rho} u_\sigma = \left(\nabla^\mu K^{[\nu\rho\sigma]}\right) S_{\nu\rho} u_\sigma. \tag{4.17}$$

Hence, if we have inert torsion this term vanishes, since the axial mode is proportional to the totally antisymmetric contortion. Moreover, Equation (4.14) recovers the form of the single-point approximation, which means that the Equation (4.9) is now valid, and so the second term of Equation (4.15) vanishes. As previously stressed, these conditions imply that the Dirac particles will follow geodesics.

Now that we have studied the two approaches, we can see which one is more appropriate in order to calculate the acceleration and trajectories of Dirac particles. First of all, it is clear that the single-point approximation of the MP method must be discarded, since it does not reflect the appropriate coupling between gravity and spin. One could think that the pole-dipole approximation is the one to follow, since it stipulates a non-geodesical behaviour and having inert torsion implies geodesical one, which is compatible with the minimally coupling prescription for Dirac fields. Nevertheless, even imposing the Pirani condition (which is controversial from the start) the set of Equations (4.14)-(4.16) is not complete, in the sense that the number of unknown quantities is higher than the number of equations. The reader might not agree with us in this point because, if we count the mentioned expressions we see that the set is completed. The question is that we have already simplified those equations, particularly the one that gives us the spin vector evolution. In the MP method, this equation is subject to an arbitrary constant, that is usually set to 1 for Dirac particles, in order to obtain the same results of the WKB approximation. So, in the end, the MP method by itself gives us an ambiguous result. On the other hand, the WKB method gives an explicit expression for the spin density tensor, that can be derived from Quantum Mechanics, and also the evolution of spin is directly given without assuming additional constraints beyond the WKB expansion. Therefore, the Pirani condition does not need to be imposed, it holds naturally by applying this method. That is why we have chosen this approximation to study the Dirac particles from now on. First of all, we will see this non-geodesical motion applied to a congruence of curves.

5 Raychaudhuri equation

One way of studying the consequences of the non-geodesical behaviour is to analyse the evolution of a congruence of the resulting curves throughout the Raychaudhuri equation. Also, this will provide more clues about the singular behaviour of these particles, and will help us to assure previous conclusions reached by the authors in [22]. It is known that Killing vectors define a static frame that will allow us to measure the dynamical quantities with respect to it [23]. Nevertheless, in general, an arbitrary spacetime will not have Killing vectors, therefore we do not have a preferred frame to measure the acceleration. In this case, the best one can do is to measure the relative acceleration of two close bodies, which is studied by the analysis of the behaviour of congruences of timelike curves.

If we observe the evolution of a congruence of curves, we will study the Raychaudhuri equation. To obtain this equation, we decompose the covariant derivative of the tangent vector of a congruence of curves, $B_{\mu\nu} = \mathring{\nabla}_\nu v_\mu$, into its antisymmetric component $\omega_{\mu\nu}$, known as *vorticity*, a traceless symmetric $\sigma_{\mu\nu}$, usually referred as *shear*, and its trace θ, also known as *expansion*, such as

$$B_{\mu\nu} = \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \tag{5.1}$$

where $h_{\mu\nu}$ is the projection of the metric into the spacial subspace orthogonal to the tangent vector. Then, it can be seen that [23]

$$\begin{aligned} v^\rho \mathring{\nabla}_\rho \theta = \frac{d\theta}{ds} = &-\frac{1}{3}\theta^2 - \sigma^{\mu\rho}\sigma_{\mu\rho} \\ &+ \omega^{\mu\rho}\omega_{\mu\rho} - \mathring{R}_{\rho\varphi} v^\rho v^\varphi + \mathring{\nabla}_\mu \left(v^\nu \mathring{\nabla}_\nu v^\mu \right), \end{aligned} \tag{5.2}$$

which is the equation under analysis.

Then, if we substitute the acceleration given in Equation (3.14) into the Raychaudhuri equation, we obtain

$$v^\rho \mathring{\nabla}_\rho \theta = \frac{d\theta}{ds} = -\frac{1}{3}\theta^2 - \Sigma^{\mu\rho}\Sigma_{\mu\rho} + \omega^{\mu\rho}\omega_{\mu\rho} - \mathring{R}_{\rho\varphi}v^\rho v^\varphi + \frac{\hbar}{4m_{esp}}\mathring{\nabla}_\mu \left(\widetilde{R}^\mu{}_{\nu\alpha\beta}\bar{b}_0\sigma^{\alpha\beta}b_0 v^\nu\right). \quad (5.3)$$

It is clear that the only difference with respect to the geodesical movement is the acceleration term. Let us analyse it in more detail:

$$\begin{aligned}\mathring{\nabla}_\mu \left(\widetilde{R}^\mu{}_{\nu\alpha\beta}\bar{b}_0\sigma^{\alpha\beta}b_0 v^\nu\right) = {} & \left(\mathring{\nabla}_\mu \widetilde{R}^\mu{}_{\nu\alpha\beta}\right)\bar{b}_0\sigma^{\alpha\beta}b_0 v^\nu + \widetilde{R}^\mu{}_{\nu\alpha\beta}\left[\mathring{\nabla}_\mu\left(\bar{b}_0\sigma^{\alpha\beta}b_0\right)\right]v^\nu \\ & + \widetilde{R}^\mu{}_{\nu\alpha\beta}\bar{b}_0\sigma^{\alpha\beta}b_0\mathring{\nabla}_\mu v^\nu, \end{aligned} \quad (5.4)$$

where we have used the Leibniz rule for the covariant derivative. Let us study the different contributions separately.

For the third term we have that:

$$\widetilde{R}^\mu{}_{\nu\alpha\beta}\bar{b}_0\sigma^{\alpha\beta}b_0\mathring{\nabla}_\mu v^\nu = \widetilde{R}^{\mu\nu}{}_{\alpha\beta}\bar{b}_0\sigma^{\alpha\beta}b_0\mathring{\nabla}_\mu v_\nu = \widetilde{R}^{\mu\nu}{}_{\alpha\beta}\bar{b}_0\sigma^{\alpha\beta}b_0\left(\frac{1}{3}\theta h_{\mu\nu} + \Sigma_{\mu\nu} + \omega_{\mu\nu}\right). \quad (5.5)$$

Since the two contracted indexes μ and ν of the Riemann tensor are antisymmetric and the tensors h and Σ are symmetric we have that:

$$\widetilde{R}^\mu{}_{\nu\alpha\beta}\bar{b}_0\sigma^{\alpha\beta}b_0\mathring{\nabla}_\mu v^\nu = \widetilde{R}^{\mu\nu}{}_{\alpha\beta}\bar{b}_0\sigma^{\alpha\beta}b_0\omega_{\mu\nu}. \quad (5.6)$$

One interesting feature is that if we consider a congruence orthonormal to an spacelike hypersurface, the shear is null, therefore this term of the Raychaudhuri equation is identically zero.

For the first and the second one we cannot find any simplification. In any case, the appearance of focal points will occur when

$$\mathring{R}_{\rho\varphi}v^\rho v^\varphi \geq A_\nu v^\nu, \quad (5.7)$$

where

$$A_\nu = \frac{\hbar}{4m_{esp}}\mathring{\nabla}_\mu\left(\widetilde{R}^\mu{}_{\nu\alpha\beta}\bar{b}_0\sigma^{\alpha\beta}b_0\right). \quad (5.8)$$

As explained at the beginning of this section, this term gives us the contribution of torsion to the relative acceleration between two spin 1/2 particles, making it a good indicator to see the difference with respect to a geodesical behaviour. Therefore, we can make a more rigorous approach to the singular behaviour of these particles. In [22] the authors claim that the appearance of n-dimensional black/white hole regions was a good criteria for the occurrence of singularities, even for the Dirac particles, given that the difference with the geodesical movement were not so strong near the event horizon. Now we can say that this will be a good criteria as long as $A_\nu \ll 1$, which is what we expect in plausible spacetimes with Dirac particles.

6 Calculations within the Reissner-Nordström geometry induced by torsion

In this section we will calculate the acceleration and trajectories of electrons in a Reissner-Nordström solution obtained by two of the authors in the framework of PG field theory of gravity, with the following vacuum action [24, 25]:

$$S=\frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\left[-\mathring{R}+\frac{d_1}{2}R_{\lambda\rho\mu\nu}R^{\mu\nu\lambda\rho}-\frac{d_1}{4}R_{\lambda\rho\mu\nu}R^{\lambda\rho\mu\nu}\right.$$
$$\left.-\frac{d_1}{2}R_{\lambda\rho\mu\nu}R^{\lambda\mu\rho\nu}+d_1R_{\mu\nu}\left(R^{\mu\nu}-R^{\nu\mu}\right)\right]. \tag{6.1}$$

The exact metric of the solution is

$$ds^2=f\left(r\right)dt^2-\frac{1}{f\left(r\right)}dr^2-r^2\left(d\theta^2+sin^2\theta d\varphi^2\right), \tag{6.2}$$

where

$$f\left(r\right)=1-\frac{2m}{r}+\frac{d_1\kappa^2}{r^2}. \tag{6.3}$$

From now on we will consider $d_1=1$, which simplifies the computations.
In order to know the total and modified connection we need to have the values of the non-vanishing torsion components, which are:

$$\begin{cases} T_{tr}{}^{t}=\frac{a(r)}{2}=\frac{\dot{f}(r)}{4f(r)},\\ T_{tr}{}^{r}=\frac{b(r)}{2}=\frac{\dot{f}(r)}{4},\\ T_{t\theta_i}{}^{\theta_j}=\delta^{\theta_j}_{\theta_i}\frac{c(r)}{2}=\delta^{\theta_j}_{\theta_i}\frac{f(r)}{4r},\\ T_{r\theta_i}{}^{\theta_j}=\delta^{\theta_j}_{\theta_i}\frac{g(r)}{2}=-\delta^{\theta_j}_{\theta_i}\frac{1}{4r},\\ T_{t\theta_i}{}^{\theta_j}=e^{a\theta_j}e^{b}{}_{\theta_i}\varepsilon_{ab}\frac{d(r)}{2}=e^{a\theta_j}e^{b}{}_{\theta_i}\varepsilon_{ab}\frac{\kappa}{2r},\\ T_{r\theta_i}{}^{\theta_j}=e^{a\theta_j}e^{b}{}_{\theta_i}\varepsilon_{ab}\frac{h(r)}{2}=-e^{a\theta_j}e^{b}{}_{\theta_i}\varepsilon_{ab}\frac{\kappa}{2rf(r)}, \end{cases} \tag{6.4}$$

where we have made the identification $\{\theta_1,\,\theta_2\}=\{\theta,\,\varphi\}$, ε_{ab} is the Levi-Civita symbol, and the dot $\dot{}$ means the derivative with respect to the radial coordinate. Also, since the definition of the torsion tensor in the mentioned article differs from our conventions, all the components are divided by 2 with respect to the ones in there.
Now, with the components of the metric and the torsion tensors, we can calculate the modified connection and therefore the Riemann tensor of Equation (3.14), in order to obtain the acceleration. Moreover, we know that the b_0 and $\bar{b}_0$ are the lowest order in $\hbar$ of the general spinor state Ψ. Then we can use that the most general form of a positive energy solution of the Dirac equation for b_0 and $\bar{b}_0$ is [26]

$$b_0=\begin{pmatrix} cos\left(\frac{\alpha}{2}\right)\\ e^{i\beta}sin\left(\frac{\alpha}{2}\right)\\ 0\\ 0\end{pmatrix}\;;\quad \bar{b}_0=\left(\,cos\left(\frac{\alpha}{2}\right),\,e^{-i\beta}sin\left(\frac{\alpha}{2}\right),\,0,\,0\,\right)\;; \tag{6.5}$$

where the angles give the direction of the spin of the particle

$$\overrightarrow{n}=\left(\,sin\left(\alpha\right)cos\left(\beta\right),\,sin\left(\alpha\right)sin\left(\beta\right),\,cos\left(\alpha\right)\,\right). \tag{6.6}$$

Before calculating the acceleration, let us use this form of the spinor to calculate the corresponding spin vector. Using Equation (3.12) we have

$$s^{\mu} = \begin{pmatrix} 0 \\ \\ -sin\left(\alpha\right)cos\left(\beta\right)\sqrt{f\left(r\right)} \\ \\ -\frac{sin(\alpha)sin(\beta)}{r} \\ \\ -\frac{cos(\alpha)csc(\theta)}{r} \end{pmatrix} \;;\quad s_{\mu} = \left(0,\, \frac{sin(\alpha)cos(\beta)}{\sqrt{f(r)}},\, rsin\left(\alpha\right)sin\left(\beta\right),\, rsin\left(\theta\right)cos\left(\alpha\right)\right). \tag{6.7}$$

With all this we can calculate the acceleration for the special case of the solution. To ease the reading of this paper, the acceleration components can be found in the Appendix A. It is worthwhile to note that the only components of the torsion tensor that contribute to the acceleration are those related to the functions $d(r)$ and $h(r)$. This is important, because if we set the κ constant to zero, any torsion component does not contribute to the acceleration. Therefore, in this case the torsion tensor is inert, since the axial vector is zero, as expected. On the other hand, The above expressions are complex and it is difficult to understand their behaviour intuitively. In this sense, it is interesting to study two relevant cases that simplify the equations:

- Low values of κ:
  If we consider a realistic physical implementation of this solution, in order to avoid naked singularities, we expect low values of the parameter $\xi = \frac{\kappa}{m^2}$. Indeed, ξ is the dimensionless parameter which controls the contribution of the torsion tensor. Therefore, if we consider the acceleration, we can see that it is a good approximation to consider only up to first order in an expansion of the acceleration in terms of ξ. These results can be found in the Appendix B.

- Asymptotic behaviour:
  It is interesting to study what happens at the asymptotic limit $r \to \infty$, in order to observe what is the leading term and compare its strengh with other effects on the particle. We obtain the following:

$$\lim_{r\to\infty} a^t \simeq \frac{m^2\xi\hbar}{2m_{esp}r}\left(\sin(\alpha)\sin(\beta)\theta'(s) + \sin(\theta)\cos(\alpha)\varphi'(s)\right), \tag{6.8}$$

$$\lim_{r\to\infty} a^r \simeq \frac{m^2\xi\hbar}{2m_{esp}r}\left(\sin(\alpha)\sin(\beta)\theta'(s) + \sin(\theta)\cos(\alpha)\varphi'(s)\right), \tag{6.9}$$

$$\begin{aligned}\lim_{r\to\infty} a^\theta \simeq\ & \frac{m\hbar}{2m_{esp}r^3}\left[-m\xi r'(s)\left(\sin(\alpha)\sin(\beta) + m^2\xi\cos(\alpha)\right)\right. \\ & + m\xi t'(s)\left(\sin(\alpha)\sin(\beta) + m^2\xi\cos(\alpha)\right) \\ & \left.- \ 2\sin(\alpha)\cos(\beta)\sin(\theta)\varphi'(s)\right],\end{aligned} \tag{6.10}$$

$$\begin{aligned}\lim_{r\to\infty} a^\varphi \simeq\ & \frac{m\hbar\csc(\theta)}{2m_{esp}r^3}\left[m\xi r'(s)\left(m^2\xi\sin(\alpha)\sin(\beta) - \cos(\alpha)\right)\right. \\ & \left.+ \ m\xi t'(s)\left(\cos(\alpha) - m^2\xi\sin(\alpha)\sin(\beta)\right) + 2\sin(\alpha)\cos(\beta)\theta'(s)\right].\end{aligned} \tag{6.11}$$

  Where we have used the viability condition (6.18), because as we will see, that is a neccesary condition for the semiclassical aproximation.

We can observe that the time and radial components follow a r^{-1} pattern, while the angular components follow a r^{-3} behaviour. Hence, in the first components the torsion effect goes asymptotically to zero at a lower rate than the strength provided by the conventional gravitational field. Meanwhile in the angular ones, it goes at a higher rate.

It is interesting to analyse the two components of the acceleration that are non-zero in GR, a^θ and a^φ, to reach a deeper understanding. They read

$$a^\theta|_{\kappa=0} = \frac{m\hbar\sin(\theta)}{2m_{esp}r^3\sqrt{1-\frac{2m}{r}}}\left(s^\varphi r'(s) + 2s^r\varphi'(s)\right), \tag{6.12}$$

and

$$a^\varphi|_{\kappa=0} = \frac{m\hbar\csc(\theta)}{2m_{esp}r^3\sqrt{1-\frac{2m}{r}}}\left(s^\theta r'(s) + 2s^r\theta'(s)\right), \tag{6.13}$$

where we have used the expression of the spin vector (6.7) to simplify the equations. As we can see, the form of the two equations is very similar, and can be made equal by establishing the identifications $\sin(\theta) \leftrightarrow \csc(\theta)$, and $\varphi \leftrightarrow \theta$. For two of them we observe that the spin-gravity coupling acts as a *cross-product force*, in the sense that the acceleration is perpendicular to the direction of the velocity and the spin vector.
Now, to measure the torsion contribution in the acceleration we shall compare the acceleration for $\kappa = 0$ and for arbitrary values of κ. In this sense, we define a new dimensionless parameter as the fraction between the acceleration for a finite value of κ and the one given by $\kappa = 0$:

$$B^\mu(\kappa) = \frac{a^\mu}{a^\mu|_{\kappa=0}}. \tag{6.14}$$

As we have stated before, the viability condition (6.18) implies that

$$\cos(\alpha)\theta'(s) - \sin(\alpha)\sin(\beta)\sin(\theta)\varphi'(s) = 0, \tag{6.15}$$

so $a^t|_{\kappa=0}$ and $= a^r|_{\kappa=0}$ vanish identically. This means that we cannot study these two components of the B^μ parameter. Nevertheless, we can still measure it in the angular coordinates. Let us explore two examples, that are shown in Figure 1. There we represent different components of B^μ in function of κ for a fixed position and two different spin and velocity directions.

As can be seen, this gives rise to some interesting features, that we would like to address. First of all, it is worthwhile to stress that there is nothing in the form of the metric or in the underlying theory that stops us from taking negative values of κ, in contrast with the usual electromagnetic version of the solution. We can observe that as we take higher absolute values for κ we find that the acceleration caused by the spacetime torsion is directed in the opposite direction of the one produced by the gravitational coupling, reaching significant differences for large κ. This is expected since we have chosen a strong coupling between spin and torsion.

Now, we go one step forward and calculate the trajectory of the particle, using Equation (3.14) and having in mind the spinor evolution equation (3.13), which can be rewritten as

$$v^\mu\tilde{\nabla}_\mu b_0 = 0. \tag{6.16}$$

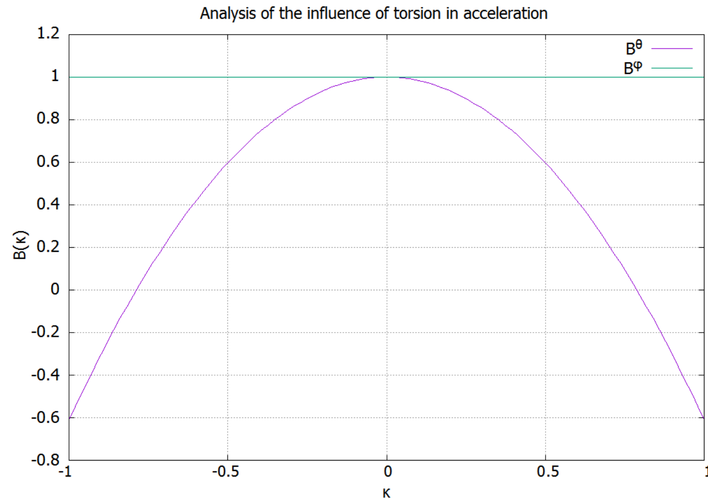


Figure 1: We have considered a black hole of 24 solar masses and a particle located near the external event horizon in the $\theta = \pi/2$ plane, at a radial distance of $2m + \varepsilon$, where $\varepsilon = m/10$. The position in φ is irrelevant because the acceleration does not depend on this coordinate. For the B_θ case, we assume that the particle has radial velocity equal to 0.8, and that the direction of the spin is in the φ direction. The rest of the velocity components are zero except for $v^t = (8.8\kappa + 0.3)^{-1/2}$. It is clear from (6.12) and (6.13) that we can only calculate the relative acceleration in the θ direction. For the B_φ case the velocity is in the θ direction, and has the same modulus as before. Again, the rest of the components are zero except for $v^t = 1.3(8.8\kappa + 0.3)^{-1/2}$. The spin has only a radial component, therefore the acceleration would be in the φ direction.

For the exact Reissner-Nordström geometry supported by torsion, we find several interesting features. First, in order to maintain the semiclassical approximation and the positive energy associated with the spinor, two conditions must be fulfilled:

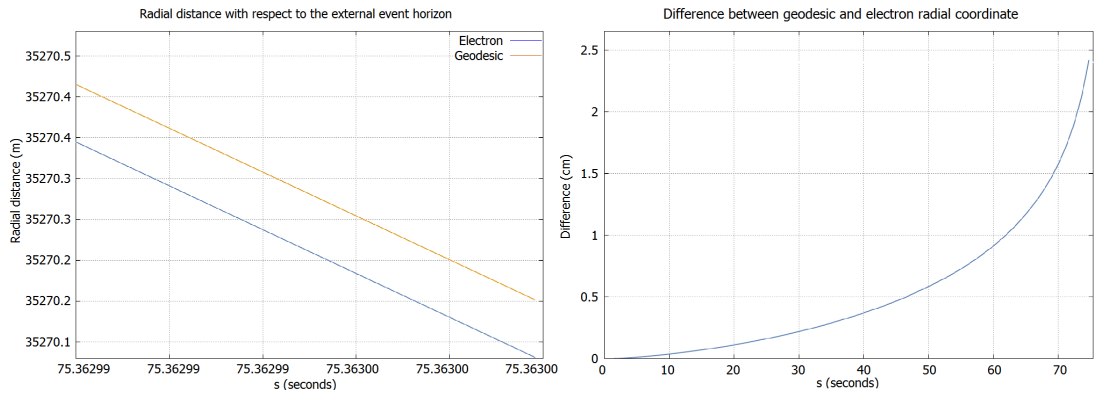
$$\dot{f}\,(r) \ll L f\,(r)\,, \tag{6.17}$$

where $L = 3.3 \cdot 10^{-8}\ m^{-1}$, so that in the units we are using the derivative of $f\,(r)$ is at least two orders of magnitude below the value of $f\,(r)$.
The other one is

$$\left(\bar{b}_0 \sigma^{r\beta} b_0\right) v_\beta = 0. \tag{6.18}$$

The first one is a consequence of the method that we are applying: if both curvature and torsion are strong then the interaction is also strong, and the WKB approximation fails. This one is a purely metric condition, since it comes from the Levi-Civita part of the Riemann tensor, so it will be the same for all the spherically symmetric solutions. The second one is the radial component of the Pirani condition, that was explained in section 4. We have solved the above equations numerically for different scenarios, obtaining the results that are shown in Figure 2.

We have chosen the same trajectories analysed in the discussion of the acceleration. That discussion shows that any difference from the geodesical behaviour in the radial coordinate would be an exclusive consequence of the torsion-spin coupling, with no presence of GR



(a) Trajectory at 35 km of the event horizon.

(b) Relative position between the two particles.

Figure 2: For this numerical computation we have used a black hole with 24 solar masses and $\kappa = 10$, with the electron located outside the external event horizon in the $\theta = \pi/2$ plane. We have assumed an electron with radial velocity of 0.9 and initial spin aligned in the φ direction. All the rest of the initial conditions are the same than the ones presented in Figure 1.

terms, since the acceleration term in this coordinate depends on κ. Indeed it is possible to have situations under which the geodesics and the trajectories of spin $1/2$ particles are distanced due to this effect, even by starting at the same point. If we are able to measure such a difference experimentally, we could have an idea of the specific values of the torsion field present in this particular geometry.

7 Conclusions

Motivated by the lack of consensus on how Dirac particles propagate in torsion theories, we review the two main formulations for this purpose and compare them. We reach the conclusion that the WKB method is more consistent for the mentioned task, since it does not need any additional condition, like the Pirani one, in order to solve the resulting equations. In addition, it seems a better approach to treat the intrinsic spin dynamic from the Dirac equation than from a classical equation like the MP one.

After that, we have written the Raychaudhuri equation for the spin particles and defined a new parameter to measure the non-geodesical behaviour. In contrast with just the acceleration given by Equation (3.14), this parameter constitutes a well-defined physical criterion in order to distinguish observationally the existence of a non-zero torsion, since it quantifies the difference of the acceleration with respect to the geodesical one measured by nearby observers.

Finally, we have applied the WKB method to a specific geometrical solution of PG gravity and analysed the results. Within the asymptotic behaviour at large distances, where the WKB approximation holds, the torsion effects are typically much smaller than the contribution given by the Levi-Civita connection. Therefore, it is interesting to find scenarios where this component is not present. In this particular case, we have found a *cross-product behaviour* of the gravitational interaction, i.e. an acceleration induced that is perpendicular to the spin direction of the particle and to its velocity when torsion is absent. Therefore differences from geodesical behaviours in other directions can only be consequence of the torsion contribution.

With this fact in mind, we have found a situation where we can appreciate qualitative differences between the geodesical movement and the trajectories of spin 1/2 particles, as shown in Figure 2. However, this different dynamics needs an important magnitude of the torsion coupling in order to be observed. To have a realistic situation that can be explained through the studied metric, we would need a neutron-star like system, where we have a large concentration of spin aligned particles due to a magnetic field inside the star. In such a case, we could try to observe the difference of angles between photons and neutrinos coming from the same source behind the neutron star. This and other studies will be analysed in future works following the computations developed in this article.

A Acceleration components

Here we present the components of the acceleration calculated following the prescription discussed in section 6.

$$\begin{aligned} a^t = & -\frac{\kappa\hbar}{2m_{esp}r^2\left(\frac{\kappa-2mr+r^2}{r^2}\right)^{3/2}}\left\{\sqrt{\frac{\kappa-2mr+r^2}{r^2}}\sin(\alpha)\cos(\beta)r'(s)\right. \\ & - \theta'(s)\left[\sin(\alpha)\sin(\beta)\left(r-m\right)+\kappa r\cos(\alpha)\right] \\ & \left. + \sin(\theta)\varphi'(s)\left[\cos(\alpha)\left(m-r\right)+\kappa r\sin(\alpha)\sin(\beta)\right]\right\} \end{aligned} \tag{A.1}$$

$$\begin{aligned} a^r = & -\frac{\hbar}{2m_{esp}r^4\left(\kappa-2mr+r^2\right)}\left\{r\sqrt{\frac{\kappa-2mr+r^2}{r^2}}\left[\theta'(s)\left(\cos(\alpha)\left(2m^2r^2-mr^3-3m\kappa r+\kappa^2\right.\right.\right.\right. \\ - & \left.\left.\kappa^2r^4+\kappa r^2\right)+\kappa r^3\sin(\alpha)\sin(\beta)(m-r)\right)+\sin(\theta)\varphi'(s)\left(\sin(\alpha)\sin(\beta)\left(-2m^2r^2+mr^3\right.\right. \\ + & \left.\left.\left.3m\kappa r-\kappa^2+\kappa^2r^4-\kappa r^2\right)+\kappa r^3\cos(\alpha)(m-r)\right)\right] \\ & \left.+\,\kappa\sin(\alpha)\cos(\beta)\left(\kappa-2mr+r^2\right)^2t'(s)\right\}, \end{aligned} \tag{A.2}$$

$$\begin{aligned} a^\theta = & -\frac{\hbar\sin(\theta)}{4m_{esp}r^7\left(\frac{\kappa-2mr+r^2}{r^2}\right)^{3/2}}\left\{-2\csc(\theta)r'(s)\left[\cos(\alpha)\left(2m^2r^2-mr^3-3m\kappa r+\kappa^2-\kappa^2r^4+\kappa r^2\right)\right.\right. \\ + & \left.\kappa r^3\sin(\alpha)\sin(\beta)(m-r)\right]-2r\left(-\kappa+2mr-r^2\right)\left[\sin(\alpha)\cos(\beta)(2mr-\kappa)\sqrt{\frac{\kappa-2mr+r^2}{r^2}}\varphi'(s)\right. \\ - & \left.\left.\kappa\csc(\theta)t'(s)\left(\sin(\alpha)\sin(\beta)(r-m)+\kappa r\cos(\alpha)\right)\right]\right\}, \end{aligned} \tag{A.3}$$

$$
\begin{aligned}
a^{\varphi} = & -\frac{\hbar \csc(\theta)}{4m_{esp}r^7\left(\frac{\kappa-2mr+r^2}{r^2}\right)^{3/2}}\Big\{2r'(s)\left[\sin(\alpha)\sin(\beta)\left(2m^2r^2-mr^3-3m\kappa r+\kappa^2-\kappa^2r^4+\kappa r^2\right)\right.\\
& -\left.\kappa r^3\cos(\alpha)(m-r)\right]+2r\left(\kappa-2mr+r^2\right)\Bigg[\sin(\alpha)\cos(\beta)(\kappa-2mr)\sqrt{\frac{\kappa-2mr+r^2}{r^2}}\theta'(s)\\
& +\kappa t'(s)\left(\cos(\alpha)(m-r)+\kappa r\sin(\alpha)\sin(\beta)\right)\Bigg]\Big\} \qquad \text{(A.4)}
\end{aligned}
$$

B Acceleration at low κ

Here we display the acceleration components at first order of the dimensionless parameter $\xi = \kappa/m^2$, as indicated in section 6.

$$
\begin{aligned}
a^t = & -\frac{\xi m^2\hbar}{2\left(m_{esp}r(r-2m)\sqrt{1-\frac{2m}{r}}\right)}\left[\sin(\alpha)\cos(\beta)\sqrt{1-\frac{2m}{r}}r'(s)\right.\\
& \left.+(m-r)\left(\sin(\alpha)\sin(\beta)\theta'(s)+\cos(\alpha)\sin(\theta)\varphi'(s)\right)\right]+O\left(\xi^2\right), \qquad \text{(B.1)}
\end{aligned}
$$

$$
\begin{aligned}
a^r = & \frac{m\hbar\sqrt{1-\frac{2m}{r}}}{2m_{esp}r^2}\left(\cos(\alpha)\theta'(s)-\sin(\alpha)\sin(\beta)\sin(\theta)\varphi'(s)\right)\\
& -\frac{\xi m^2\hbar}{4\left(m_{esp}r^4\sqrt{1-\frac{2m}{r}}\right)}\left[\theta'(s)\left(2r^2\sin(\alpha)\sin(\beta)(m-r)+\cos(\alpha)(2r-5m)\right)\right.\\
& +\sin(\theta)\varphi'(s)\left(2r^2\cos(\alpha)(m-r)+\sin(\alpha)\sin(\beta)(5m-2r)\right)\\
& \left.+2r\sin(\alpha)\cos(\beta)\sqrt{1-\frac{2m}{r}}(r-2m)t'(s)\right]+O\left(\xi^2\right), \qquad \text{(B.2)}
\end{aligned}
$$

$$
\begin{aligned}
a^{\theta} = & -\frac{m\hbar}{2m_{esp}r^4}\left(\frac{\cos(\alpha)r'(s)}{\sqrt{1-\frac{2m}{r}}}+2r\sin(\alpha)\cos(\beta)\sin(\theta)\varphi'(s)\right)\\
& +\frac{m^2\hbar\xi}{4m_{esp}r^5(r-2m)\sqrt{1-\frac{2m}{r}}}\left[r'(s)\left(2r^2\sin(\alpha)\sin(\beta)(m-r)+\cos(\alpha)(2r-3m)\right)\right.\\
& \left.+r\sin(\alpha)(r-2m)\left(2\cos(\beta)\sin(\theta)\sqrt{1-\frac{2m}{r}}\varphi'(s)-2\sin(\beta)(m-r)t'(s)\right)\right]\\
& +O\left(\xi^2\right), \qquad \text{(B.3)}
\end{aligned}
$$

$$
\begin{aligned}
a^{\varphi} =\ & \frac{m\hbar \sin(\alpha)\csc(\theta)}{2m_{esp}r^4}\left(\frac{\sin(\beta)r'(s)}{\sqrt{1-\frac{2m}{r}}}+2r\cos(\beta)\theta'(s)\right) \\
+\ & \frac{m^2\hbar\xi\csc(\theta)}{4m_{esp}r^5\sqrt{1-\frac{2m}{r}}(r-2m)}\Bigg[r'(s)\left(2r^2\cos(\alpha)(m-r)+\sin(\alpha)\sin(\beta)(3m-2r)\right) \\
+\ & r(r-2m)\left(-2\sin(\alpha)\cos(\beta)\sqrt{1-\frac{2m}{r}}\theta'(s)-2\cos(\alpha)(m-r)t'(s)\right)\Bigg] \\
+\ & O\left(\xi^2\right).
\end{aligned}
\tag{B.4}
$$

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Chapter 3

Singularities and stability conditions

3.1 Stability and singular geometries

Besides the study of fundamental symmetries and exact vacuum solutions provided by a particular theory of gravitation, it is crucial to analyse other consistency properties of the new framework, like the occurrence of space-time singularities and physical instabilities. In this sense, it is expected to achieve an extension of the standard results and theorems referred to these issues in GR, when the presence of a space-time torsion is assumed.

Primarily, from a mathematical point of view, it is a well-known fact that under certain regular conditions every pseudo-Riemannian manifold inevitably develops space-time singularities, in a way that its respective geodesics cannot be extended to arbitrary values of the affine parameter [7, 69]. This geodesic incompleteness can be trivially and independently classified as timelike, null or spacelike [1], in function of the nature of the distinct geodesic curves defined within the differentiable manifold. Then, the standard singularity theorems focus on three kinds of critical conditions, in order to characterize the generic situations under which the system irremediably develops singular points:

- Global consistency on the causal structure.
- Energy constraint.
- Gravity strong enough to trap a region.

[1] On account of the physical irrelevance of spacelike geodesics, only timelike and null geodesic completeness are really interesting from a physical point of view.

First, the requirement of a coherent causal structure demands the absence of closed timelike curves, in order to ensure a well-defined chronological order where every event is appropriately preceded by a cause. In addition, it ensures the existence of a family of hypersurfaces orthogonal to a set of congruences, for the case of timelike geodesics as well as for null geodesics, which means the vanishing of the vorticity tensor associated with these types of congruences [70].

The second kind of condition underlays from a restriction on curvature that produces a significant convergence of neighbouring geodesics. It may be expressed in terms of the following inequality involving the Ricci tensor and an arbitrary non-spacelike vector v^μ:

$$R_{\mu\nu}v^\mu v^\nu \geq 0\,, \tag{3.1}$$

which in turn, for timelike geodesics, can be trivially related to the energy-momentum tensor via the Einstein field equations of GR:

$$T_{\mu\nu}v^\mu v^\nu \geq T^\mu{}_\mu v^\nu v_\nu\,, \tag{3.2}$$

or, weakly, for null geodesics:

$$T_{\mu\nu}v^\mu v^\nu \geq 0\,. \tag{3.3}$$

Indeed, for ordinary matter satisfying the relations above, it is straightforward to appreciate the attractive nature of gravity in terms of the Raychaudhuri equation [71]. Namely, for a congruence of timelike geodesics parametrized by proper time τ, the propagation equation reads:

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{3} - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}v^\mu v^\nu\,, \tag{3.4}$$

and for a congruence of null geodesics with affine parameter p:

$$\frac{d\theta}{dp} = -\frac{\theta^2}{2} - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}v^\mu v^\nu\,, \tag{3.5}$$

where θ, $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ denote the expansion scalar, shear and vorticity, respectively. Therefore, since the shear and vorticity tensors are purely spatial, the evolution of a set of congruences with vanishing vorticity is characterized by an irremediable decrease of the expansion scalar and a natural convergence of geodesics.

Finally, by the third kind of condition, there must exist a closed spacelike surface such that the respective ingoing and outgoing null geodesics orthogonal to it converge and thus they get trapped inside a succession of surfaces of smaller area. This surface is then called closed trapped surface and is a reflection of the mentioned attractive behaviour of gravity, provided by the energy conditions above, when the density of matter reaches a critical value.

Thereby, different combinations and possibilities involving these principal conditions give rise to a breakdown of the geodesic completeness present in the space-time manifold. For example, the original version of the Penrose theorem includes the following relation among the cited general conditions [72]:

Theorem 1. *Every space-time characterized by a pseudo-Riemannian manifold $\mathcal{M}$ endowed with a metric tensor g cannot be null geodesically complete if:*

i) $R_{\mu\nu}v^{\mu}v^{\nu} \geq 0$ *for all null vectors* v^{μ}.

ii) There is a non-compact Cauchy surface $\mathcal{H}$ in $\mathcal{M}$ (i.e. $\mathcal{H}$ is intersected by every inextensible and non-spacelike curve exactly once).

iii) There is a closed trapped surface $\mathcal{J}$ in $\mathcal{M}$.

Likewise, the posterior Hawking theorem considers the following composition [73]:

Theorem 2. *Space-time is not timelike geodesically complete if:*

i) $R_{\mu\nu}v^{\mu}v^{\nu} \geq 0$ *for every non-spacelike vector* v^{μ}.

ii) There exists a compact spacelike three-surface $\mathcal{S}$ without edge.

iii) The unit normals to $\mathcal{S}$ are everywhere converging (or everywhere diverging) on $\mathcal{S}$.

It is important to note that these general conditions lead to a geodesic incompleteness even without the requirement of solving the gravitational field equations and, furthermore, without the presence of a particular space-time symmetry. It means that any kind of deviation from a special symmetry existing in a system cannot prevent the appearance of singularities. On the other hand, their geometrical implications can be trivially applied in two essential situations: the final configuration after the gravitational collapse of stars and other massive bodies, as well as the initial state of the present universe.

Additional types of singular events uncovered by the concept of geodesic incompleteness, such as particular divergences of scalar invariants defined from the curvature tensor, may appear within a space-time manifold. The existence of this wide variety of space-time singularities evinces, however, that GR is not a complete gravitational theory, since it loses its consistency and predictability in such points. In this regard, it is expected that these issues may be regularized in the realm of a quantum field theory of gravity.

In the same way, the inclusion of higher order corrections into the gravitational action may potentially introduce fundamental instabilities, namely ghosts and/or tachyons that violate unitary and causality, respectively. From a mathematical point of view, the presence of ghost fields in the particle spectrum of the theory is related to Lagrangian densities with negative kinetic terms, which leads to a classical Hamiltonian unbounded from below and to a set of states with negative norm in the quantum regime, whereas the existence of negative mass terms indicates a tachyonic behaviour characterized by an inadmissible propagation faster than light.

In order to prevent these problems, at an initial approach, the Lagrangian coefficients must satisfy the appropriate restrictions in the linear field approximation. For this purpose, a linearization procedure is applied over the gravitational action around a particular background space-time and the resulting dynamic terms are analysed.

This procedure has been carried out by different authors in the framework of quadratic PG theory around the Minkowski vacuum by the spin-projection operator formalism, which allows the corresponding geometrical degrees of freedom to be split into different spin modes in a systematic way and enables the evaluation of the signs of the poles of the propagators and of their associated residues. The resulting states are then denoted as J^P, where J and P refer to spin and parity, respectively.

In general, by considering the possible massive character of torsion, the analysis also distinguishes between the completely massive case and the case with zero-mass modes. The latter involves additional gauge symmetries in the linearized regime that reduce the available physical degrees of freedom of the particle spectrum, which means that these cases must be examined separately [74].

