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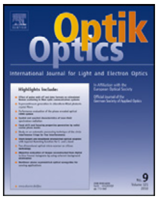

A simple depolarization criterion for light

R. Espinosa-Luna [a,*], E. Bernabeu [a,b], G. Atondo-Rubio [c], S. Hinojosa-Ruíz [d]

[a] Centro de Investigaciones en Óptica, A.C., Loma del Bosque 115, Colonia Lomas del Campestre, 37150 León, Guanajuato, Mexico
[b] Departamento de Óptica, Universidad Complutense de Madrid, Facultad de Ciencias Físicas, Ciudad Universitaria s/n, 28040 Madrid, Spain
[c] Escuela de Ciencias Físico-Matemáticas, Universidad Autónoma de Sinaloa, Ciudad Universitaria s/n, 80010 Culiacán, Sinaloa, Mexico
[d] Facultad de Física, Universidad Autónoma de Zacatecas, Avenida Preparatoria 301, Fraccionamiento Progreso, 98060 Zacatecas, Zacatecas, Mexico

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ABSTRACT

A simple depolarization criterion for light is proposed. This criterion is based on the depolarization part derived from the degree of polarization formulation. Some reported Mueller matrices are employed to test its reliability and usefulness. Results prove that the criterion proposed can be employed as the first step to test the physical consistency of Mueller matrices.

1. Introduction

The light emerging S^o of an optical system for an incident Stokes vector S^i can be expressed in terms of intensities, through the relation [1]

$$S^o = MS^i \Rightarrow \begin{pmatrix} s_0^o \\ s_1^o \\ s_2^o \\ s_3^o \end{pmatrix} = \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{pmatrix} s_0^i \\ s_1^i \\ s_2^i \\ s_3^i \end{pmatrix}$$

$$= \begin{pmatrix} m_{00}s_0^i + m_{01}s_1^i + m_{02}s_2^i + m_{03}s_3^i \\ m_{10}s_0^i + m_{11}s_1^i + m_{12}s_2^i + m_{13}s_3^i \\ m_{20}s_0^i + m_{21}s_1^i + m_{22}s_2^i + m_{23}s_3^i \\ m_{30}s_0^i + m_{31}s_1^i + m_{32}s_2^i + m_{33}s_3^i \end{pmatrix} \qquad (1)$$

where, M is named the Mueller matrix of the system, represented as a 4×4 matrix of real elements, and S is the Stokes vector. S represents any polarization state of light, including partial and unpolarized light. It is defined in terms of the orthogonal components of the electric field vector (E_p, E_s) as [1]

$$S = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} \left\langle E_p E_p^* \right\rangle + \left\langle E_s E_s^* \right\rangle \\ \left\langle E_p E_p^* \right\rangle - \left\langle E_s E_s^* \right\rangle \\ \left\langle E_p E_s^* \right\rangle + \left\langle E_s E_p^* \right\rangle \\ \pm i \left(\left\langle E_p E_s^* \right\rangle - \left\langle E_s E_p^* \right\rangle \right) \end{pmatrix}, \qquad (2a)$$

where the angular brackets represent temporal averages, * indicates complex conjugation, the upper (lower) sign in the right hand side of s_3 corresponds to a description of polarization states as looking to the source (propagation direction) and its factor, i, represents the imaginary number $\left(i^2 = -1\right)$. The Stokes vectors can also be written in terms of the azimuthal $\left(0 \le \psi \le \pi\right)$ and the ellipticity $\left(-\pi/4 \le \chi \le \pi/4\right)$ angles of the polarization ellipse of the wave, respectively [1]:

$$S = \langle s_0 \rangle \begin{pmatrix} 1 \\ \cos(2\chi)\cos\left(2\psi\right) \\ \cos(2\chi)\sin\left(2\psi\right) \\ \sin(2\chi) \end{pmatrix} \qquad (2b)$$

An optical system can modify the polarization state of the incident light in several ways. For example, a totally polarized incident beam of light can emerge as totally polarized, partially polarized (partially depolarized) or as totally depolarized. In principle, the same possibilities exist for output light when partially or totally unpolarized incident light is employed in the appropriate optical

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[*] Corresponding author.
E-mail address: reluna@cio.mx (R. Espinosa-Luna).

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systems. In this work, attention will be focused only to the case where totally polarized light is employed for the incidence and to systems which partially depolarize light.

In this sense, a very important property associated to an optical system is its capability to depolarize light, which is measured by using some of the following depolarization scalar metrics. The depolarization index *DI*(*M*) and its physical realizable limits are defined by [2,3]:

$$0 \le DI(M) = \frac{\left\{\sum_{j,k=0}^{3} m_{jk}^2 - m_{00}^2\right\}^{1/2}}{\sqrt{3}m_{00}} \le 1 \tag{3}$$

DI(*M*) is directly related to the Mueller matrix elements only. This means this metric can be applied only to the Mueller matrix elements associated to the optical system under study and not to the outgoing beam of light emerging from the system under consideration.

The degree of polarization, *DoP*(*M*,*S*), and its physical realizable limits have been defined by [4–6]:

$$0 \le DoP(M,S) = \frac{\sqrt{(s_1^o)^2 + (s_2^o)^2 + (s_3^o)^2}}{s_0^o} = \frac{\left[\sum_{j=1}^{3}(m_{j0}s_0^i + m_{j1}s_1^i + m_{j2}s_2^i + m_{j3}s_3^i)^2\right]^{\frac{1}{2}}}{m_{00}s_0^i + m_{01}s_1^i + m_{02}s_2^i + m_{03}s_3^i} \le 1. \tag{4}$$

DoP(*M*,*S*) is directly related to both, the Mueller matrix elements of the system under study and the incident Stokes vector. The *DoP*(*M*,*S*) usually is measured directly from the Stokes vector emerging of the system and the measured value is associated to the outgoing light; however, it is inherently related to the optical response of the system, as can be noted from Eq. (4). The upper limit associated to both, *DI*(*M*) and *DoP*(*M*,*S*), means the optical system does not depolarize, the lower limit is associated to total depolarization, and the medium limits correspond to partial depolarization.

The diattenuation, *D*(*M*), and the polarizance parameters, *P*(*M*), are defined by [7]

$$0 \le D(M) = \frac{\sqrt{m_{01}^2 + m_{02}^2 + m_{03}^2}}{m_{00}} \le 1 \tag{5}$$

and

$$0 \le P(M) = \frac{\sqrt{m_{10}^2 + m_{20}^2 + m_{30}^2}}{m_{00}} \le 1 \tag{6}$$

respectively.

The *Q*(*M*) metric and its physical realizable bounds are defined as [8,9]

$$0 \le Q(M) = \frac{\sum_{j=1,k=0}^{3} m_{jk}^2}{\sum_{k=0}^{3} m_{0k}^2} = \frac{3[DI(M)]^2 - [D(M)]^2}{1 + [D(M)]^2} = \frac{\left\{\sum_{j,k=1}^{3} m_{jk}^2\right\}/m_{00}^2 + [P(M)]^2}{1 + [D(M)]^2} \le 3, \tag{7}$$

where $Q(M) = 0$ for a totally depolarizing optical system; $0 < Q(M) < 1$ for a partially depolarizing optical system; if $1 \le Q(M) < 3$ and $0 < DI(M) < 1$ the system partially depolarizes also, but if $DI(M) = 1$, it is a non-depolarizing diattenuating optical system; and $Q(M) = 3$ for a non-depolarizing non-diattenuating optical system, respectively [8,9].