The results for the viability of the theory with massive torsion are shown in table 3.1 [75–77], where the quadratic Lagrangian (1.27) has been rewritten by the Gauss-Bonnet theorem and by a reformulation of its coefficients to simplify the expressions of the propagators:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_m - \lambda\tilde{R} + \frac{1}{6}\left(2p+q-6r\right)\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} + \frac{1}{6}\left(2p+q\right)\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu} \\
&+ \frac{2}{3}\left(p-q\right)\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\mu\rho\nu} + \left(s+t\right)\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + \left(s-t\right)\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} \\
&+ \frac{1}{12}\left(4a+b+3\lambda\right)T_{\lambda\mu\nu}T^{\lambda\mu\nu} + \frac{1}{6}\left(2a-b+3\lambda\right)T_{\lambda\mu\nu}T^{\mu\lambda\nu} \\
&+ \frac{1}{3}\left(2c-a-3\lambda\right)T^{\lambda}{}_{\lambda\nu}T^{\mu}{}_{\mu}{}^{\nu}\,. \qquad (3.6)
\end{aligned}$$

The same analysis can be achieved by canceling the fourth order pole in all the spin sectors for the case with massless torsion [78]. In this sense, the Lagrangian acquires the following structure:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_m - R + \frac{1}{6}\left(q-4r\right)\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} + \frac{1}{6}\left(2r+q\right)\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu} \\ &+ \frac{2}{3}\left(r-q\right)\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\mu\rho\nu} - 2r\tilde{R}_{\mu\nu}\left(\tilde{R}^{\mu\nu} - \tilde{R}^{\nu\mu}\right) .\end{aligned} \tag{3.7}$$

Alternative works focusing on the occurrence of the mentioned extra gauge symmetries related to a torsion field with zero-mass modes and neglecting the requirement of vanishing fourth order poles reach completely different stability conditions [79–82], which means that there still exists an important disagreement around these models. Furthermore, as previously stressed, they are not developed as perturbative analyses around any specific curved background which may be induced by the presence of a dynamical torsion, but on a rigid flat space-time where the possible effects of the torsion field are completely neglected. In this sense, the stability conditions of the PG theory is still an open issue.

Parameter relations			Particle content
$p=0$	$a+b=0$	$s+t=0$	$2^+,0^+,0^-$
$p=0$	$a+b=0$	$s-2r=0$	$1^-,0^+,0^-$
$p=0$	$a+b=0$	$r-2s=0$	$2^+,1^-,0^-$
$p=0$	$a+c=0$	$s-2r=0$	$1^+,0^+,0^-$
$p=0$	$a+c=0$	$r-2s=0$	$2^+,1^+,0^-$
$p=0$	$a+\lambda=0$	$s+t=0$	$1^+,0^+,0^-$
$p=0$	$a+\lambda=0$	$r-2s=0$	$1^+,1^-,0^-$
$q=0$	$a+b=0$	$2p-2r+s=0$	$2^-,1^-,0^+$
$q=0$	$a+c=0$	$2p-2r+s=0$	$2^-,1^+,0^+$
$2r+t=0$	$a+c=0$	$2p-2r+s=0$	$2^-,0^+,0^-$
$p=0$	$a+c=0$	$a+\lambda=0$	$1^+,0^-$
$p=0$	$s+t=0$	$2r+t=0$	$0^+,0^-$

Table 3.1: Conditions for a ghost and tachyon-free linearized PG theory. The additional constraints for the available spin modes are given by: $2^+ : 2p-2r+s>0, a\lambda(a+\lambda)<0$; $2^- : p<0, a>0$; $1^+ : 2r+t>0, ab(a+b)<0$; $1^- : p+s+t<0, ac(a+c)>0$; $0^+ : p-r+2s>0, c\lambda(c-\lambda)>0$; $0^- : q<0, b>0$.

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Singularities and n-dimensional black holes in torsion theories

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Abstract. In this work we have studied the singular behaviour of gravitational theories with non symmetric connections. For this purpose we introduce a new criteria for the appearance of singularities based on the existence of black/white hole regions of arbitrary codimension defined inside a spacetime of arbitrary dimension. We discuss this prescription by increasing the complexity of the particular torsion theory under study. In this sense, we start with Teleparallel Gravity, then we analyse Einstein-Cartan theory, and finally dynamical torsion models.

Keywords: GR black holes, modified gravity

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Contents

1 Introduction

In a physical theory, a singularity is commonly known as a "place" where some of the variables used in the description of the system diverge. For example, we find this in the singularity in $r = 0$ of the Coulombian potential $V = K\frac{q}{r}$. This kind of behaviour appears mainly because the theory is not valid in the considered region or we have assumed a simplification. In the previous example the singularity arises due to the fact that we are considering the charged particle as a point and neglecting the quantum effects.

In General Relativity (GR), one might expect to observe singularities when the components of the tensors that describe the curvature of the spacetime diverge. This means that the curvature is higher than $\frac{1}{l_p^2}$, where l_p is the Planck length, so we need to have into account the quantum effects, which are not considered in this theory. However, there are situations where this behaviour is given as a result of the chosen coordinates. This is the case of the "singularity" in $r = 2M$ in the Schwarzschild metric. For this reason, another criteria, proposed by Penrose [1], is used to define a singularity: geodesic incompleteness. The physical interpretation of this condition is the existence of free falling observers that appear or disappear out of nothing. This is "strange" enough to consider it a sufficient condition to assure that there is a singularity.

Already in the first solutions of Einstein equations there are "places" where the components of the curvature tensors diverge, like in $r = 0$ in the Schwarzschild metric and $t = 0$ in the Friedmann-Lemaître-Robertson-Walker (FLWR) metric, but it was thought that this was a consequence of the excessive symmetry of the solutions, as it occurs in many situations in classical mechanics or electromagnetism. The first attempt of proving a singularity theorem was made by Raychaudhuri [2] in 1955, in an article where he introduced his famous equation, which is essential in the later development of singularity theorems. Ten years later, Penrose formulated the first singularity theorem that does not assume any symmetry [1] (for a recent review see [3]). It is also the first to use geodesic incompleteness in the definition of a singularity. This theorem showed that the singularity in $r = 0$ of the Schwarzschild metric is also present under non symmetrical gravitational collapses.

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The same happens with the singularity in $t = 0$ of the FLRW metric, but this time is a consequence of a theorem stated by Hawking a year later [4], which predicts that, under three physically realistic conditions, all past directed timelike geodesics have finite length, therefore every particle of the Universe (hence the Universe itself) had a beginning. The mentioned conditions are that the action of the Ricci tensor over a timelike vector is greater or equal than cero, which it is interpreted as the attractive nature of gravity, that the Universe is globally hyperbolic and there is an hypersurface with positive initial expansion. Although we have said that the conditions are physically realistic, since it was measured the accelerated expansion of the Universe [5], the convergence condition fails.

In general, all singularity theorems follow the same pattern, made explicit by Senovilla in [6]:

Theorem 1.1. *(Pattern singularity "theorem"). If the spacetime satisfies:*

1) A condition on the curvature.

2) A causality condition.

3) An appropriate initial and/or boundary condition.

Then there are null or timelike inextensible incomplete geodesics.

Let us stop for a moment and analyse the configuration of the theorems. When the singularity theorems are derived, no assumptions are made on the underlying physical theory, that is, the one that links the matter and energy content with the structure of the spacetime. This means that they are valid, not only for GR, but for all the modifications that change the Einstein-Hilbert action. It is worth mentioning that the first condition can be reformulated using the field equations of the theory, obtaining what is known as the *energy conditions*. These conditions are dependent of the considered theory, therefore they will differ from one to another, e.g., in GR they are formulated in terms of the energy-stress tensor only, while in f(R) theories there are some extra terms related to the curvature [7]. Since we are working in a Lorentzian manifold, we have to endow it with an affine structure, which is implicitly assumed to be the Levi-Civita one, as it is postulated in GR, given by the Christoffel symbols [8],

$$\mathring{\Gamma}^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right). \tag{1.1}$$

This is the unique connection that is covariantly conserved [9], $\mathring{\nabla}_\rho g_{\mu\nu} = 0$, and symmetric, $\mathring{\Gamma}^{\rho}_{\mu\nu} = \mathring{\Gamma}^{\rho}_{\nu\mu}$.

A metric has $D(D+1)/2$ components in a D-dimensional Lorentzian manifold, as it is a symmetric 2-covariant tensor. On the other hand, a general connection has D^3 components which are, in principle, completely independent degrees of freedom. Out of the D^3 components, $D^2(D-1)/2$ reside in the antisymmetric part

$$T^{\rho}_{\mu\nu} \equiv \Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu}, \tag{1.2}$$

which is known as *torsion*. The rest of degrees of freedom, $D^2(D+1)/2$, are encoded in the *non-metricity* tensor

$$M_{\rho\mu\nu} = \nabla_\rho g_{\mu\nu}. \tag{1.3}$$

One might wonder if it is possible to modify the gravitational theory by setting these tensors to be different from zero, i.e. postulating a connection that it is not Levi-Civita. Certainly it is, although we have to take into account some considerations:

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1. Every connection assigns to a curve γ a different acceleration, given by

$$a^\nu = v^\mu \nabla_\mu v^\nu, \tag{1.4}$$

where $v^\nu = \frac{dx^\mu}{ds}$ is the four-velocity of the curve γ, parametrised by its proper time as $\gamma(s) = x^\mu(s)$. If acceleration is to keep a meaning [10], it is necessary that the same metric is considered all along the curve. In other words, the connection must parallel-transport the metric, that is

$$v^\rho \nabla_\rho g_{\mu\nu} = 0 \tag{1.5}$$

for every vector field v^ρ, which is equivalent to the metricity condition ($M_{\rho\mu\nu} = 0$). This is why we will only consider connections that fulfill this condition from now on, although there has been work done in modified theories that set the non-metricity tensor different from zero, like in [9] (for a review of these theories see [11]).

2. A different connection does not necessarily leads to a theory with a different phenomenology, since the action may be invariant or differ only by a divergence term under this change, therefore leaving the field equations unchanged. This is the case of a spacetime with linear vector distortion [9] or teleparallel Gravity (TEGR) [10].

The latter case deserves some attention, as it is one of the simplest cases of this kind of theories, while at the same time, it is a good example to first apply the methods that we will use in more complicated ones. But first, let us review the singularity theorems in GR.

2 Singularity theorems in General Relativity

It seems logical that since we are generalizing the singularity theorems of GR, we introduce in this section the most general ones. This is the case of two recent theorems due to Senovilla and Galloway [12], that predict the occurrence of singularities, i.e. incomplete geodesics, based on the existence of trapped submanifolds of arbitrary co-dimension. The main key of the demonstration is, like in almost every singularity theorem, finding the conditions for the appearance of *focal* and/or *conjugate* points.

Let us consider a family of geodesics $\gamma_s(t)$, where $T^\mu = \left(\frac{\partial}{\partial t}\right)^\mu$ is the tangent vector to the family and $X^\mu = \left(\frac{\partial}{\partial s}\right)^\mu$ is the orthogonal deviation vector (that represents the displacement towards an infinitesimally near geodesic). These vectors follow the orthogonal deviation equation

$$T^\mu \nabla_\mu \left(T^\nu \nabla_\nu X^\rho\right) = -R_{\mu\nu\lambda}^{\ \ \ \ \rho} X^\nu T^\mu T^\lambda. \tag{2.1}$$

A solution X^μ of this equation is called a *Jacobi field on* γ. With this established we can see what we understand by conjugate and focal points:

Definition 2.1. *Let γ be a geodesic emanating from p (orthogonal to a spacelike submanifold Σ). Then a point q is conjugate (focal) along γ to the the point p (of the spacelike hypersurface Σ) if there exists a non-zero Jacobi field on γ that vanishes at p and q (does not vanish at Σ and vanishes at q).*

The problem of whether this kind of points will appear or not can be addressed in two different ways. In the physics orientated literature [8, 13] it is studied by means of the Raychaudhuri equation, which gives us the evolution of the expansion in a congruence

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of curves (not necessarily geodesics). To obtain this equation, we decompose the covariant derivative of the tangent vector of a congruence of curves, $B_{\mu\nu} = \mathring{\nabla}_\nu v_\mu$, into its antisymmetric $\omega_{\mu\nu}$, known as *vorticity*, traceless symmetric $\sigma_{\mu\nu}$, usually referred as *shear*, and trace part θ, also known as *expansion*, such as

$$B_{\mu\nu} = \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \tag{2.2}$$

where $h_{\mu\nu}$ is the projection of the metric into the spacial subspace orthogonal to the tangent vector. Then, it can be seen that [13]

$$\begin{aligned} v^\rho \mathring{\nabla}_\rho \theta = \frac{d\theta}{ds} = &-\frac{1}{3}\theta^2 - \sigma^{\mu\rho}\sigma_{\mu\rho} \\ &+ \omega^{\mu\rho}\omega_{\mu\rho} - \mathring{R}_{\rho\varphi} v^\rho v^\varphi + \mathring{\nabla}_\mu \left(v^\nu \mathring{\nabla}_\nu v^\mu \right), \end{aligned} \tag{2.3}$$

which is the so called Raychaudhuri equation. With that, we can predict under what circumstances the expansion goes to minus infinity, which is the equivalent of having a conjugate/focal point [8].

On the other hand, in the mathematical literature [14] this is solved in the context of variational calculus, by using the so-called *Hessian* form. It is based on the idea that the set of all piecewise smooth curve segments $\gamma : [0,\, b] \longrightarrow M$ from a submanifold P (that clearly includes the case $P = p$) to a point q, $\Omega\,(P,\, q)$, can be treated as a manifold.

There is an explicit expression for this form, but before we write it we have to familiarize ourselves with the notation. Let Σ be a spacelike submanifold of arbitrary co-dimension, then we can define [12]:

- n_μ: future directed vector, perpendicular to the spacelike submanifold Σ.
- $\overrightarrow{e}_A$: vector fields tangent to Σ.
- γ: geodesic curve tangent to n^μ at Σ.
- u: affine parameter along γ, taking $u = 0$ at Σ.
- N^μ: geodesic vector field tangent to γ, having $\left. N_\mu \right|_{u=0} = n_\mu$.
- $\overrightarrow{E}_A$: vector fields that are the parallel transport of $\overrightarrow{e}_A$ along γ (using the Levi-Civita connection), satisfying that $\left. \overrightarrow{E}_A \right|_{u=0} = \overrightarrow{e}_A$.
- $P^{\mu\nu} \equiv \gamma^{AB} E^\mu_A E^\nu_B$, where γ^{AB} is the inverse of the first fundamental form of Σ in the spacetime, $\gamma_{AB} = g_{\mu\nu} e^\mu_A e^\nu_B$. In $u = 0$, $P^{\mu\nu}$ is just the proyector to Σ.
- Expansion of the submanifold Σ along $\overrightarrow{n}$: $\theta\,(\overrightarrow{n}) \equiv n_\mu H^\mu = \gamma^{AB} K_{AB}\,(\overrightarrow{n})$, where $\overrightarrow{H}$ is the *mean curvature vector* of the submanifold Σ, and $K_{AB}\,(\overrightarrow{n})$ is the contraction of the *shape tensor* $\overrightarrow{K}_{AB}$ with the one-form n_μ [14]. If $\theta\,(\overrightarrow{n}) < 0$ for all posible normal vectors, Σ is said to be a future trapped submanifold.

Now we can express the Hessian of two vector fields $V,\, W \in T_\gamma\,(\Omega\,(\Sigma,\, q))$ as

$$\begin{aligned} I_\gamma\,(V,\, W) = \int_0^b \Big[&\left(N^\mu \nabla_\mu V^\nu \right) \left(N^\rho \nabla_\rho W_\nu \right) - \\ &- N_\mu R^\mu_{\ \nu\rho\sigma} V^\nu N^\rho W^\sigma \Big] du + K_{AB}\,(\overrightarrow{n})\, v^A w^B, \end{aligned} \tag{2.4}$$

where $\overrightarrow{v} = \overrightarrow{V}\,(u = 0)$, $v_A = v_\mu e^\mu_A$ is the part of $\overrightarrow{v}$ tangent to Σ , and the same for $\overrightarrow{W}$ and $\overrightarrow{w}$ [12].

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The reader might be wondering what is the connection between the Hessian and the conjugate and focal points. The next theorem clears all doubts [15].

Theorem 2.2. *Let Σ be a spacelike submanifold and γ a causal curve orthogonal to Σ, then the submanifold Σ does not have focal points along γ if and only if the Hessian is semi-positive definite, having $I_\gamma(V, V) = 0$ only if $\overrightarrow{V}$ is proportional to $\overrightarrow{N}$ on γ.*

To assure the appearance of focal points to a hypersurface of arbitrary co-dimension Σ, Senovilla and Galloway develop a curvature condition.

Proposition 2.3. *Let Σ be a spacelike submanifold of co-dimension m in a Lorentzian manifold of dimension n, and let n_μ be a future-pointing normal to Σ. If $\theta(\overrightarrow{n}) \equiv (m-n)\,c < 0$, and the curvature tensor satisfies the inequality*

$$R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} \geq 0 \tag{2.5}$$

along γ, then there is a point focal to Σ along γ at or before $q = \gamma\left(u = \frac{1}{c}\right)$, given that the curve had arrived so far.

This condition can be interpreted as a manifestation of the attractive character of gravity.

Based on this focalisation theorem, Senovilla and Galloway prove a generalisation of the Penrose and Hawking-Penrose theorem. The first result predicts the incompleteness of null geodesics:

Theorem 2.4. *Let (M, g) contain a non-compact Cauchy hypersurface S and a closed future trapped submanifold Σ of arbitrary co-dimension. If the curvature condition holds along every future directed null geodesic emanating orthogonally from Σ, then (M, g) is future null geodesically incomplete.*

The second theorem is based on the Hawking-Penrose lemma, which is valid for arbitrary dimension, that states that this three conditions cannot all hold:

- Every inextensible causal geodesic contains a pair of conjugate points.
- There are not closed timelike curves (chronology condition).
- there is an achronal set Σ such that $E^+(\Sigma)$ is compact.

It is an established result [8, 13] that the first statement holds if $R_{\mu\nu} v^\mu v^\nu \geq 0$ for every non-spacelike vector v^μ. When applied to timelike vectors it is known as the *timelike convergence condition*, while in the case of null ones it is called the *null convergence condition*. Using the Einstein field equations we can rewrite these conditions in terms of the energy momentum tensor $T_{\mu\nu}$. The equivalent of the timelike convergence is the *strong energy condition*, $T_{\mu\nu} v^\mu v^\nu \geq \frac{1}{2} T$, and for the null one the weak energy condition, $T_{\mu\nu} v^\mu v^\nu \geq 0$, where T is the trace of the energy-momentum tensor.

Now we can review the generalization of the H-P theorem:

Theorem 2.5. *If the chronology, generic, timelike and null convergence conditions hold and there is a closed future trapped submanifold Σ of arbitrary co-dimension such that the curvature condition holds along every null geodesic emanating orthogonally from Σ, then the spacetime is causal geodesically incomplete.*

Sketch of the proof. First of all, it has to be proven that the existence of closed trapped submanifolds leads to the existence of an achronal set with the properties mentioned in the lemma [12]. Once the H-P lemma is proved, this theorem can be easily deduced, as it is explained in [13].

3 Black hole regions

We know from experience, e.g. the Schwarzschild metric, that the existence of incomplete null geodesics leads to the appearance of *black holes*, that are regions of the spacetime that once an observer enters them, it cannot leave. This applies to all timelike and null curves, not just geodesics. This is usually known as the *cosmic censorship conjecture*, which is a concept that Penrose introduced in 1969. It basically states that singularities cannot be *naked*, that means that they cannot be seen by an outside observer. However, how can we express this concept mathematically? The answer lies in the concept of *conformal compactification*, which can be defined as [18]:

Definition 3.1. *Let* (M, g) *and* $\left(\tilde{M}, \tilde{g}\right)$ *be two spacetimes. Then* $\left(\tilde{M}, \tilde{g}\right)$ *is said to be a conformal compactification of* M *if and only if the following properties are met:*

1. M *is an open submanifold of* $\tilde{M}$ *with smooth boundary* $\partial\tilde{M} = \mathcal{J}$*. This boundary is usually denoted* conformal infinity*.*
2. *There exists a smooth scalar field* Ω *on* $\tilde{M}$*, such that* $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ *on* M*, and so that* $\Omega = 0$ *and its gradient* $d\Omega \neq 0$ *on* $\mathcal{J}$*.*

If additionally, every null geodesic in M *acquires a future and a past endpoint on* $\mathcal{J}$*, the spacetime is called* asymptotically simple*. Also, if the Ricci tensor is zero in a neighbourhood of* $\mathcal{J}$ *the spacetime is said to be* asymptotically empty*.*

In a conformal compactification, $\mathcal{J}$ is composed by two null hypersurfaces, $\mathcal{J}^+$ and $\mathcal{J}^-$, known as *future null infinity* and *past null infinity* respectively.

In order to establish the definition of black hole, we need to introduce two more concepts [8]:

Definition 3.2. *A spacetime* (M, g) *is said to be* asymptotically flat *if there is an asymptotically empty spacetime* (M', g') *and a neighbourhood* $\mathcal{U}'$ *of* $\mathcal{J}'$*, such that* $\mathcal{U}' \cap M'$ *is isometric to an open set* $\mathcal{U}$ *of* M*.*

Definition 3.3. *Let* (M, g) *be an asymptotically flat spacetime with conformal compactification* $\left(\tilde{M}, \tilde{g}\right)$*. Then* M *is called* (future) strongly asymptotically predictable *if there is an open region* $\tilde{V} \subset \tilde{M}$*, with* $\overline{J^-(\mathcal{J}^+) \cap M} \subset \tilde{V}$*, such that* $\tilde{V}$ *is globally hyperbolic.*

This definition does not require the condition of the endpoints of the null geodesics, meaning that this kind of spacetimes can be singular. Nevertheless, if a spacetime is asymptotically predictable, then the singularities are not naked, i.e. are not visible from $\mathcal{J}^+$.

Now we can establish what we understand by a black hole:

Definition 3.4. *A strongly asymptotically predictable spacetime* (M, g) *is said to contain a black hole if* M *is not contained in* $J^-(\mathcal{J}^+)$*. The* black hole region*,* B*, is defined to be* $B = M - J^-(\mathcal{J}^+)$ *and its boundary,* ∂B*, is known as the* event horizon*.*

Intuitively, we think that a particle in a closed trapped surface cannot scape to $\mathcal{J}^+$, meaning that it is part of the black hole region of the spacetime. Nevertheless, this is not true in general. In the next proposition we establish the conditions that ensure the existence of black holes when we have a closed future trapped submanifold of arbitrary co-dimension:

Proposition 3.5. *Let (M, g) be a strongly asymptotically predictable spacetime of dimension n, and Σ a closed future trapped submanifold of arbitrary co-dimension m in M. If the curvature condition holds along every future directed null geodesic emanating orthogonally from Σ, then Σ cannot intersect $J^-(\mathcal{J}^+)$, i.e. Σ is in the black hole region B of M.*[1]

Proof. This proof is similar to the one of proposition 12.2.2 by Wald [8]. Let us suppose that Σ intersects $J^-(\mathcal{J}^+)$. Then, in the conformal compactification $\tilde{M}$, we would have that $J^+(\Sigma) \cap \mathcal{J}^+ \neq \emptyset$. On other hand, we know that the spatial infinity i^0, the point of the compactification where the future (past) complete spacelike geodesics end (begin), is not in the causal future of any point in M. Therefore it follows trivially that $i^0 \notin J^+(\Sigma)$. Since M is strongly asymptotically predictable, there is a globally hyperbolic region $\tilde{V}$ in the compactification such that $\overline{J^-(\mathcal{J}^+) \cap M} \subset \tilde{V}$. From basic topology we have that the intersection of two closed sets is closed, therefore $\Lambda = \Sigma \cap \left(\overline{J^-(\mathcal{J}^+) \cap M}\right)$ is closed, where clearly $\Lambda \subset \Sigma$ and $\Lambda \subset \tilde{V}$. In addition, a closed subset of a compact is also compact, so from the compactness of Σ we deduce that Λ is compact. It is an standard result of Lorentzian geometry that, in a globally hyperbolic space, the causal future of a compact set is closed [8], so, in this case, we have that $J^+(\Lambda)$ is closed in $\tilde{V}$. This means that it contains all of its limit points, therefore since $i^0 \notin J^+(\Lambda)$, there is an open neighbourhood of i^0 that does not intersect $J^+(\Lambda)$, and so, an open region of $\mathcal{J}^+$ that does not intersect $J^+(\Lambda)$. It is known that a connected set cannot contain a subset with no boundary (except for the empty set and the set itself) [19]. As we have already proved, $J^+(\Lambda) \cap \mathcal{J}^+$ is not equal to $\mathcal{J}^+$. Since $\mathcal{J}^+$ is connected, it follows that there must be a point $q \in \mathcal{J}^+$ in $\partial J^+(\Lambda)$. In the proof of the generalised Penrose theorem we used that in a globally hyperbolic spacetime $\partial J^+(\Lambda) = E^+(\Lambda)$, so in the compactification $\tilde{M}$ there is a null geodesic γ connecting $p \in \Lambda \subset \Sigma$ with q. Furthermore, using the theorem 51 of O'Neill [14], we see that this null geodesic must be orthogonal to Σ and not contain any focal point of Σ before q, as otherwise we would have that $q \in I^+(\Lambda)$ and therefore $q \notin E^+(\Lambda)$. With respect to the metric g of M, γ is also a null geodesic orthogonal to Σ with no focal point of Σ, but now γ is future complete [8]. Although, since Σ is future trapped one has $\theta(\overrightarrow{n}) \equiv (m-n)\, c < 0$ for any future-pointing null normal one-form n_μ [12]. Now, let $(m-n)\, C$ be the maximum value of all possible $\theta(\overrightarrow{n})$ on the compact Σ. Then, using the proposition 3.5, we have that every null geodesic emanating orthogonally from Σ will have a focal point at or before the affine parameter reaches the value $\frac{1}{C}$. This clearly leads to a contradiction, therefore the assumption is false. $\square$

This proposition will help us to study the singularities in theories of gravitation that include torsion. But first, let us introduce the main aspects of these theories.

[1] Analogously, it can be defined a past strongly asymptotically predictable space time, and then the proposition would predict the existence of white hole regions, $B = M - J^+(\mathcal{J}^-)$, that are regions that particles cannot enter, only exit.

4 General aspects of theories with torsion

In this section, we introduce the geometrical background of gravitational theories that allow a non symmetric connection that still fulfills the metricity condition. The interesting fact about these theories is that they appear naturally as a gauge theory of the Poincaré Group [20], making their formalism closer to that of the Standard Model of Particles, and hence making it a good candidate to explore the quantization of gravity.

Since the connection is not necessarily symmetric, the torsion can be different from zero. For an arbitrary connection, that meets the metricity condition, there exists a relation between it and the Levi-Civita connection

$$\mathring{\Gamma}^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} - K^{\rho}{}_{\mu\nu}, \tag{4.1}$$

where

$$K^{\rho}{}_{\mu\nu} = \frac{1}{2}\left(T^{\rho}{}_{\mu\nu} - T_{\mu}{}^{\rho}{}_{\nu} - T_{\nu}{}^{\rho}{}_{\mu}\right) \tag{4.2}$$

is the *contortion* tensor.

Since the curvature tensors depend on the connection, there is a relation between the ones defined throughout the Levi-Civita connection and the general ones. For the Riemann tensor we have [21]

$$\begin{aligned}\mathring{R}^{\sigma}{}_{\mu\nu\rho} = R^{\sigma}{}_{\mu\nu\rho} - \mathring{\nabla}_{\nu}K^{\sigma}{}_{\mu\rho} + \mathring{\nabla}_{\rho}K^{\sigma}{}_{\mu\nu} -& \\ - K^{\sigma}{}_{\alpha\nu}K^{\alpha}{}_{\mu\rho} + K^{\sigma}{}_{\alpha\rho}K^{\alpha}{}_{\mu\nu},&\end{aligned} \tag{4.3}$$

where the upper index $^{\circ}$ denotes the Levi-Civita quantities. By contraction we can obtain the expression for the Ricci tensor

$$\begin{aligned}\mathring{R}_{\mu\rho} = R_{\mu\rho} - \mathring{\nabla}_{\sigma}K^{\sigma}{}_{\mu\rho} + \mathring{\nabla}_{\rho}K^{\sigma}{}_{\mu\sigma} -& \\ - K^{\sigma}{}_{\alpha\sigma}K^{\alpha}{}_{\mu\rho} + K^{\sigma}{}_{\alpha\rho}K^{\alpha}{}_{\mu\sigma},&\end{aligned} \tag{4.4}$$

and the scalar curvature

$$\begin{aligned}\mathring{R} = g^{\mu\rho}\mathring{R}_{\mu\rho} = R - \mathring{\nabla}^{\rho}K^{\sigma}{}_{\sigma\rho} -& \\ - K^{\sigma}{}_{\alpha\sigma}K^{\alpha\rho}{}_{\rho} + K^{\alpha}{}_{\sigma\rho}K^{\sigma}{}_{\mu\alpha}.&\end{aligned} \tag{4.5}$$

All the theories that we will consider from now on will follow these geometrical properties, the only change would be the underlying physical theory.

5 Singularities in Teleparallel Gravity

TEGR is a degenerate case of the Poincaré gauge theories, since it is a gauge theory of the translation group only. Any gauge theory including these transformations will differ from the usual internal gauge models in many ways, the most significant being the presence of a tetrad field [22]. Given a nontrivial tetrad h^{a}_{μ}, it is possible to define a connection known as Weitzenböck connection

$$\Gamma^{\rho}{}_{\mu\nu} = h_{a}{}^{\rho}\partial_{\mu}h^{a}{}_{\mu}, \tag{5.1}$$

that presents torsion, but no curvature. With this tetrad field we can also construct the Levi-Civita connection, taking into account that the metric can be expressed as

$$g_{\mu\nu} = \eta_{ab}h^{a}{}_{\mu}h^{b}{}_{\nu}, \tag{5.2}$$

where η_{ab} is the Lorentz-Minkowski metric, and using the usual definition, as seen in equation (1.1).

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The relation between these two connections is given by equation (4.1). The Lagrangian density of this gravitational theory can be written as

$$\mathcal{L} = \frac{hc^4}{16\pi G} S^{\mu\nu\rho} T_{\mu\nu\rho}, \tag{5.3}$$

where $h = det\left(h^a{}_\mu\right)$, and

$$S^{\mu\nu\rho} = -S^{\mu\rho\nu} \equiv \frac{1}{2}\left(K^{\nu\rho\mu} + g^{\mu\rho} T^{\sigma\nu}{}_{\sigma} + g^{\mu\nu} T^{\sigma\rho}{}_{\sigma}\right), \tag{5.4}$$

which is usually known as *superpotential*.

Using the relation between the Weitzenböck and the Levi-Civita connection in equation (4.1) we can express this Lagrangian as

$$\mathcal{L} = \mathring{\mathcal{L}} - \partial_\mu \left(\frac{hc^4}{8\pi G} T^{\nu\mu}{}_{\nu}\right), \tag{5.5}$$

where $\mathring{\mathcal{L}}$ is the Einstein-Hilbert Lagrangian of GR. Since they are equal except for a total divergence, the same field equations arise. Therefore it is a theory equivalent to GR, as it can be seen for example when one studies the junction conditions [36].

The field equations can be obtained by taking variations of the Lagrangian. Expressing them in pure spacetime form, we have

$$\partial_\sigma \left(h S_\mu{}^{\sigma\nu}\right) - \frac{4\pi G}{c^4}\left(h t_\mu{}^{\nu}\right) = 0, \tag{5.6}$$

where

$$h t_\mu{}^{\nu} = \frac{hc^4}{4\pi G} \Gamma^{\sigma}{}_{\rho\mu} S_\sigma{}^{\rho\nu} + \delta_\mu{}^{\nu} \mathcal{L} \tag{5.7}$$

is the canonical *energy-momentum pseudotensor* of the gravitational field. Although this is the simplest framework for a theory with torsion, it is helpful for introducing the methods that we will use in more general cases. In that sense, the next considerations are general, and can be applied in all the theories of gravitation.

In GR we have considered geodesic incompleteness as a criterium of the appearance of singularities, based on the fact that causal geodesics are the trajectories of free-falling observers. Therefore, we wish to modify this criteria by terms of these trayectories in the theory that we are considering. We will say that our spacetime is singular if the domain of the affine parameter of at least one curve that follow any free-falling observer (including photons) is different from $\mathbb{R}$. For spacetimes in which we can define a conformal boundary, as the ones considered in section 3, this can be stated in the following way:

Definition 5.1. *A spacetime (M, g), endowed with a conformal compactification, is said to be singular if at least one non-spacelike curve has an endpoint outside the conformal infinity.*

Before continuing, it is useful to define two important classes of curves, which coincide in the case of the Levi-Civita connection [24]:

- **Autoparallel curves**: these are the curves in which its tangent vector v^μ is parallel transported to itself, that is:

$$v^\mu \nabla_\mu v^\nu = 0. \tag{5.8}$$

The differential equation of the autoparallels is, under a suitable choice of the affine parameter:

$$\frac{dv^\mu}{dt} + \Gamma^\mu_{\rho\sigma} v^\rho v^\sigma = 0, \tag{5.9}$$

which only takes into account the symmetric part of the connection.

- **Extremal curves**: these are the ones that extremise the length with respect to the metric of the manifold. It is worth mentioning that the length only depends on the metric, and not on the torsion. In order to see what are the equations of these curves we recall a standard result from Lorentzian geometry, that can be used as a definition:

  Theorem 5.2. *Let γ be a smooth timelike curve connecting two points $p, q \in M$. Then the necessary and sufficient condition that γ locally maximizes the length between p and q over smooth one parameter variations is that γ is a geodesic with no point conjugate to p between p and q.*

  Then, the differential equations of these curves are the same of the Levi-Civita geodesics:

$$\frac{dv^\mu}{dt} + \mathring{\Gamma}^\mu_{\rho\sigma} v^\rho v^\sigma = 0, \tag{5.10}$$

The trajectories of free-falling observers in theories different from GR do not follow these curves in general. Nevertheless, in TEGR they do. The equation of motion for free falling observers, scalar particles, is [22]:

$$\frac{dv_\mu}{dt} + \Gamma_{\rho\sigma\mu} v^\rho v^\sigma = 0, \tag{5.11}$$

which is equivalent to

$$\frac{dv^\mu}{dt} + \mathring{\Gamma}^\mu_{\rho\sigma} v^\rho v^\sigma = 0. \tag{5.12}$$

Therefore they follow extremal curves, which are the autoparallels of the Levi-Civita connection.

It is particular interesting to discuss this issue for photons. It has been stated that Maxwell equations do not couple to torsion in the minimal approach. However, in TEGR the electromagnetic field is able to couple to torsion without violating gauge invariance [10]. Using the relation between the Levi-Civita and the Weitzenböck connection, one can verify that the teleparallel version of Maxwell's equations are completely equivalent with the usual Maxwell's equations in the context of GR. This means that they move according to the geodesic equation of GR, and so the causal structure is the same as in GR.

This discussion is more general. In fact, the equivalence between TEGR and GR means that all the singularity theorems developed in GR apply to this theory also. Therefore, the causal convergence and the curvature condition remain the same, although the expression for the Riemann and Ricci tensor change as discussed in the previous section, specifically in equations (4.3), (4.4).

6 Singularities in Einstein-Cartan theory

The Einstein-Cartan (EC) theory of gravitation is the most recognised theory that includes torsion [23, 24]. The main reason to introduce this theory is the fact that it allows to consider massive spinning fields in a natural way, while maintaining all the experimental success of GR. This theory arises when searching for a gravitational Lagrangian linear in the curvature term $R^{\lambda}{}_{\rho\mu\nu}$. The geometrical structure is the one analysed in section 4.

The field equations are obtained by varying the Lagrangian of this theory with respect to the metric and the contortion:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa \Sigma_{\mu\nu} \tag{6.1}$$

and

$$S_{\mu\nu\rho} = \kappa \tau_{\mu\nu\rho}, \tag{6.2}$$

where

$$S_{\mu\nu}{}^{\rho} = T_{\mu\nu}{}^{\rho} + \delta^{\rho}_{\mu} T_{\nu\sigma}{}^{\sigma} - \delta^{\rho}_{\nu} T_{\mu\sigma}{}^{\sigma} \tag{6.3}$$

is the modified torsion tensor. At this point, we might wonder what are the trajectories of the free-falling observers, in order to establish some singularity theorems.

Since it is impossible to perform the minimally coupling prescription for the Maxwell's field while maintaining the $U(1)$ gauge invariance, the Maxwell equations are the same as in GR. Therefore, they move following null extremal curves, and so the causal structure is determined by the metric structure, just like in GR. Also, from the minimally couple procedure, it follows that particles with no spin, represented by scalar fields, do not feel torsion as well, since the covariant derivative of a scalar field is just its partial derivative. This means that the test particles follow the geodesics of the Levi-Civita connection, which allow us to generalise trivially the singularity theorems. Just like in TEGR, the causal convergence and the curvature conditions remain the same, it just changes the expression for the Levi-Civita Riemann and Ricci tensors, as given by equations (4.3), (4.4).

In any case, even for trajectories decoupled from torsion, energy conditions are modified. Although the curvature condition is the same as in GR, these conditions change due to the fact that the field equations are different. Since equation (6.2) is purely algebraic we can substitute everywhere spin with torsion. Now we split the Einstein tensor into the Levi-Civita ($\mathring{G}^{\mu\nu}$) part and the rest, and we change the torsion terms by means of equation (6.2), obtaining

$$\mathring{G}^{\mu\nu} = \kappa \tilde{\sigma}^{\mu\nu}, \tag{6.4}$$

where $\tilde{\sigma}^{\mu\nu}$ is the combined energy-momentum tensor

$$\begin{aligned}\tilde{\sigma}_{\mu\rho} = \Sigma_{\mu\rho} - \nabla_{\sigma} K^{\sigma}{}_{\mu\rho} + \nabla_{\rho} K^{\sigma}{}_{\mu\sigma} - K^{\sigma}{}_{\alpha\sigma} K^{\alpha}{}_{\mu\rho} + K^{\sigma}{}_{\alpha\rho} K^{\alpha}{}_{\mu\sigma} + \\ + \frac{1}{2} g_{\mu\rho} \Big(\nabla^{\alpha} K^{\sigma}{}_{\sigma\alpha} + K^{\sigma}{}_{\alpha\sigma} K^{\alpha\rho}{}_{\rho} - K^{\alpha}{}_{\sigma\rho} K^{\sigma}{}_{\mu\alpha} \Big).\end{aligned} \tag{6.5}$$

Now, by using equation (6.4) we can write the energy conditions. The strong energy condition can be expressed as

$$\tilde{\sigma}_{\mu\nu} v^{\mu} v^{\nu} \geq \frac{1}{2} \tilde{\sigma}, \tag{6.6}$$

where $\tilde{\sigma} = g_{\mu\nu} \tilde{\sigma}^{\mu\nu}$. And for the weak energy condition we have

$$\tilde{\sigma}_{\mu\nu} v^{\mu} v^{\nu} \geq 0. \tag{6.7}$$

It is interesting noting that when the torsion is zero, one recovers the energy conditions of GR, as one would expect, since the contortion tensor involved in equation (6.5) also vanishes.

So far we have analysed the singular behaviour of photons and spinless particles, but it is more interesting to study the behaviour of spinning fields. This question has already been addressed in the literature, mainly following two approaches. The first one is to study the singular behaviour of particular cosmological models using the energy conditions and the modified Raychaudhuri equation for non symmetric connection derived by Stewart and Hajicek [25] (for a review of this approach see [24]). These studies try to obtain plausible cosmological models that are singularity free. Nevertheless, they come to the conclusion that it is necesary to have regions with high spin density to observe a behaviour different from GR, and to avoid the singularities. On the other hand, Esposito [26] proved a singularity theorem for EC theory based on the incompleteness of autoparallel curves. He considers this criteria to be sufficient to establish the singular character of a spacetime.