All the previous scalar metrics ensure an unphysical sense to values outside the limits considered therein. That unphysical sense can be associated to several factors like noise or calibration errors for experimentally determined Mueller matrices, among many other factors. In the case of theoretically or numerically determined Mueller matrices, the unphysical sense may be associated to unphysical considerations in the model employed or to average mistakes, for example. In practice, it is very important to test when a Mueller matrix has been correctly determined before going far away with specialized analysis techniques like the polar decomposition [7]. In this work we propose a simple, direct criterion to test the physical realizability of Mueller matrices subject to be analyzed by powerful, time-demanding, techniques.

2. Mathematical model

It has been established that the Stokes parameters are related through the relation [1,4]:

$$s_0^2 \ge s_1^2 + s_2^2 + s_3^2, \tag{8a}$$

where for a perfectly polarized beam of light [1,4]

$$s_0^2 = s_1^2 + s_2^2 + s_3^2, \tag{8b}$$

for a perfectly unpolarized light [1,4]

$$s_0^2 > 0 \quad \text{and} \quad s_1 = s_2 = s_3 = 0, \tag{8c}$$

and for a partially polarized light [1,4]

$$s_0^2 > s_1^2 + s_2^2 + s_3^2, \tag{8d}$$

Eq. (8) are based in the premise that the optical field, represented by Stokes vectors, can be decomposed into unpolarized and polarized independent portions, respectively [1,4]:

$$S = S^{\text{un}} + S^{\text{pol}} \tag{9a}$$

where

$$S^{un} = \begin{pmatrix} s_0 - \sqrt{s_1^2 + s_2^2 + s_3^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{9b}$$

is the unpolarized portion, and the polarized part is given as

$$S^{pol} = \begin{pmatrix} \sqrt{s_1^2 + s_2^2 + s_3^2} \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} \tag{9c}$$

The degree of polarization of the original wave is [1,4]:

$$0 \le DoP = \frac{I_{\text{pol}}}{I_{\text{tot}}} = \frac{\sqrt{s_1^2 + s_2^2 + s_3^2}}{s_0} \le 1 \tag{10}$$

This treatment is universally applied just as expressed by Eqs. (9) and (10). Observe that Eq. (10) can be written explicitly as

$$s_0^2 = s_1^2 + s_2^2 + s_3^2 + d^2, \tag{11a}$$

or

$$d^2 = s_0^2 - s_1^2 - s_2^2 - s_3^2 \tag{11b}$$

In this work, we propose the following limits for Eq. (11b)

$$d^2 = \begin{cases} 0 \text{ for total polarization} \\ > 0 \text{ for total or partial depolarization} \\ < 0 \text{ for non-physical sense (erroneous calculus/measurement)} \end{cases}. \tag{11c}$$

By using Eq. (2a), it can be easily verified that Eq. (11a) also implies that

$$4\left\langle E_p E_p^* \right\rangle \left\langle E_s E_s^* \right\rangle = 4\left\langle E_p E_s^* \right\rangle \left\langle E_s E_p^* \right\rangle + d^2 \tag{12a}$$

or

$$a^2 = \left\langle E_p E_p^* \right\rangle \left\langle E_s E_s^* \right\rangle - \left\langle E_p E_s^* \right\rangle \left\langle E_s E_p^* \right\rangle \geq 0. \tag{12b}$$

Observe that $a^2 = d^2/4$ must be associated to a depolarization term in the classical Stokes formalism (a and d are constant numbers). Note also that Eq. (12b) is fulfilled for any totally polarized state, since $d^2 = 4a^2 = 0$. This can be easily proved for a monochromatic plane wave:

$$\begin{aligned} a^2 = \left\langle E_p E_p^* \right\rangle \left\langle E_s E_s^* \right\rangle - \left\langle E_p E_s^* \right\rangle \left\langle E_s E_p^* \right\rangle = \frac{1}{T}\int_0^T E_p \\ \times \exp\left[i(k\cdot r - \omega t)\right] E_p \exp\left[-i(k\cdot r-\omega t)\right] \\ \times dt \frac{1}{T}\int_0^T E_s \exp\left[i\left(k\cdot r - \omega t + \delta\right)\right] E_s \exp\left[-i\left(k\cdot r - \omega t + \delta\right)\right] \\ \times dt - \frac{1}{T}\int_0^T E_p \exp\left[i(k\cdot r-\omega t)\right] E_s \exp\left[-i\left(k\cdot r - \omega t+\delta\right)\right] \\ \times dt \frac{1}{T}\int_0^T E_s \exp\left[i\left(k\cdot r - \omega t + \delta\right)\right] E_p \exp\left[-i(k\cdot r - \omega t)\right] dt = 0 \end{aligned} \tag{13}$$

where δ is any arbitrary spatial phase difference between the orthogonal components of the electric field vector.

Note also the following relation holds

$$4 \left\langle E_p E_p^* \right\rangle \left\langle E_s E_s^* \right\rangle \geq s_2^2 + s_3^2 \tag{14}$$

where the equality is fulfilled for a totally polarized beam of light.

Observe that by illuminating an optical system with totally polarized Stokes vectors, Eq. (11) can be employed as a simple, direct criterion to test the physical consistency of determined Mueller matrices.

3. Results

To exemplify the benefits of the criterion proposed here, a graphical analysis of the depolarized component associated to some reported Mueller matrices is presented. A totally polarized incident beam of light has been considered in all the cases reported by the original authors.

Example 1. Experimental calibration Mueller matrix for a rotating retarder polarimeter [1]

$$\begin{bmatrix} 0.978 & 0 & 0.003 & 0.005 \\ 0 & 1.000 & -0.007 & 0.006 \\ 0 & 0.007 & 0.999 & -0.007 \\ 0.005 & -0.003 & -0.002 & 0.994 \end{bmatrix} \tag{15}$$

By applying the depolarizing scalar metrics, Eqs. (3)–(15), results: $DI(M) = 1.020$, $DoP(M, S_M) \approx 1.025$, $D(M) = 0.006$, $P(M) = 0.005$, and $Q(M) = 3.122$, where S_M is the incident Stokes vector that maximizes the degree of polarization. The physical meaning associated to the depolarization scalar metrics, Eqs. (3) and (4) and Eq. (7), is just that Eq. (15) corresponds to an unphysically consistent system. The same conclusion has been obtained for this system by considering the existence of negative eigenvalues for the corresponding coherency matrix [1]: $\left[1.986 \quad -0.016 \quad -0.007 \quad -0.005\right]$, where a more complexity is required. Now, we show the same conclusion can be obtained also through Eq. (11) by calculating the depolarizing component. The depolarizing component, Eq. (11b), can be represented graphically by scanning over all the incident Stokes vectors on the Poincaré sphere to calculate the output polarized Stokes vectors and by applying Eq. (2b), see Fig. 1. Note that the depolarizing component has negative values, which is interpreted in this work as a non-physical or non-consistent Mueller matrix determination, Eq. (11c).

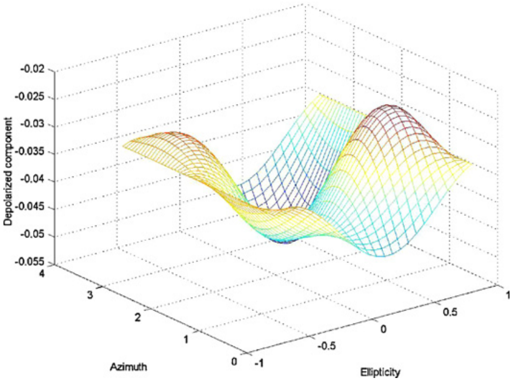

Fig. 1. Depolarized component as a function of the incident state of polarization parametrized by the ellipsometric angles χ and ψ. The negative values are associated to a non-consistent Mueller matrix determination, Eq. (15).