In those two approaches, the argument is based on the modified Raychaudhuri equation for non-symmetric metric connections. The main difference comes, as one would expect, from a change in the antisymmetric part of the decomposition mentioned in equation (2.2), since now $B_{\mu\nu}$ is defined throughout the total connection. Then, the equation can be expressed as follows:

$$\begin{aligned} v^\rho \nabla_\rho \theta = \frac{d\theta}{d\tau} = & -\frac{1}{3}\theta^2 - \sigma^{\mu\rho}\sigma_{\mu\rho} \\ & + \left(\omega^{\mu\rho} + S^{\mu\nu}\right)\left(\omega_{\mu\rho} + S_{\mu\nu}\right) - R_{\rho\varphi} v^\rho v^\varphi, \end{aligned} \tag{6.8}$$

where $S_{\mu\nu}$ is a tensor that is usually defined through the following relation with the modified torsion tensor [26]

$$S_{\mu\nu}{}^{\rho} = S_{\mu\nu} v^\rho. \tag{6.9}$$

The problem with this reasoning is that the spin particles do not follow in general autoparallels curves of the total connection, so there might be situations where there is incompleteness of this kind of curves but no singular spin trajectories. Nevertheless, one could have the curiosity of knowing which is the Raychadhuri equation for the spin test particles. We will know study how we can expressing, making an study valid for all the Poincaré gauge theories of gravity.

All the analysis of these trajectories up to this point, which are reviewed in [27], have a thing in common: after some algebra, they can all be expressed in the form

$$\begin{aligned} a^\mu &= v^\rho \mathring{\nabla}_\rho v_\mu \\ &= C\left(\frac{\hbar}{m}\right) f\left(\widehat{R}^{\mu}{}_{\lambda\rho\sigma} s^{\rho\sigma} v^\lambda + K_{\rho\sigma}{}^{\mu} v^\rho v^\sigma\right), \end{aligned} \tag{6.10}$$

where C is a constant, m is the mass of the particle, and we have made explicit the Planck constant. The tensor $s^{\rho\sigma}$ is the internal spin tensor, related to the spin s^μ of the particle by

$$s^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} v_\nu s_{\rho\sigma}, \tag{6.11}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor, which is normalised with the square root of the metric, as it is usual in a Lorentzian manifold. The *function* f represents a linear combination of different contractions of the tensors involved in the expression, depending on the analysis chosen. The connection with respect it is calculated the Riemmann tensor in brackets is also dependent of the analysis, but it is always one constructed with the Levi Civita and linear combinations of torsion related quantities.

When writing the Raychaudhuri equation we choose to make it with respect to the Levi-Civita connection, since it is analogous to the expression in terms of the total connection, and in this way we avoid introducing new terms to the decomposition in equation (2.2). With that in mind we have that

$$v^{\rho}\mathring{\nabla}_{\rho}\theta = \frac{d\theta}{ds} = -\frac{1}{3}\theta^2 - \sigma^{\mu\rho}\sigma_{\mu\rho} + \omega^{\mu\rho}\omega_{\mu\rho} - \mathring{R}_{\rho\varphi}v^{\rho}v^{\varphi} + C\left(\frac{\hbar}{m}\right)\mathring{\nabla}_{\mu}\left(\widehat{R}^{\mu}{}_{\lambda\rho\sigma}s^{\rho\sigma}v^{\lambda} + K_{\rho\sigma}{}^{\mu}v^{\rho}v^{\sigma}\right). \tag{6.12}$$

Using this equation we could predict the appearance of focal/conjugate points in a congruence of this timelike curves, just by imposing a generalised curvature condition

$$\mathring{R}_{\rho\varphi}v^{\rho}v^{\varphi} - C\left(\frac{\hbar}{m}\right)\mathring{\nabla}_{\mu}\left[f\left(\widehat{R}^{\mu}{}_{\lambda\rho\sigma}s^{\rho\sigma}v^{\lambda} + K_{\rho\sigma}{}^{\mu}v^{\rho}v^{\sigma}\right)\right] \geq 0, \tag{6.13}$$

that must hold for every timelike vector v^{μ}. Nevertheless, there are some issues with this approach. First of all, this is only valid for congruences of curves that have the same spin orientation for all the test particles, hence limiting the analysis. On the other hand, we know for the singularity theorems in GR that the existence of focal/conjugate points is not a sufficient condition for the appearance of singularities. We also need global conditions that allow us to reach a contradiction with the completeness of the curves. Since we are considering non-geodesical behaviour, the theorems that allow us to make that contradiction are no longer valid, and we cannot predict the singularities. That is why in this article we propose another approach, based on the result of the appearance of black/white hole regions in an arbitrary Lorentzian manifold.

It is clear from the previous analysis that spinning particles do not follow extremal curves. However, independently of how torsion affects these particles, they will follow timelike curves, since they are massive and we assume that locally (in a normal neighbourhood of a point) nothing can be faster than light (null geodesics). Hence, it would be interesting to see under what circumstances we have non-geodesical timelike singularities. For that, we recover the definition of an n-dimensional black/white hole given in section 3. From this definition, we conclude that if this kind of structures exist in our spacetime, we would have timelike curves (including non-geodesics) that do not have endpoints in the conformal infinity, since for the case of black holes, the spacetime M is not contained in $J^{-}(\mathcal{J}^{+})$, while for white holes, M is not contained in $J^{+}(\mathcal{J}^{-})$. This is exactly the extended definition of singularity that we have given in section 5. Considering this, we establish the following theorem:

Theorem 6.1. *Let (M, g) be a strongly asymptotically predictable spacetime of dimension n, and Σ a closed future trapped submanifold of arbitrary co-dimension m in M. If the curvature condition holds along every future directed null geodesic emanating orthogonally from Σ, then some timelike curves in M would not have endpoints in the conformal infinity, hence M is a singular spacetime (definition 5.1).*

From a physical point of view, one might wonder if one of the incomplete timelike curves actually represents the trajectory of a spin particle. From equation (6.10), which represents the non-geodesical behaviour, we see that the only possible way that all the trajectories have endpoints in the conformal infinity is that there are huge values of the curvature and torsion near the event horizon, which in a physically plausible scenario it is not possible. This is why we consider it a more physically relevant theorem for the singular behaviour of the spin particles, since it is strongly related to the actual trajectories.

7 Singularities in dynamical torsion theories

So far, the two torsion theories that we have analysed are part of a set of theories known as Poincaré Gauge Gravity (PG) [24, 28]. The reason why there are many theories under this premise is because we can construct a large number of invariants from the curvature and torsion tensors, and therefore a general gravitational Lagrangian has the complicated form of a sum of all available invariants of proper dimension. The coefficients in the sum can be arranged to obtain different gravitational theories (for some criteria on the election and stability of a large class of these PG Lagrangians see [28–31]).

In this section, we will study a PG theory of gravity, hence it has the same geometrical background explained in section 4, with the following vacuum Lagrangian [32]:

$$\begin{aligned} S = \frac{c^4}{16\pi G} \int d^4x\sqrt{-g}\Big[&2\Lambda - \mathring{R} - \\ &- \frac{1}{2}\left(2c_1 + c_2\right) R_{\lambda\rho\mu\nu}R^{\mu\nu\lambda\rho} + c_1 R_{\lambda\rho\mu\nu}R^{\lambda\rho\mu\nu} + \\ &+ c_2 R_{\lambda\rho\mu\nu}R^{\lambda\mu\rho\nu} + d_1 R_{\mu\nu}\left(R^{\mu\nu} - R^{\nu\mu}\right)\Big]. \end{aligned} \tag{7.1}$$

One interesting feature about this theory is that if we set the torsion to be zero, we recover GR. Then, it is expected to involve slight modifications to the standard theory in terms of the torsion tensor alone.

The field equations can be derived from this action by performing variations with respect to the gauge potentials A_μ, which are related to the translations and Lorentz rotations generators of the Poincaré group in the following way:

$$A_\mu = e^a{}_\mu P_a + \omega^{ab}{}_\mu J_{ab}, \tag{7.2}$$

where $e^a{}_\mu$ is the vierbein field and $\omega^{ab}{}_\mu$ is the spin connection [32]. The generators P_a and J_{ab} follow the usual commutation relations:

$$[P_a,\, P_b] = 0, \tag{7.3}$$

$$[P_a,\, J_{bc}] = i\eta_{a[b}P_{c]}, \tag{7.4}$$

$$[J_{ab},\, J_{cd}] = \frac{i}{2}\left(\eta_{ad}J_{bc} + \eta_{cb}J_{ad} - \eta_{db}J_{ac} - \eta_{ac}J_{bd}\right). \tag{7.5}$$

With that procedure we obtain the field equations:

$$\begin{aligned} \mathring{G}_\mu{}^\nu = &-\frac{1}{2}\Lambda\delta_\mu{}^\nu + 2c_1 T1_\mu{}^\nu + c_2 T2_\mu{}^\nu - \\ &- \left(2c_1 + c_2\right) T3_\mu{}^\nu + d_1\left(H1_\mu{}^\nu - H2_\mu{}^\nu\right) \end{aligned} \tag{7.6}$$

and

$$2c_1 C1_{[\mu\lambda]}{}^\nu - c_2 C2_{[\mu\lambda]}{}^\nu + \left(2c_1 + c_2\right) C3_{[\mu\lambda]}{}^\nu - d_1\left(Y1_{[\mu\lambda]}{}^\nu - Y2_{[\mu\lambda]}{}^\nu\right) = 0, \tag{7.7}$$

where the functions T, H, C, Y depend on the Riemann and torsion tensor and their contractions:

$$
\begin{aligned}
T1_{\mu}{}^{\nu} &\equiv R_{\lambda\rho\mu\sigma}R^{\lambda\rho\nu\sigma} - \frac{1}{4}\delta_{\mu}{}^{\nu}R_{\lambda\rho\alpha\sigma}R^{\lambda\rho\alpha\sigma}, \\
T2_{\mu}{}^{\nu} &\equiv R_{\lambda\rho\mu\sigma}R^{\lambda\nu\rho\sigma} + R_{\lambda\rho\sigma\mu}R^{\lambda\sigma\rho\nu} - \frac{1}{2}\delta_{\mu}{}^{\nu}R_{\lambda\rho\alpha\sigma}R^{\lambda\alpha\rho\sigma}, \\
T3_{\mu}{}^{\nu} &\equiv R_{\lambda\rho\mu\sigma}R^{\nu\sigma\lambda\rho} - \frac{1}{4}\delta_{\mu}{}^{\nu}R_{\lambda\rho\alpha\sigma}R^{\alpha\sigma\lambda\rho}, \\
H1_{\mu}{}^{\nu} &\equiv R^{\nu}{}_{\lambda\mu\rho}R^{\lambda\rho} + R_{\lambda\mu}R^{\lambda\nu} - \frac{1}{2}\delta_{\mu}{}^{\nu}R_{\lambda\rho}R^{\lambda\rho}, \\
H2_{\mu}{}^{\nu} &\equiv R^{\nu}{}_{\lambda\mu\rho}R^{\rho\lambda} + R_{\lambda\mu}R^{\nu\lambda} - \frac{1}{2}\delta_{\mu}{}^{\nu}R_{\lambda\rho}R^{\rho\lambda}, \\
C1_{\mu}{}^{\lambda\nu} &\equiv \mathring{\nabla}_{\rho}R_{\mu}{}^{\lambda\rho\nu} + K^{\lambda}{}_{\sigma\rho}R_{\mu}{}^{\sigma\rho\nu} - K^{\sigma}{}_{\mu\rho}R_{\sigma}{}^{\lambda\rho\nu}, \\
C2_{\mu}{}^{\lambda\nu} &\equiv \mathring{\nabla}_{\rho}\left(R_{\mu}{}^{\nu\lambda\rho} - R_{\mu}{}^{\rho\lambda\nu}\right) + K^{\lambda}{}_{\sigma\rho}\left(R_{\mu}{}^{\nu\sigma\rho} - R_{\mu}{}^{\rho\sigma\nu}\right) - K^{\sigma}{}_{\mu\rho}\left(R_{\sigma}{}^{\nu\lambda\rho} - R_{\sigma}{}^{\rho\lambda\nu}\right), \\
C3_{\mu}{}^{\lambda\nu} &\equiv \mathring{\nabla}_{\rho}R^{\rho\nu\lambda}{}_{\mu} + K^{\lambda}{}_{\sigma\rho}R^{\rho\nu\sigma}{}_{\mu} - K^{\sigma}{}_{\mu\rho}R^{\rho\nu\lambda}{}_{\sigma}, \\
Y1_{\mu}{}^{\lambda\nu} &\equiv \delta_{\mu}{}^{\nu}\mathring{\nabla}_{\rho}R^{\lambda\rho} - \mathring{\nabla}_{\mu}R^{\lambda\nu} + \delta_{\mu}{}^{\nu}K^{\lambda}{}_{\sigma\rho}R^{\sigma\rho} + K^{\rho}{}_{\mu\rho}R^{\lambda\nu} - K^{\nu}{}_{\mu\rho}R^{\lambda\rho} - K^{\lambda}{}_{\rho\mu}R^{\rho\nu}, \\
Y2_{\mu}{}^{\lambda\nu} &\equiv \delta_{\mu}{}^{\nu}\mathring{\nabla}_{\rho}R^{\rho\lambda} - \mathring{\nabla}_{\mu}R^{\nu\lambda} + \delta_{\mu}{}^{\nu}K^{\lambda}{}_{\sigma\rho}R^{\rho\sigma} + K^{\rho}{}_{\mu\rho}R^{\nu\lambda} - K^{\nu}{}_{\mu\rho}R^{\rho\lambda} - K^{\lambda}{}_{\rho\mu}R^{\nu\rho}.
\end{aligned}
\tag{7.8}
$$

As we have explained, the only difference between this theory and EC are the fields equations. This means that the curvature conditions remain the same, and so does the proposition about the appearance of black holes. Nevertheless, the energy conditions change.

In equation (7.6) we have already isolated the Levi-Civita Einstein tensor $\mathring{G}$, therefore we can consider the right side of the equation as an effective energy-momentum tensor

$$\mathring{G}_{\mu\nu} = \mathcal{T}_{\mu\nu}. \tag{7.9}$$

This leads us to the energy conditions for this theory:

- Strong energy condition:
$$\mathcal{T}_{\mu\nu}v^{\mu}v^{\nu} \geq \frac{1}{2}\mathcal{T} \tag{7.10}$$
for every timelike vector v^{μ}.

- Weak energy condition:
$$\mathcal{T}_{\mu\nu}v^{\mu}v^{\nu} \geq 0 \tag{7.11}$$
for every null vector v^{μ}.

These conditions depend on some intricate functions of the curvature tensor, and it makes us think that probably it is better in this case (and also in EC) to evaluate the conditions directly calculating the torsion-free Riemann and Ricci tensor of the considered metric. However, expressing them in this form makes us realise of some curious facts about the theory.

It is interesting to note that in GR a vacuum solution always meets the energy conditions. In this theory though, the situation is different. For example, we can arrange the coefficients in a way that the spacetime contains a closed trapped submanifold of codimension 2 (closed trapped surface) and yet be a singularity free spacetime. This is impossible for a vacuum solution in GR (if the generic condition holds), since in this kind of solutions the Ricci tensor is identically zero.

Let us now explore a specific case. First, we set all the coefficients to zero except for d_1. Observing the field equations, we see that the second one can be solved by setting the Ricci tensor to be zero. In that case, the first equation is just:

$$\mathring{G}_{\mu\nu} = 0, \tag{7.12}$$

which is the vacumm field equation in GR. This means that flat Ricci solutions ($R_{\mu\nu} = 0$) recover the same metrics that GR. However, this is not true for an arbitrary connection, since the equations that relate the Ricci tensor with the Levi-Civita one must hold. Therefore, this statement would be true for connections that follow the equation

$$\mathring{\nabla}_\sigma K^\sigma{}_{\mu\rho} - \mathring{\nabla}_\rho K^\sigma{}_{\mu\sigma} + K^\sigma{}_{\alpha\sigma} K^\alpha{}_{\mu\rho} - K^\sigma{}_{\alpha\rho} K^\alpha{}_{\mu\sigma} = 0. \tag{7.13}$$

At first sight, one might think that the only solution to this equation is a zero contortion tensor, hence obtaining a torsion-free spacetime. However, let us for example take $K^0{}_{10} = -K^1{}_{00} = 1$ and the rest to be zero. Then it is easy to see that the previous equation holds. Therefore, with a suitable connection we can recover all the metrics of the vacuum solutions of GR in a torsion theory.

The interesting fact is that, although the metrics are the same as in GR, and hence very well known spacetimes that describe satisfactorily many physical situations, the underlying theory is different, and so the matter and energy content and the motion of particles will differ from GR. Nevertheless, as we have seen, we can still apply the GR singularity theorems to scalar fields and photons, and the black hole formalism for the rest of particles. Since the metric is the same, the conditions of the appearance of timelike and null singularities and black/white hole regions would be the same as in GR. So in this case, we can establish that the presence of torsion does not change the singular behaviour of the spacetime.

Although this was a rather special case, it is possible to recover some famous metrics with a more general election of the coefficients. This is the case of a recent solution by two of the authors [32], where a Reissner-Norström solution is found setting the coefficients to be $c_1 = -d_1/4$ and $c_2 = -d_1/2$. Since this is a black hole solution, we can study the singular behaviour of spin particles within this framework.

8 Conclusions

In this work we have studied how to extend the tools used in GR to deduce the appearance of singularities to theories of gravitation that include torsion. In order to study that, we have first reviewed two modern singularity theorems by Senovilla and Galloway. For our purposes, the interesting part about these theorems is the curvature condition that they obtain to predict the existence of focal points of a spacelike submanifold. We have used that result to prove the proposition 3.5, that gives us the necessary conditions for the appearance of black/white hole regions of arbitrary dimension in a spacetime. With that established, we have analysed three particular theories. In the case of TEGR we have obtained equivalent results to GR, although the expression for the curvature tensors change, as one might expect. In EC theory we have seen that for minimally coupled scalar fields and photons we can use the results proved in GR. For the rest of particles, we consider the existence of black/white hole as an indicator of the singular character of their trajectories. In this case we also obtain

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their energy conditions. For the dynamical torsion example we have made a similar analysis of that of EC theory. We have obtained the same geometrical results, although the energy conditions change, leading to some interesting behaviours. For instance, we have shown that in a vacuum solution we can have a violation of the energy conditions, something that cannot happen in GR or EC theory. Furthermore, we analyse a particular Lagrangian and obtain that we can reproduce all the metric structure of the vacuum solutions of GR in theories with torsion.

The formalism that we have developed can be used in other modified gravity theories, as long as the inner structure is a Lorentzian manifold, using the following considerations. As we have already discussed, a minimally coupled scalar field in these theories will follow timelike geodesics, so we can use the singularity theorems of GR that are based on incomplete timelike geodesics, such as the Hawking theorem. On the other hand, we have been using the fact that in the theories that we have considered, photons follow null geodesics. This is not necessarily true for all the torsion theories, since in some of them we can couple the Maxwell equations to torsion non-minimally and still preserve the gauge invariance [33]. Nevertheless, this would mean that we can still use the black hole formalism, because they would not follow spacelike curves. Here we can see how powerful this result is, because it allows us to predict the singular behaviour of any non-spacelike curve, which includes coupled photons, spinning particles or non-minimal coupled fields.

Moreover, we have used the cosmic censorship as a plausible condition in torsion theories. In any case, it would be very interesting to study the possible creation of naked singularities in these theories under physical realistic conditions [35], and to test with concrete examples if spinning particles would reach the black/white hole regions. In order to conclude if the spin can advert singularities in torsion theories, it is useful to work in the semiclassical limit of the Dirac wave function via the WKB approximation, as treated in [34]. Using the equation of motion given by Audretsch we can simulate numerically the movement of spin particles around the event horizon. Work is in progress along this line.

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Stability in quadratic torsion theories

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Abstract We revisit the definition and some of the characteristics of quadratic theories of gravity with torsion. We start from a Lagrangian density quadratic in the curvature and torsion tensors. By assuming that General Relativity should be recovered when the torsion vanishes and investigating the behaviour of the vector and pseudo-vector torsion fields in the weak-gravity regime, we present a set of *necessary* conditions for the stability of these theories. Moreover, we explicitly obtain the gravitational field equations using the Palatini variational principle with the metricity condition implemented via a Lagrange multiplier.

1 Introduction

General relativity (GR) radically changed our understanding of the universe. The predictions of this elegant theory have been confirmed up to the date [1,2]. In order to fit extragalactic and cosmological observational data, however, the presence of a non-vanishing cosmological constant and six times more dark matter than ordinary matter have to be assumed in this framework [3]. In addition, the observed value of this cosmological constant differs greatly from the value expected for the vacuum energy. On the other hand, while the strong and electroweak forces are renormalisable gauge theories, that is not the case for GR, and the compatibility of GR with the quantum realm is still a matter of debate. Given this situation, there has been a renewed interest in alternative theories of gravity, which modify the predictions of GR.

A particular approach to formulating alternative theories of gravity involves an extension of the geometrical treatment that covers the microscopic properties of matter [4]. It should be noted that the mass is not enough to characterize particles at the quantum level given that they have another independent *label*, that is, the spin. Whereas at macroscopic scales the energy-momentum tensor is enough to describe the source of gravity, a description of the spacetime distribution of the spin density is needed at microscopic scales. Moreover, there are macroscopic configurations that may also need a description of the spin distribution, as super-massive objects (e.g. black holes or neutron stars with nuclear polarisation). In this spirit, a new geometrical concept should be related to the spin distribution in the same way that spacetime curvature is related to the energy-momentum distribution. Torsion is a natural candidate for this purpose [4,5] and an important advantage of a theory of gravity with torsion is that it can be formulated as a gauge theory [6–8].

Since 1924 many authors have considered theories of gravity in a Riemann–Cartan U_4 spacetime. In this manifold the non-vanishing torsion can be coupled to the intrinsic spin density of matter and, in this way, the spin part of the Poincaré group can change the geometry of the manifold as the energy-momentum tensor does it. The first attempt to introduce torsion in a theory of gravity was the Einstein–Cartan theory, which is a reformulation of GR in a U_4 spacetime. In this theory the scalar curvature of the Einstein–Hilbert action is constructed from a U_4 connection instead of using the Christoffel symbols. However, the resulting theory was not completely satisfactory because the field equations relate the torsion and its source in an algebraic way and, therefore, torsion is not dynamical. Hence the torsion field vanishes in vacuum and the Einstein–Cartan theory collapses to GR except for unobservable corrections to the energy-momentum tensor [4]. In order to obtain a theory with propagating torsion, we need to consider an action that is at least quadratic in the curvature tensors [4,6–11]. Moreover, an important advantage of adding quadratic terms $\mathcal{R}^2$ to the Einstein–Hilbert action is the possibility of making the theory renormalisable [9]. In addition, it can be shown [4,6] that, considering a gauge description, the torsion and curvature tensors correspond to the field-strength tensors of the gauge potentials of

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the Poincaré group ($e_\mu^{\ a}, w_\mu^{\ ab}$), which are the vierbein and the local Lorentz connection, respectively. Thus, a pure $\mathcal{R}^2$ gauge theory of gravity has some resemblance to electroweak and strong theories.

From an experimental point of view there have been many attempts to detect torsion or to set an upper bound to its gravitational effects. One of the most debated attempts was the use of the Gravity Probe B experiment to measure torsion effects [12]. Nevertheless, this experiment was criticized because torsion will never couple to the gyroscopes installed in the satellite [13]. Therefore, this probe cannot measure the gravitational effects due to torsion. On the other hand, other unsuccessful experiments aimed to constrain torsion with accurate measurements on the perihelion advance and the orbital geodetic effect of a satellite [14]. The experimental difficulty is the need of dealing with elementary particles with spin to obtain a maximal coupling with torsion.

In this paper we present a self-contained introduction to quadratic theories of gravity with torsion in the geometrical approach (gauge treatment is not considered). We partly recover well-known results about the stability of these theories using simple methods. Therefore, we simplify the existent mathematical treatment and reinforce the critical discussion as regards some controversial results published in the literature.

The paper is organized as follows: In Sect. 2 we present a general introduction to the basic concepts on general affine geometries and introduce the conventions used throughout the paper. In Sect. 3 we present our main results. In the first place, we consider a Lagrangian density quadratic in the curvature and torsion tensors. In Sect. 3.1 we discuss the different methods presented in the literature to obtain the field equations and explicitly derive them in the Palatini formalism. In Sect. 3.2 we obtain conditions on the parameters of the Lagrangian necessary to avoid large deviations from GR and instabilities. Then, in Sect. 3.3, we analyse the Lagrangian density with the aim of setting necessary conditions for avoiding ghost and tachyon instabilities. The conclusions are summarized in Sect. 4. We relegate some calculations and further comments to the appendices: in Appendix A we include the Gauss–Bonnet term in Riemann–Cartan geometries; in Appendix B we include detailed expressions necessary to obtain the equations of the dynamics using the Palatini formalism; in Appendix C we discuss the source terms of these equations; and, in Appendix D, we include relevant expressions for the study of the vector and pseudo-vector torsion fields around Minkowski.

2 Basic concepts and conventions

The geometric structure of a manifold can be catalogued by the properties of the affine connection. A general affine connection $\tilde{\Gamma}$ provides three main characteristics: curvature, torsion, and non-metricity. Combinations of these quantities in the affine connection generate the geometric structure [5]. In GR it is assumed that the spacetime geometry is described by a Riemannian manifold, thus the affine connection reduces to the so-called Levi-Civita connection and the gravitational effects are only produced by the consequent curvature in terms of the metric tensor alone. Nevertheless, in a general geometrical theory of gravity the gravitational effects are generated by the whole connection, which involves a post-Riemannian approach described by curvature, torsion and non-metricity. In this scheme, there are many ways to deal with torsion and non-metricity due to different conventions. For that reason, it is important to set the conventions and definitions used throughout this work. Thus, the notation assumed for the symmetric and the antisymmetric part of a tensor A is

$$A_{(\mu_1 \ldots \mu_s)} \equiv \frac{1}{s!} \sum_{\pi \in P(s)} A_{\pi(\mu_1)\cdots\pi(\mu_s)}, \tag{1}$$

$$A_{[\mu_1 \ldots \mu_s]} \equiv \frac{1}{s!} \sum_{\pi \in P(s)} \mathrm{sgn}(\pi) A_{\pi(\mu_1)\cdots\pi(\mu_s)}, \tag{2}$$

respectively, where $P(s)$ is the set of all the permutations of $1, \ldots, s$ and $\mathrm{sgn}(\pi)$ is positive for even permutations whereas it is negative for odd permutations.

In the first place, the Cartan torsion is defined as the antisymmetric part of the affine connection as [4,15–17]

$$T^{\mu}_{\cdot\nu\sigma} \equiv \tilde{\Gamma}^{\mu}_{\cdot[\nu\sigma]}. \tag{3}$$

Note that a dot appears below the index μ to indicate the position that it takes when it is lowered with the metric. As the difference of two connections transforms as a tensor, the Cartan torsion is a tensor. Thus, from now on we call it just the torsion and emphasise that it cannot be eliminated with a suitable change of coordinates.

In the second place, non-metricity can also be described by a third rank tensor. This is

$$Q_{\rho\mu\nu} \equiv \tilde{\nabla}_\rho g_{\mu\nu}, \tag{4}$$

where $\tilde{\nabla}$ is the covariant derivative defined from the affine connection $\tilde{\Gamma}$. The non-metricity tensor is usually split into a trace vector $\omega_\rho \equiv \frac{1}{4} Q_{\rho\nu\cdot}^{\ \ \nu}$, called the Weyl vector [18], and a traceless part $\overline{Q}_{\rho\mu\nu}$,

$$Q_{\rho\mu\nu} = \omega_\rho g_{\mu\nu} + \overline{Q}_{\rho\mu\nu}. \tag{5}$$

It should be noted that there are manifolds with non-metricity where the cancellation of the ω_ρ or the traceless part of Q are demanded.

Since the general connection $\tilde{\Gamma}$ is asymmetric in the last two indices, a convention is needed for the covariant derivative of a tensor. Let $A^{\mu_1\cdots\mu_r}_{\cdot\;\cdots\;\cdot\;\nu_1\cdots\nu_s}$ be the components of a tensor type (r, s), then

$$\begin{aligned}\tilde{\nabla}_\rho A^{\mu_1\cdots\mu_r}_{\cdot\;\cdots\;\cdot\;\nu_1\cdots\nu_s} \equiv&\ \partial_\rho A^{\mu_1\cdots\mu_r}_{\cdot\;\cdots\;\cdot\;\nu_1\cdots\nu_s} \\ &+ \sum_{i=1}^{r} \tilde{\Gamma}^{\mu_i}_{\cdot\lambda\rho} A^{\mu_1\cdot\lambda\cdot\mu_r}_{\cdot\;\cdots\;\cdot\;\nu_1\cdots\nu_s} \\ &- \sum_{j=1}^{s} \tilde{\Gamma}^{\lambda}_{\cdot\nu_j\rho} A^{\mu_1\cdots\mu_r}_{\cdot\;\cdots\;\cdot\;\nu_1\cdot\lambda\cdot\nu_s}. \end{aligned} \tag{6}$$

It is important to emphasise the syntax of the lower indices in the affine connections, that is, the index ρ of the derivative is written in the last position in the affine connection.

Using the definitions presented in this section, the general connection $\tilde{\Gamma}$ is written as [4,15,19]

$$\tilde{\Gamma}^{\mu}_{\cdot\nu\sigma} = \Gamma^{\mu}_{\cdot\nu\sigma} + W^{\mu}_{\cdot\nu\sigma}, \tag{7}$$

with $\Gamma^{\mu}_{\cdot\nu\sigma}$ the Levi-Civita connection,

$$\Gamma^{\mu}_{\cdot\nu\sigma} = \frac{1}{2} g^{\mu\rho} \Delta^{\alpha\beta\gamma}_{\sigma\nu\rho} \partial_\alpha g_{\beta\gamma}, \tag{8}$$

which is expressed in a compact form by the permutation tensor [20]

$$\Delta^{\alpha\beta\gamma}_{\sigma\nu\rho} = \delta_\sigma^{\ \alpha}\delta_\nu^{\ \beta}\delta_\rho^{\ \gamma} + \delta_\nu^{\ \alpha}\delta_\rho^{\ \beta}\delta_\sigma^{\ \gamma} - \delta_\rho^{\ \alpha}\delta_\sigma^{\ \beta}\delta_\nu^{\ \gamma}, \tag{9}$$

and the additional tensor $W^{\mu}_{\cdot\nu\sigma}$ defined by the following expression:

$$W^{\mu}_{\cdot\nu\sigma} = K^{\mu}_{\cdot\nu\sigma} + \frac{1}{2}\left(Q^{\mu}_{\cdot\nu\sigma} - Q_{\sigma\cdot\nu}^{\ \mu} - Q_{\nu\cdot\sigma}^{\ \mu}\right), \tag{10}$$

where $K^{\mu}_{\cdot\nu\sigma}$ is called the contortion tensor,

$$K^{\mu}_{\cdot\nu\sigma} = T^{\mu}_{\cdot\nu\sigma} - T_{\nu\cdot\sigma}^{\ \mu} - T_{\sigma\cdot\nu}^{\ \mu}. \tag{11}$$

Note that $Q_{\rho\mu\nu}$ is symmetric in the last two indices, while $T^{\mu}_{\cdot\nu\sigma}$ is antisymmetric in these indices. However, the contortion, $K^{\mu}_{\cdot\nu\sigma}$, is antisymmetric in the first pair of indices. This property ensures the existence of a metric-compatible connection when the non-metricity tensor vanishes.

Furthermore, it is useful to write the torsion through its three irreducible components. These are [19]

(i) the trace vector $T^{\mu}_{\cdot\nu\mu} \equiv T_\nu$;
(ii) the pseudo-trace axial vector $S^\nu \equiv \epsilon^{\alpha\beta\sigma\nu} T_{\alpha\beta\sigma}$;
(iii) the tensor $q^{\alpha}_{\cdot\beta\sigma}$, which satisfies $q^{\alpha}_{\cdot\beta\alpha} = 0$ and $\epsilon^{\alpha\beta\sigma\nu} q_{\alpha\beta\sigma} = 0$.

Thus, the torsion field can be rewritten as

$$T^{\alpha}_{\cdot\beta\mu} = \frac{1}{3}(T_\beta\delta^{\alpha}_{\ \mu} - T_\mu\delta^{\alpha}_{\ \beta}) + \frac{1}{6} g^{\alpha\sigma}\epsilon_{\sigma\beta\mu\nu}S^\nu + q^{\alpha}_{\cdot\beta\mu}. \tag{12}$$

The introduction of these new geometrical degrees of freedom leads to the generalisation of the usual definition of the curvature tensor in the Riemann spacetime, $[\nabla_\rho, \nabla_\sigma]V^\mu = R^{\mu}_{\cdot\nu\rho\sigma}V^\nu$, by the following commutative relations associated with a connection $\tilde{\Gamma}$:

$$[\tilde{\nabla}_\rho, \tilde{\nabla}_\sigma]V^\mu = \tilde{R}^{\mu}_{\cdot\nu\rho\sigma}V^\nu + 2T^{\alpha}_{\cdot\rho\sigma}\tilde{\nabla}_\alpha V^\mu, \tag{13}$$

where the curvature tensor reads

$$\tilde{R}^{\mu}_{\cdot\nu\rho\sigma} = \partial_\rho\tilde{\Gamma}^{\mu}_{\cdot\nu\sigma} - \partial_\sigma\tilde{\Gamma}^{\mu}_{\cdot\nu\rho} + \tilde{\Gamma}^{\mu}_{\cdot\lambda\rho}\tilde{\Gamma}^{\lambda}_{\cdot\nu\sigma} - \tilde{\Gamma}^{\mu}_{\cdot\lambda\sigma}\tilde{\Gamma}^{\lambda}_{\cdot\nu\rho}. \tag{14}$$

Using Eq. (7), the curvature tensor can be rewritten as

$$\begin{aligned}\tilde{R}^{\mu}_{\cdot\nu\rho\sigma} =&\ R^{\mu}_{\cdot\nu\rho\sigma} + \nabla_\rho W^{\mu}_{\cdot\nu\sigma} - \nabla_\sigma W^{\mu}_{\cdot\nu\rho} + W^{\mu}_{\cdot\lambda\rho}W^{\lambda}_{\cdot\nu\sigma} \\ &- W^{\mu}_{\cdot\lambda\sigma}W^{\lambda}_{\cdot\nu\rho},\end{aligned} \tag{15}$$

with $R^{\mu}_{\cdot\nu\rho\sigma}$ the curvature tensor of the Riemann spacetime, commonly called the Riemann tensor, and ∇ the covariant derivative constructed from the Levi-Civita connection.

On the other hand, the generalisation of the two Bianchi identities can be computed from Eq. (14). Taking into account Eq. (3), the new Bianchi identities are

$$\tilde{R}^{\mu}_{\cdot[\nu\rho\sigma]} = 2\tilde{\nabla}_{[\rho}T^{\mu}_{\cdot\nu\sigma]} - 4T^{\lambda}_{\cdot[\nu\rho}T^{\mu}_{\cdot\sigma]\lambda}, \tag{16}$$

$$\tilde{\nabla}_{[\mu|}\tilde{R}^{\alpha}_{\cdot\beta|\nu\rho]} = -2T^{\lambda}_{\cdot[\mu\nu|}\tilde{R}^{\alpha}_{\cdot\beta|\rho]\lambda}. \tag{17}$$

Moreover, it is well known that not all the components of the curvature tensor (14) are independent. By definition, this tensor is antisymmetric in the last pair of indices $\tilde{R}^{\mu}_{\cdot\nu\rho\sigma} = \tilde{R}^{\mu}_{\cdot\nu[\rho\sigma]}$. A simple calculation using Eq. (15) shows that

$$\tilde{R}_{(\mu\nu)\rho\sigma} = \nabla_{[\rho}Q_{\sigma]\mu\nu} + T^{\lambda}_{\cdot\rho\sigma}Q_{\lambda\mu\nu}. \tag{18}$$

Thus, when the connection is set to be metric-compatible, the curvature tensor is also antisymmetric in the first pair of indices. The symmetry of the curvature tensor under the exchange of pair of indices depends on the torsion and non-metricity tensors. In general, for non-trivial values for those tensors, this symmetry does not hold. However, there are particular conditions under which the exchange symmetry is recovered for non-trivial values.

From now on we consider a metric-compatible connection, focusing our attention only on curvature and torsion. We denote by a hat the objects constructed from a metric-compatible connection with torsion:

$$\widehat{\Gamma} \equiv \tilde{\Gamma}\Big|_{Q=0}. \tag{19}$$

All the conventions and identities that we have already presented are, of course, still valid. The Ricci tensor and the scalar curvature are obtained with the usual contractions, $\widehat{R}_{\mu\nu} = \widehat{R}^{\sigma}_{\cdot\mu\sigma\nu}$ and $\widehat{R} = g^{\mu\nu}\widehat{R}_{\mu\nu}$. However, the absence of symmetry in the exchange of pair of indices in Eq. (14) allows the Ricci tensor $\widehat{R}_{\mu\nu}$ to be non-symmetric. Indeed, the antisymmetric part of this tensor is

$$\widehat{R}_{[\mu\nu]} = \widehat{\nabla}_{\rho}(T^{\rho}_{\cdot\mu\nu} + \delta^{\rho}_{\ \mu}T_{\nu} - \delta^{\rho}_{\ \nu}T_{\mu}) - 2T_{\rho}T^{\rho}_{\cdot\mu\nu}\,. \tag{20}$$

In view of this identity, a modified torsion tensor can be defined

$$\overset{\star}{T}{}^{\rho}_{\cdot\mu\nu} \equiv T^{\rho}_{\cdot\mu\nu} + \delta^{\rho}_{\ \mu}T_{\nu} - \delta^{\rho}_{\ \nu}T_{\mu}, \tag{21}$$

and a modified covariant derivative can be introduced,

$$\overline{\nabla}_{\rho} \equiv \widehat{\nabla}_{\rho} - 2T_{\rho}. \tag{22}$$

Hence the antisymmetric part of the Ricci tensor is rewritten as

$$R_{[\mu\nu]} = \overline{\nabla}_{\rho}\overset{\star}{T}{}^{\rho}_{\cdot\mu\nu}. \tag{23}$$

It should be stressed the importance of this modified derivative for vectors, since $\partial_{\mu}(\sqrt{-g}A^{\mu}) = \sqrt{-g}\,\overline{\nabla}_{\mu}A^{\mu}$, for any vector A^{μ}.

3 Quadratic theory of gravity

As we have already argued in the introduction, we are going to consider an action that is quadratic in the curvature tensor, in order to obtain a theory with propagating torsion [4,6–11]. Excluding parity violating pieces, a total of six independent scalars can be formed from the curvature tensor (14) and its contractions. In addition, three other scalars can be constructed from the torsion tensor (3). On the other hand, the Gauss–Bonnet action is known to lead to a total divergence in a 4-dimensional Riemannian manifold and, therefore, it does not produce any contribution through the variational process of the action. It is worth noting that the Gauss–Bonnet Lagrangian does not contribute to the field equations even in a Riemann–Cartan geometry [6,21].[1] Therefore, the terms $\widehat{R}^2$, $\widehat{R}_{\nu\sigma}\widehat{R}^{\sigma\nu}$, and $\widehat{R}_{\mu\nu\rho\sigma}\widehat{R}^{\rho\sigma\mu\nu}$ in the Lagrangian density are not independent. Throughout this work, we are going to consider the quadratic Lagrangian density from Poincaré gauge theory of gravity, as written in Refs. [6,7,10,11]. This is

$$\begin{aligned}\mathcal{L}_g = &-\lambda\widehat{R} + \frac{1}{12}(4a+b+3\lambda)T_{\mu\nu\rho}T^{\mu\nu\rho}\\ &+\frac{1}{6}(-2a+b-3\lambda)T_{\mu\nu\rho}T^{\nu\rho\mu}\\ &+\frac{1}{3}(-a+2c-3\lambda)T^{\lambda}_{\cdot\mu\lambda}T^{\cdot\mu\rho}_{\rho}\\ &+\frac{1}{6}(2p+q)\widehat{R}_{\mu\nu\rho\sigma}\widehat{R}^{\mu\nu\rho\sigma}\\ &+\frac{1}{6}(2p+q-6r)\widehat{R}_{\mu\nu\rho\sigma}\widehat{R}^{\rho\sigma\mu\nu}\\ &+\frac{2}{3}(p-q)\widehat{R}_{\mu\nu\rho\sigma}\widehat{R}^{\mu\rho\nu\sigma}\\ &+(s+t)\widehat{R}_{\nu\sigma}\widehat{R}^{\nu\sigma} + (s-t)\widehat{R}_{\nu\sigma}\widehat{R}^{\sigma\nu}\,,\end{aligned} \tag{24}$$

with $\lambda, a, b, c, p, q, r, s$ and t the free parameters of the theory. The particular combinations of the parameters that appear in the Lagrangian density have been chosen for convenience without loss of generality. Note that the scalar curvature is also included, which is the only term present in the Einstein–Cartan theory. The procedure to obtain the field equations of this Lagrangian density is summarized in Sect. 3.1. In addition, parity violating pieces can also be assumed in a natural way in the Lagrangian density leading to interesting results; see Refs. [8,22].

In this work we are interested in the stability of theories of gravity with dynamical torsion that avoid large deviations from the predictions of GR where this theory is satisfactory. In this spirit, we focus on quadratic theories, because that is the minimal modification leading to dynamical torsion, and we will not assume that all the components obtained by the irreducible decomposition of the torsion necessarily propagate. In order to study the stability of the theory, we will focus on two regimes where the metric and torsion degrees of freedom completely decoupled from each other through the consideration of the following conditions:

(a) *GR must be recovered when the torsion vanishes.*
(b) *The theory must be stable in the weak-gravity regime.*

Note that condition (a) implies both that the general relativistic predictions will be recovered when the torsion is small and that the theory is stable at least when the torsion vanishes. This condition will be imposed in Sect. 3.2 by means of the geometrical structure of the manifold, whereas the second condition will be investigated in Sect. 3.3 considering the propagation of the torsion modes in a Minkowki space. Both conditions have been studied separately in the literature using different approaches; see Refs. [6–8].

3.1 Field equations

The field equations of the Lagrangian density (24) have to be obtained, as usual, from a variational principle where the

[1] We include the definition of the Gauss–Bonnet action in the presence of the torsion and check this property in Appendix A, since incompatible definitions are used throughout the literature.

action is extremised with respect to the dynamical variables. However, different sets of dynamical variables can be chosen and different field equations will be obtained accordingly. On one hand, the metric and the affine connection can be taken as completely independent variables. Then the field equations are obtained from varying the action with respect $g^{\mu\nu}$ and $\tilde{\Gamma}^{\sigma}_{\cdot\mu\nu}$. This is called the Palatini formalism.[2] On the other hand, the connection can be taken to be metric-compatible from the beginning. Hence, the field equations are obtained varying with respect to g and T, or to g and K. This procedure is sometimes called the metric or Hilbert variational method. The Palatini and Hilbert methods are known to differ only on the constraint on the symmetric part of the connection $\tilde{\Gamma}_{(s)}{}^{\sigma}_{\cdot\mu\nu} = \Gamma^{\sigma}_{\cdot\mu\nu} - T_{\nu\cdot\sigma}^{\;\;\mu} - T_{\sigma\cdot\nu}^{\;\;\mu}$; that is, they differ on a Lagrange multiplier for the metricity condition, see Refs. [23,24]. Therefore, the two methods coincide without imposing the Lagrange multiplier when after solving the field equations the related quantity turns out to be zero. In addition, a third method consists in treating the theory as a gauge theory. This may be seen as being more natural, since the variables are the gauge potentials $(e_{\mu}^{\;a}, w_{\mu}^{\;ab})$. The field equations in this formalism can been found in Refs. [8,10].

Let us use the Palatini formalism with the metricity condition implemented as a constraint via a Lagrange multiplier Λ to obtain the field equations. The total Lagrangian density of the theory can by written as

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_M + \Lambda_{\nu\mu\cdot}^{\;\;\;\rho}\tilde{\nabla}_\rho g^{\mu\nu}, \tag{25}$$

with $\mathcal{L}_g$ from Eq. (24), $\mathcal{L}_M$ the Lagrangian density for matter fields minimally coupled to gravity, and $\Lambda_{\nu\mu\cdot}^{\;\;\;\rho}$ a Lagrange multiplier. The use of the Lagrange multipliers in theories of gravity has been studied in Refs. [20,25,26]. For the sake of simplicity, we rewrite the Lagrangian density $\mathcal{L}_g$ as

$$\begin{aligned}\mathcal{L}_g = &-\lambda\,\delta_\alpha^{\;\gamma} g^{\beta\delta}\tilde{R}^{\alpha}_{\cdot\beta\gamma\delta} + \underset{T}{f}{}_{\lambda\alpha}^{\;\;\eta\rho\beta\gamma}\, T^{\lambda}_{\cdot\eta\rho}T^{\alpha}_{\cdot\beta\gamma}\\ &+ \underset{R}{f}{}_{\lambda\alpha}^{\;\;\eta\rho\sigma\beta\gamma\delta}\,\tilde{R}^{\lambda}_{\cdot\eta\rho\sigma}\tilde{R}^{\alpha}_{\cdot\beta\gamma\delta},\end{aligned} \tag{26}$$

with the permutation tensors $\underset{T}{f}{}_{\lambda\alpha}^{\;\;\eta\rho\beta\gamma}$ and $\underset{R}{f}{}_{\lambda\alpha}^{\;\;\eta\rho\sigma\beta\gamma\delta}$ defined in Appendix B. This decomposition factorizes $\mathcal{L}_g$ in parts depending purely on the metric and parts depending on the connection—those are the permutation tensors, and the curvature tensors and the torsion tensors, respectively; thus, the application of the Euler–Lagrange equations is straightforward. The field equations for the Lagrangian density (25) are

$$\tilde{\mathcal{E}}_{\mu\nu} - (\tilde{\nabla}_\kappa - 2T_\kappa)\Lambda_{\nu\mu\cdot}^{\;\;\;\kappa} - \frac{1}{2}\Lambda_{\mu\nu\cdot}^{\;\;\;\kappa} g^{\alpha\beta}\tilde{\nabla}_\kappa g_{\alpha\beta} = \tilde{\tau}_{\mu\nu}, \tag{27}$$

$$\tilde{\mathcal{P}}_\tau^{\cdot\mu\nu} + 2\Lambda_\tau^{\cdot\mu\nu} = \tilde{\Sigma}_\tau^{\cdot\mu\nu}, \tag{28}$$

$$\tilde{\nabla}_\rho g^{\mu\nu} = 0. \tag{29}$$

Note that the metricity condition is obtained as a field equation from the variation of the action with respect to the Lagrange multiplier. The definitions used in the above equations are

$$\tilde{\mathcal{E}}_{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}\mathcal{L}_g}{\partial g^{\mu\nu}}, \tag{30}$$

$$\tilde{\mathcal{P}}_\tau^{\cdot\mu\nu} \equiv \frac{\partial\mathcal{L}_g}{\partial\tilde{\Gamma}^{\tau}_{\cdot\mu\nu}} - \frac{1}{\sqrt{-g}}\partial_\kappa\left(\sqrt{-g}\frac{\partial\mathcal{L}_g}{\partial(\partial_\kappa\tilde{\Gamma}^{\tau}_{\cdot\mu\nu})}\right). \tag{31}$$

The tensor $\tilde{\mathcal{E}}_{\mu\nu}$ could be considered as the generalisation of the Einstein tensor for the Lagrangian density $\mathcal{L}_g$, as it contains the dynamical information of the metric. Analogously, the tensor $\tilde{\mathcal{P}}_\tau^{\cdot\mu\nu}$ is the generalisation of the Palatini tensor. The source tensors are the energy-momentum tensor

$$\tilde{\tau}_{\mu\nu} \equiv -\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}\mathcal{L}_M(g,\tilde{\Gamma},\Psi)}{\partial g^{\mu\nu}}, \tag{32}$$

and the hypermomentum tensor

$$\tilde{\Sigma}_\tau^{\cdot\mu\nu} \equiv -\frac{\partial\mathcal{L}_M(g,\tilde{\Gamma},\Psi)}{\partial\tilde{\Gamma}^{\tau}_{\cdot\mu\nu}}, \tag{33}$$

as defined in Refs. [20,27].

Now, taking into account the expression of $\mathcal{L}_g$ in Eq. (26), the generalized Einstein and Palatini tensors are

$$\begin{aligned}\tilde{\mathcal{E}}_{\mu\nu} = &-\lambda\tilde{G}_{(\mu\nu)} + \left(\frac{\partial \underset{T}{f}{}_{\lambda\alpha}^{\;\;\eta\rho\beta\gamma}}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\underset{T}{f}{}_{\lambda\alpha}^{\;\;\eta\rho\beta\gamma}\right)T^{\lambda}_{\cdot\eta\rho}T^{\alpha}_{\cdot\beta\gamma}\\ &+\left(\frac{\partial \underset{R}{f}{}_{\lambda\alpha}^{\;\;\eta\rho\sigma\beta\gamma\delta}}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\underset{R}{f}{}_{\lambda\alpha}^{\;\;\eta\rho\sigma\beta\gamma\delta}\right)\tilde{R}^{\lambda}_{\cdot\eta\rho\sigma}\tilde{R}^{\alpha}_{\cdot\beta\gamma\delta},\end{aligned} \tag{34}$$

where $\tilde{G}_{(\mu\nu)}$ is the symmetric part of the Einstein tensor, and

$$\begin{aligned}\tilde{\mathcal{P}}_\tau^{\cdot\mu\nu} = &-2\lambda\left[\overset{\star}{T}{}^{\nu\mu\cdot}_{\;\;\;\;\sigma} + \delta_\sigma^{\;\nu}\left(\tilde{\nabla}_\lambda g^{\mu\lambda} + \frac{1}{2}g^{\alpha\beta}\tilde{\nabla}^\mu g_{\alpha\beta}\right)\right.\\ &\left.- \tilde{\nabla}_\sigma g^{\mu\nu} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\tilde{\nabla}_\sigma g_{\alpha\beta}\right]\\ &+2\underset{T}{f}{}_{\lambda\alpha}^{\;\;\eta\rho\beta\gamma}T^{\lambda}_{\cdot\eta\rho}\frac{\partial T^{\alpha}_{\cdot\beta\gamma}}{\partial\tilde{\Gamma}^{\tau}_{\cdot\mu\nu}} + 2\underset{R}{f}{}_{\lambda\alpha}^{\;\;\eta\rho\sigma\beta\gamma\delta}\tilde{R}^{\lambda}_{\cdot\eta\rho\sigma}\frac{\partial\tilde{R}^{\alpha}_{\cdot\beta\gamma\delta}}{\partial\tilde{\Gamma}^{\tau}_{\cdot\mu\nu}}\\ &-\frac{2}{\sqrt{-g}}\partial_\kappa\left(\sqrt{-g}\,\underset{R}{f}{}_{\lambda\alpha}^{\;\;\eta\rho\sigma\beta\gamma\delta}\tilde{R}^{\lambda}_{\cdot\eta\rho\sigma}\frac{\partial\tilde{R}^{\alpha}_{\cdot\beta\gamma\delta}}{\partial\left(\partial_\kappa\tilde{\Gamma}^{\tau}_{\cdot\mu\nu}\right)}\right),\end{aligned} \tag{35}$$

[2] It should be stressed that, for the Palatini method, the general connection $\tilde{\Gamma}$ should be considered. Then the conditions of metricity and of being torsion-free must be implemented via Lagrange multipliers.

respectively. The full expressions of these tensors in terms of the free parameters of the Lagrangian density are shown in Appendix B.

As the metricity condition has arisen as a field equation, from now on we can consider a metric-compatible connection $\widehat{\Gamma}$. Then the field equations (27) and (28) reduce to

$$\widehat{\mathcal{E}}_{\mu\nu} - \overline{\nabla}_\kappa \Lambda_{\nu\mu\cdot}^{\ \ \ \kappa} = \widehat{\tau}_{\mu\nu} \tag{36}$$

$$\widehat{\mathcal{P}}_\tau^{\cdot\mu\nu} + 2\Lambda_\tau^{\cdot\mu\nu} = \widehat{\Sigma}_\tau^{\cdot\mu\nu} . \tag{37}$$

To obtain the final expression for the field equations, the Lagrange multiplier Λ must be solved from Eqs. (36) and (37). To this end, note that a generic third rank tensor A can always be written as

$$A_{\alpha\beta\gamma} = \Delta^{\mu\nu\rho}_{\beta\alpha\gamma}\left(A_{\mu(\nu\rho)} - A_{[\mu\nu]\rho}\right) \tag{38}$$

where $\Delta^{\mu\nu\rho}_{\beta\alpha\gamma}$ is defined in Eq. (9). As $\Lambda_{\nu\mu\cdot}^{\ \ \ \rho}$ is symmetric in the first two indices, we can solve from Eq. (36)

$$\Lambda_{\mu\nu\rho} = \frac{1}{2}\Delta^{\alpha\beta\gamma}_{\nu\mu\rho}\left(\widehat{\Sigma}_{\alpha(\beta\gamma)} - \widehat{\mathcal{P}}_{\alpha(\beta\gamma)}\right) . \tag{39}$$

Thus, the field equations become

$$\widehat{\mathcal{E}}_{\mu\nu} - \frac{1}{2}\Delta^{\alpha\beta\gamma}_{\nu\mu\kappa}\overline{\nabla}^\kappa\left(\widehat{\Sigma}_{\alpha(\beta\gamma)} - \widehat{\mathcal{P}}_{\alpha(\beta\gamma)}\right) = \widehat{\tau}_{\mu\nu} , \tag{40}$$

$$\Delta^{\alpha\beta\gamma}_{\nu\mu\kappa}\left(\widehat{\Sigma}_{[\alpha\beta]\gamma} - \widehat{\mathcal{P}}_{[\alpha\beta]\gamma}\right) = 0. \tag{41}$$

These are the general expressions of the field equations of any theory of gravity with metricity and torsion. This set of equations is obviously equivalent to the equations obtained from a Hilbert variational principle over the variables (g, K) or (g, T), as can easily be checked. Now, taking into account the calculations showed in Appendix B for the Lagrangian density (24), these equations are

$$\begin{aligned}
&-\lambda\left(\widehat{G}_{(\mu\nu)} - 2\overline{\nabla}^\kappa \overset{\star}{T}_{(\mu\nu)\kappa}\right)\\
&\quad + \frac{1}{12}(4a+b+3\lambda)\left(2T_{\alpha\beta\mu}T^{\alpha\beta\cdot}_{\ \ \ \nu} - T_{\mu\alpha\beta}T_\nu^{\cdot\alpha\beta}\right.\\
&\quad \left. - \frac{1}{2}g_{\mu\nu}T_{\alpha\beta\rho}T^{\alpha\beta\rho}\right)\\
&\quad + \frac{1}{6}(-2a+b-3\lambda)\left(T_{\alpha\beta\mu}T^{\beta\alpha\cdot}_{\ \ \ \nu} - \frac{1}{2}g_{\mu\nu}T_{\alpha\beta\rho}T^{\beta\rho\alpha}\right)\\
&\quad + \frac{1}{3}(-a+2c-3\lambda)\left(T_\mu T_\nu - \frac{1}{2}g_{\mu\nu}T_\alpha T^\alpha\right)\\
&\quad + \frac{1}{6}(2p+q)\left[2\widehat{R}_{\alpha\beta\lambda\mu}\widehat{R}^{\alpha\beta\lambda\cdot}_{\ \ \ \ \nu} - \frac{1}{2}g_{\mu\nu}\widehat{R}_{\alpha\beta\lambda\sigma}\widehat{R}^{\alpha\beta\lambda\sigma}\right.\\
&\quad \left. - 4\overline{\nabla}^\kappa\left(\overline{\nabla}^\lambda \widehat{R}_{\kappa(\mu\nu)\lambda} + T_{(\mu}^{\cdot\ \lambda\beta}\widehat{R}_{\nu)\kappa\lambda\beta}\right)\right]\\
&\quad + \frac{1}{6}(2p+q-6r)\left[2\widehat{R}_{\alpha(\mu|\beta\lambda}\widehat{R}^{\beta\lambda\alpha\cdot}_{\ \ \ \ \ |\nu)}\right.\\
&\quad - \frac{1}{2}g_{\mu\nu}\widehat{R}_{\alpha\beta\lambda\sigma}\widehat{R}^{\lambda\sigma\alpha\beta}\\
&\quad \left. - 4\overline{\nabla}^\kappa\left(\overline{\nabla}^\lambda \widehat{R}_{\lambda(\mu\nu)\kappa} + T_{(\mu|}^{\cdot\ \lambda\beta}\widehat{R}_{\lambda\beta|\nu)\kappa}\right)\right]\\
&\quad + \frac{2}{3}(p-q)\left[2\widehat{R}_{\alpha(\mu|\beta\lambda}\widehat{R}^{\alpha\beta\cdot\lambda}_{\ \ \ \ \ |\nu)} + \widehat{R}_{\alpha\lambda\sigma\mu}\widehat{R}^{\alpha\sigma\lambda}_{\ \ \ \ \nu}\right.\\
&\quad - \widehat{R}_{\mu\alpha\lambda\sigma}\widehat{R}_\nu^{\cdot\lambda\alpha\sigma} - \frac{1}{2}g_{\mu\nu}\widehat{R}_{\alpha\beta\lambda\sigma}\widehat{R}^{\alpha\beta\lambda\sigma}\\
&\quad - 2\overline{\nabla}^\kappa\left(\overline{\nabla}^\lambda \widehat{R}_{\kappa(\mu\nu)\lambda} - 2T_\kappa^{\cdot\lambda\beta}\widehat{R}_{\beta(\mu\nu)\lambda}\right.\\
&\quad \left.\left. + 2T_{(\mu}^{\ \ \cdot\lambda\beta}\widehat{R}_{\nu)\beta\lambda\kappa} - 2T_{(\mu|}^{\ \ \cdot\lambda\beta}\widehat{R}_{\kappa\beta\lambda|\nu)}\right)\right]\\
&\quad + (s+t)\left[\widehat{R}_{\mu\cdot}^{\ \lambda}\widehat{R}_{\nu\lambda} + \widehat{R}^\lambda_{\cdot\mu}\widehat{R}_{\lambda\nu} - \frac{1}{2}g_{\mu\nu}\widehat{R}_{\alpha\beta}\widehat{R}^{\alpha\beta}\right.\\
&\quad + \overline{\nabla}^\kappa\left(g_{\mu\nu}\overline{\nabla}^\lambda \widehat{R}_{\kappa\lambda} + \overline{\nabla}_\kappa \widehat{R}_{(\mu\nu)} - \overline{\nabla}_{(\mu}\widehat{R}_{\nu)\kappa}\right.\\
&\quad - \overline{\nabla}_{(\mu|}\widehat{R}_{\kappa|\nu)} + \frac{1}{2}T_{(\mu|\kappa\cdot}^{\ \ \ \ \ \lambda}\widehat{R}_{|\nu)\lambda} - \frac{1}{2}T_{\kappa(\mu\cdot}^{\ \ \ \ \lambda}\widehat{R}_{\nu)\lambda}\\
&\quad \left.\left. - \frac{1}{2}T_{(\mu\nu)\cdot}^{\ \ \ \ \ \lambda}\widehat{R}_{\kappa\lambda}\right)\right]\\
&\quad + (s-t)\left[\widehat{R}_{\mu\cdot}^{\ \lambda}\widehat{R}_{\lambda\nu} + \widehat{R}^\lambda_{\cdot\mu}\widehat{R}_{\nu\lambda} - \frac{1}{2}g_{\mu\nu}\widehat{R}_{\alpha\beta}\widehat{R}^{\beta\alpha}\right.\\
&\quad + \overline{\nabla}^\kappa\left(g_{\mu\nu}\overline{\nabla}^\lambda \widehat{R}_{\lambda\kappa}\right.\\
&\quad + \overline{\nabla}_\kappa \widehat{R}_{(\mu\nu)} - \overline{\nabla}_{(\mu}\widehat{R}_{\nu)\kappa}\\
&\quad - \overline{\nabla}_{(\mu|}\widehat{R}_{\kappa|\nu)} + \frac{1}{2}T_{(\mu|\kappa\cdot}^{\ \ \ \ \ \lambda}\widehat{R}_{\lambda|\nu)}\\
&\quad \left.\left. - \frac{1}{2}T_{\kappa(\mu|\cdot}^{\ \ \ \ \ \lambda}\widehat{R}_{\lambda|\nu)} - \frac{1}{2}T_{(\mu\nu)\cdot}^{\ \ \ \ \ \lambda}\widehat{R}_{\lambda\kappa}\right)\right]\\
&= \widehat{\tau}_{\mu\nu} + \frac{1}{2}\Delta^{\alpha\beta\gamma}_{\nu\mu\kappa}\overline{\nabla}^\kappa\widehat{\Sigma}_{\alpha(\beta\gamma)}
\end{aligned} \tag{42}$$

and

$$\begin{aligned}
&-2\lambda\overset{\star}{T}_{\nu\mu\tau} + \frac{1}{6}(4a+b+3\lambda)T_{[\tau\mu]\nu}\\
&\quad - \frac{1}{6}(-2a+b-3\lambda)\left(T_{[\mu\tau]\nu} + T_{\nu\mu\tau}\right)\\
&\quad + \frac{1}{3}(-a+b-3\lambda)g_{\nu[\tau}T_{\mu]}\\
&\quad + \frac{2}{3}(2p+q)\left(\overline{\nabla}^\kappa\widehat{R}_{\tau\mu\nu\kappa} - T_\nu^{\cdot\lambda\kappa}\widehat{R}_{\tau\mu\lambda\kappa}\right)\\
&\quad + \frac{2}{3}(2p+q-6r)\left(\overline{\nabla}^\kappa\widehat{R}_{\nu\kappa\tau\mu} - T_\nu^{\cdot\lambda\kappa}\widehat{R}_{\lambda\kappa\tau\mu}\right)\\
&\quad + \frac{4}{3}(p-q)\left(\overline{\nabla}^\kappa\widehat{R}_{\kappa[\tau\mu]\nu} - \overline{\nabla}^\kappa\widehat{R}_{\nu[\tau\mu]\kappa} - 2T_\nu^{\cdot\lambda\kappa}\widehat{R}_{\kappa[\tau\mu]\lambda}\right)\\
&\quad + (s+t)\left(2g_{\nu[\tau}\overline{\nabla}^\kappa\widehat{R}_{\mu]\kappa} - 2\overline{\nabla}_{[\tau}\widehat{R}_{\mu]\nu} + T_{\nu\cdot[\tau}^{\ \ \lambda}\widehat{R}_{\mu]\lambda}\right)\\
&\quad + (s-t)\left(2g_{\nu[\tau|}\overline{\nabla}^\kappa\widehat{R}_{\kappa|\mu]} - 2\overline{\nabla}_{[\tau|}\widehat{R}_{\nu|\mu]} + T_{\nu\cdot[\tau|}^{\ \ \lambda}\widehat{R}_{\lambda|\mu]}\right)\\
&= \widehat{\Sigma}_{[\tau\mu]\nu} .
\end{aligned} \tag{43}$$

For an interpretation of the right sides of both field equations, see Appendix C.

3.2 Reduction to GR

We want to obtain a theory which reduces to GR when the torsion vanishes. Thus, the theory will not only be stable in this regime, but it will also deviate only slightly from the predictions of GR when the torsion is small. Note that when the torsion is set to zero, the usual Riemannian structure is recovered. Therefore, the Riemann tensor is now symmetric under the exchange of the first and the second pair of indices and the Ricci tensor is symmetric. From the first Bianchi identity (16), it follows that

$$R_{\mu\nu\rho\sigma}\left(R^{\mu\nu\rho\sigma}-2R^{\mu\rho\nu\sigma}\right)=0 \quad \text{for} \quad T^{\alpha}_{\cdot\beta\gamma}=0. \tag{44}$$

Then, when $T=0$, the Lagrangian density (24) becomes

$$\mathcal{L}_g\big|_{T=0}=-\lambda\, R+(p-r)\, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+2\,s\, R_{\mu\nu}R^{\mu\nu}\,. \tag{45}$$

From this expression, it is clear that GR is recovered when $T=0$ if and only if $p=r$ and $s=0$. This is the only choice of parameters that leads to GR when the torsion vanishes.

Note that the same conclusion can be extracted from a different and longer approach. That is, considering the field equations (42) and (43), it can be concluded that this is the only choice of parameters that produce the Einstein equations of GR when the torsion vanishes. The same conclusion was reached in Ref. [8].

3.3 Stability in Minkowski spacetime

It is well known that the Lagrangian density (24) contains, along with the usual graviton 2^+, up to six new modes or torsions. These are 2^+, 2^-, 1^+, 1^-, 0^+ and 0^-, in the representation S^P where S is the spin and P is the parity of the mode. A physically meaningful restriction is to demand the theory to be stable in all the S^P sectors; see Refs. [6,7,28–30]. Quadratic theories in the curvature and torsion tensors are usually treated as a gauge theory, hence the variables considered are the gauge potentials of the Poincaré group $(e_\mu^{\ a}, w_\mu^{\ ab})$. Then the stability analysis is made through the construction of the spin projection operators.

In this work, however, we consider the metric formulation. We will examine the decoupling limit between the torsion and curvature degrees of freedom. Thus, in view of Eq. (15), we focus on the case where $g_{\mu\nu}=\eta_{\mu\nu}$, with $\eta_{\mu\nu}$ the Minkowski metric. For the sake of simplicity, we do not consider the purely tensor component of the torsion in Eq. (12). As the only torsion components compatible with a Friedmann–Lemaître–Robertson–Walker (FLRW) universe are the vectorial T^i and pseudo-vectorial S^i components [31], we assume that they are the minimum non-vanishing components that should be taken into account in this framework. In the spirit of investigating only slight modifications of GR, we assume that they are the only non-vanishing torsion components for a minimal modification over the FLRW background. Under these considerations, we will now impose the absence of ghost and tachyon instabilities for the theory given by the Lagrangian density (24). The quadratic Riemann and torsion terms that appear in this Lagrangian density are computed in Appendix D.

As we consider only the vector and pseudo-vector torsion components in Minkowski spacetime, the Lagrangian density (24) reduces in this regime to an ordinary vector and pseudo-vector field theory in flat spacetime. A general quadratic action for a vector A^μ in flat spacetime comes from [32–34]

$$\mathcal{L}=\alpha\partial_\mu A_\nu\partial^\mu A^\nu+\beta\partial_\mu A_\nu\partial^\nu A^\mu+\gamma\partial_\mu A^\mu\partial_\nu A^\nu-\mathcal{V}, \tag{46}$$

where $\mathcal{V}$ is a possible potential for A^μ. However, not all the kinetic terms are independent from each other. The terms with factors β and γ are related by

$$\int\sqrt{-g}\,d^4x\,(\nabla_\mu A^\mu)^2=\int\sqrt{-g}\,d^4x\,\big(\nabla_\mu A_\nu\nabla^\nu A^\mu +R_{\mu\nu}A^\mu A^\nu\big)\,, \tag{47}$$

as can be seen from Eq. (13). Thus, in flat spacetime these terms are related by a total derivative. On the other hand, as is well known, the Hamiltonian density of a system is obtained by performing a Legendre transformation. For this vector system, it is

$$\mathcal{H}=\pi^\mu\dot{A}_\mu-\mathcal{L}, \tag{48}$$

where $\dot{A}_\mu\equiv\partial_0A_\mu$ are the generalized velocities and π^μ the canonical momenta defined as $\pi^\mu\equiv\frac{\partial\mathcal{L}}{\partial\dot{A}^\mu}$. The canonical momenta of the Lagrangian density (46) are

$$\pi^\mu=2\alpha\dot{A}^\mu+2\beta\eta^{\mu\nu}\partial_\nu A^0+2\gamma\eta^{\mu0}\partial_\alpha A^\alpha, \tag{49}$$

or written in terms of the components of the four-vector,

$$\pi^0=2(\alpha+\beta+\gamma)\dot{A}^0+2\gamma\partial_iA^i, \tag{50}$$

$$\pi^i=2\alpha\dot{A}^i-2\beta\delta^{ij}\partial_jA^0\,. \tag{51}$$

Then, performing the Legendre transformation (48), the Hamiltonian density reads

$$\mathcal{H}=\frac{(\pi^0-2\gamma\partial_iA^i)^2}{4(\alpha+\beta+\gamma)}-\frac{(\pi^i+2\beta\partial_iA_0)^2}{4\alpha}+\frac{\beta}{2}F_{ij}F^{ij} +\alpha(\partial_iA_0)^2-(\alpha+\beta)(\partial_iA_j)^2-\gamma(\partial_iA^i)^2+\mathcal{V}\,, \tag{52}$$

with $F_{ij}=2\partial_{[i}A_{j]}$. Unfortunately, the kinetic energy of this system is unbounded from below and, therefore, suffers from ghost-type instabilities whatever the signs of α, β and γ are. This behaviour confirms that vector theories suffer from ghost-type instabilities if all the degrees of freedom of the four-vector A^μ propagates (see Refs. [32,33]). Hence, a necessary condition for the absence of this kind of instabilities is to make the scalar mode non-dynamical. Alternatively, the vector degrees of freedom can be frozen and propagate only the scalar mode, but this corresponds to a scalar theory rather than a vectorial one. To remove the scalar mode, the free parameters of the theory must be chosen in such a way that the canonical momenta given in Eq. (50) vanish. Since $\partial_0 A^0$ and $\partial_i A^i$ are independent quantities, the only possibility to cancel out the contribution of $\partial_i A^i$ to the canonical momenta of the scalar mode is to set $\gamma = 0$. In addition, $\alpha + \beta = 0$ is also needed to remove the contributions of the two remaining kinetic terms in the Lagrangian density (46) to the dynamics of the scalar mode. With these conditions, the kinetic terms in the vector Lagrangian density becomes a Maxwell-type $F_{\mu\nu}F^{\mu\nu}$ that only propagates the spatial degrees of freedom of the four-vector A^μ. This conclusion is in agreement with the well-known fact that the only ghost-free vector theory in flat spacetime is the Maxwell–Proca Lagrangian density. Then the Hamiltonian density can be positive-defined with $\alpha = -\beta < 0$. For a more detailed discussion on this item see Ref. [34].

Back to the Lagrangian density (24), when the metric corresponds to the Minkowski spacetime the expression reduces to

$$\begin{aligned}\mathcal{L}_g = &\frac{16}{9}(p+s+t)\partial_\mu T_\nu \partial^\mu T^\nu + \frac{16}{9}(p-2r)\partial_\mu T_\nu \partial^\nu T^\mu \\ &+ \frac{16}{9}(p-r+5s-t)\partial_\mu T^\mu \partial_\nu T^\nu - \frac{1}{9}t\partial_\mu S_\nu \partial^\nu S^\mu \\ &+ \frac{1}{9}(2r+t)\partial_\mu S_\nu \partial^\mu S^\nu + \frac{1}{18}(3q-4r)\partial_\mu S^\mu \partial_\nu S^\nu \\ &+ \frac{8}{27}(p-q-3t)\varepsilon^{\mu\nu\rho\sigma}\partial_\rho T_\mu \partial_\nu S_\sigma - \mathcal{V}(T,S), \quad (53)\end{aligned}$$

where $\mathcal{V}(T, S)$ are potential-type terms of the torsion fields; see Appendix D. As discussed previously, the free parameters p, q, r, s and t must be carefully selected to produce ghost-free kinetic terms, i.e. Maxwell-type kinetic terms for the trace four-vector T^μ and pseudo-trace four-vector S^μ. After suitable integrations by parts the expression above simplifies to

$$\begin{aligned}\mathcal{L}_g = &\frac{8}{9}(p+s+t)F_{\mu\nu}(T)F^{\mu\nu}(T) \\ &+ \frac{1}{18}(2r+t)F_{\mu\nu}(S)F^{\mu\nu}(S) + \frac{1}{6}q\partial_\mu S^\mu \partial_\nu S^\nu \\ &+ \frac{16}{3}(p-r+2s)\partial_\mu T^\mu \partial_\nu T^\nu - \mathcal{V}(T,S) . \quad (54)\end{aligned}$$

Since we have two dynamical fields, there are two canonical momenta. These are

$$\begin{aligned}\pi_T^\mu \equiv \frac{\partial \mathcal{L}_g}{\partial(\partial_0 T_\mu)} = &\frac{32}{9}(p+s+t)F^{0\mu}(T) \\ &+ \frac{32}{3}\eta^{0\mu}(p-r+2s)\partial_\alpha T^\alpha, \quad (55)\end{aligned}$$

$$\pi_S^\mu \equiv \frac{\partial \mathcal{L}_g}{\partial(\partial_0 S_\mu)} = \frac{2}{9}(2r+t)F^{0\mu}(S) + \frac{1}{3}\eta^{0\mu}q\partial_\alpha S^\alpha . \quad (56)$$

Written in terms of the scalar and vectorial degrees of freedom of the four-vectors we have

$$\pi_T^0 = \frac{32}{3}(p-r+2s)\partial_\alpha T^\alpha , \quad (57)$$

$$\pi_T^i = \frac{32}{9}(p+s+t)\left(\dot{T}^i - \partial^i T^0\right), \quad (58)$$

$$\pi_S^0 = \frac{1}{3}q\,\partial_\alpha S^\alpha, \quad (59)$$

$$\pi_S^i = \frac{2}{9}(2r+t)\left(\dot{S}^i - \partial^i S^0\right) . \quad (60)$$

As here we have two fields with their own kinetic terms, we need to ensure that neither of them introduces a ghost. Thus, to remove the scalar T^0 and pseudo-scalar S^0 degrees of freedom, we consider $p-r+2s = 0$ and $q = 0$, respectively. Then the Hamiltonian density reads

$$\begin{aligned}\mathcal{H}_g = &-\frac{9}{64}\frac{(\pi_T^i)^2}{(p+s+t)} - \frac{8}{9}(p+s+t)F_{ij}(T)F^{ij}(T) \\ &-\frac{9}{4}\frac{(\pi_S^i)^2}{2r+t} - \frac{1}{18}(2r+t)F_{ij}(S)F^{ij}(S) \\ &+\pi_T^i\partial_i T_o + \pi_S^i\partial_i S_o + \mathcal{V}(T,S) . \quad (61)\end{aligned}$$

The kinetic energy can be bounded from below with the extra conditions of $p+s+t < 0$ and $2r+t < 0$ for the vectorial and pseudo-vectorial torsion fields, respectively. These conditions are summarised in Table 1.

On the other hand, we now require the absence of tachyon instabilities. In the first place, we consider the weak torsion fields regime, that is, the regime where the quadratic terms in torsion fields lead the evolution of the potential. Thus, the potential in the Lagrangian density (54) takes the form

$$\mathcal{V}(T,S) = -\frac{2}{3}(c+3\lambda)T_\mu T^\mu - \frac{1}{24}(b+3\lambda)S_\mu S^\mu + \mathcal{O}(3); \quad (62)$$

see Appendix D. Note that the mass terms in an action for a vector field comes from a potential of type $V(\phi) \propto \frac{1}{2}m^2\phi_\mu\phi^\mu$. Hence, the roles of the masses m^2 for the vector and pseudo-vector torsion fields are played by the combinations of the coupling constants b, c and λ. For these combinations, the correct sign must be taken for the spatial components to avoid tachyon-like instabilities. In our convention, $\phi_\mu\phi^\mu = \phi_0^2 - \boldsymbol{\phi}^{\,2}$, then the combinations $c+3\lambda$ and $b+3\lambda$ must be positive to ensure a well-behaved vector and pseudo-vector sector, respectively (see Table 1). In summary, with these simple arguments we have found a set of conditions for the ghost and tachyon stability of the Lagrangian density (24) at the decoupling limit and the weak torsion regime, summarized in Table 1.

Table 1 Conditions over the free parameters of the Lagrangian density (24) for stability and reduction to GR when the torsion vanishes

	T^μ	S^μ	Description
Ghost-free	$p-r+2s=0$ $p+s+t<0$	$q=0$ $2r+t<0$	To remove the scalar/pseudo-scalar mode and to ensure a well-posed kinetic term
Tachyon-free (Weak torsion)	$c+3\lambda>0$	$b+3\lambda>0$	To have a positive-defined quadratic potential $\mathcal{V}^{(2)}$
Tachyon-free (General torsion)	$p+3s=0$ $c+3\lambda>0$	$p+3s=0$ $b+3\lambda>0$	To cancel $\mathcal{V}^{(4)}$ and to make $\mathcal{V}^{(2)}$ positive-defined
Reduction to GR when $T^\alpha_{\cdot\mu\nu}=0$	$p-r=0$ $s=0$		

In Refs. [6,7], Sezgin and Nieuwenhuizen provided a detailed analysis of the stability of the Lagrangian density (24) for the weak torsion field regime. These two articles were the first systematic stability analysis of this kind of theories, made with the spin projectors formalism, and they are a key reference point in this issue. The conclusions they showed for the 1^- torsions are compatible with those obtained here. Their ghost-free condition is the same we have obtained here, and the tachyon-free condition is compatible. On the other hand, for the 1^+ sector both conclusions are, however, incompatible. While the condition obtained for a well-defined kinetic term for S^μ in this section is $2r+t<0$, they claim that $2r+t>0$ is needed. It is worth noting that other authors have suggested that the analysis carried out by Sezgin and Nieuwenhuizen is not restrictive enough to ensure a ghost- and tachyon-free spectrum; see Refs. [28,29]. In fact, in Ref. [28] the authors pointed out that they even obtain a different expression of the spin projector operator for the pseudo-vector mode. Furthermore, they argue the relevance of considering the additional condition for the absence of p^{-4} poles in all spin sectors, which is not done in the analysis of Refs. [6,7]. In Ref. [35], Fabbri analyses the stability of the most general quadratic gravitational action with torsion and Dirac fields by demanding, in addition, a consistent decoupling between curvature and torsion that preserves continuity in the torsionless limit, concluding that the only non-vanishing component of the torsion is given by the pseudo-vector mode and that parity-violating terms are not allowed in the Lagrangian density. Nevertheless, due to some lack of clarity in the existing literature, a deeper analysis of the origins of these differences is not available yet.