Example 2. Mueller matrix associated to a biological tissue (vegetal leave) [10]:

$$\begin{bmatrix} 1.0000 & 0.0269 & -0.0021 & -0.0018 \\ 0.0101 & 0.3236 & -0.0087 & -0.0023 \\ 0.0008 & -0.0024 & -0.3276 & 0.0009 \\ 0.0026 & 0.0023 & -0.0029 & -0.2754 \end{bmatrix} \tag{16}$$

By applying a similar procedure to Eq. (16), it can be proved that: $DI(M) = 0.3103$, $DoP(M, S_M) \approx 0.323$, $D(M) = 0.027$, $P(M) = 0.010$, and $Q(M) = 0.2879$. Now, the physical meaning associated to the depolarization scalar metrics, Eqs. (3), (4) and (7), is just that Eq. (16) corresponds to a physical consistent or realizable depolarizing system. The same conclusion has been obtained for this system by considering the existence of positive eigenvalues for the corresponding coherency matrix [10]: $\left[\approx 0.5 \quad \approx 0.19 \quad \approx 0.18 \quad \approx 0.15\right]$. Now, we show the same conclusion can be obtained also through Eq. (11b), by calculating the depolarizing component, see Fig. 2. In this case, the depolarizing component has positive values only, which is interpreted in this work as a physical or consistent Mueller matrix determination, in accordance with Eq. (11c).

It must be pointed out here that a powerful and beautiful graphical analysis has been reported previously by other authors, where

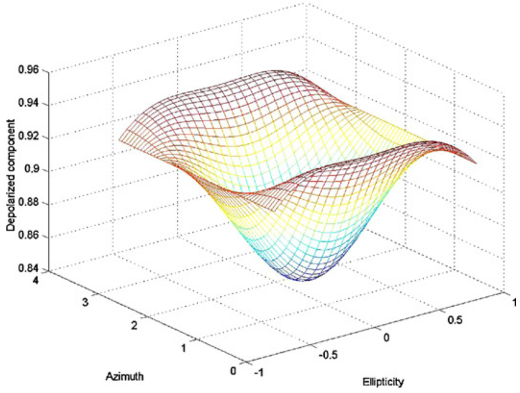

Fig. 2. Depolarized component as a function of the incident state of polarization parametrized by the ellipsometric angles χ and ψ. The positive values are associated to a consistent Mueller matrix determination, Eq. (16).

a lot of polarization surface and maps have been employed for the analysis of depolarization properties of Mueller matrices [5] and where totally polarized incident states have been considered. In the following lines, we will try to explain that the method presented here is analogous, but clearly simpler than the method reported in Ref. [5].

In that work, it has been pointed out that a graphical plot of the numerator surface must not to protrude from the denominator surface of *DoP* for any incident polarized state; otherwise the $DoP \leq 1$ would fail, meaning the associated Mueller matrix in unphysical [5]. This is interpreted in terms of the model we present here as $d^2 > 1$, according to Eq. (11c).

It is also noted there that for points of tangency between numerator and denominator surfaces, where the numerator equals the denominator, $DoP = 1$ [5]. This situation is equivalent to $d^2 = 0$, Eq. (11c).

Finally, when the numerator surface is contained inside the denominator surfaces, $DoP \leq 1$, then the associated Mueller matrix is physically consistent and depolarization occurs [5]. This condition is equivalent to $d^2 > 1$, Eq. (11c).

To our knowledge, the method presented here is valid at least for any physical system that depolarizes incident light totally polarized and for any normalized Mueller matrix.

4. Conclusions

A simple depolarization criterion for Mueller matrices has been proposed. This criterion has been based in the depolarization part derived from the degree of polarization formulation. Some reported Mueller matrices have been employed to test its reliability and usefulness. This method is simpler than other graphical method reported previously by other authors. Results proved that it can be employed as the first step to test the physical consistency of Mueller matrices.

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