Let us now go beyond the weak torsion regime when analysing the potential $\mathcal{V}$. Thus, higher orders in the potential can dominate its evolution. The highest order that appears in the potential is quartic, symbolically $\mathcal{V}^{(4)}$,

$$\begin{aligned}\mathcal{V}^{(4)}(T,S) = &-\frac{64}{27}(p-r+2s)T_\alpha T^\alpha T_\beta T^\beta\\ &-\frac{1}{108}(p-r+2s)S_\alpha S^\alpha S_\beta S^\beta\\ &-\frac{8}{81}(2p+3q-4r+2s)T_\alpha S^\alpha T_\beta S^\beta\\ &-\frac{8}{81}(p+r+4s)T_\alpha T^\alpha S_\beta S^\beta. \end{aligned}\tag{63}$$

As there are terms mixing the vector and pseudo-vector fields, we note that the potential can be diagonalized in the following basis:

$$\mathcal{V}^{(4)} = \begin{pmatrix} T_\alpha T^\alpha \\ S_\alpha S^\alpha \\ T_\alpha S^\alpha \end{pmatrix} \mathbb{V}^{(4)} \left(T_\alpha T^\alpha \;\; S_\alpha S^\alpha \;\; T_\alpha S^\alpha \right), \tag{64}$$

with $\mathbb{V}^{(4)}$ a 3×3 matrix. The eigenvalues of $\mathcal{V}^{(4)}$ are

$$\lambda_1 = -\frac{8}{81}(2p+3q-4r+2s), \tag{65}$$

$$\lambda_2 = -\frac{79}{72}\left(p-r+2s+\sqrt{A}\right), \tag{66}$$

$$\lambda_3 = -\frac{79}{72}\left(p-r+2s-\sqrt{A}\right), \tag{67}$$

with

$$\begin{aligned}A = \frac{1}{711^2}\Big(&586249p^2 - 1168402pr + 586249r^2\\ &+ 2349092ps - 2332708rs + 2357284s^2\Big). \end{aligned}\tag{68}$$

For a positive-defined quadratic form, the three eigenvalues must be positive. Since we are only interested in the vector and pseudo-vector torsion degrees of freedom, we can assume $p-r+2s=0$ and $q=0$, which are the conditions found for making the scalar and pseudo-scalar mode non-dynamic, respectively. Then the expressions of the eigenvalues reduce to

$$\lambda_1 = \frac{16}{81}(p+3s), \tag{69a}$$

$$\lambda_2 = -\frac{8}{81}(p+3s), \tag{69b}$$

$$\lambda_3 = \frac{8}{81}(p+3s). \tag{69c}$$

Table 2 Compatibility of the stability conditions studied in this paper. In the first column we show necessary conditions for a theory propagating vector or pseudo-vector torsion to be stable. Those conditions have to be implemented (at least) by the inequality contained in the second column when the vector mode propagates and by the conditions of the last column when the pseudo-vector also propagates

Summary	T^{μ}	S^{μ}
$p = r = s = 0$	$c + 3\lambda > 0$	$q = 0$
$t < 0$		$b + 3\lambda > 0$

It is easy to see that these eigenvalues cannot be positive at the same time for any combination of p and s. Hence, the quartic order in the potential in Eq. (61) is unstable and, therefore, this order must be removed to obtain a stable theory. This can be done taking $3s + p = 0$. Furthermore, the third order in the potential is not present once we consider that GR is recovered when the torsion vanishes. Therefore, when we take $p = r$, $s = 0$ and $3s + p = 0$, there are only quadratic terms in the potential. Thus, the potential is stable under the same conditions as those obtained in the weak torsion field approximation with the additional constraint of $p + 3s = 0$; see Table 1.

On the other hand, we should stress that the stability analysis developed in the literature is usually made using a weak curvature approximation for the metric. However, our stability analysis is made in the limit where the degrees of freedom of the torsion are completely decoupled from those of the metric. For this purpose, we have considered that GR is recovered when $T = 0$ and we have investigated the stability of the torsion in Minkowski flat spacetime, assuming that only the vector and pseudo-vector modes propagate. These conditions are combined and summarized in Table 2. Therefore, we expect that the conditions obtained, which are found to be necessary and sufficient for the stability in this regime, are necessary but no longer sufficient conditions for the stability of the theory when both curvature and torsion are present.

4 Summary

In this work we have investigated a quadratic and parity preserving action with curvature and torsion [6,7,10,11] in order to obtain a stable theory of gravity with dynamical torsion. For this purpose, we have analysed two regimes where the degrees of freedom of the metric and those of the torsion are completely decoupled. The assumptions made in those regimes are also motivated by looking for theories of which the predictions are expected not to be in great disagreement with those of GR.

On the one hand, we have assumed that the theory reduces to GR when the torsion vanishes. This implies the stability of the metric degrees of freedom in the regime where there are no torsion modes. Therefore, we have imposed the requirement that the only term independent of the torsion is contained in the scalar curvature $\widehat{R}$, obtaining two conditions for the parameters of the general quadratic Lagrangian.

On the other hand, we have investigated the stability of the torsion when the metric is flat, following an approach that differs from the usual techniques used in the literature. We have focussed attention on the stability of the vector and psuedo-vector torsion components in Minkowski because they are the only components that propagate in a FLRW spacetime [31] from the torsion irreducible decomposition. Therefore, it is not necessary to consider the purely tensor component if we are interested in "minimal" modifications of the predictions of GR. We have studied the stability of these fields analysing the Hamiltonian formulation of the theory to ensure a ghost and tachyon-free spectrum in this regime. Thus, we have obtained several conditions for the parameters of the general quadratic action with propagating torsion that we have summarized in Table 1. Moreover, we have contrasted the conditions obtained in the weak torsion limit of this regime with those already presented in the literature [6,7,28,29]. As we have discussed in detail, the disagreement with the conclusions of Ref. [6,7] regarding the pseudo-vector field may be due to the arguments exposed in Refs. [28,29]. It should be stressed that, after the first approach, we have gone beyond the weak torsion approximation, obtaining the general conditions for the stability of the vector and pseudo-vector torsion fields in Minkowski spacetime.

In summary, we have found the most general subfamily of the Lagrangian density (24) that is stable in both decoupling regimes. This is described by

$$\begin{aligned}\mathcal{L}_g = & -\lambda\widehat{R} + \frac{1}{12}(4a + b + 3\lambda)T_{\mu\nu\rho}T^{\mu\nu\rho} \\ & + \frac{1}{6}(-2a + b - 3\lambda)T_{\mu\nu\rho}T^{\nu\rho\mu} \\ & + \frac{1}{3}(-a + 2c - 3\lambda)T^{\lambda}_{\cdot\mu\lambda}T_{\rho}^{\cdot\mu\rho} + 2t\widehat{R}_{\mu\nu}\widehat{R}^{[\mu\nu]}\ , \qquad (70)\end{aligned}$$

where $b + 3\lambda > 0$, $c + 3\lambda > 0$, and $t < 0$, and we restrict our study to theories where only the vector and pseudo-vector torsion components of the irreducible decomposition propagate.

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Appendices

Appendix A: The Gauss–Bonnet term in Riemann–Cartan geometries

We have noted that there is no agreement about the expression of the Gauss–Bonnet term in a Riemann–Cartan manifold throughout the literature, probably due to several misprints. Therefore, in this appendix, we present the correct expression for the Gauss–Bonnet action. This is

$$S_{GB} = \int d^4x\sqrt{-g}\left(\widehat{R}^2 - 4\widehat{R}_{\nu\sigma}\widehat{R}^{\sigma\nu} + \widehat{R}_{\mu\nu\rho\sigma}\widehat{R}^{\rho\sigma\mu\nu}\right) . \tag{A.1}$$

One can easily check that this is the correct order of the indices focussing attention on the vectorial and pseudo-vectorial torsion fields in the weak curvature approximation. In this regime we have

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$
$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} . \tag{A.2}$$

Let us now prove that, order by order in the fields $h_{\alpha\beta}$, T_α and S_α, the term (A.1) leads to a total divergence. The expressions of $R^{\mu}_{\cdot\nu\rho\sigma}$, $R_{\nu\sigma}$ and R in terms of h are well known in linearized gravity [36]. These are

$$R^{\mu}_{\cdot\nu\rho\sigma} = \frac{1}{2}\left(\partial_\rho\partial_\nu h^{\mu}_{\ \sigma} + \partial^\mu\partial_\sigma h_{\nu\rho} - \partial^\rho\partial^\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h^{\mu}_{\ \rho}\right), \tag{A.3}$$

$$R_{\nu\sigma} = \frac{1}{2}\left(\partial_\mu\partial_\nu h^{\mu}_{\ \sigma} + \partial_\sigma\partial_\mu h^{\mu}_{\ \nu} - \Box h_{\sigma\nu} - \partial_\sigma\partial_\nu h\right), \tag{A.4}$$

$$R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h , \tag{A.5}$$

with $\Box = \partial_\mu\partial^\mu$. Then, from Eq. (15), it is clear that in the action (A.1) will appear a Gauss–Bonnet term for the Levi-Civita connection, terms quadratic in torsion and a term mixing torsion and h terms. This action can be expressed as

$$S_{GB} = S^{(1)}_{GB}(\partial h) + S^{(2)}_{GB}(\partial T, \partial S, T, S) + S^{(3)}_{GB}(\partial h, \partial T, T, S). \tag{A.6}$$

The first term on the r.h.s. of this equation is known to be invariant. Nevertheless, this invariance can be proven with an explicit calculation from Eqs. (A.3), (A.4) and (A.5) with the appropriate boundary conditions on h. The second term is calculated with the results of Appendix D. It can be seen that

$$\begin{aligned} S^{(2)}_{GB} = \int d^4x\sqrt{-g}\Bigg[&\frac{32}{9}(\partial_\rho T_\nu\partial^\nu T^\rho - \partial_\alpha T^\alpha\partial_\beta T^\beta) \\ &- \frac{2}{9}(\partial_\alpha S^\alpha\partial_\beta S^\beta - \partial_\alpha S_\beta\partial^\beta S^\alpha) + \frac{64}{27}\partial_\alpha\left(T^\alpha T_\beta T^\beta\right) \\ &+ \frac{4}{27}\partial_\alpha\left(T^\alpha S_\beta S^\beta + 2S^\alpha T_\beta S^\beta\right) \\ &+ \frac{8}{9}\epsilon^{\mu\nu\rho\sigma}\partial_\nu S_\sigma\partial_\mu T_\rho\Bigg] . \end{aligned} \tag{A.7}$$

After integration by parts, the expression above leads to a total divergence. Taking the torsion to be zero at the boundary of $\mathcal{U}_4$, $S^{(2)}_{GB}$ is identically zero. Finally, the third term on the r.h.s. of Eq. (A.6), $S^{(3)}_{GB}(\partial h, \partial T, T, S)$, is analysed using Eqs. (D.27), (D.28) and (D.29) for the torsion part and (A.3), (A.4) and (A.5) for the metric dependent part. Thus,

$$\begin{aligned} S^{(3)}_{GB} = \int d^4x\sqrt{-g}\Bigg[&4\Box h\partial_\alpha T^\alpha - 4\partial_\mu\partial_\nu h^{\mu\nu}\partial_\alpha T^\alpha \\ &+ \frac{32}{3}\left(\partial_\sigma\partial_\mu h^{\sigma\mu}\partial_\alpha T^\alpha - \Box h\partial_\alpha T^\alpha\right) \\ &+ \frac{8}{3}\left(\partial_\rho\partial_\nu h\partial^\rho T^\nu - \partial_\mu\partial_\nu h^{\mu\sigma}\partial_\sigma T^\nu\right)\Bigg] . \end{aligned} \tag{A.8}$$

Note that there are no mixing terms between ∂h and ∂S or ST, as expected from parity conservation. After some algebraical manipulations and integration by parts, the equation for $S^{(3)}_{GB}$ vanishes. Hence, we have checked the invariance of an action upon addition of the action (A.1) in the weak curvature limit. As was pointed by Nieh [21], the Gauss–Bonnet term will remain invariant even in a curved non-flat metric $g_{\mu\nu}$. But, for this work, the invariance in weak field limit is sufficient.

Appendix B: Variations in the Palatini formalism

The Palatini formalism for varying the action consists in taking the metric $g^{\mu\nu}$ and the generic connection $\tilde{\Gamma}^{\sigma}_{\cdot\alpha\beta}$ as the dynamical variables. So, it is useful to rewrite the action in terms of those variables. Some useful well-known relations for considering that variation are

$$g_{\mu\alpha}\delta g^{\alpha\nu} = -g^{\alpha\nu}\delta g_{\mu\alpha}, \qquad \delta\sqrt{-g} = -\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu} . \tag{B.9}$$

Thus, one can easily obtain

$$\partial_\mu\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\tilde{\nabla}_\mu g_{\alpha\beta} + \sqrt{-g}\tilde{\Gamma}^{\alpha}_{\cdot\alpha\mu} . \tag{B.10}$$

Let us know consider the variation of the action written in terms of the Lagrangian density (26). This is

$$\begin{aligned} \mathcal{L}_g = &-\lambda\,\delta_\alpha^{\ \gamma} g^{\beta\delta}\tilde{R}^{\alpha}_{\cdot\beta\gamma\delta} + f_{T\ \lambda\alpha}^{\ \ \eta\rho\beta\gamma}\, T^{\lambda}_{\cdot\eta\rho}T^{\alpha}_{\cdot\beta\gamma} \\ &+ f_{R\ \lambda\alpha}^{\ \ \eta\rho\sigma\beta\gamma\delta}\,\tilde{R}^{\lambda}_{\cdot\eta\rho\sigma}\tilde{R}^{\alpha}_{\cdot\beta\gamma\delta}, \end{aligned} \tag{B.11}$$

where the permutation tensors are

$$
\begin{aligned}
f_{T\,\lambda\alpha}^{\ \ \eta\rho\beta\gamma} =& \frac{1}{12}(4a+b+3\lambda)g_{\lambda\alpha}g^{\eta\beta}g^{\rho\gamma} \\
&+\frac{1}{6}(-2a+b-3\lambda)\delta_{\lambda}^{\ \gamma}\delta_{\alpha}^{\ \eta}g^{\rho\beta} \\
&+\frac{1}{3}(-a+2c-3\lambda)\delta_{\lambda}^{\ \rho}\delta_{\alpha}^{\ \gamma}g^{\eta\beta},
\end{aligned}
\tag{B.12}
$$

$$
\begin{aligned}
f_{R\,\lambda\alpha}^{\ \ \eta\rho\sigma\beta\gamma\delta} =& \frac{1}{6}(2p+q)g_{\lambda\alpha}g^{\eta\beta}g^{\rho\gamma}g^{\sigma\delta} \\
&+\frac{1}{6}(2p+q-6r)\delta_{\lambda}^{\ \gamma}\delta_{\alpha}^{\ \rho}g^{\eta\delta}g^{\sigma\beta} \\
&+\frac{2}{3}(p-q)g_{\lambda\alpha}g^{\eta\gamma}g^{\rho\beta}g^{\sigma\delta} \\
&+(s+t)\delta_{\lambda}^{\ \rho}\delta_{\alpha}^{\ \gamma}g^{\eta\beta}g^{\sigma\delta} \\
&+(s-t)\delta_{\lambda}^{\ \rho}\delta_{\alpha}^{\ \gamma}g^{\eta\delta}g^{\sigma\beta}.
\end{aligned}
\tag{B.13}
$$

In order to compute the complete generalized Einstein tensor in Eq. (34), the following expressions are needed:

$$
\begin{aligned}
\frac{\partial f_{R\,\lambda\alpha}^{\ \ \eta\rho\sigma\beta\gamma\delta}}{\partial g^{\mu\nu}} =& \frac{1}{6}(2p+q)\left(\delta_{\mu}^{\ \eta}\delta_{\nu}^{\ \beta}g_{\lambda\alpha}g^{\rho\gamma}g^{\sigma\delta}\right. \\
&+\delta_{\mu}^{\ \rho}\delta_{\nu}^{\ \gamma}g_{\lambda\alpha}g^{\eta\beta}g^{\sigma\delta}+\delta_{\mu}^{\ \sigma}\delta_{\nu}^{\ \eta}g_{\lambda\alpha}g^{\eta\beta}g^{\rho\gamma} \\
&\left.-g_{\alpha\mu}g_{\lambda\nu}g^{\eta\beta}g^{\rho\gamma}g^{\sigma\delta}\right) \\
&+\frac{1}{6}(2p+q-6r)\left(\delta_{\mu}^{\ \eta}\delta_{\nu}^{\ \delta}\delta_{\lambda}^{\ \gamma}\delta_{\alpha}^{\ \rho}g^{\sigma\beta}\right. \\
&\left.+\delta_{\mu}^{\ \sigma}\delta_{\nu}^{\ \beta}\delta_{\lambda}^{\ \gamma}\delta_{\alpha}^{\ \rho}g^{\eta\delta}\right) \\
&+\frac{2}{3}(p-q)\left(-g_{\lambda\mu}g_{\alpha\nu}g^{\eta\gamma}g^{\rho\beta}g^{\sigma\delta}\right. \\
&+\delta_{\mu}^{\ \eta}\delta_{\nu}^{\ \gamma}g_{\lambda\alpha}g^{\rho\beta}g^{\sigma\delta}+\delta_{\mu}^{\ \rho}\delta_{\nu}^{\ \beta}g_{\lambda\alpha}g^{\nu\gamma}g^{\sigma\delta} \\
&\left.+\delta_{\mu}^{\ \sigma}\delta_{\nu}^{\ \delta}g_{\lambda\alpha}g^{\nu\gamma}g^{\rho\beta}\right) \\
&+(s+t)\left(\delta_{\mu}^{\ \eta}\delta_{\nu}^{\ \beta}\delta_{\lambda}^{\ \rho}\delta_{\alpha}^{\ \gamma}g^{\sigma\delta}\right. \\
&\left.+\delta_{\mu}^{\ \sigma}\delta_{\nu}^{\ \delta}\delta_{\lambda}^{\ \rho}\delta_{\alpha}^{\ \gamma}g^{\eta\beta}\right) \\
&+(s-t)\left(\delta_{\mu}^{\ \eta}\delta_{\nu}^{\ \delta}\delta_{\lambda}^{\ \rho}\delta_{\alpha}^{\ \gamma}g^{\sigma\beta}\right. \\
&\left.+\delta_{\mu}^{\ \sigma}\delta_{\nu}^{\ \beta}\delta_{\lambda}^{\ \rho}\delta_{\alpha}^{\ \gamma}g^{\eta\delta}\right),
\end{aligned}
\tag{B.14}
$$

$$
\begin{aligned}
\frac{\partial f_{T\,\lambda\alpha}^{\ \ \eta\rho\beta\gamma}}{\partial g^{\mu\nu}} =& \frac{1}{12}(4a+b+3\lambda)\left(-g_{\lambda\mu}g_{\alpha\nu}g^{\eta\beta}g^{\rho\gamma}\right. \\
&\left.+\delta_{\mu}^{\ \eta}\delta_{\nu}^{\ \beta}g_{\lambda\alpha}g^{\rho\gamma}+\delta_{\mu}^{\ \rho}\delta_{\nu}^{\ \gamma}g_{\lambda\alpha}g^{\eta\beta}\right) \\
&+\frac{1}{6}(2p+q-6r)\delta_{\lambda}^{\ \gamma}\delta_{\alpha}^{\ \eta}\delta_{\mu}^{\ \rho}\delta_{\nu}^{\ \beta} \\
&+\frac{1}{3}(-a+2c-3\lambda)\delta_{\lambda}^{\rho}\delta_{\alpha}^{\gamma}\delta_{\mu}^{\ \eta}\delta_{\nu}^{\beta}.
\end{aligned}
\tag{B.15}
$$

For the calculation of the generalized Palatini tensor in Eq. (35), we need the following expressions:

$$
\begin{aligned}
\frac{\partial \tilde{R}^{\alpha}_{\cdot\beta\gamma\delta}}{\partial \tilde{\Gamma}^{\tau}_{\cdot\mu\nu}} =& \tilde{\Gamma}^{\alpha}_{\cdot\tau\gamma}\delta_{\beta}^{\ \mu}\delta_{\delta}^{\ \nu}-\tilde{\Gamma}^{\alpha}_{\cdot\tau\delta}\delta_{\beta}^{\ \mu}\delta_{\gamma}^{\ \nu}+\tilde{\Gamma}^{\mu}_{\cdot\beta\delta}\delta_{\tau}^{\ \alpha}\delta_{\gamma}^{\ \nu} \\
&-\tilde{\Gamma}^{\mu}_{\cdot\beta\gamma}\delta_{\tau}^{\ \alpha}\delta_{\delta}^{\ \nu},
\end{aligned}
\tag{B.16}
$$

$$
\frac{\partial T^{\alpha}_{\cdot\beta\gamma}}{\partial \tilde{\Gamma}^{\tau}_{\cdot\mu\nu}} = \frac{1}{2}\left(\delta^{\alpha}_{\tau}\delta^{\mu}_{\beta}\delta^{\nu}_{\gamma}-\delta^{\alpha}_{\tau}\delta^{\nu}_{\beta}\delta^{\mu}_{\gamma}\right),
\tag{B.17}
$$

$$
\frac{\partial \tilde{R}^{\alpha}_{\cdot\beta\gamma\delta}}{\partial\left(\partial_{\kappa}\tilde{\Gamma}^{\tau}_{\cdot\mu\nu}\right)} = \delta^{\kappa}_{\gamma}\delta^{\alpha}_{\tau}\delta^{\mu}_{\beta}\delta^{\nu}_{\delta}-\delta^{\kappa}_{\delta}\delta^{\alpha}_{\tau}\delta^{\mu}_{\beta}\delta^{\nu}_{\gamma}.
\tag{B.18}
$$

Then, taking into account the definition of the torsion and curvature tensors, Eqs. (3) and (14), respectively, the generalized Einstein and Palatini tensors of the quadratic Lagrangian density (24) read

$$
\begin{aligned}
\tilde{\mathcal{E}}_{\mu\nu} =& -\lambda\tilde{G}_{(\mu\nu)}+\frac{1}{12}(4a+b+3\lambda)\left(2T_{\alpha\beta\mu}T^{\alpha\beta\cdot}_{\ \ \ \nu}\right. \\
&\left.-T_{\mu\alpha\beta}T_{\nu}^{\cdot\alpha\beta}-\frac{1}{2}g_{\mu\nu}T_{\alpha\beta\rho}T^{\alpha\beta\rho}\right) \\
&+\frac{1}{6}(-2a+b-3\lambda)\left(T_{\alpha\beta\mu}T^{\beta\alpha\cdot}_{\ \ \ \nu}\right. \\
&\left.-\frac{1}{2}g_{\mu\nu}T_{\alpha\beta\rho}T^{\beta\rho\alpha}\right) \\
&+\frac{1}{3}(-a+2c-3\lambda)\left(T_{\mu}T_{\nu}-\frac{1}{2}g_{\mu\nu}T_{\alpha}T^{\alpha}\right) \\
&+\frac{1}{6}(2p+q)\left(2\tilde{R}_{\alpha\beta\lambda\mu}\tilde{R}^{\alpha\beta\lambda\cdot}_{\ \ \ \ \nu}-\tilde{R}_{\mu\alpha\lambda\sigma}\tilde{R}_{\nu}^{\cdot\alpha\lambda\sigma}\right. \\
&\left.+\tilde{R}_{\alpha\mu\lambda\sigma}\tilde{R}^{\alpha\cdot\lambda\sigma}_{\ \nu}-\frac{1}{2}g_{\mu\nu}\tilde{R}_{\alpha\beta\lambda\sigma}\tilde{R}^{\alpha\beta\lambda\sigma}\right) \\
&+\frac{1}{6}(2p+q-6r)\left(2\tilde{R}_{\alpha}(\mu|\beta\lambda\tilde{R}^{\beta\lambda\alpha\cdot}_{|\nu)}\right. \\
&\left.-\frac{1}{2}g_{\mu\nu}\tilde{R}_{\alpha\beta\lambda\sigma}\tilde{R}^{\lambda\sigma\alpha\beta}\right) \\
&+\frac{2}{3}(p-q)\left(2\tilde{R}_{\alpha(\mu|\beta\lambda}\tilde{R}^{\alpha\beta\cdot\lambda}_{\ \ \ \ |\nu)}+\tilde{R}_{\alpha\lambda\sigma\mu}\tilde{R}^{\alpha\sigma\lambda}_{\ \ \ \ \nu}\right. \\
&\left.-\tilde{R}_{\mu\alpha\lambda\sigma}\tilde{R}_{\nu}^{\cdot\lambda\alpha\sigma}-\frac{1}{2}g_{\mu\nu}\tilde{R}_{\alpha\beta\lambda\sigma}\tilde{R}^{\alpha\lambda\beta\sigma}\right) \\
&+(s+t)\left(\tilde{R}_{\mu\cdot}^{\ \lambda}\tilde{R}_{\nu\lambda}+\tilde{R}^{\lambda}_{\cdot\mu}\tilde{R}_{\lambda\nu}-\frac{1}{2}g_{\mu\nu}\tilde{R}_{\alpha\beta}\tilde{R}^{\alpha\beta}\right) \\
&+(s-t)\left(\tilde{R}_{\mu\cdot}^{\ \lambda}\tilde{R}_{\lambda\nu}+\tilde{R}^{\lambda}_{\cdot\mu}\tilde{R}_{\nu\lambda}-\frac{1}{2}g_{\mu\nu}\tilde{R}_{\alpha\beta}\tilde{R}^{\beta\alpha}\right),
\end{aligned}
\tag{B.19}
$$

$$
\begin{aligned}
\tilde{\mathcal{P}}_{\tau}^{\cdot\mu\nu} =& -2\lambda\left[\overset{\star}{T}{}^{\nu\mu\cdot}_{\ \ \ \sigma}+\delta_{\sigma}^{\ \nu}\left(\tilde{\nabla}_{\lambda}g^{\mu\lambda}+\frac{1}{2}g^{\alpha\beta}\tilde{\nabla}^{\mu}g_{\alpha\beta}\right)\right. \\
&\left.-\tilde{\nabla}_{\sigma}g^{\mu\nu}-\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\tilde{\nabla}_{\sigma}g_{\alpha\beta}\right] \\
&+\frac{1}{6}(4a+b+3\lambda)T_{\tau}^{\cdot\mu\nu} \\
&+\frac{1}{6}(-2a+b-3\lambda)\left(T^{\mu\nu\cdot}_{\ \ \ \tau}-T^{\nu\mu\cdot}_{\ \ \ \tau}\right) \\
&+\frac{1}{3}(-a+b-3\lambda)\left(\delta_{\tau}^{\ \nu}T^{\mu}-\delta_{\tau}^{\ \mu}T^{\nu}\right) \\
&+\frac{2}{3}(2p+q)\left[\left(\tilde{\nabla}_{\kappa}-2T_{\kappa}\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}g^{\alpha\beta}\tilde{\nabla}_\kappa g_{\alpha\beta}\Big)\,\tilde{R}_\tau^{\cdot\mu\nu\kappa}-T^{\nu}_{\cdot\lambda\kappa}\tilde{R}_\tau^{\cdot\mu\lambda\kappa}\Big]\\
&+\frac{2}{3}(2p+q-6r)\Big[\Big(\tilde{\nabla}_\kappa-2T_\kappa\\
&+\frac{1}{2}g^{\alpha\beta}\tilde{\nabla}_\kappa g_{\alpha\beta}\Big)\,\tilde{R}^{[\nu\kappa]\cdot\mu}_{\tau}-T^{\nu}_{\cdot\lambda\kappa}\tilde{R}^{[\lambda\kappa]\cdot\mu}_{\tau}\Big]\\
&+\frac{8}{3}(p-q)\Big[\Big(\tilde{\nabla}_\kappa-2T_\kappa+\frac{1}{2}g^{\alpha\beta}\tilde{\nabla}_\kappa g_{\alpha\beta}\Big)\,\tilde{R}_\tau^{\cdot[\kappa\nu]\mu}\\
&-T^{\nu}_{\cdot\lambda\kappa}\tilde{R}_\tau^{\cdot[\kappa\lambda]\mu}\Big]\\
&+(s+t)\Big[2\delta_\tau^{\ \nu}\Big(\tilde{\nabla}_\kappa-2T_\kappa+\frac{1}{2}g^{\alpha\beta}\tilde{\nabla}_\kappa g_{\alpha\beta}\Big)\,\tilde{R}^{\mu\kappa}\\
&-2\Big(\tilde{\nabla}_\tau-2T_\tau+\frac{1}{2}g^{\alpha\beta}\tilde{\nabla}_\tau g_{\alpha\beta}\Big)\,\tilde{R}^{\mu\nu}+T^{\nu}_{\cdot\lambda\tau}\tilde{R}^{\mu\lambda}\Big]\\
&+(s-t)\Big[2\delta_\tau^{\ \nu}\Big(\tilde{\nabla}_\kappa-2T_\kappa+\frac{1}{2}g^{\alpha\beta}\tilde{\nabla}_\kappa g_{\alpha\beta}\Big)\,\tilde{R}^{\kappa\mu}\\
&-2\Big(\tilde{\nabla}_\tau-2T_\tau+\frac{1}{2}g^{\alpha\beta}\tilde{\nabla}_\tau g_{\alpha\beta}\Big)\,\tilde{R}^{\nu\mu}+T^{\nu}_{\cdot\lambda\tau}\tilde{R}^{\lambda\mu}\Big].
\end{aligned}
\tag{B.20}
$$

Appendix C: Source tensors

In order to understand the r.h.s. of the field equations (42) and (43), it is necessary to make a distinction between the Hilbert definition of the energy-momentum tensor and the definition carried through in Eq. (32). Hilbert's definition is made in a Riemannian $\mathcal{V}_4$ spacetime and, therefore, there is a dependence of the matter Lagrangian density on ∂g introduced by the Levi-Civita connection. This definition is

$$
\begin{aligned}
\tau_{\mu\nu}&\equiv-\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}_M(g,\partial g,\Psi)}{\delta g^{\mu\nu}}\\
&=-\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{-g}\mathcal{L}_M}{\partial g^{\mu\nu}}-\partial^\kappa\frac{\partial\sqrt{-g}\mathcal{L}_M}{\partial(\partial^\kappa g^{\mu\nu})}\right).
\end{aligned}
\tag{C.21}
$$

Nevertheless, in the Palatini formalism this dependence on ∂g does not exist, since the matter Lagrangian depends on g and $\tilde{\Gamma}$ as independent variables. Therefore, the energy-momentum tensor is as in Eq. (32). This is

$$
\tilde{\tau}_{\mu\nu}\equiv-\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}\mathcal{L}_M(g,\tilde{\Gamma},\Psi)}{\partial g^{\mu\nu}}.
\tag{C.22}
$$

There is a clear difference between the two definitions.

However, when the metricity condition is implemented, the connection $\tilde{\Gamma}$ becomes $\widehat{\Gamma}=\Gamma+K$ and, therefore, there appears a dependence on ∂g in the definition (32). The term $\Delta^{\alpha\beta\gamma}_{\nu\mu\kappa}\overline{\nabla}^\kappa\,\widehat{\Sigma}_{\alpha(\beta\gamma)}$ in the r.h.s. of Eq. (42) takes into account this new dependence, which is not present in the original definition of $\widehat{\tau}_{\mu\nu}$. To check the consistency of this argument, let us take

$$
\begin{aligned}
\frac{\delta\sqrt{-g}\mathcal{L}_M(g,\partial g,T,\Psi)}{\delta g^{\mu\nu}}&=\left(\frac{\partial\sqrt{-g}\mathcal{L}_M}{\partial g^{\mu\nu}}-\partial^\kappa\frac{\partial\sqrt{-g}\mathcal{L}_M}{\partial(\partial^\kappa g^{\mu\nu})}\right)\\
&=\left(\frac{\partial\sqrt{-g}\mathcal{L}_M}{\partial g^{\mu\nu}}\right.\\
&\qquad\left.-\partial^\kappa\frac{\partial\sqrt{-g}\mathcal{L}_M}{\partial\widehat{\Gamma}_\alpha^{\cdot(\beta\gamma)}}\frac{\partial\widehat{\Gamma}_\alpha^{\cdot(\beta\gamma)}}{\partial(\partial^\kappa g^{\mu\nu})}\right),
\end{aligned}
\tag{C.23}
$$

where different tensors have been defined in Eqs. (8), (32) and (33). This leads to

$$
-\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}_M(g,\partial g,T,\Psi)}{\delta g^{\mu\nu}}=\widehat{\tau}_{\mu\nu}+\frac{1}{2}\Delta^{\alpha\beta\gamma}_{\nu\mu\kappa}\overline{\nabla}^\kappa\,\widehat{\Sigma}_{\alpha(\beta\gamma)}\,.
\tag{C.24}
$$

The r.h.s. of Eq. (C.24) is exactly the expression on the r.h.s. of Eq. (42), while the l.h.s. is similar to Hilbert's definition of the energy-momentum tensor (C.21). Indeed $\widehat{\tau}_{\mu\nu}+\frac{1}{2}\Delta^{\alpha\beta\gamma}_{\nu\mu\kappa}\overline{\nabla}^\kappa\,\widehat{\Sigma}_{\alpha(\beta\gamma)}$ is the generalisation of Hilbert's definition of the energy-momentum tensor to the Riemann–Cartan U_4 spacetime.

On the other hand, $\Sigma_{[\tau\mu]\nu}$ is related to the contortion tensor, which is the remaining part of the connection, see Ref. [27]. Thus, the r.h.s. of Eq. (43) corresponds to the spin distribution tensor

$$
S_\sigma^{\cdot\mu\nu}\equiv-\frac{\partial\mathcal{L}_M(g,\partial g,T,\Psi)}{\partial K^\sigma_{\cdot\mu\nu}},
\tag{C.25}
$$

as defined in Refs. [4,27].

Appendix D: Vector and pseudo-vector torsion in the weak-gravity regime

In this appendix we are going to take the vector T^μ and pseudo-vector S^μ torsion components as the only non-vanishing torsion fields and calculate the expressions needed for the analysis carried out in Sect. 3.3.

Assuming that the only non-vanishing components of the torsion tensor in the decomposition (12) are the vector T_μ and pseudo-vector S_μ torsion components, the expression for the contortion tensor (11) can be rewritten as

$$
K^\mu_{\cdot\nu\sigma}=\frac{2}{3}g^{\mu\lambda}(T_\nu g_{\lambda\sigma}-T_\lambda g_{\nu\sigma})+\frac{1}{6}g^{\mu\alpha}\epsilon_{\alpha\nu\sigma\gamma}S^\gamma\,.
\tag{D.26}
$$

Under this assumption, the curvature tensor (15) takes the form

$$\begin{aligned}\widehat{R}^{\mu}_{\cdot\nu\rho\sigma} &= R^{\mu}_{\cdot\nu\rho\sigma} + \frac{2}{3}\left[\nabla_\rho(\delta^{\mu}{}_{\sigma}T_\nu - \eta_{\nu\sigma}T^\mu)\right.\\&\quad\left. - \nabla_\sigma(\delta^{\mu}{}_{\rho}T_\nu - \eta_{\nu\rho}T^\mu)\right]\\&\quad + \frac{4}{9}\left[(T_\sigma T_\nu - \eta_{\nu\sigma}T_\alpha T^\alpha)\delta^{\mu}{}_{\rho}\right.\\&\quad\left. - (T_\rho T_\nu - \eta_{\nu\rho}T_\beta T^\beta)\delta^{\mu}{}_{\sigma} + T^\mu(T_\rho\eta_{\nu\sigma} - T_\sigma\eta_{\nu\rho})\right]\\&\quad + \frac{1}{6}\eta^{\mu\alpha}\left(\epsilon_{\alpha\nu\sigma\beta}\nabla_\rho S^\beta - \epsilon_{\alpha\nu\rho\beta}\nabla_\sigma S^\beta\right)\\&\quad + \frac{1}{36}\eta^{\mu\alpha}\eta^{\lambda\delta}\left(\epsilon_{\alpha\lambda\rho\tau}\epsilon_{\delta\nu\sigma\gamma}S^\tau S^\gamma - \epsilon_{\alpha\lambda\sigma\tau}\epsilon_{\delta\nu\rho\gamma}S^\tau S^\gamma\right)\\&\quad - \frac{1}{9}\left[T^\alpha S^\gamma(\delta^{\mu}_{\sigma}\epsilon_{\alpha\nu\rho\gamma} - \delta^{\mu}_{\rho}\epsilon_{\alpha\nu\sigma\gamma})\right.\\&\quad + 2T^\mu S^\gamma\epsilon_{\rho\nu\sigma\gamma} - 2T_\nu S^\gamma\eta^{\mu\alpha}\epsilon_{\alpha\sigma\rho\gamma}\\&\quad\left. + \eta^{\mu\alpha}T^\lambda S^\gamma(\eta_{\nu\sigma}\epsilon_{\alpha\lambda\rho\gamma} - \eta_{\nu\rho}\epsilon_{\alpha\lambda\sigma\gamma})\right].\end{aligned} \tag{D.27}$$

The Ricci tensor is obtained by the usual contraction $\widehat{R}^{\mu}_{\cdot\nu\mu\sigma}$,

$$\begin{aligned}\widehat{R}_{\nu\sigma} &= R_{\nu\sigma} - \frac{2}{3}\left(2\nabla_\sigma T_\nu + \nabla_\alpha T^\alpha\eta_{\nu\sigma}\right)\\&\quad + \frac{8}{9}\left(T_\nu T_\sigma - T_\beta T^\beta\eta_{\nu\sigma}\right) + \frac{1}{6}\epsilon_{\alpha\nu\sigma\beta}\nabla^\alpha S^\beta\\&\quad - \frac{1}{36}\eta^{\mu\alpha}\eta^{\lambda\delta}\epsilon_{\alpha\lambda\sigma\beta}\epsilon_{\delta\nu\mu\gamma}S^\beta S^\gamma,\end{aligned} \tag{D.28}$$

and the scalar curvature $\widehat{R} = \eta^{\nu\sigma}\widehat{R}_{\nu\sigma}$,

$$\widehat{R} = R - 4\nabla_\alpha T^\alpha - \frac{8}{3}T_\beta T^\beta - \frac{1}{6}S_\beta S^\beta\,. \tag{D.29}$$

As we want to get a set of stability condition on the parameters of the theory when $g_{\mu\nu} = \eta_{\mu\nu}$, we take the expression of the curvature tensors (D.27), (D.28) and (D.29) to compute the scalars in the Lagrangian density (24). These are

$$\begin{aligned}\widehat{R}^2\Big|_{g=\eta} &= 16\partial_\alpha T^\alpha\partial_\beta T^\beta + \frac{64}{3}\partial_\alpha T^\alpha T_\beta T^\beta\\&\quad + \frac{8}{6}\partial_\alpha T^\alpha S_\beta S^\beta + \frac{8}{9}T_\alpha T^\alpha S_\beta S^\beta\\&\quad + \frac{1}{36}S_\alpha S^\alpha S_\beta S^\beta + \frac{64}{9}T_\alpha T^\alpha T_\beta T^\beta,\end{aligned} \tag{D.30}$$

$$\begin{aligned}\widehat{R}_{\nu\sigma}\widehat{R}^{\nu\sigma}\Big|_{g=\eta} &= \frac{16}{9}\partial_\mu T_\nu\partial^\mu T^\nu + \frac{32}{9}\partial_\alpha T^\alpha\partial_\beta T^\beta\\&\quad + \frac{1}{18}(\partial_\alpha S_\beta\partial^\alpha S^\beta - \partial_\alpha S_\beta\partial^\beta S^\alpha)\\&\quad + \frac{4}{9}\epsilon^{\mu\nu\rho\sigma}\partial_\mu S_\sigma\partial_\rho T_\nu\\&\quad + \frac{160}{27}\partial_\alpha T^\alpha T_\beta T^\beta - \frac{64}{27}\partial_\mu T_\nu T^\mu T^\nu\\&\quad + \frac{10}{27}\partial_\alpha T^\alpha S_\beta S^\beta - \frac{4}{27}\partial_\mu T_\nu S^\mu S^\nu\\&\quad + \frac{64}{27}T_\alpha T^\alpha T_\beta T^\beta\\&\quad + \frac{1}{108}S_\alpha S^\alpha S_\beta S^\beta + \frac{16}{81}T_\alpha T^\alpha S_\beta S^\beta\\&\quad + \frac{8}{81}T_\alpha S^\alpha T_\beta S^\beta\,,\end{aligned} \tag{D.31}$$

$$\begin{aligned}&\widehat{R}_{\nu\sigma}\widehat{R}^{\sigma\nu}\Big|_{g=\eta}\\&= \frac{48}{9}\partial_\alpha T^\alpha\partial_\beta T^\beta\\&\quad - \frac{1}{18}(\partial_\alpha S_\beta\partial^\alpha S^\beta - \partial_\alpha S_\beta\partial^\beta S^\alpha)\\&\quad + \frac{4}{9}\epsilon^{\mu\nu\rho\sigma}\partial_\mu S_\sigma\partial_\nu T_\rho + \frac{160}{27}\partial_\alpha T^\alpha T_\beta T^\beta\\&\quad - \frac{64}{27}\partial_\alpha T_\beta T^\beta T^\alpha + \frac{10}{27}\partial_\alpha T^\alpha S_\beta S^\beta\\&\quad - \frac{4}{27}\partial_\mu T_\nu S^\mu S^\nu + \frac{64}{27}T_\alpha T^\alpha T_\beta T^\beta\\&\quad + \frac{1}{108}S_\alpha S^\alpha S_\beta S^\beta\\&\quad + \frac{16}{81}T_\alpha T^\alpha S_\beta S^\beta + \frac{8}{81}T_\alpha S^\alpha T_\beta S^\beta,\end{aligned} \tag{D.32}$$

$$\begin{aligned}&\widehat{R}_{\mu\nu\rho\sigma}\widehat{R}^{\mu\nu\rho\sigma}\Big|_{g=\eta}\\&= \frac{32}{9}\partial_\rho T_\nu\partial^\rho T^\nu + \frac{16}{9}\partial_\alpha T^\alpha\partial_\beta T^\beta\\&\quad + \frac{2}{9}\partial_\alpha S_\beta\partial^\alpha S^\beta + \frac{1}{9}\partial_\alpha S^\alpha\partial_\beta S^\beta\\&\quad + \frac{8}{9}\epsilon^{\mu\nu\rho\sigma}\partial_\nu S_\sigma\partial_\rho T_\mu\\&\quad - \frac{128}{27}\partial_\rho T_\nu T^\rho T^\nu + \frac{128}{27}\partial_\alpha T^\alpha T_\beta T^\beta\\&\quad + \frac{8}{27}\partial_\alpha\left(T^\alpha S_\beta S^\beta - S^\alpha T_\beta S^\beta\right)\\&\quad + \frac{8}{9}\partial_\alpha S^\alpha T_\beta S^\beta - \frac{8}{9}\partial_\alpha S_\beta T^\alpha S^\beta\\&\quad + \frac{64}{27}T_\alpha T^\alpha T_\beta T^\beta + \frac{1}{108}S_\alpha S^\alpha S_\beta S^\beta\\&\quad + \frac{24}{81}T_\alpha T^\alpha S_\beta S^\beta + \frac{48}{81}T_\alpha S^\alpha T_\beta S^\beta,\end{aligned} \tag{D.33}$$

$$\begin{aligned}&\widehat{R}_{\mu\nu\rho\sigma}\widehat{R}^{\rho\sigma\mu\nu}\Big|_{g=\eta}\\&= \frac{32}{9}\partial_\rho T_\nu\partial^\nu T^\rho + \frac{16}{9}\partial_\alpha T^\alpha\partial_\beta T^\beta\\&\quad - \frac{2}{9}(\partial_\alpha S_\beta\partial^\alpha S^\beta - \partial_\alpha S^\alpha\partial_\beta S^\beta)\\&\quad - \frac{8}{9}\epsilon^{\mu\nu\rho\sigma}\partial_\nu S_\sigma\partial_\mu T_\rho\\&\quad - \frac{128}{27}\partial_\rho T_\nu T^\rho T^\nu + \frac{128}{27}\partial_\alpha T^\alpha T_\beta T^\beta\\&\quad - \frac{8}{27}\partial_\alpha S_\beta T^\alpha S^\beta + \frac{16}{27}\partial_\alpha T_\beta S^\alpha S^\beta\\&\quad + \frac{64}{27}T_\alpha T^\alpha T_\beta T^\beta + \frac{1}{108}S_\alpha S^\alpha S_\beta S^\beta\\&\quad - \frac{8}{81}T_\alpha T^\alpha S_\beta S^\beta + \frac{32}{81}T_\alpha S^\alpha T_\beta S^\beta,\end{aligned} \tag{D.34}$$

$$\widehat{R}_{\mu\nu\rho\sigma}\widehat{R}^{\mu\rho\nu\sigma}\Big|_{g=\eta}$$

$$= \frac{8}{9}\partial_\rho T_\nu \partial^\nu T^\rho + \frac{8}{9}\partial_\rho T_\nu \partial^\rho T^\nu + \frac{8}{9}\partial_\alpha T^\alpha \partial_\beta T^\beta - \frac{1}{6}\partial_\alpha S^\alpha \partial_\beta S^\beta + \frac{8}{9}\epsilon^{\mu\nu\rho\sigma}\partial_\nu S_\sigma \partial_\rho T_\mu + \frac{32}{27}\partial_\alpha T^\alpha T_\beta T^\beta - \frac{32}{27}\partial_\alpha T_\beta T^\beta T^\alpha + \frac{4}{27}\partial_\alpha T^\alpha S_\beta S^\beta - \frac{4}{27}\partial_\alpha T_\beta S^\alpha S^\beta - \frac{12}{27}\partial_\alpha S^\alpha T_\beta S^\beta + \frac{32}{27}T_\alpha T^\alpha T_\beta T^\beta + \frac{1}{216}S_\alpha S^\alpha S_\beta S^\beta - \frac{16}{81}T_\alpha S^\alpha T_\beta S^\beta + \frac{4}{81}T_\alpha T^\alpha S_\beta S^\beta . \tag{D.35}$$

Note that there are no terms $\partial T \partial S$, ∂STT, or STT , as expected from parity conservation. On the other hand, it is also possible to compute the pure torsion squared terms via Eq. (12). These are,

$$T_{\mu\nu\rho}T^{\mu\nu\rho} = \frac{2}{3}T_\mu T^\mu + \frac{1}{6}S_\nu S^\nu , \tag{D.36}$$

$$T_{\mu\nu\rho}T^{\nu\rho\mu} = -\frac{1}{3}T_\mu T^\mu + \frac{1}{6}S_\nu S^\nu , \tag{D.37}$$

$$T^{\lambda}_{\cdot\mu\lambda}T^{\cdot\mu\rho}_{\rho} = T_\mu T^\mu . \tag{D.38}$$

In view of these calculations, the potential that appears in Eq. (53) is

$$\mathcal{V}(T,S) = -\frac{2}{3}(c+3\lambda)T_\alpha T^\alpha - \frac{1}{24}(b+3\lambda)S_\alpha S^\alpha - \frac{12}{27}q\,\partial_\alpha S^\alpha T_\beta T^\beta - \frac{8}{81}(3r-4p-2q)\partial_\alpha S_\beta T^\alpha S^\beta - \frac{64}{81}(q-5p+6r-6s)\partial_\alpha T_\beta T^\alpha T^\beta - \frac{64}{81}(5p-q+6r+15s)\partial_\alpha T^\alpha T_\beta T^\beta - \frac{8}{81}(p+2q-3s)\partial_\alpha T_\beta S^\alpha S^\beta - \frac{4}{81}(2p-2q+15s)\partial_\alpha T^\alpha S_\beta S^\beta - \frac{64}{27}(p-r+2s)T_\alpha T^\alpha T_\beta T^\beta - \frac{1}{108}(p-r+2s)S_\alpha S^\alpha S_\beta S^\beta - \frac{8}{81}(p+r+4s)T_\alpha T^\alpha S_\beta S^\beta - \frac{8}{81}(2p+3q-4r+2s)T_\alpha S^\alpha T_\beta S^\beta . \tag{D.39}$$

Note that the parameter t does not appear in the expression of the potential, since the antisymmetric part of the Ricci tensor does not give rise to potential-type terms for the vector and pseudo-vector torsion degrees of freedom.

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Chapter 4

Einstein-Yang-Mills systems

4.1 Introduction to Einstein-Yang-Mills theory

Einstein-Yang-Mills (EYM) theory constitutes the general framework to describe the nature as well as the interaction of non-Abelian gauge fields and conventional gravitation. Thus, this theory describes the phenomenology of YM fields [83], such as the electro-weak model or the strong nuclear force associated with quantum chromodynamics, in the presence of a curved space-time and it represents the most natural generalization of the Einstein-Maxwell theory.

Therefore, in principle, the search for non-Abelian systems and heterogeneous BHs in the framework of GR would be admissible. Nevertheless, according to the no-hair conjecture, the structure of a stationary BH is completely determined by its mass, its orbital angular momentum and its Abelian charge, which means a strong conjectural restriction on the possible existence of BH configurations endowed with YM fields. Despite this assumption, a large number of EYM BHs were systematically found out and classified as counterexamples that manifestly violated it, showing up a notable and richer structure than the ones expected from the Abelian sector [84].

From a mathematical point of view, a gauge field over a pseudo-Riemannian manifold $\mathcal{M}$ is associated with a Lie group $\mathcal{G}$ and is described by a connection 1-form A in the principal bundle $\mathcal{P}\left(\mathcal{M},\mathcal{G}\right)$, which takes values on the Lie algebra:

$$A_\mu = A^a_\mu\, T_a\,, \tag{4.1}$$

with T_a the respective generators of such Lie algebra, which satisfy the following completeness relation:

$$[T_a, T_b] = i f_{abc}\, T^c\,, \tag{4.2}$$

where the coefficients f_{abc} are the so called structure constants.

The gauge connection (4.1) defines a covariant derivative on the tangent bundle of $\mathcal{G}$ and a 2-form gauge curvature F, which constitutes the YM propagating field playing the role of carrier of the interaction:

$$D_\mu = \nabla_\mu - i\,[A_\mu, \cdot\,]\ , \tag{4.3}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu, A_\nu]\ . \tag{4.4}$$

Then, by considering an arbitrary vector v^λ, the following commutation relation is satisfied:

$$[D_\mu, D_\nu]\ v^\lambda = R^\lambda{}_{\rho\mu\nu}\, v^\rho - i\left[F_{\mu\nu}, v^\lambda\right]\ . \tag{4.5}$$

The behaviour of these components under a gauge transformation $S \in \mathcal{G}$:

$$A_\mu \to A'_\mu = S^{-1} A_\mu S + i S^{-1} \partial_\mu S\,, \tag{4.6}$$

$$F_{\mu\nu} \to F'_{\mu\nu} = S^{-1} F_{\mu\nu} S\,, \tag{4.7}$$

allows the construction of minimal coupling actions by the standard procedure:

$$S = -\frac{1}{16\pi}\int \left(R - tr\,F_{\mu\nu}F^{\mu\nu}\right)\sqrt{-g}\,d^4x\,. \tag{4.8}$$

The metric tensor and the gauge connection represent the main variables in this approach and their variations lead to the general EYM equations:

$$D_\mu\, F^{\mu\nu} = 0\,, \tag{4.9}$$

$$G_{\mu\nu} = 8\pi T_{\mu\nu}\,, \tag{4.10}$$

where $T_{\mu\nu} = \frac{1}{4\pi}\,tr\left(\frac{1}{4} g_{\mu\nu} F_{\lambda\rho}F^{\lambda\rho} - F_{\mu\rho}F_\nu{}^\rho\right)$ is the energy-momentum tensor associated with the YM field. Furthermore, the divergenceless of the Einstein tensor implies the same for the energy-momentum tensor of the YM field, which also satisfies the following identity from its propagating equation (4.9):

$$D_\mu D_\nu\, F^{\mu\nu} = 0\,. \tag{4.11}$$

Indeed, this quantity is proportional to the contraction of the commutator of the covariant derivatives and the gauge curvature, which vanishes identically since the Ricci tensor constitutes a symmetric quantity in Riemannian geometry and the commutator of two field strength tensors is zero:

$$\frac{1}{2}\left[D_\mu, D_\nu\right] F^{\mu\nu} = R_{\mu\nu} F^{\mu\nu} - \frac{i}{2}\left[F_{\mu\nu}, F^{\mu\nu}\right] . \tag{4.12}$$

In order to solve the EYM field equations, it is possible to simplify the problem by applying a suitable set of internal gauge transformations that preserve the invariance of the gauge connection under space-time symmetries [85]. Specifically, the change in the gauge potential $A_\mu \to A'_\mu = A_\mu - \mathcal{L}_\xi A_\mu$ under the infinitesimal coordinate transformation $x^\mu \to x'^\mu = x^\mu + \xi^\mu$ can be compensated by the following infinitesimal gauge transformation:

$$A'_\mu \to \hat{A}_\mu = A'_\mu + \partial_\mu \omega - i\left[A_\mu, \omega\right] , \tag{4.13}$$

where $\partial_\mu \omega - i\left[A_\mu, \omega\right] = \mathcal{L}_\xi A_\mu$ implies the equality $\hat{A}_\mu = A_\mu$ in the present gauge. Such a gauge condition represents a strong constraint for the covariant component of the connection that can be solved in special cases, like the one given by the gauge group $SU(2)$ in the presence of a static and spherically symmetric space-time (see Appendix C for a detailed resolution within this context). In this sense, the final expression for the gauge connection acquires the following structure [86]:

$$\begin{aligned} A &= p(r)\,\tau_3\,dt + u(r)\,\tau_3\,dr + \left(v(r)\,\tau_1 + w(r)\,\tau_2\right) d\theta_1 \\ &+ \left(\cot\theta_1\,\tau_3 + v(r)\,\tau_2 - w(r)\,\tau_1\right) \sin\theta_1 d\theta_2 , \end{aligned} \tag{4.14}$$

with p, u, v and w four arbitrary functions depending on r and $\{\tau_i\}_{i=1,2,3}$ the generators related to $SU(2)$, which obey the standard commutation relations:

$$\left[\tau_i, \tau_j\right] = i\epsilon_{ijk}\tau^k . \tag{4.15}$$

In addition, besides the strong simplification provided by this symmetry condition, the remaining group of residual gauge transformations still preserves this ansatz and may restrict even more the number of degrees of freedom involved in the problem. In this case, the gauge transformation $V_1 = e^{i\alpha(r)\tau_3}$ with $\alpha'(r) = u(r)$ allows the vanishing of the spatial component A^3_r without changing the structure of (4.14).

Thereby, the corresponding gauge curvature associated with this simple ansatz reads:

$$
\begin{aligned}
F &= p'(r)\tau_3\, dr\wedge dt + \Big(v^2(r)+w^2(r)-1\Big)\sin\theta_1\tau_3\, d\theta_1\wedge d\theta_2 \\
&+ p(r)\left(v(r)\tau_2 - w(r)\tau_1\right)\, dt\wedge d\theta_1 - p(r)\left(v(r)\tau_1 + w(r)\tau_2\right)\sin\theta_1\, dt\wedge d\theta_2 \\
&+ \left(v'(r)\tau_1 + w'(r)\tau_2\right)\, dr\wedge d\theta_1 + \left(v'(r)\tau_2 - w'(r)\tau_1\right)\sin\theta_1\, dr\wedge d\theta_2\,. \qquad (4.16)
\end{aligned}
$$

By taking into account the EYM equations and setting the third component in the Lie algebra of the expression $D_\mu F^{\mu 1}$ equal to zero, it is straightforward to obtain the following restriction involving the spatial components of the gauge connection:

$$v(r) = k\, w(r)\,, \tag{4.17}$$

where k is a constant. Then, this expression must be fulfilled in order to satisfy the mentioned EYM field equation. Furthermore, in virtue of this constraint, it is possible to apply a new residual gauge transformation $V_2 = e^{ik\tau_3}$ that vanishes the components $A^1_{\theta_1}$ and $A^2_{\theta_2}$, which means a new decrease in the number of degrees of freedom contained in the gauge potential.

The two remaining components $p(r)$ and $w(r)$ can be directly related to the electric and magnetic components of the YM field. Indeed, the standard definition of such components:

$$E_\mu = F_{\mu\nu}u^\nu\,, \tag{4.18}$$

$$B_\mu = \frac{\sqrt{-g}}{2}\epsilon^{\lambda\rho}{}_{\mu\nu}F_{\lambda\rho}u^\nu\,, \tag{4.19}$$

particularized in the rest frame of reference, gives rise to the following outcome for a static and spherically symmetric space-time with line element (2.6):

$$E_1 = p'(r)\,\tau_3\,, \tag{4.20}$$

$$E_2 = p(r)w(r)\,\tau_1\,, \tag{4.21}$$

$$E_3 = p(r)w(r)\sin\theta_1\,\tau_2\,, \tag{4.22}$$

$$B_1 = \frac{1-w^2(r)}{r^2}\sqrt{\frac{\Psi_1(r)}{\Psi_2(r)}}\,\tau_3\,, \tag{4.23}$$

$$B_2 = -\, w'(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,\tau_1\,, \tag{4.24}$$

$$B_3 = -\, w'(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,\sin\theta_1\,\tau_2\,. \tag{4.25}$$

Therefore, the case $p(r) = 0$ describes a purely magnetic configuration and then constitutes a special system within this theory. In fact, two types of purely magnetic non-Abelian solutions to the static and spherically symmetric EYM field equations can be found: a self-gravitating solitonic solution derived by Bartnik and McKinnon (BK) and a hairy BH solution discovered by Bizon [87, 88]. Consequently, they are related to different boundary conditions, namely to regular and singular conditions, respectively.

First, by introducing a pair of free parameters a, b and the ADM mass M, the BK solution acquires the following structure for the degrees of freedom contained in the metric tensor and the gauge field:

$$m(r) = \begin{cases} 2b^2r^3 - \frac{8b^3r^5}{5} + \mathcal{O}(r^6)\,, & \text{if} \quad 0 \leq r \ll \bar{r} \\ \\ M - \frac{a^2}{r^3} + \mathcal{O}(r^{-4})\,, & \text{if} \quad \bar{r} \ll r\,, \end{cases} \tag{4.26}$$

$$w(r) = \begin{cases} 1 - br^2 + \left(\frac{3b^2}{10} - \frac{4b^3}{5}\right) r^4 + \mathcal{O}(r^6)\,, & \text{if} \quad 0 \leq r \ll \bar{r} \\ \\ \pm\left(1 - \frac{a}{r} + \left(\frac{3a^2-6aM}{4r^2}\right) + \mathcal{O}(r^{-3})\right)\,, & \text{if} \quad \bar{r} \ll r\,, \end{cases} \tag{4.27}$$

where the metric functions have been redefined for convenience in the following way:

$$\Psi_1(r) = \sigma^2(r)\Psi_2(r)\,, \tag{4.28}$$

$$\Psi_2(r) = 1 - \frac{2m(r)}{r}\,, \tag{4.29}$$

and the function $\sigma(r)$ is completely determined by the EYM equation:

$$\frac{\sigma'(r)}{\sigma(r)} = \frac{2w'^{\,2}(r)}{r}\,. \tag{4.30}$$

Then, it is possible to match the two regions at the point $\bar{r}$ by a numerical extension and to obtain a globally regular solution for a discrete family of values of

n	a_n	b_n	M_n
1	0.8933	0.4537	0.8286
2	8.8638	0.6517	0.9713
3	58.9290	0.6970	0.9953
4	366.2000	0.7048	0.9992
5	2246.8000	0.7061	0.9998

Table 4.1: Parameters of the BK solution.

the parameters, which can be labeled by a natural number n (see table 4.1 to check the first values of the family $\{a_n, b_n, M_n\}$, from $n = 1$ to $n = 5$).

The geometry shows three principal regions that evince the richer structure provided by the interaction between gravity and non-Abelian gauge fields: a high density region characterized by an intense YM field strength, a near-field zone where the metric is approximately a Reissner-Nordström (RN) type with unit magnetic charge and a far-field region where this charge decays asymptotically to zero and the metric resembles the Schwarzschild geometry with mass M_n. The balance between the attractive component of the gravitational field and the repulsive forces applied by the $SU(2)$ field allows the existence of this equilibrium configuration and prevents the formation of singularities in the space-time.

As can be seen from Fig. 4.1, the contrast existing between these zones becomes even more remarkable for higher values of n, where it is worthwhile to stress the fast increase of the ADM mass and the corresponding transition to heavier self-gravitating states within this context. Furthermore, this parameter also provides the number of nodes of the function $w(r)$, which moreover is bounded within the interval $[-1, 1]$.

The BH configuration also constitutes a discrete family of solutions and presents the same geometrical pattern with the cited transition zones, but it includes the existence of a regular event horizon provided by the following singular boundary conditions:

$$\Psi_2(r) = \begin{cases} \frac{r_h^2 - \left(w_h^2 - 1\right)^2}{r_h^3}\,(r - r_h) + \mathcal{O}\left((r - r_h)^2\right)\,, & \text{if} \quad r \approx r_h \\ \\ 1 - \frac{2M}{r} + \frac{2a^2}{3r^4} + \mathcal{O}(r^{-5})\,, & \text{if} \quad r_h \ll r\,, \end{cases} \tag{4.31}$$

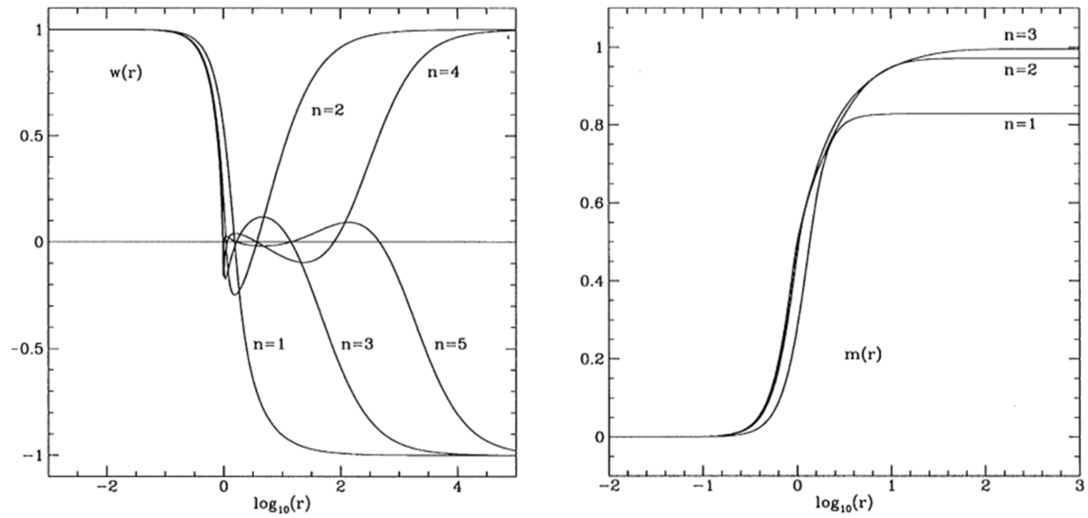

Figure 4.1: Obtained from [84]. $w(r)$ and the effective mass $m(r)$ for the lowest BK solution.

$$w(r) = \begin{cases} w_h + \frac{w_h r_h^3 \left(w_h^2 - 1\right)}{r_h^2 - \left(w_h^2 - 1\right)^2} (r - r_h) + \mathcal{O}\left((r - r_h)^2\right) , & \text{if} \quad r \approx r_h \\ \\ \pm \left(1 - \frac{a}{r} + \frac{3a^2 - 6aM}{4r^2} + \mathcal{O}(r^{-3})\right) , & \text{if} \quad r_h \ll r , \end{cases} \tag{4.32}$$

where the parameter r_h indicates the location of the event horizon of the solution and w_h the value of function $w(r)$ at $r = r_h$. In addition, the interior region in the vicinity $r \approx 0$ of the essential singularity can be numerically matched by three distinct types of local solutions (see [84] for further details on interior solutions).

The existence of these regular and BH solutions constitutes the first example of a self-gravitating system coupled to a non-Abelian gauge field and also the first manifest violation of the no-hair conjecture in the framework of the EYM theory. Nevertheless, a large class of analytical and numerical studies have been shown their instability under small spherically symmetric perturbations and, furthermore, in the nonlinear regime [89–92], which strongly questions its validity as a viable configuration in nature.

On the other hand, from a theoretical point of view, a large variety of extended solutions to the EYM equations have also been systematically found by different authors, including the application of higher rank non-Abelian groups or the incorporation of a cosmological constant and external fields into the general action, such as the dilaton and the Higgs fields [93–99], which represents a considerable number of examples and theoretical illustrations within this field.

All these solutions can be formally classified by attending to the algebraic properties of the YM field. In the same way that the Petrov classification of the conventional gravitational field describes the algebraic symmetries of the Weyl tensor [100], the Carmeli method establishes for the YM field an analogous result according to its distinct eigenspinors and eigenvalues [101, 102]. Specifically, this method takes into account an eigenspinor equation for the following gauge invariant spinors defined from the non-Abelian gauge field [1]:

$$\eta_{ABCD} = \xi_{(ABCD)} \,, \tag{4.34}$$

$$\xi_{ABCD} = \frac{1}{4} \epsilon^{\dot{E}\dot{F}} \epsilon^{\dot{G}\dot{H}} \left(f_{A\dot{E}B\dot{F}} \, f_{C\dot{G}D\dot{H}} \right) \,, \tag{4.35}$$

where $f_{A\dot{B}C\dot{D}} = \tau^{\mu}_{A\dot{B}} \, \tau^{\nu}_{C\dot{D}} \, F_{\mu\nu}$ is the spinor equivalent to the YM strength field tensor written in terms of the generalizations of the unit and Pauli matrices, which establish the correspondence between spinors and tensors, whereas the dotted and undotted indices run from $\dot{1}$ to $\dot{2}$ and 1 to 2, respectively.

These quantities define, among others, the following invariants of the YM field:

$$P = \xi_{AB}{}^{AB} \,, \tag{4.36}$$

$$G = \eta_{ABCD} \, \eta^{ABCD} \,, \tag{4.37}$$

$$H = \eta_{AB}{}^{CD} \, \eta_{CD}{}^{EF} \, \eta_{EF}{}^{AB} \,, \tag{4.38}$$

where the parameter P relates to the previous spinor fields by means of the expression:

$$\xi_{ABCD} = \eta_{ABCD} + \frac{P}{6} \left(\epsilon_{AC} \epsilon_{BD} + \epsilon_{AD} \epsilon_{BC} \right) \,. \tag{4.39}$$

In such a case, by introducing a symmetrical spinor ϕ_{AB}, the corresponding equation $\eta_{AB}{}^{CD} \, \phi_{CD} = \lambda' \, \phi_{AB}$ provides the set of eigenspinors and eigenvalues of the spinor field ξ_{ABCD} by the relation $\lambda = \lambda' + P/3$, where the root λ' can be directly computed by the characteristic polynomial:

---

[1] The YM spinor field satisfies the following symmetry properties:

$$\xi_{ABCD} = \xi_{BACD} \,, \quad \xi_{ABCD} = \xi_{ABDC} \,, \quad \xi_{ABCD} = \xi_{CDAB} \,. \tag{4.33}$$

$$p(\lambda') = \lambda'^{\,3} - G\lambda'/2 - H/3\,. \tag{4.40}$$

Hence, the classification scheme of the principal spinor ξ_{ABCD} reduces to the alternative and simpler classification of the totally symmetric spinor η_{ABCD}. The distinct families of eigenspinors and eigenvalues allow the YM field to be categorized in a systematic way, which in fact improves the physical understanding of the EYM solutions. In table 4.2, the possible algebraic symmetry types are displayed, according to the number of degenerate eigenvalues and linearly independent spinors, as well as to the value of the invariant P. Note that, for each symmetry type, it is possible to express the quantity ξ_{ABCD} in terms of up to four arbitrary one-index spinors L_A, M_A, N_A and K_A.

This algebraic structure points out the existence of different degenerate levels, in the sense that possible transitions with a consequent loss of generality can occur (see diagram 4.2).

The particularization to the purely magnetic $SU(2)$ gauge field in a static and spherically symmetric geometry can then be accomplished by computing the components of the associated YM spinors and their gauge invariants. By considering the line element defined by Expression (2.6) and the gauge curvature (4.16) with $p(r) = v(r) = 0$, these components are:

$$\eta_{1111} = \xi_{1111}\,, \tag{4.41}$$

$$\eta_{2222} = \xi_{2222}\,, \tag{4.42}$$

$$\eta_{1122} = \frac{1}{12r^2}\left(\frac{\left(w^2(r)-1\right)^2}{r^2} - \Psi_2(r)\, w'^{\,2}(r)\right)\,, \tag{4.43}$$

$$\xi_{1111} = \xi_{2222}\,, \tag{4.44}$$

$$\xi_{2222} = \frac{1}{4r^2}\left(\Psi_2(r)\, w'^{\,2}(r) - \frac{\left(w^2(r)-1\right)^2}{r^2}\right)\,, \tag{4.45}$$

$$\xi_{1122} = \frac{1}{4r^2}\left(\Psi_2(r)\, w'^{\,2}(r) + \frac{\left(w^2(r)-1\right)^2}{r^2}\right)\,, \tag{4.46}$$

$$\xi_{1212} = -\,\frac{\Psi_2(r)\, w'^{\,2}(r)}{4r^2}\,, \tag{4.47}$$

I_P	$\lambda_i \neq \lambda_j \,\forall\, i,j=1,2,3$	$3\ \phi_{AB}$ l.i.	$\xi_{ABCD} = L_{(A}M_B N_C K_{D)} - \frac{P}{3}\epsilon_{A(C}\epsilon_{D)B}$
I_0	$\lambda_i \neq \lambda_j \,\forall\, i,j=1,2,3$	$3\ \phi_{AB}$ l.i.	$\xi_{ABCD} = L_{(A}M_B N_C K_{D)}$
II_P	$\lambda_1 \neq \lambda_2 = \lambda_3$	$2\ \phi_{AB}$ l.i.	$\xi_{ABCD} = L_{(A}L_B M_C N_{D)} - \frac{P}{3}\epsilon_{A(C}\epsilon_{D)B}$
II_0	$\lambda_1 \neq \lambda_2 = \lambda_3$	$2\ \phi_{AB}$ l.i.	$\xi_{ABCD} = L_{(A}L_B M_C N_{D)}$
D_P	$\lambda_1 \neq \lambda_2 = \lambda_3$	$3\ \phi_{AB}$ l.i.	$\xi_{ABCD} = L_{(A}L_B M_C M_{D)} - \frac{P}{3}\epsilon_{A(C}\epsilon_{D)B}$
D_0	$\lambda_1 \neq \lambda_2 = \lambda_3$	$3\ \phi_{AB}$ l.i.	$\xi_{ABCD} = L_{(A}L_B M_C M_{D)}$
III_P	$\lambda_1 = \lambda_2 = \lambda_3$	$1\ \phi_{AB}$ l.i.	$\xi_{ABCD} = L_{(A}L_B L_C M_{D)} - \frac{P}{3}\epsilon_{A(C}\epsilon_{D)B}$
III_0	$\lambda_1 = \lambda_2 = \lambda_3$	$1\ \phi_{AB}$ l.i.	$\xi_{ABCD} = L_{(A}L_B L_C M_{D)}$
IV_P	$\lambda_1 = \lambda_2 = \lambda_3$	$2\ \phi_{AB}$ l.i.	$\xi_{ABCD} = L_{(A}L_B L_C L_{D)} - \frac{P}{3}\epsilon_{A(C}\epsilon_{D)B}$
IV_0	$\lambda_1 = \lambda_2 = \lambda_3$	$2\ \phi_{AB}$ l.i.	$\xi_{ABCD} = L_{(A}L_B L_C L_{D)}$
0_P	$\lambda_1 = \lambda_2 = \lambda_3$	$3\ \phi_{AB}$ l.i.	$\xi_{ABCD} = -\frac{P}{3}\epsilon_{A(C}\epsilon_{D)B}$
0_0	$\lambda_1 = \lambda_2 = \lambda_3 = 0$	$3\ \phi_{AB}$ l.i.	$\xi_{ABCD} = 0$

Table 4.2: Carmeli types for the YM field.

and the characteristic polynomial for the spinor η_{ABCD} acquires the following form:

$$\begin{aligned} p(\lambda') &= \lambda'^{\,3} - \frac{\lambda'}{12r^4}\left(\Psi_2(r)\, w'^{\,2}(r) - \frac{\left(w^2(r)-1\right)^2}{r^2}\right)^2 \\ &+ \frac{1}{108r^6}\left(\Psi_2(r)\, w'^{\,2}(r) - \frac{\left(w^2(r)-1\right)^2}{r^2}\right)^3 . \end{aligned} \tag{4.48}$$

Hence, in general, the eigenspinor equation $\eta_{AB}{}^{CD}\,\phi_{CD} = \lambda'\,\phi_{AB}$ gives rise to a simple eigenvalue:

$$\lambda'_1 = -\frac{1}{3r^2}\left(\Psi_2(r)\, w'^{\,2}(r) - \frac{\left(w^2(r)-1\right)^2}{r^2}\right), \tag{4.49}$$

and to a degenerate eigenvalue:

$$\lambda'_2 = \lambda'_3 = \frac{1}{6r^2}\left(\Psi_2(r)\, w'^{\,2}(r) - \frac{\left(w^2(r)-1\right)^2}{r^2}\right). \tag{4.50}$$

The transformation $\lambda = \lambda' + P/3$ can then be applied by the calculation of this parameter and, consequently, the classification of the YM field can be achieved:

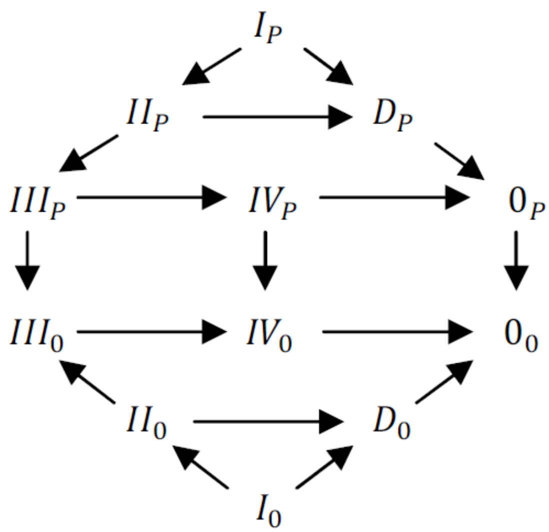

Figure 4.2: Diagram of classification of YM fields.

$$\lambda_1 = \frac{\left(w^2(r)-1\right)^2}{2r^4}\,, \tag{4.51}$$

$$\lambda_2 = \lambda_3 = \frac{\Psi_2(r)\, w'^{\,2}(r)}{2r^2}\,, \tag{4.52}$$

with:

$$P = \frac{1}{2r^2}\left(2\Psi_2(r)\, w'^{\,2}(r) + \frac{\left(w^2(r)-1\right)^2}{r^2}\right)\,. \tag{4.53}$$

i) $\lambda_1 \neq \lambda_2 = \lambda_3$

In this case, there exist three linearly independent eigenspinors and a non-vanishing gauge invariant P:

$$B_\phi \;=\; \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}\,. \tag{4.54}$$

Therefore, the YM field constitutes a type D_P and is associated with isolated gravitating systems. Note that the embedded RN solution with unit magnetic charge, given by the condition $w(r) = 0$, also belongs to this class of symmetry.

ii) $\lambda_1 = \lambda_2 = \lambda_3$

In this case, there exist only one degenerate eigenvalue on account of the relation:

$$\Psi_2(r)\, w'^{\,2}(r) = \frac{\left(w^2(r) - 1\right)^2}{r^2}\,, \tag{4.55}$$

which additionally involves the vanishing of the completely symmetric spinor η_{ABCD} and the simplification of the gauge invariant:

$$P = \frac{3\left(w^2(r) - 1\right)^2}{2r^4}\,. \tag{4.56}$$

The number of linearly independent eigenspinors is once again three, which means that the YM field reduces to a type 0_P if $w(r) \neq \pm 1$ or to a type 0_0 if $w(r) = \pm 1$, namely if the YM field vanishes identically and the solution coincides with the Schwarzschild solution:

$$B_\phi \;=\; \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}. \tag{4.57}$$

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Einstein–Yang–Mills–Lorentz black holes

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Abstract Different black hole solutions of the coupled Einstein–Yang–Mills equations have been well known for a long time. They have attracted much attention from mathematicians and physicists since their discovery. In this work, we analyze black holes associated with the gauge Lorentz group. In particular, we study solutions which identify the gauge connection with the spin connection. This *ansatz* allows one to find exact solutions to the complete system of equations. By using this procedure, we show the equivalence between the Yang–Mills–Lorentz model in curved space-time and a particular set of extended gravitational theories.

1 Introduction

The dynamical interacting system of equations related to non-abelian gauge theories defined on a curved space-time is known as Einstein–Yang–Mills (EYM) theory. Thus, this theory describes the phenomenology of Yang–Mills fields [1] interacting with the gravitational attraction, such as the electro-weak model or the strong nuclear force associated with quantum chromodynamics. The EYM model constitutes a paradigmatic example of the non-linear interactions related to gravitational phenomenology. Indeed, the evolution of a spherical symmetric system obeying these equations can be very rich. Its dynamics is opposite to the one predicted by other models, such as the ones provided by the Einstein–Maxwell (EM) equations, whose static behaviour is enforced by a version of the Birkhoff theorem.

For instance, in the four-dimensional space-time, the EYM equations associated with the gauge group $SU(2)$ support a discrete family of static self-gravitating solitonic solutions, found by Bartnik and McKinnon [2]. There are also *hairy* black hole (BH) solutions, as was shown by Bizon [3–5]. They are known as colored black holes and can be labeled by the number of nodes of the exterior Yang–Mills field configuration. Although the Yang–Mills fields do not vanish completely outside the horizon, these solutions are characterized by the absence of a global charge. This feature is opposite to the one predicted by the standard BH uniqueness theorems related to the EM equations, whose solutions can be classified solely with the values of the mass, (electric and/or magnetic) charge and angular momentum evaluated at infinity. In any case, the EYM model also supports the Reissner–Nordström BH as an embedded abelian solution with global electric and/or magnetic charge [6]. It is also interesting to mention that there are a larger variety of solutions associated with different generalizations of the EYM equations extended with dilaton fields, higher curvature corrections, Higgs fields, merons or cosmological constants (see [7,8] and the references therein).

In this work, we are interested in finding solutions of the EYM equations associated with the Lorentz group as gauge group. The main motivation for considering such a gauge symmetry is offered by the spin connection dynamics. This connection is introduced for a consistent description of spinor fields defined on curved space-times. Although general coordinate transformations do not have spinor representations [9], they can be described by the representations associated with the Lorentz group. In addition, they can be used to define alternative theories of gravity [10].

We shall impose the requirement that the spin connection is dynamical and its evolution is determined by the Yang–Mills action related to the $SO(1, n-1)$ symmetry, where n is the number of dimensions of the space-time. In order to complete the EYM equations, we shall assume that gravitation is described by the metric of a Lorentzian manifold. We shall find vacuum analytic solutions to the EYM system by choosing a particular *ansatz*, which will relate the spin connection to the gauge connection. Therefore, this assumption provides additional gravitational degrees of freedom besides the ones given by the standard case, so that all the BH con-

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figurations found by this approach are not associated with an internal symmetry group and they do not carry any classical hair (i.e. they constitute a class of non-hairy BH solutions in a pure gravity model).

This work is organized in the following way. First, in Sect. 2, we present basic features of the EYM model. In Sect. 3, we show the general results based on the Lorentz group taking as a starting point the spin connection, which yields exact solutions to the EYM equations in vacuum. The expressions of the field for the Schwarzschild–de Sitter metric in a four-dimensional space-time are shown in Sect. 4, where we also remark some properties of particular the solutions in higher-dimensional space-times. Finally, we classify the Yang–Mills field configurations through Carmeli method in Sect. 5, and we present the conclusions obtained from our analysis in Sect. 6.

2 EYM equations associated with the Lorentz group

The dynamics of a non-abelian gauge theory defined on a four-dimensional Lorentzian manifold is described by the following EYM action:

$$S = -\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\,R + \alpha\int \mathrm{d}^4x\sqrt{-g}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}), \tag{1}$$

where $A_\mu = A^a_\mu T^a$, $[A_\mu, A_\nu] = i f^{abc} A^a_\mu A^b_\nu T^c$, and $F_{\mu\nu} = F^a_{\mu\nu} T^a$, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$. Unless otherwise specified, we will use Planck units throughout this work ($G = c = \hbar = 1$), the signature $(+,-,-,-)$ is used for the metric tensor, and Greek letters denote covariant indices, whereas Latin letters stand for Lorentzian indices. The above action is called pure EYM, since it is related to its simplest form, without any additional field or matter content (see [8] for more complex systems).

The EYM equations can be derived from this action by performing variations with respect to the gauge connection:

$$\left(D_\mu\, F^{\mu\nu}\right)^a = 0, \tag{2}$$

and the metric tensor:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{3}$$

where the energy-momentum tensor associated with the Yang–Mills field configuration is given by

$$T_{\mu\nu} = 4\alpha\,\mathrm{tr}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\lambda\rho}F^{\lambda\rho}\right). \tag{4}$$

As we have commented, the first non-abelian solution with matter fields was found numerically by Bartnik and McKinnon for the case of a four-dimensional space-time and a $SU(2)$ gauge group [2]. We are interested in solving the above system of equations when the gauge symmetry is associated with the Lorentz group $SO(1,3)$. In this case, we can write the gauge connection as $A_\mu = A^{ab}{}_\mu\, J_{ab}$, where the generators of the gauge group J_{ab}, can be written in terms of the Dirac gamma matrices: $J_{ab} = i[\gamma_a, \gamma_b]/8$. In such a case, it is straightforward to deduce the commutative relations of the Lorentz generators:

$$[J_{ab}, J_{cd}] = \frac{i}{2}\,(\eta_{ad}\, J_{bc} + \eta_{cb}\, J_{ad} - \eta_{db}\, J_{ac} - \eta_{ac}\, J_{bd})\,. \tag{5}$$

3 EYM-Lorentz *ansatz*

The above set of equations constitutes a complicated system involving a large number of degrees of freedom, which interact not only under the regular gravitational attraction but also under the non-abelian gauge interaction. It is not simple to find its solutions. We propose the following *ansatz*, which identifies the gauge connection with the spin connection:

$$A^{ab}{}_\mu = e^a{}_\lambda\, e^{b\rho}\, \Gamma^\lambda{}_{\rho\mu} + e^a{}_\lambda\, \partial_\mu\, e^{b\lambda}, \tag{6}$$

with $e^a{}_\lambda$ the tetrad field [11,12], that is defined through the metric tensor $g_{\mu\nu} = e^a{}_\mu\, e^b{}_\nu\, \eta_{ab}$; and $\Gamma^\lambda{}_{\rho\mu}$ is the affine connection of a semi-Riemannian manifold V_4.

By using the antisymmetric property of the gauge connection with respect to the Lorentz indices: $A^{ab}{}_\mu = -A^{ba}{}_\mu$, we can write the field strength tensor as

$$F^{ab}{}_{\mu\nu} = \partial_\mu A^{ab}{}_\nu - \partial_\nu A^{ab}{}_\mu + A^a{}_{c\mu}\, A^{cb}{}_\nu - A^a{}_{c\nu}\, A^{cb}{}_\mu. \tag{7}$$

Then, by taking into account the orthogonal property of the tetrad field $e_a{}^\lambda\, e^a{}_\rho = \delta^\lambda_\rho$, the field strength tensor takes the form [13,14]

$$F^{ab}{}_{\mu\nu} = e^a{}_\lambda\, e^{b\rho}\, R^\lambda{}_{\rho\mu\nu}, \tag{8}$$

where $R^\lambda{}_{\rho\mu\nu}$ are the components of the Riemann tensor.

Thus, $F_{\mu\nu} = e^a{}_\lambda\, e^{b\rho}\, R^\lambda{}_{\rho\mu\nu}\, J_{ab}$ represents a gauge curvature and we can express the pure EYM equations (2) and (3) in terms of geometrical quantities. On the one hand, Eq. (2) can be written as

$$\left(D_\mu\, F^{\mu\nu}\right)^{ab} = e^a{}_\lambda\, e^b{}_\rho\, \nabla_\mu\, R^{\mu\nu\lambda\rho} = 0, \tag{9}$$

whereas, on the other hand, the standard Einstein equation given by Eq. (3) has the following gravitational correction to the Einstein tensor:

$$T_{\mu\nu} = 2\alpha\left(R^{\sigma\omega}{}_{\mu\rho}R_{\sigma\omega\nu}{}^\rho - \frac{1}{4}g_{\mu\nu}R_{\sigma\omega\lambda\rho}R^{\sigma\omega\lambda\rho}\right), \tag{10}$$

which replaces Eq. (4).

4 Solutions of the EYM-Lorentz *ansatz*

The EYM-Lorentz *ansatz* described above reduces the problem to a pure gravitational system and simplifies the search for particular solutions. According to the second Bianchi identity for a semi-Riemannian manifold, the components of the Riemann tensor satisfy

$$\nabla_{[\mu}\, R_{\lambda\rho]}{}^{\sigma\nu} = 0. \tag{11}$$

By contracting this expression with the metric tensor:

$$\nabla_{[\mu}\, R_{\lambda\rho]}{}^{\mu\nu} = 0. \tag{12}$$

By using the symmetries of the Riemann tensor:

$$\nabla_{\mu}\, R^{\mu\nu}{}_{\lambda\rho} + \nabla_{\rho}\, R_{\lambda}{}^{\nu} - \nabla_{\lambda}\, R_{\rho}{}^{\nu} = 0, \tag{13}$$

with $R_{\lambda}{}^{\nu}$ the components of the Ricci tensor. Then, taking into account (9), we finally obtain

$$\nabla_{[\lambda} R_{\rho]\nu} = 0. \tag{14}$$

The integrability condition $R_{[\mu\nu|\lambda|}{}^{\sigma} R_{\rho]\sigma} = 0$ for this expression is known to have as only solutions [15]:

$$R_{\mu\nu} = b\, g_{\mu\nu}, \tag{15}$$

where b is a constant.

First, we shall analyze the case of a space-time characterized by four dimensions. In such a case, $T_{\mu\nu}$ is trace-free and the solutions are scalar-flat. From the expression of this tensor in terms of the Weyl and Ricci tensors, the Einstein equations are

$$R_{\mu\nu} - 16\pi\alpha\, C_{\mu\lambda\nu\rho} R^{\lambda\rho} = 0, \tag{16}$$

where $C_{\mu\lambda\nu\rho} = R_{\mu\lambda\nu\rho} - \left(g_{\mu[\nu} R_{\rho]\lambda} - g_{\lambda[\nu} R_{\rho]\mu}\right) + R g_{\mu[\nu} g_{\rho]\lambda}/3$.

Therefore, by using (15) and the condition $C_{\mu\lambda\nu}{}^{\lambda} = 0$, the only solutions are vacuum solutions defined by $R_{\mu\nu} = 0$ [16,17]. Hence, for empty space, $T_{\mu\nu} = 0$ and all the equations are satisfied for well-known solutions [18], such as the Schwarzschild or Kerr metric. We can also add a cosmological constant in the Lagrangian and generalize the standard solutions to de Sitter or anti-de Sitter asymptotic space-times, depending on the sign of such a constant. Note that these solutions are generally supported for a large variety of different field models and gravitational theories [19,20].

It is worthwhile to stress that these conclusions contrast with the ones given by other classical BH solutions in higher derivative gravity, where the approach assumes the requirement of the metric formalism and it leads to a different system of variational equations [21]. Indeed, whereas the gauge and the Palatini formalisms are found to be equivalent by requiring the presence of a metric-compatible connection [22], it is shown that the latter also implies the metric formalism but the opposite is not true for theories endowed with this type of higher order curvature corrections in the Lagrangian [23]. Then it is expected that alternative vacuum solutions may also arise in the framework of the higher derivative gravity [24].

On the other hand, although the EYM theory typically involves gauging internal degrees of freedom associated with fields coupled to gravity, our solutions are also compatible with other gauge gravitational theories, such as Poincaré Gauge Gravity (PGG) [25–27]. This theory is based on the Poincaré group, which is also known as the inhomogeneous Lorentz group. Within this model, the external degrees of freedom (rotations and translations) are gauged and the connection is defined by $A_\mu = e^{a}{}_{\mu} P_a + \left(e^{a}{}_{\lambda}\, e^{b\rho}\, \Gamma^{\lambda}{}_{\rho\mu} + e^{a}{}_{\lambda}\, \partial_\mu\, e^{b\lambda}\right) J_{ab}$, where P_a are the generators of the translation group. The equations corresponding to the Lagrangian (1) in PGG are the same than the previous system of equations [22]. However, PGG is less constrained than a purely quadratic YM field strength.

Once the metric solution is fixed by the particular boundary conditions, the EYM-Lorentz *ansatz* defined by Eq. (6) determines the solution of the Yang–Mills field configuration. In order to characterize such a configuration, it is interesting to establish the form of the electric $E_\mu = F_{\mu\nu}\, u^\nu$, and magnetic field $B_\mu = *F_{\mu\nu}\, u^\nu$, as measured by an observer moving with four-velocity u^ν. In particular, for the Schwarzschild–de Sitter solution, one may find the following electric and magnetic *projections* of the Yang–Mills field strength tensor in the rest frame of reference [28]:

$$E_r = \frac{\frac{4M}{r^3} + \frac{2\Lambda}{3}}{\sqrt{1 - \frac{2M}{r} - \frac{\Lambda}{3}\, r^2}}\, J_{01}, \tag{17}$$

$$E_\theta = -\,2r \left(\frac{M}{r^3} - \frac{\Lambda}{3}\right) J_{02}, \tag{18}$$

$$E_\phi = -\,2r\, \sin\theta \left(\frac{M}{r^3} - \frac{\Lambda}{3}\right) J_{03}, \tag{19}$$

$$B_r = \frac{\frac{4M}{r^3} + \frac{2\Lambda}{3}}{\sqrt{1 - \frac{2M}{r} - \frac{\Lambda}{3}\, r^2}}\, J_{23}, \tag{20}$$

$$B_\theta = 2r \left(\frac{M}{r^3} - \frac{\Lambda}{3}\right) J_{13}, \tag{21}$$

$$B_\phi = -\,2r\, \sin\theta \left(\frac{M}{r^3} - \frac{\Lambda}{3}\right) J_{12}. \tag{22}$$

It is straightforward to check that the above solution verifies

$$\mathrm{tr}\left(\vec{E}^2 + \vec{B}^2\right) = 0 \tag{23}$$

and

$$\mathrm{tr}\left(\vec{E} \cdot \vec{B}\right) = 0. \tag{24}$$

It is also interesting to remark that the family of solutions provided by the EYM-Lorentz *ansatz* is not restricted to the signature $(+, -, -, -)$. It is also valid for the Euclidean case $(+,+,+,+)$. For the latter signature, the corresponding gauge group is $SO\,(4)$ and the associated generators satisfy the following commutation relations:

$$[J_{ab}, J_{cd}] = \frac{i}{2}\,(\delta_{ad}\,J_{bc} + \delta_{cb}\,J_{ad} - \delta_{db}\,J_{ac} - \delta_{ac}\,J_{bd})\,. \tag{25}$$

The above solutions can also be generalized to a space-time with an arbitrarily higher number of dimensions. For the n-dimensional case, the assumption of the *ansatz* (6) in the EYM equations (2), (3) and (4) is equivalent to work with the following gravitational action in the Palatini formalism:

$$S = \int \mathrm{d}^n x\sqrt{-g}\,\left\{-\frac{1}{16\pi}R + 2^{\tilde{n}/2-3}\alpha\,R_{\lambda\rho\mu\nu}R^{\lambda\rho\mu\nu}\right\}, \tag{26}$$

where $\tilde{n}=n$ and $\tilde{n}=n-1$ for even and odd n.

In such a case, the quadratic Yang–Mills correction takes the form of the one associated with a cosmological constant, in a similar way to certain solutions of modified gravity theories, as the Boulware–Deser solution in Gauss–Bonnet gravity [29]. For instance, for a de Sitter geometry, the Riemann curvature tensor is given by

$$R_{\lambda\rho\mu\nu} = \frac{2\Lambda}{(n-2)(n-3)}\,\left(g_{\lambda\mu}\,g_{\rho\nu} - g_{\lambda\nu}\,g_{\rho\mu}\right). \tag{27}$$

In this case, the geometrical correction associated with the Yang–Mills configuration given by Eq. (10) takes the form

$$T_{\mu\nu} = -\,2^{\tilde{n}/2}\alpha\,\Lambda^2\frac{(n-1)\,(n-4)}{(n-2)^2\,(n-3)^2}\,g_{\mu\nu}. \tag{28}$$

Therefore, $T_{\mu\nu}=0$ is a particular result associated with the four-dimensional space-time.

On the other hand, the equivalence between the Yang–Mills–Lorentz model in curved space-time and a pure gravitational theory is not restricted to Einstein gravity. For example, in the five-dimensional case, we can study the gravitational model defined by the following action in the Palatini formalism:

$$S_G = \int \mathrm{d}^5 x\sqrt{-g}\,\left\{\alpha_0 + \alpha_1 R + \alpha_2 R^2 - 4\alpha_3 R_{\mu\nu}R^{\mu\nu} + \alpha_4 R_{\lambda\rho\mu\nu}R^{\lambda\rho\mu\nu}\right\}. \tag{29}$$

The above expression includes not only the cosmological constant (proportional to α_0) and the Einstein–Hilbert term (proportional to α_1), but also quadratic contributions of the curvature tensor (proportional to α_2, α_3 and α_4). In this case, the addition of the Yang–Mills action under the restriction of the Lorentz *ansatz* (6) is equivalent to work with the same gravitational model given by Eq. (29) with the following redefinition of α_4:

$$\alpha_4^{YM} = \alpha_4 + \frac{\alpha}{2}. \tag{30}$$

It is particularly interesting to consider the model with $\alpha_2=\alpha_3=\alpha_4^{YM}$. In such a case, the higher order contribution in the equivalent gravitational system is proportional to the Gauss–Bonnet term. As is well known, this latter term reduces to a topological surface contribution for $n=4$, but it is dynamical for $n\geq 5$. In particular, according to the Boulware–Deser solution, the metric associated with the corresponding equations takes the simple form

$$\mathrm{d}s^2 = A^2(r)\,\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{A^2(r)} - r^2 d\Omega_3^2, \tag{31}$$

where $d\Omega_3^2$ is the metric of a unitary three-sphere, and $A^2(r)$ is given by

$$A^2(r) = 1 + \frac{r^2}{4\Upsilon} + \sigma\frac{r^2}{4\Upsilon}\sqrt{1+\frac{16\Upsilon M}{r^4}+\frac{4\Upsilon\Lambda}{3}}, \tag{32}$$

with $\alpha_0/\alpha_1=-2\Lambda$, $\alpha_2/\alpha_1=\Upsilon$, and $\sigma=1$ or $\sigma=-1$. Therefore, from the EYM point of view, the Yang–Mills field contribution modifies the metric solution in a very non-trivial way. We can study the limit $\Upsilon\to 0$ in the Boulware–Deser metric. It is interesting to note that it does not necessarily mean a weak coupling regime of the EYM interaction, since $\alpha_4^{YM}\to 0$ does not imply $\alpha\to 0$. It is convenient to distinguish between the branch $\sigma=-1$ and $\sigma=1$. The first choice recovers the Schwarzschild–de Sitter solution for $\Upsilon=0$:

$$A^2_{\sigma=-1}(r) \simeq 1 - \frac{2M}{r^2}\left(1-\frac{2\Lambda\Upsilon}{3}\right) - \frac{\Lambda}{6}\left(1-\frac{\Lambda\Upsilon}{3}\right)r^2 + \frac{8M^2\Upsilon}{r^6}. \tag{33}$$

When this metric is deduced from the equations corresponding to a pure gravitational theory, the new contributions from finite values of Υ are usually interpreted as short distance corrections of high-curvature terms in the Einstein–Hilbert action. From the EYM model point of view, these corrections originate with the Yang–Mills contribution interacting with the gravitational attraction.

On the other hand, the metric solution takes the following form in the EYM weak coupling limit for the value $\sigma=1$:

$$A^2_{\sigma=1}(r) \simeq 1 + \frac{2M}{r^2}\left(1-\frac{2\Lambda\Upsilon}{3}\right) + \frac{\Lambda}{6}\left(1+\frac{3}{\Lambda\Upsilon}-\frac{\Lambda\Upsilon}{3}\right)r^2 - \frac{8M^2\Upsilon}{r^6}. \tag{34}$$

The corresponding geometry does not recover the Schwarzschild–de Sitter limit when $\Upsilon\to 0$, and it shows ghost instabilities.

5 Carmeli classification of the Yang–Mills field configurations

In the same way that the Petrov classification of the gravitational field describes the possible algebraic symmetries of the Weyl tensor through the problem of finding their eigenvalues and eigenbivectors [30], the Carmeli classification analyzes the symmetries of Yang–Mills fields configurations [31].

Let ξ_{ABCD} be the gauge invariant spinor defined by $\xi_{ABCD} = \frac{1}{4}\epsilon^{\dot{E}\dot{F}}\epsilon^{\dot{G}\dot{H}}\mathrm{tr}\left(f_{A\dot{E}B\dot{F}}\, f_{C\dot{G}D\dot{H}}\right)$, with $f_{A\dot{B}C\dot{D}} = \tau^{\mu}_{A\dot{B}}\,\tau^{\nu}_{C\dot{D}}\,F_{\mu\nu}$ the spinor equivalent to the Yang–Mills strength field tensor written in terms of the generalizations of the unit and Pauli matrices, which establish the correspondence between spinors and tensors. Let ϕ_{AB} be a symmetrical spinor. Then, by studying the eigenspinor equation $\xi_{AB}{}^{CD}\,\phi_{CD} = \lambda\,\phi_{AB}$, we can classify Yang–Mills field configurations in a systematic way.

This analysis can be applied to any of the EYM-Lorentz solutions but, for simplicity, we will illustrate the computation for the EYM solution related to the Schwarzschild metric in four dimensions. We find the following invariants of the Yang–Mills field:

$$P = \xi_{AB}{}^{AB} = \frac{3M^2}{4r^6}, \tag{35}$$

$$G = \eta_{ABCD}\,\eta^{ABCD} = \frac{3M^4}{32r^{12}}, \tag{36}$$

$$H = \eta_{AB}{}^{CD}\,\eta_{CD}{}^{EF}\,\eta_{EF}{}^{AB} = \frac{3M^6}{256r^{18}}, \tag{37}$$

$$S = \xi_{ABCD}\,\xi^{ABCD} = \frac{9M^4}{32r^{12}}, \tag{38}$$

$$F = \xi_{AB}{}^{CD}\,\xi_{CD}{}^{EF}\,\xi_{EF}{}^{AB} = \frac{33M^6}{256r^{18}}, \tag{39}$$

where η_{ABCD} is the totally symmetric spinor $\xi_{(ABCD)}$, and ξ_{ABCD} satisfies the equalities $\xi_{ABCD} = \xi_{BACD} = \xi_{ABDC} = \xi_{CDAB}$. Then the characteristic polynomial $p(\lambda') = \lambda'^3 - G\lambda'/2 - H/3$ associated with eigenspinor equation of η_{ABCD} provides directly the eigenvalues of the corresponding ξ_{ABCD}. By taking $\lambda = \lambda' + P/3$, we obtain the following results:

$$\lambda_1 = \frac{M^2}{2r^6}, \tag{40}$$

$$\lambda_{2,3} = \frac{M^2}{8r^6}. \tag{41}$$

Thus, there are two different eigenvalues: the first one is simple, whereas the second one is double. There are three distinct eigenspinors and the corresponding Yang–Mills field is of type D_P, which is associated with the Yang–Mills configurations of isolated massive objects.

6 Conclusions

In this work, we have studied the EYM theory associated with a $SO(1, n-1)$ gauge symmetry, where n is the number of dimensions associated with the space-time. In particular, we have derived analytical expressions for a large variety of BH solutions. For this analysis, we have used an *ansatz* that identifies the gauge connection with the spin connection. We have shown that this *ansatz* allows one to interpret different known metric solutions corresponding to pure gravitational systems, in terms of equivalent EYM models. We have demonstrated that this analytical method can also be applied successfully to the study of fundamental BH configurations. Such configurations usually differ from the given by the standard case, so that they are useful to improve the understanding of the resulting approach by showing the similarities and differences with respect to the present in other quadratic gravity theories (see [32] and the references therein for a recent overview and additional BH solutions).

For the analysis of the corresponding Yang–Mills model with Lorentz gauge symmetry in curved space-time, we have used the appropriate procedure in order to solve the equivalent gravitational equations, which governs the dynamics of pure gravitational systems associated with the proper gravitational theory. In particular, we have derived the solutions for the Schwarzschild–de Sitter geometry in a four-dimensional space-time and for the Boulware–Deser metric in the five-dimensional case. For these solutions, we have specified the corresponding pure gravitational theories. The algebraic symmetries associated with the Yang–Mills configuration related to a given solution can be classified by following the Carmeli method. We have explicitly shown the equivalence with the Petrov classification for the Schwarzschild metric in four dimensions.

In addition, numerical results obtained for these gravitational systems can be extrapolated to the EYM-Lorentz model by following our prescription. Through the gravitational analogy, one can also deduce the stability properties of the EYM solutions or the gravitational collapse associated with such a system. Here, we have limited the EYM-Lorentz *ansatz* to the analysis of spherical and static BH configurations, but it can be used to study other types of solutions. For example, by using the same *ansatz*, gravitational plane waves in modified theories of gravity may be interpreted as EYM-Lorentz waves. We consider that all these ideas deserve further investigation in future work.

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Correspondence between Einstein-Yang-Mills-Lorentz systems and dynamical torsion models

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In the framework of Einstein-Yang-Mills theories, we study the gauge Lorentz group and establish a particular correspondence between this case and a certain class of theories with torsion within Riemann-Cartan space-times. This relation is specially useful in order to simplify the problem of finding exact solutions to the Einstein-Yang-Mills equations. The applicability of the method is divided into two approaches: one associated with the Lorentz group $SO(1,n-1)$ of the space-time rotations, and another one with its subgroup $SO(n-2)$. Solutions for both cases are presented by the explicit use of this correspondence and, interestingly, for the last one by imposing on our ansatz the same kind of rotation and reflection symmetry properties as for a nonvanishing space-time torsion. Although these solutions were found in previous literature by a different approach, our method provides an alternative way to obtain them, and it may be used in future research to find other exact solutions within this theory.

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I. INTRODUCTION

Research of the Einstein-Yang-Mills (EYM) model has shown it to be a field of successful results. In the same way that we can find solutions in general relativity (GR) with Abelian gauge bosons [1], we can also find more general solutions in the presence of non-Abelian vector fields with a large number of interesting properties, despite the nonhair conjecture [2]. The first non-Abelian solution in the presence of curved space-time was found numerically by Bartnik and McKinnon in the four-dimensional static spherically symmetric EYM-$SU(2)$ theory [3]. It is a particlelike system, unlike the Abelian case given by the $U(1)$ gauge group of the Einstein-Maxwell theory, where such a distribution is prohibited, but the same EYM model does also contain a black hole configuration [4].

Increasing the number n of dimensions of the space-time, new exact solutions for the EYM-$SO(n-2)$ case were found by the Wu-Yang ansatz [5]. In our work, we arrive to the same result by making use of a spin connection–like ansatz with Yang-Mills (YM) charge and applying the standard class of symmetry conditions as those assigned to the fundamental geometrical quantities of a Riemann-Cartan (RC) manifold (i.e., curvature and torsion).

From a mathematical point of view, any gauge field over a pseudo-Riemannian manifold $\mathcal{M}$ (i.e., coupled to gravity) is associated with a Lie group $\mathcal{G}$ and is expressed by a connection 1-form A in the principal bundle $\mathcal{P}(\mathcal{M},\mathcal{G})$, which takes values on the Lie algebra. This gauge connection defines a covariant derivative on the tangent bundle of $\mathcal{G}$ and the subsequent 2-form gauge curvature F, which constitutes the physical field playing the role of carrier of an interaction (i.e., the YM field if $\mathcal{G}$ is a non-Abelian Lie group) [6],

$$D_\mu = \nabla_\mu + \frac{i}{\sigma}[A_\mu, \cdot], \tag{1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{i}{\sigma}[A_\mu, A_\nu], \tag{2}$$

where σ is related to the coupling constant.

Then, the following commutating relation is satisfied:

$$[D_\mu, D_\nu]v^\lambda = R^\lambda{}_{\rho\mu\nu}v^\rho + \frac{i}{\sigma}[F_{\mu\nu}, v^\lambda], \tag{3}$$

where $R^\lambda{}_{\rho\mu\nu} = \partial_\mu\Gamma^\lambda_{\rho\nu} - \partial_\nu\Gamma^\lambda_{\rho\mu} + \Gamma^\lambda_{\omega\mu}\Gamma^\omega_{\rho\nu} - \Gamma^\lambda_{\omega\nu}\Gamma^\omega_{\rho\mu}$ are the components of the Riemann tensor derived from the Levi-Civita connection and v^λ is an arbitrary vector.

Their behavior under a gauge transformation $S \in \mathcal{G}$ allows us to construct minimal coupling actions. In terms of their components, it is given by the following rules:

$$A_\mu \to A'_\mu = S^{-1}A_\mu S - i\sigma S^{-1}\partial_\mu S, \tag{4}$$

$$F_{\mu\nu} \to F'_{\mu\nu} = S^{-1}F_{\mu\nu}S. \tag{5}$$

On the other hand, RC space-times incorporate the notion of torsion as the antisymmetric part of the affine connection on the manifold,

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$$T^{\lambda}{}_{\mu\nu} = 2\tilde{\Gamma}^{\lambda}{}_{[\mu\nu]}. \tag{6}$$

Note that the notation with a tilde refers to elements defined within the RC manifold and with the absence of a tilde to elements defined within the torsion-free pseudo-Riemannian manifold. Additionally, according to the correspondence used by our method, the same convention applies to quantities depending on torsionlike components (i.e., corrections in the gauge potentials that are referred to internal symmetry groups and have similar algebraic symmetries in analogy to the torsion tensor).

Although the affine connection does not transform like a tensor under a general change of coordinates, its antisymmetric part does (i.e., torsion is a third-rank tensor, and it cannot be locally vanished if it has not associated an absolute zero value). Furthermore, whereas curvature is related to the rotation of a vector along an infinitesimal path over the space-time, torsion is related to the translation and has deep geometrical implications, such as breaking infinitesimal parallelograms on the manifold [7].

Thus, unlike the torsion-free case where the geometry is completely described by the metric (i.e., the affine connection corresponds to the Levi-Civita connection), the presence of torsion introduces independent characteristics and modifies the expression of the affine connection in the following form:

$$\tilde{\Gamma}^{\lambda}{}_{\rho\mu} = \Gamma^{\lambda}{}_{\rho\mu} + K^{\lambda}{}_{\rho\mu}, \tag{7}$$

where $K^{\lambda}{}_{\rho\mu} = \frac{1}{2}(T^{\lambda}{}_{\rho\mu} - T_{\mu}{}^{\lambda}{}_{\rho} + T_{\rho\mu}{}^{\lambda})$ is the so-called contortion tensor and fulfills $K^{\lambda}{}_{\rho\mu} = -K_{\rho}{}^{\lambda}{}_{\mu}$, in order to preserve the metricity condition $\tilde{\nabla}_{\lambda} g_{\mu\nu} = 0$ (i.e., the total covariant derivative of the metric tensor vanishes identically).

One of the most fundamental aspects of introducing these new geometrical characteristics within a physical theory of space-time and matter beyond GR is its main role as a dynamical field if higher order curvature and torsion terms are included in the Lagrangian. Whereas the so-called Einstein-Cartan theory only incorporates first-order corrections in the Lagrangian, and therefore no propagating torsion is allowed, second-order corrections describe a Lagrangian with dynamical torsion depending on ten parameters [8,9].

In the present work, we use these notions about the EYM theory and the quadratic gravitation theory with propagating torsion to bridge the gap between both in a very special case. Indeed, under a simple class of additional restrictions, we shall see that our assumptions allow us to obtain different classes of exact solutions to the EYM equations and to study other possible configurations in such a case. In this sense, the primary starting point of our analysis is based on the study of noncompact Lie groups. Although these constructions are related to nonunitary theories, one interesting aspect of this type of group is the possibility of establishing a correspondence between the theory under study and a set of modified theories of gravity with propagating space-time torsion, as is developed in this work. Indeed, the standard theory of gravity and the larger part of its extensions belong to this group. Following our discussion, we establish original dynamical constraints in order to simplify and to classify all the possible solutions derived by the approach described in the manuscript.

This paper is organized as follows. Section II presents the general EYM-Lorentz field equations, as well as these equations under the spin connection–like ansatz and its association with a particular quadratic gravitational theory of second order in the curvature term with dynamical torsion. The general expressions for the metric and the torsion tensor under rotations and reflections in the static spherically symmetric space-time are shown in Sec. III. We apply these particular conditions and find the respective solutions for the torsionlike and torsionless cases in Sec. IV. Finally, the conclusions obtained from our analysis are presented in Sec. V.

II. EYM-LORENTZ ANSATZ AND CONDITIONS

We will use Planck units throughout this work ($G = c = \hbar = 1$) and consider for our study the following Lagrangian:

$$S = -\frac{1}{16\pi} \int (R - \mathrm{tr} F_{\mu\nu} F^{\mu\nu}) \sqrt{-g} d^n x, \tag{8}$$

where the minimal coupling is assumed. Note that depending on the character of the gauge formalism and its corresponding group of transformations assumed by the approach, this action can be framed either on a modified gravity model or on a system of interaction between gauge fields and regular gravity. Specifically, gauging external or internal degrees of freedom is related to a large class of gauge gravity models based on space-time symmetries and to YM theories, respectively. In the present case, we consider both analyses with the external $SO(1, n-1)$ group and the internal $SO(n-2)$, in order to obtain a class of general constraints that allows us to classify their possible solutions under the appropriate correspondence conditions.

Therefore, the general equations derived from this action by performing variations with respect to the metric tensor and the gauge connection of the groups under consideration are

$$(D_{\mu} F^{\mu\nu})^{ab} = 0, \tag{9}$$

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{10}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu} = \frac{1}{4\pi} \mathrm{tr}(\frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho} - F_{\mu\rho} F_{\nu}{}^{\rho})$, whereas latin a, b and greek μ, ν indices run from 0 to $n-1$ and refer to an anholonomic and coordinate basis, respectively. Furthermore, the divergencelessness of the Einstein tensor implies the following conservation law:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{11}$$

These field equations typically constitute a complicated nonlinear system of equations, and additional constraints are usually required in order to simplify the problem and to focus on particular cases. Then, by taking into account these lines, we assign the following spin connection–like ansatz to the gauge connection:

$$A^{ab}{}_\mu = Q(e^a{}_\lambda e^{b\rho}\tilde{\Gamma}^\lambda{}_{\rho\mu} + e^a{}_\lambda \partial_\mu e^{b\lambda}). \tag{12}$$

This expression usually represents a spin connection on a RC space-time (i.e., a curved space-time with torsion), so it can be regarded as the gauge field generated by local Lorentz transformations in such a case. Alternatively, under the EYM framework associated with internal gauge groups, it is always possible to select any particular ansatz in order to describe the respective YM field, so in this formalism we will start from the same mathematical expression and find embedded non-Abelian $SO(n-2)$ solutions.

The gauge connection can be written as $A_\mu = A^{ab}{}_\mu J_{ab}$, where $J_{ab} = i[\gamma_a, \gamma_b]/8$ are the generators of the Lorentz gauge group, which satisfy the following commutative relations:

$$[J_{ab}, J_{cd}] = \frac{i}{2}(\eta_{ad}J_{bc} + \eta_{cb}J_{ad} - \eta_{db}J_{ac} - \eta_{ac}J_{bd}). \tag{13}$$

By using the antisymmetric property of the gauge connection with respect to the Lorentz indices, $A^{ab}{}_\mu = -A^{ba}{}_\mu$, we can write the field strength tensor as

$$F^{ab}{}_{\mu\nu} = \partial_\mu A^{ab}{}_\nu - \partial_\nu A^{ab}{}_\mu + \frac{1}{\sigma}(A^a{}_{c\mu}A^{cb}{}_\nu - A^a{}_{c\nu}A^{cb}{}_\mu). \tag{14}$$

Finally, by taking into account the orthogonal property of the tetrad field $e_a{}^\lambda e^a{}_\rho = \delta^\lambda_\rho$ and setting $\sigma = Q$, the field strength tensor takes the form [10,11]

$$F^{ab}{}_{\mu\nu} = Q e^a{}_\lambda e^b{}_\rho \tilde{R}^{\lambda\rho}{}_{\mu\nu}, \tag{15}$$

where $\tilde{R}^\lambda{}_{\rho\mu\nu}$ coincides with the general expression of the components of the Riemann tensor over a RC space-time.

Rewriting the above action under the spin connection–like ansatz, it turns out that it coincides with the following quadratic gravity action in presence of torsion:

$$S = -\frac{1}{16\pi}\int (R - 2^{\tilde{n}/2-3}Q^2\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu})\sqrt{-g}d^n x, \tag{16}$$

with $\tilde{n} = n$ and $\tilde{n} = n-1$ for even and odd n.

Therefore, Eqs. (9) and (10) for such a case can be expressed in terms of geometrical quantities, respectively, as follows:

$$\partial_\rho \tilde{R}_\mu{}^{\lambda\nu\rho} + \Gamma^\rho{}_{\omega\rho}\tilde{R}_\mu{}^{\lambda\nu\omega} + \tilde{\Gamma}^\lambda{}_{\omega\rho}\tilde{R}_\mu{}^{\omega\nu\rho} - \tilde{\Gamma}^\omega{}_{\mu\rho}\tilde{R}_\omega{}^{\lambda\nu\rho} = 0, \tag{17}$$

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 2^{\tilde{n}/2}Q^2(g_{\mu\nu}\tilde{R}_{\lambda\rho\omega\tau}\tilde{R}^{\lambda\rho\omega\tau} - 4\tilde{R}^{\lambda\rho}{}_{\mu\omega}\tilde{R}_{\lambda\rho\nu}{}^\omega). \tag{18}$$

Thus, if a certain class of space-time symmetries are imposed, then not only the condition $\mathcal{L}_\xi g_{\mu\nu} = 0$ must be satisfied, but also $\mathcal{L}_\xi T^\lambda{}_{\mu\nu} = 0$ (i.e., the Lie derivative in the direction of the Killing field ξ on $T^\lambda{}_{\mu\nu}$ vanishes) in order to preserve the reasonable curvature and torsion symmetries.

III. SPHERICAL AND REFLECTION SYMMETRIES

The metric of a n-dimensional static spherically symmetric space-time can be written as

$$ds^2 = A(r)dt^2 - \frac{dr^2}{B(r)} - r^2 d\Omega^2_{n-2}, \tag{19}$$

where $d\Omega_{n-2} = d\theta_1^2 + \sum_{i=2}^{n-2}\prod_{j=1}^{i-1}\sin^2\theta_j d\theta_i^2$, with $0 \leq \theta_{n-2} \leq 2\pi$ and $0 \leq \theta_k \leq \pi$, $1 \leq k \leq n-3$. We assume $n \geq 4$.

Then, it can be shown that the only nonvanishing components of $T^\lambda{}_{\mu\nu}$ are [12,13]

$$\begin{aligned}
T^t{}_{tr} &= a(r);\\
T^r{}_{tr} &= b(r);\\
T^{\theta_k}{}_{t\theta_l} &= \delta^{\theta_k}{}_{\theta_l} c(r);\\
T^{\theta_k}{}_{r\theta_l} &= \delta^{\theta_k}{}_{\theta_l} g(r);\\
T^{\theta_k}{}_{t\theta_l} &= e^{a\theta_k}e^b{}_{\theta_l}\epsilon_{ab}d(r), \qquad \text{if } n=4;\\
T^{\theta_k}{}_{r\theta_l} &= e^{a\theta_k}e^b{}_{\theta_l}\epsilon_{ab}h(r), \qquad \text{if } n=4;\\
T^t{}_{\theta_k\theta_l} &= \epsilon_{kl}k(r)\sin\theta_1, \qquad \text{if } n=4;\\
T^r{}_{\theta_k\theta_l} &= \epsilon_{kl}l(r)\sin\theta_1, \qquad \text{if } n=4;\\
T^{\theta_k}{}_{\theta_l\theta_m} &= e^{a\theta_k}e^b{}_{\theta_l}e^c{}_{\theta_m}\epsilon_{abc}f(r), \qquad \text{if } n=5,
\end{aligned} \tag{20}$$

where a, b, c, d, g, h, k, and l are arbitrary functions depending only on r; $k, l = 1, 2$, and ϵ_{ab}, ϵ_{abc} are the totally antisymmetric Levi-Civita symbol of second and third order, respectively.

Therefore, in addition to the two functions associated with the metric, for $n = 4$ dimensions, there are still a total number of eight unknown independent functions to solve the field equations. Furthermore, imposing reflection symmetry [i.e., $O(3)$ spherical symmetry] requires that $d(r)$, $h(r)$, $k(r)$, and $l(r)$ vanish, so that the number reduces to four.

IV. SOLUTIONS

In order to categorize all the possible solutions, we can rewrite Eq. (17) in the following form:

$$\nabla_\rho R_\mu{}^{\lambda\nu\rho} + \nabla_\rho T_\mu{}^{\lambda\nu\rho} + K^\lambda{}_{\omega\rho}\tilde{R}_\mu{}^{\omega\nu\rho} - K^\omega{}_{\mu\rho}\tilde{R}_\omega{}^{\lambda\nu\rho} = 0, \quad (21)$$

where $T^\lambda{}_{\rho\mu\nu} = \nabla_\mu K^\lambda{}_{\rho\nu} - \nabla_\nu K^\lambda{}_{\rho\mu} + K^\lambda{}_{\sigma\mu}K^\sigma{}_{\rho\nu} - K^\lambda{}_{\sigma\nu}K^\sigma{}_{\rho\mu}$ coincides with the torsion contribution to the curvature tensor of the RC space-time, so that we can distinguish between the torsion-free and the torsion parts if it is required.

On the other hand, according to the second Bianchi identity for a pseudo-Riemannian manifold, the components of the Riemann tensor in such a manifold satisfy

$$\nabla_{[\lambda|}R^\omega{}_{\rho|\mu\nu]} = 0. \quad (22)$$

By contracting this expression with the metric tensor and considering the above form of the mentioned field equation, it is straightforward to obtain the following condition for our model:

$$2\nabla_{[\mu}R_{\nu]\rho} = \nabla_\lambda T_{\mu\nu\rho}{}^\lambda + 2K^\omega{}_{[\mu|\lambda}\tilde{R}_{\nu]\omega\rho}{}^\lambda. \quad (23)$$

In addition, the conservation law (11) turns out to be equivalent to the following expression:

$$\frac{1}{2}\nabla_\nu R + \nabla_\lambda T_{\mu\nu}{}^{\mu\lambda} + K^\omega{}_{\mu\lambda}\tilde{R}_{\nu\omega}{}^{\mu\lambda} - K^\omega{}_{\nu\lambda}\tilde{R}_\omega{}^\lambda = 0. \quad (24)$$

These expressions are shown as generic conditions of this model, and they will allow us to classify all the possible configurations in the most important cases.

Before distinguishing between torsionless and nonvanishing torsionlike cases, let us summarize the respective assumptions that allow us to establish and to obtain the distinct classes of solutions according to our discussion. The starting point is the mapping defined in Eq. (12), which coincides with the well-known spin connection of a given space-time. This quantity has typically been used in order to describe appropriately the dynamics of the fermion fields on a general space-time. It has also been used in the most important gauge theories of gravity, such as the well-known Lorentz gauge gravity or the Poincaré gauge gravity, since it gives rise to a Lorentz gauge curvature which is proportional to the Riemann tensor, as is shown in Eq. (15).

Continuing with our analysis, when the nonvanishing torsionlike $O(n-1)$ symmetric and the purely magnetic cases are considered in a n-dimensional static and spherically symmetric space-time, the system of equations given by Eq. (17) and Eq. (18) together with the constraints (23) and (24) will allow us to find the mentioned embedded $SO(n-2)$ solutions. It is straightforward to check the dimension of this gauge group by computing the independent connection components of the solutions, giving rise to a dimension of $(n-2)(n-3)/2$, as expected.

A. Torsionless case

For the torsionless $SO(1,n-1)$ case, the following constraint is satisfied:

$$\nabla_{[\lambda}R_{\rho]\nu} = 0, \quad (25)$$

with

$$[\nabla_{[\mu}, \nabla_{\nu|}]R_{\lambda|\rho]} = -R_{[\mu\nu|\lambda}{}^\omega R_{|\rho]\omega}. \quad (26)$$

Thus, the existence of the integrability condition $R_{[\mu\nu|\lambda}{}^\omega R_{|\rho]\omega} = 0$ allows us to solve this equation and obtain the following solutions [14]:

$$R_{\mu\nu} = b g_{\mu\nu}, \quad (27)$$

where b is a constant.

Therefore, the only possible geometries for this torsionless case correspond to Einstein manifolds. Note that the tracelessness of the torsion-free Einstein tensor in four dimensions implies that $b = 0$, so these solutions satisfy $R_{\mu\nu} = 0$ (i.e., the space-time is Ricci-flat). On the other hand, by increasing the number of dimensions, the corrections to the gravitational field act as a cosmological constant in the Einstein equations [15].

B. Nonvanishing torsionlike case

The condition (23) equal to zero enables the existence of Einstein manifold solutions even for the case of an external symmetry group $SO(1,n-1)$ in the presence of a nonvanishing space-time torsion. However, other geometries are allowed according to the generic conditions (23) and (24).

Particularly, for a n-dimensional static spherically symmetric space-time, if we simplify the problem using the previous considerations and restrict to the internal gauge group $SO(n-2)$, it is possible to find the following purely magnetic black hole solutions to the resulting EYM equations with $O(n-1)$ symmetric torsionlike tensor (rotation and reflection symmetric):

$$T^t{}_{tr} = \frac{A'(r)}{2A(r)}, \qquad T^{\theta_k}{}_{r\theta_k} = -\frac{1}{r},$$
$$T^r{}_{tr} = T^{\theta_k}{}_{t\theta_k} = T^{\theta_k}{}_{\theta_l\theta_m} = 0, \quad (28)$$

with

$$A(r) = B(r) = \begin{cases} 1 - \frac{2m}{r^2} - \frac{2Q^2 ln(r)}{r^2}, & \text{if } n = 5 \\ 1 - \frac{2m}{r^{n-3}} - 2^{\tilde{n}/2-2}\frac{(n-3)Q^2}{(n-5)r^2}, & \text{if } n \neq 5. \end{cases} \tag{29}$$

Although these geometries are asymptotically flat, for $n = 5$ and $n \geq 6$ dimensions their Arnowitt-Deser-Misner (ADM) mass [16] diverges as $ln(r)$ and r^{n-5}, respectively. Nevertheless, solutions with finite ADM mass are found by including higher-order terms of the YM hierarchy in the Lagrangian [17,18].

The nonvanishing components of the field strength tensor are

$$F^{ab}{}_{\theta_i\theta_j} = Q e_{\theta_i}{}^a e_{\theta_j}{}^b \tilde{R}^{\theta_i\theta_j}{}_{\theta_i\theta_j}, \tag{30}$$

with $\tilde{R}^{\theta_i\theta_j}{}_{\theta_i\theta_j} = -\frac{1}{r^2}$.

For $n = 4$ dimensions, the system reduces to the EYM-$SO(2)$ case, which is indeed equivalent to the magnetic Einstein-Maxwell solution because of the isomorphism between $SO(2)$ and the $U(1)$ group. On the other hand, for $n \geq 5$ dimensions the existence of these EYM-$SO(n-2)$ solutions describes the coupling of a nontrivial YM magnetic field to gravity.

It has also been shown by different ways that these solutions have a number of interesting properties, and they are compatible with the existence of a cosmological constant and Maxwell fields, as well as with other modified theories of gravity, such as Gauss-Bonnet gravity [5,19].

This work completes our previous study on EYM theory presented in [15]. More general solutions may be found using this method, especially for $n = 4$ dimensions since the $\mathcal{L}_\xi T^\lambda{}_{\mu\nu} = 0$ condition allows a richer structure than for any other number of dimensions.

V. DISCUSSION AND SUMMARY

In this article, we have presented a new method to find exact solutions to the EYM-Lorentz theory based on the correspondence between the EYM system and a certain class of quadratic gravity theories in the presence of torsion, under the restriction introduced by the spin connection–like ansatz. The available configurations can be categorized into the torsionless and the nonvanishing torsionlike cases, according to general conditions. For the torsionless branch, it is shown that the only possible geometries correspond to Einstein manifolds associated with the external group $SO(1, n-1)$, whereas for the nonvanishing torsionlike branch, the method allows us to distinguish the mentioned external group of symmetries from the internal $SO(n-2)$, and other families of embedded solutions emerge. These solutions describe a sort of purely magnetic black hole with YM charge, and they were found earlier by different approaches [5,19].

Note that these results are derived from similar mathematical expressions, but they refer to completely different approaches. Namely, from a gauge-theoretical approach, it is a well-known fact that the presence of a space-time torsion requires gauging the external degrees of freedom consisting of rotations and translations in a way that both curvature and torsion are inexorably related to the rotation and the translation noncompact groups, respectively [8,9]. Furthermore, as previously stressed, the displacement of a vector along an infinitesimal path in a RC manifold involves a breaking of the consequent parallelograms defined on such a manifold, in a way that its translational closure failure proportionally depends on the torsion tensor [7]. Therefore, the embedding of the $SO(n-2)$ group corresponds to a distinct configuration where the resulting gauge connections are accordingly related to an internal symmetry group, and the additional torsionlike degrees of freedom contained in the latter do not represent a space-time torsion but a third-rank tensor with similar algebraic symmetries that provides a purely magnetic black hole solution to the variational equations. Indeed, it is straightforward to check from the nonvanishing torsionlike components of this solution that the corresponding $SO(n-2)$ gauge connection and its associated field strength tensor can be written as $A_\mu = A^{\tilde{a}\tilde{b}}{}_\mu J_{\tilde{a}\tilde{b}}$ and $F_{\mu\nu} = F^{\tilde{a}\tilde{b}}{}_{\mu\nu} J_{\tilde{a}\tilde{b}}$, respectively, with $\tilde{a}, \tilde{b} = 2, \ldots, n-1$. Thus, it is clear that these quantities are connected to the mentioned gauge group instead of an external symmetry group related to the space-time rotations or translations.

On the other hand, further implications arise when considering the coupling with matter fields. For instance, if we study the dynamics of a Dirac fermion within the solution given by Eqs. (28) and (29), the behavior is completely different than that which occurred in the presence of a space-time torsion, where the fermion would irremediably suffer the associated spin connection. However, in the first case, the fermion would interact with the $SO(n-2)$ gauge interaction depending on its particular multiplet representation (for $n > 4$) or charge (for $n = 4$). In the simplest case, it could even be a singlet ($n > 4$) or neutral (zero charge), so it would not interact with the new gauge force.

This fact contrasts with some publications that do not bear in mind these fundamental relations, and wrongly try to identify the space-time torsion with YM or electromagnetic fields (see Fallacy 9 on page 267 of reference [20]). Thus, our $SO(n-2)$ solution is not covered by this sort of fallacy, in the same way as the Mazharimousavi-Halilsoy solution since both solutions coincide and represent the same type of configuration.

Finally, it is worthwhile to stress that distinct classes of EYM-Lorentz systems that are physically meaningful may be found using our ansatz, especially in $n = 4$ dimensions because the $\mathcal{L}_\xi T^\lambda{}_{\mu\nu} = 0$ condition could allow for more complex solutions. Additionally, for the development of this aim, the general condition $\tilde{\nabla}_\lambda g_{\mu\nu} = 0$ still holds, but it could be also possible to deal with the same analysis relaxing this restriction in order to find different EYM systems related to this geometrical property. Within this framework, an interesting and simple case might arise from the Weyl-Cartan geometry, where the nonmetricity condition is expressed as $\tilde{\nabla}_\lambda g_{\mu\nu} = w_\lambda g_{\mu\nu}$ so that the number of irreducible decomposition pieces of nonmetricity reduces to the Weyl 1-form w [21]. Further research following these lines of study will be addressed in the future.

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Conclusions

In this thesis, we have researched the possible implications derived by the existence of an antisymmetric component of the affine connection in the universe. The resulting configuration can be naturally described by a RC manifold endowed with curvature and torsion. This quantity may potentially introduce a large number of physical effects in the space-time and reveal new features of the gravitational field, beyond the conventional approach of GR. By attending to the conservation laws of the material tensors, the spin density tensor arises as a natural source of torsion and allows the introduction of an antisymmetric component of the energy-momentum tensor into the geometrical scheme. In addition, the dynamical character of torsion is subject to the presence of the corresponding kinetic terms in the gravitational action (e.g. higher order curvature terms), although it is possible to formulate theories provided with a non-propagating torsion bound by spinning sources, like the EC theory, among others. These general foundations can also be systematized as a gauge approach to gravity in the framework of the PG theory, leading to an appropriate correspondence between the gauge potentials and the field strength tensors, that gives rise to the expected conservation laws for the energy-momentum and spin density tensors.

Encouraged by the consistency of the mentioned approach, we have focused on different aspects of the space-time torsion. Since we noticed the existence of certain models that provide, in the realm of teleparallelism, an equivalent description of gravitation such as the one given by GR, we concentrated on the investigation of new dynamical aspects arising from torsion. It is worthwhile to stress the extensive work available along these lines, where the search for particular vacuum solutions has provided new insights into the different roles assumed by torsion. However, it was shown that various of these solutions present underlying problems, like the existence of an underdetermined geometry or the absence of an axial mode that propagates in a reasonable way at large distances, which involves a fundamental difficulty in measuring the possible effects occurred on Dirac particles minimally coupled to torsion. In this sense, one of the most important results achieved in this thesis is the finding of a new exact vacuum solution with a non-vanishing axial mode that behaves as a Coulomb-like potential and provides an explicit RN geometry. Hence, it constitutes a new geometrical configuration that is compatible with the Newtonian limit and, furthermore, which can yield dynamical effects on matter via the metric and torsion tensors. Indeed, the existing correspondence between spin and torsion involves specific effects on the behaviour of those spinning particles coupled to this geometric quantity. Such a behaviour can be quantified for the case of spin 1/2 particles minimally coupled to torsion by means of the WKB method, especially within post-Riemannian geometries induced by an axial component of the torsion tensor. By performing a numerical analysis, we have noticed interesting differences between the geodesic motion and the trajectories of these kinds of particles within the RN space-time provided by torsion, in a way that all the possible deviations from the standard case of GR are completely switched off in absence of a dynamical axial mode. Their magnitude depends on the value of both the spin charge associated with the source of torsion and the coupling constant that determines the fundamental strength of the interaction. In any case, it is expected to note significant effects only at extremely high densities of spinning matter, such as neutron stars or specific BHs characterized by an intense torsion field.

On the other hand, additional implications of the space-time torsion have also been analysed in this work. In particular, we have addressed the extension of the singularity theorems of GR to the case of RC manifolds endowed with torsion. First, it is straightforward to note that the notion of geodesic incompleteness can be generalized for the mentioned case by modifying the standard energy condition via the vierbein equation induced by the curvature and torsion tensors. The new gravitational action can give rise to the violation of such a condition and avoid the occurrence of singularities, for both cases of non-propagating torsion coupled to matter fields and dynamical torsion. Furthermore, the previously stressed differences between the geodesic motion of ordinary matter uncoupled to torsion and the trajectories of the rest of spinning matter mean that our analyses must also deal with a possible incompleteness of non-geodesic curves. For this purpose, we establish a new theorem based on a class of conditions general enough to involve the existence of curves with endpoints outside the conformal infinity of the RC manifold and, therefore, characterized by a singular behaviour.

Additionally, we have analysed the stability of torsion in a Minkowski space-time. In a first approximation, we have assumed the vanishing of its purely tensor component, in order to focalize our study on a homogeneous and isotropic case. Then, we obtain the corresponding ghost and tachyon-free conditions associated with the axial and trace vectorial modes in both their weak and general regimes. We note the existence of a high degree of compatibility with respect to the standard stability models presented in the literature, although some observations must be remarked. First, the algebraic conditions for the 1^+ sector related to the axial mode are just the opposite to the resulting ones from those models. Moreover, as mentioned above, our starting assumptions involve an important restriction on the possible values of the Lagrangian coefficients, which means that the release of several of these constraints allows the number of viable PG Lagrangians to be extended. Specifically, it is significant to note that the omission of a perturbative analysis around a specific curved background may change the concluding results, since the presence of a dynamical torsion in the metric tensor involves additional interaction terms in the field equations, even in the weak-field limit derived from these equations. In this sense, we consider that this dichotomy deserves further investigation in future work.

All these results concerning the torsion field have been obtained by applying the gauge principles to the external degrees of freedom consisting of rotations and translations. Nevertheless, it is possible to establish a particular correspondence between this approach and the EYM theory of internal symmetry groups. Concretely, we define torsionlike components related to the embedding of the special orthogonal group, which match the consequent EYM action to the one given by a PG Lagrangian defined by the combination of the torsion-free Ricci scalar and the square of the RC curvature. This mathematical relation can be used to impose the same types of torsion symmetries to the new components and to reduce the complexity of the EYM equations notably. The advantages of this correspondence are stressed by the obtention of a set of purely magnetic BH solutions derived by rotation and reflection symmetric torsionlike components, which were found in previous literature by the application of the Wu-Yang ansatz. A simple analysis of the non-Abelian sector shows a divergence of the ADM mass, which suggests the search of alternative EYM configurations with viable mass and energy parameters.

Finally, it is also worthwhile to point out some additional prospects of research, as the possible implication of the torsion field at cosmological scales (e.g. constituting an extra geometrical quantity in the framework of a cosmological perturbation theory [103]) or its contribution in the establishment of post-Riemannian axisymmetric configurations. In this regard, the Newman-Janis algorithm may be useful to construct different classes of PG rotating BH solutions from their corresponding non-rotating counterparts [104], but its applicability has been questioned recently for the case of modified theories of gravity, on account of the introduction of unsuitable pathologies in the metric tensor [105]. Therefore, we appreciate that the development of deeper analyses following these lines must also be addressed in posterior works.

To conclude, it is gratifying to note that the results presented in this thesis provide new insights concerning post-Riemannian geometry and matter fields. This fundamental relation constitutes the foundations of gravitation and hence the present work represents an attempt to elucidate the possible existence of different, still unknown, aspects and properties of the gravitational field.

Appendix A

Expressions of the Poincaré gauge field equations

The Lagrangian (1.27) imposes the vanishing of the tensors $X1_\mu{}^\nu$ and $X2_\mu{}^{\lambda\nu}$ in vacuum, whose expressions can be written as:

$$\begin{aligned} X1_\mu{}^\nu &= -2\tilde{G}^\nu{}_\mu + 4a_2 T1_\mu{}^\nu + 2a_4 T2_\mu{}^\nu + 4a_3 T3_\mu{}^\nu + 2a_5 H1_\mu{}^\nu \\ &+ 2a_6 H2_\mu{}^\nu + \alpha\, I1_\mu{}^\nu + \beta\, I2_\mu{}^\nu + \gamma\, I3_\mu{}^\nu\,, \end{aligned} \tag{A.1}$$

$$\begin{aligned} X2_\mu{}^{\lambda\nu} &= \overset{\star}{T}_\mu{}^{\lambda\nu} + 4a_2 C1_\mu{}^{\lambda\nu} - 2a_4 C2_\mu{}^{\lambda\nu} - 4a_3 C3_\mu{}^{\lambda\nu} - 2a_5 Y1_\mu{}^{\lambda\nu} \\ &- 2a_6 Y2_\mu{}^{\lambda\nu} - \alpha\, Z1_\mu{}^{\lambda\nu} - \beta\, Z2_\mu{}^{\lambda\nu} - \gamma\, Z3_\mu{}^{\lambda\nu}\,, \end{aligned} \tag{A.2}$$

where it is given the explicit dependence with the following geometrical quantities:

$$\tilde{G}_\mu{}^\nu = \tilde{R}_\mu{}^\nu - \frac{\tilde{R}}{2}\delta_\mu{}^\nu\,, \tag{A.3}$$

$$T1_\mu{}^\nu = \tilde{R}_{\lambda\rho\mu\sigma}\tilde{R}^{\lambda\rho\nu\sigma} - \frac{1}{4}\delta_\mu{}^\nu \tilde{R}_{\lambda\rho\tau\sigma}\tilde{R}^{\lambda\rho\tau\sigma}\,, \tag{A.4}$$

$$T2_\mu{}^\nu = \tilde{R}_{\lambda\rho\mu\sigma}\tilde{R}^{\lambda\nu\rho\sigma} + \tilde{R}_{\lambda\rho\sigma\mu}\tilde{R}^{\lambda\sigma\rho\nu} - \frac{1}{2}\delta_\mu{}^\nu \tilde{R}_{\lambda\rho\tau\sigma}\tilde{R}^{\lambda\tau\rho\sigma}\,, \tag{A.5}$$

$$T3_\mu{}^\nu = \tilde{R}_{\lambda\rho\mu\sigma}\tilde{R}^{\nu\sigma\lambda\rho} - \frac{1}{4}\delta_\mu{}^\nu \tilde{R}_{\lambda\rho\tau\sigma}\tilde{R}^{\tau\sigma\lambda\rho}\,, \tag{A.6}$$

$$H1_\mu{}^\nu = \tilde{R}^\nu{}_{\lambda\mu\rho}\tilde{R}^{\lambda\rho} + \tilde{R}_{\lambda\mu}\tilde{R}^{\lambda\nu} - \frac{1}{2}\delta_\mu{}^\nu\tilde{R}_{\lambda\rho}\tilde{R}^{\lambda\rho}\,, \tag{A.7}$$

$$H2_\mu{}^\nu = \tilde{R}^\nu{}_{\lambda\mu\rho}\tilde{R}^{\rho\lambda} + \tilde{R}_{\lambda\mu}\tilde{R}^{\nu\lambda} - \frac{1}{2}\delta_\mu{}^\nu\tilde{R}_{\lambda\rho}\tilde{R}^{\rho\lambda}\,, \tag{A.8}$$

$$I1_\mu{}^\nu = 4\left(\nabla_\lambda T_\mu{}^{\nu\lambda} + K_{\lambda\rho\mu}T^{\lambda\rho\nu} - \frac{1}{4}\delta_\mu{}^\nu T_{\lambda\rho\sigma}T^{\lambda\rho\sigma}\right)\,, \tag{A.9}$$

$$I2_\mu{}^\nu = 2\left(\nabla_\lambda T^{\lambda\nu}{}_\mu - \nabla_\lambda T^{\nu\lambda}{}_\mu + K_{\lambda\rho\mu}\left(T^{\nu\rho\lambda} + T^{\rho\lambda\nu}\right) - \frac{1}{2}\delta_\mu{}^\nu T_{\lambda\rho\sigma}T^{\rho\lambda\sigma}\right)\,, \tag{A.10}$$

$$I3_\mu{}^\nu = -2\left(\nabla_\mu T^\lambda{}_\lambda{}^\nu + K^\nu{}_{\lambda\mu}T^\rho{}_\rho{}^\lambda + \frac{1}{2}\delta_\mu{}^\nu\left(T^\lambda{}_{\lambda\sigma}T^\rho{}_\rho{}^\sigma - 2\nabla_\lambda T^\rho{}_\rho{}^\lambda\right)\right)\,, \tag{A.11}$$

$$\overset{\star}{T}_\mu{}^{\lambda\nu} = \delta_\mu{}^\nu g^{\lambda\sigma}T^\rho{}_{\rho\sigma} - g^{\lambda\nu}T^\rho{}_{\rho\mu} - g^{\lambda\sigma}T^\nu{}_{\mu\sigma}\,, \tag{A.12}$$

$$C1_\mu{}^{\lambda\nu} = \nabla_\rho\tilde{R}_\mu{}^{\lambda\rho\nu} + K^\lambda{}_{\sigma\rho}\tilde{R}_\mu{}^{\sigma\rho\nu} - K^\sigma{}_{\mu\rho}\tilde{R}_\sigma{}^{\lambda\rho\nu}\,, \tag{A.13}$$

$$C2_\mu{}^{\lambda\nu} = \nabla_\rho\left(\tilde{R}_\mu{}^{\nu\lambda\rho} - \tilde{R}_\mu{}^{\rho\lambda\nu}\right) + K^\lambda{}_{\sigma\rho}\left(\tilde{R}_\mu{}^{\nu\sigma\rho} - \tilde{R}_\mu{}^{\rho\sigma\nu}\right) - K^\sigma{}_{\mu\rho}\left(\tilde{R}_\sigma{}^{\nu\lambda\rho} - \tilde{R}_\sigma{}^{\rho\lambda\nu}\right)\,, \tag{A.14}$$

$$C3_\mu{}^{\lambda\nu} = \nabla_\rho\tilde{R}^{\rho\nu\lambda}{}_\mu + K^\lambda{}_{\sigma\rho}\tilde{R}^{\rho\nu\sigma}{}_\mu - K^\sigma{}_{\mu\rho}\tilde{R}^{\rho\nu\lambda}{}_\sigma\,, \tag{A.15}$$

$$Y1_\mu{}^{\lambda\nu} = \delta_\mu{}^\nu\nabla_\rho\tilde{R}^{\lambda\rho} - \nabla_\mu\tilde{R}^{\lambda\nu} + \delta_\mu{}^\nu K^\lambda{}_{\sigma\rho}\tilde{R}^{\sigma\rho} + K^\rho{}_{\mu\rho}\tilde{R}^{\lambda\nu} - K^\nu{}_{\mu\rho}\tilde{R}^{\lambda\rho} - K^\lambda{}_{\rho\mu}\tilde{R}^{\rho\nu}\,, \tag{A.16}$$

$$Y2_\mu{}^{\lambda\nu} = \delta_\mu{}^\nu\nabla_\rho\tilde{R}^{\rho\lambda} - \nabla_\mu\tilde{R}^{\nu\lambda} + \delta_\mu{}^\nu K^\lambda{}_{\sigma\rho}\tilde{R}^{\rho\sigma} + K^\rho{}_{\mu\rho}\tilde{R}^{\nu\lambda} - K^\nu{}_{\mu\rho}\tilde{R}^{\rho\lambda} - K^\lambda{}_{\rho\mu}\tilde{R}^{\nu\rho}\,, \tag{A.17}$$

$$Z1_\mu{}^{\lambda\nu} = 4T^{\lambda\nu}{}_\mu\,, \tag{A.18}$$

$$Z2_{\mu}{}^{\lambda\nu} = 2\left(T^{\nu\lambda}{}_{\mu} - T^{\lambda\nu}{}_{\mu}\right) , \tag{A.19}$$

$$Z3_{\mu}{}^{\lambda\nu} = g^{\lambda\nu}T^{\rho}{}_{\rho\mu} - \delta_{\mu}{}^{\nu}g^{\lambda\sigma}T^{\rho}{}_{\rho\sigma} . \tag{A.20}$$

Appendix B

Torsion and curvature collineations

The introduction of additional degrees of freedom into the affine connection demands a generalization of the standard symmetry conditions, in order to extend this notion to the whole geometric structure provided by the new gravitational framework. In particular, it is possible to reach a fundamental symmetry constraint involving the torsion field, similar to the one existing in Riemannian geometry for the metric tensor.

Let be $W^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n}$ an arbitrary world tensor defined within a RC manifold. Then, it is possible to construct its Lie derivative in the direction of a Killing vector ξ in terms of the LC connection as follows:

$$\begin{aligned} \mathcal{L}_\xi W^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} &= W^{\mu_1 \cdots \mu_m}{}_{\lambda \cdots \nu_n} \nabla_{\nu_1} \xi^\lambda + \cdots + W^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \lambda} \nabla_{\nu_n} \xi^\lambda \\ &- W^{\lambda \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} \nabla_\lambda \xi^{\mu_1} - \cdots - W^{\mu_1 \cdots \lambda}{}_{\nu_1 \cdots \nu_n} \nabla_\lambda \xi^{\mu_m} \\ &+ \xi^\lambda \nabla_\lambda W^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} \,. \end{aligned} \tag{B.1}$$

It is straightforward to note that the torsion-free covariant derivative and the Lie derivative commute when the latter is applied with respect to an arbitrary Killing vector ξ. Indeed, the respective commutator acting on a general world tensor can be easily computed in the following way:

$$[\nabla_\rho, \mathcal{L}_\xi] \, W^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} = \nabla_\rho \, \mathcal{L}_\xi W^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} - \mathcal{L}_\xi \nabla_\rho W^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} \,, \tag{B.2}$$

where:

$$\begin{aligned}
\nabla_\rho \mathcal{L}_\xi W^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n} &= \nabla_\rho W^{\mu_1\cdots\mu_m}{}_{\lambda\cdots\nu_n}\nabla_{\nu_1}\xi^\lambda + \cdots + \nabla_\rho W^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\lambda}\nabla_{\nu_n}\xi^\lambda \\
&- \nabla_\rho W^{\lambda\cdots\mu_m}{}_{\nu_1\cdots\nu_n}\nabla_\lambda\xi^{\mu_1} - \cdots - \nabla_\rho W^{\mu_1\cdots\lambda}{}_{\nu_1\cdots\nu_n}\nabla_\lambda\xi^{\mu_m} \\
&+ W^{\mu_1\cdots\mu_m}{}_{\lambda\cdots\nu_n}\nabla_\rho\nabla_{\nu_1}\xi^\lambda + \cdots + W^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\lambda}\nabla_\rho\nabla_{\nu_n}\xi^\lambda \\
&- W^{\lambda\cdots\mu_m}{}_{\nu_1\cdots\nu_n}\nabla_\rho\nabla_\lambda\xi^{\mu_1} - \cdots - W^{\mu_1\cdots\lambda}{}_{\nu_1\cdots\nu_n}\nabla_\rho\nabla_\lambda\xi^{\mu_m} \\
&+ \nabla_\rho\,\xi^\lambda\nabla_\lambda W^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n} + \xi^\lambda\nabla_\rho\nabla_\lambda W^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n}\,, \qquad \text{(B.3)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_\xi \nabla_\rho W^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n} &= \nabla_\rho W^{\mu_1\cdots\mu_m}{}_{\lambda\cdots\nu_n}\nabla_{\nu_1}\xi^\lambda + \cdots + \nabla_\rho W^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\lambda}\nabla_{\nu_n}\xi^\lambda \\
&- \nabla_\rho W^{\lambda\cdots\mu_m}{}_{\nu_1\cdots\nu_n}\nabla_\lambda\xi^{\mu_1} - \cdots - \nabla_\rho W^{\mu_1\cdots\lambda}{}_{\nu_1\cdots\nu_n}\nabla_\lambda\xi^{\mu_m} \\
&+ \xi^\lambda\nabla_\lambda\nabla_\rho W^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n} + \nabla_\lambda W^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n}\nabla_\rho\,\xi^\lambda\,. \qquad \text{(B.4)}
\end{aligned}$$

Thus, this quantity acquires a very compact form:

$$\begin{aligned}
\left[\nabla_\rho, \mathcal{L}_\xi\right] W^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n} &= W^{\mu_1\cdots\mu_m}{}_{\lambda\cdots\nu_n}\nabla_\rho\nabla_{\nu_1}\xi^\lambda + \cdots + W^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\lambda}\nabla_\rho\nabla_{\nu_n}\xi^\lambda \\
&- W^{\lambda\cdots\mu_m}{}_{\nu_1\cdots\nu_n}\nabla_\rho\nabla_\lambda\xi^{\mu_1} - \cdots - W^{\mu_1\cdots\lambda}{}_{\nu_1\cdots\nu_n}\nabla_\rho\nabla_\lambda\xi^{\mu_m} \\
&+ \xi^\lambda\left[\nabla_\rho, \nabla_\lambda\right] W^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n}\,, \qquad \text{(B.5)}
\end{aligned}$$

with:

$$\begin{aligned}
\left[\nabla_\rho, \nabla_\lambda\right] W^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n} &= R^{\mu_1}{}_{\sigma\rho\lambda}W^{\sigma\cdots\mu_m}{}_{\nu_1\cdots\nu_n} + \cdots + R^{\mu_m}{}_{\sigma\rho\lambda}W^{\mu_1\cdots\sigma}{}_{\nu_1\cdots\nu_n} \\
&- R^{\sigma}{}_{\nu_1\rho\lambda}W^{\mu_1\cdots\mu_m}{}_{\sigma\cdots\nu_n} - \cdots - R^{\sigma}{}_{\nu_n\rho\lambda}W^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\sigma}\,. \qquad \text{(B.6)}
\end{aligned}$$

Likewise, the Ricci identity for a Killing vector ξ takes the following simple form:

$$\nabla_\rho\nabla_\sigma\xi^{\mu_k} = \xi^\lambda R^{\mu_k}{}_{\sigma\rho\lambda}\,, \qquad \text{(B.7)}$$

$$\nabla_\rho\nabla_{\nu_l}\xi^{\sigma} = \xi^\lambda R^{\sigma}{}_{\nu_l\rho\lambda}\,, \qquad \text{(B.8)}$$

for all $k = 1, ..., m$ and $l = 1, ..., n$, which means the vanishing of the commutator above.

Thereby, by taking the Lie derivative of Expression (B.6), it turns out that the Riemann tensor is only preserved in the direction of Killing fields (i.e. the vanishing of the Lie derivative of the metric tensor implies the vanishing of the Lie derivative of the Riemann tensor):

$$\mathcal{L}_\xi R^\lambda{}_{\rho\mu\nu} = 0\,. \tag{B.9}$$

These properties can be easily extended to the case of quantities depending on torsion. Specifically, the notion of a general covariant derivative endowed with torsion requires that the latter satisfies the same symmetry condition as the metric tensor:

$$\mathcal{L}_\xi T^\lambda{}_{\mu\nu} = 0\,, \tag{B.10}$$

in order to maintain the corresponding commutation relations:

$$\left[\tilde{\nabla}_\rho, \mathcal{L}_\xi\right] = 0\,. \tag{B.11}$$

Therefore, the application of the Lie derivative to the generalized commutation relations of covariant derivatives within a RC manifold:

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\, v^\lambda = \tilde{R}^\lambda{}_{\rho\mu\nu}\, v^\rho + T^\rho{}_{\mu\nu}\, \tilde{\nabla}_\rho v^\lambda\,, \tag{B.12}$$

involves the preserving of the RC curvature along an arbitrary Killing vector provided the fulfillment of the condition (B.10):

$$\mathcal{L}_\xi \tilde{R}^\lambda{}_{\rho\mu\nu} = 0\,. \tag{B.13}$$

Appendix C

SU(2) gauge connection in static and spherically symmetric space-times

In the context of EYM theory, the gauge condition $\partial_\mu \omega - i\left[A_\mu, \omega\right] = \mathcal{L}_\xi A_\mu$ constitutes a strong symmetry constraint to simplify the expression of the gauge connection, especially when it concerns gravitational systems and fields endowed with a high degree of symmetry. In this sense, the particular case of non-Abelian $SU(2)$ fields coupled to a four-dimensional static and spherically symmetric space-time represents a fundamental configuration where such a constraint can be fulfilled.

Hence, we start from the expression of the line element:

$$ds^2 = \Psi_1(r)\, dt^2 - \frac{dr^2}{\Psi_2(r)} - r^2 \left(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2\right) , \tag{C.1}$$

and the respective family of Killing vectors satisfying the Lie algebra of the rotation group $SO(3)$:

$$\xi^\mu_{(1)} = \cos\theta_2 \delta^\mu_{\theta_1} - \cot\theta_1 \sin\theta_2 \delta^\mu_{\theta_2} , \tag{C.2}$$

$$\xi^\mu_{(2)} = -\sin\theta_2 \delta^\mu_{\theta_1} - \cot\theta_1 \cos\theta_2 \delta^\mu_{\theta_2} , \tag{C.3}$$

$$\xi^\mu_{(3)} = \delta^\mu_{\theta_2} . \tag{C.4}$$

Let $\eta = \left[\xi_{(m)}, \xi_{(n)}\right]$ be the resulting Killing field from the commutation relations of the Killing vectors above (i.e. $\eta = \epsilon_{mn}{}^{p}\,\xi_{(p)}$, with $m, n, p = 1, 2, 3$). Then, the original gauge condition can be expressed in the following way:

$$\epsilon_{mn}{}^{p}\left(\partial_\mu \omega_{(p)} - i\left[A_\mu, \omega_{(p)}\right]\right) = \mathcal{L}_{\left[\xi_{(m)},\xi_{(n)}\right]} A_\mu\,, \tag{C.5}$$

where:

$$\mathcal{L}_\eta A_\mu = \mathcal{L}_{\xi_{(m)}}\left(\partial_\mu \omega_{(n)} - i\left[A_\mu, \omega_{(n)}\right]\right) - \mathcal{L}_{\xi_{(n)}}\left(\partial_\mu \omega_{(m)} - i\left[A_\mu, \omega_{(m)}\right]\right)\,. \tag{C.6}$$

By expanding and rearranging terms, it is straightforward to obtain the following consistency constraint involving the gauge variables $\omega_{(m)} = \omega^i_{(m)}\tau_i$ alone:

$$\mathcal{L}_{\xi_{(m)}}\omega_{(n)} - \mathcal{L}_{\xi_{(n)}}\omega_{(m)} - i\left[\omega_{(m)}, \omega_{(n)}\right] - \epsilon_{mn}{}^{p}\omega_{(p)} = 0\,. \tag{C.7}$$

As is shown, these variables take values in the Lie algebra, so that it is possible to impose a general transformation law for these quantities under gauge transformations $S \in SU(2)$:

$$\omega_{(m)} \to \omega'_{(m)} = S^{-1}\omega_{(m)}S + i\xi^\mu_{(m)}S^{-1}\partial_\mu S\,. \tag{C.8}$$

Indeed, this transformation rule preserves the symmetry gauge condition, which means that every pair $\left(A_\mu, \omega_{(m)}\right)$ that is a solution of the mentioned expression can be trivially changed to another pair $\left(A'_\mu, \omega'_{(m)}\right)$ by the action of S and still satisfy this condition:

$$S^{-1}\left(\partial_\mu \omega_{(m)} - i\left[A_\mu, \omega_{(m)}\right]\right)S = S^{-1}\mathcal{L}_{\xi_{(m)}}A_\mu S\,. \tag{C.9}$$

Therefore, it is possible to consider $\omega_{(3)} = \Phi(\theta_2)f^i(r,\theta_1)\,\tau_i$ and to perform a gauge transformation $S_1 = e^{if^i(r,\theta_1)\,\tau_i\int\Phi(\theta_2)d\theta_2}$ without modifying our general requirements, which implies the vanishing of this component and the consequent conservation rule for the potential:

$$\partial_{\theta_2}A_\mu = 0\,. \tag{C.10}$$

In addition, Equation (C.7) allows us to find the rest of the components, which must fulfill the following system of equations:

$$\partial_{\theta_2}\omega_{(1)} - \omega_{(2)} = 0\,, \tag{C.11}$$

$$\partial_{\theta_2}\omega_{(2)} + \omega_{(1)} = 0\,, \tag{C.12}$$

$$\begin{aligned}\left[\omega_{(1)}, \omega_{(2)}\right] &= \cos\theta_2\partial_{\theta_1}\omega_{(2)} + \sin\theta_2\partial_{\theta_1}\omega_{(1)} \\ &+ \cot\theta_1\left(\cos\theta_2\partial_{\theta_2}\omega_{(1)} - \sin\theta_2\partial_{\theta_2}\omega_{(2)}\right)\,. \end{aligned}\tag{C.13}$$

It is straightforward to note that the solution of Equations (C.11) and (C.12) can be expressed as follows:

$$\omega_{(1)} = X(\theta_1)\cos\theta_2 + Y(\theta_1)\sin\theta_2\,, \tag{C.14}$$

$$\omega_{(2)} = Y(\theta_1)\cos\theta_2 - X(\theta_1)\sin\theta_2\,, \tag{C.15}$$

where X and Y are arbitrary functions, defined on the Lie algebra, whose possible dependence on the coordinate r has been omitted. Such a restriction simplifies the problem notably and it is compatible, as outlined below, with the existence of an ansatz solution for the gauge connection.

In general, the gauge transformation rules (C.8) fix the respective transformations of the mentioned functions:

$$X \to X^{'} = S^{-1}XS + iS^{-1}\partial_{\theta_1}S\,, \tag{C.16}$$

$$Y \to Y^{'} = S^{-1}YS - i\cot\theta_1 S^{-1}\partial_{\theta_2}S\,. \tag{C.17}$$

Once again, it is possible to consider X as a function independent of θ_1 and to apply a new gauge transformation $S_2 = e^{iX\theta_1}$, which involves the vanishing of this function. Then, we are led to deal only with the function Y; indeed, because of our previous choice to vanish $\omega_{(3)}$ it is not possible to perform a new gauge transformation depending on θ_2 which cancels this function simultaneously.

Therefore, Equation (C.13) reduces to the following differential condition:

$$\frac{dY(\theta_1)}{d\theta_1} + Y(\theta_1)\cot\theta_1 = 0\,, \tag{C.18}$$

whose general solution can be written in the following way:

$$Y(\theta_1) = \frac{c^i\tau_i}{\sin\theta_1}\,, \tag{C.19}$$

with c^i three integration constants, which can also be simplified by the application of an additional gauge transformation $S_3 = e^{i\left((c^2/c^3)\tau_1 - (c^1/c^3)\tau_2\right)}$, in order to cancel c^1 and c^2.

In summary, our analyses lead to the following structure for the gauge variables:

$$\omega_{(1)} = \frac{\sin\theta_2}{\sin\theta_1}\, c^3 \tau_3\,, \tag{C.20}$$

$$\omega_{(2)} = \frac{\cos\theta_2}{\sin\theta_1}\, c^3 \tau_3\,, \tag{C.21}$$

$$\omega_{(3)} = 0\,. \tag{C.22}$$

By substituting these factors in the general symmetry condition for the gauge connection:

$$c^i \partial_\mu \left(\frac{\sin\theta_2}{\sin\theta_1}\right) + \epsilon^i{}_{jk} A^j_\mu c^k \left(\frac{\sin\theta_2}{\sin\theta_1}\right) = \partial_{\theta_1} A^i_\mu \cos\theta_2 + A^i_{\theta_1} \partial_\mu \cos\theta_2 - A^i_{\theta_2} \partial_\mu \left(\cot\theta_1 \sin\theta_2\right)\,, \tag{C.23}$$

$$c^i \partial_\mu \left(\frac{\cos\theta_2}{\sin\theta_1}\right) + \epsilon^i{}_{jk} A^j_\mu c^k \left(\frac{\cos\theta_2}{\sin\theta_1}\right) = -\,\partial_{\theta_1} A^i_\mu \sin\theta_2 - A^i_{\theta_1} \partial_\mu \sin\theta_2 - A^i_{\theta_2} \partial_\mu \left(\cot\theta_1 \cos\theta_2\right)\,. \tag{C.24}$$

Thus, the resulting system of equations can be trivially solved and it shows that, in the present gauge, the YM connection is described by the following ansatz:

$$\begin{aligned} A &= p(r)\,\tau_3\, dt + u(r)\,\tau_3\, dr + \left(v(r)\,\tau_1 + w(r)\,\tau_2\right) d\theta_1 \\ &+ \left(\cot\theta_1\, \tau_3 + v(r)\,\tau_2 - w(r)\,\tau_1\right) \sin\theta_1 d\theta_2\,, \end{aligned} \tag{C.25}$$

where p, u, v and w are four arbitrary functions depending on the coordinate r.

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